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Compactness Via Proximity Spaces

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1444 AH

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

هُوَ الَّذِي خَلَقَ السَّمَاوَاتِ وَالْأَرْضَ فِي سِتَّةِ أَيَّامٍ ثُمَّ
اسْتَوَىٰ عَلَى الْعَرْشِ ۚ يَعْلَمُ مَا يَلْجُ فِي الْأَرْضِ وَمَا يَخْرُجُ
مِنْهَا وَمَا يَنْزِلُ مِنَ السَّمَاءِ وَمَا يَعْرُجُ فِيهَا ۖ وَهُوَ مَعَكُمْ
أَيْنَ مَا كُنْتُمْ ۚ وَاللَّهُ بِمَا تَعْمَلُونَ بَصِيرٌ

صدق الله العلي العظيم

سورة الحديد (4)

DEDICATION

To the symbol of sacrifice and giving

My father

To the source of kindness

My mother

To my second half

My wife and children

To my support in life

My brothers and sisters

I dedicate the fruit of my humble effort

Ghassan Adnan

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Praise be to God, Lord of the worlds, and prayers and peace be upon the most honorable of the prophets and messengers, our Master Muhammad, his family, his companions, and those who followed them with kindness until the Day of Judgment, and after...

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Ghassan Adnan

Approval of Scientific Supervisor

I certify that this dissertation “**Compactness Via Proximity Spaces**” by student “**Ghassan Adnan Qahtan**” was prepared under my supervision at the University of Babylon, Faculty of Education for Pure Sciences, in a partial fulfillment of the requirements for the degree of Doctor of Philosophy in Education / Mathematics.

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Abstract

This study aims to build new functions and spaces in the central topological spaces through proximity theory, and to study all the topological concepts that can be studied in it. As well as the relationships between those spaces within this space. Where some basic concepts of proximity theory were reviewed, as well as the concept of C -set and the algebraic properties of that set and the construction of central topological spaces, which represents the first stage of this work.

The second stage: The concept of function in the central topological spaces was introduced by studying their mathematical properties as well as the concept of continuity and equivalence and the effect of those functions on the central axioms of the class. Several functions have been identified, including:

- Shrink central function
- Reduction central function
- Stronger central function
- Weakly central function

The classification of the above functions was adopted according to the presence of proximity spaces in the domain and the corresponding domain, where the first and second class were defined assuming that the function was defined from the proximity space to the proximity space, while the third class was defined from the proximity space to a set whose relationship was defined through the domain relationship, while the last class was defined from a set to the proximity space and the relationship was defined in the same manner as the third category.

The third stage: The concept of C -compact spaces and some classifications of that space were introduced (LC - compact space, AC - compact space and HC - closed space).

The fourth stage: The concept of C - lindelof spaces and some classifications of that space were introduced (LC - lindelof spaces, AC - lindelof spaces and HLC - closed spaces).

The five stage: The concept of C - countable compact spaces and some classifications of that space were introduced (LC - countable compact space, AC - countable compact space and HC - countable closed space).

In the last three stages, the topological properties were studied by means of the functions constructed in the first stage, the heredity properties of those spaces, the relationship of those spaces to the axioms of central separation and the mutual influence between them, as well as the union and intersection of those spaces.

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List of symbols

Symbols	Description
\mathcal{C}_A	Center set
\mathcal{C}_X	Universal center
\mathcal{C}_\emptyset	Empty center
\mathcal{C}_X^A	$\{\langle X, B \rangle : \langle A, B \rangle \in \mathcal{C}_A\}$
$\mathcal{C}_X^{(A)}$	Complements center of \mathcal{C}_A
\leq_c	Center subsets
$=_c$	Center equal to
\neq_c	Not center equal
$\mathbb{P}_c(X)$	Family of all center set of (X, δ)
γ_c	Union center
λ_c	Intersection center
$cl_c(\mathcal{C}_A)$	Center closure of \mathcal{C}_A
$int_c(\mathcal{C}_A)$	Center interior of \mathcal{C}_A
$A\bar{\delta}B$	A is not related with B

Introduction

Introduction

The first pillar of the study presented in this thesis is the use of the concept of proximity relation, this type of relationship was identified at the beginning of the last century by F. Riesz (1909) [10], It was rediscovered and axiomatized by V. A. Efremovič(1934) [9], under the name of infinitesimal space , Where he used the concepts of proximity space to put forward his theory known as “theory of enchainment” which did not see the light and does not realize the development at that time.

Where the idea of the concept of proximity crystallized through metric space, this is because the metric space (X, d) , we may define it as a relationship δ on $P(X)$ as follows:

$$A\delta B \text{ if and only if } D(A, B) = 0,$$

$$\text{Where } D(A, B) = \inf \{d(a, b): a \in A, b \in B\}.$$

This definition reflects a nearness between A and B . The concept of nearness was initiated by F. Riesz in the year 1908.

After a period of time, the definition of proximity neighbourhoods was provided by V. A. Efremovič [9], An equivalent set of axioms is obtained to obtain the proximity space. Proximity theory continued to develop rapidly by many researchers it has been translated into many research papers that have been directly and indirectly applied and in different fields. We review some lines of research in this field.

- In 1986, Künzi [7] studied the concept of proximity within a topological space by coarsest quasi-proximity. In the same year, a research paper was published by Mee-Kyoung [8], that quasi-proximity spaces that it causes quasi-uniformity It has the same feature and quasi-uniformity It is a structure that has the same closure in topological spaces.

- Also uses Latecki, Longin and Frank Prokop in 1995 [10], semi-proximity spaces To provide a relationship between topological concepts digital image processing in R^n .
- In 2019 , researcher Yiezi [12], using proximity spaces To extract new spaces called i -topological proximity spaces. Then the concept of focal focal functions was introduced through the concept of focal set.

The second and fundamental pillar of our work is the center topological spaces. In 2019, researchers D. A. Abdulsada, L. A. Al-Swidi, presented a concept for the first time, center set via proximity relation, through this concept, a new space called central topological spaces is built [1] . Several concepts were introduced within this space such as (\mathcal{C} -open, \mathcal{C} -closed, \mathcal{C} -neighborhood, \mathcal{C} -closure, \mathcal{C} -interior) sets [2], Where the same researchers studied the central separation axioms in center topological spaces such as ($\mathfrak{S}_{\mathcal{C}0}$ -space, $\mathfrak{S}_{\mathcal{C}1}$ -space, $\mathfrak{S}_{\mathcal{C}2}$ -space, \mathcal{C} -regular, \mathcal{C} -normal, $\mathfrak{S}_{\mathcal{C}3}$ -space and $\mathfrak{S}_{\mathcal{C}4}$ -space)[3,4,5].

This dissertation consists of five chapters:

In **chapter one**, the most important basics of proximity space are given, \mathcal{C} – set and \mathcal{C} – topological spaces, by presenting the most important algebraic and topological properties of these concepts.

Chapter two consists of four sections. In section one; we introduced the concept of sharing central function in proximity spaces. We studied the most important properties of these functions, continuity, and its relationship to the separation axioms in \mathcal{C} – topological spaces. In section two; we introduced the concept of reduction central function in proximity spaces. We studied the most important properties of these functions, continuity, and its relationship to the separation axioms in \mathcal{C} – topological spaces. For the third and fourth sections, we have introduced two types of functions called stronger central function and

weaker central function , in which these two functions are characterized by having a single relationship on either X or Y space. As in the first section, these functions and their continuity are studied in the central topological spaces.

Chapter three consists of four sections. In section one; we introduced the concept of \mathcal{C} – compact spaces in central topological spaces and studied their topological properties and their relationship to the central separation axioms. In the other sections of this chapter, we introduce the concepts (\mathcal{LC} – compact space, \mathcal{AC} – compact space and \mathcal{HC} – closed space) and the topological properties of each of these spaces.

In Chapter four, we give several characterizations of Center lindelofness, where the concept of \mathcal{C} – lindelof spaces within the central topological spaces was introduce and the topological properties of this space and its relationship to the axioms of separation were studied. Also in section two we introduce new concept, namely \mathcal{LC} - lindelof space and its relationship to the axioms of separation were studied. In section three and fore we introduce new concepts namely \mathcal{AC} – lindelof and \mathcal{HLC} – closed, and we study their properties.

Chapter five consists of four sections. In section one; we introduced the concept of \mathcal{C} – countable compact spaces in central topological spaces and studied their topological properties and their relationship to the central separation axioms. In the other sections of this chapter, we introduce the concepts (\mathcal{LC} – countable compact space, \mathcal{AC} – countable compact space and \mathcal{HC} – countable closed space) and the topological properties of each of these spaces.

CHAPTER

ONE

**Proximity Space &
Center topological
Spaces**

Chapter One: Center Set & \mathcal{C} -Topological space

In this chapter, we deal with the most important special mathematical tools in the process of building a thesis, as they are the mainstay of our work. The main aspect of the proximity relationship includes its definition, its most important characteristics and the construction processes that were the basic building block for building the concept of \mathcal{C} - set, and which was the starting point for the process of building modern concepts that was adopted in this thesis. The first chapter contains two parts:

The first topic deals with the concept of proximity space and its most important characteristics, theories and examples. The second topic dealt with the concept of \mathcal{C} - set and its algebraic properties, as well as the topological concepts through the concept of \mathcal{C} - topological spaces.

1.1 Proximity Space

In this part, we review the concept of proximity space and its basic properties, also give some examples and basic theories that we need in our work.

Definition 1.1.1[7].

A binary relation δ on the power set of X is called an Efremoviô proximity on X if and only if it satisfies the following axioms for each $A, B, C, E \subseteq X$:

- B1.** $A\delta B$ implies $B\delta A$;
- B2.** $(A \cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$;
- B3.** $A\delta B$ implies $A \neq \emptyset, B \neq \emptyset$;
- B4.** $A \cap B \neq \emptyset$ implies $A\delta B$.
- B5.** $A\bar{\delta}B$ implies that there exists a subset E such that $A\bar{\delta}E$ and $X - E\bar{\delta}B$;

The pair (X, δ) is called a proximity space. In addition, if δ also satisfies:

B6. $\forall x, y \in X: (\{x\} \delta \{y\} \text{ implies } x = y)$, then the pair (X, δ) is called a separated proximity space.

Note, strictly speaking one should use the notation $A \delta B$ ($(A, B) \in \delta$) or $A \bar{\delta} B$ ($(A, B) \notin \delta$), when the sets A and B are either near (δ) each other or far ($\bar{\delta}$) each other.

Let (X, δ) be a proximity space. For a subset A of X define $cl(A) = \{x \in X : \{x\} \delta A\}$. Then ' cl ' is a Kuratowski closure operator on X . The topology obtained by this operator is called the topology induced by δ , and is denoted by T_δ .

Note that for subsets A and B of a proximity space (X, δ) , $A \delta B$ if and only if $cl_{T_\delta}(A) \delta cl_{T_\delta}(B) \neq \emptyset$.

Definition 1.1.2 [7].

If on a set X , there is a topology T and a proximity δ such that $T = T_\delta$, then T and δ are said to be compatible.

Examples 1.1.3 .

- ❖ Let X be a non-empty set. For $A, B \in \mathbb{P}(X)$, $A \delta B$ if and only if $A \cap B \neq \emptyset$. δ is a separated proximity. The proximity defined in this way is called discrete proximity and (X, δ) called discrete proximity space.
- ❖ Let X be any non-empty set and define $A \delta B$ iff $A \neq \emptyset$ and $B \neq \emptyset$. δ is a proximity on X . If X contains two or more points, then δ is not a separated proximity. δ defined in this way is called the trivial proximity and (X, δ) called trivial proximity space.
- ❖ Let (X, d) be a pseudometric space. Define $A \delta B$ iff $d(A, B) = 0$,
Where, $d(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\}$. So δ is a proximity on (X, d) . δ defined in this way is called the pseudometric proximity and (X, d, δ) called pseudo-metric proximity space.

- ❖ Consider a normal space (X, T) . Define $A\delta B$ iff $cl_{T_\delta}(A) \cap cl_{T_\delta}(B) \neq \emptyset$. δ is a proximity space on X . δ defined in this way is called the normal proximity and (X, T, δ) called normal proximity space.

The following theorem shows the most important properties of the proximity relationship, which has an important mathematical content in the construction process.

Theorem 1.1.4[7]:

Let (X, δ) be a proximity space. Then

- (a) If $A\delta B$ and $B \subseteq C$, then $A\delta C$;
- (b) If $A\bar{\delta}B$ and $C \subseteq B$, then $A\bar{\delta}C$;
- (c) If there exists a point $x \in X$ such that $A\delta\{x\}$ and $\{x\}\delta B$, then $A\delta B$;
- (d) $A\bar{\delta}\emptyset$ for every $A \subseteq X$;

The axiom B5 has basic equivalents for the purpose of simplifying its concept, as shown in the following proposition.

Proposition 1.1.5[7]:

If δ is a proximity relation on a set X , then the axiom (B5) in (definition 1.1.1) is equivalent to each of the following statements:

- (B5') If $A\bar{\delta}B$, then there are sets C and D such that $A\bar{\delta}C$, $B\bar{\delta}D$ and $C \cup D = X$;
- (B5'') If $A\bar{\delta}B$, then there are sets C and D such that $A\bar{\delta}X - C$, $X - D\bar{\delta}B$ and $C\bar{\delta}D$;
- (B5''') If $A\bar{\delta}B$, then there are sets C and D such that $C \cap D = \emptyset$, $A\bar{\delta}X - C$ and $B\bar{\delta}X - D$.

Now we will review the definition of neighborhoods in proximity spaces and their properties and characteristics.

Definition 1.1.6 [7].

Let (X, δ) be a proximity space. We say that the sets $A, B \subset X$ are in the relation \ll and write $A \ll B$ if $A\bar{\delta}X - B$. When $A \ll B$, we call B a proximity or δ -neighborhood of A .

Neighborhoods have properties and we will notice that they are synonymous with neighborhoods properties in usual topology, as shown in the following Proposition, which has an important influence.

Proposition 1.1.7[8].

Let (X, δ) be a proximity space. Then the relation \ll satisfies the following properties:

- O1.** $X \ll X$;
- O2.** If $A \ll B$, then $A \subset B$;
- O3.** $A \subset B \ll C \subset D$ implies $A \ll D$;
- O4.** $A \ll B$ implies $X - B \ll X - A$;
- O5.** $A \ll B_k$ is true for $k = 1, 2, \dots, n$ if and only if $A \ll \bigcap_{k=1}^n B_k$;
- O6.** If $A \ll B$, then there exists a set $C \subset X$ such that $A \ll C \ll B$.

If δ is a separated proximity, then

- O7.** $\{x\} \ll X - \{y\}$ if and only if $x \neq y$.

Corollary 1.1.8[8]:

If $A_k \ll B_k$ for $k = 1, 2, \dots, n$, for $\bigcap_{k=1}^n B_k \neq \emptyset$ then:

- 1. $\bigcup_{k=1}^n A_k \ll \bigcap_{k=1}^n B_k$.
- 2. $\bigcup_{k=1}^n A_k \ll \bigcup_{k=1}^n B_k$.

$$3. \bigcap_{k=1}^n A_k \ll \bigcap_{k=1}^n B_k.$$

$$4. \bigcap_{k=1}^n A_k \ll \bigcup_{k=1}^n B_k.$$

We will now go through the process of creating a topology from proximity space (X, δ) and this topology is produced by introducing the definition of a closed set.

Definition 1.1.9[11]:

Let (X, δ) be a proximity space. A subset $F \subset X$ is defined to be **closed** if and only if $\{x\}\delta F$ implies $x \in F$. τ_δ is the family of complements of all the sets defined in such a way, such that τ_δ is a topology on the set X .

Proposition 1.1.10[7, 11]:

If G is a subset of a proximity space (X, δ) , then G is open in topology τ_δ if and only if $\{x\}\delta \bar{X} - G$ for every $x \in G$.

Proposition 1.1.11[7]:

If A and B are subsets of a proximity space (X, δ) , then $A\bar{\delta}B$ implies:

- (a) $cl_{\tau_\delta}(B) \subset X - A$,
- (b) $B \subset int_{\tau_\delta}(X - A)$.

Where the closure and the interior are taken with respect to the topology τ_δ .

Proposition 1.1.12[7]:

Let (X, δ) be a proximity space. For any subsets A and B of X , then:

- (a) $A \ll B$ implies $cl_{\tau_\delta}(A) \ll B$;
- (b) $A \ll B$ implies $A \ll int_{\tau_\delta}(B)$.

Proposition 1.1.13[7]:

The intersection of all δ -neighborhoods of a set A is equal to the τ_δ -closure of the set A .

Proposition 1.1.14[7]:

The topology τ_δ generated by a proximity relation δ on a space X is regular.

Now we review the concept of continuity in the space of proximity and the most important basic characteristics of continuity that have been adopted in our work through the following definition and theories.

Definition 1.1.15[7]:

Let (X, δ_X) and (Y, δ_Y) be two proximity spaces. The mapping $f : (X, \delta_X) \rightarrow (Y, \delta_Y)$ is said to be **proximally** or δ -continuous if $A\delta_X B$ implies $f(A)\delta_Y f(B)$ for every two sets $A, B \subset X$.

Theorem 1.1.16[7]:

A mapping $f : (X, \delta_X) \rightarrow (Y, \delta_Y)$ of a proximity space (X, δ_X) into a proximity space (Y, δ_Y) is δ -continuous if and only if for every two sets $P, Q \subset Y$, $P\overline{\delta_Y} Q$ implies $f^{-1}(P)\overline{\delta_X} f^{-1}(Q)$.

Corollary 1.1.17[7]:

Let $f : X \rightarrow (Y, \delta_Y)$ be a mapping from a set X on a proximity space (Y, δ_Y) . Then $\delta_X = f^{-1}(\delta_Y)$ is the coarsest proximity on X for which f is a δ -continuous mapping.

Theorem 1.1.18[7]:

Let (X, δ_X) be a proximity space, $f : (X, \delta_X) \rightarrow Y$ a given mapping and δ_Y be a relation on the set Y defined by: $A\delta_Y B$ if and only if $f^{-1}(A)\delta_X f^{-1}(B)$. Whenever δ_Y , defined in this way, is a proximity on Y , then $\delta_Y = f(\delta_X)$. This is the case if f is surjective and if for $A, B \subset Y$, $f^{-1}(A)\overline{\delta_X} f^{-1}(B)$, there exist $C, D \subset Y$ such that $C \cap D = \emptyset$, $f^{-1}(A)\overline{\delta_X} f^{-1}(Y - C)$ and $f^{-1}(B)\overline{\delta_X} f^{-1}(Y - D)$, then δ_Y is proximity on Y .

1.2 Center Sets Theory

In this part, we review the concept of the center sets and present some of the algebraic properties of these sets.

Definition 1.2.1[1]:

Let (X, δ) be a proximity space and $A \subseteq X$. A center set of A is defined by

$$\mathcal{C}_A = \{\langle A, B \rangle : B \subseteq X \text{ and } A\delta B\}$$

Example 1.2.2:

Let $X = \{a, b\}$ and δ discrete proximity, then the following center sets are:

$$\mathcal{C}_\emptyset = \emptyset;$$

$$\mathcal{C}_{\{a\}} = \{\langle \{a\}, \{a\} \rangle, \langle \{a\}, X \rangle\};$$

$$\mathcal{C}_{\{b\}} = \{\langle \{b\}, \{b\} \rangle, \langle \{b\}, X \rangle\};$$

$$\mathcal{C}_X = \{\langle X, \{a\} \rangle, \langle X, \{b\} \rangle, \langle X, X \rangle\}.$$

Example 1.2.3:

Let $X = \{1, 2, 3\}$ and d defined by

$$d: \left\{ \begin{array}{l} d(1,1) = d(2,2) = d(3,3) = 0 \\ d(1,2) = d(2,1) = d(2,3) = d(3,2) = 1 \\ d(1,3) = d(3,1) = 4 \end{array} \right\}, \text{ then } (X, d) \text{ is pseudometric}$$

space. Define δ pseudo-metric proximity, then the following center sets are:

$$\mathcal{C}_\emptyset = \emptyset;$$

$$\mathcal{C}_{\{1\}} = \{\langle\{1\}, \{1\}\rangle, \langle\{1\}, \{1,2\}\rangle, \langle\{1\}, \{1,3\}\rangle, \langle\{1\}, X\rangle\};$$

$$\mathcal{C}_{\{2\}} = \{\langle\{2\}, \{2\}\rangle, \langle\{2\}, \{1,2\}\rangle, \langle\{2\}, \{2,3\}\rangle, \langle\{2\}, X\rangle\};$$

$$\mathcal{C}_{\{3\}} = \{\langle\{3\}, \{3\}\rangle, \langle\{3\}, \{1,3\}\rangle, \langle\{3\}, \{2,3\}\rangle, \langle\{3\}, X\rangle\};$$

$$\mathcal{C}_{\{1,2\}} = \left\{ \begin{array}{l} \langle\{1,2\}, \{1\}\rangle, \langle\{1,2\}, \{2\}\rangle, \langle\{1,2\}, \{1,2\}\rangle, \\ \langle\{1,2\}, \{1,3\}\rangle, \langle\{1,2\}, \{2,3\}\rangle, \langle\{1,2\}, X\rangle \end{array} \right\};$$

$$\mathcal{C}_{\{1,3\}} = \left\{ \begin{array}{l} \langle\{1,3\}, \{1\}\rangle, \langle\{1,3\}, \{3\}\rangle, \langle\{1,3\}, \{1,2\}\rangle, \\ \langle\{1,3\}, \{1,3\}\rangle, \langle\{1,3\}, \{2,3\}\rangle, \langle\{1,3\}, X\rangle \end{array} \right\};$$

$$\mathcal{C}_{\{2,3\}} = \left\{ \begin{array}{l} \langle\{2,3\}, \{2\}\rangle, \langle\{2,3\}, \{3\}\rangle, \langle\{2,3\}, \{1,2\}\rangle, \\ \langle\{2,3\}, \{1,3\}\rangle, \langle\{2,3\}, \{2,3\}\rangle, \langle\{2,3\}, X\rangle \end{array} \right\};$$

$$\mathcal{C}_X = \{\langle X, \{1\}\rangle, \langle X, \{2\}\rangle, \langle X, \{3\}\rangle, \langle X, \{1,2\}\rangle, \langle X, \{1,3\}\rangle, \langle X, \{2,3\}\rangle, \langle X, X\rangle\}.$$

Example 1.2.4:

Let $X = \{a, b, c\}$ and $T = \{\emptyset, \{a\}, \{b, c\}, X\}$, then (X, T) is normal space.

Define δ normal proximity, then the following are center sets,

$$\mathcal{C}_\emptyset = \emptyset;$$

$$\mathcal{C}_{\{a\}} = \{\langle\{a\}, \{a\}\rangle, \langle\{a\}, \{a, b\}\rangle, \langle\{a\}, \{a, c\}\rangle, \langle\{a\}, X\rangle\};$$

$$\mathcal{C}_{\{b\}} = \left\{ \begin{array}{l} \langle\{b\}, \{b\}\rangle, \langle\{b\}, \{c\}\rangle, \langle\{b\}, \{a, b\}\rangle, \langle\{b\}, \{a, c\}\rangle, \\ \langle\{b\}, \{b, c\}\rangle, \langle\{b\}, X\rangle \end{array} \right\};$$

$$\mathcal{C}_{\{c\}} = \left\{ \begin{array}{l} \langle\{c\}, \{b\}\rangle, \langle\{c\}, \{c\}\rangle, \langle\{c\}, \{a, b\}\rangle, \langle\{c\}, \{a, c\}\rangle, \\ \langle\{c\}, \{b, c\}\rangle, \langle\{c\}, X\rangle \end{array} \right\};$$

$$\mathcal{C}_{\{a,b\}} = \left\{ \begin{array}{l} \langle\{a, b\}, \{a\}\rangle, \langle\{a, b\}, \{b\}\rangle, \langle\{a, b\}, \{c\}\rangle, \langle\{a, b\}, \{a, b\}\rangle, \\ \langle\{a, b\}, \{a, c\}\rangle, \langle\{a, b\}, \{b, c\}\rangle, \langle\{a, b\}, X\rangle \end{array} \right\};$$

$$\mathcal{C}_{\{a,c\}} = \left\{ \begin{array}{l} \langle\{a, c\}, \{a\}\rangle, \langle\{a, c\}, \{b\}\rangle, \langle\{a, c\}, \{c\}\rangle, \langle\{a, c\}, \{a, b\}\rangle, \\ \langle\{a, c\}, \{a, c\}\rangle, \langle\{a, c\}, \{b, c\}\rangle, \langle\{a, c\}, X\rangle \end{array} \right\};$$

$$\mathcal{C}_{\{b,c\}} = \left\{ \begin{array}{l} \langle\{b, c\}, \{b\}\rangle, \langle\{b, c\}, \{c\}\rangle, \langle\{b, c\}, \{a, b\}\rangle, \\ \langle\{b, c\}, \{a, c\}\rangle, \langle\{b, c\}, \{b, c\}\rangle, \langle\{b, c\}, X\rangle \end{array} \right\};$$

$$\mathcal{C}_X = \{\langle X, \{a\} \rangle, \langle X, \{b\} \rangle, \langle X, \{c\} \rangle, \langle X, \{a, b\} \rangle, \langle X, \{a, c\} \rangle, \langle X, \{b, c\} \rangle, \langle X, X \rangle\}.$$

We review the algebraic relationships of the center set through the following definition.

Definition 1.2.5. [1]

For two center sets \mathcal{C}_A and \mathcal{C}_B over a proximity space (X, δ) , we say that

1. \mathcal{C}_A is a **center subset** of \mathcal{C}_B if and only if for each $\langle A, C \rangle \in \mathcal{C}_A$, implies $\langle B, C \rangle \in \mathcal{C}_B$. We write $\mathcal{C}_A \preceq_c \mathcal{C}_B$. In (Example 1.2.4) $\mathcal{C}_{\{a\}} \preceq_c \mathcal{C}_{\{a,b\}}$.
2. \mathcal{C}_A is a **center equal** to \mathcal{C}_B if and only if \mathcal{C}_A is a center subset of \mathcal{C}_B and \mathcal{C}_B is a center subset of \mathcal{C}_A . We write $\mathcal{C}_A =_c \mathcal{C}_B$. In (Example 1.2.4) $\mathcal{C}_{\{b\}} =_c \mathcal{C}_{\{c\}} =_c \mathcal{C}_{\{b,c\}}$ and $\mathcal{C}_{\{a,b\}} =_c \mathcal{C}_{\{a,c\}} =_c \mathcal{C}_X$.
3. $\mathcal{C}_X = \{\langle X, A \rangle : \emptyset \neq A \subseteq X\}$ is **universe center**.
4. $\mathcal{C}_\emptyset = \emptyset$ is **Null center** set.
5. The **center complement** of \mathcal{C}_A is define by:

$$Cop. \mathcal{C}_A = \{\langle X, B \rangle \in \mathcal{C}_X : \langle A, B \rangle \notin \mathcal{C}_A\}$$

Clearly, $Cop. \mathcal{C}_X = \mathcal{C}_\emptyset$ and $Cop. \mathcal{C}_\emptyset = \mathcal{C}_X$. We denote $cop. \mathcal{C}_A$ by $\mathcal{C}_X^{(A)}$.

6. The **center difference** is define by:

$$\mathcal{C}_A - \mathcal{C}_B = \{\langle A, C \rangle : \langle A, C \rangle \in \mathcal{C}_A \text{ and } \langle B, C \rangle \notin \mathcal{C}_B\}.$$

7. **Center union** of \mathcal{C}_A and \mathcal{C}_B is define by:

$$\mathcal{C}_A \vee_c \mathcal{C}_B = \{\langle A \cup B, C \rangle : \langle A, C \rangle \in \mathcal{C}_A \text{ or } \langle B, C \rangle \in \mathcal{C}_B\}$$

8. **Center intersection** of \mathcal{C}_A and \mathcal{C}_B is define by:

$$\mathcal{C}_A \wedge_c \mathcal{C}_B = \{\langle A \cup B, C \rangle : \langle A, C \rangle \in \mathcal{C}_A \text{ and } \langle B, C \rangle \in \mathcal{C}_B\}$$

Example 1.2.6:

Let (X, δ) be a proximity space such that $X = \{1,2,3\}$ and δ is discrete proximity. Then the following are center sets:

$$\mathcal{C}_{\{1\}} = \{\langle \{1\}, \{1\} \rangle, \langle \{1\}, \{1,2\} \rangle, \langle \{1\}, \{1,3\} \rangle, \langle \{1\}, X \rangle\}$$

$$\mathcal{C}_{\{2\}} = \{\langle\{2\}, \{2\}\rangle, \langle\{2\}, \{1,2\}\rangle, \langle\{2\}, \{2,3\}\rangle, \langle\{2\}, X\rangle\}$$

$$\mathcal{C}_{\{1\}} \wedge_{\mathcal{C}} \mathcal{C}_{\{2\}} = \{\langle\{1,2\}, \{1,2\}\rangle, \langle\{1,2\}, X\rangle\}$$

$$\mathcal{C}_{\{1\}} \vee_{\mathcal{C}} \mathcal{C}_{\{2\}} = \left\{ \begin{array}{l} \langle\{1,2\}, \{1\}\rangle, \langle\{1,2\}, \{2\}\rangle, \langle\{1,2\}, \{1,2\}\rangle, \langle\{1,2\}, \{1,3\}\rangle, \\ \langle\{1,2\}, \{2,3\}\rangle, \langle\{1,2\}, X\rangle \end{array} \right\}$$

$$\text{cop. } \mathcal{C}_{\{1\}} = \{\langle X, \{2\}\rangle, \langle X, \{3\}\rangle, \langle X, \{2,3\}\rangle\}$$

$$\text{cop. } \mathcal{C}_{\{2\}} = \{\langle X, \{1\}\rangle, \langle X, \{3\}\rangle, \langle X, \{1,3\}\rangle\}.$$

The following theorem show the most of the algebraic properties of the Center union, Center intersection and center difference between the center sets in (X, δ) .

Proposition 1.2.7[1]

Suppose that \mathcal{C}_A and \mathcal{C}_B are center sets of (X, δ) , then:

1. $\mathcal{C}_A \vee_{\mathcal{C}} \mathcal{C}_A = \mathcal{C}_A$
2. $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_A = \mathcal{C}_A$
3. $\mathcal{C}_A \vee_{\mathcal{C}} \mathcal{C}_{\emptyset} = \mathcal{C}_A$
4. $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_{\emptyset} = \mathcal{C}_{\emptyset}$
5. $\mathcal{C}_A \vee_{\mathcal{C}} \text{Cop. } \mathcal{C}_A = \mathcal{C}_X$
6. $\mathcal{C}_A \wedge_{\mathcal{C}} \text{Cop. } \mathcal{C}_A = \mathcal{C}_{\emptyset}$
7. $\text{Cop. } (\mathcal{C}_A \vee_{\mathcal{C}} \mathcal{C}_B) = \text{Cop. } \mathcal{C}_A \wedge_{\mathcal{C}} \text{Cop. } \mathcal{C}_B$
8. $\text{Cop. } (\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B) = \text{Cop. } \mathcal{C}_A \vee_{\mathcal{C}} \text{Cop. } \mathcal{C}_B$
9. $\text{Cop. } (\text{Cop. } (\mathcal{C}_A)) =_{\mathcal{C}} \mathcal{C}_A.$
10. $(\mathcal{C}_A \vee_{\mathcal{C}} \mathcal{C}_B) \vee_{\mathcal{C}} \mathcal{C}_C = \mathcal{C}_A \vee_{\mathcal{C}} (\mathcal{C}_B \vee_{\mathcal{C}} \mathcal{C}_C).$
11. $(\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B) \wedge_{\mathcal{C}} \mathcal{C}_C = \mathcal{C}_A \wedge_{\mathcal{C}} (\mathcal{C}_B \wedge_{\mathcal{C}} \mathcal{C}_C).$
12. $(\mathcal{C}_A \vee_{\mathcal{C}} \mathcal{C}_B) \wedge_{\mathcal{C}} \mathcal{C}_C = (\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_C) \vee_{\mathcal{C}} (\mathcal{C}_B \wedge_{\mathcal{C}} \mathcal{C}_C).$
13. $(\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B) \vee_{\mathcal{C}} \mathcal{C}_C = (\mathcal{C}_A \vee_{\mathcal{C}} \mathcal{C}_C) \wedge_{\mathcal{C}} (\mathcal{C}_B \vee_{\mathcal{C}} \mathcal{C}_C).$
14. $\mathcal{C}_A - \mathcal{C}_B =_{\mathcal{C}} \mathcal{C}_A \wedge_{\mathcal{C}} \text{cop. } \mathcal{C}_B$
15. $(\mathcal{C}_B - \mathcal{C}_A) \vee_{\mathcal{C}} (\mathcal{C}_C - \mathcal{C}_A) = (\mathcal{C}_B \vee_{\mathcal{C}} \mathcal{C}_C) - \mathcal{C}_A.$

Definition 1.2.8[1]

Let (X, δ) be a proximity space and $\{\mathcal{C}_{A_i} : i \in I\}$ be a family of center sets.

Then

1. $\forall_C \mathcal{C}_{A_i} = \{\langle \cup A_i, C \rangle : \langle A_i, C \rangle \in \mathcal{C}_{A_i} \text{ for some } i \in I\}$.
2. $\lambda_C \mathcal{C}_{A_i} = \{\langle \cup A_i, C \rangle : \langle A_i, C \rangle \in \mathcal{C}_{A_i} \text{ for each } i \in I\}$.

Proposition 1.2.9 [1]:

If $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ and $\{\mathcal{C}_{A_i} : i \in I\}$ be a center set, then:

1. $\mathcal{C}_A \preceq_C \mathcal{C}_B$ and $\mathcal{C}_B \preceq_C \mathcal{C}_C \Rightarrow \mathcal{C}_A \preceq_C \mathcal{C}_C$
2. $\mathcal{C}_{A_i} \preceq_C \mathcal{C}_B$ for each $i \in I \Rightarrow \forall_C \mathcal{C}_{A_i} \preceq_C \mathcal{C}_B$
3. $\mathcal{C}_B \preceq_C \mathcal{C}_{A_i}$ for each $i \in I \Rightarrow \mathcal{C}_B \preceq_C \lambda_C \mathcal{C}_{A_i}$
4. $Cop.(\forall_C \mathcal{C}_{A_i}) = \lambda_C (Cop. \mathcal{C}_{A_i})$
5. $Cop.(\lambda_C \mathcal{C}_{A_i}) = \forall_C (Cop. \mathcal{C}_{A_i})$
6. $\mathcal{C}_A \preceq_C \mathcal{C}_B \Rightarrow Cop. \mathcal{C}_B \preceq_C Cop. \mathcal{C}_A$

We now present the definition of the central point as well as the concept of belonging through the following definition.

Definition 1.2.10[1]:

Let (X, δ) be a proximity space and $\{x\}, B \subseteq X$, such that $\{x\} \delta B$. Then $x_B = \{\{\{x\}, B\}\}$ is called a **center point** in (X, δ) . And if \mathcal{C}_A center set in (X, δ) . Then $x_B \in \mathcal{C}_A$ if and only if $\langle A, B \rangle \in \mathcal{C}_A$.

Proposition 1.2.11[1]:

Let $\{\mathcal{C}_{A_i} : i \in I\}$ be a family of center sets in (X, δ) . Then

1. If $x_B \in \mathcal{C}_{A_i}$ for each $i \in I$, then $x_B \in \lambda_C \mathcal{C}_{A_i}$.
2. If $\exists i \in I$ such that $x_B \in \mathcal{C}_{A_i}$, then $x_B \in \forall_C \mathcal{C}_{A_i}$.
3. $\mathcal{C}_{A_1} \preceq_C \mathcal{C}_{A_2}$ iff for each $x_B \in \mathcal{C}_{A_1} \Rightarrow x_B \in \mathcal{C}_{A_2}$.
4. $(\forall_C)_{x_B \in \mathcal{C}_A} x_B = \mathcal{C}_A$

Definition 1.2.12[1]:

Let (X, δ) be a proximity space, then $\wp(X)$ is a family of all center set, center subset, universe center, empty set, center complement of all center sets, also the center union and center intersection of the center sets with each of them.

Any center set that will be mentioned in this dissertation is one of the sets mentioned above.

1.3 \mathcal{C} - Topological Space

In this part, we review the concept of central topological spaces and introduce some important topological concepts (\mathcal{C} -open, \mathcal{C} -closed, \mathcal{C} -neighborhood and the separation axioms of center topological space) with their properties. It will be a starting point for us to introduce new concepts in these spaces.

Definition 1.3.1[2]:

Let (X, δ) be a proximity space and $\mathfrak{T}_{\mathcal{C}} \subseteq \wp(X)$, then $\mathfrak{T}_{\mathcal{C}}$ is said to be a center topology (\mathcal{C} -topology) if the following hold:

1. $\mathcal{C}_{\emptyset}, \mathcal{C}_X \in \mathfrak{T}_{\mathcal{C}}$
2. $\{\mathcal{C}_{A_i} : i \in I\} \in \mathfrak{T}_{\mathcal{C}} \Rightarrow \bigvee_{\mathcal{C}} \{\mathcal{C}_{A_i} : i \in I\} \in \mathfrak{T}_{\mathcal{C}}$.
3. $\mathcal{C}_{A_1}, \mathcal{C}_{A_2} \in \mathfrak{T}_{\mathcal{C}} \Rightarrow \mathcal{C}_{A_1} \wedge_{\mathcal{C}} \mathcal{C}_{A_2} \in \mathfrak{T}_{\mathcal{C}}$

The triplet $(X, \delta, \mathfrak{T}_{\mathcal{C}})$ is called a center topological space (\mathcal{C} -topological space) and the members of $\mathfrak{T}_{\mathcal{C}}$ are said to be center open (\mathcal{C} -open).

$\mathfrak{T}_{\mathcal{C}}$ is called indiscrete \mathcal{C} -topology if $\mathfrak{T}_{\mathcal{C}} = \{\mathcal{C}_X, \mathcal{C}_{\emptyset}\}$ and called discrete \mathcal{C} -topology if $\mathfrak{T}_{\mathcal{C}} = \wp(X)$.

Definition 1.3.2[2]:

Let $(X, \delta, \mathfrak{T}_c)$ be a \mathcal{C} -topological space and \mathcal{C}_A be center open, then $\mathcal{C}_X^{(A)} = \text{cop. } \mathcal{C}_A$, called center closed (\mathcal{C} -closed).

Clearly, if $\mathcal{C}_X^{(A)}$ is \mathcal{C} -closed, then $\text{cop. } (\mathcal{C}_X^{(A)}) =_c \mathcal{C}_A$ is center open.

Proposition 1.3.3[2]:

Let $(X, \delta, \mathfrak{T}_c)$ be \mathcal{C} -topological space and \mathfrak{H}_c be a collection of all \mathcal{C} -closed sets of $(X, \delta, \mathfrak{T}_c)$. Then

1. $\mathcal{C}_X, \mathcal{C}_\emptyset \in \mathfrak{H}_c$
2. $\{\mathcal{C}_X^{(A_i)} : i \in I\} \in \mathfrak{H}_c \implies \bigwedge_c \{\mathcal{C}_X^{(A_i)} : i \in I\} \in \mathfrak{H}_c$.
3. $\mathcal{C}_X^{(A_1)}, \mathcal{C}_X^{(A_2)} \in \mathfrak{H}_c \implies \mathcal{C}_X^{(A_1)} \vee_c \mathcal{C}_X^{(A_2)} \in \mathfrak{H}_c$.
4. $\mathfrak{T}_c = \{\mathcal{C}_A : \text{cop. } (\mathcal{C}_A) \in \mathfrak{H}_c\}$.
5. If \mathfrak{T}_c indiscrete \mathcal{C} -topology, then $\mathfrak{T}_c = (\mathfrak{H}_c)^c$, where $(\mathfrak{H}_c)^c$ family of all center complement center closed
6. If \mathfrak{T}_c non-indiscrete \mathcal{C} -topology on (X, δ) , then:
 $\mathfrak{T}_c^\# = (\mathfrak{H}_c)^c$ is \mathcal{C} -topology.

It is not necessary $\mathfrak{T}_c = \mathfrak{T}_c^\#$, but for each $\text{cop. } \mathcal{C}_X^{(A)} \in \mathfrak{T}_c^\#$, there is $\mathcal{C}_A \in \mathfrak{T}_c$ so that $\text{cop. } \mathcal{C}_X^{(A)} =_c \mathcal{C}_A$ and for each $\mathcal{C}_A \in \mathfrak{T}_c$, there is $\text{cop. } \mathcal{C}_X^{(A)} \in \mathfrak{T}_c^\#$ so that $\mathcal{C}_A =_c \text{cop. } \mathcal{C}_X^{(A)}$.

Example 1.3.4:

Let (X, δ) be a proximity space such that $X = \{a, b, c\}$ and δ is discrete proximity. Then

$$\mathfrak{T}_c = \{\mathcal{C}_\emptyset, \mathcal{C}_{\{b\}}, \mathcal{C}_X^{\{b\}}, \mathcal{C}_X\}$$

Where,

$$\mathcal{C}_\emptyset = \emptyset$$

$$\mathcal{C}_{\{b\}} = \{\langle\{b\}, \{b\}\rangle, \langle\{b\}, \{a, b\}\rangle, \langle\{b\}, \{b, c\}\rangle, \langle\{b\}, X\rangle\}$$

$$\mathcal{C}_X^{\{b\}} = \{\langle X, \{b\}\rangle, \langle X, \{a, b\}\rangle, \langle X, \{b, c\}\rangle, \langle X, X\rangle\}$$

$$\mathcal{C}_X = \{\langle X, \{a\}\rangle, \langle X, \{b\}\rangle, \langle X, \{c\}\rangle, \langle X, \{a, b\}\rangle, \langle X, \{a, c\}\rangle, \langle X, \{b, c\}\rangle, \langle X, X\rangle\}$$

is a \mathcal{C} -topology on (X, δ) . Hence:

$\mathfrak{S}_{\mathcal{C}} = \{ \mathcal{C}_{\emptyset}, \mathcal{C}_X^{\{b\}}, \mathcal{C}_X \}$ is the collection of all \mathcal{C} -closed sets where,

$$\mathcal{C}_X^{\{b\}} = \{\langle X, \{a\}\rangle, \langle X, \{c\}\rangle, \langle X, \{a, c\}\rangle\}.$$

Note that $\mathcal{C}_X^{\{b\}}$ is a \mathcal{C} -closed and

$$\begin{aligned} \mathcal{C}_{\{b\}} \neq_c \text{cop. } \mathcal{C}_X^{\{b\}} &= \text{cop. } \{\langle X, \{a\}\rangle, \langle X, \{c\}\rangle, \langle X, \{a, c\}\rangle\} = \\ \{\langle X, \{b\}\rangle, \langle X, \{a, b\}\rangle, \langle X, \{b, c\}\rangle, \langle X, X\rangle\} &\neq \mathcal{C}_{\{b\}}. \end{aligned}$$

The following theorem and example gives the concept of subspaces from central topological spaces.

Theorem 1.3.5[2]:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be \mathcal{C} -topological space and \mathcal{C}_Y be a center set of X , then

$\mathfrak{S}_{\mathcal{C}_X^Y} = \{ \mathcal{C}_X^Y \wedge_c \mathcal{C}_U : \mathcal{C}_U \in \mathfrak{S}_{\mathcal{C}} \}$ is \mathcal{C} -topological space of \mathcal{C}_X^Y .

Example 1.3.6:

Let $X = \{a, b\}$ and δ discrete proximity, then

$\mathfrak{S}_{\mathcal{C}} = \{ \mathcal{C}_{\emptyset}, \mathcal{C}_{\{a\}}, \mathcal{C}_X^{\{a\}}, \mathcal{C}_X \}$ is \mathcal{C} -topological space and assume $Y = \{b\}$, then

$\mathcal{C}_X^Y = \{\langle X, b\rangle, \langle X, X\rangle\}$. So, $\mathfrak{S}_{\mathcal{C}_X^Y} = \{ \mathcal{C}_{\emptyset}, \{\langle X, X\rangle\}, \mathcal{C}_X^Y \}$ is \mathcal{C} -topological space on \mathcal{C}_X^Y and $(\mathcal{C}_X^Y, \delta, \mathfrak{S}_{\mathcal{C}_X^Y})$ is center subspace of $(X, \delta, \mathfrak{S}_{\mathcal{C}})$.

Theorem 1.3.7[2]:

Let $(\mathcal{C}_X^Y, \delta, \mathfrak{S}_{\mathcal{C}_X^Y})$ be a subspace of $(X, \delta, \mathfrak{S}_{\mathcal{C}})$. $\mathcal{C}_X^{(B)}$ is \mathcal{C} -closed in \mathcal{C}_X^Y iff there exists a \mathcal{C} -closed set $\mathcal{C}_X^{(F)}$ in X , such that $\mathcal{C}_X^{(B)} = \mathcal{C}_X^Y \wedge_c \mathcal{C}_X^{(F)}$.

Now we present the definition of \mathcal{C} -closure through the following definition.

Definition 1.3.8[2]:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -topological space and \mathcal{C}_B be a center set.

Then the **\mathcal{C} -closure** of \mathcal{C}_B , denoted by $cl_{\mathcal{C}}(\mathcal{C}_B)$ is:

$$cl_{\mathcal{C}}(\mathcal{C}_B) = \wedge_{\mathcal{C}} \left\{ \mathcal{C}_X^{(A)} : \mathcal{C}_X^{(A)} \text{ is } \mathcal{C} - \text{closed and } \mathcal{C}_B \preceq_{\mathcal{C}} \mathcal{C}_X^{(A)} \right\}$$

Theorem 1.3.9[2]:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -topological space and \mathcal{C}_A be a center set, then center point $x_H \in cl_{\mathcal{C}}(\mathcal{C}_A)$ if and only if for every \mathcal{C} -open set \mathcal{C}_U , $x_H \in \mathcal{C}_U$, then $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_U \neq \mathcal{C}_{\emptyset}$.

Theorem 1.3.10[2]:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -topological space, and let \mathcal{C}_A be a center set, then for any \mathcal{C} -open set \mathcal{C}_G , $\mathcal{C}_G \wedge_{\mathcal{C}} cl_{\mathcal{C}}(\mathcal{C}_A) \preceq_{\mathcal{C}} cl_{\mathcal{C}}(\mathcal{C}_G \wedge_{\mathcal{C}} \mathcal{C}_A)$.

The following theorem, reviews the main properties of \mathcal{C} -closure set in central topological spaces.

Theorem 1.3.11[2].

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -topological space, \mathcal{C}_B and \mathcal{C}_E are center sets over (X, δ) . Then the following hold:

1. $cl_{\mathcal{C}}(\mathcal{C}_B)$ is the smallest \mathcal{C} -closed set such that $\mathcal{C}_B \preceq_{\mathcal{C}} cl_{\mathcal{C}}(\mathcal{C}_B)$.
2. $\mathcal{C}_B \preceq_{\mathcal{C}} cl_{\mathcal{C}}(\mathcal{C}_B)$
3. $\mathcal{C}_X^{(B)}$ is a \mathcal{C} -closed set if and only if $\mathcal{C}_X^{(B)} = cl_{\mathcal{C}}(\mathcal{C}_X^{(B)})$.
4. $cl_{\mathcal{C}}(\mathcal{C}_{\emptyset}) = \mathcal{C}_{\emptyset}$ and $cl_{\mathcal{C}}(\mathcal{C}_X) = \mathcal{C}_X$.
5. $cl_{\mathcal{C}}(cl_{\mathcal{C}}(\mathcal{C}_B)) = cl_{\mathcal{C}}(\mathcal{C}_B)$
6. $\mathcal{C}_B \preceq_{\mathcal{C}} \mathcal{C}_E$ implies $cl_{\mathcal{C}}(\mathcal{C}_B) \preceq_{\mathcal{C}} cl_{\mathcal{C}}(\mathcal{C}_E)$.

7. $\text{cl}_{\mathcal{C}}(\mathcal{C}_B) \vee_{\mathcal{C}} \text{cl}_{\mathcal{C}}(\mathcal{C}_E) = \text{cl}_{\mathcal{C}}(\mathcal{C}_B \vee_{\mathcal{C}} \mathcal{C}_E)$.
8. $\text{cl}_{\mathcal{C}}(\mathcal{C}_B \wedge_{\mathcal{C}} \mathcal{C}_E) \leq_{\mathcal{C}} \text{cl}_{\mathcal{C}}(\mathcal{C}_B) \wedge_{\mathcal{C}} \text{cl}_{\mathcal{C}}(\mathcal{C}_E)$.

Now we present the definition of \mathcal{C} - neighborhood in \mathcal{C} -topological space through the following definition.

Definition 1.3.12[2]:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -topological space , \mathcal{C}_A be a center set and x_B be a center point. Then \mathcal{C}_A is said to be a \mathcal{C} -neighborhood of x_K , if there exists a \mathcal{C} -open set \mathcal{C}_U such that $x_K \in \mathcal{C}_U \leq_{\mathcal{C}} \mathcal{C}_A$.

Theorem 1.3.13[2]:

1. If a center set \mathcal{C}_A of \mathcal{C} -topological space is \mathcal{C} -open set, then is a \mathcal{C} -neighborhood of each of its center points.
2. If \mathcal{C}_A is a \mathcal{C} -neighborhood of each of its points, then, there exists \mathcal{C} -open set \mathcal{C}_{A_1} such that $\mathcal{C}_A =_{\mathcal{C}} \mathcal{C}_{A_1}$.
3. If a center set \mathcal{C}_A a subset of \mathcal{C}_X is open if and only if it is a \mathcal{C} -neighborhood of each of its centers points.

Example 1.3.14.

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -topological space such that

$$X = \{a, b\}, \mathfrak{S}_{\mathcal{C}} = \{\mathcal{C}_X, \mathcal{C}_{\emptyset}, \mathcal{C}_X^{\{a\}}\} \text{ where } \delta \text{ is discrete proximity:}$$

$$\mathcal{C}_{\emptyset} = \emptyset,$$

$$\mathcal{C}_X = \{\{\langle X, \{a\} \rangle\}, \{\langle X, \{b\} \rangle\}, \{\langle X, X \rangle\}\} \text{ and}$$

$$\mathcal{C}_X^{\{a\}} = \{\{\langle X, \{a\} \rangle\}, \{\langle X, X \rangle\}\}$$

Now, let $\mathcal{C}_{\{a\}} = \{\{\langle \{a\}, \{a\} \rangle\}, \{\langle \{a\}, X \rangle\}\}$,

then it is \mathcal{C} -neighborhood of each its center point, but is not \mathcal{C} -open set.

Proposition 1.3.15[2]:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -topological space, then:

1. Each center point x_H has a \mathcal{C} -neighborhood.
2. if \mathcal{C}_A and \mathcal{C}_B are \mathcal{C} -neighborhoods of some x_H , then $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ is also a \mathcal{C} -neighborhood of x_H .
3. if \mathcal{C}_A is a \mathcal{C} -neighborhood of x_H and $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_B$, then \mathcal{C}_B is also a \mathcal{C} -neighborhood of x_H .

The following definition, introduces the concept of the set of \mathcal{C} -interior points in \mathcal{C} -topological space.

Definition 1.3.16[2]:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -topological space, \mathcal{C}_A be a center set and x_B be a center point, x_B is said to be an \mathcal{C} -interior point of \mathcal{C}_A iff \mathcal{C}_A is a \mathcal{C} -neighborhood of x_B , that is, iff there exists a \mathcal{C} -open set \mathcal{C}_U , $x_B \in \mathcal{C}_U \preccurlyeq_{\mathcal{C}} \mathcal{C}_A$. The set of all \mathcal{C} -interior points of \mathcal{C}_A is called \mathcal{C} -interior of \mathcal{C}_A and denoted by $\text{int}_{\mathcal{C}}(\mathcal{C}_A)$.

Theorem 1.3.17[2]:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -topological space and \mathcal{C}_A be a center set, then:

$$\text{int}_{\mathcal{C}}(\mathcal{C}_A) = \bigvee_{\mathcal{C}} \{ \mathcal{C}_G : \mathcal{C}_G \text{ is } \mathcal{C}\text{-open and } \mathcal{C}_G \preccurlyeq_{\mathcal{C}} \mathcal{C}_A \}.$$

The following theorem, reviews the main properties of \mathcal{C} -interior set in central topological spaces.

Theorem 1.3.18[2]:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -topological space, \mathcal{C}_A and \mathcal{C}_B are center sets, then:

1. $\text{int}_{\mathcal{C}}(\mathcal{C}_A) = \mathcal{C}_{\emptyset}$ or $\text{int}_{\mathcal{C}}(\mathcal{C}_A) \subseteq \mathcal{C}_A$.
2. $\text{int}_{\mathcal{C}}(\mathcal{C}_A)$ is \mathcal{C} -open
3. $\text{int}_{\mathcal{C}}(\mathcal{C}_{\emptyset}) = \mathcal{C}_{\emptyset}$ and $\text{int}_{\mathcal{C}}(\mathcal{C}_X) = \mathcal{C}_X$.

4. $\text{int}_{\mathcal{C}}(\mathcal{C}_A) \leq_{\mathcal{C}} \mathcal{C}_A$.
5. If \mathcal{C}_A is a \mathcal{C} -open set, then $\mathcal{C}_A =_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_A)$.
6. If $\mathcal{C}_A = \text{int}_{\mathcal{C}}(\mathcal{C}_A)$, then \mathcal{C}_A is a \mathcal{C} -open set.
7. $\text{int}_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_A)) = \text{int}_{\mathcal{C}}(\mathcal{C}_A)$
8. $\mathcal{C}_A \leq_{\mathcal{C}} \mathcal{C}_B$ implies $\text{int}_{\mathcal{C}}(\mathcal{C}_A) \leq_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_B)$.
9. $\text{int}_{\mathcal{C}}(\mathcal{C}_A) \wedge_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_B) = \text{int}_{\mathcal{C}}(\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B)$.
10. $\text{int}_{\mathcal{C}}(\mathcal{C}_A) \vee_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_B) \leq_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_A \vee_{\mathcal{C}} \mathcal{C}_B)$.

Theorem 1.3.19[2]:

Let \mathcal{C}_A be a center set of \mathcal{C} -topological space over a \mathcal{C} -topological space $(X, \delta, \mathfrak{S}_{\mathcal{C}})$. Then

1. $\text{int}_{\mathcal{C}}(\text{cop. } \mathcal{C}_A) = \text{cop.}(\text{cl}_{\mathcal{C}}(\mathcal{C}_A))$.
2. $\text{cl}_{\mathcal{C}}(\text{cop. } \mathcal{C}_A) = \text{cop.}(\text{int}_{\mathcal{C}}(\mathcal{C}_A))$.
3. $\text{int}_{\mathcal{C}}(\mathcal{C}_A) = \text{cop.}(\text{cl}_{\mathcal{C}}(\text{cop. } \mathcal{C}_A))$.
4. $\text{cl}_{\mathcal{C}}(\mathcal{C}_A) = \text{cop.}(\text{int}_{\mathcal{C}}(\text{cop. } \mathcal{C}_A))$.

Now, we review the concepts of the separation axioms in center topological spaces ($\mathfrak{S}_{\mathcal{C}0}$ -space, $\mathfrak{S}_{\mathcal{C}1}$ -space, $\mathfrak{S}_{\mathcal{C}2}$ -space, \mathcal{C} -regular, \mathcal{C} -normal, $\mathfrak{S}_{\mathcal{C}3}$ -space and $\mathfrak{S}_{\mathcal{C}4}$ -space) with the characteristics of these concepts are introduced.

Definition 1.3.20 [2]:

A \mathcal{C} -topological space $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ called $\mathfrak{S}_{\mathcal{C}0}$ -space if and only if for each x_K, y_H , center points such that $x_K \neq_{\mathcal{C}} y_H$, there exists at least one center open set \mathcal{C}_A or \mathcal{C}_B such that $x_K \in \mathcal{C}_A, y_H \notin \mathcal{C}_A$ or $y_H \in \mathcal{C}_B, x_K \notin \mathcal{C}_B$.

Definition 1.3.21[3]:

A \mathcal{C} -topological space $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ called $\mathfrak{S}_{\mathcal{C}1}$ -space if and only if for each x_{B1}, y_{B2} , center points such that $x_{B1} \neq_{\mathcal{C}} y_{B2}$, there exists center open sets \mathcal{C}_{A1} and \mathcal{C}_{A2} such that $x_{B1} \in \mathcal{C}_{A1}, y_{B2} \notin \mathcal{C}_{A1}$ and $y_{B2} \in \mathcal{C}_{A2}, x_{B1} \notin \mathcal{C}_{A2}$.

Definition 1.3.22[3]:

A \mathcal{C} -topological space $(X, \delta, \mathfrak{T}_{\mathcal{C}})$ called $\mathfrak{T}_{\mathcal{C}2}$ -space if and only if for each x_{B1}, y_{B2} , center points such that $x_{B1} \neq_{\mathcal{C}} y_{B2}$, there exist center open sets \mathcal{C}_{A1} and \mathcal{C}_{A2} such that $x_{B1} \in \mathcal{C}_{A1}$, $y_{B2} \in \mathcal{C}_{A2}$ and $\mathcal{C}_{A1} \wedge_{\mathcal{C}} \mathcal{C}_{A2} = \mathcal{C}_{\emptyset}$.

Proposition 1.3.23 [3]:

1. If $(X, \delta, \mathfrak{T}_{\mathcal{C}})$ is $\mathfrak{T}_{\mathcal{C}1}$ -space, then $(X, \delta, \mathfrak{T}_{\mathcal{C}})$ is $\mathfrak{T}_{\mathcal{C}0}$ -space.
2. If $(X, \delta, \mathfrak{T}_{\mathcal{C}})$ is $\mathfrak{T}_{\mathcal{C}2}$ -space, then $(X, \delta, \mathfrak{T}_{\mathcal{C}})$ is $\mathfrak{T}_{\mathcal{C}1}$ -space.

Example 1.3.24

Let $(X, \delta, \mathfrak{T}_{\mathcal{C}})$ be a \mathcal{C} -topological space where, $X = \{1,2\}$ and δ is desecrate proximity. Then

$$\mathfrak{T}_{\mathcal{C}} = \{\mathcal{C}_{\emptyset}, \{(X, \{1\})\}, \{(X, \{1\}), (X, \{2\})\}, \mathcal{C}_X\}$$

Is $\mathfrak{T}_{\mathcal{C}0}$ -space but not $\mathfrak{T}_{\mathcal{C}1}$ -space.

Example 1.3.25:

Let $(X, \delta, \mathfrak{T}_{\mathcal{C}})$ be a \mathcal{C} -topological space where, X is infinite and $\mathfrak{T}_{\mathcal{C}} = \mathcal{C}_{\emptyset} \cup \{\mathcal{C}_A : \text{cop. } \mathcal{C}_A \text{ is finite}\}$.

Then $(X, \delta, \mathfrak{T}_{\mathcal{C}})$ is $\mathfrak{T}_{\mathcal{C}1}$ -space, since if $x_B \neq_{\mathcal{C}} y_E$,

then $x_B \in \mathcal{C}_X - y_E$, $y_E \notin \mathcal{C}_X - y_E$ and $y_E \in \mathcal{C}_X - x_B$, $x_B \notin \mathcal{C}_X - x_B$.

$(X, \delta, \mathfrak{T}_{\mathcal{C}})$ is not $\mathfrak{T}_{\mathcal{C}2}$ -space, because if possible, there exists two center open sets $\mathcal{C}_G, \mathcal{C}_H$ such that $\mathcal{C}_G \wedge_{\mathcal{C}} \mathcal{C}_H = \mathcal{C}_{\emptyset}$, then $\text{cop. } (\mathcal{C}_G \wedge_{\mathcal{C}} \mathcal{C}_H) = \mathcal{C}_X$.

Hence $\text{cop. } \mathcal{C}_G \vee_{\mathcal{C}} \text{cop. } \mathcal{C}_H = \mathcal{C}_X$. But $\text{cop. } \mathcal{C}_G$ and $\text{cop. } \mathcal{C}_H$ are finite sets and so their union is also finite which is a contradiction since X is infinite.

Theorem 1.3.26[3]:

Let $(X, \delta, \mathfrak{T}_{\mathcal{C}})$ be a $\mathfrak{T}_{\mathcal{C}1}$ -topological space. Then, for each center point x_B there exists \mathcal{C} -open set \mathcal{C}_G such that $\text{cop. } x_B =_{\mathcal{C}} \mathcal{C}_G$.

Proposition 1.3.27[3]:

Let $(X, \delta, \mathfrak{J}_c)$ be a \mathcal{C} -topological space and \mathcal{C}_Y be a center set of X . If $(X, \delta, \mathfrak{J}_c)$ is a \mathfrak{J}_{c1} -space, then $(\mathcal{C}_X^Y, \delta, \mathfrak{J}_{c_X^Y})$ is a \mathfrak{J}_{c1} -space.

Proposition 1.3.28[3]:

Let $(X, \delta, \mathfrak{J}_c)$ be a \mathcal{C} -topological space and \mathcal{C}_Y be a center set of X . If $(X, \delta, \mathfrak{J}_c)$ is a \mathfrak{J}_{c2} -space, then $(\mathcal{C}_X^Y, \delta, \mathfrak{J}_{c_X^Y})$ is a \mathfrak{J}_{c2} -space.

Theorem 1.3.29[3]:

If $(X, \delta, \mathfrak{J}_c)$ is a center \mathfrak{J}_{c2} -space and for any $x_{B1}, y_{B2} \in \mathcal{C}_X$ such that $x_{B1} \neq y_{B2}$, then there exists the center closed sets \mathcal{C}_{F1} and \mathcal{C}_{F2} such that $x_{B1} \in \mathcal{C}_{F1}, y_{B2} \notin \mathcal{C}_{F1}$ and $x_{B1} \notin \mathcal{C}_{F2}, y_{B2} \in \mathcal{C}_{F2}$ and $\mathcal{C}_{F1} \vee_{\mathcal{C}} \mathcal{C}_{F2} = \mathcal{C}_X$.

Definition 1.3.30[3]:

A \mathcal{C} -topological space $(X, \delta, \mathfrak{J}_c)$ called \mathcal{C} -regular space if and only if $\mathcal{C}_X^{(F)}$ a center closed set in $(X, \delta, \mathfrak{J}_c)$ and $x_B \in \mathcal{C}_X$ such that $x_B \notin \mathcal{C}_X^{(F)}$, then there exists the center open sets \mathcal{C}_{A1} and \mathcal{C}_{A2} such that $x_B \in \mathcal{C}_{A1}, \mathcal{C}_X^{(F)} \leq_{\mathcal{C}} \mathcal{C}_{A2}$ and $\mathcal{C}_{A1} \wedge_{\mathcal{C}} \mathcal{C}_{A2} = \mathcal{C}_{\emptyset}$.

In the following theory, we give the properties of the \mathcal{C} -regular spaces in the middle.

Theorem 1.3.31[3]:

Let $(X, \delta, \mathfrak{J}_c)$ be a \mathcal{C} -topological space. At that point the accompanying articulations are equivalent:

1. $(X, \delta, \mathfrak{J}_c)$ is \mathcal{C} -regular.
2. If \mathcal{C}_A is center open in $(X, \delta, \mathfrak{J}_c)$ and $x_B \in \mathcal{C}_A$, then there is the center open \mathcal{C}_E , $x_B \in \mathcal{C}_E$ and $x_B \in cl_{\mathcal{C}} \mathcal{C}_E \leq_{\mathcal{C}} \mathcal{C}_A$.
3. If $x_B \in \mathcal{C}_X$ center point, then, x_B has a center neighborhood base \mathcal{C}_A such that $\mathcal{C}_X^{(F)} \leq_{\mathcal{C}} \mathcal{C}_A$, where $\mathcal{C}_X^{(F)}$ is center closed.

The following theory explains that the regularity of the center is a central hereditary property:

Theorem 1.3.32[3]:

If $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -regular space. Then $(\mathcal{C}_X^Y, \delta, \mathfrak{S}_{\mathcal{C}_X^Y})$ is a \mathcal{C} -regular space where $(\mathcal{C}_X^Y, \delta, \mathfrak{S}_{\mathcal{C}_X^Y})$ is center subspace of $(X, \delta, \mathfrak{S}_{\mathcal{C}})$.

Theorem 1.3.33[3]:

A space $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} -regular if and only if for each $x_B \in \mathcal{C}_X$ and for each center closed set \mathcal{C}_A in $(X, \delta, \mathfrak{S}_{\mathcal{C}})$, $x_B \notin \mathcal{C}_A$, there exists two center open sets $\mathcal{C}_{A_1}, \mathcal{C}_{A_2}$ in $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ such that $x_B \in \mathcal{C}_{A_1}$, $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_{A_2}$ and $cl_{\mathcal{C}} \mathcal{C}_{A_1} \wedge_{\mathcal{C}} cl_{\mathcal{C}} \mathcal{C}_{A_2} = \mathcal{C}_{\emptyset}$.

Definition 1.3.34[3]:

A \mathcal{C} -topological space called $\mathfrak{S}_{\mathcal{C}3}$ -space, if and only if is \mathcal{C} -regular space and a $\mathfrak{S}_{\mathcal{C}1}$ -space.

Definition 1.3.35[3]:

A \mathcal{C} -topological space $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is called, \mathcal{C} -normal space if and only if for each two center closed sets $\mathcal{C}_X^{(F)}, \mathcal{C}_X^{(E)}$ such that $\mathcal{C}_X^{(F)} \wedge_{\mathcal{C}} \mathcal{C}_X^{(E)} = \mathcal{C}_{\emptyset}$, there exist two center open sets $\mathcal{C}_{A_1}, \mathcal{C}_{A_2}$ such that $\mathcal{C}_X^{(F)} \preccurlyeq_{\mathcal{C}} \mathcal{C}_{A_1}$, $\mathcal{C}_X^{(E)} \preccurlyeq_{\mathcal{C}} \mathcal{C}_{A_2}$ and $\mathcal{C}_{A_1} \wedge_{\mathcal{C}} \mathcal{C}_{A_2} = \mathcal{C}_{\emptyset}$.

Definition 1.3.36[3]:

A \mathcal{C} -topological space called $\mathfrak{S}_{\mathcal{C}4}$ -space, if and only if is \mathcal{C} -normal space and a $\mathfrak{S}_{\mathcal{C}1}$ -space.

The following theory gives the advantages or characteristics of \mathcal{C} -normal space.

Theorem 1.3.37[3]:

A \mathcal{C} -topological space $(X, \delta, \mathfrak{S}_c)$ is \mathcal{C} -normal if and only if for each center closed set $\mathcal{C}_X^{(F)}$ and center open set \mathcal{C}_E such that $\mathcal{C}_X^{(F)} \leq_c \mathcal{C}_E$, there exists the center open set \mathcal{C}_H , $\mathcal{C}_X^{(F)} \leq_c \mathcal{C}_H$ such that $\mathcal{C}_X^{(F)} \leq_c \mathcal{C}_H \leq_c cl_c \mathcal{C}_H \leq_c \mathcal{C}_E$.

Theorem 1.3.38 [3]:

Every center closed subspace of a \mathcal{C} -normal space is \mathcal{C} -normal.

CHAPTER TWO

Central Continuous Function

Chapter Two: Central Continuous Function

This chapter consists of four sections. In section one, are clarified the basic concepts of **shrink central continuous function**. In section two, the concept of the center set, with an explanation of the most important algebraic properties is presented.

2.1 Shrink Central Continuous Function

In this section, we define of a special type of functions based on the concept of proximity relation is presented in conjunction with central set theory for the purpose of finding a special definition of continuity between center topological spaces and the transfer of topological properties, where we call shrink central continuous function.

2.1.1 Shrink Central Function

In this section, we introduce the definition of sharing central function and its most important properties.

Definition 2.1.1.1:

Let $(X, \delta_X), (Y, \delta_Y)$ are proximity spaces, a function $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$, is called shrink function if the image for center set \mathcal{C}_A of X is $\mathcal{C}_{f(A)} = \{\langle f(A), B \rangle : f(A) \delta_Y B, \text{ for each } B \subseteq Y\}$, and the inverse of shrink function f is of form $\mathcal{C}_{f^{-1}(B)} = \{\langle f^{-1}(B), H \rangle : f^{-1}(B) \delta_X H, \text{ for each } H \subseteq X\}$.

Example 2.1.1.2:

Let $X = \{1, 2\}$ and δ_X be a discrete proximity, define function from X into X such that $f(1) = 2, f(2) = 1$, then the center set in X as $\mathcal{C}_X, \mathcal{C}_\emptyset, \mathcal{C}_{\{1\}}$ and $\mathcal{C}_{\{2\}}$, therefore $f(\mathcal{C}_\emptyset) =_c \mathcal{C}_\emptyset, f(\mathcal{C}_X) =_c \{\langle f(X), B \rangle : f(X) \delta_Y B\}$
 $=_c \{\langle X, \{1\} \rangle, \langle X, \{2\} \rangle, \langle X, X \rangle\} =_c \mathcal{C}_X, f(\mathcal{C}_{\{1\}}) =_c \{\langle \{2\}, \{2\} \rangle, \langle \{2\}, X \rangle\} =_c \mathcal{C}_{\{2\}}$

and $f(\mathcal{C}_{\{2\}}) =_c \{\{\{1\}, \{1\}\}, \{\{1\}, X\}\} =_c \mathcal{C}_{\{1\}}$. And $f^{-1}(\mathcal{C}_{\emptyset}) =_c \mathcal{C}_{\emptyset}$,
 $f^{-1}(\mathcal{C}_X) =_c \mathcal{C}_X$, $f^{-1}(\mathcal{C}_{\{1\}}) =_c \mathcal{C}_{\{2\}}$ and $f^{-1}(\mathcal{C}_{\{2\}}) =_c \mathcal{C}_{\{1\}}$

Definition 2.1.1.3:

Let f be a shrink function from proximity space (X, δ_X) into proximity space (Y, δ_Y) , the image of center point x_A in \mathcal{C}_X is of the form, $f(x_A) =_c \{\{\{f(x)\}, f(A)\}\}$. And the inverse image of center point y_B in \mathcal{C}_Y is of the form $f^{-1}(y_B) =_c \{\{\{f^{-1}(y)\}, f^{-1}(B)\}\}$.

For example 2.1.1.2 the center point in \mathcal{C}_X as : 1_X , $1_{\{1\}}$, 2_X and $2_{\{2\}}$, then
 $f(1_X) =_c \{\{\{f(1)\}, f(X)\}\} =_c \{\{\{2\}, X\}\} =_c 2_X$, $f(2_X) =_c \{\{\{f(2)\}, f(X)\}\}$
 $=_c \{\{\{1\}, X\}\} =_c 1_X$, $f(1_{\{1\}}) =_c \{\{\{f(1)\}, f(\{1\})\}\} =_c \{\{\{2\}, \{2\}\}\} =_c 2_{\{2\}}$
and $f(2_{\{2\}}) =_c \{\{\{f(2)\}, f(\{2\})\}\} =_c \{\{\{1\}, \{1\}\}\} =_c 1_{\{1\}}$.

Proposition 2.1.1.4:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ - continuous shrink function and \mathcal{C}_B be center set in X , if $x_A \in \mathcal{C}_B$ then $f(x_A) \in \mathcal{C}_{f(B)}$.

Proof:

Let $x_A \in \mathcal{C}_B$, imply that $A\delta_X B$, but f is δ - continuous, so $f(A)\delta_Y f(B)$.
Hence $f(x_A) \in \mathcal{C}_{f(B)}$.

Proposition 2.1.1.5:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be bijective and f^{-1} is δ - continuous shrink function and \mathcal{C}_V center set in Y , if $y_A \in \mathcal{C}_V$ then $f^{-1}(y_A) \in \mathcal{C}_{f^{-1}(V)}$.

Proof:

Let $y_A \in \mathcal{C}_V$, imply that $A\delta_Y V$, but f^{-1} is δ -continuous, so $f^{-1}(A)\delta_X f^{-1}(B)$. Hence $f^{-1}_c(y_A) \in \mathcal{C}_{f^{-1}(V)}$.

Proposition 2.1.1.6:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ -homeomorphism shrink function. Hence the following is correct:

- i. If $\mathcal{C}_A \preceq_c \mathcal{C}_B$, then $\mathcal{C}_{f(A)} \preceq_c \mathcal{C}_{f(B)}$, for any center sets $\mathcal{C}_A, \mathcal{C}_B$ in X .
- ii. If $\mathcal{C}_{Y_1} \preceq_c \mathcal{C}_{Y_2}$, then $\mathcal{C}_{f^{-1}(Y_1)} \preceq_c \mathcal{C}_{f^{-1}(Y_2)}$, for any center sets $\mathcal{C}_{Y_1}, \mathcal{C}_{Y_2}$ in Y .

Proof:

- i. Let $y_V \in \mathcal{C}_{f(A)}$, imply that $f(A)\delta_Y V$, by using the defined injection function and δ -continuous of f^{-1} we get $A\delta_X f^{-1}(V)$, then $B\delta_X f^{-1}(V)$, by using the onto and δ -continuous of f we get $f(B)\delta_Y V$. Hence $y_V \in \mathcal{C}_{f(B)}$.
- ii. Let $x_A \in \mathcal{C}_{f^{-1}(Y_1)}$, imply that $A\delta_X f^{-1}(Y_1)$, by using the onto and δ -continuous of f we get $Y_1\delta_Y f(A)$, then $Y_2\delta_Y f(A)$, by using 1-1 and δ -continuous of f^{-1} we get $f^{-1}(Y_2)\delta_X A$. By definition (2.1.1.1) we get $x_A \in \mathcal{C}_{f^{-1}(Y_2)}$.

Proposition 2.1.1.7:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ -homeomorphism shrink function and $\mathcal{C}_A, \mathcal{C}_B \preceq_c \mathcal{C}_X$, then the following are true:

- i. If $\mathcal{C}_B =_c \mathcal{C}_{f(A)}$, then $\mathcal{C}_A \preceq_c \mathcal{C}_{f^{-1}(B)}$.
- ii. If $\mathcal{C}_A =_c \mathcal{C}_{f^{-1}(B)}$, then $\mathcal{C}_{f(A)} \preceq_c \mathcal{C}_B$.

Proof:

- i. Let $x_H \in \mathcal{C}_A$ such that $H \subseteq X$ imply that $A\delta_X H$, since f is δ -continuous, then $f(A)\delta_Y f(H)$. But $\mathcal{C}_B =_c \mathcal{C}_{f(A)}$ then $f(H)\delta_Y B$, since f^{-1} is δ -continuous, then $f^{-1}(f(H))\delta_X f^{-1}(B)$, since $f^{-1}(f(H)) \subseteq H$, then $H\delta_X f^{-1}(B)$. Hence $x_H \in \mathcal{C}_{f^{-1}(B)}$.
- ii. Let $y_K \in \mathcal{C}_{f(A)}$ such that $K \subseteq Y$, then $f(A)\delta_Y K$, since f^{-1} is δ -continuous, then $f^{-1}(f(A))\delta_X f^{-1}(K)$ and since $f^{-1}(f(A)) = A$, then $A\delta_X f^{-1}(K)$. But $\mathcal{C}_A =_c \mathcal{C}_{f^{-1}(B)}$, then $f^{-1}(K)\delta_X f^{-1}(B)$, since f is δ -continuous, then $f(f^{-1}(K))\delta_Y f(f^{-1}(B))$ and $f(f^{-1}(K)) = K$, then $K\delta_Y B$. Thus $y_A \in \mathcal{C}_B$.

Proposition 2.1.1.8:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ -homeomorphism shrink function, $\{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\}$ and $\{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}$ are families of center subsets of \mathbb{C}_X and \mathbb{C}_Y respectively, where Δ and Λ are arbitrary index sets, then:

1. $f_c(\vee \{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\}) =_c \vee \{\mathcal{C}_{f(A_\alpha)}, \alpha \in \Delta\}$
2. $f_c(\wedge \{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\}) =_c \wedge \{\mathcal{C}_{f(A_\alpha)}, \alpha \in \Delta\}$
3. $f_c^{-1}(\vee \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}) =_c \vee \{\mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$
4. $f_c^{-1}(\wedge \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}) =_c \wedge \{\mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$

Proof 1:

Let $y_K \in f_c(\vee \{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\})$, iff for some $\alpha_0 \in \Delta$, $x \in X, H \subseteq X$ such that $\{x\}\delta_X H$, $A_{\alpha_0}\delta_X H$ such that $f_c(x_H) =_c y_K$, by proposition (2.1.1.4), then $f_c(x_H) \in \mathcal{C}_{f(A_{\alpha_0})}$ for some $\alpha_0 \in \Delta$, because $f(A_{\alpha_0})\delta_Y f(H)$, therefore $f(A_{\alpha_0})\delta_Y K$. thus $y_K \in \mathcal{C}_{f(A_{\alpha_0})}$ for some $\alpha_0 \in \Delta$, Hence $y_K \in \vee \{\mathcal{C}_{f(A_\alpha)}, \alpha \in \Delta\}$.

Conversely, let $y_K \in \mathcal{Y} \{ \mathcal{C}_{f(A_\alpha)}, \alpha \in \Delta \}$, then for some $\alpha_0 \in \Delta$, $y \in Y, K \subseteq Y$ such that $\{y\} \delta_Y K$ and $f_c(A_{\alpha_0}) \delta_Y K$, by proposition (2.1.1.5), then $f_c^{-1}(y_K) \in \mathcal{C}_{f^{-1}(f(A_{\alpha_0}))}$, therefore $f^{-1}(f(A_{\alpha_0})) \delta_X f^{-1}(K)$, since f is onto, then $A_{\alpha_0} \delta_X f^{-1}(K)$, thus $f^{-1}(K) \in \mathcal{C}_{A_{\alpha_0}}$ for some $\alpha_0 \in \Delta$, by proposition (2.1.1.4), then $f_c(f_c^{-1}(y_K)) \in f_c(\mathcal{Y} \{ \mathcal{C}_{A_\alpha}, \alpha \in \Delta \})$. Hence $y_K \in f_c(\mathcal{Y} \{ \mathcal{C}_{A_\alpha}, \alpha \in \Delta \})$.

Proof 2:

Let $y_K \in f_c(\mathcal{A} \{ \mathcal{C}_{A_\alpha}, \alpha \in \Delta \})$, then for each $\alpha \in \Delta$, $\exists x \in X, H \subseteq X$ such that $\{x\} \delta_X H$, $A_\alpha \delta_X H$ such that $f_c(x_H) =_c y_K$, by proposition (2.1.1.4), then $f_c(x_H) \in \mathcal{C}_{f(A_\alpha)}$ for each $\alpha \in \Delta$, because $f(A_\alpha) \delta_Y f(H) \quad \forall \alpha \in \Delta$, therefore $y_K \in \mathcal{C}_{f(A_\alpha)}$ for each $\alpha \in \Delta$, Hence $y_K \in \mathcal{A} \{ \mathcal{C}_{f(A_\alpha)}, \alpha \in \Delta \}$.

Conversely, let $y_K \in \mathcal{A} \{ \mathcal{C}_{f(A_\alpha)}, \alpha \in \Delta \}$, then for each $\alpha \in \Delta$, $\exists y \in Y, K \subseteq Y$ such that $\{y\} \delta_Y K$ and $f(A_\alpha) \delta_Y K$, by proposition (2.1.1.5), then $f^{-1}(f(A_\alpha)) \delta_X f^{-1}(K)$, this mean $f_c^{-1}(y_K) \in \mathcal{C}_{f^{-1}(f(A_\alpha))}$, since f is (1-1), then $A_\alpha \delta_X f^{-1}(K) \quad \forall \alpha \in \Delta$, therefore $f^{-1}(y_K) \in \mathcal{C}_{A_\alpha}$ for each $\alpha \in \Delta$, thus $f^{-1}(y_K) \in \mathcal{A} \{ \mathcal{C}_{A_\alpha}, \alpha \in \Delta \}$. by proposition (2.1.1.4), then $y_K \in f(\mathcal{A} \{ \mathcal{C}_{A_\alpha}, \alpha \in \Delta \})$.

Proof 3:

Let $x_H \in f_c^{-1}(\mathcal{Y} \{ \mathcal{C}_{B_\alpha}, \alpha \in \Lambda \})$, then for some $\alpha_0 \in \Lambda$, there exists $y \in Y, K \subseteq Y$ such that $\{y\} \delta_Y K$, $B_{\alpha_0} \delta_Y K$ and $f_c^{-1}(y_K) = x_H$, by proposition (2.1.1.5), then $f_c^{-1}(y_K) \in \mathcal{C}_{f^{-1}(B_{\alpha_0})}$, this mean $f^{-1}(B_{\alpha_0}) \delta_X f^{-1}(K)$, then for some $\alpha_0 \in \Lambda$, $x_H \in \mathcal{C}_{f^{-1}(B_{\alpha_0})}$. Hence $x_H \in \mathcal{Y} \{ \mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda \}$.

Conversely, let $x_H \in \mathcal{Y} \{ \mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda \}$, then for some $\alpha_0 \in \Lambda$, $x \in X, H \subseteq X$ such that $\{x\} \delta_X H$ and $f^{-1}(B_{\alpha_0}) \delta_X H$, by proposition (2.1.1.4), then $f(f^{-1}(B_{\alpha_0})) \delta_Y f(H)$. But $f(f^{-1}(B_{\alpha_0})) \subseteq B_{\alpha_0}$, therefore $B_{\alpha_0} \delta_Y f(H)$, then

$f_c(x_H) \in \mathcal{C}_{B_{\alpha_0}}$ for some $\alpha_0 \in \Lambda$, thus $f_c(x_H) \in \bigvee \{\mathcal{C}_{B_{\alpha}}, \alpha \in \Lambda\}$, by proposition (2.1.1.5), then $x_H \in f_c^{-1}(\bigvee \{\mathcal{C}_{B_{\alpha}}, \alpha \in \Lambda\})$.

Proof 4:

Let $x_H \in f_c^{-1}(\bigwedge \{\mathcal{C}_{B_{\alpha}}, \alpha \in \Lambda\})$, then for each $\alpha \in \Lambda$, $\exists y \in Y, K \subseteq Y$ such that $\{y\} \delta_Y K, B_{\alpha} \delta_Y K$ and $f_c^{-1}(y_K) = x_H$, by proposition (2.1.1.5), then $f_c^{-1}(y_K) \in \mathcal{C}_{f^{-1}(B_{\alpha})}$, this mean $f^{-1}(B_{\alpha}) \delta_X f^{-1}(K)$, then for each $\alpha \in \Lambda$, $x_H \in \mathcal{C}_{f^{-1}(B_{\alpha})}$. Hence $x_H \in \bigwedge \{\mathcal{C}_{f^{-1}(B_{\alpha})}, \alpha \in \Lambda\}$.

Conversely, let $x_H \in \bigwedge \{\mathcal{C}_{f^{-1}(B_{\alpha})}, \alpha \in \Lambda\}$, then for each $\alpha \in \Lambda$, $x \in X, H \subseteq X$ such that $\{x\} \delta_X H$ and $f^{-1}(B_{\alpha}) \delta_X H$, by proposition (2.1.1.4), then $f_c(x_H) \in \mathcal{C}_{f(f^{-1}(B_{\alpha}))}$, this mean $f(f^{-1}(B_{\alpha})) \delta_Y f(H)$. But $f(f^{-1}(B_{\alpha})) \subseteq B_{\alpha}$, therefore $B_{\alpha} \delta_Y f(H)$, then $f_c(x_H) \in \mathcal{C}_{B_{\alpha}}$ for each $\alpha \in \Lambda$, thus $f_c(x_H) \in \bigwedge \{\mathcal{C}_{B_{\alpha}}, \alpha \in \Lambda\}$, by proposition (2.1.1.5), then $x_H \in f_c^{-1}(\bigwedge \{\mathcal{C}_{B_{\alpha}}, \alpha \in \Lambda\})$

2.1.2 Shrink Central Continuous Function

In this section, we introduce the definition of Shrink Central Continuous Function and its most important properties. And the relationship of this type of functions with the separation axioms in \mathcal{C} -topological space.

Definition 2.1.2.1:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a shrink function and $(X, \delta_X, \mathfrak{C}_X), (Y, \delta_Y, \mathfrak{C}_Y)$ are \mathcal{C} -topological spaces, we call that $f_c: (X, \delta_X, \mathfrak{C}_X) \rightarrow (Y, \delta_Y, \mathfrak{C}_Y)$ is a shrink \mathcal{C} -continuous function at the center point x_A with respect to the function f , if for each $\mathcal{C}_V \in \mathfrak{C}_Y$ such that $f(A) \delta_Y V$, there exists $\mathcal{C}_U \in \mathfrak{C}_X$ with $U \delta_X A$ satisfy that, for each $f(U) \delta_Y K, K \subseteq Y$, then $V \delta_Y K$. And f_c is called a shrink \mathcal{C} -continuous, if its shrink \mathcal{C} -continuous function at each center point in \mathcal{C}_X .

Example 2.1.2.2:

Let $(X, \delta_X, \mathfrak{T}_{\mathcal{C}X})$ and $(Y, \delta_Y, \mathfrak{T}_{\mathcal{C}Y})$ are \mathcal{C} – topological space such that $X = \{1, 2\}$, $Y = \{a, b\}$ and δ_X, δ_Y are discrete proximity, where $\mathfrak{T}_{\mathcal{C}X} = \{\mathcal{C}_X, \mathcal{C}_\emptyset, \mathcal{C}_{\{1\}}, \mathcal{C}_X^{\{\{1\}\}}\}$ and $\mathfrak{T}_{\mathcal{C}Y} = \{\mathcal{C}_Y, \mathcal{C}_\emptyset, \mathcal{C}_{\{a\}}, \mathcal{C}_Y^{\{\{a\}\}}\}$ are \mathcal{C} – topologies. Define f from (X, δ_X) into (Y, δ_Y) such that $f(1) = a$ and $f(2) = b$. Then $f_{\mathcal{C}}$ is shrink \mathcal{C} – continuous function at the center point $1_{\{1\}}$.

Now we present some equivalent theorems for the definition of shrink \mathcal{C} – continuous function in \mathcal{C} -topological space.

Proposition 2.1.2.3:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism shrink function, then $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{T}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{T}_{\mathcal{C}Y})$ is shrink \mathcal{C} – continuous function if and only if

1. $\mathcal{C}_{f^{-1}(V)} \in \mathfrak{T}_{\mathcal{C}X}$, for each $\mathcal{C}_V \in \mathfrak{T}_{\mathcal{C}Y}$.
2. $\mathcal{C}_{f^{-1}(F)}$ is \mathcal{C} –closed subset of \mathcal{C}_X , for each \mathcal{C}_F is \mathcal{C} –closed subset of \mathcal{C}_Y .
3. $\text{cl}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(B)}) \leq_{\mathcal{C}} f^{-1}(\text{cl}_{\mathcal{C}}(\mathcal{C}_B))$, center set subset of \mathcal{C}_Y .
4. $f_{\mathcal{C}}(\text{cl}_{\mathcal{C}}(\mathcal{C}_A)) \leq_{\mathcal{C}} \text{cl}_{\mathcal{C}}(\mathcal{C}_{f(A)})$, for each \mathcal{C}_A center set subset of \mathcal{C}_X .
5. $f_{\mathcal{C}}^{-1}(\text{int}_{\mathcal{C}}(\mathcal{C}_B)) \leq_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(B)})$, for each \mathcal{C}_B center set subset of \mathcal{C}_Y .
6. $\text{int}_{\mathcal{C}}(\mathcal{C}_{f(A)}) \leq_{\mathcal{C}} f_{\mathcal{C}}(\text{int}_{\mathcal{C}}\mathcal{C}_A)$, for each \mathcal{C}_A center set subset of \mathcal{C}_X .

Proof 1:

Let $f_{\mathcal{C}}$ is shrink \mathcal{C} – continuous function and $\mathcal{C}_V \in \mathfrak{T}_{\mathcal{C}Y}$, and let $A \subseteq X$ such that $A\delta_X f^{-1}(V)$, by δ – continuous of f , then $f(A)\delta_Y f^{-1}(V)$, but $f(f^{-1}(V)) \subseteq V$, then $f(A)\delta_Y V$, since $f_{\mathcal{C}}$ is shrink \mathcal{C} – continuous, there exists $\mathcal{C}_U \in \mathfrak{T}_{\mathcal{C}X}$ such that $A\delta_X U$ and $\mathcal{C}_{f(U)} \leq_{\mathcal{C}} \mathcal{C}_V$, by proposition (2.1.1.6), then $\mathcal{C}_{f^{-1}(f(U))} \leq_{\mathcal{C}} \mathcal{C}_{f^{-1}(V)}$, then $\mathcal{C}_U \leq_{\mathcal{C}} \mathcal{C}_{f^{-1}(V)}$. Hence $\mathcal{C}_{f^{-1}(V)} \in \mathfrak{T}_{\mathcal{C}X}$.

Conversely, let $A \subseteq X$ and $\mathcal{C}_V \in \mathfrak{J}_{\mathcal{C}Y}$ such that $f(A)\delta_Y V$, by assumption, then $\mathcal{C}_{f^{-1}(V)} \in \mathfrak{J}_{\mathcal{C}X}$ and $\mathcal{C}_{f(f^{-1}(V))} \leq_c \mathcal{C}_V$, but $f(A)\delta_Y V$, then $f^{-1}(f(A))\delta_X f^{-1}(V)$, therefore $A\delta_X f^{-1}(V)$, put $\mathcal{C}_U =_c \mathcal{C}_{f^{-1}(V)} \in \mathfrak{J}_{\mathcal{C}X}$. Hence f_c is shrink \mathcal{C} – continuous.

Proof 2:

Let f_c is shrink \mathcal{C} – continuous function and \mathcal{C}_F is \mathcal{C} –closed in Y , then $\mathcal{C}_Y^{(F)} \in \mathfrak{J}_{\mathcal{C}Y}$ since f_c is shrink \mathcal{C} – continuous and by part 1 , then $\mathcal{C}_X^{(f^{-1}(F))} \in \mathfrak{J}_{\mathcal{C}X}$. Hence $\mathcal{C}_{f^{-1}(F)}$ is \mathcal{C} –closed in X .

Conversely, let $\mathcal{C}_V \in \mathfrak{J}_{\mathcal{C}Y}$, then $\mathcal{C}_Y^{(V)}$ is \mathcal{C} –closed in Y , by hypotheses, then $\mathcal{C}_X^{(f^{-1}(V))}$ is \mathcal{C} –closed in X . Therefore $\mathcal{C}_{f^{-1}(V)} \in \mathfrak{J}_{\mathcal{C}X}$. Hence f_c is shrink \mathcal{C} – continuous.

Proof 3:

Let f_c is shrink \mathcal{C} – continuous function, since $\text{cl}_c(\mathcal{C}_B)$ is \mathcal{C} –closed subset of \mathcal{C}_Y , by part 2 , then $f^{-1}(\text{cl}_c(\mathcal{C}_B))$ is \mathcal{C} –closed in X , since $\text{cl}_c(f^{-1}(\text{cl}_c(\mathcal{C}_B))) =_c f^{-1}(\text{cl}_c(\mathcal{C}_B))$, $\mathcal{C}_B \leq_c \text{cl}_c(\mathcal{C}_B)$, then $\mathcal{C}_{f^{-1}(B)} \leq_c f^{-1}(\text{cl}_c(\mathcal{C}_B))$, therefore $\text{cl}_c(\mathcal{C}_{f^{-1}(B)}) \leq_c f^{-1}(\text{cl}_c(\mathcal{C}_B))$.

Conversely, let \mathcal{C}_F be any \mathcal{C} –closed in Y , so that $\text{cl}_c(\mathcal{C}_F) =_c \mathcal{C}_F$. By hypothesis $\text{cl}_c(\mathcal{C}_{f^{-1}(F)}) \leq_c f^{-1}(\text{cl}_c(\mathcal{C}_F)) =_c \mathcal{C}_{f^{-1}(F)}$. But $\mathcal{C}_{f^{-1}(F)} \leq_c \text{cl}_c(\mathcal{C}_{f^{-1}(F)})$, therefore $\text{cl}_c(\mathcal{C}_{f^{-1}(F)}) =_c \mathcal{C}_{f^{-1}(F)}$ and $\mathcal{C}_{f^{-1}(F)}$ is \mathcal{C} –closed in X , by part 2, then f_c is shrink \mathcal{C} – continuous function.

Proof 4:

Let f_c is shrink \mathcal{C} – continuous function and \mathcal{C}_A center set in X , since $\mathcal{C}_{f(A)}$ center set in Y and by part 3, then $\text{cl}_c(\mathcal{C}_{f^{-1}(f(A))}) \preceq_c f_c^{-1}(\text{cl}_c(\mathcal{C}_{f(A)}))$, since f is (1 – 1) function, then $\text{cl}_c(\mathcal{C}_A) \preceq_c f_c^{-1}(\text{cl}_c(\mathcal{C}_{f(A)}))$, by proposition 2.1.1.6 (1), then $f_c(\text{cl}_c(\mathcal{C}_A)) \preceq_c \text{cl}_c(\mathcal{C}_{f(A)})$.

Conversely, let \mathcal{C}_B be an arbitrary \mathcal{C} –closed in Y , then $\mathcal{C}_{f^{-1}(B)} \preceq_c \mathcal{C}_X$. By hypothesis $f_c(\text{cl}_c(\mathcal{C}_{f^{-1}(B)})) \preceq_c \text{cl}_c(\mathcal{C}_{f(f^{-1}(B))})$, therefore $f_c(\text{cl}_c(\mathcal{C}_{f^{-1}(B)})) \preceq_c \text{cl}_c(\mathcal{C}_B)$, by proposition 2.1.1.6 (2), then $\text{cl}_c(\mathcal{C}_{f^{-1}(B)}) \preceq_c f_c^{-1}(\text{cl}_c(\mathcal{C}_B))$. But \mathcal{C}_B is \mathcal{C} –closed, then $\text{cl}_c(\mathcal{C}_B) =_c \mathcal{C}_B$, there fore $\text{cl}_c(\mathcal{C}_{f^{-1}(B)}) \preceq_c \mathcal{C}_{f^{-1}(B)}$. Hence $\mathcal{C}_{f^{-1}(B)}$ is \mathcal{C} –closed in X , by part 2, then f_c is shrink \mathcal{C} – continuous.

Proof 5:

Let f_c is shrink \mathcal{C} – continuous function, since $\text{int}_c(\mathcal{C}_B)$ is \mathcal{C} –open in Y , by part 1 , then $f_c^{-1}(\text{int}_c(\mathcal{C}_B))$ is \mathcal{C} –open $\preceq_c \mathcal{C}_X$, therefore $\text{int}_c(f_c^{-1}(\text{int}_c(\mathcal{C}_B))) =_c f_c^{-1}(\text{int}_c(\mathcal{C}_B))$, since $\text{int}_c(\mathcal{C}_B) \preceq_c \mathcal{C}_B$ and by part 4, then $f_c^{-1}(\text{int}_c(\mathcal{C}_B)) \preceq_c \mathcal{C}_{f^{-1}(B)}$. Hence $f_c^{-1}(\text{int}_c(\mathcal{C}_B)) \preceq_c \text{int}_c(\mathcal{C}_{f^{-1}(B)})$.

Conversely, let \mathcal{C}_U be any \mathcal{C} –open set in Y , so that $\text{int}_c(\mathcal{C}_U) =_c \mathcal{C}_U$. By hypothesis $\mathcal{C}_{f^{-1}(U)} =_c f^{-1}(\text{int}_c(\mathcal{C}_U)) \preceq_c \text{int}_c(\mathcal{C}_{f^{-1}(U)})$, therefore $\text{int}_c(\mathcal{C}_{f^{-1}(U)}) =_c \mathcal{C}_{f^{-1}(U)}$. Hence $\mathcal{C}_{f^{-1}(U)}$ is \mathcal{C} – open in X , by part 1, then f_c is shrink \mathcal{C} – continuous function.

Proof 6:

Let f_c is shrink \mathcal{C} – continuous function and \mathcal{C}_A center set in X , since $\mathcal{C}_{f(A)}$ center set in Y by part (5) , then $f_c^{-1}(\text{int}_c(\mathcal{C}_{f(A)})) \preceq_c \text{int}_c(\mathcal{C}_{f^{-1}(f(A))})$, since f

is $(1 - 1)$ function, then $f_c^{-1}(\text{int}_c(\mathcal{C}_{f(A)})) \preceq_c \text{int}_c(\mathcal{C}_A)$, by proposition 2.1.2.3(6), then $\text{int}_c(\mathcal{C}_{f(A)}) \preceq_c f_c(\text{int}_c(\mathcal{C}_A))$.

Conversely, let \mathcal{C}_U be an arbitrary \mathcal{C} -open in Y , then $\mathcal{C}_{f^{-1}(U)} \preceq_c \mathcal{C}_X$. By hypothesis $\text{int}_c(\mathcal{C}_{f(f^{-1}(U))}) \preceq_c f_c(\text{int}_c(\mathcal{C}_{f^{-1}(U)}))$, therefore $\text{int}_c(\mathcal{C}_U) \preceq_c f_c(\text{int}_c(\mathcal{C}_{f^{-1}(U)}))$, by proposition 2.1.2.3(5), then $f_c^{-1}(\text{int}_c(\mathcal{C}_U)) \preceq_c \text{int}_c(\mathcal{C}_{f^{-1}(U)})$. But \mathcal{C}_U is \mathcal{C} -open, then $\text{int}_c(\mathcal{C}_U) =_c \mathcal{C}_U$, therefore $\mathcal{C}_{f^{-1}(U)} \preceq_c \text{int}_c(\mathcal{C}_{f^{-1}(U)})$. Hence $\mathcal{C}_{f^{-1}(U)}$ is \mathcal{C} -open in X , by proposition 2.1.2.3(1), then f_c is shrink \mathcal{C} -continuous.

Proposition 2.1.2.4:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$, $g: (Y, \delta_Y) \rightarrow (Z, \delta_Z)$ be a δ -homeomorphism shrink functions. If $f_c: (X, \delta_X, \mathfrak{I}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{I}_{cY})$ and $g_c: (Y, \delta_Y, \mathfrak{I}_{cY}) \rightarrow (Z, \delta_Z, \mathfrak{I}_{cZ})$ are shrink \mathcal{C} -continuous functions, then $g_c \circ f_c$ is shrink \mathcal{C} -continuous.

Proof:

Let \mathcal{C}_A center set in Z , since g_c is shrink \mathcal{C} -continuous, by proposition 2.1.2.3(1), then $\mathcal{C}_{g^{-1}(A)}$ is \mathcal{C} -open in Y , and since f_c is shrink \mathcal{C} -continuous, by proposition 2.1.2.3(1), then $\mathcal{C}_{f^{-1}(g^{-1}(A))}$ is \mathcal{C} -open in X , therefore $\mathcal{C}_{(g \circ f)^{-1}(A)}$ is \mathcal{C} -open in X . Hence $g_c \circ f_c$ is shrink \mathcal{C} -continuous.

Definition 2.1.2.5:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a shrink function, we call that $f_c: (X, \delta_X, \mathfrak{I}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{I}_{cY})$ is shrink \mathcal{C} -open function if $\mathcal{C}_{f(A)}$ \mathcal{C} -open subset of \mathcal{C}_Y for each \mathcal{C}_A \mathcal{C} -open subset of \mathcal{C}_X . And f_c is said to be shrink \mathcal{C} -closed function if $\mathcal{C}_{f(F)}$ \mathcal{C} -closed subset of \mathcal{C}_Y for each \mathcal{C}_F \mathcal{C} -closed subset of \mathcal{C}_X .

Example 2.1.2.6:

Let $(X, \delta_X, \mathfrak{T}_{\mathcal{C}X})$ and $(Y, \delta_Y, \mathfrak{T}_{\mathcal{C}Y})$ are \mathcal{C} – topological space such that $X = \{1,2,3\}$, $Y = \{a, b, c\}$ and δ_X, δ_Y are discrete proximity, where $\mathfrak{T}_{\mathcal{C}X} = \{\mathcal{C}_X, \mathcal{C}_\emptyset, \mathcal{C}_{\{1\}}, \mathcal{C}_{\{2\}}, \mathcal{C}_{\{1,2\}}\}$ and $\mathfrak{T}_{\mathcal{C}Y} = \{\mathcal{C}_Y, \mathcal{C}_\emptyset, \mathcal{C}_{\{a\}}, \mathcal{C}_{\{c\}}, \mathcal{C}_{\{a,c\}}\}$ are \mathcal{C} – topologies on X, Y respectively. Define f from (X, δ_X) into (Y, δ_Y) such that $f(1) = c$ and $f(2) = a, f(3) = b$ Then $f_{\mathcal{C}}$ is shrink \mathcal{C} – open function.

Proposition 2.1.2.7:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism shrink function, then $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{T}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{T}_{\mathcal{C}Y})$ is shrink \mathcal{C} – open function if and only if

1. $f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_E)) \preceq_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_{f(E)})$, for each \mathcal{C}_E center set subset of \mathcal{C}_X .
2. $\text{cl}_{\mathcal{C}}(\mathcal{C}_{f(E)}) \preceq_{\mathcal{C}} f_{\mathcal{C}}(\text{cl}_{\mathcal{C}}(\mathcal{C}_E))$, for each \mathcal{C}_E center set subset of \mathcal{C}_X .

Proof 1:

Let $f_{\mathcal{C}}$ is shrink \mathcal{C} – open function and \mathcal{C}_E center set in X , since $\text{int}_{\mathcal{C}}(\mathcal{C}_E)$ \mathcal{C} – open set in X , then $f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_E))$ \mathcal{C} – open $\preceq_{\mathcal{C}} \mathcal{C}_Y$, since $\text{int}_{\mathcal{C}}(\mathcal{C}_E) \preceq_{\mathcal{C}} \mathcal{C}_E$ and by proposition 2.1.1.6(1) , then $f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_E)) \preceq_{\mathcal{C}} \mathcal{C}_{f(E)}$. Hence

$$f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_E)) \preceq_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_{f(E)}).$$

Conversely, let \mathcal{C}_U be an arbitrary \mathcal{C} –open in X , then $\mathcal{C}_U =_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_U)$, therefore $\mathcal{C}_{f(U)} =_{\mathcal{C}} f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_U))$. By hypothesis $f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_U)) \preceq_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_{f(U)})$, then $\mathcal{C}_{f(U)} \preceq_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_{f(U)})$, thus $\mathcal{C}_{f(U)}$ is \mathcal{C} –open in Y . Hence $f_{\mathcal{C}}$ is shrink \mathcal{C} – open function.

Proof 2:

Let $f_{\mathcal{C}}$ is shrink \mathcal{C} – closed function and \mathcal{C}_E center set in X , since $\text{cl}_{\mathcal{C}}(\mathcal{C}_E)$ \mathcal{C} – closed set in X , then $f_{\mathcal{C}}(\text{cl}_{\mathcal{C}}(\mathcal{C}_E))$ \mathcal{C} – closed in Y , since $\mathcal{C}_E \preceq_{\mathcal{C}} \text{cl}_{\mathcal{C}}(\mathcal{C}_E)$ and by

proposition 2.1.1.6(1), then $\mathcal{C}_{f(E)} \leq_c f_c(\text{cl}_c(\mathcal{C}_E))$. Hence $\text{cl}_c(\mathcal{C}_{f(E)}) \leq_c f_c(\text{cl}_c(\mathcal{C}_E))$.

Conversely, let \mathcal{C}_F be an arbitrary \mathcal{C} -closed in X , then $\mathcal{C}_F =_c \text{cl}_c(\mathcal{C}_F)$. By hypothesis $\text{cl}_c(\mathcal{C}_{f(F)}) \leq_c f_c(\text{cl}_c(\mathcal{C}_F)) =_c \mathcal{C}_{f(F)}$, therefore $\text{cl}_c(\mathcal{C}_{f(F)}) \leq_c \mathcal{C}_{f(F)}$, thus $\mathcal{C}_{f(F)}$ is \mathcal{C} -closed in Y . Hence f_c is shrink \mathcal{C} -closed function.

Definition 2.1.2.8:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a bijective shrink function, we call that $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ is shrink \mathcal{C} -homeomorphism function iff f_c and f_c^{-1} are shrink \mathcal{C} -continuous functions.

Remarks 2.1.2.9:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ and $g: (Y, \delta_Y) \rightarrow (Z, \delta_Z)$ are δ -homeomorphism shrink function,

1. If $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ is shrink \mathcal{C} -homeomorphism function, then f_c^{-1} is also shrink \mathcal{C} -homeomorphism function.
2. If $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ and $g_c: (Y, \delta_Y, \mathfrak{S}_{cY}) \rightarrow (Z, \delta_Z, \mathfrak{S}_{cZ})$ are shrink \mathcal{C} -homeomorphism function, then $g_c \circ f_c$ is shrink \mathcal{C} -homeomorphism function.

Proposition 2.1.2.10:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ -homeomorphism shrink function, then $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ is

1. Shrink \mathcal{C} -homeomorphism function if and only if for each \mathcal{C}_B center set subset of \mathcal{C}_Y , then $\text{cl}_c(\mathcal{C}_{f^{-1}(B)}) =_c f_c^{-1}(\text{cl}_c(\mathcal{C}_B))$.
2. Shrink \mathcal{C} -homeomorphism function if and only if for each \mathcal{C}_B center set subset of \mathcal{C}_Y , then $\text{int}(\mathcal{C}_{f^{-1}(B)}) =_c f_c^{-1}(\text{int}_c(\mathcal{C}_B))$

Proof :

1. By propositions 2.1.2.3(3) and 2.1.2.7(2).
2. By propositions 2.1.2.3(5) and 2.1.2.7(1).

We now introduce some relationships between the concept of shrink \mathcal{C} – continuity and separation axioms in \mathcal{C} – topological space.

Proposition 2.1.2.11:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism shrink function and $f_c: (X, \delta_X, \mathfrak{S}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ be a shrink \mathcal{C} – open function if $(X, \delta_X, \mathfrak{S}_{\mathcal{C}X})$

is $\mathfrak{S}_{\mathcal{C}2}$ – space, then $(Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ is $\mathfrak{S}_{\mathcal{C}2}$ – space.

Proof:

Let f_c is shrink \mathcal{C} – open function, $(X, \delta_X, \mathfrak{S}_{\mathcal{C}X})$ is $\mathfrak{S}_{\mathcal{C}2}$ – space and let y_{K_1}, y_{K_2} are center points in Y such that $y_{K_1} \neq_c y_{K_2}$, since f is bijective function, there exist x_{H_1}, x_{H_2} two center points in X such that $x_{H_1} \neq_c x_{H_2}$, where $f_c(x_{H_1}) =_c y_{K_1}$ and $f_c(x_{H_2}) =_c y_{K_2}$. But $(X, \delta_X, \mathfrak{S}_{\mathcal{C}X})$ is $\mathfrak{S}_{\mathcal{C}2}$ – space, there exist center open sets \mathcal{C}_{A_1} and \mathcal{C}_{A_2} in \mathcal{C}_X such that $x_{H_1} \in \mathcal{C}_{A_1}$, $x_{H_2} \in \mathcal{C}_{A_2}$ and $\mathcal{C}_{A_1} \wedge_c \mathcal{C}_{A_2} =_c \mathcal{C}_\emptyset$, this mean $A_1 \delta_X H_1$ and $A_2 \delta_X H_2$, by proposition 2.1.1.4, then $f_c(x_{H_1}) \in \mathcal{C}_{f(A_1)}$ and $f_c(x_{H_2}) \in \mathcal{C}_{f(A_2)}$, therefore $y_{K_1} \in \mathcal{C}_{f(A_1)}$ and $y_{K_2} \in \mathcal{C}_{f(A_2)}$. Since f_c is shrink \mathcal{C} – open function, then $\mathcal{C}_{f(A_1)}$ and $\mathcal{C}_{f(A_2)}$ are center open sets in \mathcal{C}_Y , by proposition 2.1.1.8(2), $\mathcal{C}_{f(A_1)} \wedge_c \mathcal{C}_{f(A_2)} =_c f_c(\mathcal{C}_{A_1} \wedge_c \mathcal{C}_{A_2}) =_c f_c(\mathcal{C}_\emptyset) =_c \mathcal{C}_\emptyset$. Hence $(Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ is $\mathfrak{S}_{\mathcal{C}2}$ – space.

Proposition 2.1.2.12:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism shrink function and $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ be a shrink \mathcal{C} – continuous function, if $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathfrak{S}_{c2} – space, then $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathfrak{S}_{c2} – space.

Proof:

Let f_c is shrink \mathcal{C} – continuous function, $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathfrak{S}_{c2} – space and let x_{H_1}, x_{H_2} are center points in X such that $x_{H_1} \neq_c x_{H_2}$, since f is bijective function, there exist y_{K_1}, y_{K_2} two center points in Y such that $y_{K_1} \neq_c y_{K_2}$, where $f_c^{-1}(y_{K_1}) =_c x_{H_1}$ and $f_c^{-1}(y_{K_2}) =_c x_{H_2}$. But $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathfrak{S}_{c2} – space, there exist center open sets \mathcal{C}_{B_1} and \mathcal{C}_{B_2} in Y such that $y_{K_1} \in \mathcal{C}_{B_1}$, $y_{K_2} \in \mathcal{C}_{B_2}$ and $\mathcal{C}_{B_1} \wedge_c \mathcal{C}_{B_2} =_c \mathcal{C}_\emptyset$, this mean $B_1 \delta_Y K_1$ and $B_2 \delta_Y K_2$, by proposition 2.1.1.5, then $f_c^{-1}(y_{K_1}) \in \mathcal{C}_{f^{-1}(B_1)}$ and $f_c^{-1}(y_{K_2}) \in \mathcal{C}_{f^{-1}(B_2)}$, therefore $x_{H_1} \in \mathcal{C}_{f^{-1}(B_1)}$ and $x_{H_2} \in \mathcal{C}_{f^{-1}(B_2)}$. Since f_c is shrink \mathcal{C} – open function, then $\mathcal{C}_{f^{-1}(B_1)}$ and $\mathcal{C}_{f^{-1}(B_2)}$ are center open sets in X , by proposition 2.1.1.8(4), $\mathcal{C}_{f^{-1}(B_1)} \wedge_c \mathcal{C}_{f^{-1}(B_2)} =_c f_c^{-1}(\mathcal{C}_{B_1} \wedge_c \mathcal{C}_{B_2}) =_c f_c^{-1}(\mathcal{C}_\emptyset) =_c \mathcal{C}_\emptyset$. Hence $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathfrak{S}_{c2} – space.

Proposition 2.1.2.13:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism shrink function and $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ be a shrink \mathcal{C} – homeomorphism function,

1. If $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathcal{C} – regular space, then $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathcal{C} – regular space.
2. If $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathcal{C} – normal space, then $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathcal{C} – normal space.

Proof 1:

Let $(X, \delta_X, \mathfrak{J}_{\mathcal{C}X})$ is \mathcal{C} – regular space, \mathcal{C}_F is \mathcal{C} – closed in Y and y_K be a center point in Y such that $y_K \notin \mathcal{C}_F$, since f is bijective function, there exist x_H center point in Y such that $f_{\mathcal{C}}(x_H) =_{\mathcal{C}} y_K$, by proposition 2.1.1.5, then $x_H \notin \mathcal{C}_{f^{-1}(F)}$, since $f_{\mathcal{C}}$ is shrink \mathcal{C} – continuous function, then $\mathcal{C}_{f^{-1}(F)}$ is \mathcal{C} – closed in X . But $(X, \delta_X, \mathfrak{J}_{\mathcal{C}X})$ is \mathcal{C} – regular space, there exist \mathcal{C} – open sets in X \mathcal{C}_{A_1} and \mathcal{C}_{A_2} such that $x_H \in \mathcal{C}_{A_1}$, $\mathcal{C}_{f^{-1}(F)} \preccurlyeq_{\mathcal{C}} \mathcal{C}_{A_2}$ and $\mathcal{C}_{A_1} \wedge_{\mathcal{C}} \mathcal{C}_{A_2} =_{\mathcal{C}} \mathcal{C}_{\emptyset}$, this mean $A_1 \delta_X H$ and $f^{-1}(F) \delta_X B \Rightarrow B \delta_X A_2$ for all B subset of X . by propositions (2.1.1.4) and (2.1.1.6), then $f_{\mathcal{C}}(x_H) \in \mathcal{C}_{f(A_1)}$, $\mathcal{C}_F \preccurlyeq_{\mathcal{C}} \mathcal{C}_{f(A_2)}$, since $f_{\mathcal{C}}^{-1}$ is shrink \mathcal{C} – continuous function, then $\mathcal{C}_{f(A_1)}$ and $\mathcal{C}_{f(A_2)}$ are \mathcal{C} – open sets in X , by proposition 2.1.1.8(2), $\mathcal{C}_{f(A_1)} \wedge_{\mathcal{C}} \mathcal{C}_{f(A_2)} =_{\mathcal{C}} \mathcal{C}_{f(\mathcal{C}_{A_1} \wedge_{\mathcal{C}} \mathcal{C}_{A_2})} =_{\mathcal{C}} \mathcal{C}_{\emptyset}$. Hence $(Y, \delta_Y, \mathfrak{J}_{\mathcal{C}Y})$ is \mathcal{C} – regular space.

Proof 2:

Let $(X, \delta_X, \mathfrak{J}_{\mathcal{C}X})$ is \mathcal{C} – normal space, $\mathcal{C}_{F_1}, \mathcal{C}_{F_2}$ are \mathcal{C} – closed sets in Y such that $\mathcal{C}_{F_1} \wedge_{\mathcal{C}} \mathcal{C}_{F_2} =_{\mathcal{C}} \mathcal{C}_{\emptyset}$, since $f_{\mathcal{C}}$ is shrink \mathcal{C} – continuous function, then $\mathcal{C}_{f^{-1}(F_1)}$ and $\mathcal{C}_{f^{-1}(F_2)}$ are \mathcal{C} – closed sets in X , since f is bijective function, then $\mathcal{C}_{f^{-1}(F_1)} \wedge_{\mathcal{C}} \mathcal{C}_{f^{-1}(F_2)} =_{\mathcal{C}} \mathcal{C}_{\emptyset}$. But $(X, \delta_X, \mathfrak{J}_{\mathcal{C}X})$ is \mathcal{C} – normal space, there exist \mathcal{C} – open sets in X \mathcal{C}_{A_1} and \mathcal{C}_{A_2} such that $\mathcal{C}_{f^{-1}(F_1)} \preccurlyeq_{\mathcal{C}} \mathcal{C}_{A_1}$, $\mathcal{C}_{f^{-1}(F_2)} \preccurlyeq_{\mathcal{C}} \mathcal{C}_{A_2}$ and $\mathcal{C}_{A_1} \wedge_{\mathcal{C}} \mathcal{C}_{A_2} =_{\mathcal{C}} \mathcal{C}_{\emptyset}$, this mean $f^{-1}(F_1) \delta_X B_1 \Rightarrow B_1 \delta_X A_1$ and $f^{-1}(F_2) \delta_X B_2 \Rightarrow B_2 \delta_X A_2$ for all B_1, B_2 subsets of X . by proposition 2.1.1.6(i), then $\mathcal{C}_{F_1} \preccurlyeq_{\mathcal{C}} \mathcal{C}_{f(A_1)}$ and $\mathcal{C}_{F_2} \preccurlyeq_{\mathcal{C}} \mathcal{C}_{f(A_2)}$, by proposition 2.1.1.8(2), then $\mathcal{C}_{f(A_1)} \wedge_{\mathcal{C}} \mathcal{C}_{f(A_2)} =_{\mathcal{C}} \mathcal{C}_{f(\mathcal{C}_{A_1} \wedge_{\mathcal{C}} \mathcal{C}_{A_2})} =_{\mathcal{C}} \mathcal{C}_{\emptyset}$. Hence $(Y, \delta_Y, \mathfrak{J}_{\mathcal{C}Y})$ is \mathcal{C} – normal space.

2.2 Reduction Central Continuous Function

In this section, we introduce a new definition of continuous functions for a special type of functions what we called reduction central continuous function on \mathcal{C} – topological spaces and show its important properties, in additional, we apply the separation axioms in these spaces.

2.2.1 Reduction Central Function

In this section, we introduce the definition of reduction central function and its most important properties.

Definition 2.2.1.1:

Let (X, δ_X) , (Y, δ_Y) are proximity spaces, a function $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$, is called reduction function if the image for center set \mathcal{C}_A of X is $f(\mathcal{C}_A) =_c \{ \langle f(A), K \rangle : \text{for each } K \subseteq Y, f(A) \delta_Y K \text{ iff } A \delta_X f^{-1}(K) \}$ and the reverse reduction function f is of form $f^{-1}(\mathcal{C}_B) =_c \{ \langle f^{-1}(B), H \rangle : \text{for each } H \subseteq X, f^{-1}(B) \delta_X H \text{ iff } B \delta_Y f(H) \}$.

Proposition 2.2.1.2:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be (1-1), δ - continuous reduction function and \mathcal{C}_B be center set in X , if $x_A \in \mathcal{C}_B$ then $f_c(x_A) \in f(\mathcal{C}_B)$.

Proof:

Let $x_A \in \mathcal{C}_B$, imply that $A \delta_X B$, by condition of f is (1-1) function we get that $A \delta_X f^{-1}(f(B))$ iff $A \delta_X B$ and by δ - continuous, $f(A) \delta_Y f(B)$. Now if $f(A) \delta_Y f(B)$ and $A \delta_X B$, given we have $A \delta_X f^{-1}(f(B))$ iff $f(A) \delta_Y f(B)$. Hence $f(x_A) \in f(\mathcal{C}_B)$.

Proposition 2.2.1.3:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be bijective, f^{-1} is δ - continuous reduction function and \mathcal{C}_V center set in Y , if $y_A \in \mathcal{C}_V$ then $f^{-1}_c(y_A) \in f^{-1}(\mathcal{C}_V)$.

Proof:

Let $y_A \in \mathcal{C}_V$ iff $A\delta_Y V$. To show that $f^{-1}(V)\delta_X f^{-1}(A)$ iff $V\delta_Y f(f^{-1}(A))$, let $f^{-1}(V)\delta_X f^{-1}(A)$, but $A\delta_Y V$ and f is onto, so $V\delta_Y f(f^{-1}(A))$. Now if $V\delta_Y f(f^{-1}(A))$ iff $A\delta_Y V$, since f is onto and by δ -continuous of f^{-1} we get $f^{-1}(A)\delta_X f^{-1}(V)$. Hence $f^{-1}_c(y_A) \in f^{-1}(\mathcal{C}_V)$.

Proposition 2.2.1.4:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ - homeomorphism reduction function. Hence the following is correct:

- i. If $\mathcal{C}_A \preceq_c \mathcal{C}_B$, then $f(\mathcal{C}_A) \preceq_c f(\mathcal{C}_B)$, for any $\mathcal{C}_A, \mathcal{C}_B \preceq_c \mathcal{C}_X$
- ii. If $\mathcal{C}_{Y_1} \preceq_c \mathcal{C}_{Y_2}$, then $f^{-1}(\mathcal{C}_{Y_1}) \preceq_c f^{-1}(\mathcal{C}_{Y_2})$, for any $\mathcal{C}_{Y_1}, \mathcal{C}_{Y_2} \preceq_c \mathcal{C}_Y$.

Proof:

- i. Let $\mathcal{C}_A \preceq_c \mathcal{C}_B$, through which we get $\forall \emptyset \neq H \subseteq X$, if $A\delta_X H$ then $B\delta_X H$. New let $y_V \in f(\mathcal{C}_A)$, so $f(A)\delta_Y V$ iff $A\delta_X f^{-1}(V)$, if $A\delta_X f^{-1}(V)$ then $B\delta_X f^{-1}(V)$, from δ - continuity and onto of f we get $f(B)\delta_Y V$, if $f(A)\delta_Y V$, by δ - continuity of f^{-1} we get $A\delta_X f^{-1}(V)$, then $B\delta_X f^{-1}(V)$ and by δ - continuity and onto of f we get $f(B)\delta_Y V$. Therefore $f(B)\delta_Y V$ iff $B\delta_X f^{-1}(V)$. So $y_V \in f(\mathcal{C}_B)$.
- ii. Let $\mathcal{C}_{Y_1} \preceq_c \mathcal{C}_{Y_2}$, so $\forall \emptyset \neq K \subseteq Y$, if $Y_1\delta_Y K$ then $Y_2\delta_Y K$, let $x_A \in f^{-1}(\mathcal{C}_{Y_1})$, imply that $f^{-1}(Y_1)\delta_X A$ iff $Y_1\delta_Y f(A)$, for if $Y_1\delta_Y f(A)$ then $Y_2\delta_Y f(A)$, by using 1-1 δ - continuity of f^{-1} we get $f^{-1}(Y_2)\delta_X A$. For $f^{-1}(Y_1)\delta_X A$, by using onto δ - continuity of f we get if $Y_1\delta_Y f(A)$ then $Y_2\delta_Y f(A)$ by using

1-1 δ - continuity of f^{-1} we get $f^{-1}(Y_2)\delta_X A$, so $f^{-1}(Y_2)\delta_X A$ iff $Y_2 \delta_Y f(A)$.

Hence $x_A \in f(\mathcal{C}_{Y_2})$.

Proposition 2.2.1.5:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ - homeomorphism reduction function and $\mathcal{C}_A, \mathcal{C}_B \preceq_c \mathcal{C}_X$, then the following are true:

- i. If $\mathcal{C}_B =_c f(\mathcal{C}_A)$, then $\mathcal{C}_A \preceq_c f^{-1}(\mathcal{C}_B)$.
- ii. If $\mathcal{C}_A =_c f^{-1}(\mathcal{C}_B)$, then $f(\mathcal{C}_A) \preceq_c \mathcal{C}_B$.

Proof:

- i. Let $x_K \in \mathcal{C}_A$ where $K \subseteq X$, then $A\delta_X K$, by δ - continuity of f we get if $A\delta_X K$ then $f(A) \delta_Y f(K)$, but $\mathcal{C}_B =_c f(\mathcal{C}_A)$, then $B\delta_Y f(K)$, by δ - continuity of f^{-1} we get $f^{-1}(B)\delta_X f^{-1}(f(K))$, since f is 1-1 function, then $f^{-1}(B)\delta_X K$. Hence $x_K \in f^{-1}(B)$.
- ii. Let $y_H \in f(\mathcal{C}_A)$, then $f(A)\delta_Y H$ iff $A\delta_X f^{-1}(H)$ where $H \subseteq X$, since $\mathcal{C}_A =_c f^{-1}(\mathcal{C}_B)$, therefore $f^{-1}(B)\delta_X f^{-1}(H)$, by δ - continuity of f , then $f(f^{-1}(B))\delta_Y f(f^{-1}(H))$, but f is onto, then $B\delta_Y H$. Hence $y_H \in \mathcal{C}_B$.

Proposition 2.2.1.6:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ - homeomorphism reduction function, $\{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\}$ and $\{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}$ are families of center subsets of \mathbb{C}_X and \mathbb{C}_Y respectively, where Δ and Λ are arbitrary index sets, then:

1. $f(\gamma_c \{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\}) =_c \gamma_c \{f(\mathcal{C}_{A_\alpha}), \alpha \in \Delta\}$
2. $f(\lambda_c \{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\}) =_c \lambda_c \{f(\mathcal{C}_{A_\alpha}), \alpha \in \Delta\}$
3. $f^{-1}(\gamma_c \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}) =_c \gamma_c \{f^{-1}(\mathcal{C}_{B_\alpha}), \alpha \in \Lambda\}$
4. $f^{-1}(\lambda_c \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}) =_c \lambda_c \{f^{-1}(\mathcal{C}_{B_\alpha}), \alpha \in \Lambda\}$

Proof 1:

Let $y_H \in f(\mathcal{V}_C \{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\})$, there exist $x_K \in \mathcal{V}_C \{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\}$ such that $f(x_K) = y_H$ and $f(K) = H$, for some $\alpha_0 \in \Delta$, $x_K \in \mathcal{C}_{A_{\alpha_0}}$ such that $A_{\alpha_0} \delta_X K$, by proposition(2.2.1.4), then $f(x_K) \in f(\mathcal{C}_{A_{\alpha_0}})$ $\alpha_0 \in \Delta$, therefore $f(A_{\alpha_0}) \delta_Y f(K) = H$ iff $A_{\alpha_0} \delta_X f^{-1}(f(K)) = f^{-1}(H)$. Thus $y_H \in f(\mathcal{C}_{A_{\alpha_0}})$ for some $\alpha_0 \in \Delta$. Hence $y_H \in \mathcal{V}_C \{f(\mathcal{C}_{A_\alpha}), \alpha \in \Delta\}$.

Conversely, let $y_H \in \mathcal{V}_C \{f(\mathcal{C}_{A_\alpha}), \alpha \in \Delta\}$, then for some $\alpha_0 \in \Delta$, $\exists y \in Y$, $\{y\} \delta_Y Y$ and $f(A_{\alpha_0}) \delta_Y H$ iff so by Definition(2.2.1.1), then $A_{\alpha_0} \delta_X f^{-1}(H)$, by proposition (2.2.1.3) $f^{-1}(y_H) \in \mathcal{C}_{A_{\alpha_0}}$ for some $\alpha_0 \in \Delta$, by proposition (2.2.1.2) and f is onto, then $y_H \in f(\mathcal{C}_{A_{\alpha_0}})$ for some $\alpha_0 \in \Delta$. Hence $y_H \in f(\mathcal{V}_C \{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\})$.

Proof 2:

It is similar to part 1.

Proof 3:

Let $x_K \in f^{-1}(\mathcal{V}_C \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\})$, there exist $y_H \in \mathcal{V}_C \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}$ such that $f^{-1}(y_H) = x_K$ and $f^{-1}(H) = K$, for some $\alpha_0 \in \Lambda$, $y_H \in \mathcal{C}_{B_{\alpha_0}}$ such that $B_{\alpha_0} \delta_Y H$, by proposition(2.2.1.3), then $f^{-1}(y_H) \in f^{-1}(\mathcal{C}_{B_{\alpha_0}})$ $\alpha_0 \in \Lambda$, therefore $f^{-1}(B_{\alpha_0}) \delta_X f^{-1}(H) = K$ iff $B_{\alpha_0} \delta_Y f(f^{-1}(H)) = f(K)$. Thus $x_K \in f^{-1}(\mathcal{C}_{B_{\alpha_0}})$ for some $\alpha_0 \in \Lambda$. Hence $x_K \in \mathcal{V}_C \{f^{-1}(\mathcal{C}_{B_\alpha}), \alpha \in \Lambda\}$.

Conversely, let $x_K \in \mathcal{V}_C \{f^{-1}(\mathcal{C}_{B_\alpha}), \alpha \in \Lambda\}$, then $x_K \in f^{-1}(\mathcal{C}_{B_{\alpha_0}})$ for some $\alpha_0 \in \Lambda$, $x \in X$, $\{x\} \delta_X X$, $K \delta_X X$ and $f^{-1}(B_{\alpha_0}) \delta_X K$, so by Definition(2.2.1.1) iff $B_{\alpha_0} \delta_Y f(H)$, by proposition (2.2.1.2) $f(x_K) \in \mathcal{C}_{B_{\alpha_0}}$ for some $\alpha_0 \in \Lambda$, by proposition (2.2.1.3) and f is onto, then $x_K \in f^{-1}(\mathcal{C}_{B_{\alpha_0}})$ for some $\alpha_0 \in \Lambda$. Hence $x_K \in f^{-1}(\mathcal{V}_C \{\mathcal{C}_{B_\alpha}, \alpha \in \Delta\})$.

Proof 4:

It is similar to part 3.

2.2.2 Reduction Central Continuous Function

In this section, we introduce the definition of reduction central continuous function and its most important properties. And the relationship of this type of functions with the separation axioms in \mathcal{C} -topological space.

Definition 2.2.2.1:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a reduction function and $(X, \delta_X, \mathfrak{I}_{\mathcal{C}X})$, $(Y, \delta_Y, \mathfrak{I}_{\mathcal{C}Y})$ are \mathcal{C} – topological spaces, we call that $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{I}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{I}_{\mathcal{C}Y})$ is a reduction \mathcal{C} – continuous function at the center point x_A with respect to the function f , if for each $\mathcal{C}_V \in \mathfrak{I}_{\mathcal{C}Y}$ such that $f(A) \delta_Y V$, there exists $\mathcal{C}_U \in \mathfrak{I}_{\mathcal{C}X}$ with $U \delta_X A$ satisfy that, for each $f(U) \delta_Y K$, $K \subseteq Y$, then $V \delta_Y K$. And $f_{\mathcal{C}}$ is called a reduction \mathcal{C} – continuous, if its reduction \mathcal{C} – continuous function at each center point in \mathcal{C}_X .

Proposition 2.2.2.2:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism reduction function, then $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{I}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{I}_{\mathcal{C}Y})$ is

1. Reduction \mathcal{C} – continuous function if and only if for each $\mathcal{C}_V \in \mathfrak{I}_{\mathcal{C}Y}$, then $f_{\mathcal{C}}^{-1}(\mathcal{C}_V) \in \mathfrak{I}_{\mathcal{C}X}$.
2. Reduction \mathcal{C} – continuous function if and only if for each \mathcal{C}_F is \mathcal{C} –closed subset of \mathcal{C}_Y , then $f_{\mathcal{C}}^{-1}(\mathcal{C}_F)$ is \mathcal{C} –closed subset of \mathcal{C}_X .
3. Reduction \mathcal{C} – continuous function if and only if for each \mathcal{C}_B center set subset of \mathcal{C}_Y , then $\text{cl}_{\mathcal{C}}(f_{\mathcal{C}}^{-1}(\mathcal{C}_B)) \preceq_{\mathcal{C}} f_{\mathcal{C}}^{-1}(\text{cl}_{\mathcal{C}}(\mathcal{C}_B))$.
4. Reduction \mathcal{C} – continuous function if and only if for each \mathcal{C}_A center set subset of \mathcal{C}_X , then $f_{\mathcal{C}}(\text{cl}_{\mathcal{C}}(\mathcal{C}_A)) \preceq_{\mathcal{C}} \text{cl}_{\mathcal{C}}(f_{\mathcal{C}}(\mathcal{C}_A))$.

5. Reduction \mathcal{C} – continuous function if and only if for each \mathcal{C}_B center set subset of \mathcal{C}_Y , then $f_c^{-1}(\text{int}_c(\mathcal{C}_B)) \preceq_c \text{int}_c(f_c^{-1}(\mathcal{C}_B))$.
6. Reduction \mathcal{C} – continuous function if and only if for each \mathcal{C}_A center set subset of \mathcal{C}_X , then $\text{int}_c(f_c(\mathcal{C}_A)) \preceq_c f_c(\text{int}_c \mathcal{C}_A)$.

Proof 1:

Let f_c is reduction \mathcal{C} – continuous function and $\mathcal{C}_V \in \mathfrak{J}_{cY}$, and let $A \subseteq X$ such that $A\delta_X f^{-1}(V)$, by δ – continuous of f , then $f(A)\delta_Y f(f^{-1}(V))$, but $f(f^{-1}(V)) \subseteq V$, then $f(A)\delta_Y V$, since f_c is reduction \mathcal{C} – continuous, there exists $\mathcal{C}_U \in \mathfrak{J}_{cX}$ such that $A\delta_X U$ and $f_c(\mathcal{C}_U) \preceq_c \mathcal{C}_V$, by proposition (2.2.1.3), then $f_c^{-1}(f_c(\mathcal{C}_U)) \preceq_c f_c^{-1}(\mathcal{C}_V)$, then $\mathcal{C}_U \preceq_c f_c^{-1}(\mathcal{C}_V)$. Hence $f_c^{-1}(\mathcal{C}_V) \in \mathfrak{J}_{cX}$. Conversely, let $A \subseteq X$ and $\mathcal{C}_V \in \mathfrak{J}_{cY}$ such that $f(A)\delta_Y V$, by assumption, then $f_c^{-1}(\mathcal{C}_V) \in \mathfrak{J}_{cX}$ and $f_c(f_c^{-1}(\mathcal{C}_V)) \preceq_c \mathcal{C}_V$, but $f(A)\delta_Y V$, then $f^{-1}(f(A))\delta_X f^{-1}(V)$, therefore $A\delta_X f^{-1}(V)$, put $\mathcal{C}_U =_c f_c^{-1}(\mathcal{C}_V) \in \mathfrak{J}_{cX}$. Hence f_c is reduction \mathcal{C} – continuous.

Proof 2:

Let f_c is reduction \mathcal{C} – continuous function and \mathcal{C}_F is \mathcal{C} –closed in Y , then $\mathcal{C}_Y^{(F)} \in \mathfrak{J}_{cY}$ since f_c is reduction \mathcal{C} – continuous and by part 1, then $f^{-1}(\mathcal{C}_Y^{(F)}) \in \mathfrak{J}_{cX}$. Hence $f_c^{-1}(\mathcal{C}_F)$ is \mathcal{C} –closed in X . Conversely, let $\mathcal{C}_V \in \mathfrak{J}_{cY}$, then $\mathcal{C}_Y^{(V)}$ is \mathcal{C} –closed in Y , by hypotheses, then $f^{-1}(\mathcal{C}_Y^{(V)})$ is \mathcal{C} –closed in X . Therefore $f_c^{-1}(\mathcal{C}_V) \in \mathfrak{J}_{cX}$. Hence f_c is reduction \mathcal{C} – continuous.

Proof 3:

Let f_c is reduction \mathcal{C} – continuous function, since $\text{cl}_c(\mathcal{C}_B)$ is \mathcal{C} –closed subset of \mathcal{C}_Y , by part 2, then $f_c^{-1}(\text{cl}_c(\mathcal{C}_B))$ is \mathcal{C} –closed in X , since $\text{cl}_c(f_c^{-1}(\text{cl}_c(\mathcal{C}_B))) =_c f_c^{-1}(\text{cl}_c(\mathcal{C}_B))$, $\mathcal{C}_B \preceq_c \text{cl}_c(\mathcal{C}_B)$, then $f_c^{-1}(\mathcal{C}_B) \preceq_c f_c^{-1}(\text{cl}_c(\mathcal{C}_B))$, therefore $\text{cl}_c(f_c^{-1}(\mathcal{C}_B)) \preceq_c f_c^{-1}(\text{cl}_c(\mathcal{C}_B))$.

Conversely, let \mathcal{C}_F be any \mathcal{C} -closed in Y , so that $\text{cl}_c(\mathcal{C}_F) =_c \mathcal{C}_F$. By hypothesis $\text{cl}_c(f_c^{-1}(\mathcal{C}_F)) \leq_c f_c^{-1}(\text{cl}_c(\mathcal{C}_F)) =_c f_c^{-1}(\mathcal{C}_F)$. But $f_c^{-1}(\mathcal{C}_F) \leq_c \text{cl}_c(f_c^{-1}(\mathcal{C}_F))$, therefore $\text{cl}_c(f_c^{-1}(\mathcal{C}_F)) =_c f_c^{-1}(\mathcal{C}_F)$ and $f_c^{-1}(\mathcal{C}_F)$ is \mathcal{C} -closed in X , by part 2, then f_c is reduction \mathcal{C} -continuous function.

Proof 4:

Let f_c is reduction \mathcal{C} -continuous function and \mathcal{C}_A center set in X , since $f_c(\mathcal{C}_A)$ center set in Y and by part 3, then $\text{cl}_c(f_c^{-1}(f_c(\mathcal{C}_A))) \leq_c f_c^{-1}(\text{cl}_c(f_c(\mathcal{C}_A)))$, since f is (1-1) function, then $\text{cl}_c(\mathcal{C}_A) \leq_c f_c^{-1}(\text{cl}_c(f_c(\mathcal{C}_A)))$, by proposition (2.2.1.3), then $f_c(\text{cl}_c(\mathcal{C}_A)) \leq_c \text{cl}_c(f_c(\mathcal{C}_A))$. Conversely, let \mathcal{C}_B be an arbitrary \mathcal{C} -closed in Y , then $f_c^{-1}(\mathcal{C}_B) \leq_c \mathcal{C}_X$. By hypothesis $f_c(\text{cl}_c(f_c^{-1}(\mathcal{C}_B))) \leq_c \text{cl}_c(f_c(f_c^{-1}(\mathcal{C}_B)))$, therefore $f_c(\text{cl}_c(f_c^{-1}(\mathcal{C}_B))) \leq_c \text{cl}_c(\mathcal{C}_B)$, by part 3, then $\text{cl}_c(f_c^{-1}(\mathcal{C}_B)) \leq_c f_c^{-1}(\text{cl}_c(\mathcal{C}_B))$. But \mathcal{C}_B is \mathcal{C} -closed, then $\text{cl}_c(\mathcal{C}_B) =_c \mathcal{C}_B$, there fore $\text{cl}_c(f_c^{-1}(\mathcal{C}_B)) \leq_c f_c^{-1}(\mathcal{C}_B)$. Hence $f_c^{-1}(\mathcal{C}_B)$ is \mathcal{C} -closed in X , by part 2, then f_c is reduction \mathcal{C} -continuous.

Proof 5:

Let f_c is reduction \mathcal{C} -continuous function, since $\text{int}_c(\mathcal{C}_B)$ is \mathcal{C} -open $\leq_c \mathcal{C}_Y$, by part 1, then $f_c^{-1}(\text{int}_c(\mathcal{C}_B))$ is \mathcal{C} -open $\leq_c \mathcal{C}_X$, therefore $\text{int}_c(f_c^{-1}(\text{int}_c(\mathcal{C}_B))) =_c f_c^{-1}(\text{int}_c(\mathcal{C}_B))$, since $\text{int}_c(\mathcal{C}_B) \leq_c \mathcal{C}_B$ and by part 4, then $f_c^{-1}(\text{int}_c(\mathcal{C}_B)) \leq_c f_c^{-1}(\mathcal{C}_B)$. Hence $f_c^{-1}(\text{int}_c(\mathcal{C}_B)) \leq_c \text{int}_c(f_c^{-1}(\mathcal{C}_B))$. Conversely, let \mathcal{C}_U be any \mathcal{C} -open set in Y , so that $\text{int}_c(\mathcal{C}_U) =_c \mathcal{C}_U$. By hypothesis $f_c^{-1}(\mathcal{C}_U) =_c f_c^{-1}(\text{int}_c(\mathcal{C}_U)) \leq_c \text{int}_c(f_c^{-1}(\mathcal{C}_U))$, therefore

$\text{int}_c(f_c^{-1}(\mathcal{C}_U)) =_c f_c^{-1}(\mathcal{C}_U)$. Hence $f_c^{-1}(\mathcal{C}_U)$ is \mathcal{C} – open in X , by part 1, then f_c is reduction \mathcal{C} – continuous function.

Proof 6:

Let f_c is reduction \mathcal{C} – continuous function and \mathcal{C}_A center set in X , since $f_c(\mathcal{C}_A)$ center set in Y by part 5, then $f_c^{-1}(\text{int}_c(f_c(\mathcal{C}_A))) \leq_c \text{int}_c(f_c^{-1}(f_c(\mathcal{C}_A)))$, since f is (1 – 1) function, then $f_c^{-1}(\text{int}_c(f_c(\mathcal{C}_A))) \leq_c \text{int}_c(\mathcal{C}_A)$, by proposition (2.2.1.3), then $\text{int}_c(f_c(\mathcal{C}_A)) \leq_c f_c(\text{int}_c(\mathcal{C}_A))$. Conversely, let \mathcal{C}_U be an arbitrary \mathcal{C} – open in Y , then $f_c^{-1}(\mathcal{C}_U) \leq_c \mathcal{C}_X$. By hypothesis $\text{int}_c(f_c(f_c^{-1}(\mathcal{C}_U))) \leq_c f_c(\text{int}_c(f_c^{-1}(\mathcal{C}_U)))$, therefore $\text{int}_c(\mathcal{C}_U) \leq_c f_c(\text{int}_c(f_c^{-1}(\mathcal{C}_U)))$, by proposition (2.2.1.3), then $f_c^{-1}(\text{int}_c(\mathcal{C}_U)) \leq_c \text{int}_c(f_c^{-1}(\mathcal{C}_U))$. But \mathcal{C}_U is \mathcal{C} – open, then $\text{int}_c(\mathcal{C}_U) =_c \mathcal{C}_U$, therefore $f_c^{-1}(\mathcal{C}_U) \leq_c \text{int}_c(f_c^{-1}(\mathcal{C}_U))$. Hence $f_c^{-1}(\mathcal{C}_U)$ is \mathcal{C} – open in X , by part 1, then f_c is reduction \mathcal{C} – continuous.

Definition 2.2.2.3:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a reduction function, we call that $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ is reduction \mathcal{C} – open function if $f_c(\mathcal{C}_A)$ \mathcal{C} – open subset of \mathcal{C}_Y for each \mathcal{C}_A \mathcal{C} – open subset of \mathcal{C}_X . And f_c is said to be reduction \mathcal{C} – closed function if $f_c(\mathcal{C}_F)$ \mathcal{C} – closed subset of \mathcal{C}_Y for each \mathcal{C}_F \mathcal{C} – closed subset of \mathcal{C}_X .

Proposition 2.2.2.4:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism reduction function, then $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ is

1. Reduction \mathcal{C} – open function if and only if for each \mathcal{C}_E center set subset of \mathcal{C}_X , then $f_c(\text{int}_c(\mathcal{C}_E)) \leq_c \text{int}_c(f_c(\mathcal{C}_E))$.

2. Reduction \mathcal{C} – closed function if and only if for each \mathcal{C}_E center set subset of \mathcal{C}_X , then $\text{cl}_{\mathcal{C}}(f_{\mathcal{C}}(\mathcal{C}_E)) \leq_{\mathcal{C}} f_{\mathcal{C}}(\text{cl}_{\mathcal{C}}(\mathcal{C}_E))$.

Proof 1:

Let $f_{\mathcal{C}}$ is reduction \mathcal{C} – open function and \mathcal{C}_E center set in X , since $\text{int}_{\mathcal{C}}(\mathcal{C}_E)$ \mathcal{C} – open set in X , then $f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_E))$ \mathcal{C} – open $\leq_{\mathcal{C}} \mathcal{C}_Y$, since $\text{int}_{\mathcal{C}}(\mathcal{C}_E) \leq_{\mathcal{C}} \mathcal{C}_E$ and by proposition (2.2.1.4), then $f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_E)) \leq_{\mathcal{C}} f_{\mathcal{C}}(\mathcal{C}_E)$. Hence $f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_E)) \leq_{\mathcal{C}} \text{int}_{\mathcal{C}}(f_{\mathcal{C}}(\mathcal{C}_E))$. Conversely, let \mathcal{C}_U be an arbitrary \mathcal{C} – open in X , then $\mathcal{C}_U =_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_U)$, therefore $f_{\mathcal{C}}(\mathcal{C}_U) =_{\mathcal{C}} f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_U))$. By hypothesis $f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_U)) \leq_{\mathcal{C}} \text{int}_{\mathcal{C}}(f_{\mathcal{C}}(\mathcal{C}_U))$, then $f_{\mathcal{C}}(\mathcal{C}_U) \leq_{\mathcal{C}} \text{int}_{\mathcal{C}}(f_{\mathcal{C}}(\mathcal{C}_U))$, thus $f_{\mathcal{C}}(\mathcal{C}_U)$ is \mathcal{C} – open in Y . Hence $f_{\mathcal{C}}$ is reduction \mathcal{C} – open function.

Proof 2:

Let $f_{\mathcal{C}}$ is reduction \mathcal{C} – closed function and \mathcal{C}_E center set in X , since $\text{cl}_{\mathcal{C}}(\mathcal{C}_E)$ \mathcal{C} – closed set in X , then $f_{\mathcal{C}}(\text{cl}_{\mathcal{C}}(\mathcal{C}_E))$ \mathcal{C} – closed in Y , since $\mathcal{C}_E \leq_{\mathcal{C}} \text{cl}_{\mathcal{C}}(\mathcal{C}_E)$ and by proposition (2.2.1.4), then $f_{\mathcal{C}}(\mathcal{C}_E) \leq_{\mathcal{C}} f_{\mathcal{C}}(\text{cl}_{\mathcal{C}}(\mathcal{C}_E))$. Hence $\text{cl}_{\mathcal{C}}(f_{\mathcal{C}}(\mathcal{C}_E)) \leq_{\mathcal{C}} f_{\mathcal{C}}(\text{cl}_{\mathcal{C}}(\mathcal{C}_E))$. Conversely, let \mathcal{C}_F be an arbitrary \mathcal{C} – closed in X , then $\mathcal{C}_F =_{\mathcal{C}} \text{cl}_{\mathcal{C}}(\mathcal{C}_F)$. By hypothesis $\text{cl}_{\mathcal{C}}(f_{\mathcal{C}}(\mathcal{C}_F)) \leq_{\mathcal{C}} f_{\mathcal{C}}(\text{cl}_{\mathcal{C}}(\mathcal{C}_F)) =_{\mathcal{C}} f_{\mathcal{C}}(\mathcal{C}_F)$, therefore $\text{cl}_{\mathcal{C}}(f_{\mathcal{C}}(\mathcal{C}_F)) \leq_{\mathcal{C}} f_{\mathcal{C}}(\mathcal{C}_F)$, thus $f_{\mathcal{C}}(\mathcal{C}_F)$ is \mathcal{C} – closed in Y . Hence $f_{\mathcal{C}}$ is reduction \mathcal{C} – closed function.

Definition 2.2.2.5:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a bijective reduction function, we call that $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{S}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ is reduction \mathcal{C} – homeomorphism function iff $f_{\mathcal{C}}$ and $f_{\mathcal{C}}^{-1}$ are reduction \mathcal{C} – continuous functions.

Proposition 2.2.2.6:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism reduction function, then $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{S}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ is

1. Reduction \mathcal{C} – homeomorphism function if and only if for each \mathcal{C}_B center set subset of \mathcal{C}_Y , then $\text{cl}_c(f_c^{-1}(\mathcal{C}_B)) =_c f_c^{-1}(\text{cl}_c(\mathcal{C}_B))$.
2. Reduction \mathcal{C} – homeomorphism function if and only if for each \mathcal{C}_B center set subset of \mathcal{C}_Y , then $\text{int}(f_c^{-1}(\mathcal{C}_B)) =_c f_c^{-1}(\text{int}_c(\mathcal{C}_B))$.

Proof :

Direct proof by proposition (2.2.2.2)

We now introduce some relationships between the concept of reduction \mathcal{C} – continuity and separation axioms in \mathcal{C} – topological space.

Proposition 2.2.2.7:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism reduction function and $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ be a reduction \mathcal{C} – open function if $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathfrak{S}_{c2} – space, then $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathfrak{S}_{c2} – space.

Proof:

Let f_c is reduction \mathcal{C} – open function, $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathfrak{S}_{c2} – space and let y_{K_1}, y_{K_2} are center points in Y such that $y_{K_1} \neq_c y_{K_2}$, since f is bijective function, there exist x_{H_1}, x_{H_2} two center points in X such that $x_{H_1} \neq_c x_{H_2}$, where $f_c(x_{H_1}) =_c y_{K_1}$ and $f_c(x_{H_2}) =_c y_{K_2}$. But $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathfrak{S}_{c2} – space, there exist center open sets in X \mathcal{C}_{A_1} and \mathcal{C}_{A_2} such that $x_{H_1} \in \mathcal{C}_{A_1}$, $x_{H_2} \in \mathcal{C}_{A_2}$ and $\mathcal{C}_{A_1} \wedge_c \mathcal{C}_{A_2} =_c \mathcal{C}_\emptyset$, this mean $A_1 \delta_X H_1$ and $A_2 \delta_X H_2$, by proposition (2.2.1.2), then $f_c(x_{H_1}) \in f_c(\mathcal{C}_{A_1})$ and $f_c(x_{H_2}) \in f_c(\mathcal{C}_{A_2})$, therefore $y_{K_1} \in f_c(\mathcal{C}_{A_1})$ and $y_{K_2} \in f_c(\mathcal{C}_{A_2})$. Since f_c is reduction \mathcal{C} – open function, then $f_c(\mathcal{C}_{A_1})$ and $f_c(\mathcal{C}_{A_2})$ are center open sets in Y , by proposition (2.2.1.6), $f_c(\mathcal{C}_{A_1}) \wedge_c f_c(\mathcal{C}_{A_2}) =_c f_c(\mathcal{C}_{A_1} \wedge_c \mathcal{C}_{A_2}) =_c f_c(\mathcal{C}_\emptyset) =_c \mathcal{C}_\emptyset$. Hence $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathfrak{S}_{c2} – space.

Proposition 2.2.2.8:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism reduction function and $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ be a reduction \mathcal{C} – continuous function, if $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathfrak{S}_{c2} – space, then $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathfrak{S}_{c2} – space.

Proof:

Let f_c is reduction \mathcal{C} – continuous function, $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathfrak{S}_{c2} – space and let x_{H_1}, x_{H_2} are center points in X such that $x_{H_1} \neq_c x_{H_2}$, since f is bijective function, there exist y_{K_1}, y_{K_2} two center points in Y such that $y_{K_1} \neq_c y_{K_2}$, where $f_c^{-1}(y_{K_1}) =_c x_{H_1}$ and $f_c^{-1}(y_{K_2}) =_c x_{H_2}$. But $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathfrak{S}_{c2} – space, there exist center open sets in Y \mathcal{C}_{B_1} and \mathcal{C}_{B_2} such that $y_{K_1} \in \mathcal{C}_{B_1}$, $y_{K_2} \in \mathcal{C}_{B_2}$ and $\mathcal{C}_{B_1} \wedge_c \mathcal{C}_{B_2} =_c \mathcal{C}_\emptyset$, this mean $B_1 \delta_Y K_1$ and $B_2 \delta_Y K_2$, by proposition (2.2.1.3), then $f_c^{-1}(y_{K_1}) \in f_c^{-1}(\mathcal{C}_{B_1})$ and $f_c^{-1}(y_{K_2}) \in f_c^{-1}(\mathcal{C}_{B_2})$, therefore $x_{H_1} \in f_c^{-1}(\mathcal{C}_{B_1})$ and $x_{H_2} \in f_c^{-1}(\mathcal{C}_{B_2})$. Since f_c is reduction \mathcal{C} – open function, then $f_c^{-1}(\mathcal{C}_{B_1})$ and $f_c^{-1}(\mathcal{C}_{B_2})$ are center open sets in X , by proposition (2.2.1.6), $f_c^{-1}(\mathcal{C}_{B_1}) \wedge_c f_c^{-1}(\mathcal{C}_{B_2}) =_c f_c^{-1}(\mathcal{C}_{B_1} \wedge_c \mathcal{C}_{B_2}) =_c f_c^{-1}(\mathcal{C}_\emptyset) =_c \mathcal{C}_\emptyset$. Hence $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathfrak{S}_{c2} – space.

Proposition 2.2.2.9:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism reduction function and $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ be a reduction \mathcal{C} – homeomorphism function, then :

1. If $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathcal{C} – regular space, then $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathcal{C} – regular space.
2. If $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathcal{C} – normal space, then $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathcal{C} – normal space.

Proof 1:

Let $(X, \delta_X, \mathfrak{J}_{CX})$ is \mathcal{C} – regular space, \mathcal{C}_F is \mathcal{C} – closed in Y and y_K be a center point in Y such that $y_K \notin \mathcal{C}_F$, since f is bijective function, there exist x_H center point in Y such that $f_c(x_H) =_c y_K$, by proposition (2.2.1.3), then $x_H \notin f_c^{-1}(\mathcal{C}_F)$, since f_c is reduction \mathcal{C} – continuous function, then $f_c^{-1}(\mathcal{C}_F)$ is \mathcal{C} – closed in X . But $(X, \delta_X, \mathfrak{J}_{CX})$ is \mathcal{C} – regular space, there exist \mathcal{C} – open sets in X \mathcal{C}_{A_1} and \mathcal{C}_{A_2} such that $x_H \in \mathcal{C}_{A_1}$, $f_c^{-1}(\mathcal{C}_F) \ll_c \mathcal{C}_{A_2}$ and $\mathcal{C}_{A_1} \wedge_c \mathcal{C}_{A_2} =_c \mathcal{C}_\emptyset$, this mean $A_1 \delta_X H$ and $f^{-1}(F) \delta_X B \Rightarrow B \delta_X A_2$ for all B subset of X . by proposition (2.2.1.2), then $f_c(x_H) \in f_c(\mathcal{C}_{A_1})$, $\mathcal{C}_F \ll_c f_c(\mathcal{C}_{A_2})$, since f_c^{-1} is reduction \mathcal{C} – continuous function, then $f_c(\mathcal{C}_{A_1})$ and $f_c(\mathcal{C}_{A_2})$ are \mathcal{C} – open sets in X , by proposition (2.2.1.6), $f_c(\mathcal{C}_{A_1}) \wedge_c f_c(\mathcal{C}_{A_2}) =_c f_c(\mathcal{C}_{A_1} \wedge_c \mathcal{C}_{A_2}) =_c \mathcal{C}_\emptyset$. Hence $(Y, \delta_Y, \mathfrak{J}_{CY})$ is \mathcal{C} – regular space.

Proof 2:

Let $(X, \delta_X, \mathfrak{J}_{CX})$ is \mathcal{C} – normal space, $\mathcal{C}_{F_1}, \mathcal{C}_{F_2}$ are \mathcal{C} – closed sets in Y such that $\mathcal{C}_{F_1} \wedge_c \mathcal{C}_{F_2} =_c \mathcal{C}_\emptyset$, since f_c is reduction \mathcal{C} – continuous function, then $f_c^{-1}(\mathcal{C}_{F_1})$ and $f_c^{-1}(\mathcal{C}_{F_2})$ are \mathcal{C} – closed sets in X , since f is bijective function, then $f_c^{-1}(\mathcal{C}_{F_1}) \wedge_c f_c^{-1}(\mathcal{C}_{F_2}) =_c \mathcal{C}_\emptyset$. But $(X, \delta_X, \mathfrak{J}_{CX})$ is \mathcal{C} – normal space, there exist \mathcal{C} – open sets in X \mathcal{C}_{A_1} and \mathcal{C}_{A_2} such that $f_c^{-1}(\mathcal{C}_{F_1}) \ll_c \mathcal{C}_{A_1}$, $f_c^{-1}(\mathcal{C}_{F_2}) \ll_c \mathcal{C}_{A_2}$ and $\mathcal{C}_{A_1} \wedge_c \mathcal{C}_{A_2} =_c \mathcal{C}_\emptyset$, this mean $f^{-1}(F_1) \delta_X B_1 \Rightarrow B_1 \delta_X A_1$ and $f^{-1}(F_2) \delta_X B_2 \Rightarrow B_2 \delta_X A_2$ for all B_1, B_2 subsets of X . by proposition (2.2.1.2), then $\mathcal{C}_{F_1} \ll_c f_c(\mathcal{C}_{A_1})$ and $\mathcal{C}_{F_2} \ll_c f_c(\mathcal{C}_{A_2})$, by proposition (2.2.1.6), then $f_c(\mathcal{C}_{A_1}) \wedge_c f_c(\mathcal{C}_{A_2}) =_c f_c(\mathcal{C}_{A_1} \wedge_c \mathcal{C}_{A_2}) =_c \mathcal{C}_\emptyset$. Hence $(Y, \delta_Y, \mathfrak{J}_{CY})$ is \mathcal{C} – normal space.

2.3 Stronger Central Continuous Function

In this section, we introduce a new definition of continuous functions for a special type of functions what we called stronger central continuous function on \mathcal{C} – topological spaces and show its important properties, in additional, we apply the separation axioms in these spaces.

2.3.1 Stronger Central Function

In this section, we present the definition of stronger central function and its most important properties.

Definition 2.3.1.1:

Let (X, δ_X) be a proximity space and $\delta_Y = f(\delta_X)$ be a proximity on Y , a function $f: (X, \delta_X) \rightarrow Y$ is called stronger function if the image for center set \mathcal{C}_A in X is $\mathcal{C}_{f(A)} = \{(f(A), B): f(A)\delta_Y B, \text{ for each } B \subseteq Y\}$, and the inverse of stronger \mathcal{C} -function f is of form $\mathcal{C}_{f^{-1}(B)} = \{(f^{-1}(B), H): f^{-1}(B)\delta_X H, \text{ for each } H \subseteq X\}$.

Proposition 2.3.1.2:

Let (X, δ_X) be a proximity space, $f: (X, \delta_X) \rightarrow Y$, a give bijective stronger function, and \mathcal{C}_B be center set in X . If $x_A \in \mathcal{C}_B$, then $f_C(x_A) \in \mathcal{C}_{f(B)}$.

Proof:

Let $x_A \in \mathcal{C}_B$, imply $A\delta_X B$. But f is (1 – 1) and by $\delta_Y = f(\delta_X)$, so we get that $A\delta_X B$ iff $f^{-1}(f(A))\delta_X f^{-1}(f(B))$ iff $f(A)\delta_Y f(B)$. Hence $f_C(x_A) \in \mathcal{C}_{f(B)}$.

Proposition 2.3.1.3:

Let (X, δ_X) are proximity space, $f: (X, \delta_X) \rightarrow Y$, give onto stronger function and \mathcal{C}_V be center set in Y , if $y_A \in \mathcal{C}_V$ then $f^{-1}_C(y_A) \in \mathcal{C}_{f^{-1}(V)}$.

Proof:

Let $y_A \in \mathcal{C}_V$, iff $A\delta_Y V$ iff $f^{-1}(A)\delta_X f^{-1}(V)$, by $\delta_Y = f(\delta_X)$, then $f^{-1}_c(y_A) \in \mathcal{C}_{f^{-1}(V)}$.

Proposition 2.3.1.4:

Let (X, δ_X) are proximity space, $f: (X, \delta_X) \rightarrow Y$, a give bijective stronger function,. Hence the following is correct:

- i. If $\mathcal{C}_A \preceq_c \mathcal{C}_B$, then $\mathcal{C}_{f(A)} \preceq_c \mathcal{C}_{f(B)}$, for any center sets $\mathcal{C}_A, \mathcal{C}_B$ in X .
- ii. If $\mathcal{C}_{Y_1} \preceq_c \mathcal{C}_{Y_2}$, then $\mathcal{C}_{f^{-1}(Y_1)} \preceq_c \mathcal{C}_{f^{-1}(Y_2)}$, for any center sets $\mathcal{C}_{Y_1}, \mathcal{C}_{Y_2}$ in Y .

Proof:

- i. Let $y_V \in \mathcal{C}_{f(A)}$ iff $f(A)\delta_Y V$ iff $f^{-1}(f(A))\delta_X f^{-1}(V)$ iff $A\delta_X f^{-1}(V)$, then $B\delta_X f^{-1}(V)$ iff $f^{-1}(f(B))\delta_X f^{-1}(V)$ iff $f(B)\delta_Y V$. Hence $y_V \in \mathcal{C}_{f(B)}$.
- ii. Let $\mathcal{C}_{Y_1} \preceq_c \mathcal{C}_{Y_2}$, then frome it we get, $\forall \emptyset \neq K \subseteq Y$, if $Y_1\delta_Y K$, imply $Y_2\delta_Y K$, let $x_A \in \mathcal{C}_{f^{-1}(Y_1)}$ iff $A\delta_X f^{-1}(Y_1)$ iff $f^{-1}(f(A))\delta_X f^{-1}(Y_1)$, because f is 1-1, iff $f(A)\delta_Y Y_1$, by $\delta_Y = f(\delta_X)$, then $f(A)\delta_Y Y_2$ iff $f^{-1}(f(A))\delta_X f^{-1}(Y_2)$ iff $A\delta_X f^{-1}(Y_2)$. Hence $x_A \in \mathcal{C}_{f^{-1}(Y_2)}$.

Proposition 2.3.1.5:

Let (X, δ_X) be a proximity space, $f: (X, \delta_X) \rightarrow Y$ give bijective stronger function, $\{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\}$ and $\{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}$ are families of center subsets of \mathbb{C}_X and \mathbb{C}_Y respectively, where Δ and Λ are arbitrary index sets, then:

1. $f_c(\vee \{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\}) =_c \vee \{\mathcal{C}_{f(A_\alpha)}, \alpha \in \Delta\}$
2. $f_c(\wedge \{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\}) =_c \wedge \{\mathcal{C}_{f(A_\alpha)}, \alpha \in \Delta\}$
3. $f_c^{-1}(\vee \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}) =_c \vee \{\mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$
4. $f_c^{-1}(\wedge \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}) =_c \wedge \{\mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$

Proof 1:

Let $y_K \in f_C(\vee \{C_{A_\alpha}, \alpha \in \Delta\})$, iff there exist $x_H \in \vee \{C_{A_\alpha}, \alpha \in \Delta\}$ such that $f_C(x_H) =_c y_K$, iff for some $\alpha_0 \in \Delta$, $x_H \in C_{A_{\alpha_0}}$ such that $A_{\alpha_0} \delta_X H$, by proposition (2.3.1.2), then $f_C(x_H) \in C_{f(A_{\alpha_0})}$ for some $\alpha_0 \in \Delta$, because $f(A_{\alpha_0}) \delta_Y f(H)$, therefore $f(A_{\alpha_0}) \delta_Y K$. thus $y_K \in C_{f(A_{\alpha_0})}$ for some $\alpha_0 \in \Delta$, Hence $y_K \in \vee \{C_{f(A_\alpha)}, \alpha \in \Delta\}$.

Conversely, let $y_K \in \vee \{C_{f(A_\alpha)}, \alpha \in \Delta\}$, then $y_K \in C_{f(A_{\alpha_0})}$ for some $\alpha_0 \in \Delta$, this mean $f_C(A_{\alpha_0}) \delta_Y K$, by proposition (2.3.1.3), then $f_C^{-1}(y_K) \in C_{f^{-1}(f(A_{\alpha_0}))}$, therefore $f^{-1}(f(A_{\alpha_0})) \delta_X f^{-1}(K)$, since f is onto, then $A_{\alpha_0} \delta_X f^{-1}(K)$, thus $f^{-1}(y_K) \in C_{A_{\alpha_0}}$ for some $\alpha_0 \in \Delta$, by proposition (2.3.1.2), then $f_C(f_C^{-1}(y_K)) \in f_C(\vee \{C_{A_\alpha}, \alpha \in \Delta\})$. Hence $y_K \in f_C(\vee \{C_{A_\alpha}, \alpha \in \Delta\})$.

Proof 2:

Let $y_K \in f_C(\wedge \{C_{A_\alpha}, \alpha \in \Delta\})$, then there exist $x_H \in \wedge \{C_{A_\alpha}, \alpha \in \Delta\}$ such that $f_C(x_H) =_c y_K$, iff for each $\alpha \in \Delta$, then $x_H \in C_{A_\alpha}$ such that $A_\alpha \delta_X H$, by proposition (2.3.1.2), then $f_C(x_H) \in C_{f(A_\alpha)}$ for each $\alpha \in \Delta$, because $f(A_\alpha) \delta_Y f(H) \quad \forall \alpha \in \Delta$, therefore $y_K \in C_{f(A_\alpha)}$ for each $\alpha \in \Delta$, Hence $y_K \in \wedge \{C_{f(A_\alpha)}, \alpha \in \Delta\}$.

Conversely, let $y_K \in \wedge \{C_{f(A_\alpha)}, \alpha \in \Delta\}$, then $y_K \in C_{f(A_\alpha)}$ for each $\alpha \in \Delta$, by proposition (2.3.1.3), then $f_C^{-1}(y_K) \in C_{f^{-1}(f(A_\alpha))}$ this mean $f^{-1}(f(A_\alpha)) \delta_X f^{-1}(K)$, since f is (1-1), then $A_\alpha \delta_X f^{-1}(K) \quad \forall \alpha \in \Delta$, therefore $f^{-1}(y_K) \in C_{A_\alpha}$ for each $\alpha \in \Delta$, thus $f^{-1}(y_K) \in \wedge \{C_{A_\alpha}, \alpha \in \Delta\}$. by proposition (2.3.1.2), then $y_K \in f_C(\wedge \{C_{A_\alpha}, \alpha \in \Delta\})$.

Proof 3:

Let $x_H \in f_C^{-1}(\mathcal{Y}\{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\})$, there exists $y_K \in \mathcal{Y}\{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}$, such that $f_C^{-1}(y_K) = x_H$ for $K \subseteq Y$, then for some $\alpha_0 \in \Lambda$, $y_K \in \mathcal{C}_{B_{\alpha_0}}$ such that $B_{\alpha_0} \delta_Y K$, by proposition (2.3.1.3), then $f_C^{-1}(y_K) \in \mathcal{C}_{f^{-1}(B_{\alpha_0})}$, this mean $f^{-1}(B_{\alpha_0}) \delta_X f^{-1}(K)$, then for some $\alpha_0 \in \Lambda$, $x_H \in \mathcal{C}_{f^{-1}(B_{\alpha_0})}$. Hence $x_H \in \mathcal{Y}\{\mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$.

Conversely, let $x_H \in \mathcal{Y}\{\mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$, then $x_H \in \mathcal{C}_{f^{-1}(B_{\alpha_0})}$ for some $\alpha_0 \in \Lambda$, by proposition (2.3.1.2), then $f_C(x_H) \in \mathcal{C}_{f(f^{-1}(B_{\alpha_0}))}$, this mean $f(f^{-1}(B_{\alpha_0})) \delta_Y f(H)$. But $f(f^{-1}(B_{\alpha_0})) \subseteq B_{\alpha_0}$, therefore $B_{\alpha_0} \delta_Y f(H)$, then $f_C(x_H) \in \mathcal{C}_{B_{\alpha_0}}$ for some $\alpha_0 \in \Lambda$, thus $f_C(x_H) \in \mathcal{Y}\{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}$, by proposition (2.3.1.3), then $x_H \in f_C^{-1}(\mathcal{Y}\{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\})$.

Proof 4:

Let $x_H \in f_C^{-1}(\mathcal{A}\{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\})$, there exists $y_K \in \mathcal{A}\{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}$, such that $f_C^{-1}(y_K) = x_H$ for $K \subseteq Y$, then for each $\alpha \in \Lambda$, $y_K \in \mathcal{C}_{B_\alpha}$ such that $B_\alpha \delta_Y K$, by proposition (2.3.1.3), then $f_C^{-1}(y_K) \in \mathcal{C}_{f^{-1}(B_\alpha)}$, this mean $f^{-1}(B_\alpha) \delta_X f^{-1}(K)$, then for each $\alpha \in \Lambda$, $x_H \in \mathcal{C}_{f^{-1}(B_\alpha)}$. Hence $x_H \in \mathcal{A}\{\mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$.

Conversely, let $x_H \in \mathcal{A}\{\mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$, then $x_H \in \mathcal{C}_{f^{-1}(B_\alpha)}$ for each $\alpha \in \Lambda$, by proposition (2.3.1.2), then $f_C(x_H) \in \mathcal{C}_{f(f^{-1}(B_\alpha))}$, this mean $f(f^{-1}(B_\alpha)) \delta_Y f(H)$. But $f(f^{-1}(B_\alpha)) \subseteq B_\alpha$, therefore $B_\alpha \delta_Y f(H)$, then $f_C(x_H) \in \mathcal{C}_{B_\alpha}$ for each $\alpha \in \Lambda$, thus $f_C(x_H) \in \mathcal{A}\{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}$, by proposition (2.3.1.3), then $x_H \in f_C^{-1}(\mathcal{A}\{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\})$.

2.3.2 Stronger Central Continuous Function

In this section, we introduce the definition of stronger central continuous function and its most important properties. And the relationship of this type of functions with the separation axioms in \mathcal{C} -topological space.

Definition2.3.2.1:

Let $f: (X, \delta_X) \rightarrow Y$ be a Stronger function and $(X, \delta_X, \mathfrak{J}_{\mathcal{C}X})$, $(Y, \delta_Y, \mathfrak{J}_{\mathcal{C}Y})$ are \mathcal{C} – topological spaces, we call that $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{J}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{J}_{\mathcal{C}Y})$ is a Stronger \mathcal{C} – continuous function at the center point x_A with respect to the function f , if for each $\mathcal{C}_V \in \mathfrak{J}_{\mathcal{C}Y}$ such that $f(A) \delta_Y V$, there exists $\mathcal{C}_U \in \mathfrak{J}_{\mathcal{C}X}$ with $U \delta_X A$ satisfy that, for each $f(U) \delta_Y K$, $K \subseteq Y$, then $V \delta_Y K$. And $f_{\mathcal{C}}$ is called a Stronger \mathcal{C} – continuous, if its Stronger \mathcal{C} – continuous function at each center point in \mathcal{C}_X .

Example2.3.2.2:

Let $(X, \delta_X, \mathfrak{J}_{\mathcal{C}X})$ and $(Y, f(\delta_X), \mathfrak{J}_{\mathcal{C}Y})$ are \mathcal{C} – topological space such that $X = \{1, 2\}$, $Y = \{a, b\}$ and δ_X is discrete proximity, where $\mathfrak{J}_{\mathcal{C}X} = \{\mathcal{C}_X, \mathcal{C}_{\emptyset}, \mathcal{C}_{\{1\}}, \mathcal{C}_X^{\{\{1\}\}}\}$ and $\mathfrak{J}_{\mathcal{C}Y} = \{\mathcal{C}_Y, \mathcal{C}_{\emptyset}, \mathcal{C}_{\{a\}}, \mathcal{C}_Y^{\{\{a\}\}}\}$ are \mathcal{C} – topologies. Define f from (X, δ_X) into Y such that $f(1) = a$ and $f(2) = b$. Then $f_{\mathcal{C}}$ is Stronger \mathcal{C} – continuous function at the center point $2_{\{2\}}$.

Now we present some equivalent theorems for the definition of stronger \mathcal{C} – continuous function in \mathcal{C} -topological space.

Proposition2.3.2.3:

Let $f: (X, \delta_X) \rightarrow Y$ be a bijective stronger function, then $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{J}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{J}_{\mathcal{C}Y})$ is

1. Stronger \mathcal{C} – continuous function if and only if for each $\mathcal{C}_V \in \mathfrak{J}_{\mathcal{C}Y}$, then $\mathcal{C}_{f^{-1}(V)} \in \mathfrak{J}_{\mathcal{C}X}$.

2. Stronger \mathcal{C} – continuous function if and only if for each \mathcal{C}_F is \mathcal{C} –closed subset of \mathcal{C}_Y , then $\mathcal{C}_{f^{-1}(F)}$ is \mathcal{C} –closed subset of \mathcal{C}_X .
3. Stronger \mathcal{C} – continuous function if and only if for each \mathcal{C}_B center set subset of \mathcal{C}_Y , then $\text{cl}_c(\mathcal{C}_{f^{-1}(B)}) \leq_c f^{-1}(\text{cl}_c(\mathcal{C}_B))$.
4. Stronger \mathcal{C} – continuous function if and only if for each \mathcal{C}_A center set subset of \mathcal{C}_X , then $f_c(\text{cl}_c(\mathcal{C}_A)) \leq_c \text{cl}_c(\mathcal{C}_{f(A)})$.
5. Stronger \mathcal{C} – continuous function if and only if for each \mathcal{C}_B center set subset of \mathcal{C}_Y , then $f_c^{-1}(\text{int}_c(\mathcal{C}_B)) \leq_c \text{int}_c(\mathcal{C}_{f^{-1}(B)})$.
6. Stronger \mathcal{C} – continuous function if and only if for each \mathcal{C}_A center set subset of \mathcal{C}_X , then $\text{int}_c(\mathcal{C}_{f(A)}) \leq_c f_c(\text{int}_c \mathcal{C}_A)$.

Proof 1:

Let f_c is Stronger \mathcal{C} – continuous function and $\mathcal{C}_V \in \mathfrak{J}_{cY}$, and let $A \subseteq X$ such that $A\delta_X f^{-1}(V)$, by $\delta_Y = f(\delta_X)$, then $f(A)\delta_Y f(f^{-1}(V))$, but $f(f^{-1}(V)) \subseteq V$, then $f(A)\delta_Y V$, since f_c is Stronger \mathcal{C} – continuous, there exists $\mathcal{C}_U \in \mathfrak{J}_{cX}$ such that $A\delta_X U$ and $\mathcal{C}_{f(U)} \leq_c \mathcal{C}_V$, by proposition (2.3.1.4), then

$\mathcal{C}_{f^{-1}(f(U))} \leq_c \mathcal{C}_{f^{-1}(V)}$, then $\mathcal{C}_U \leq_c \mathcal{C}_{f^{-1}(V)}$. Hence $\mathcal{C}_{f^{-1}(V)} \in \mathfrak{J}_{cX}$.

Conversely, let $A \subseteq X$ and $\mathcal{C}_V \in \mathfrak{J}_{cY}$ such that $f(A)\delta_Y V$, by assumption, then $\mathcal{C}_{f^{-1}(V)} \in \mathfrak{J}_{cX}$ and $\mathcal{C}_{f(f^{-1}(V))} \leq_c \mathcal{C}_V$, but $f(A)\delta_Y V$, then $f^{-1}(f(A))\delta_X f^{-1}(V)$, therefore $A\delta_X f^{-1}(V)$, put $\mathcal{C}_U =_c \mathcal{C}_{f^{-1}(V)} \in \mathfrak{J}_{cX}$. Hence f_c is Stronger \mathcal{C} – continuous.

Proof 2:

Let f_c is Stronger \mathcal{C} – continuous function and \mathcal{C}_F is \mathcal{C} –closed in Y , then $\mathcal{C}_Y^{(F)} \in \mathfrak{J}_{cY}$ since f_c is Stronger \mathcal{C} – continuous and by part 1, then $\mathcal{C}_X^{(f^{-1}(F))} \in \mathfrak{J}_{cX}$. Hence $\mathcal{C}_{f^{-1}(F)}$ is \mathcal{C} –closed in X .

Conversely, let $\mathcal{C}_V \in \mathfrak{J}_{\mathcal{C}Y}$, then $\mathcal{C}_Y^{(V)}$ is \mathcal{C} -closed in Y , by hypotheses, then $\mathcal{C}_X^{(f^{-1}(V))}$ is \mathcal{C} -closed in X . Therefore $\mathcal{C}_{f^{-1}(V)} \in \mathfrak{J}_{\mathcal{C}X}$. Hence f_c is Stronger \mathcal{C} -continuous.

Proof 3:

Let f_c is Stronger \mathcal{C} -continuous function, since $\text{cl}_{\mathcal{C}}(\mathcal{C}_B)$ is \mathcal{C} -closed subset of \mathcal{C}_Y , by part (2), then $f^{-1}(\text{cl}_{\mathcal{C}}(\mathcal{C}_B))$ is \mathcal{C} -closed in X , since $\text{cl}_{\mathcal{C}}(f^{-1}(\text{cl}_{\mathcal{C}}(\mathcal{C}_B))) =_{\mathcal{C}} f^{-1}(\text{cl}_{\mathcal{C}}(\mathcal{C}_B))$, $\mathcal{C}_B \preceq_{\mathcal{C}} \text{cl}_{\mathcal{C}}(\mathcal{C}_B)$, then $\mathcal{C}_{f^{-1}(B)} \preceq_{\mathcal{C}} f^{-1}(\text{cl}_{\mathcal{C}}(\mathcal{C}_B))$, therefore $\text{cl}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(B)}) \preceq_{\mathcal{C}} f^{-1}(\text{cl}_{\mathcal{C}}(\mathcal{C}_B))$.

Conversely, let \mathcal{C}_F be any \mathcal{C} -closed in Y , so that $\text{cl}_{\mathcal{C}}(\mathcal{C}_F) =_{\mathcal{C}} \mathcal{C}_F$. By hypothesis $\text{cl}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(F)}) \preceq_{\mathcal{C}} f^{-1}(\text{cl}_{\mathcal{C}}(\mathcal{C}_F)) =_{\mathcal{C}} \mathcal{C}_{f^{-1}(F)}$. But $\mathcal{C}_{f^{-1}(F)} \preceq_{\mathcal{C}} \text{cl}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(F)})$, therefore $\text{cl}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(F)}) =_{\mathcal{C}} \mathcal{C}_{f^{-1}(F)}$ and $\mathcal{C}_{f^{-1}(F)}$ is \mathcal{C} -closed in X , by part 2, then f_c is Stronger \mathcal{C} -continuous function.

Proof 4:

Let f_c is Stronger \mathcal{C} -continuous function and \mathcal{C}_A center set in X , since $\mathcal{C}_{f(A)}$ center set in Y and by part (3), then $\text{cl}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(f(A))}) \preceq_{\mathcal{C}} f_c^{-1}(\text{cl}_{\mathcal{C}}(\mathcal{C}_{f(A)}))$, since f is (1-1) function, then $\text{cl}_{\mathcal{C}}(\mathcal{C}_A) \preceq_{\mathcal{C}} f_c^{-1}(\text{cl}_{\mathcal{C}}(\mathcal{C}_{f(A)}))$, by proposition (2.3.1.4), then $f_c(\text{cl}_{\mathcal{C}}(\mathcal{C}_A)) \preceq_{\mathcal{C}} \text{cl}_{\mathcal{C}}(\mathcal{C}_{f(A)})$.

Conversely, let \mathcal{C}_B be an arbitrary \mathcal{C} -closed in Y , then $\mathcal{C}_{f^{-1}(B)} \preceq_{\mathcal{C}} \mathcal{C}_X$. By hypothesis $f_c(\text{cl}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(B)}) \preceq_{\mathcal{C}} \text{cl}_{\mathcal{C}}(\mathcal{C}_{f(f^{-1}(B))})$, therefore $f_c(\text{cl}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(B)}) \preceq_{\mathcal{C}} \text{cl}_{\mathcal{C}}(\mathcal{C}_B)$, by proposition (2.3.1.4), then $\text{cl}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(B)}) \preceq_{\mathcal{C}} f_c^{-1}(\text{cl}_{\mathcal{C}}(\mathcal{C}_B))$. But \mathcal{C}_B is \mathcal{C} -closed, then $\text{cl}_{\mathcal{C}}(\mathcal{C}_B) =_{\mathcal{C}} \mathcal{C}_B$, there fore $\text{cl}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(B)}) \preceq_{\mathcal{C}} \mathcal{C}_{f^{-1}(B)}$. Hence $\mathcal{C}_{f^{-1}(B)}$ is \mathcal{C} -closed in X , by part 2, then f_c is Stronger \mathcal{C} -continuous.

Proof 5:

Let f_c is Stronger \mathcal{C} – continuous function, since $\text{int}_c(\mathcal{C}_B)$ is \mathcal{C} – open $\preceq_c \mathcal{C}_Y$, by part 1 , then $f_c^{-1}(\text{int}_c(\mathcal{C}_B))$ is \mathcal{C} – open $\preceq_c \mathcal{C}_X$, therefore $\text{int}_c(f_c^{-1}(\text{int}_c(\mathcal{C}_B))) =_c f_c^{-1}(\text{int}_c(\mathcal{C}_B))$, since $\text{int}_c(\mathcal{C}_B) \preceq_c \mathcal{C}_B$ and by part 4, then $f_c^{-1}(\text{int}_c(\mathcal{C}_B)) \preceq_c \mathcal{C}_{f^{-1}(B)}$. Hence $f_c^{-1}(\text{int}_c(\mathcal{C}_B)) \preceq_c \text{int}_c(\mathcal{C}_{f^{-1}(B)})$.

Conversely, let \mathcal{C}_U be any \mathcal{C} – open set in Y , so that $\text{int}_c(\mathcal{C}_U) =_c \mathcal{C}_U$. By hypothesis $\mathcal{C}_{f^{-1}(U)} =_c f^{-1}(\text{int}_c(\mathcal{C}_U)) \preceq_c \text{int}_c(\mathcal{C}_{f^{-1}(U)})$, therefore $\text{int}_c(\mathcal{C}_{f^{-1}(U)}) =_c \mathcal{C}_{f^{-1}(U)}$. Hence $\mathcal{C}_{f^{-1}(U)}$ is \mathcal{C} – open in X , by part 1, then f_c is Stronger \mathcal{C} – continuous function.

Proof 6:

Let f_c is Stronger \mathcal{C} – continuous function and \mathcal{C}_A center set in X , since $\mathcal{C}_{f(A)}$ center set in Y by part 5 , then $f_c^{-1}(\text{int}_c(\mathcal{C}_{f(A)})) \preceq_c \text{int}_c(\mathcal{C}_{f^{-1}(f(A))})$, since f is $(1 - 1)$ function, then $f_c^{-1}(\text{int}_c(\mathcal{C}_{f(A)})) \preceq_c \text{int}_c(\mathcal{C}_A)$, by proposition (2.3.1.4) , then $\text{int}_c(\mathcal{C}_{f(A)}) \preceq_c f_c(\text{int}_c(\mathcal{C}_A))$.

Conversely, let \mathcal{C}_U be an arbitrary \mathcal{C} – open in Y , then $\mathcal{C}_{f^{-1}(U)} \preceq_c \mathcal{C}_X$. By hypothesis $\text{int}_c(\mathcal{C}_{f(f^{-1}(U))}) \preceq_c f_c(\text{int}_c(\mathcal{C}_{f^{-1}(U)}))$, therefore $\text{int}_c(\mathcal{C}_U) \preceq_c f_c(\text{int}_c(\mathcal{C}_{f^{-1}(U)}))$, by proposition (2.3.1.4) , then $f_c^{-1}(\text{int}_c(\mathcal{C}_U)) \preceq_c \text{int}_c(\mathcal{C}_{f^{-1}(U)})$. But \mathcal{C}_U is \mathcal{C} – open, then $\text{int}_c(\mathcal{C}_U) =_c \mathcal{C}_U$, therefore $\mathcal{C}_{f^{-1}(U)} \preceq_c \text{int}_c(\mathcal{C}_{f^{-1}(U)})$. Hence $\mathcal{C}_{f^{-1}(U)}$ is \mathcal{C} – open in X , by part 1, then f_c is Stronger \mathcal{C} – continuous.

We now introduce some relationships between the concept of stronger \mathcal{C} – continuity and separation axioms in \mathcal{C} – topological space.

Proposition 2.3.2.4:

Let $f: (X, \delta_X) \rightarrow Y$ be a bijective stronger \mathcal{C} – function and $f_c: (X, \delta_X, \mathfrak{S}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ be a stronger \mathcal{C} – continuous function, if $(Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ is $\mathfrak{S}_{\mathcal{C}2}$ – space, then $(X, \delta_X, \mathfrak{S}_{\mathcal{C}X})$ is $\mathfrak{S}_{\mathcal{C}2}$ – space.

Proof:

Let f_c is stronger \mathcal{C} – continuous function, $(Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ is $\mathfrak{S}_{\mathcal{C}2}$ – space and let x_{H_1}, x_{H_2} are center points in X such that $x_{H_1} \neq_c x_{H_2}$, since f is bijective function, there exist y_{K_1}, y_{K_2} two center points in Y such that $y_{K_1} \neq_c y_{K_2}$, where $f_c^{-1}(y_{K_1}) =_c x_{H_1}$ and $f_c^{-1}(y_{K_2}) =_c x_{H_2}$. But $(Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ is $\mathfrak{S}_{\mathcal{C}2}$ – space, there exist \mathcal{C}_{B_1} and \mathcal{C}_{B_2} center open sets in Y such that $y_{K_1} \in \mathcal{C}_{B_1}$, $y_{K_2} \in \mathcal{C}_{B_2}$ and $\mathcal{C}_{B_1} \wedge_c \mathcal{C}_{B_2} =_c \mathcal{C}_\emptyset$, this mean $B_1 \delta_Y K_1$ and $B_2 \delta_Y K_2$, by proposition (2.3.1.3), then $f_c^{-1}(y_{K_1}) \in \mathcal{C}_{f^{-1}(B_1)}$ and $f_c^{-1}(y_{K_2}) \in \mathcal{C}_{f^{-1}(B_2)}$, therefore $x_{H_1} \in \mathcal{C}_{f^{-1}(B_1)}$ and $x_{H_2} \in \mathcal{C}_{f^{-1}(B_2)}$. By proposition (2.3.2.3), then $\mathcal{C}_{f^{-1}(B_1)}$ and $\mathcal{C}_{f^{-1}(B_2)}$ are center open sets in X , by proposition (2.3.1.5), then $\mathcal{C}_{f^{-1}(B_1)} \wedge_c \mathcal{C}_{f^{-1}(B_2)} =_c f_c^{-1}(\mathcal{C}_{B_1} \wedge_c \mathcal{C}_{B_2}) =_c f_c^{-1}(\mathcal{C}_\emptyset) =_c \mathcal{C}_\emptyset$. Hence $(X, \delta_X, \mathfrak{S}_{\mathcal{C}X})$ is $\mathfrak{S}_{\mathcal{C}2}$ – space.

Proposition 2.3.2.5:

Let $f: (X, \delta_X) \rightarrow Y$ be a bijective stronger function and $f_c: (X, \delta_X, \mathfrak{S}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ be a stronger \mathcal{C} – homeomorphism function, then

1. If $(X, \delta_X, \mathfrak{S}_{\mathcal{C}X})$ is \mathcal{C} – regular space, then $(Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ is \mathcal{C} – regular space.

2. If $(X, \delta_X, \mathfrak{J}_{\mathcal{C}X})$ is \mathcal{C} – normal space, then $(Y, \delta_Y, \mathfrak{J}_{\mathcal{C}Y})$ is \mathcal{C} – normal space.

Proof 1:

Let $(X, \delta_X, \mathfrak{J}_{\mathcal{C}X})$ is \mathcal{C} – regular space, \mathcal{C}_F is \mathcal{C} – closed in Y and y_K be a center point in Y such that $y_K \notin \mathcal{C}_F$, since f is bijective function, there exist x_H center point in Y such that $f_{\mathcal{C}}(x_H) =_{\mathcal{C}} y_K$, by proposition (2.3.1.3), then $x_H \notin \mathcal{C}_{f^{-1}(F)}$, since $f_{\mathcal{C}}$ is stronger \mathcal{C} – continuous function, then $\mathcal{C}_{f^{-1}(F)}$ is \mathcal{C} – closed in X . But $(X, \delta_X, \mathfrak{J}_{\mathcal{C}X})$ is \mathcal{C} – regular space, there exist \mathcal{C}_{A_1} and \mathcal{C}_{A_2} are \mathcal{C} – open sets in X such that $x_H \in \mathcal{C}_{A_1}$, $\mathcal{C}_{f^{-1}(F)} \leq_{\mathcal{C}} \mathcal{C}_{A_2}$ and $\mathcal{C}_{A_1} \wedge_{\mathcal{C}} \mathcal{C}_{A_2} =_{\mathcal{C}} \mathcal{C}_{\emptyset}$, this mean $A_1 \delta_X H$ and $f^{-1}(F) \delta_X B \Rightarrow B \delta_X A_2$ for all B subset of X . by propositions (2.3.1.2) and (2.3.1.4), then $f_{\mathcal{C}}(x_H) \in \mathcal{C}_{f(A_1)}$, $\mathcal{C}_F \leq_{\mathcal{C}} \mathcal{C}_{f(A_2)}$, since $f_{\mathcal{C}}^{-1}$ is stronger \mathcal{C} – continuous function, then $\mathcal{C}_{f(A_1)}$ and $\mathcal{C}_{f(A_2)}$ are \mathcal{C} – open sets in X , by proposition (2.3.1.5), $\mathcal{C}_{f(A_1)} \wedge_{\mathcal{C}} \mathcal{C}_{f(A_2)} =_{\mathcal{C}} \mathcal{C}_{f(\mathcal{C}_{A_1} \wedge_{\mathcal{C}} \mathcal{C}_{A_2})} =_{\mathcal{C}} \mathcal{C}_{\emptyset}$. Hence $(Y, \delta_Y, \mathfrak{J}_{\mathcal{C}Y})$ is \mathcal{C} – regular space.

Proof 2:

Let $(X, \delta_X, \mathfrak{J}_{\mathcal{C}X})$ is \mathcal{C} – normal space, $\mathcal{C}_{F_1}, \mathcal{C}_{F_2}$ are \mathcal{C} – closed sets in Y such that $\mathcal{C}_{F_1} \wedge_{\mathcal{C}} \mathcal{C}_{F_2} =_{\mathcal{C}} \mathcal{C}_{\emptyset}$, since $f_{\mathcal{C}}$ is stronger \mathcal{C} – continuous function, then $\mathcal{C}_{f^{-1}(F_1)}$ and $\mathcal{C}_{f^{-1}(F_2)}$ are \mathcal{C} – closed sets in X , since f is bijective function, then $\mathcal{C}_{f^{-1}(F_1)} \wedge_{\mathcal{C}} \mathcal{C}_{f^{-1}(F_2)} =_{\mathcal{C}} \mathcal{C}_{\emptyset}$. But $(X, \delta_X, \mathfrak{J}_{\mathcal{C}X})$ is \mathcal{C} – normal space, there exist \mathcal{C} – open sets in X \mathcal{C}_{A_1} and \mathcal{C}_{A_2} such that $\mathcal{C}_{f^{-1}(F_1)} \leq_{\mathcal{C}} \mathcal{C}_{A_1}$, $\mathcal{C}_{f^{-1}(F_2)} \leq_{\mathcal{C}} \mathcal{C}_{A_2}$ and $\mathcal{C}_{A_1} \wedge_{\mathcal{C}} \mathcal{C}_{A_2} =_{\mathcal{C}} \mathcal{C}_{\emptyset}$, this mean $f^{-1}(F_1) \delta_X B_1 \Rightarrow B_1 \delta_X A_1$ and $f^{-1}(F_2) \delta_X B_2 \Rightarrow B_2 \delta_X A_2$ for all B_1, B_2 subsets of X . by proposition (2.3.1.2), then $\mathcal{C}_{F_1} \leq_{\mathcal{C}} \mathcal{C}_{f(A_1)}$ and $\mathcal{C}_{F_2} \leq_{\mathcal{C}} \mathcal{C}_{f(A_2)}$, by proposition (2.3.1.5), then $\mathcal{C}_{f(A_1)} \wedge_{\mathcal{C}} \mathcal{C}_{f(A_2)} =_{\mathcal{C}} \mathcal{C}_{f(\mathcal{C}_{A_1} \wedge_{\mathcal{C}} \mathcal{C}_{A_2})} =_{\mathcal{C}} \mathcal{C}_{\emptyset}$. Hence $(Y, \delta_Y, \mathfrak{J}_{\mathcal{C}Y})$ is \mathcal{C} – normal space.

2.4 Weaker Central Continuous Function

In this section, we introduce a new definition of continuous functions for a special type of functions what we called weaker central continuous function on \mathcal{C} – topological spaces and show its important properties, in additional, we apply the separation axioms in these spaces.

2.4.1 Weaker Central Function

In this section, we introduce the definition of weaker central function and its most important properties.

Definition 2.4.1.1:

Let (Y, δ_Y) be a proximity space and $\delta_X = f^{-1}(\delta_Y)$ be a proximity on X , a function $f: X \rightarrow (Y, \delta_Y)$ is called weaker function if the image for center set \mathcal{C}_A in X is $\mathcal{C}_{f(A)} = \{(f(A), B): f(A)\delta_Y B, \text{ for each } B \subseteq Y\}$, and the inverse of weaker \mathcal{C} -function f is of form $\mathcal{C}_{f^{-1}(B)} = \{(f^{-1}(B), H): f^{-1}(B)\delta_X H, \text{ for each } H \subseteq X\}$.

Proposition 2.4.1.2:

Let (Y, δ_Y) be a proximity space, $f: X \rightarrow (Y, \delta_Y)$ a give weaker \mathcal{C} -function, and \mathcal{C}_A be center set in X . If $x_B \in \mathcal{C}_A$ then $f_C(x_B) \in \mathcal{C}_{f(A)}$.

Proof:

Let $x_B \in \mathcal{C}_A$, imply $A\delta_X B$. And by specifying δ_X , we have $f(A)\delta_Y f(B)$, therefore $f(x_B) \in \mathcal{C}_{f(A)}$.

Proposition 2.4.1.3:

Let (Y, δ_Y) be a proximity space, $f: X \rightarrow (Y, \delta_Y)$ a give surjective weaker \mathcal{C} -function and \mathcal{C}_V be center set in Y , if $y_A \in \mathcal{C}_V$ then $f^{-1}_C(y_A) \in \mathcal{C}_{f^{-1}(V)}$.

Proof:

Let $y_A \in \mathcal{C}_V$, iff $A\delta_Y V$ iff $f(f^{-1}(A))\delta_Y f(f^{-1}(V))$ because f is surjective iff by $\delta_X = f^{-1}(\delta_Y)$, then $f^{-1}(A)\delta_X f^{-1}(V)$, therefore $f^{-1}(y_A) \in \mathcal{C}_{f^{-1}(V)}$.

Proposition 2.4.1.4:

Let (Y, δ_Y) be a proximity space, $f: X \rightarrow (Y, \delta_Y)$ a give surjective weaker \mathcal{C} - function. Hence the following is correct:

- i. If $\mathcal{C}_A \preceq_{\mathcal{C}} \mathcal{C}_B$, then $\mathcal{C}_{f(A)} \preceq_{\mathcal{C}} \mathcal{C}_{f(B)}$, for any center sets $\mathcal{C}_A, \mathcal{C}_B$ in X .
- ii. If $\mathcal{C}_{Y_1} \preceq_{\mathcal{C}} \mathcal{C}_{Y_2}$, then $\mathcal{C}_{f^{-1}(Y_1)} \preceq_{\mathcal{C}} \mathcal{C}_{f^{-1}(Y_2)}$, for any center sets $\mathcal{C}_{Y_1}, \mathcal{C}_{Y_2}$ in Y .

Proof:

- i. Let $\mathcal{C}_A \preceq_{\mathcal{C}} \mathcal{C}_B$, and from it we get $\forall H \subseteq X$, $A\delta_X H$, imply $B\delta_X H$. For any $\emptyset \neq K \subseteq Y$ and $f(A)\delta_Y K$. But f is surjective, there exist $H \subseteq X$ such that $f(H) = K$, so $f(A)\delta_Y f(H)$ iff $A\delta_X H$ because $\delta_X = f^{-1}(\delta_Y)$, then $H\delta_X B$ iff $K = f(H)\delta_Y f(B)$ because $\delta_X = f^{-1}(\delta_Y)$. Therefore $\mathcal{C}_{f(A)} \preceq_{\mathcal{C}} \mathcal{C}_{f(B)}$.
- ii. Let $\mathcal{C}_{Y_1} \preceq_{\mathcal{C}} \mathcal{C}_{Y_2}$ and from it we get $\forall \emptyset \neq K \subseteq Y$, if $Y_1\delta_Y K$, imply $Y_2\delta_Y K$, let H be non-empty subset of X such that $f^{-1}(Y_1)\delta_X H$ iff $f(f^{-1}(Y_1))\delta_Y f(H)$ because $\delta_X = f^{-1}(\delta_Y)$ iff $Y_1\delta_Y f(H)$, imply $Y_2\delta_Y f(H)$ iff $f(f^{-1}(Y_2))\delta_Y f(H)$ iff $f^{-1}(Y_2)\delta_X H$. Therefore $\mathcal{C}_{f^{-1}(Y_1)} \preceq_{\mathcal{C}} \mathcal{C}_{f^{-1}(Y_2)}$.

Proposition 2.4.1.5:

Let (Y, δ_Y) be a proximity space, $f: X \rightarrow (Y, \delta_Y)$ a give bijective weaker \mathcal{C} - function, $\{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\}$ and $\{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}$ are families of center subsets of \mathbb{C}_X and \mathbb{C}_Y respectively, where Δ and Λ are arbitrary index sets, then:

$$1. f_{\mathcal{C}}(\vee \{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\}) =_{\mathcal{C}} \vee \{\mathcal{C}_{f(A_\alpha)}, \alpha \in \Delta\}$$

2. $f_c(\wedge \{C_{A_\alpha}, \alpha \in \Delta\}) =_c \wedge \{C_{f(A_\alpha)}, \alpha \in \Delta\}$
3. $f_c^{-1}(\vee \{C_{B_\alpha}, \alpha \in \Lambda\}) =_c \vee \{C_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$
4. $f_c^{-1}(\wedge \{C_{B_\alpha}, \alpha \in \Lambda\}) =_c \wedge \{C_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$

Proof 1:

Let $y_K \in f_c(\vee \{C_{A_\alpha}, \alpha \in \Delta\})$, iff there exist $x_H \in \vee \{C_{A_\alpha}, \alpha \in \Delta\}$ such that $f_c(x_H) =_c y_K$, iff for some $\alpha_0 \in \Delta$, $x_H \in C_{A_{\alpha_0}}$ such that $A_{\alpha_0} \delta_X H$, by proposition (2.4.1.2), then $f_c(x_H) \in C_{f(A_{\alpha_0})}$ for some $\alpha_0 \in \Delta$, because $f(A_{\alpha_0}) \delta_Y f(H)$, therefore $f(A_{\alpha_0}) \delta_Y K$. Thus $y_K \in C_{f(A_{\alpha_0})}$ for some $\alpha_0 \in \Delta$, Hence $y_K \in \vee \{C_{f(A_\alpha)}, \alpha \in \Delta\}$.

Conversely, let $y_K \in \vee \{C_{f(A_\alpha)}, \alpha \in \Delta\}$, then $y_K \in C_{f(A_{\alpha_0})}$ for some $\alpha_0 \in \Delta$, this mean $f_c(A_{\alpha_0}) \delta_Y K$, by proposition (2.4.1.3), then $f_c^{-1}(y_K) \in C_{f^{-1}(f(A_{\alpha_0}))}$, therefore $f^{-1}(f(A_{\alpha_0})) \delta_X f^{-1}(K)$, since f is onto, then $A_{\alpha_0} \delta_X f^{-1}(K)$, thus $f^{-1}(y_K) \in C_{A_{\alpha_0}}$ for some $\alpha_0 \in \Delta$, by proposition (2.4.1.2), then $f_c(f_c^{-1}(y_K)) \in f_c(\vee \{C_{A_\alpha}, \alpha \in \Delta\})$. Hence $y_K \in f_c(\vee \{C_{A_\alpha}, \alpha \in \Delta\})$.

Proof 2:

Let $y_K \in f_c(\wedge \{C_{A_\alpha}, \alpha \in \Delta\})$, then there exist $x_H \in \wedge \{C_{A_\alpha}, \alpha \in \Delta\}$ such that $f_c(x_H) =_c y_K$, iff for each $\alpha \in \Delta$, then $x_H \in C_{A_\alpha}$ such that $A_\alpha \delta_X H$, by proposition (2.4.1.2), then $f_c(x_H) \in C_{f(A_\alpha)}$ for each $\alpha \in \Delta$, because $f(A_\alpha) \delta_Y f(H) \quad \forall \alpha \in \Delta$, therefore $y_K \in C_{f(A_\alpha)}$ for each $\alpha \in \Delta$, Hence $y_K \in \wedge \{C_{f(A_\alpha)}, \alpha \in \Delta\}$.

Conversely, let $y_K \in \wedge \{C_{f(A_\alpha)}, \alpha \in \Delta\}$, then $y_K \in C_{f(A_\alpha)}$ for each $\alpha \in \Delta$, by proposition (2.4.1.3), then $f_c^{-1}(y_K) \in C_{f^{-1}(f(A_\alpha))}$ this mean $f^{-1}(f(A_\alpha)) \delta_X f^{-1}(K)$, since f is (1-1), then $A_\alpha \delta_X f^{-1}(K) \quad \forall \alpha \in \Delta$, therefore

$f^{-1}(y_K) \in \mathcal{C}_{A_\alpha}$ for each $\alpha \in \Delta$, thus $f^{-1}(y_K) \in \bigwedge \{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\}$. by proposition (2.4.1.2), then $y_K \in f(\bigwedge \{\mathcal{C}_{A_\alpha}, \alpha \in \Delta\})$.

Proof 3:

Let $x_H \in f_C^{-1}(\bigvee \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\})$, there exists $y_K \in \bigvee \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}$, such that $f_C^{-1}(y_K) = x_H$ for $K \subseteq Y$, then for some $\alpha_0 \in \Lambda$, $y_K \in \mathcal{C}_{B_{\alpha_0}}$ such that $B_{\alpha_0} \delta_Y K$, by proposition (2.4.1.3), then $f_C^{-1}(y_K) \in \mathcal{C}_{f^{-1}(B_{\alpha_0})}$, this mean $f^{-1}(B_{\alpha_0}) \delta_X f^{-1}(K)$, then for some $\alpha_0 \in \Lambda$, $x_H \in \mathcal{C}_{f^{-1}(B_{\alpha_0})}$. Hence $x_H \in \bigvee \{\mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$.

Conversely, let $x_H \in \bigvee \{\mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$, then $x_H \in \mathcal{C}_{f^{-1}(B_{\alpha_0})}$ for some $\alpha_0 \in \Lambda$, by proposition (2.4.1.2), then $f_C(x_H) \in \mathcal{C}_{f(f^{-1}(B_{\alpha_0}))}$, this mean $f(f^{-1}(B_{\alpha_0})) \delta_Y f(H)$. But $f(f^{-1}(B_{\alpha_0})) \subseteq B_{\alpha_0}$, therefore $B_{\alpha_0} \delta_Y f(H)$, then $f_C(x_H) \in \mathcal{C}_{B_{\alpha_0}}$ for some $\alpha_0 \in \Lambda$, thus $f_C(x_H) \in \bigvee \{\mathcal{C}_{B_{\alpha_0}}, \alpha \in \Lambda\}$, by proposition (2.4.1.3), then $x_H \in f_C^{-1}(\bigvee \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\})$.

Proof 4:

Let $x_H \in f_C^{-1}(\bigwedge \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\})$, there exists $y_K \in \bigwedge \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}$, such that $f_C^{-1}(y_K) = x_H$ for $K \subseteq Y$, then for each $\alpha \in \Lambda$, $y_K \in \mathcal{C}_{B_\alpha}$ such that $B_\alpha \delta_Y K$, by proposition (2.4.1.3), then $f_C^{-1}(y_K) \in \mathcal{C}_{f^{-1}(B_\alpha)}$, this mean $f^{-1}(B_\alpha) \delta_X f^{-1}(K)$, then for each $\alpha \in \Lambda$, $x_H \in \mathcal{C}_{f^{-1}(B_\alpha)}$. Hence $x_H \in \bigwedge \{\mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$.

Conversely, let $x_H \in \bigwedge \{\mathcal{C}_{f^{-1}(B_\alpha)}, \alpha \in \Lambda\}$, then $x_H \in \mathcal{C}_{f^{-1}(B_\alpha)}$ for each $\alpha \in \Lambda$, by proposition (2.4.1.2), then $f_C(x_H) \in \mathcal{C}_{f(f^{-1}(B_\alpha))}$, this mean $f(f^{-1}(B_\alpha)) \delta_Y f(H)$. But $f(f^{-1}(B_\alpha)) \subseteq B_\alpha$, therefore $B_\alpha \delta_Y f(H)$, then $f_C(x_H) \in \mathcal{C}_{B_\alpha}$ for each $\alpha \in \Lambda$, thus $f_C(x_H) \in \bigwedge \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\}$, by proposition (2.4.1.3), then $x_H \in f_C^{-1}(\bigwedge \{\mathcal{C}_{B_\alpha}, \alpha \in \Lambda\})$.

2.4.2 Weaker Central Continuous Function

In this section, we introduce the definition of weaker central continuous function and its most important properties. And the relationship of this type of functions with the separation axioms in \mathcal{C} -topological space.

Definition2.4.2.1:

Let $f: X \rightarrow (Y, \delta_Y)$ be a weaker function and $(X, \delta_X, \mathfrak{J}_{\mathcal{C}X})$, $(Y, \delta_Y, \mathfrak{J}_{\mathcal{C}Y})$ are \mathcal{C} – topological spaces, we call that $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{J}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{J}_{\mathcal{C}Y})$ is a weaker \mathcal{C} – continuous function at the center point x_A with respect to the function f , if for each $\mathcal{C}_V \in \mathfrak{J}_{\mathcal{C}Y}$ such that $f(A) \delta_Y V$, there exists $\mathcal{C}_U \in \mathfrak{J}_{\mathcal{C}X}$ with $U \delta_X A$ satisfy that, for each $f(U) \delta_Y K$, $K \subseteq Y$, then $V \delta_Y K$. And $f_{\mathcal{C}}$ is called a weaker \mathcal{C} – continuous, if its weaker \mathcal{C} – continuous function at each center point in X .

Now we present some equivalent theorems for the definition of weaker \mathcal{C} – continuous function in \mathcal{C} -topological space.

Proposition2.4.2.2:

Let $f: X \rightarrow (Y, \delta_Y)$ be a bijective weaker \mathcal{C} –function, then $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{J}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{J}_{\mathcal{C}Y})$ is

1. Weaker \mathcal{C} – continuous function if and only if for each $\mathcal{C}_V \in \mathfrak{J}_{\mathcal{C}Y}$, then $\mathcal{C}_{f^{-1}(V)} \in \mathfrak{J}_{\mathcal{C}X}$.
2. Weaker \mathcal{C} – continuous function if and only if for each \mathcal{C}_F is \mathcal{C} –closed subset of \mathcal{C}_Y , then $\mathcal{C}_{f^{-1}(F)}$ is \mathcal{C} –closed subset of \mathcal{C}_X .
3. Weaker \mathcal{C} – continuous function if and only if for each \mathcal{C}_B center set subset of \mathcal{C}_Y , then $\text{cl}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(B)}) \leq_{\mathcal{C}} f^{-1}(\text{cl}_{\mathcal{C}}(\mathcal{C}_B))$.
4. Weaker \mathcal{C} – continuous function if and only if for each \mathcal{C}_A center set subset of \mathcal{C}_X , then $f_{\mathcal{C}}(\text{cl}_{\mathcal{C}}(\mathcal{C}_A)) \leq_{\mathcal{C}} \text{cl}_{\mathcal{C}}(\mathcal{C}_{f(A)})$.
5. Weaker \mathcal{C} – continuous function if and only if for each \mathcal{C}_B center set subset of \mathcal{C}_Y , then $f_{\mathcal{C}}^{-1}(\text{int}_{\mathcal{C}}(\mathcal{C}_B)) \leq_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(B)})$.

6. Weaker \mathcal{C} – continuous function if and only if for each \mathcal{C}_A center set subset of \mathcal{C}_X , then $\text{int}_{\mathcal{C}}(\mathcal{C}_{f(A)}) \preceq_{\mathcal{C}} f_{\mathcal{C}}(\text{int}_{\mathcal{C}}\mathcal{C}_A)$.

Proof 1:

Let $f_{\mathcal{C}}$ is weaker \mathcal{C} – continuous function, $\mathcal{C}_V \in \mathfrak{J}_{\mathcal{C}Y}$ and $A \subseteq X$ such that $A\delta_X f^{-1}(V)$, by $\delta_X = f^{-1}(\delta_Y)$, then $f(A)\delta_Y f^{-1}(V)$, but $f(f^{-1}(V)) \subseteq V$, then $f(A)\delta_Y V$, since $f_{\mathcal{C}}$ is weaker \mathcal{C} – continuous, there exists $\mathcal{C}_U \in \mathfrak{J}_{\mathcal{C}X}$ such that $A\delta_X U$ and $\mathcal{C}_{f(U)} \preceq_{\mathcal{C}} \mathcal{C}_V$, by proposition (2.4.1.3) , then $\mathcal{C}_{f^{-1}(f(U))} \preceq_{\mathcal{C}} \mathcal{C}_{f^{-1}(V)}$, then $\mathcal{C}_U \preceq_{\mathcal{C}} \mathcal{C}_{f^{-1}(V)}$. Hence $\mathcal{C}_{f^{-1}(V)} \in \mathfrak{J}_{\mathcal{C}X}$.

Conversely, let $A \subseteq X$ and $\mathcal{C}_V \in \mathfrak{J}_{\mathcal{C}Y}$ such that $f(A)\delta_Y V$, by assumption, then $\mathcal{C}_{f^{-1}(V)} \in \mathfrak{J}_{\mathcal{C}X}$ and $\mathcal{C}_{f(f^{-1}(V))} \preceq_{\mathcal{C}} \mathcal{C}_V$, but $f(A)\delta_Y V$, then $f^{-1}(f(A))\delta_X f^{-1}(V)$, therefore $A\delta_X f^{-1}(V)$, put $\mathcal{C}_U =_{\mathcal{C}} \mathcal{C}_{f^{-1}(V)} \in \mathfrak{J}_{\mathcal{C}X}$. Hence $f_{\mathcal{C}}$ is weaker \mathcal{C} – continuous.

Proof 2: Let $f_{\mathcal{C}}$ is weaker \mathcal{C} – continuous function and \mathcal{C}_F is \mathcal{C} –closed in Y , then $\mathcal{C}_Y^{(F)} \in \mathfrak{J}_{\mathcal{C}Y}$ since $f_{\mathcal{C}}$ is weaker \mathcal{C} – continuous and by part 1 , then $\mathcal{C}_X^{(f^{-1}(F))} \in \mathfrak{J}_{\mathcal{C}X}$. Hence $\mathcal{C}_{f^{-1}(F)}$ is \mathcal{C} –closed in X .

Conversely, let $\mathcal{C}_V \in \mathfrak{J}_{\mathcal{C}Y}$, then $\mathcal{C}_Y^{(V)}$ is \mathcal{C} –closed in Y , by hypotheses, then $\mathcal{C}_X^{(f^{-1}(V))}$ is \mathcal{C} –closed in X . Therefore $\mathcal{C}_{f^{-1}(V)} \in \mathfrak{J}_{\mathcal{C}X}$. Hence $f_{\mathcal{C}}$ is weaker \mathcal{C} – continuous.

Proof 3:

Let $f_{\mathcal{C}}$ is weaker \mathcal{C} – continuous function, since $\text{cl}_{\mathcal{C}}(\mathcal{C}_B)$ is \mathcal{C} –closed subset of \mathcal{C}_Y , by part (2) , then $f^{-1}(\text{cl}_{\mathcal{C}}(\mathcal{C}_B))$ is \mathcal{C} –closed in X , since

$\text{cl}_C(f^{-1}(\text{cl}_C(\mathcal{C}_B))) =_C f^{-1}(\text{cl}_C(\mathcal{C}_B))$, $\mathcal{C}_B \preceq_C \text{cl}_C(\mathcal{C}_B)$, then $\mathcal{C}_{f^{-1}(B)} \preceq_C f^{-1}(\text{cl}_C(\mathcal{C}_B))$, therefore $\text{cl}_C(\mathcal{C}_{f^{-1}(B)}) \preceq_C f^{-1}(\text{cl}_C(\mathcal{C}_B))$.

Conversely, let \mathcal{C}_F be any \mathcal{C} – closed in Y , so that $\text{cl}_C(\mathcal{C}_F) =_C \mathcal{C}_F$. By hypothesis $\text{cl}_C(\mathcal{C}_{f^{-1}(F)}) \preceq_C f^{-1}(\text{cl}_C(\mathcal{C}_F)) =_C \mathcal{C}_{f^{-1}(F)}$. But $\mathcal{C}_{f^{-1}(F)} \preceq_C \text{cl}_C(\mathcal{C}_{f^{-1}(F)})$, therefore $\text{cl}_C(\mathcal{C}_{f^{-1}(F)}) =_C \mathcal{C}_{f^{-1}(F)}$ and $\mathcal{C}_{f^{-1}(F)}$ is \mathcal{C} – closed in X , by part 2, then f_C is weaker \mathcal{C} – continuous function.

Proof 4:

Let f_C is weaker \mathcal{C} – continuous function and \mathcal{C}_A center set in X , since $\mathcal{C}_{f(A)}$ center set in Y and by part (3), then $\text{cl}_C(\mathcal{C}_{f^{-1}(f(A))}) \preceq_C f_C^{-1}(\text{cl}_C(\mathcal{C}_{f(A)}))$, since f is $(1 - 1)$ function, then $\text{cl}_C(\mathcal{C}_A) \preceq_C f_C^{-1}(\text{cl}_C(\mathcal{C}_{f(A)}))$, by proposition (2.4.1.3) , then $f_C(\text{cl}_C(\mathcal{C}_A)) \preceq_C \text{cl}_C(\mathcal{C}_{f(A)})$.

Conversely, let \mathcal{C}_B be an arbitrary \mathcal{C} – closed in Y , then $\mathcal{C}_{f^{-1}(B)} \preceq_C \mathcal{C}_X$. By hypothesis $f_C(\text{cl}_C(\mathcal{C}_{f^{-1}(B)})) \preceq_C \text{cl}_C(\mathcal{C}_{f(f^{-1}(B))})$, therefore $f_C(\text{cl}_C(\mathcal{C}_{f^{-1}(B)})) \preceq_C \text{cl}_C(\mathcal{C}_B)$, by proposition (2.4.1.3) , then $\text{cl}_C(\mathcal{C}_{f^{-1}(B)}) \preceq_C f_C^{-1}(\text{cl}_C(\mathcal{C}_B))$. But \mathcal{C}_B is \mathcal{C} – closed, then $\text{cl}_C(\mathcal{C}_B) =_C \mathcal{C}_B$, there fore $\text{cl}_C(\mathcal{C}_{f^{-1}(B)}) \preceq_C \mathcal{C}_{f^{-1}(B)}$. Hence $\mathcal{C}_{f^{-1}(B)}$ is \mathcal{C} – closed in X , by part 2, then f_C is weaker \mathcal{C} – continuous.

Proof 5:

Let f_C is weaker \mathcal{C} – continuous function, since $\text{int}_C(\mathcal{C}_B)$ is \mathcal{C} – open $\preceq_C \mathcal{C}_Y$, by part 1 , then $f_C^{-1}(\text{int}_C(\mathcal{C}_B))$ is \mathcal{C} – open $\preceq_C \mathcal{C}_X$, therefore $\text{int}_C(f_C^{-1}(\text{int}_C(\mathcal{C}_B))) =_C f_C^{-1}(\text{int}_C(\mathcal{C}_B))$, since $\text{int}_C(\mathcal{C}_B) \preceq_C \mathcal{C}_B$ and by part 4, then $f_C^{-1}(\text{int}_C(\mathcal{C}_B)) \preceq_C \mathcal{C}_{f^{-1}(B)}$. Hence $f_C^{-1}(\text{int}_C(\mathcal{C}_B)) \preceq_C \text{int}_C(\mathcal{C}_{f^{-1}(B)})$.

Conversely, let \mathcal{C}_U be any \mathcal{C} – open set in Y , so that $\text{int}_{\mathcal{C}}(\mathcal{C}_U) =_{\mathcal{C}} \mathcal{C}_U$. By hypothesis $\mathcal{C}_{f^{-1}(U)} =_{\mathcal{C}} f^{-1}(\text{int}_{\mathcal{C}}(\mathcal{C}_U)) \leq_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(U)})$, therefore $\text{int}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(U)}) =_{\mathcal{C}} \mathcal{C}_{f^{-1}(U)}$. Hence $\mathcal{C}_{f^{-1}(U)}$ is \mathcal{C} – open in X , by part 1, then $f_{\mathcal{C}}$ is weaker \mathcal{C} – continuous function.

Proof 6:

Let $f_{\mathcal{C}}$ is weaker \mathcal{C} – continuous function and \mathcal{C}_A center set in X , since $\mathcal{C}_{f(A)}$ center set in Y by part 5 , then $f_{\mathcal{C}}^{-1}(\text{int}_{\mathcal{C}}(\mathcal{C}_{f(A)})) \leq_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(f(A))})$, since f is (1 – 1) function, then $f_{\mathcal{C}}^{-1}(\text{int}_{\mathcal{C}}(\mathcal{C}_{f(A)})) \leq_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_A)$, by proposition (2.4.1.3) , then $\text{int}_{\mathcal{C}}(\mathcal{C}_{f(A)}) \leq_{\mathcal{C}} f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_A))$.

Conversely, let \mathcal{C}_U be an arbitrary \mathcal{C} – open in Y , then $\mathcal{C}_{f^{-1}(U)} \leq_{\mathcal{C}} \mathcal{C}_X$. By hypothesis $\text{int}_{\mathcal{C}}(\mathcal{C}_{f(f^{-1}(U))}) \leq_{\mathcal{C}} f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(U)}))$, therefore $\text{int}_{\mathcal{C}}(\mathcal{C}_U) \leq_{\mathcal{C}} f_{\mathcal{C}}(\text{int}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(U)}))$, by proposition (2.4.1.3) , then $f_{\mathcal{C}}^{-1}(\text{int}_{\mathcal{C}}(\mathcal{C}_U)) \leq_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(U)})$. But \mathcal{C}_U is \mathcal{C} – open, then $\text{int}_{\mathcal{C}}(\mathcal{C}_U) =_{\mathcal{C}} \mathcal{C}_U$, therefore $\mathcal{C}_{f^{-1}(U)} \leq_{\mathcal{C}} \text{int}_{\mathcal{C}}(\mathcal{C}_{f^{-1}(U)})$. Hence $\mathcal{C}_{f^{-1}(U)}$ is \mathcal{C} – open in X , by part 1, then $f_{\mathcal{C}}$ is weaker \mathcal{C} – continuous.

We now introduce some relationships between the concept of weaker \mathcal{C} – continuity and separation axioms in \mathcal{C} – topological space.

Proposition 2.4.2.3:

Let $f: X \rightarrow (Y, \delta_Y)$ be a bijective weaker \mathcal{C} – function and $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{S}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ be a weaker \mathcal{C} – continuous function, if $(Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ is $\mathfrak{S}_{\mathcal{C}2}$ – space, then $(X, \delta_X, \mathfrak{S}_{\mathcal{C}X})$ is $\mathfrak{S}_{\mathcal{C}2}$ – space.

Proof:

Let f_c is weaker \mathcal{C} – continuous function, $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathfrak{S}_{c2} – space and let x_{H_1}, x_{H_2} are center points in X such that $x_{H_1} \neq_c x_{H_2}$, since f is bijective function, there exist y_{K_1}, y_{K_2} two center points in Y such that $y_{K_1} \neq_c y_{K_2}$, where $f_c^{-1}(y_{K_1}) =_c x_{H_1}$ and $f_c^{-1}(y_{K_2}) =_c x_{H_2}$. But $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathfrak{S}_{c2} – space, there exist \mathcal{C}_{B_1} and \mathcal{C}_{B_2} center open sets in Y such that $y_{K_1} \in \mathcal{C}_{B_1}$, $y_{K_2} \in \mathcal{C}_{B_2}$ and $\mathcal{C}_{B_1} \wedge_c \mathcal{C}_{B_2} =_c \mathcal{C}_\emptyset$, this mean $B_1 \delta_Y K_1$ and $B_2 \delta_Y K_2$, by proposition (2.4.1.2), then $f_c^{-1}(y_{K_1}) \in \mathcal{C}_{f^{-1}(B_1)}$ and $f_c^{-1}(y_{K_2}) \in \mathcal{C}_{f^{-1}(B_2)}$, therefore $x_{H_1} \in \mathcal{C}_{f^{-1}(B_1)}$ and $x_{H_2} \in \mathcal{C}_{f^{-1}(B_2)}$. By proposition (2.4.2.2), then $\mathcal{C}_{f^{-1}(B_1)}$ and $\mathcal{C}_{f^{-1}(B_2)}$ are center open sets in X , by proposition (2.4.1.4), then $\mathcal{C}_{f^{-1}(B_1)} \wedge_c \mathcal{C}_{f^{-1}(B_2)} =_c f_c^{-1}(\mathcal{C}_{B_1} \wedge_c \mathcal{C}_{B_2}) =_c f_c^{-1}(\mathcal{C}_\emptyset) =_c \mathcal{C}_\emptyset$. Hence $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathfrak{S}_{c2} – space.

Proposition 2.4.2.4:

Let $f: X \rightarrow (Y, \delta_Y)$ be a bijective weaker \mathcal{C} – function and $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ be a weaker \mathcal{C} – homeomorphism function, then

1. If $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathcal{C} – regular space, then $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathcal{C} – regular space.
2. If $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathcal{C} – normal space, then $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathcal{C} – normal space.

Proof 1:

Let $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathcal{C} – regular space, \mathcal{C}_F is \mathcal{C} – closed in Y and y_K be a center point in Y such that $y_K \notin \mathcal{C}_F$, since f is bijective function, there exist x_H center point in X such that $f_c(x_H) =_c y_K$, by proposition (2.4.1.2), then $x_H \notin \mathcal{C}_{f^{-1}(F)}$, since f_c is weaker \mathcal{C} – continuous function, then $\mathcal{C}_{f^{-1}(F)}$ is \mathcal{C} – closed in X . But $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathcal{C} – regular space, there exist \mathcal{C}_{A_1} and \mathcal{C}_{A_2} are \mathcal{C} – open sets

in X such that $x_H \in \mathcal{C}_{A_1}$, $\mathcal{C}_{f^{-1}(F)} \preceq_c \mathcal{C}_{A_2}$ and $\mathcal{C}_{A_1} \wedge_c \mathcal{C}_{A_2} =_c \mathcal{C}_\emptyset$, this mean $A_1 \delta_X H$ and $f^{-1}(F) \delta_X B \Rightarrow B \delta_X A_2$ for all B subset of X . by propositions (2.4.1.2) and (2.4.1.3), then $f_c(x_H) \in \mathcal{C}_{f(A_1)}$, $\mathcal{C}_F \preceq_c \mathcal{C}_{f(A_2)}$, since f_c^{-1} is weaker \mathcal{C} – continuous function, then $\mathcal{C}_{f(A_1)}$ and $\mathcal{C}_{f(A_2)}$ are \mathcal{C} – open sets in X , by proposition (2.4.1.4), $\mathcal{C}_{f(A_1)} \wedge_c \mathcal{C}_{f(A_2)} =_c \mathcal{C}_{f(\mathcal{C}_{A_1} \wedge_c \mathcal{C}_{A_2})} =_c \mathcal{C}_\emptyset$. Hence $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathcal{C} – regular space.

Proof 2:

Let $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathcal{C} – normal space, $\mathcal{C}_{F_1}, \mathcal{C}_{F_2}$ are \mathcal{C} – closed sets in Y such that $\mathcal{C}_{F_1} \wedge_c \mathcal{C}_{F_2} =_c \mathcal{C}_\emptyset$, since f_c is weaker \mathcal{C} – continuous function, then $\mathcal{C}_{f^{-1}(F_1)}$ and $\mathcal{C}_{f^{-1}(F_2)}$ are \mathcal{C} – closed sets in X , since f is bijective function, then $\mathcal{C}_{f^{-1}(F_1)} \wedge_c \mathcal{C}_{f^{-1}(F_2)} =_c \mathcal{C}_\emptyset$. But $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathcal{C} – normal space, there exist \mathcal{C} – open sets in X \mathcal{C}_{A_1} and \mathcal{C}_{A_2} such that $\mathcal{C}_{f^{-1}(F_1)} \preceq_c \mathcal{C}_{A_1}$, $\mathcal{C}_{f^{-1}(F_2)} \preceq_c \mathcal{C}_{A_2}$ and $\mathcal{C}_{A_1} \wedge_c \mathcal{C}_{A_2} =_c \mathcal{C}_\emptyset$, this mean $f^{-1}(F_1) \delta_X B_1 \Rightarrow B_1 \delta_X A_1$ and $f^{-1}(F_2) \delta_X B_2 \Rightarrow B_2 \delta_X A_2$ for all B_1, B_2 subsets of X . by proposition (2.4.1.2), then $\mathcal{C}_{F_1} \preceq_c \mathcal{C}_{f(A_1)}$ and $\mathcal{C}_{F_2} \preceq_c \mathcal{C}_{f(A_2)}$, by proposition (2.4.1.4), then $\mathcal{C}_{f(A_1)} \wedge_c \mathcal{C}_{f(A_2)} =_c \mathcal{C}_{f(\mathcal{C}_{A_1} \wedge_c \mathcal{C}_{A_2})} =_c \mathcal{C}_\emptyset$. Hence $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathcal{C} – normal space.

CHAPTER

THREE

Center Compactness

Chapter Three: Center Compactness

In this chapter we give several characterizations of Center Compactness, where the concept of \mathcal{C} – compact spaces within the central topological spaces was introduced and the topological properties of this space and its relationship to the axioms of separation were studied. Also in section two we introduce new concept, namely \mathcal{LC} - compact space and its relationship to the axioms of separation were studied. In section three and fore we introduce new concepts namely \mathcal{AC} – compact and \mathcal{HC} – closed, and we study their properties.

3.1 Center Compactness

In this section, we introduce a new type of compactness in center topological space has been presented, that is called a \mathcal{C} – compact. Furthermore, the relationship between this concept and the separation axioms in center topological space had been studied.

Definitions 3.1.1:

Let $\mathcal{H} = \{\mathcal{C}_{A_i}; i \in I\}$ be the collection of non-empty center sets define on (X, δ) is called is \mathcal{C} – cover of the center set \mathcal{C}_X if $X = \bigcup_{i \in I} A_i$ and $\forall H \subseteq X$ such that $\langle X, H \rangle \in \mathcal{C}_X, \exists i_0 \in I. \langle A_{i_0}, H \rangle \in \mathcal{C}_{A_{i_0}}$. And this collection is called finite \mathcal{C} – cover if I is finite set.

Also is called \mathcal{C} – cover of the non-empty center set \mathcal{C}_A if $A = \bigcup_{i \in I} A_i$ and $\forall H \subseteq X$ such that $\langle A, H \rangle \in \mathcal{C}_A, \exists i_0 \in I. \langle A_{i_0}, H \rangle \in \mathcal{C}_{A_{i_0}}$. Where I be any non-empty index. And the mention is said to the family \mathcal{H} is \mathcal{C} – open cover, if each members is \mathcal{C} – open. In other words, if \mathcal{C}_{A_i} is \mathcal{C} – open set for all i , then the $\{\mathcal{C}_{A_i}, i \in I\}$ said to be \mathcal{C} – open cover.

Definition 3.1.2:

Let $(X, \delta, \mathfrak{S}_C)$ be a \mathcal{C} -topological space, the universal set \mathcal{C}_X (a non-empty center set \mathcal{C}_A) is called \mathcal{C} – compact space if each \mathcal{C} – open cover $\{\mathcal{C}_{U_\alpha}, \alpha \in \Gamma\}$ of \mathcal{C}_X (\mathcal{C}_A) has finite Δ subset of Γ such that $\forall H \subseteq X$, there exist $\alpha_0 \in \Delta, U_{\alpha_0} \delta H$.

Remarks and examples 3.1.3:

- i. The indiscrete –*topological space* , then every center sets are \mathcal{C} – compact space.
- ii. The discrete –*topological space* , then every center sets are \mathcal{C} – compact space if and only if it's finite.
- iii. If $A \subseteq B$ then $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_B$ because if $\langle A, H \rangle \in \mathcal{C}_A$, then $A \delta H$, but $A \subseteq B$, thus $B \delta H$, that mean $\langle B, H \rangle \in \mathcal{C}_B$, so $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_B$. From this fact we conclude that every center set is center subset of universal center set \mathcal{C}_X .

Theorem 3.1.4 :

Each \mathcal{C} – closed subset of \mathcal{C} –compact *topological space* $(X, \delta, \mathfrak{S}_C)$ is \mathcal{C} – compact set.

Proof:

Let \mathcal{C}_A be any non-empty \mathcal{C} – closed set in \mathcal{C} – compact space $(X, \delta, \mathfrak{S}_C)$. And $\mathcal{H} = \{\mathcal{C}_{A_\lambda}, \lambda \in \Lambda\}$ be \mathcal{C} – open cover to \mathcal{C}_A , so $\mathcal{C}_X^{(A)}$ be \mathcal{C} – open set. Thus for each $B \subseteq X$, $\forall x \in X$ such that $\{x\} \delta B$ and $\exists \lambda_0 \in \Lambda, B \delta A_{\lambda_0}$, from the concept of \mathcal{C} – compact space of \mathcal{C}_X , then there exists finite subset I of Λ say $\{\mathcal{C}_{A_1}, \mathcal{C}_{A_2}, \dots, \mathcal{C}_{A_n}, \mathcal{C}_X^{(A)}\}$ such that $\forall H \subseteq X$, there exist $\lambda_{i^0} \in I, A_{\lambda_{i^0}} \delta H$, Hence \mathcal{C}_A is \mathcal{C} – compact set.

Theorem 3.1.5:

Each \mathcal{C} – compact subset of $\mathfrak{S}_{\mathcal{C}2-}$ space $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} – closed set.

Proof:

Let \mathcal{C}_A be any non-empty \mathcal{C} – compact set, so by Proposition (1.2.11 part 4) we get that $\mathcal{C}_A =_{\mathcal{C}} \bigvee_{x_{\alpha B \lambda} \in \mathcal{C}_A} x_{\alpha B \lambda}$, then for any $\alpha, \lambda, x_{\alpha} \in X$ such that $\{x_{\alpha}\} \delta B_{\lambda}$ and $B_{\lambda} \delta A$, and for the fact that $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_X^{(A)} =_{\mathcal{C}} \mathcal{C}_{\emptyset}$ we get for any $y \in X, \{y\} \delta B, B \bar{\delta} A$ and for each α and $\lambda, y_B \neq x_{\alpha B \lambda}$. But $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is $\mathfrak{S}_{\mathcal{C}2-}$ space, so there exists two disjoint \mathcal{C} – open set $\mathcal{C}_{U_{\lambda}}$ and $\mathcal{C}_{V_{\lambda}}$ containing $x_{\alpha B \lambda}$ and y_B respectively, thus $\mathcal{C}_A =_{\mathcal{C}} \bigvee_{x_{\alpha B \lambda} \in \mathcal{C}_A} x_{\alpha B \lambda} \leq_{\mathcal{C}} \bigvee_{\mathcal{C}} \{\mathcal{C}_{U_{\lambda}}, x_{\alpha B \lambda} \in \mathcal{C}_{U_{\lambda}}\}$, therefore by \mathcal{C} – compactness then exists finite index n $\alpha_1, \alpha_2, \dots, \alpha_n, \lambda_1, \lambda_2, \dots, \lambda_n$ such that $\mathcal{C}_A \leq_{\mathcal{C}} \bigvee_{i=1}^n \{\mathcal{C}_{U_{\lambda_i}}, x_{\alpha_i B \lambda_i} \in \mathcal{C}_{U_{\lambda_i}}\}$. And $\mathcal{C}_V =_{\mathcal{C}} \bigwedge_{i=1}^n \mathcal{C}_{V_{\lambda_i}}$ is \mathcal{C} – open center set containing y_B such that $\forall y \in X, \forall B \subseteq X, \{y\} \delta B$ and $B \delta V$, so $B \bar{\delta} A$ and $V \bar{\delta} A$. Hence \mathcal{C}_A is \mathcal{C} – closed center set.

Definition 3.1.6:

A family of center sets $\{\mathcal{C}_{A_{\lambda}}, \lambda \in \Lambda\}$ will be said to have the a finite intersection property if then exists finite index Λ_0 subset of Λ and a non-empty subset H of X such that $\langle A_{\lambda_i}, H \rangle \in \mathcal{C}_{A_{\lambda_i}} \forall i \in \Lambda_0$ and will denoted by \mathcal{C}_{FIP} .

Theorem 3.1.7:

A \mathcal{C} – topological space $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} – compact if and only if any family $\mathcal{F} = \{\mathcal{C}_{F_{\lambda}}, \lambda \in \Lambda\}$ of center \mathcal{C} -closed sets having the property \mathcal{C}_{FIP} , satisfy that there exists anon-empty subset K of X such that $\langle F_{\lambda}, K \rangle \in \mathcal{C}_{F_{\lambda}} \forall \lambda \in \Lambda$.

Proof:

Suppose $(X, \delta, \mathfrak{S}_C)$ is \mathcal{C} – compact space and $\mathcal{F} = \{\mathcal{C}_{F_\lambda}, \lambda \in \Lambda\}$ be a family of \mathcal{C} -closed center sets with property \mathcal{C}_{FIP} . Let us if possible that for each non-empty subsets H of X , $\forall \lambda \in \Lambda$ such that $F_\lambda \bar{\delta} H$. From this fact we get that for some $\lambda \in \Lambda$, $\forall x \in X$, $\{x\} \delta X$ and $\{x\} \bar{\delta} F_\lambda$. So then exists $\lambda_1, \lambda_2, \dots, \lambda_n$ with property $\mathcal{C}_X =_C \bigvee_{i=1}^n \mathcal{C}_X^{(F_{\lambda_i})}$, through at this we get that for each non-empty subset K of X , that for each $i = 1, 2, \dots, n$ $F_{\lambda_i} \bar{\delta} K$, so the family \mathcal{F} do not have a property \mathcal{C}_{FIP} , which contradiction with assumption. Therefore there exists a non-empty H subset of X satisfy that, $\forall \lambda \in \Lambda$, $F_\lambda \delta H$.

Conversely, if possible that \mathcal{C}_X is not \mathcal{C} – compact, then there exists \mathcal{C} – open cover say $\mathcal{H} = \{\mathcal{C}_{A_\lambda}, \lambda \in \Lambda\}$ has no finite sub cover, then for each finite subset Λ_0 of Λ , that is $\bigvee_{\lambda \in \Lambda_0} \mathcal{C}_{A_\lambda} \neq_C \mathcal{C}_X$, then there exists a non-empty subset H of X such that $\langle A_\lambda, H \rangle \in \mathcal{C}_X^{(A_\lambda)}$ for each $\lambda \in \Lambda_0$. But \mathcal{H} is not \mathcal{C} – open cover of \mathcal{C}_X , so for this fact we get that for each non-empty subset K of X we have for each $\lambda \in \Lambda$, $X \bar{\delta} K$, this contradiction with assumption because each $\mathcal{C}_X^{(A_\lambda)}$ is \mathcal{C} – closed. Therefore $(X, \delta, \mathfrak{S}_C)$ is \mathcal{C} – compact center space.

The relation between \mathcal{C} – compact and center separation axioms has been presented by some theorems and properties as below.

Theorem 3.1.8:

If $(X, \delta, \mathfrak{S}_C)$ be a \mathcal{C} – topological space then:

- a) If \mathcal{C}_A and \mathcal{C}_B are \mathcal{C} – compact, then $\mathcal{C}_{A \cup B}$ is \mathcal{C} – compact.
- b) If \mathcal{C}_A is \mathcal{C} – closed and \mathcal{C}_B is \mathcal{C} – compact in \mathfrak{S}_{C2-} space X , then $\mathcal{C}_A \wedge \mathcal{C}_B$ is \mathcal{C} – compact.

- c) If \mathcal{C}_A and \mathcal{C}_B are \mathcal{C} – compact sets in $\mathfrak{S}_{\mathcal{C}_2}$ – space X , then $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ is \mathcal{C} – compact.

Proof:

- a) Let $\mathcal{H} = \{\mathcal{C}_{A_\lambda}, \lambda \in \Lambda\}$ be any \mathcal{C} – open cover of $\mathcal{C}_{A \cup B}$, but $A, B \subseteq A \cup B$, thus the family \mathcal{H} be \mathcal{C} – open cover of \mathcal{C}_A and \mathcal{C}_B . But \mathcal{C}_A and \mathcal{C}_B are \mathcal{C} – compactness, so there exists finite subsets Λ_1 and Λ_2 of Λ such that $\mathcal{C}_A \preceq_{\mathcal{C}} \bigvee_{\lambda \in \Lambda_1} \mathcal{C}_{A_\lambda}$ and $\mathcal{C}_B \preceq_{\mathcal{C}} \bigvee_{\alpha \in \Lambda_2} \mathcal{C}_{A_\alpha}$, then we get that
- $$\mathcal{C}_{A \cup B} \preceq_{\mathcal{C}} \left(\bigvee_{\lambda \in \Lambda_1} \mathcal{C}_{A_\lambda} \right) \bigvee_{\mathcal{C}} \left(\bigvee_{\alpha \in \Lambda_2} \mathcal{C}_{A_\alpha} \right)$$
- $$=_{\mathcal{C}} \bigvee_{\gamma \in (\Lambda_1 \cup \Lambda_2)} \mathcal{C}_{A_\gamma}. \text{ But } \Lambda_1 \cup \Lambda_2 \text{ is finite subset of } \Lambda. \text{ Hence } \mathcal{C}_{A \cup B} \text{ is } \mathcal{C} \text{ – compact center set.}$$
- b) and c) are directly by theorem 3.1.4 and the fact the \mathcal{C} – intersection of two \mathcal{C} – closed center set is \mathcal{C} – closed center set.

Theorem 3.1.9:

If $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} – topological space then:

- a) Every \mathcal{C} – compact $\mathfrak{S}_{\mathcal{C}_2}$ – space is \mathcal{C} – regular.
 b) Every \mathcal{C} – compact $\mathfrak{S}_{\mathcal{C}_2}$ – space is \mathcal{C} – normal.
 c) If \mathcal{C}_A is \mathcal{C} – compact subset of \mathcal{C} – regular, then for each \mathcal{C} – open $\mathcal{C}_A \preceq_{\mathcal{C}} \mathcal{C}_U$, there is an \mathcal{C} – open \mathcal{C}_V with $\mathcal{C}_A \preceq_{\mathcal{C}} \mathcal{C}_V \preceq_{\mathcal{C}} cl_{\mathcal{C}}(\mathcal{C}_V) \preceq_{\mathcal{C}} \mathcal{C}_U$

Proof:

- a) Let x_B be a center point such that $A \bar{\delta} B$, with $\mathcal{C}_X^{(A)}$ is \mathcal{C} – open and $x_B \in \mathcal{C}_X^{(A)}$. Then for each $y_H \in \mathcal{C}_A$ we get that $H \bar{\delta} B$. And by $\mathfrak{S}_{\mathcal{C}_2}$ – space we have two disjoint center sets \mathcal{C}_{U_y} and \mathcal{C}_{V_y} with property that $U_y \delta B$ and $V_y \delta H$, then the collection of center \mathcal{C} – open set $\xi = \{\mathcal{C}_{V_y}; \text{ for each } y_B \in \mathcal{C}_A\}$ is \mathcal{C} – open cover. But \mathcal{C}_A is \mathcal{C} – closed of

\mathcal{C} – compact and by theorem 3.1.4 we have that \mathcal{C}_A is \mathcal{C} – compact so, there exists y_1, y_2, \dots, y_n such that for each non empty K subset of X , $K\delta A$

and $K\delta V_{y_i}$, put $\mathcal{C}_V = \bigcap_{i=1}^n \mathcal{C}_{V_{y_i}}$ and $\mathcal{C}_U = \bigcup_{i=1}^n \mathcal{C}_{U_{y_i}}$ having the property

that for each non-empty subset K of X that is $U\bar{\delta}K$ and $V\bar{\delta}K$. But $\mathcal{C}_A \preceq_{\mathcal{C}} \mathcal{C}_V$ and $x_B \in \mathcal{C}_U$, thus $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} – regular space.

b) Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} –compact $\mathfrak{S}_{\mathcal{C}2}$ – space and let $\mathcal{C}_H, \mathcal{C}_K$ are disjoint \mathcal{C} –closed sets, so by part (a) that $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} – regular, then for each $x_B \in \mathcal{C}_H$ there exists two disjoint center \mathcal{C} – open sets \mathcal{C}_{U_x} and \mathcal{C}_{V_x} such that $B\delta U_x$ and $\mathcal{C}_K \preceq_{\mathcal{C}} \mathcal{C}_{V_x}$, Therefore the collection of \mathcal{C} – open set $\{\mathcal{C}_{U_x}, \text{ for each } x_B \in \mathcal{C}_H\}$ is \mathcal{C} – cover of \mathcal{C}_H but by theorem 3.1.4 \mathcal{C}_H is \mathcal{C} –compact then there exists finite center point $x_{i_{B_i}}$ of \mathcal{C}_H such that

$\mathcal{C}_H \preceq_{\mathcal{C}} \bigcap_{i=1}^n \mathcal{C}_{U_{x_i}}$. Put $\mathcal{C}_U = \bigcup_{i=1}^n \mathcal{C}_{U_{x_i}}$ and $\mathcal{C}_V = \bigcap_{i=1}^n \mathcal{C}_{V_{x_i}}$. The \mathcal{C} –

open center sets \mathcal{C}_U and \mathcal{C}_V having the property that for each a non-empty subset M of X satisfy that $U\bar{\delta}M$ and $V\bar{\delta}M$. Hence $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} –normal space.

Proposition 3.1.10:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism shrink function and $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{S}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ be a shrink \mathcal{C} – homeomorphism function, then If \mathcal{C}_A is \mathcal{C} – compact set for each \mathcal{C}_A subset of \mathcal{C}_X , then $\mathcal{C}_{f(A)}$ is \mathcal{C} – compact set subset of \mathcal{C}_Y .

Proof:

Let \mathcal{C}_A is \mathcal{C} – compact set in X and let $\{\mathcal{C}_{U_{\alpha}}, \alpha \in \Delta\}$ be \mathcal{C} –open cover of $\mathcal{C}_{f(A)}$, then $\mathcal{C}_{f(A)} = \bigcap_{\alpha \in \Delta} \mathcal{C}_{U_{\alpha}}$, this mean, for each $\emptyset \neq K \subseteq Y, K\delta_Y f(A)$

iff there exists $\alpha_o \in \Delta$, $K\delta_Y U_{\alpha_o}$, since f_c^{-1} is shrink \mathcal{C} -continuous function, then $f^{-1}(K)\delta_X f^{-1}(f(A))$ iff $f^{-1}(K)\delta_X f^{-1}(U_{\alpha_o})$, since f is onto function, then $f^{-1}(K)\delta_X A$, implies $f^{-1}(K)\delta_X f^{-1}(U_{\alpha_o})$ for some $\alpha_o \in \Delta$, since f_c is shrink \mathcal{C} -continuous function, then $\{\mathcal{C}_{f^{-1}(U_{\alpha})}, \alpha \in \Delta\}$ is \mathcal{C} -open cover of \mathcal{C}_A . But

\mathcal{C}_A is \mathcal{C} -compact set, therefore $\mathcal{C}_A =_c \bigvee_{i=1}^n \{\mathcal{C}_{f^{-1}(U_i)}, i \in \Delta\}$, then $\forall H \subseteq X$,

$A\delta_X H$ iff $H\delta_X f^{-1}(U_i)$ such that $i = 1, 2, \dots, n$, since f is δ -continuous function, then $f(A)\delta_Y f(H)$ iff $f(H)\delta_Y f(f^{-1}(U_i))$, but $f(f^{-1}(U_i)) \subseteq U_i$, then $f(A)\delta_Y f(H)$ iff $f(H)\delta_Y U_i$ for some $i = 1, 2, \dots, n$. Thus

$\mathcal{C}_{f(A)} =_c \bigvee_{i=1}^n \{\mathcal{C}_{U_i}, i = 1, 2, \dots, n\}$. Hence $\mathcal{C}_{f(A)}$ is \mathcal{C} -compact set in Y .

Proposition 3.1.11:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a reduction δ -homeomorphism reduction function and $f_c: (X, \delta_X, \mathfrak{I}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{I}_{\mathcal{C}Y})$ be a reduction \mathcal{C} -homeomorphism function, if \mathcal{C}_A is \mathcal{C} -compact set for each \mathcal{C}_A subset of \mathcal{C}_X , then $f_c(\mathcal{C}_A)$ is \mathcal{C} -compact set subset of \mathcal{C}_Y .

Proof:

Let \mathcal{C}_A is \mathcal{C} -compact set in X and let $\{\mathcal{C}_{U_{\alpha}}, \alpha \in \Delta\}$ be \mathcal{C} -open cover of $f_c(\mathcal{C}_A)$, then $f_c(\mathcal{C}_A) =_c \bigvee_{\alpha \in \Delta} \{\mathcal{C}_{U_{\alpha}}, \alpha \in \Delta\}$, this mean, for each $\emptyset \neq K \subseteq Y$, $K\delta_Y f(A)$ iff there exists $\alpha_o \in \Delta$, $K\delta_Y U_{\alpha_o}$, since f_c^{-1} is reduction \mathcal{C} -continuous function, then $f^{-1}(K)\delta_X f^{-1}(f(A))$ iff $f^{-1}(K)\delta_X f^{-1}(U_{\alpha_o})$, since f is onto function, then $f^{-1}(K)\delta_X A$, implies $f^{-1}(K)\delta_X f^{-1}(U_{\alpha_o})$ for some $\alpha_o \in \Delta$, since f_c is reduction \mathcal{C} -continuous function, then $\{f_c(\mathcal{C}_{U_{\alpha}}), \alpha \in \Delta\}$ is \mathcal{C} -open

cover of \mathcal{C}_A . But \mathcal{C}_A is \mathcal{C} -compact set, therefore $\mathcal{C}_A =_c \bigvee_{i=1}^n \{f_c^{-1}(\mathcal{C}_{U_i}), i \in \Delta\}$

$\subseteq \Delta\}$, then $\forall H \subseteq X, A\delta_X H$ iff $H\delta_X f^{-1}(U_i)$ such that $i = 1, 2, \dots, n$, since f is δ -continuous function, then $f(A)\delta_Y f(H)$ iff $f(H)\delta_Y f(f^{-1}(U_i))$, but $f(f^{-1}(U_i)) \subseteq U_i$, then $f(A)\delta_Y f(H)$ iff $f(H)\delta_Y U_i$ for some $i = 1, 2, \dots, n$. Thus

$f(C_A) =_{\mathcal{C}} \bigcup_{i=1}^n \{C_{U_i}, i = 1, 2, \dots, n\}$. Hence $f_{\mathcal{C}}(C_A)$ is \mathcal{C} -compact set in Y .

3.2 \mathcal{LC} - Compact:

In this section we introduce and study notion of \mathcal{LC} - compact space in \mathcal{C} -topological space, and its relationship to the axioms of separation were studied.

Definition 3.2.1:

A \mathcal{C} -topological space $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is said to be \mathcal{LC} - compact space if and only if every center point x_A in X has at least one \mathcal{C} -neighborhood C_N whose $cl_{\mathcal{C}}(C_N)$ is \mathcal{C} -compact.

Theorem 3.2.2:

Every \mathcal{C} -compact space $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is \mathcal{LC} - compact space.

Proof:

Let $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ be \mathcal{C} -compact space and x_A any center point in C_X , then $A\delta_X X$, then C_X is \mathcal{C} -neighborhood of x_A , but $cl_{\mathcal{C}}(C_X) = C_X$ is \mathcal{C} -compact. Thus $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is \mathcal{LC} - compact space.

The following example shows the fact that the converse of the above theorem is not true.

Example 3.2.3

Let $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ be any \mathcal{C} -topological space such that X is infinite set, then $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is not \mathcal{C} -compact space. But its \mathcal{LC} - compact space. For let x_A be

any \mathcal{C} – point in X , then $\{x_A\}$ is \mathcal{C} – neighborhood of x_A whose $cl_{\mathcal{C}}(\{x_A\}) = \{x_A\}$. Also $\{x_A\}$ is a \mathcal{C} –compact for all x_A \mathcal{C} – point in X , being finite.

Theorem 3.2.4:

Every \mathcal{C} – closed subspace of \mathcal{LC} - compact space is \mathcal{LC} - compact.

Proof:

Let \mathcal{C}_A be \mathcal{C} – closed subspace of \mathcal{LC} – compact space $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ and x_B \mathcal{C} – point in X such that $B\delta_x A$, and $B\delta_x X$. Since $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is \mathcal{LC} – compact space, there exists \mathcal{C}_N \mathcal{C} – neighborhood of x_B such that $cl_{\mathcal{C}}(\mathcal{C}_N)$ is \mathcal{C} – compact. But the $\mathcal{C}_N \wedge_{\mathcal{C}} \mathcal{C}_A$ is \mathcal{C} – neighborhood of x_B and $cl_{\mathcal{C}}(\mathcal{C}_N \wedge_{\mathcal{C}} \mathcal{C}_A)$ is \mathcal{C} – closed subset of \mathcal{C} –compact set $cl_{\mathcal{C}}(\mathcal{C}_N)$ and is thus itself \mathcal{C} –compact.

Proposition 3.2.5:

Every \mathcal{C} – open subspace of $\mathfrak{S}_{\mathcal{C}2-}$ space \mathcal{LC} - compact space is \mathcal{LC} – compact.

Proof:

Let \mathcal{C}_V be a \mathcal{C} – open subspace of $\mathfrak{S}_{\mathcal{C}2-}$ space \mathcal{LC} – compact $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ and x_B \mathcal{C} – point in X such that $B\delta_x V$ and $B\delta_x X$, there exist \mathcal{C}_{N_x} \mathcal{C} – open with $N_x \delta_x B$ such that $cl_{\mathcal{C}}(\mathcal{C}_{N_x})$ is \mathcal{C} –compact. But $\{x_B\}$ is \mathcal{C} –compact set in X , $\mathcal{C}_V \wedge_{\mathcal{C}} \mathcal{C}_{N_x}$ is \mathcal{C} – open such that $B\delta_x(V \cup N_x)$ and $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is $\mathfrak{S}_{\mathcal{C}2-}$ space and \mathcal{LC} – compact space. So there exists an \mathcal{C} – open set \mathcal{C}_U in X such that $cl_{\mathcal{C}}(\mathcal{C}_U)$ is \mathcal{C} –compact, but $B\delta_x U$ and $\mathcal{C}_U \approx_{\mathcal{C}} cl_{\mathcal{C}}(\mathcal{C}_U) \Rightarrow \forall H\delta_x U \Rightarrow H\delta_x(V \cup N_x) \Rightarrow H\delta_x V$ This implies that $cl_{\mathcal{C}}(\mathcal{C}_U)$ is \mathcal{C} –compact (\mathcal{C} – neighborhood) of x_B subset of \mathcal{C}_V . Hence \mathcal{C}_V is \mathcal{LC} – compact.

Proposition 3.2.6:

Let $(X, \delta_x, \mathfrak{T}_{\mathcal{C}2-})$ be a $\mathfrak{T}_{\mathcal{C}2-}$ space . If \mathcal{C}_A and \mathcal{C}_B are \mathcal{LC} – compact spaces, then $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ is \mathcal{LC} – compact space.

Proof:

Let $x_H \in \mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ such that $H\delta_X A$ and $H\delta_X B$ for all $\emptyset \neq H \subseteq X$. And let \mathcal{C}_{N_1} and \mathcal{C}_{N_2} are \mathcal{C} – compact such that $N_1\delta_X H$ in \mathcal{C}_A and $N_2\delta_X H$ in \mathcal{C}_B , by theorem (3.1.8), then $\mathcal{C}_{N_1} \wedge_{\mathcal{C}} \mathcal{C}_{N_2}$ is \mathcal{C} – compact and \mathcal{C} – neighborhood of x_H . Thus $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ is \mathcal{LC} – compact space.

Proposition 3.2.7:

A $\mathfrak{T}_{\mathcal{C}2-}$ space $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – compact if and only if each of its \mathcal{C} – points is an \mathcal{C} – interior point of some \mathcal{C} – compact subspace of $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$.

Proof:

If $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – compact, then for all x_A in X there exists \mathcal{C}_N such that $A\delta_X N$ and $cl_{\mathcal{C}}(\mathcal{C}_N)$ is \mathcal{C} – compact, it follows that $cl_{\mathcal{C}}(\mathcal{C}_N)$ is a \mathcal{C} – compact (\mathcal{C} – neighborhood) of x_A is an \mathcal{C} – interior point. Thus $\forall x_A$ is an \mathcal{C} – interior point of some \mathcal{C} – compact.

Conversely, let x_A is \mathcal{C} – point in X be arbitrary. By hypothesis, there exists an \mathcal{C} – compact \mathcal{C}_H such that $A\delta_X H$, then \mathcal{C}_H is a \mathcal{C} – neighborhood of x_A . Since $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is $\mathfrak{T}_{\mathcal{C}2-}$ space, implies \mathcal{C}_H is \mathcal{C} – closed, so that $cl_{\mathcal{C}}(\mathcal{C}_H) = \mathcal{C}_H$, thus x_A has a \mathcal{C} – neighborhood whose $cl_{\mathcal{C}}(\mathcal{C}_H)$ is \mathcal{C} – compact. And so $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – compact.

Proposition 3.2.8:

Every \mathcal{LC} – compact $\mathfrak{S}_{\mathcal{C}2-}$ space is \mathcal{C} – regular space.

Proof:

Let $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is \mathcal{LC} – compact $\mathfrak{S}_{\mathcal{C}2-}$ space, x_A be any \mathcal{C} – point in X and \mathcal{C}_F any \mathcal{C} – closed set in X such that $A\overline{\delta_X}F$ for each $A \subseteq X$. Since $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is $\mathfrak{S}_{\mathcal{C}2-}$ space, then \mathcal{C}_F is \mathcal{C} – compact and for each $y \in X$ and $B\delta_X F$, then there exists \mathcal{C}_G and \mathcal{C}_{H_y} \mathcal{C} – open sets such that $A\delta_X G$, $F\delta_X H_y$ and $\mathcal{C}_G \wedge_{\mathcal{C}} \mathcal{C}_{H_y} = \mathcal{C}_\emptyset$, the family of all $\{\mathcal{C}_{H_{y_i}}, y_B \in \mathcal{C}_F\}$ \mathcal{C} – open cover of \mathcal{C} – compact \mathcal{C}_{H_y} and hence a finite subfamily $\mathcal{C}_{H_{y_1}}, \mathcal{C}_{H_{y_2}}, \dots, \mathcal{C}_{H_{y_n}}$ such that $\mathcal{C}_{H_y} \leq_{\mathcal{C}} \bigvee_{i=1}^n \mathcal{C}_{H_{y_i}}$. Hence $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is \mathcal{C} – regular space.

Proposition 3.2.9:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – continuous onto shrink function and $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{S}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ be a shrink \mathcal{C} – continuous function, then If $(X, \delta_X, \mathfrak{S}_{\mathcal{C}X})$ is \mathcal{LC} – compact space, then $(Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ is \mathcal{LC} – compact space.

Proof:

Let $B \subseteq Y$ and $y \in Y$ such that $\{y\}\delta_Y B$, but $B\delta_Y Y$. Since f is onto, then there exists $A \subseteq X$ such that $\{x\}\delta_X A$ and $f_{\mathcal{C}}(x_A) = y_B$. But $(X, \delta_X, \mathfrak{S}_{\mathcal{C}X})$ is \mathcal{LC} – compact space, then there exists \mathcal{C}_N nbd for x_A such that for each $A \subseteq X$ such that $A\delta_X N$ and $cl_{\mathcal{C}}(\mathcal{C}_N)$ is \mathcal{C} – compact set. Also $f_{\mathcal{C}}$ is shrink \mathcal{C} – continuous function and by proposition (3.1.10), we get for each $B \subseteq Y$, $\{y\}\delta_Y B$, then $B\delta_Y f(N)$. Hence $(Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ is \mathcal{LC} – compact space

3.3 \mathcal{AC} - Compact:

In this section we introduce and study notion of \mathcal{AC} – compact space in \mathcal{C} –topological space, and its relationship to the axioms of separation were studied.

Definition 3.3.1:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -topological space, the universal set \mathcal{C}_X (a non-empty center set \mathcal{C}_A) is called \mathcal{AC} – compact space if each \mathcal{C} – open cover $\{\mathcal{C}_{U_{\alpha}}, \alpha \in \Gamma\}$ of \mathcal{C}_X (\mathcal{C}_A) has finite Δ subset of Γ such that $\forall H \subseteq X$, there exist $\alpha_0 \in \Delta$, $cl_{\mathcal{C}}(U_{\alpha_0})\delta H$.

Proposition 3.3.2:

Every \mathcal{C} – compact space is an \mathcal{AC} – compact space.

Proof:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be \mathcal{C} – compact space and $\mathcal{H} = \{\mathcal{C}_{A_{\lambda}}, \lambda \in \Lambda\}$ be a \mathcal{C} – open cover of \mathcal{C}_X . Then there exists finite I subset of Λ such that $\forall H \subseteq X$, there exist $\lambda_0 \in I$, $A_{\lambda_0}\delta H$. Now since $\mathcal{C}_{A_{\lambda}} \preceq_{\mathcal{C}} cl_{\mathcal{C}}(\mathcal{C}_{A_{\lambda}})$. We have for each $\lambda \in \Lambda$, $H\delta A_{\lambda}$, then $H\delta cl(\mathcal{C}_{A_{\lambda}})$. Therefore there exist $\lambda_0 \in I$, $cl(\mathcal{C}_{A_{\lambda_0}})\delta H$. Hence $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{AC} – compact space.

Remark 3.3.3:

Every \mathcal{C} – compact set is \mathcal{AC} – compact set.

The following example shows that the conversely of the above remark is not true.

Example 3.3.4:

Let $X = \{1, 2, 3, \dots\}$ with δ be discrete proximity, define a \mathcal{C} –topology $\mathfrak{S}_{\mathcal{C}}$ on \mathcal{C}_X as follows:

$$\mathfrak{S}_C = \{\mathcal{C}_{\{1\}}, \mathcal{C}_{\{1,2\}}, \mathcal{C}_{\{1,2,3\}}, \dots\} \vee_C \{\mathcal{C}_\emptyset\}.$$

Now we will show that $(X, \delta, \mathfrak{S}_C)$ is an \mathcal{AC} – compact space.

But $(X, \delta, \mathfrak{S}_C)$ is not \mathcal{C} – compact space.

For each $\emptyset \neq \mathcal{C}_{A_\alpha} \in \mathfrak{S}_C$, we have $cl_C(\mathcal{C}_{A_\alpha}) =_C \mathcal{C}_X$.

Now, let $\mathcal{H} = \{\mathcal{C}_{A_\alpha}, \lambda \in \Lambda\}$ be a \mathcal{C} – open cover of \mathcal{C}_X .

This means $\mathcal{C}_X =_C \vee_C \{\mathcal{C}_{A_\alpha}, \lambda \in \Lambda\}$.

Then there exists $\alpha_o \in \Lambda$ such that $\mathcal{C}_{A_{\alpha_o}} \in \mathfrak{S}_C$.

But $cl_C(\mathcal{C}_{A_\alpha}) =_C \mathcal{C}_X$. Thus $(X, \delta, \mathfrak{S}_C)$ is an \mathcal{AC} – compact space.

Now we will show that $(X, \delta, \mathfrak{S}_C)$ is not \mathcal{C} – compact space.

Let $\Gamma = \{\mathcal{C}_{\{1\}}, \mathcal{C}_{\{1,2\}}, \mathcal{C}_{\{1,2,3\}}, \dots\}$ be a \mathcal{C} – open cover of \mathcal{C}_X , it is clear that Γ dose not have a finite subcover of \mathcal{C}_X .

Proposition 3.3.5:

Let $(X, \delta, \mathfrak{S}_C)$ be a \mathcal{C} –Regular space, then $(X, \delta, \mathfrak{S}_C)$ is \mathcal{C} – compact space if and only if is \mathcal{AC} – compact space.

Proof:

If $(X, \delta, \mathfrak{S}_C)$ is \mathcal{C} – compact space, then by Proposition (3.3.2) $(X, \delta, \mathfrak{S}_C)$ is \mathcal{AC} – compact space. Conversely, let $\mathcal{H} = \{\mathcal{C}_{U_\lambda}, \lambda \in \Lambda\}$ be a \mathcal{C} – open cover of \mathcal{C}_X . For each $x \in X$, $A \subseteq X$ and $\{x\} \delta A$, there exists λ_x such that $A \delta U_{\lambda_x}$, but $(X, \delta, \mathfrak{S}_C)$ is \mathcal{C} –Regular space, then there exists \mathcal{C} –open set $\mathcal{C}_{H_{\lambda_x}}$ such that for each $x \in X$, $H_{\lambda_x} \subseteq X, \{x\} \delta H_{\lambda_x}$ and $\forall K \subseteq X, K \delta cl(H_{\lambda_x})$, then $K \delta U_{\lambda_x}$. Now, $\mathcal{F} = \{\mathcal{C}_{H_{\lambda_x}}, x \in X\}$ is a \mathcal{C} – open cover of \mathcal{C}_X . But $(X, \delta, \mathfrak{S}_C)$ is \mathcal{AC} – compact space, then there exists finite I subset of X such that for each

$x \in X, \forall K \subseteq X, K \delta X$ and $K \delta cl(H_{\lambda x})$ for some $x \in I$, therefore $K \delta cl_{\tau}(H_{\lambda x})$ for some $x \in I$. Hence $(X, \delta, \mathfrak{J}_C)$ is \mathcal{C} – compact space.

Proposition 3.3.6:

Let $(X, \delta, \mathfrak{J}_C)$ be a \mathcal{C} –topological space and $\mathcal{C}_A, \mathcal{C}_B$ are center sets in X such that \mathcal{C}_A is \mathcal{AC} – compact set and $\mathcal{C}_A \preceq_C \mathcal{C}_B \preceq_C cl_C(\mathcal{C}_A)$, then \mathcal{C}_B is \mathcal{AC} – compact set.

Proof:

Let $\mathcal{H} = \{\mathcal{C}_{U_{\lambda}}, \lambda \in \Lambda\}$ be a \mathcal{C} – open cover of \mathcal{C}_B ,

\Rightarrow For each $y \in X, \{y\} \delta B$, for some $\lambda_o \in \Lambda, \forall \emptyset \neq K \subseteq X, K \delta B$, then $K \delta U_{\lambda_o}$.

But $\forall K \delta A$, then $K \delta B$,

\Rightarrow for some $\lambda_o \in \Lambda, \forall \emptyset \neq K \subseteq X, K \delta A$, then $K \delta U_{\lambda_o}$.

And \mathcal{C}_A is \mathcal{AC} – compact set,

\Rightarrow there exists finite I subset of Λ such that $\forall \emptyset \neq K \subseteq X, K \delta A$, then for some $\lambda_o \in I, K \delta cl(U_{\lambda_o})$.

$\Rightarrow \forall \emptyset \neq K \subseteq X, K \delta cl(A)$ iff for some $\lambda_o \in I, K \delta cl(U_{\lambda_o})$.

But $\mathcal{C}_B \preceq_C cl_C(\mathcal{C}_A)$, then some $\lambda_o \in I, K \delta B$, then $K \delta cl(U_{\lambda_o})$.

Hence \mathcal{C}_B is \mathcal{AC} – compact set.

Corollary 3.3.7:

Let $(X, \delta, \mathfrak{J}_C)$ be a \mathcal{C} –topological space and \mathcal{C}_A be a \mathcal{AC} – compact set in X , then $cl_C(\mathcal{C}_A)$ is also \mathcal{AC} – compact set.

Proposition 3.3.8:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism shrink function and $f_c: (X, \delta_X, \mathfrak{C}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{C}_{cY})$ be a shrink \mathcal{C} – continuous function, then If \mathcal{C}_A is \mathcal{AC} – compact set for each \mathcal{C}_A subset of \mathcal{C}_X , then $\mathcal{C}_{f(A)}$ is \mathcal{AC} – compact set subset of \mathcal{C}_Y .

Proof:

Suppose \mathcal{C}_A is \mathcal{AC} – compact set in X and $\mathcal{H} = \{\mathcal{C}_{U_\lambda}, \lambda \in \Lambda\}$ be a \mathcal{C} – open cover of $\mathcal{C}_{f(A)}$ such that for each $B \subseteq X, \forall x \in X$ such that $\{x\} \delta B$ and $\exists \lambda \in \Lambda, B \delta A_{\lambda \circ}$. By proposition (2.1.1.8), then $\mathcal{C}_A \ll_c f^{-1}(\mathcal{C}_{f(A)}) \ll_c f^{-1}(\bigvee_{\lambda} \{\mathcal{C}_{U_\lambda}\}) =_c \bigvee_{\lambda} \{\mathcal{C}_{f^{-1}(U_\lambda)}\}$. Therefore $\Gamma = \{\mathcal{C}_{f^{-1}(U_\lambda)}, \lambda \in \Lambda\}$ is \mathcal{C} – open cover of \mathcal{C}_A such that for each $K \subseteq X, \forall x \in A$ such that $\{x\} \delta K$ and $\exists \lambda \in \Lambda$ and $K \delta f^{-1}(U_{\lambda \circ})$. But \mathcal{C}_A is \mathcal{AC} – compact set, then Γ has a finite subfamily I such that for some $\lambda \circ \in I, \forall K \subseteq X, \forall x \in A$ such that $\{x\} \delta K$ and $\exists \lambda \circ \in I, K \delta f^{-1}(U_{\lambda \circ})$. Whose center closure \mathcal{C}_A and by proposition (2.1.1.8), that is to say:

$$\begin{aligned} \mathcal{C}_A \ll_c cl_c \left(\bigvee_{i=1}^n \mathcal{C}_{f^{-1}(U_{\lambda_i})} \right) &= cl_c \left(f^{-1} \left(\bigvee_{i=1}^n \mathcal{C}_{U_{\lambda_i}} \right) \right) \\ &\ll_c f^{-1} \left(cl_c \left(\bigvee_{i=1}^n \mathcal{C}_{U_{\lambda_i}} \right) \right). \text{ Accordingly and by Proposition (2.1.1.6), then} \\ \mathcal{C}_{f(A)} \ll_c f \left(f^{-1} \left(cl_c \left(\bigvee_{i=1}^n \mathcal{C}_{U_{\lambda_i}} \right) \right) \right) \\ &\ll_c cl_c \left(\bigvee_{i=1}^n \mathcal{C}_{U_{\lambda_i}} \right) =_c \bigvee_{i=1}^n cl_c(\mathcal{C}_{U_{\lambda_i}}). \end{aligned}$$

Thus $\mathcal{C}_{f(A)}$ is \mathcal{AC} – compact set.

Proposition 3.3.9:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a reduction δ – homeomorphism reduction function and $f_C: (X, \delta_X, \mathfrak{I}_{C_X}) \rightarrow (Y, \delta_Y, \mathfrak{I}_{C_Y})$ be a reduction \mathcal{C} – homeomorphism function, if C_A is \mathcal{AC} – compact set for each C_A subset of C_X , then $f_C(C_A)$ is \mathcal{AC} – compact set subset of C_Y .

Proof:

We prove the same way to prove Proposition (3.3.8).

3.4 \mathcal{HC} - closedness:

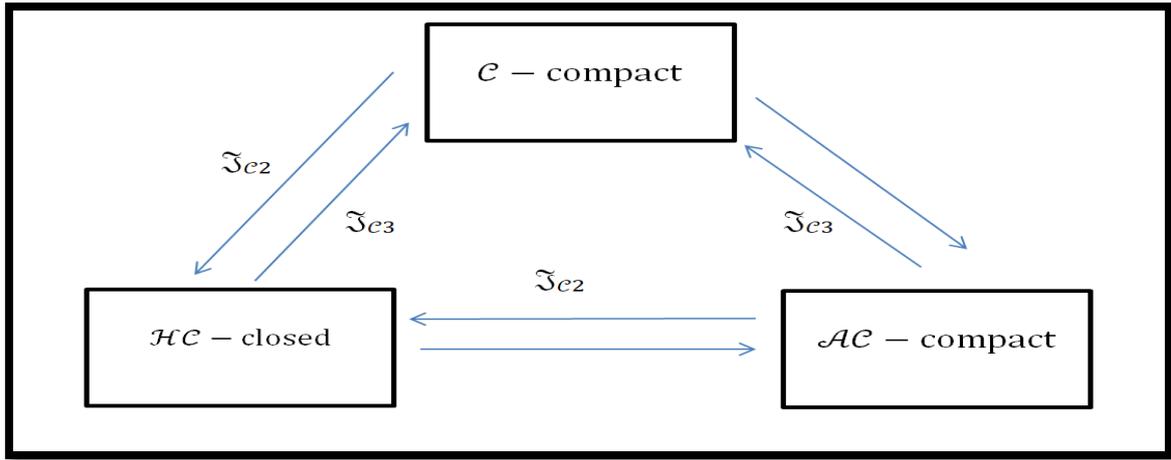
In this section, we give several characterization of \mathcal{C} – compact in center topological space, nearly \mathcal{HC} – closed space. Furthermore, the relationship between this concept and the separation axioms in center topological space had been studied.

Definition 3.4.1:

Let $(X, \delta, \mathfrak{I}_C)$ be \mathfrak{I}_{C_2} – space, the universal set C_X (a non-empty center set C_A) is called \mathcal{HC} – closed space if each \mathcal{C} – open cover $\{C_{U_\alpha}, \alpha \in \Gamma\}$ of C_X (C_A) has finite Δ subset of Γ such that $\forall H \subseteq X$, there exist $\alpha_0 \in \Delta, cl_C(U_{\alpha_0}) \delta H$.

Remarks 3.4.2:

- i. Every \mathcal{HC} – closed space is \mathcal{AC} – compact space.
- ii. Every \mathcal{AC} – compact space and \mathfrak{I}_{C_2} – space is \mathcal{HC} – closed space.
- iii. Every \mathcal{HC} – closed space and \mathfrak{I}_{C_3} – space is \mathcal{C} – compact space.
- iv. See the following diagram



The following example shows that the conversely of the above remark (i) is not true.

Example 3.4.3:

Let $(X, \delta, \mathfrak{T}_C)$ be a \mathcal{C} -topological space,

Where $X = \{a, b\}$, $A\delta B$ iff $A \cap B \neq \emptyset$

And $\mathfrak{T}_C = \{ \mathcal{C}_\emptyset, \{ \langle X, \{a\} \rangle \}, \{ \langle X, \{b\} \rangle \}, \{ \langle X, \{a\} \rangle, \langle X, \{b\} \rangle \}, \{ \langle X, \{a\} \rangle, \langle X, X \rangle \},$

$\{ \langle X, \{b\} \rangle, \langle X, X \rangle \}, \{ \langle X, X \rangle \}, \{ \langle \{a\}, \{a\} \rangle, \langle \{a\}, X \rangle \}, \{ \langle \{b\}, \{b\} \rangle, \langle \{b\}, X \rangle \}, \mathcal{C}_X \}$

Then $(X, \delta, \mathfrak{T}_C)$ is \mathcal{AC} – compact space.

But is not \mathfrak{T}_{C2} – *Space*. Hence it is not \mathcal{HC} – closed space.

Theorem 3.4.4:

Every \mathcal{C} – compact space and \mathfrak{T}_{C2} – *space* is \mathcal{HC} – closed space.

Proof:

Let $(X, \delta, \mathfrak{T}_C)$ be \mathcal{C} – compact space and \mathfrak{T}_{C2} – *space* and $\Gamma = \{ \mathcal{C}_{U_\alpha}, \alpha \in \Gamma \}$ be \mathcal{C} – open cover of $(X, \delta, \mathfrak{T}_C)$, then there exists finite I subset of Γ such

that $\forall x \in X$, $\forall H \subseteq X$, $\{x\} \delta H$, then for some $\alpha_o \in I$, $H \delta U_{\alpha_o}$, now since $\forall H \subseteq X$, $H \delta U_{\alpha_i}$, then $H \delta cl(U_{\alpha_i})$. Hence $(X, \delta, \mathfrak{S}_c)$ is \mathcal{HC} – closed space.

Theorem 3.4.5:

Let $(X, \delta, \mathfrak{S}_c)$ be \mathcal{C} – *Regular* space and \mathfrak{S}_{c2} – *space*, then $(X, \delta, \mathfrak{S}_c)$ is \mathcal{HC} – closed space if and only if it is \mathcal{C} – compact space.

Proof:

Since $(X, \delta, \mathfrak{S}_c)$ is \mathcal{HC} – closed space, then $(X, \delta, \mathfrak{S}_c)$ is \mathcal{AC} – compact space, but its \mathcal{C} – *Regular* space, then by Proposition (3.3.5) we have $(X, \delta, \mathfrak{S}_c)$ is \mathcal{C} – compact space.

Conversely, since $(X, \delta, \mathfrak{S}_c)$ is \mathcal{C} – compact space and \mathcal{C} – *Regular* space, and then by Proposition (3.3.5) we have $(X, \delta, \mathfrak{S}_c)$ is \mathcal{HC} – closed space.

Proposition 3.4.6:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism shrink function and $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ be a shrink \mathcal{C} – continuous function. If $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathcal{HC} – closed space, then $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathcal{HC} – closed space.

Proof:

Direct from define of \mathcal{HC} – closed space and by Proposition (3.3.8).

CHAPTER

FOUR

Center Indelofness

Chapter Four: Center Lindelofness

In this chapter, we give several characterizations of Center lindelofness, where the concept of \mathcal{C} – lindelof spaces within the central topological spaces was introduce and the topological properties of this space and its relationship to the axioms of separation were studied. Also in section two we introduce new concept, namely \mathcal{LC} - lindelof space and its relationship to the axioms of separation were studied. In section three and fore we introduce new concepts namely \mathcal{AC} – lindelof and \mathcal{HLC} – closed, and we study their properties.

4.1 Center Lindelofness

In this section, we introduce a new type of lindelofness in center topological space has been presented, that is called a \mathcal{C} – lindelof. Furthermore, the relationship between this concept and the separation axioms in center topological space had been studied.

Definition 4.1.1:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -topological space, the universal set \mathcal{C}_X (a non-empty center set \mathcal{C}_A) is called \mathcal{C} – lindelof space if each \mathcal{C} – open cover $\{\mathcal{C}_{U_{\alpha}}, \alpha \in \Gamma\}$ of \mathcal{C}_X (\mathcal{C}_A) has countable Δ subset of Γ such that $\forall H \subseteq X$, there exist $\alpha_0 \in \Delta, U_{\alpha_0} \delta H$.

Remarks and examples 4.1.2:

- i. The indiscrete *–topological space* , then every center sets are \mathcal{C} – lindelof space.
- ii. The discrete *–topological space* , then every center sets are \mathcal{C} – lindelof space if and only if it's countable.
- iii. Every \mathcal{C} – compact space is \mathcal{C} – lindelof space.

Theorem 4.1.3 :

Each \mathcal{C} – closed subset of \mathcal{C} – lindelof *topological space* $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} – lindelof set.

Proof:

Let \mathcal{C}_A be any non-empty \mathcal{C} – closed set in \mathcal{C} – lindelof space $(X, \delta, \mathfrak{S}_{\mathcal{C}})$. And $\mathcal{H} = \{\mathcal{C}_{A_\lambda}, \lambda \in \Lambda\}$ be \mathcal{C} – open cover to \mathcal{C}_A , so $\mathcal{C}_X^{(A)}$ be \mathcal{C} – open set. Thus for each $B \subseteq X, \forall x \in X$ such that $\{x\} \delta B$ and $\exists \lambda_0 \in \Lambda, B \delta A_{\lambda_0}$, from the concept of \mathcal{C} – lindelof space of \mathcal{C}_X , then there exists countable subset I of Λ say $\{\mathcal{C}_{A_1}, \mathcal{C}_{A_2}, \dots, \mathcal{C}_X^{(A)}\}$ such that $\forall H \subseteq X$, there exist $\lambda_{i_0} \in I, A_{\lambda_0} \delta H$, therefore for each $H \subseteq A, \forall x \in A$ there exist $\lambda_{i_0} \in I, A_{\lambda_0} \delta H$, Hence \mathcal{C}_A is \mathcal{C} – lindelof set.

Theorem 4.1.4:

Each \mathcal{C} – lindelof subset of $\mathfrak{S}_{\mathcal{C}2-}$ space $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} – closed set.

Proof:

Let \mathcal{C}_A be any non-empty \mathcal{C} – lindelof set, so by Proposition (1.2.11 part 4) we get that $\mathcal{C}_A = \bigcup_{\mathcal{C} x_{\alpha B_\lambda} \in \mathcal{C}_A} x_{\alpha B_\lambda}$, then for any $\alpha_0, \lambda_0, x_{\alpha_0} \in X$ such that $\{x_{\alpha_0}\} \delta B_{\lambda_0}$ and $B_{\lambda_0} \delta A$, and for the fact that $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_X^{(A)} =_{\mathcal{C}} \mathcal{C}_\emptyset$ we get for any $y \in X, \{y\} \delta B, B \bar{\delta} A$ and for each α and $\lambda, y_B \neq x_{\alpha B_\lambda}$. But $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is $\mathfrak{S}_{\mathcal{C}2-}$ space, so there exists two disjoint \mathcal{C} – open set \mathcal{C}_{U_λ} and \mathcal{C}_{V_λ} containing $x_{\alpha B_\lambda}$ and y_B respectively, thus $\mathcal{C}_A = \bigcup_{\mathcal{C} x_{\alpha B_\lambda} \in \mathcal{C}_A} x_{\alpha B_\lambda} \preceq_{\mathcal{C}} \bigcup_{\mathcal{C}} \{\mathcal{C}_{U_\lambda}, x_{\alpha B_\lambda} \in \mathcal{C}_{U_\lambda}\}$, therefore by \mathcal{C} – lindelofness then exists countable index $\alpha_1, \alpha_2, \dots, \lambda_1, \lambda_2, \dots$, such that $\mathcal{C}_A \preceq_{\mathcal{C}} \bigcup_{i=1}^{\infty} \{\mathcal{C}_{U_{\lambda_i}}, x_{\alpha_i B_{\lambda_i}} \in \mathcal{C}_{U_{\lambda_i}}\}$. And

$\mathcal{C}_V = \bigcap_{i=1}^{\infty} \mathcal{C}_{V\lambda_i}$ is \mathcal{C} – open center set containing y_B such that $\forall y \in X, \forall B \subseteq X, \{y\} \delta B$ and $B \delta V$, so $B \bar{\delta} A$ and $V \bar{\delta} A$. Hence \mathcal{C}_A is \mathcal{C} – closed center set.

The relation between \mathcal{C} –lindelof and center separation axioms has been presented by some theorems and properties as below:

Theorem 4.1.5:

If $(X, \delta, \mathfrak{S}_c)$ be a \mathcal{C} –topological space then:

- a) If \mathcal{C}_A and \mathcal{C}_B are \mathcal{C} –lindelof, then $\mathcal{C}_{A \cup B}$ is \mathcal{C} – lindelof.
- b) If $(X, \delta, \mathfrak{S}_c)$ is $\mathfrak{S}_{\mathcal{C}2-}$ space, then $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ is \mathcal{C} – lindelof.
- c) If \mathcal{C}_A is \mathcal{C} –closed and \mathcal{C}_B is \mathcal{C} – lindelof in $\mathfrak{S}_{\mathcal{C}2-}$ space X , then $\mathcal{C}_A \wedge \mathcal{C}_B$ is \mathcal{C} – lindelof.
- d) If \mathcal{C}_A and \mathcal{C}_B are \mathcal{C} – lindelof sets in $\mathfrak{S}_{\mathcal{C}2-}$ space X , then $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ is \mathcal{C} – lindelof.

Proof:

- a) Let $\mathcal{H} = \{\mathcal{C}_{A_\lambda}, \lambda \in \Lambda\}$ be any \mathcal{C} – open cover of $\mathcal{C}_{A \cup B}$, but $A, B \subseteq A \cup B$, thus the family \mathcal{H} be \mathcal{C} – open cover of \mathcal{C}_A and \mathcal{C}_B . But \mathcal{C}_A and \mathcal{C}_B are \mathcal{C} –lindelofness, so there exists countable subsets Λ_1 and Λ_2 of Λ such that $\mathcal{C}_A \leq_{\mathcal{C}} \bigvee_{\lambda \in \Lambda_1} \mathcal{C}_{A_\lambda}$ and $\mathcal{C}_B \leq_{\mathcal{C}} \bigvee_{\alpha \in \Lambda_2} \mathcal{C}_{A_\alpha}$, then we get that

$$\mathcal{C}_{A \cup B} \leq_{\mathcal{C}} \left(\bigvee_{\lambda \in \Lambda_1} \mathcal{C}_{A_\lambda} \right) \vee_{\mathcal{C}} \left(\bigvee_{\alpha \in \Lambda_2} \mathcal{C}_{A_\alpha} \right)$$

$$=_{\mathcal{C}} \bigvee_{\gamma \in (\Lambda_1 \cup \Lambda_2)} \mathcal{C}_{A_\gamma}. \text{ But } \Lambda_1 \cup \Lambda_2 \text{ is countable subset of } \Lambda. \text{ Hence } \mathcal{C}_{A \cup B}$$

is \mathcal{C} –lindelof set.

- b), c) and d) are directly by theorem 4.1.3 and the fact the \mathcal{C} – intersection of two \mathcal{C} –closed set is \mathcal{C} – closed set.

Theorem 4.1.6:

If $(X, \delta, \mathfrak{S}_C)$ be a \mathcal{C} –topological space then:

- a) Every \mathcal{C} – lindelof \mathfrak{S}_{C_2} – space is \mathcal{C} –regular.
- b) Every \mathcal{C} – lindelof \mathfrak{S}_{C_2} – space is \mathcal{C} –normal.
- c) If \mathcal{C}_A is \mathcal{C} – lindelof subset of \mathcal{C} –regular, then for each \mathcal{C} –open $\mathcal{C}_A \preceq_C \mathcal{C}_U$, there is an \mathcal{C} –open \mathcal{C}_V with $\mathcal{C}_A \preceq_C \mathcal{C}_V \preceq_C cl_C(\mathcal{C}_V) \preceq_C \mathcal{C}_U$

Proof:

- a) Let x_B be a center point such that $A\bar{\delta}B$, but $\mathcal{C}_X^{(A)}$ is \mathcal{C} – open and $x_B \in \mathcal{C}_X^{(A)}$. Then for each $y_H \in \mathcal{C}_A$ we get that $H\bar{\delta}B$. And by \mathfrak{S}_{C_2} – space we have two disjoint center sets \mathcal{C}_{U_y} and \mathcal{C}_{V_y} with property that $U_y\delta B$ and $V_y\delta H$, then the collection of center \mathcal{C} – open set $\xi = \{\mathcal{C}_{V_y}; \text{for each } y_B \in \mathcal{C}_A\}$ is \mathcal{C} – open cover. But \mathcal{C}_A is \mathcal{C} – closed of \mathcal{C} – lindelof and by theorem 4.1.2 we have that \mathcal{C}_A is \mathcal{C} – lindelof, so there exists y_1, y_2, \dots such that for each non empty K subset of X , $K\delta A$ and $K\delta V_{y_i}$, put $\mathcal{C}_V =_C \bigcup_{i=1}^{\infty} \mathcal{C}_{V_{y_i}}$ and $\mathcal{C}_U =_C \bigwedge_{i=1}^{\infty} \mathcal{C}_{U_{y_i}}$ having the property that for each non-empty subset K of X that is $U\bar{\delta}K$ and $V\bar{\delta}K$. But $\mathcal{C}_A \preceq_C \mathcal{C}_V$ and $x_B \in \mathcal{C}_U$, thus $(X, \delta, \mathfrak{S}_C)$ is \mathcal{C} – regular space.

- b) Let $(X, \delta, \mathfrak{S}_C)$ be a \mathcal{C} –lindelof \mathfrak{S}_{C_2} – space and let $\mathcal{C}_H, \mathcal{C}_K$ are disjoint \mathcal{C} –closed sets, so by part (a) that $(X, \delta, \mathfrak{S}_C)$ is \mathcal{C} – regular, then for each $x_B \in \mathcal{C}_H$ there exists two disjoint center \mathcal{C} – open sets \mathcal{C}_{U_x} and \mathcal{C}_{V_x} such that $B\delta U_x$ and $\mathcal{C}_K \preceq_C \mathcal{C}_{V_x}$, Therefore the collection of \mathcal{C} – open set $\{\mathcal{C}_{U_x}, \text{for each } x_B \in \mathcal{C}_H\}$ is \mathcal{C} – cover of \mathcal{C}_H but by theorem 4.1.3 \mathcal{C}_H is \mathcal{C} –lindelof then there exists countable center point $x_{i_{B_i}}$ of \mathcal{C}_H such that

$\mathcal{C}_H \leq_c \bigvee_{i=1}^{\infty} \mathcal{C}_{U_{x_i}}$. Put $\mathcal{C}_U =_c \bigvee_{i=1}^{\infty} \mathcal{C}_{U_{x_i}}$ and $\mathcal{C}_V =_c \bigwedge_{i=1}^{\infty} \mathcal{C}_{V_{x_i}}$. The \mathcal{C} – open sets \mathcal{C}_U and \mathcal{C}_V having the property that for each a non-empty subset M of X satisfy that $U\bar{\delta}M$ and $V\bar{\delta}M$. Hence $(X, \delta, \mathfrak{S}_c)$ is \mathcal{C} – normal space.

Proposition 4.1.7:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism function and $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ be a shrink \mathcal{C} – homeomorphism function, then If \mathcal{C}_A is \mathcal{C} – lindelof set for each \mathcal{C}_A subset of \mathcal{C}_X , then $\mathcal{C}_{f(A)}$ is \mathcal{C} – lindelof set subset of \mathcal{C}_Y .

Proof:

Let \mathcal{C}_A is \mathcal{C} – lindelof set in X and let $\{\mathcal{C}_{U_\alpha}, \alpha \in \Delta\}$ be \mathcal{C} – open cover of $\mathcal{C}_{f(A)}$, then $\mathcal{C}_{f(A)} =_c \bigvee_{\alpha \in \Delta} \mathcal{C}_{U_\alpha}$, this mean, for each $\emptyset \neq K \subseteq Y, K\delta_Y f(A)$ iff there exists $\alpha_o \in \Delta, K\delta_Y U_{\alpha_o}$, since f_c^{-1} is shrink \mathcal{C} – continuous function, then $f^{-1}(K)\delta_X f^{-1}(f(A))$ iff $f^{-1}(K)\delta_X f^{-1}(U_{\alpha_o})$, since f is onto function, then $f^{-1}(K)\delta_X A$, implies $f^{-1}(K)\delta_X f^{-1}(U_{\alpha_o})$ for some $\alpha_o \in \Delta$, since f_c is shrink \mathcal{C} – continuous function, then $\{\mathcal{C}_{f^{-1}(U_\alpha)}, \alpha \in \Delta\}$ is \mathcal{C} – open cover of \mathcal{C}_A . But \mathcal{C}_A is \mathcal{C} – lindelof set, therefore $\mathcal{C}_A =_c \bigvee_{i=1}^{\infty} \{\mathcal{C}_{f^{-1}(U_i)}, i \in \Delta\}$, then $\forall H \subseteq X, A\delta_X H$ iff $H\delta_X f^{-1}(U_i)$ such that $i = 1, 2, \dots$, since f is δ – continuous function, then $f(A)\delta_Y f(H)$ iff $f(H)\delta_Y f(f^{-1}(U_i))$, but $f(f^{-1}(U_i)) \subseteq U_i$, then $f(A)\delta_Y f(H)$ iff $f(H)\delta_Y U_i$ for some $i = 1, 2, \dots$. Thus $\mathcal{C}_{f(A)} =_c \bigvee_{i=1}^{\infty} \{\mathcal{C}_{U_i}, i = 1, 2, \dots\}$. Hence $\mathcal{C}_{f(A)}$ is \mathcal{C} – lindelof set in Y .

Proposition 4.1.8:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a reduction δ – homeomorphism function and $f_c: (X, \delta_X, \mathfrak{C}_X) \rightarrow (Y, \delta_Y, \mathfrak{C}_Y)$ be a reduction \mathcal{C} – homeomorphism function, if \mathcal{C}_A is \mathcal{C} – lindelof set for each \mathcal{C}_A subset of \mathcal{C}_X , then $f_c(\mathcal{C}_A)$ is \mathcal{C} – lindelof set subset of \mathcal{C}_Y .

Proof:

Let \mathcal{C}_A is \mathcal{C} – lindelof set in X and let $\{\mathcal{C}_{U_\alpha}, \alpha \in \Delta\}$ be \mathcal{C} –open cover of $f_c(\mathcal{C}_A)$, then $f_c(\mathcal{C}_A) = \bigcup_{\mathcal{C}} \{\mathcal{C}_{U_\alpha}, \alpha \in \Delta\}$, this mean, for each $\emptyset \neq K \subseteq Y$, $K \delta_Y f(A)$ iff there exists $\alpha_0 \in \Delta$, $K \delta_Y U_{\alpha_0}$, since f_c^{-1} is reduction \mathcal{C} –continuous function, then $f^{-1}(K) \delta_X f^{-1}(f(A))$ iff $f^{-1}(K) \delta_X f^{-1}(U_{\alpha_0})$, since f is onto function, then $f^{-1}(K) \delta_X A$, implies $f^{-1}(K) \delta_X f^{-1}(U_{\alpha_0})$ for some $\alpha_0 \in \Delta$, since f_c is reduction \mathcal{C} –continuous function, then $\{f_c(\mathcal{C}_{U_\alpha}), \alpha \in \Delta\}$ is \mathcal{C} –open cover of \mathcal{C}_A . But \mathcal{C}_A is \mathcal{C} – lindelof set, therefore $\mathcal{C}_A = \bigcup_{i=1}^{\infty} \{f_c^{-1}(\mathcal{C}_{U_i}), i \in \Delta\} \subseteq \Delta$, then $\forall H \subseteq X$, $A \delta_X H$ iff $H \delta_X f^{-1}(U_i)$ such that $i = 1, 2, \dots$, since f is δ –continuous function, then $f(A) \delta_Y f(H)$ iff $f(H) \delta_Y f(f^{-1}(U_i))$, but $f(f^{-1}(U_i)) \subseteq U_i$, then $f(A) \delta_Y f(H)$ iff $f(H) \delta_Y U_i$ for some $i = 1, 2, \dots$. Thus $f(\mathcal{C}_A) = \bigcup_{i=1}^{\infty} \{\mathcal{C}_{U_i}, i = 1, 2, \dots\}$. Hence $f_c(\mathcal{C}_A)$ is \mathcal{C} – lindelof set in Y .

4.2 \mathcal{LC} - Lindelof:

In this section we introduce and study notion of \mathcal{LC} - lindelof space in \mathcal{C} – topological space, and its relationship to the axioms of separation were studied.

Definition 4.2.1:

A \mathcal{C} -topological space $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is said to be \mathcal{LC} - lindelof space if and only if every center point x_A in X has at least one \mathcal{C} – neighborhood \mathcal{C}_N whose $cl_{\mathcal{C}}(\mathcal{C}_N)$ is \mathcal{C} –lindelof.

Theorem 4.2.2:

Every \mathcal{C} –lindelof space $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is \mathcal{LC} - lindelof space.

Proof:

Let $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ be \mathcal{C} –lindelof space and x_A any center point x_A in X , there exist \mathcal{C}_X such that $A\delta_X X$, it is \mathcal{C} – neighborhood of x_A where $cl_{\mathcal{C}}(\mathcal{C}_X) = \mathcal{C}_X$ is \mathcal{C} –lindelof. Thus $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is \mathcal{LC} - lindelof space.

The following example shows the fact that the converse of the above theorem is not true.

Example 4.2.3

Let $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ be any \mathcal{C} –topological space such that X is uncountable set, then $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is not \mathcal{C} –lindelof space. But its \mathcal{LC} - lindelof space. For let x_A be any \mathcal{C} –point in X , then $\{x_A\}$ is \mathcal{C} – neighborhood of x_A whose $cl_{\mathcal{C}}(\{x_A\}) = \{x_A\}$. Also $\{x_A\}$ is a \mathcal{C} –lindelof for all x_A \mathcal{C} –point in X , being countable.

Theorem 4.2.4:

Every \mathcal{LC} - compact space $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is \mathcal{LC} - lindelof space.

Theorem 4.2.5:

Every \mathcal{LC} - compact space $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} - lindelof space.

Theorem 4.2.6:

Every \mathcal{C} - closed subspace of \mathcal{LC} - lindelof space is \mathcal{LC} - lindelof.

Proof:

Let $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} - lindelof space, \mathcal{C}_F be \mathcal{C} - closed subspace of \mathcal{LC} - lindelof space $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ and x_A \mathcal{C} - point in X such that $A\delta_X F$, then $A\delta_X X$. Since $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} - lindelof space, there exists \mathcal{C}_V \mathcal{C} - neighborhood of x_A such that $cl_{\mathcal{C}}(\mathcal{C}_V)$ is \mathcal{C} - lindelof. But the $\mathcal{C}_N \wedge_{\mathcal{C}} \mathcal{C}_F$ is \mathcal{C} - neighborhood of x_A in \mathcal{C}_F . Therefore $cl_{\mathcal{C}}(\mathcal{C}_N \wedge_{\mathcal{C}} \mathcal{C}_F)$ is \mathcal{C} - closed subset of \mathcal{C} - lindelof set $cl_{\mathcal{C}}(\mathcal{C}_N)$ and is thus it self \mathcal{C} - lindelof. This prove that \mathcal{C}_F is \mathcal{LC} - lindelof.

Theorem 4.2.7:

A \mathcal{C} - regular space is \mathcal{LC} - lindelof space if and only if it has a center basis of \mathcal{C} - closed \mathcal{C} - lindelof neighbourhoods.

Proof:

Let $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{C} - regular \mathcal{LC} - lindelof space. For every $A \subseteq X$ such that $A\delta_X X$. Choose a \mathcal{C} - nbd \mathcal{C}_U such that $A\delta_X U$ and $cl_{\mathcal{C}}(\mathcal{C}_U) \ll_{\mathcal{C}} \mathcal{C}_U$. And let \mathcal{C}_V a \mathcal{C} - lindelof such that $A\delta_X V$. By theorem (4.2.6), $(\mathcal{C}_U \wedge_{\mathcal{C}} \mathcal{C}_V)$ is \mathcal{C} - lindelof. Thus $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ has a center basis of \mathcal{C} - closed \mathcal{C} - lindelof neighbourhoods. The proof of the converse is easy and omitted.

Corollary 4.2.8:

- i. Every \mathcal{C} - open subspace of \mathcal{C} - regular \mathcal{LC} - lindelof space is \mathcal{LC} - lindelof.

- ii. Every \mathcal{C} – open subspace of \mathcal{C} – regular \mathcal{C} - lindelof space is \mathcal{LC} – lindelof.

Proposition 4.2.9:

Let $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ be a $\mathfrak{T}_{\mathcal{C}2-}$ space . If \mathcal{C}_A and \mathcal{C}_B are \mathcal{LC} – lindelof spaces, then $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ is \mathcal{LC} – lindelof space.

Proof:

Let $x_H \in \mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ such that $K\delta_X A$ and $K\delta_X B$ for all $\emptyset \neq k \subseteq X$. And let \mathcal{C}_{N_1} and \mathcal{C}_{N_2} are \mathcal{C} –lindelof such that $N_1\delta_X H$ in \mathcal{C}_A and $N_2\delta_X H$ in \mathcal{C}_B , by theorem (4.1.5), then $\mathcal{C}_{N_1} \wedge_{\mathcal{C}} \mathcal{C}_{N_2}$ is \mathcal{C} – lindelof and \mathcal{C} – neighborhood of x_H . Thus $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ is \mathcal{LC} – lindelof space.

Proposition 4.2.10:

A $\mathfrak{T}_{\mathcal{C}2-}$ space $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – lindelof if and only if each of its \mathcal{C} – points is an \mathcal{C} –interior point of some \mathcal{C} – lindelof subspace of $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$.

Proof:

If $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – lindelof, then for all x_B in X there exists \mathcal{C}_M such that $B\delta_X M$ and $cl_{\mathcal{C}}(\mathcal{C}_M)$ is \mathcal{C} – lindelof, it follows that $cl_{\mathcal{C}}(\mathcal{C}_M)$ is a \mathcal{C} – lindelof (\mathcal{C} – neighborhood) of x_B is an \mathcal{C} –interior point. Thus $\forall x_B$ is an \mathcal{C} –interior point of some \mathcal{C} – lindelof.

Conversely, let x_B is \mathcal{C} – point in X be arbitrary. By hypothesis, there exists an \mathcal{C} – lindelof \mathcal{C}_U such that $B\delta_X U$, then \mathcal{C}_U is a \mathcal{C} – neighborhood of x_B . Since $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is $\mathfrak{T}_{\mathcal{C}2-}$ space, implies \mathcal{C}_U is \mathcal{C} –closed, so that $cl_{\mathcal{C}}(\mathcal{C}_U) = \mathcal{C}_U$, thus x_B has a \mathcal{C} – neighborhood whose $cl_{\mathcal{C}}(\mathcal{C}_U)$ is \mathcal{C} – lindelof. And so $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – lindelof.

Proposition 4.2.11:

Every \mathcal{LC} – lindelof $\mathfrak{T}_{\mathcal{C}2-}$ space is \mathcal{C} – regular space.

Proof:

Let $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – lindelof $\mathfrak{T}_{\mathcal{C}2-}$ space, x_A be any \mathcal{C} – point in X and \mathcal{C}_F any \mathcal{C} – closed set in X such that $A\overline{\delta_X}F$ for each $A \subseteq X$. Since $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is $\mathfrak{T}_{\mathcal{C}2-}$ space, then \mathcal{C}_F is \mathcal{C} – lindelof and for each $y_B \in \mathcal{C}_F$, then there exists \mathcal{C}_G and \mathcal{C}_{H_y} \mathcal{C} – open sets such that $A\delta_X G$, $F\delta_X H_y$ and $\mathcal{C}_G \wedge_{\mathcal{C}} \mathcal{C}_{H_y} = \mathcal{C}_\emptyset$, the family of all $\{\mathcal{C}_{H_{y_i}}, y_B \in \mathcal{C}_F\}$ \mathcal{C} – open cover of \mathcal{C} – compact \mathcal{C}_{H_y} and hence a countable subfamily $\mathcal{C}_{H_{y_1}}, \mathcal{C}_{H_{y_2}}, \dots$ such that $\mathcal{C}_{H_y} \ll_{\mathcal{C}} \bigcup_{i=1}^{\infty} \mathcal{C}_{H_{y_i}}$. Hence $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{C} – regular space.

Proposition 4.2.12:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – continuous onto function and $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{T}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{T}_{\mathcal{C}Y})$ be a shrink \mathcal{C} – continuous function, then If $(X, \delta_X, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – lindelof space, then $(Y, \delta_Y, \mathfrak{T}_{\mathcal{C}Y})$ is \mathcal{LC} – lindelof space.

Proof:

Let $y_B \in \mathcal{C}_Y$ such that for each $B \subseteq Y$, $\forall y \in Y$, $\{y\}\delta_Y B$, $B\delta_Y Y$. Since f is onto, then there exists $A \subseteq X$ such that $\{x\}\delta_X A$ and $f_{\mathcal{C}}(x_A) = y_B$. But $(X, \delta_X, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – lindelof space, then there exists \mathcal{C}_N nbd for x_A such that for each $A \subseteq X$ such that $A\delta_X N$ and $cl_{\mathcal{C}}(\mathcal{C}_N)$ is \mathcal{C} – lindelof set. Also $f_{\mathcal{C}}$ is shrink \mathcal{C} – continuous function and by proposition (4.1.6), we get for each $B \subseteq Y$, $\{y\}\delta_Y B$, then $B\delta_Y f(N)$. Hence $(Y, \delta_Y, \mathfrak{T}_{\mathcal{C}Y})$ is \mathcal{LC} – lindelof space

4.3 \mathcal{AC} - Lindelof:

In this section we introduce and study notion of \mathcal{AC} – lindelof space in \mathcal{C} – topological space, and its relationship to the axioms of separation were studied.

Definition 4.3.1:

Let $(X, \delta, \mathfrak{S}_C)$ be a \mathcal{C} -topological space, the universal set \mathcal{C}_X (a non-empty center set \mathcal{C}_A) is called \mathcal{AC} – lindelof space if each \mathcal{C} – open cover $\{\mathcal{C}_{G_\alpha}, \alpha \in \Gamma\}$ of \mathcal{C}_X (\mathcal{C}_A) has countable Δ subset of Γ such that $\forall H \subseteq X$, there exist $\alpha_0 \in \Delta, cl_C(G_{\alpha_0}) \delta H$.

Proposition 4.3.2:

Every \mathcal{C} – lindelof space is an \mathcal{AC} – lindelof space.

Proof:

Let $(X, \delta, \mathfrak{S}_C)$ be \mathcal{C} – lindelof space and $\mathcal{H} = \{\mathcal{C}_{G_\lambda}, \lambda \in \Lambda\}$ be a \mathcal{C} – open cover of \mathcal{C}_X . Then there exists countable I subset of Λ such that $\forall H \subseteq X$, there exist $\lambda_0 \in I, G_{\lambda_0} \delta H$. Now since $\mathcal{C}_{G_\lambda} \preceq_C cl_C(\mathcal{C}_{G_\lambda})$. We have for each $\lambda \in \Lambda, H \delta G_\lambda$, then $H \delta cl(G_\lambda)$. Therefore there exist $\lambda_0 \in I, cl(A_{\lambda_0}) \delta H$. Hence $(X, \delta, \mathfrak{S}_C)$ is \mathcal{AC} – lindelof space.

Remark 4.3.3:

Every \mathcal{C} – lindelof set is \mathcal{AC} – lindelof set.

Proposition 4.3.4:

Let $(X, \delta, \mathfrak{S}_C)$ be a \mathcal{C} – Regular space, then $(X, \delta, \mathfrak{S}_C)$ is \mathcal{C} – lindelof space if and only if is \mathcal{AC} – lindelof space.

Proof:

If $(X, \delta, \mathfrak{S}_C)$ is \mathcal{C} – lindelof space, then by Proposition (4.3.2) $(X, \delta, \mathfrak{S}_C)$ is \mathcal{AC} – lindelof space. Conversely, let $\mathcal{H} = \{C_{U_\lambda}, \lambda \in \Lambda\}$ be a \mathcal{C} – open cover of \mathcal{C}_X . For each $x \in X$, $A \subseteq X$ and $\{x\} \delta A$, there exists λ_x such that $A \delta U_{\lambda_x}$, but $(X, \delta, \mathfrak{S}_C)$ is \mathcal{C} – Regular space, then there exists \mathcal{C} – open set $C_{H_{\lambda_x}}$ such that for each $x \in X$, $H_{\lambda_x} \subseteq X, \{x\} \delta H_{\lambda_x}$ and $\forall K \subseteq X, K \delta cl(H_{\lambda_x})$, then $K \delta U_{\lambda_x}$. Now, $\mathcal{F} = \{C_{H_{\lambda_x}}, x \in X\}$ is a \mathcal{C} – open cover of \mathcal{C}_X . But $(X, \delta, \mathfrak{S}_C)$ is \mathcal{AC} – lindelof space, then there exists countable I subset of X such that for each $x \in X$, $\forall K \subseteq X, K \delta X$ and $K \delta cl(H_{\lambda_x})$ for some $x \in I$, therefore $K \delta cl_\tau(H_{\lambda_x})$ for some $x \in I$. Hence $(X, \delta, \mathfrak{S}_C)$ is \mathcal{C} – lindelof space.

Proposition 4.3.5:

Let $(X, \delta, \mathfrak{S}_C)$ be a \mathcal{C} – topological space and $\mathcal{C}_A, \mathcal{C}_B$ are center sets in X such that \mathcal{C}_A is \mathcal{AC} – lindelof set and $\mathcal{C}_A \leq_c \mathcal{C}_B \leq_c cl_C(\mathcal{C}_A)$, then \mathcal{C}_B is \mathcal{AC} – lindelofset.

Proof:

Let $\mathcal{H} = \{C_{U_\lambda}, \lambda \in \Lambda\}$ be a \mathcal{C} – open cover of \mathcal{C}_B ,

\Rightarrow For each $y \in X$, $\{y\} \delta B$, for some $\lambda_o \in \Lambda$, $\forall \emptyset \neq K \subseteq X, K \delta B$, then $K \delta U_{\lambda_o}$. But $\forall K \delta A$, then $K \delta B$,

\Rightarrow for some $\lambda_o \in \Lambda$, $\forall \emptyset \neq K \subseteq X, K \delta A$, then $K \delta U_{\lambda_o}$.

And \mathcal{C}_A is \mathcal{AC} – lindelof set,

\Rightarrow there exists countable I subset of Λ such that $\forall \emptyset \neq K \subseteq X, K \delta A$, then for some $\lambda_o \in I$, $K \delta cl(U_{\lambda_o})$.

$\Rightarrow \forall \emptyset \neq K \subseteq X, K \delta cl(A)$ iff for some $\lambda_o \in I$, $K \delta cl(U_{\lambda_o})$.

But $\mathcal{C}_B \preceq_c cl_c(\mathcal{C}_A)$, then some $\lambda_o \in I$, $K\delta B$, then $K\delta cl(U_{\lambda_o})$.

Hence \mathcal{C}_B is \mathcal{AC} – lindelof set.

Corollary 4.3.6:

Let $(X, \delta, \mathfrak{S}_c)$ be a \mathcal{C} –topological space and \mathcal{C}_A be a \mathcal{AC} – lindelof set in X , then $cl_c(\mathcal{C}_A)$ is also \mathcal{AC} – lindelof set.

Proposition 4.3.7:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism function and $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ be a shrink \mathcal{C} – continuous function, then If \mathcal{C}_A is \mathcal{AC} – lindelof set for each \mathcal{C}_A subset of \mathcal{C}_X , then $\mathcal{C}_{f(A)}$ is \mathcal{AC} – lindelof set subset of \mathcal{C}_Y .

Proof:

Suppose \mathcal{C}_A is \mathcal{AC} – lindelof set in X and $\mathcal{H} = \{\mathcal{C}_{U_\lambda}, \lambda \in \Lambda\}$ be a \mathcal{C} – open cover of $\mathcal{C}_{f(A)}$ such that for each $B \subseteq X$, $\forall x \in X$ such that $\{x\}\delta B$ and $\exists \lambda_o \in \Lambda, B\delta A_{\lambda_o}$. By proposition (2.1.1.8), then $\mathcal{C}_A \preceq_c f^{-1}(\mathcal{C}_{f(A)}) \preceq_c f^{-1}(\bigvee_\lambda \{\mathcal{C}_{U_\lambda}\}) =_c \bigvee_\lambda \{\mathcal{C}_{f^{-1}(U_\lambda)}\}$. Therefore $\Gamma = \{\mathcal{C}_{f^{-1}(U_\lambda)}, \lambda \in \Lambda\}$ is \mathcal{C} – open cover of \mathcal{C}_A such that for each $K \subseteq X$, $\forall x \in A$ such that $\{x\}\delta K$ and $\exists \lambda_o \in \Lambda$ and $K\delta f^{-1}(U_{\lambda_o})$. But \mathcal{C}_A is \mathcal{AC} – lindelof set, then Γ has a countable subfamily I such that for some $\lambda_o \in I$, $\forall K \subseteq X$, $\forall x \in A$ such that $\{x\}\delta K$ and $\exists \lambda_o \in I, K\delta f^{-1}(U_{\lambda_o})$. Whose center closure \mathcal{C}_A and by proposition (2.1.1.8), that is to say:

$$\mathcal{C}_A \preceq_c cl_c\left(\bigvee_{i=1}^{\infty} \mathcal{C}_{f^{-1}(U_{\lambda_i})}\right) =_c cl_c\left(f^{-1}\left(\bigvee_{i=1}^{\infty} \mathcal{C}_{U_{\lambda_i}}\right)\right).$$

$$\begin{aligned} &\leq_c f^{-1}(cl_c(\bigvee_{i=1}^{\infty} \mathcal{C}_{U_{\lambda_i}})). \text{ Accordingly and by Proposition (2.1.1.6), then} \\ \mathcal{C}_{f(A)} &\leq_c f(f^{-1}(cl_c(\bigvee_{i=1}^{\infty} \mathcal{C}_{U_{\lambda_i}}))). \\ &\leq_c cl_c\left(\bigvee_{i=1}^{\infty} \mathcal{C}_{U_{\lambda_i}}\right) =_c \bigvee_{i=1}^{\infty} cl_c(\mathcal{C}_{U_{\lambda_i}}). \end{aligned}$$

Thus $\mathcal{C}_{f(A)}$ is \mathcal{AC} -lindelof set.

Proposition 4.3.8:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a reduction δ – homeomorphism function and $f_c: (X, \delta_X, \mathfrak{I}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{I}_{\mathcal{C}Y})$ be a reduction \mathcal{C} – homeomorphism function, if \mathcal{C}_A is \mathcal{AC} – lindelof set for each \mathcal{C}_A subset of \mathcal{C}_X , then $f_c(\mathcal{C}_A)$ is \mathcal{AC} – lindelof set subset of \mathcal{C}_Y .

Proof:

We prove the same way to prove Proposition (3.3.7).Type equation here.

4.4 \mathcal{LC} - Closedness:

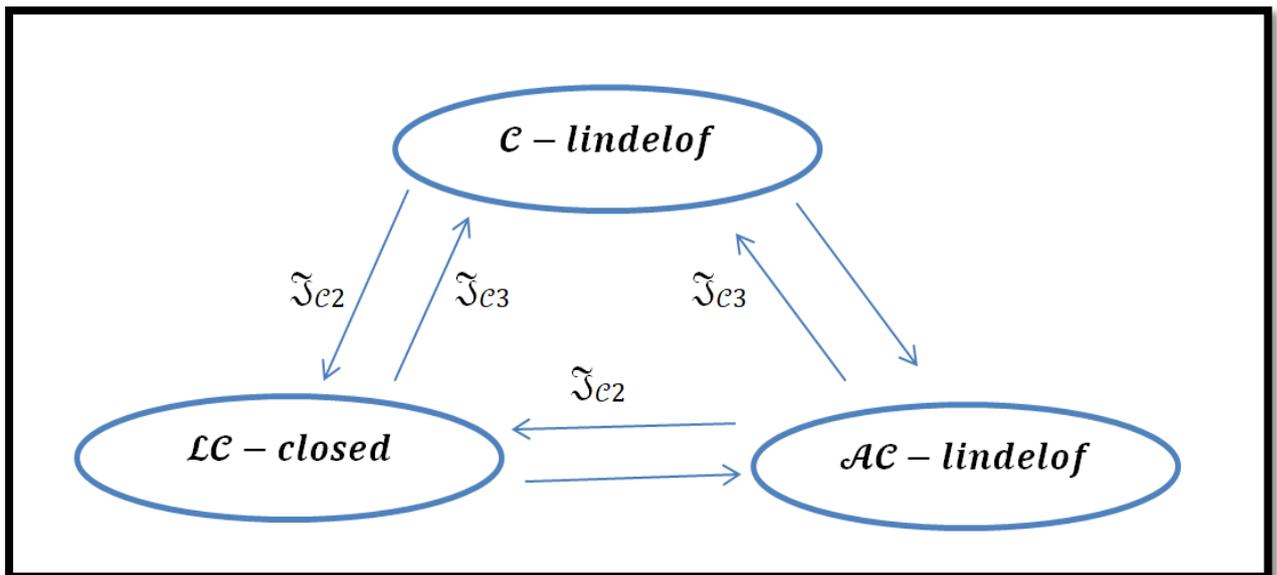
In this section, we give several characterization of \mathcal{C} – lindelof in center topological space, nearly \mathcal{LC} – closed space. Furthermore, the relationship between this concept and the separation axioms in center topological space had been studied.

Definition 4.4.1:

Let $(X, \delta, \mathfrak{I}_{\mathcal{C}})$ be $\mathfrak{I}_{\mathcal{C}2}$ – space, the universal set \mathcal{C}_X (a non-empty center set \mathcal{C}_A) is called \mathcal{LC} – closed space if each \mathcal{C} – open cover $\{\mathcal{C}_{U_{\alpha}}, \alpha \in \Gamma\}$ of \mathcal{C}_X (\mathcal{C}_A) has countable Δ subset of Γ such that $\forall H \subseteq X$, there exist $\alpha_0 \in \Delta, cl_c(U_{\alpha_0}) \delta H$.

Remarks 4.4.2:

- i. Every \mathcal{LC} – closed space is \mathcal{AC} – lindelof space.
- ii. Every \mathcal{AC} – lindelof space and $\mathfrak{T}_{\mathcal{C}2}$ – space is \mathcal{LC} – closed space.
- iii. Every \mathcal{LC} – closed space and $\mathfrak{T}_{\mathcal{C}3}$ – space is \mathcal{C} – lindelof space.
- iv. See the following diagram



The following example shows that the conversely of the above remark (i) is not true.

Example 4.4.3:

Let $(X, \delta, \mathfrak{T}_{\mathcal{C}})$ be a \mathcal{C} -topological space,

Where $X = \{a, b\}$, $A\delta B$ iff $A \cap B \neq \emptyset$

And $\mathfrak{T}_{\mathcal{C}} = \{ \mathcal{C}_{\emptyset}, \{ \langle X, \{a\} \rangle \}, \{ \langle X, \{b\} \rangle \}, \{ \langle X, \{a\} \rangle, \langle X, \{b\} \rangle \}, \{ \langle X, \{a\} \rangle, \langle X, X \rangle \},$

$\{ \langle X, \{b\} \rangle, \langle X, X \rangle \}, \{ \langle X, X \rangle \}, \{ \{ \langle a, \{a\} \rangle, \langle a, X \rangle \}, \{ \langle b, \{b\} \rangle, \langle b, X \rangle \}, \mathcal{C}_X \}$

Then $(X, \delta, \mathfrak{T}_{\mathcal{C}})$ is \mathcal{AC} – lindelof space.

But is not $\mathfrak{S}_{\mathcal{C}2}$ – Space. Hence it is not \mathcal{LC} – closed space.

Theorem 4.4.4:

Every \mathcal{C} – lindelof space and $\mathfrak{S}_{\mathcal{C}2}$ – space is \mathcal{LC} – closed space.

Proof:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be \mathcal{C} – lindelof space and $\mathfrak{S}_{\mathcal{C}2}$ – space and $\Gamma = \{C_{U_{\alpha}}, \alpha \in \Gamma\}$ be \mathcal{C} – open cover of $(X, \delta, \mathfrak{S}_{\mathcal{C}})$, then there exists countable I subset of Γ such that $\forall x \in X, \forall H \subseteq X, \{x\} \delta H$, then for some $\alpha_0 \in I, H \delta U_{\alpha_0}$, now since $\forall H \subseteq X, H \delta U_{\alpha_i}$, then $H \delta cl(U_{\alpha_i})$. Hence $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{LC} – closed space.

Theorem 4.4.5:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be \mathcal{C} – Regular space and $\mathfrak{S}_{\mathcal{C}2}$ – space, then $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{LC} – closed space if and only if it is \mathcal{C} – lindelof space.

Proof:

Since $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{LC} – closed space, then $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{AC} – lindelof space, but its \mathcal{C} – Regular space, then by Proposition (4.3.4) we have $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} – lindelof space.

Conversely, since $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} – lindelof space and \mathcal{C} – Regular space, and then by Proposition (4.3.4) we have $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{LC} – closed space.

Proposition 4.4.6:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism function and $f_{\mathcal{C}}: (X, \delta_X, \mathfrak{S}_{\mathcal{C}X}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ be a shrink \mathcal{C} – continuous function. If $(X, \delta_X, \mathfrak{S}_{\mathcal{C}X})$ is \mathcal{LC} – closed space, then $(Y, \delta_Y, \mathfrak{S}_{\mathcal{C}Y})$ is \mathcal{LC} – closed space.

Proof:

Direct from define of \mathcal{LC} – closed space and by Proposition (4.3.7).

CHAPTER

FIVE

**Center Countable
Compactness**

Chapter Five: Center Countable Compactness

In this chapter, we give several characterizations of center countable compactness, where the concept of \mathcal{C} – countable compact spaces within the central topological spaces was introduced and the topological properties of this space and its relationship to the axioms of separation were studied. Also in section two we introduce new concept, namely \mathcal{LC} - countable compact space and its relationship to the axioms of separation were studied. In section three and fore we introduce new concepts namely \mathcal{AC} – countable compact and \mathcal{HC} – countable closed, and we study their properties.

5.1 Center Countable Compactness

In this section, we introduce a new type of compactness in center topological space has been presented, that is called a \mathcal{C} – countable compact space. Furthermore, the relationship between this concept and the separation axioms in center topological space had been studied.

Definition 5.1.1:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} -topological space, the universal set \mathcal{C}_X (a non-empty center set \mathcal{C}_A) is called \mathcal{C} – countable compact space if each \mathcal{C} – countable open cover $\{\mathcal{C}_{U_{\alpha}}, \alpha \in \Gamma\}$ of \mathcal{C}_X (\mathcal{C}_A) has finite Δ subset of Γ such that $\forall H \subseteq X$, there exist $\alpha_0 \in \Delta, U_{\alpha_0} \delta H$.

Remarks 5.1.2:

- i. Every \mathcal{C} – compact space is \mathcal{C} – countable compact space.
- ii. A \mathcal{C} – countable compact space is \mathcal{C} – compact space if and only if it is \mathcal{C} – lindelof space.

Theorem 5.1.3 :

Each \mathcal{C} – closed subset of \mathcal{C} – countable compact topological space $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} – countable compact set.

Proof:

Let \mathcal{C}_A be any non-empty \mathcal{C} – closed set in \mathcal{C} – countable compact space $(X, \delta, \mathfrak{S}_{\mathcal{C}})$. And $\mathcal{H} = \{\mathcal{C}_{A_\lambda}, \lambda \in \Lambda\}$ be \mathcal{C} – countable open cover to \mathcal{C}_A , so $\mathcal{C}_X^{(A)}$ be \mathcal{C} – open set. Thus for each $B \subseteq X, \forall x \in X$ such that $\{x\} \delta B$ and $\exists \lambda_0 \in \Lambda, B \delta A_{\lambda_0}$, from the concept of \mathcal{C} – countable compact space of \mathcal{C}_X , then there exists finite subset I of Λ say $\{\mathcal{C}_{A_1}, \mathcal{C}_{A_2}, \dots, \mathcal{C}_{A_n}, \mathcal{C}_X^{(A)}\}$ such that $\forall H \subseteq X$, there exist $\lambda_{i^0} \in I, A_{\lambda_{i^0}} \delta H$, Hence \mathcal{C}_A is \mathcal{C} – countable compact set.

Theorem 5.1.4:

Each \mathcal{C} – countable compact subset of $\mathfrak{S}_{\mathcal{C}2-}$ space $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} – closed set.

Proof:

Let \mathcal{C}_A be any non-empty \mathcal{C} – countable compact set, so by Proposition (1.2.11 part 4) we get that $\mathcal{C}_A = {}_{\mathcal{C}}\bigvee_{x_{\alpha B_\lambda} \in \mathcal{C}_A} x_{\alpha B_\lambda}$, then for any $\alpha_0, \lambda_0, x_{\alpha_0} \in X$ such that $\{x_{\alpha_0}\} \delta B_{\lambda_0}$ and $B_{\lambda_0} \delta A$, and for the fact that $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_X^{(A)} = {}_{\mathcal{C}}\mathcal{C}_\emptyset$ we get for any $y \in X, \{y\} \delta B, B \bar{\delta} A$ and for each α and $\lambda, y_B \neq x_{\alpha B_\lambda}$. But $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is $\mathfrak{S}_{\mathcal{C}2-}$ space, so there exists two disjoint \mathcal{C} – open set \mathcal{C}_{U_λ} and \mathcal{C}_{V_λ} containing $x_{\alpha B_\lambda}$ and y_B respectively, thus $\mathcal{C}_A = {}_{\mathcal{C}}\bigvee_{x_{\alpha B_\lambda} \in \mathcal{C}_A} x_{\alpha B_\lambda} \preceq_{\mathcal{C}} \bigvee_{\mathcal{C}} \{\mathcal{C}_{U_\lambda}, x_{\alpha B_\lambda} \in \mathcal{C}_{U_\lambda}\}$, therefore by \mathcal{C} – countable compactness then exists finite index $\alpha_1, \alpha_2, \dots, \alpha_n, \lambda_1, \lambda_2, \dots, \lambda_n$ such that $\mathcal{C}_A \preceq_{\mathcal{C}} \bigvee_{i=1}^n \{\mathcal{C}_{U_{\lambda_i}}, x_{\alpha_i B_{\lambda_i}} \in \mathcal{C}_{U_{\lambda_i}}\}$. And

$\mathcal{C}_V = \bigcap_{i=1}^n \mathcal{C}_{V_{\lambda_i}}$ is \mathcal{C} – open center set containing y_B such that $\forall y \in X, \forall B \subseteq X, \{y\} \delta B$ and $B \delta V$, so $B \bar{\delta} A$ and $V \bar{\delta} A$. Hence \mathcal{C}_A is \mathcal{C} – closed center set.

Theorem 5.1.5:

If $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} – topological space then:

- a) If \mathcal{C}_A and \mathcal{C}_B are \mathcal{C} – countable compact, then $\mathcal{C}_{A \cup B}$ is \mathcal{C} – countable compact.
- b) If \mathcal{C}_A is \mathcal{C} – closed and \mathcal{C}_B is \mathcal{C} – countable compact in $\mathfrak{S}_{\mathcal{C}2-}$ space X , then $\mathcal{C}_A \wedge \mathcal{C}_B$ is \mathcal{C} – countable compact.
- c) If \mathcal{C}_A and \mathcal{C}_B are \mathcal{C} – countable compact sets in $\mathfrak{S}_{\mathcal{C}2-}$ space X , then $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ is \mathcal{C} – countable compact.

Proof:

- a) Let $\mathcal{H} = \{\mathcal{C}_{A_{\lambda}}, \lambda \in \Lambda\}$ be any \mathcal{C} – countable open cover of $\mathcal{C}_{A \cup B}$, but $A, B \subseteq A \cup B$, thus the family \mathcal{H} be \mathcal{C} – countable open cover of \mathcal{C}_A and \mathcal{C}_B . But \mathcal{C}_A and \mathcal{C}_B are \mathcal{C} – countable compact, so there exists finite subsets Λ_1 and Λ_2 of Λ such that $\mathcal{C}_A \preceq_{\mathcal{C}} \bigcup_{\lambda \in \Lambda_1} \mathcal{C}_{A_{\lambda}}$ and $\mathcal{C}_B \preceq_{\mathcal{C}} \bigcup_{\alpha \in \Lambda_2} \mathcal{C}_{A_{\alpha}}$, then we get that $\mathcal{C}_{A \cup B} \preceq_{\mathcal{C}} \left(\bigcup_{\lambda \in \Lambda_1} \mathcal{C}_{A_{\lambda}} \right) \bigcup_{\mathcal{C}} \left(\bigcup_{\alpha \in \Lambda_2} \mathcal{C}_{A_{\alpha}} \right) =_{\mathcal{C}} \bigcup_{\gamma \in (\Lambda_1 \cup \Lambda_2)} \mathcal{C}_{A_{\gamma}}$. But $\Lambda_1 \cup \Lambda_2$ is finite subset of Λ . Hence $\mathcal{C}_{A \cup B}$ is \mathcal{C} – countable compact center set.
- b) and c) are directly by theorem 5.1.3 and the fact the \mathcal{C} – intersection of two \mathcal{C} – closed center set is \mathcal{C} – closed center set.

Theorem 5.1.6:

If $(X, \delta, \mathfrak{I}_C)$ be a \mathcal{C} – topological space :

- a) Every \mathcal{C} – countable compact \mathfrak{I}_{C_2} – space is \mathcal{C} – regular.
- b) Every \mathcal{C} – countable compact \mathfrak{I}_{C_2} – space is \mathcal{C} – normal.

Proof:

- a) Let x_B be a center point such that $A\bar{\delta}B$, but $\mathcal{C}_X^{(A)}$ is \mathcal{C} – open and $x_B \in \mathcal{C}_X^{(A)}$. Then for each $y_H \in \mathcal{C}_A$ we get that $H\bar{\delta}B$. And by \mathfrak{I}_{C_2} – space we have two disjoint center sets \mathcal{C}_{U_y} and \mathcal{C}_{V_y} with property that $U_y\delta B$ and $V_y\delta H$, then the collection of center \mathcal{C} – open set $\xi = \{\mathcal{C}_{V_y}; \text{for each } y_B \in \mathcal{C}_A\}$ is countable \mathcal{C} – open cover. But \mathcal{C}_A is \mathcal{C} – closed of \mathcal{C} – countable compact and by theorem 5.1.3 we have that \mathcal{C}_A is \mathcal{C} – countable compact so, there exists y_1, y_2, \dots, y_n such that for each non empty K subset of X , $K\delta A$ and $K\delta V_{y_i}$, put $\mathcal{C}_V = \bigcup_{i=1}^n \mathcal{C}_{V_{y_i}}$ and $\mathcal{C}_U = \bigcup_{i=1}^n \mathcal{C}_{U_{y_i}}$ having the property that for each non-empty subset K of X that is $U\bar{\delta}K$ and $V\bar{\delta}K$. But $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_V$ and $x_B \in \mathcal{C}_U$, thus $(X, \delta, \mathfrak{I}_C)$ is \mathcal{C} – regular space.

- b) Let $(X, \delta, \mathfrak{I}_C)$ be a \mathcal{C} – countable compact \mathfrak{I}_{C_2} – space and let $\mathcal{C}_H, \mathcal{C}_K$ are disjoint \mathcal{C} – closed sets, so by part (a) that $(X, \delta, \mathfrak{I}_C)$ is \mathcal{C} – regular, then for each $x_B \in \mathcal{C}_H$ there exists two disjoint center \mathcal{C} – open sets \mathcal{C}_{U_x} and \mathcal{C}_{V_x} such that $B\delta U_x$ and $\mathcal{C}_K \preccurlyeq_{\mathcal{C}} \mathcal{C}_{V_x}$, Therefore the collection of \mathcal{C} – open set $\{\mathcal{C}_{U_x}, \text{for each } x_B \in \mathcal{C}_H\}$ is \mathcal{C} – cover of \mathcal{C}_H but by theorem 5.1.3 \mathcal{C}_H is \mathcal{C} – countable compact then there exists finite center point $x_{i_{B_i}}$ of \mathcal{C}_H such

that $\mathcal{C}_H \leq_c \bigvee_{i=1}^n \mathcal{C}_{U_{x_i}}$. Put $\mathcal{C}_U =_c \bigvee_{i=1}^n \mathcal{C}_{U_{x_i}}$ and $\mathcal{C}_V =_c \bigwedge_{i=1}^n \mathcal{C}_{V_{x_i}}$. The \mathcal{C} – open sets \mathcal{C}_U and \mathcal{C}_V having the property that for each a non-empty subset M of X satisfy that $U\bar{\delta}M$ and $V\bar{\delta}M$. Hence $(X, \delta, \mathfrak{S}_c)$ is \mathcal{C} – normal space.

Proposition 5.1.7:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism shrink function and $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ be a shrink \mathcal{C} – homeomorphism function, then If \mathcal{C}_A is \mathcal{C} – countable compact set for each \mathcal{C}_A subset of \mathcal{C}_X , then $\mathcal{C}_{f(A)}$ is \mathcal{C} – countable compact set subset of \mathcal{C}_Y .

Proof:

Let \mathcal{C}_A is \mathcal{C} – countable compact set in X and let $\{\mathcal{C}_{U_\alpha}, \alpha \in \Delta\}$ be countable \mathcal{C} – open cover of $\mathcal{C}_{f(A)}$, then $\mathcal{C}_{f(A)} =_c \bigvee_{\alpha \in \Delta} \mathcal{C}_{U_\alpha}$, this mean, for each $\emptyset \neq K \subseteq Y$, $K\delta_Y f(A)$ iff there exists $\alpha_o \in \Delta$, $K\delta_Y U_{\alpha_o}$, since f_c^{-1} is shrink \mathcal{C} – continuous function, then $f^{-1}(K)\delta_X f^{-1}(f(A))$ iff $f^{-1}(K)\delta_X f^{-1}(U_{\alpha_o})$, since f is onto function, then $f^{-1}(K)\delta_X A$, implies $f^{-1}(K)\delta_X f^{-1}(U_{\alpha_o})$ for some $\alpha_o \in \Delta$, since f_c is shrink \mathcal{C} – continuous function, then $\{\mathcal{C}_{f^{-1}(U_\alpha)}, \alpha \in \Delta\}$ is \mathcal{C} – open cover of \mathcal{C}_A . But \mathcal{C}_A is \mathcal{C} – countable compact set, therefore

$\mathcal{C}_A =_c \bigvee_{i=1}^n \{\mathcal{C}_{f^{-1}(U_i)}, i \in \Delta \subseteq \Delta\}$, then $\forall H \subseteq X$, $A\delta_X H$ iff $H\delta_X f^{-1}(U_i)$ such that

$i = 1, 2, \dots, n$, since f is δ – continuous function, then $f(A)\delta_Y f(H)$ iff $f(H)\delta_Y f(f^{-1}(U_i))$, but $f(f^{-1}(U_i)) \subseteq U_i$, then $f(A)\delta_Y f(H)$ iff $f(H)\delta_Y U_i$ for

some $i = 1, 2, \dots, n$. Thus $\mathcal{C}_{f(A)} =_c \bigvee_{i=1}^n \{\mathcal{C}_{U_i}, i = 1, 2, \dots, n\}$. Hence $\mathcal{C}_{f(A)}$ is

\mathcal{C} – countable compact set in Y .

Proposition 5.1.8:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a reduction δ – homeomorphism reduction function and $f_c: (X, \delta_X, \mathfrak{C}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{C}_{cY})$ be a reduction \mathcal{C} – homeomorphism function, if \mathcal{C}_A is \mathcal{C} – countable compact set for each \mathcal{C}_A subset of \mathcal{C}_X , then $f_c(\mathcal{C}_A)$ is \mathcal{C} – countable compact set subset of \mathcal{C}_Y .

Proof:

Let \mathcal{C}_A is \mathcal{C} – countable compact set in X and let $\{\mathcal{C}_{U_\alpha}, \alpha \in \Delta\}$ be countable \mathcal{C} – open cover of $f_c(\mathcal{C}_A)$, then $f_c(\mathcal{C}_A) =_c \bigvee_{\alpha \in \Delta} \mathcal{C}_{U_\alpha}$, this mean, for each $\emptyset \neq K \subseteq Y$, $K \delta_Y f(A)$ iff there exists $\alpha_0 \in \Delta$, $K \delta_Y U_{\alpha_0}$, since f_c^{-1} is reduction \mathcal{C} – continuous function, then $f^{-1}(K) \delta_X f^{-1}(f(A))$ iff $f^{-1}(K) \delta_X f^{-1}(U_{\alpha_0})$, since f is onto function, then $f^{-1}(K) \delta_X A$, implies $f^{-1}(K) \delta_X f^{-1}(U_{\alpha_0})$ for some $\alpha_0 \in \Delta$, since f_c is reduction \mathcal{C} – continuous function, then $\{f_c(\mathcal{C}_{U_\alpha}), \alpha \in \Delta\}$ is \mathcal{C} – open cover of \mathcal{C}_A . But \mathcal{C}_A is \mathcal{C} – countable compact set, therefore

$\mathcal{C}_A =_c \bigvee_{i=1}^n \{f_c^{-1}(\mathcal{C}_{U_i}), i \in \Delta \subseteq \Delta\}$, then $\forall H \subseteq X$, $A \delta_X H$ iff $H \delta_X f^{-1}(U_i)$ such

that $i = 1, 2, \dots, n$, since f is δ – continuous function, then $f(A) \delta_Y f(H)$ iff $f(H) \delta_Y f(f^{-1}(U_i))$, but $f(f^{-1}(U_i)) \subseteq U_i$, then $f(A) \delta_Y f(H)$ iff $f(H) \delta_Y U_i$ for some

$i = 1, 2, \dots, n$. Thus $f(\mathcal{C}_A) =_c \bigvee_{i=1}^n \{\mathcal{C}_{U_i}, i = 1, 2, \dots, n\}$. Hence $f_c(\mathcal{C}_A)$ is \mathcal{C} –

countable compact set in Y .

5.2 \mathcal{LC} - Countable Compact:

In this section we introduce and study notion of \mathcal{LC} - countable compact space in \mathcal{C} -topological space, and its relationship to the axioms of separation were studied.

Definition 5.2.1:

A \mathcal{C} -topological space $(X, \delta_x, \mathfrak{J}_{\mathcal{C}X})$ is said to be \mathcal{LC} - countable compact space if and only if every center point x_A in X has at least one \mathcal{C} - neighborhood \mathcal{C}_N whose $cl_{\mathcal{C}}(\mathcal{C}_N)$ is \mathcal{C} - countable compact.

Theorem 5.2.2:

Every \mathcal{C} - countable compact space $(X, \delta_x, \mathfrak{J}_{\mathcal{C}X})$ is \mathcal{LC} - countable compact space.

Proof:

Let $(X, \delta_x, \mathfrak{J}_{\mathcal{C}X})$ be \mathcal{C} - countable compact space and x_A any center point in \mathcal{C}_X , then $A\delta_X X$, then \mathcal{C}_X is \mathcal{C} - neighborhood of x_A , but $cl_{\mathcal{C}}(\mathcal{C}_X) = \mathcal{C}_X$ is \mathcal{C} - countable compact. Thus $(X, \delta_x, \mathfrak{J}_{\mathcal{C}X})$ is \mathcal{LC} - countable compact space.

Theorem 5.2.3:

Every \mathcal{C} - closed subspace of \mathcal{LC} - countable compact space is \mathcal{LC} - countable compact.

Proof:

Let \mathcal{C}_A be \mathcal{C} - closed subspace of \mathcal{LC} - countable compact space $(X, \delta_x, \mathfrak{J}_{\mathcal{C}X})$ and x_B \mathcal{C} - point in X such that $B\delta_X A$, and $B\delta_X X$. Since $(X, \delta_x, \mathfrak{J}_{\mathcal{C}X})$ is \mathcal{LC} - countable compact space, there exists \mathcal{C}_N \mathcal{C} - neighborhood of x_B such that $cl_{\mathcal{C}}(\mathcal{C}_N)$ is \mathcal{C} - countable compact. But the

$\mathcal{C}_N \wedge_{\mathcal{C}} \mathcal{C}_A$ is \mathcal{C} – neighborhood of x_B and $cl_{\mathcal{C}}(\mathcal{C}_N \wedge_{\mathcal{C}} \mathcal{C}_A)$ is \mathcal{C} – closed subset of \mathcal{C} – countable compact set $cl_{\mathcal{C}}(\mathcal{C}_N)$ and is thus itself \mathcal{C} – countable compact.

Proposition 5.2.4:

Every \mathcal{C} – open subspace of $\mathfrak{S}_{\mathcal{C}2-}$ space $\mathcal{L}\mathcal{C}$ - countable compact space is $\mathcal{L}\mathcal{C}$ – countable compact.

Proof:

Let \mathcal{C}_V be a \mathcal{C} – open subspace of $\mathfrak{S}_{\mathcal{C}2-}$ space $\mathcal{L}\mathcal{C}$ – countable compact $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ and x_B \mathcal{C} – point in X such that $B\delta_X V$ and $B\delta_X X$, there exist \mathcal{C}_{N_x} \mathcal{C} – open with $N_x \delta_X B$ such that $cl_{\mathcal{C}}(\mathcal{C}_{N_x})$ is \mathcal{C} – countable compact. But $\{x_B\}$ is \mathcal{C} – countable compact set in X , $\mathcal{C}_V \wedge_{\mathcal{C}} \mathcal{C}_{N_x}$ is \mathcal{C} – open such that $B\delta_X(V \cup N_x)$ and $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ is $\mathfrak{S}_{\mathcal{C}2-}$ space and $\mathcal{L}\mathcal{C}$ – countable compact space. So there exists an \mathcal{C} – open set \mathcal{C}_U in X such that $cl_{\mathcal{C}}(\mathcal{C}_U)$ is \mathcal{C} – countable compact, but $B\delta_X U$ and $\mathcal{C}_U \preccurlyeq_{\mathcal{C}} cl_{\mathcal{C}}(\mathcal{C}_U) \Rightarrow \forall H\delta_X U \Rightarrow H\delta_X(V \cup N_x) \Rightarrow H\delta_X V$ This implies that $cl_{\mathcal{C}}(\mathcal{C}_U)$ is \mathcal{C} – countable compact (\mathcal{C} – neighborhood) of x_B subset of \mathcal{C}_V . Hence \mathcal{C}_V is $\mathcal{L}\mathcal{C}$ – countable compact.

Proposition 5.2.5:

Let $(X, \delta_x, \mathfrak{S}_{\mathcal{C}X})$ be a $\mathfrak{S}_{\mathcal{C}2-}$ space . If \mathcal{C}_A and \mathcal{C}_B are $\mathcal{L}\mathcal{C}$ – countable compact spaces, then $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ is $\mathcal{L}\mathcal{C}$ – countable compact space.

Proof:

Let $x_H \in \mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ such that $K\delta_X A$ and $K\delta_X B$ for all $\emptyset \neq k \subseteq X$. And let \mathcal{C}_{N_1} and \mathcal{C}_{N_2} are \mathcal{C} – countable compact such that $N_1 \delta_X H$ in \mathcal{C}_A and $N_2 \delta_X H$ in \mathcal{C}_B , by theorem (5.1.6), then $\mathcal{C}_{N_1} \wedge_{\mathcal{C}} \mathcal{C}_{N_2}$ is \mathcal{C} – countable compact and \mathcal{C} – neighborhood of x_H . Thus $\mathcal{C}_A \wedge_{\mathcal{C}} \mathcal{C}_B$ is $\mathcal{L}\mathcal{C}$ – countable compact space.

Proposition 5.2.6:

A $\mathfrak{T}_{\mathcal{C}2-}$ space $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – countable compact if and only if each of its \mathcal{C} – points is an \mathcal{C} –interior point of some \mathcal{C} – countable compact subspace of $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$.

Proof:

If $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – countable compact, then for all x_A in X there exists \mathcal{C}_N such that $A\delta_X N$ and $cl_{\mathcal{C}}(\mathcal{C}_N)$ is \mathcal{C} – countable compact, it follows that $cl_{\mathcal{C}}(\mathcal{C}_N)$ is a \mathcal{C} – countable compact (\mathcal{C} – neighborhood) of x_A is an \mathcal{C} – interior point. Thus $\forall x_A$ is an \mathcal{C} –interior point of some \mathcal{C} – countable compact.

Conversely, let x_A is \mathcal{C} – point in X be arbitrary. By hypothesis, there exists an \mathcal{C} – countable compact \mathcal{C}_H such that $A\delta_X H$, then \mathcal{C}_H is a \mathcal{C} – neighborhood of x_A . Since $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is $\mathfrak{T}_{\mathcal{C}2-}$ space, implies \mathcal{C}_H is \mathcal{C} – closed, so that $cl_{\mathcal{C}}(\mathcal{C}_H) = \mathcal{C}_H$, thus x_A has a \mathcal{C} – neighborhood whose $cl_{\mathcal{C}}(\mathcal{C}_H)$ is \mathcal{C} – countable compact. And so $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – countable compact.

Proposition 5.2.7:

Every \mathcal{LC} – countable compact $\mathfrak{T}_{\mathcal{C}2-}$ space is \mathcal{C} – regular space.

Proof:

Let $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – countable compact $\mathfrak{T}_{\mathcal{C}2-}$ space, x_A be any \mathcal{C} – point in X and \mathcal{C}_F any \mathcal{C} – closed set in X such that $A\overline{\delta_X} F$ for each $A \subseteq X$. Since $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is $\mathfrak{T}_{\mathcal{C}2-}$ space, then \mathcal{C}_F is \mathcal{C} – countable compact and for each $y \in X$ and $B\delta_X F$, then there exists \mathcal{C}_G and \mathcal{C}_{H_y} \mathcal{C} – open sets such that $A\delta_X G$, $F\delta_X H_y$ and $\mathcal{C}_G \wedge_{\mathcal{C}} \mathcal{C}_{H_y} = \mathcal{C}_{\emptyset}$, the family of all $\{\mathcal{C}_{H_{y_i}}, y_B \in \mathcal{C}_F\}$ countable \mathcal{C} – open cover of \mathcal{C} – countable compact \mathcal{C}_{H_y} and hence a finite subfamily

$\mathcal{C}_{H_{y_1}}, \mathcal{C}_{H_{y_2}}, \dots, \mathcal{C}_{H_{y_n}}$ such that $\mathcal{C}_{H_y} \ll_c \bigvee_{i=1}^n \mathcal{C}_{H_{y_i}}$. Hence $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{C} – regular space.

Proposition 5.2.8:

Let $f: (X, \delta_x) \rightarrow (Y, \delta_y)$ be a δ – continuous onto shrink function and $f_c: (X, \delta_x, \mathfrak{T}_{\mathcal{C}X}) \rightarrow (Y, \delta_y, \mathfrak{T}_{\mathcal{C}Y})$ be a shrink \mathcal{C} – continuous function, then If $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – countable compact space, then $(Y, \delta_y, \mathfrak{T}_{\mathcal{C}Y})$ is \mathcal{LC} – countable compact space.

Proof:

Let $B \subseteq Y$ and $y \in Y$ such that $\{y\}\delta_y B$, but $B\delta_y Y$. Since f is onto, then there exists $A \subseteq X$ such that $\{x\}\delta_x A$ and $f_c(x_A) = y_B$. But $(X, \delta_x, \mathfrak{T}_{\mathcal{C}X})$ is \mathcal{LC} – countable compact space, then there exists \mathcal{C}_N nbd for x_A such that for each $A \subseteq X$ such that $A\delta_x N$ and $cl_{\mathcal{C}}(\mathcal{C}_N)$ is \mathcal{C} – countable compact set. Also f_c is shrink \mathcal{C} – continuous function and by proposition (5.1.8), we get for each $B \subseteq Y$, $\{y\}\delta_y B$, then $B\delta_y f(N)$. Hence $(Y, \delta_y, \mathfrak{T}_{\mathcal{C}Y})$ is \mathcal{LC} – countable compact space

5.3 \mathcal{AC} - Countable Compact:

In this section, we introduce and study the notion of \mathcal{AC} – countable compact space in \mathcal{C} –topological space, and its relationship to the axioms of separation were studied.

Definition 5.3.1:

Let $(X, \delta, \mathfrak{T}_{\mathcal{C}})$ be a \mathcal{C} -topological space, the universal set \mathcal{C}_X (a non-empty center set \mathcal{C}_A) is called \mathcal{AC} – countable compact space if each countable \mathcal{C} – open cover $\{\mathcal{C}_{U_\alpha}, \alpha \in \Gamma\}$ of \mathcal{C}_X (\mathcal{C}_A) has finite Δ subset of Γ such that $\forall H \subseteq X$, there exist $\alpha_0 \in \Delta, cl_{\mathcal{C}}(U_{\alpha_0})\delta H$.

Proposition 5.3.2:

Every \mathcal{C} – countable compact space is an \mathcal{AC} – countable compact space.

Proof:

Let $(X, \delta, \mathfrak{T}_c)$ be \mathcal{C} – countable compact space and $\mathcal{H} = \{C_{A_\lambda}, \lambda \in \Lambda\}$ be a countable \mathcal{C} – open cover of C_X . Then there exists finite I subset of Λ such that $\forall H \subseteq X$, there exist $\lambda_0 \in I$, $A_{\lambda_0} \delta H$. Now since $C_{A_\lambda} \leq_c cl_C(C_{A_\lambda})$. We have for each $\lambda \in \Lambda$, $H \delta A_\lambda$, then $H \delta cl(A_\lambda)$. Therefore there exist $\lambda_0 \in I$, $cl(A_{\lambda_0}) \delta H$. Hence $(X, \delta, \mathfrak{T}_c)$ is \mathcal{AC} – countable compact space.

Remark 5.3.3:

Every \mathcal{C} – countable compact set is \mathcal{AC} – countable compact set.

The following example shows that the conversely of the above remark is not true.

Example 5.3.4:

Let $X = \{x_1, x_2, x_3, \dots\}$ with δ be discrete proximity, define a \mathcal{C} – topology \mathfrak{T}_c on C_X as follows:

$$\mathfrak{T}_c = \{C_{\{x_1\}}, C_{\{x_1, x_2\}}, C_{\{x_1, x_2, x_3\}}, \dots\} \cup_C \{C_\emptyset\}.$$

Now we will show that $(X, \delta, \mathfrak{T}_c)$ is an \mathcal{AC} – countable compact space.

But $(X, \delta, \mathfrak{T}_c)$ is not \mathcal{C} – countable compact space.

For each $\emptyset \neq C_{A_\alpha} \in \mathfrak{T}_c$, we have $cl_C(C_{A_\alpha}) =_c C_X$.

Now, let $\mathcal{H} = \{C_{A_\alpha}, \lambda \in \Lambda\}$ be a countable \mathcal{C} – open cover of C_X .

This means $\mathcal{C}_X = \bigcup_{\mathcal{C}} \{\mathcal{C}_{A_\alpha}, \lambda \in \Lambda\}$, then there exists $\alpha_0 \in \Lambda$ such that $\mathcal{C}_{A_{\alpha_0}} \in \mathfrak{S}_{\mathcal{C}}$.

But $cl_{\mathcal{C}}(\mathcal{C}_{A_{\alpha_0}}) =_{\mathcal{C}} \mathcal{C}_X$. Thus $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is an \mathcal{AC} – countable compact space.

Now we will show that $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is not \mathcal{C} – countable compact space.

Let $\Gamma = \{\mathcal{C}_{\{x_1\}}, \mathcal{C}_{\{x_1, x_2\}}, \mathcal{C}_{\{x_1, x_2, x_3\}}, \dots\}$ be a countable \mathcal{C} – open cover of \mathcal{C}_X , it is clear that Γ dose not have a finite subcover of \mathcal{C}_X .

Proposition 5.3.5:

Let $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ be a \mathcal{C} –Regular space, then $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} – countable compact space if and only if is \mathcal{AC} – countable compact space.

Proof:

If $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} – countable compact space, then by Proposition (5.3.2) $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{AC} – countable compact space. Conversely, let $\mathcal{H} = \{\mathcal{C}_{U_\lambda}, \lambda \in \Lambda\}$ be a countable \mathcal{C} – open cover of \mathcal{C}_X . For each $x \in X$, $A \subseteq X$ and $\{x\} \delta A$, there exists λ_x such that $A \delta U_{\lambda_x}$, but $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} –Regular space, then there exists \mathcal{C} –open set $\mathcal{C}_{H_{\lambda_x}}$ such that for each $x \in X$, $H_{\lambda_x} \subseteq X, \{x\} \delta H_{\lambda_x}$ and $\forall K \subseteq X, K \delta cl(H_{\lambda_x})$, then $K \delta U_{\lambda_x}$. Now, $\mathcal{F} = \{\mathcal{C}_{H_{\lambda_x}}, x \in X\}$ is a countable \mathcal{C} – open cover of \mathcal{C}_X . But $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{AC} – countable compact space, then there exists finite I subset of X such that for each $x \in X$, $\forall K \subseteq X, K \delta X$ and $K \delta cl(H_{\lambda_x})$ for some $x \in I$, therefore $K \delta cl_{\tau}(H_{\lambda_x})$ for some $x \in I$. Hence $(X, \delta, \mathfrak{S}_{\mathcal{C}})$ is \mathcal{C} – countable compact space.

Proposition 5.3.6:

Let $(X, \delta, \mathfrak{S}_C)$ be a C – topological space and $\mathcal{C}_A, \mathcal{C}_B$ are center sets in X such that \mathcal{C}_A is \mathcal{AC} – countable compact set and $\mathcal{C}_A \preceq_C \mathcal{C}_B \preceq_C cl_C(\mathcal{C}_A)$, then \mathcal{C}_B is \mathcal{AC} – countable compact set.

Proof:

Let $\mathcal{H} = \{\mathcal{C}_{U_\lambda}, \lambda \in \Lambda\}$ be a countable C – open cover of \mathcal{C}_B ,

\Rightarrow For each $y \in X, \{y\} \delta B$, for some $\lambda_o \in \Lambda, \forall \emptyset \neq K \subseteq X, K \delta B$, then $K \delta U_{\lambda_o}$.

But $\forall K \delta A$, then $K \delta B$,

\Rightarrow for some $\lambda_o \in \Lambda, \forall \emptyset \neq K \subseteq X, K \delta A$, then $K \delta U_{\lambda_o}$.

And \mathcal{C}_A is \mathcal{AC} – countable compact set,

\Rightarrow there exists finite I subset of Λ such that $\forall \emptyset \neq K \subseteq X, K \delta A$, then for some $\lambda_o \in I, K \delta cl(U_{\lambda_o})$.

$\Rightarrow \forall \emptyset \neq K \subseteq X, K \delta cl(A)$ iff for some $\lambda_o \in I, K \delta cl(U_{\lambda_o})$.

But $\mathcal{C}_B \preceq_C cl_C(\mathcal{C}_A)$, then some $\lambda_o \in I, K \delta B$, then $K \delta cl(U_{\lambda_o})$.

Hence \mathcal{C}_B is \mathcal{AC} – countable compact set.

Corollary 5.3.7:

Let $(X, \delta, \mathfrak{S}_C)$ be a C – topological space and \mathcal{C}_A be a \mathcal{AC} – countable compact set in X , then $cl_C(\mathcal{C}_A)$ is also \mathcal{AC} – countable compact set.

Proposition 5.3.8:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism shrink function and $f_c: (X, \delta_X, \mathfrak{C}_X) \rightarrow (Y, \delta_Y, \mathfrak{C}_Y)$ be a shrink \mathcal{C} – continuous function, then If \mathcal{C}_A is \mathcal{AC} – countable compact set for each \mathcal{C}_A subset of \mathcal{C}_X , then $\mathcal{C}_{f(A)}$ is \mathcal{AC} –countable compact set subset of \mathcal{C}_Y .

Proof:

Suppose \mathcal{C}_A is \mathcal{AC} – countable compact set in X and $\mathcal{H} = \{\mathcal{C}_{U_\lambda}, \lambda \in \Lambda\}$ be a countable \mathcal{C} – open cover of $\mathcal{C}_{f(A)}$ such that for each $B \subseteq X, \forall x \in X$ such that $\{x\} \delta B$ and $\exists \lambda_0 \in \Lambda, B \delta A_{\lambda_0}$. By proposition (2.1.1.8), then $\mathcal{C}_A \preceq_c f^{-1}(\mathcal{C}_{f(A)}) \preceq_c f^{-1}(\bigcup_{\lambda} \mathcal{C}_{U_\lambda}) =_c \bigcup_{\lambda} \{\mathcal{C}_{f^{-1}(U_\lambda)}\}$. Therefore $\Gamma = \{\mathcal{C}_{f^{-1}(U_\lambda)}, \lambda \in \Lambda\}$ is countable \mathcal{C} – open cover of \mathcal{C}_A such that for each $K \subseteq X, \forall x \in A$ such that $\{x\} \delta K$ and $\exists \lambda_0 \in \Lambda$ and $K \delta f^{-1}(U_{\lambda_0})$. But \mathcal{C}_A is \mathcal{AC} – countable compact set, then Γ has a finite subfamily I such that for some $\lambda_0 \in I, \forall K \subseteq X, \forall x \in A$ such that $\{x\} \delta K$ and $\exists \lambda_0 \in I, K \delta f^{-1}(U_{\lambda_0})$. Whose center closure \mathcal{C}_A and by proposition (2.1.1.8), that is to say:

$$\begin{aligned} \mathcal{C}_A &\preceq_c cl_c \left(\bigcup_{i=1}^n \mathcal{C}_{f^{-1}(U_{\lambda_i})} \right) =_c cl_c \left(f^{-1} \left(\bigcup_{i=1}^n \mathcal{C}_{U_{\lambda_i}} \right) \right) \\ &\preceq_c f^{-1} \left(cl_c \left(\bigcup_{i=1}^n \mathcal{C}_{U_{\lambda_i}} \right) \right). \text{ Accordingly and by Proposition (2.1.1.6), then} \\ \mathcal{C}_{f(A)} &\preceq_c f \left(f^{-1} \left(cl_c \left(\bigcup_{i=1}^n \mathcal{C}_{U_{\lambda_i}} \right) \right) \right) \\ &\preceq_c cl_c \left(\bigcup_{i=1}^n \mathcal{C}_{U_{\lambda_i}} \right) =_c \bigcup_{i=1}^n cl_c(\mathcal{C}_{U_{\lambda_i}}). \end{aligned}$$

Thus $\mathcal{C}_{f(A)}$ is \mathcal{AC} –countable compact set.

Proposition 5.3.9:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a reduction δ – homeomorphism reduction function and $f_c: (X, \delta_X, \mathfrak{I}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{I}_{cY})$ be a reduction \mathcal{C} – homeomorphism function, if \mathcal{C}_A is \mathcal{AC} – countable compact set for each \mathcal{C}_A subset of \mathcal{C}_X , then $f_c(\mathcal{C}_A)$ is \mathcal{AC} – countable compact set subset of \mathcal{C}_Y .

Proof:

We prove the same way to prove Proposition (5.3.8).

5.4 \mathcal{HC} - Countable closedness:

In this section, we give several characterization of \mathcal{C} – countable compact in center topological space, nearly \mathcal{HC} – countable closed space. Furthermore, the relationship between this concept and the separation axioms in center topological space had been studied.

Definition 5.4.1:

Let $(X, \delta, \mathfrak{I}_c)$ be \mathfrak{I}_{c2} – space, the universal set \mathcal{C}_X (a non-empty center set \mathcal{C}_A) is called \mathcal{HC} – countable closed space if each countable \mathcal{C} – open cover $\{\mathcal{C}_{U_\alpha}, \alpha \in \Gamma\}$ of \mathcal{C}_X (\mathcal{C}_A) has finite Δ subset of Γ such that $\forall H \subseteq X$, there exist $\alpha_0 \in \Delta, cl_c(U_{\alpha_0}) \supseteq H$.

Remarks 5.4.2:

- i. Every \mathcal{HC} – countable closed space is \mathcal{AC} – countable compact space.
- ii. Every \mathcal{AC} – countable compact space and \mathfrak{I}_{c2} – space is \mathcal{HC} – countable closed space.
- iii. Every \mathcal{HC} – countable closed space and \mathfrak{I}_{c3} – space is \mathcal{C} – countable compact space.

The following example shows that the conversely of the above remark (i) is not true.

Example 5.4.3:

Let $(X, \delta, \mathfrak{T}_C)$ be a \mathcal{C} -topological space,

Where $X = \{a, b\}$, $A\delta B$ iff $A \cap B \neq \emptyset$

And $\mathfrak{T}_C = \{ \mathcal{C}_\emptyset, \{\langle X, \{a\}\rangle\}, \{\langle X, \{b\}\rangle\}, \{\langle X, \{a\}\rangle, \langle X, \{b\}\rangle\}, \{\langle X, \{a\}\rangle, \langle X, X\rangle\},$

$\{\langle X, \{b\}\rangle, \langle X, X\rangle\}, \{\langle X, X\rangle\}, \{\{\langle a, \{a\}\rangle, \langle a, X\rangle\}, \{\{\langle b, \{b\}\rangle, \langle b, X\rangle\}, \mathcal{C}_X\}$

Then $(X, \delta, \mathfrak{T}_C)$ is \mathcal{AC} – countable compact space.

But is not \mathfrak{T}_{C_2} – *Space*. Hence it is not \mathcal{HC} – countable closed space.

Theorem 5.4.4:

Every \mathcal{C} – countable compact space and \mathfrak{T}_{C_2} – *space* is \mathcal{HC} – countable closed space.

Proof:

Let $(X, \delta, \mathfrak{T}_C)$ be \mathcal{C} – countable compact space and \mathfrak{T}_{C_2} – *space* and $\Gamma = \{\mathcal{C}_{U_\alpha}, \alpha \in \Gamma\}$ be countable \mathcal{C} – open cover of $(X, \delta, \mathfrak{T}_C)$, then there exists finite I subset of Γ such that $\forall x \in X, \forall H \subseteq X, \{x\}\delta H$, then for some $\alpha_0 \in I$, $H\delta U_{\alpha_0}$, now since $\forall H \subseteq X, H\delta U_{\alpha_i}$, then $H\delta cl(U_{\alpha_i})$. Hence $(X, \delta, \mathfrak{T}_C)$ is \mathcal{HC} – countable closed space.

Theorem 5.4.5:

Let $(X, \delta, \mathfrak{T}_C)$ be \mathcal{C} – *Regular* space and \mathfrak{T}_{C_2} – *space*, then $(X, \delta, \mathfrak{T}_C)$ is \mathcal{HC} – countable closed space if and only if it is \mathcal{C} – countable compact space.

Proof:

Since $(X, \delta, \mathfrak{S}_c)$ is \mathcal{HC} – countable closed space, then $(X, \delta, \mathfrak{S}_c)$ is \mathcal{AC} – countable compact space, but its \mathcal{C} – *Regular* space, then by Proposition (5.3.5) we have $(X, \delta, \mathfrak{S}_c)$ is \mathcal{C} – countable compact space.

Conversely, since $(X, \delta, \mathfrak{S}_c)$ is \mathcal{C} – countable compact space and \mathcal{C} – *Regular* space, and then by Proposition (5.3.5) we have $(X, \delta, \mathfrak{S}_c)$ is \mathcal{HC} – countable closed space.

Proposition 5.4.6:

Let $f: (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a δ – homeomorphism shrink function and $f_c: (X, \delta_X, \mathfrak{S}_{cX}) \rightarrow (Y, \delta_Y, \mathfrak{S}_{cY})$ be a shrink \mathcal{C} – continuous function. If $(X, \delta_X, \mathfrak{S}_{cX})$ is \mathcal{HC} – countable closed space, then $(Y, \delta_Y, \mathfrak{S}_{cY})$ is \mathcal{HC} – countable closed space.

Proof:

Direct from define of \mathcal{HC} – countable closed space and by Proposition (5.3.8).

Conclusion and future work

Through this study related to the proximity theory, which was limited to the \mathcal{C} –set in the center topological space with the presence of the proximity space. We found types of functions using the \mathcal{C} –set and the effect was obvious by studying the transition behavior of items and sets for each of these types of functions and obtained significant results, In addition to the possibility of applying these results to some of the research presented before In light of the new definitions presented for compactness and some of its types in this study and the results presented therein. Also, we can add several studies, including:

1. With the concept of the \mathcal{C} –sets present in the central topological spaces, it is possible to further study the concept of \mathcal{C} – net, \mathcal{C} – filter and \mathcal{C} – ideal.
2. Expanding the concept of compactness and its types and studying the relationship between them and the topological concepts \mathcal{C} – net, \mathcal{C} – filter and \mathcal{C} – ideal.
3. Studying some relationships between the concept of \mathcal{C} – compactness and the central sequences after this concept is presented.

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المستخلص

تهدف هذه الدراسة إلى بناء دوال وفضاءات جديدة في الفضاءات التوبولوجية المركزية من خلال نظرية القرب ، ودراسة جميع المفاهيم التوبولوجية التي يمكن دراستها فيه. كذلك العلاقات التي تربط بين تلك الفضاءات ضمن هذا الفضاء. حيث تم استعراض بعض المفاهيم الأساسية لنظرية القرب وكذلك مفهوم C - set والخواص الجبرية لتلك المجموعة وعرض بناء الفضاءات التوبولوجية المركزية والتي تمثل المرحلة الأولى من هذا العمل.

المرحلة الثانية : تم تقديم مفهوم الدالة في الفضاءات التوبولوجية المركزية مع دراسة خواصها الرياضية وكذلك مفهوم الاستمرارية والتكافؤ وتأثير تلك الدوال على بديهيات الفصل المركزية. حيث تم تعريف عدة دوال منها :

- Shrink central function
- Reduction central function
- Stronger central function
- Weakly central function

اعتمد تصنيف الدوال اعلاه حسب تواجد فضاءات القرب في المجال والمجال المقابل حيث عرف الصنف الاول والثاني بافتراض ان الدالة معرفة من فضاء القرب الى فضاء القرب اما الصنف الثالث فعرف من فضاء القرب الى مجموعة تم تعريف علاقتها من خلال علاقة المجال اما الصنف الاخير فتم تعريفه من مجموعة الى فضاء القرب وتم تعريف العلاقة بنفس اسلوب الصنف الثالث.

المرحلة الثالثة: تم تقديم مفهوم الفضاءات C - compact وبعض من تصنيفات ذلك الفضاء (LC - compact و AC - compact و HC - closed).

المرحلة الرابعة : تم تقديم مفهوم الفضاءات C - lindelof وبعض من تصنيفات ذلك الفضاء (LC - lindelof و AC - lindelof و HLC - closed).

المرحلة الخامسة: تم تقديم مفهوم الفضاءات C - countable compact وبعض من تصنيفات ذلك الفضاء (LC - countable compact و AC - countable compact و HC - countable compact (closed).

حيث تم في المراحل الثلاثة الاخيرة دراسة الخواص التوبولوجية بواسطة الدوال التي تم بنائها في المرحلة الاولى والخاصية الوراثية لتلك الفضاءات وعلاقة تلك الفضاءات مع بديهيات الفصل المركزية والتاثير المتبادل بينهما وكذلك اتحاد وتقاطع تلك الفضاءات.

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التراص باستخدام فضاءات القرب

رسالة مقدمة الى

مجلس كلية التربية للعلوم الصرفة في جامعة بابل كجزء من متطلبات نيل درجة
الدكتوراه فلسفة في التربية / الرياضيات

من قبل

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2022 م

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