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Algorithms for Various Dominations in Graphs

A Dissertation

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Pure Sciences in University of Babylon in Partial Fulfillment of
The Requirements for the Degree of Doctor of Philosophy in
Education / Mathematics*

By

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

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Dedication

To the memory of my parents.

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In the Name of Allah, the Merciful, the Beneficent. Alhamdulillah for everything I have, Alhamdulillah for everything I had and Alhamdulillah for everything I will have. Praise be to the Lord of all worlds. Prayers and peace be upon our Prophet, Muhammad, and his family.

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Abstract

Various applications of domination and recent work on different forms of domination in graphs have given rise to our interest in exploring in this study new special types of graph domination.

As most of the work discussed in domination focuses more on setting the conditions for the dominant group to come up with a new concept of domination. In this thesis, the two new parameter of domination in graph that called even sum domination (ESDS) and inverse even sum domination are introduced, and another new parameter which is edge even sum dominating set (ESEDS) Many of properties of these numbers are been discussed especially on the certain graphs as cycle, path, complete, complete bipartite, star, null, and wheel. Moreover, a linear-time algorithm for the ESDS and ESEDS domination problems in wheels is written and the NP-complete of ES domination problem is proved. In addition, the corona operation of a cycle graph and one of the certain graphs such as wheel, path, cycle, complete, star, complete bipartite, and null graph for even sum domination. Also, for each case there are some figures are been introduced to illustrated these cases. Moreover, the fuzzy even sum domination in strong graph is defended and studied for some certain graphs.

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List of Symbols

G	Graph label
$G(n, m)$	Graph of order n and size m
$V(G)$	The set of all vertices in a graph G
$E(G)$	The set of all edges in a graph G
\bar{G}	Complement of graph G
$ V(G) $	number of vertices in a graph G
$ E(G) $	number of edges in a graph G
$ A $	cardinality of a set A
$\lceil a \rceil$	least integer not less than a
$\lfloor a \rfloor$	greatest integer not exceeding a
$a = b \pmod{x}$	$a = mx + b$, where a, b, m and x are integers
$\deg_G(v)$ or $\deg(v)$	degree of vertex v in a graph G
$\delta(G)$	minimum degree of vertices in G
$\Delta(G)$	maximum degree of vertices in G
$d(v, u)$	distance between v and u in a graph G
$\text{diam}(G)$	diameter of a graph G
$N_G(v)$ or $N(v)$	open neighborhood of v in a graph G
$N_G[v]$ or $N[v]$	closed neighborhood of v in a graph G
$G[H]$ or $\langle H \rangle$	induced subgraph of a graph G
$G_1 + G_2$	join of G_1 and G_2
$G_1 \times G_2$	Cartesian product of G_1 and G_2
$G_1 \odot G_2$	corona of G_1 and G_2
$P_n \times P_m$	grid graph
P_n	path graph
C_n	cycle graph
N_n	null graph
K_n	complete graph
$K_{n,m}$	complete bipartite graph
S_n	star graph
W_n	wheel graph
$\gamma_{es}(G)$	even sum domination number of a graph G
$\gamma_{es}(G)$ -set	even sum dominating set (ESDS)

$\gamma_{es}^{-1}(G)$	Inverse even sum domination number of a graph G
$\gamma'_{es}(G)$ -set	even sum edge dominating set (ESEDs)
$G = (\sigma, \mu)$	a fuzzy graph of a graph G(V, E)
$deg_E(v)$	Degree of effective
$FESDS$	fuzzy even sum dominating set
$MFESDS$	<i>minimal</i> fuzzy even sum dominating set
$M^m FESDS$	minimum fuzzy even sum dominating set
γ_{fes}	fuzzy even sum domination number

Publications

[1] Even sum domination in graphs with algorithm

Ihsan M. Rasheed, Ahmed A. Omran

Cite as: AIP Conference Proceedings 2386, 060016 (2022);

<https://doi.org/10.1063/5.0067152>

[2] Even Sum Domination of Corona of the Cycle Graph with Some Certain Graphs

Ihsan M. Rasheed, Ahmed A. Omran

(Accepted)

Introduction

The Graph theory was based by Leonhard Euler in (1736) as a solution to the problem of the seven Königsberg bridges. From the years 1736 to 1936, the concept of graph was used in many scientific fields, but under different names, as models of many real-world problems; for more details see the historic book by Wilson et al. [52].

The domination in graph theory is a model for many problems in the operations research. For example, we state the following fire station issue. Suppose a province has decided to build some fire stations such that must serve all of the cities in the province. Moreover, the fire stations are to be located in some cities so that every city either has a fire station or is a neighbor of a city which has a fire station. Now, to save money the province wants to build the minimum number of these stations with the above requirements.

Before and after there were computers, there were many kinds of algorithms. But now that there are computers, there are even more algorithms, and algorithms lie at the core of computing. An algorithm is any well-defined computational steps that takes set of values, as input and produces set of values as output. An algorithm is thus a succession of computational procedure that convert the input into the output.

For a long time many theorems for the domination problem have been determined, but the first algorithmic result on this topic was given by Cockayne et al. in (1975). They gave a linear time algorithm for this problem by using a labeling the vertices in trees. In general the domination problem is NP-complete. This is by Garey and Johnson, at about the same time, they constructed the first proof that the domination problem is NP-complete for general graphs. Since then, many algorithmic results are calculated for variations of the domination problem in various types of graphs.

Chapter One

Basic Definitions and Concepts

Chapter One

Basic Definitions and Concepts

1.1 Introduction

In this chapter, the relevant definitions, basic concepts, required background for this dissertation are introduced. Two sections in this chapter. The first section contains some basic definitions and notations of graph theory that we need in thesis. Therefore, the second section gives definition of domination number with some types and properties of domination number.

1.2 Introduction to graph theory

The city of Königsberg in Prussia (now Kaliningrad, Russia) was located on the Pregel River. The river divides the city into four separate mainland, connected by seven bridges as illustrated by the map in Fig.1.1 (a, b). The problem is to answer the question: Is it possible to find a path that pass across the seven bridges once only once at each bridge, and then return to the starting point?

The puzzle could not be proved, until the Swiss mathematician Leonhard Euler (1707-1783) in 1736 thought about this question, Euler was proved that the question has no solution.

The key to Euler's answer was in a very simple abstraction of the riddle. Let us redraw our diagram of the city of Kaliningrad by representing each of the mainland as a vertex (or point, node) and representing each bridge as an edge (or line, link) connecting the vertices corresponding to the mainland as shown in Fig.1.1 (a, b, c). This mathematical structure is called *graph*. The way he used to solve it is considered starting point of graph theory.

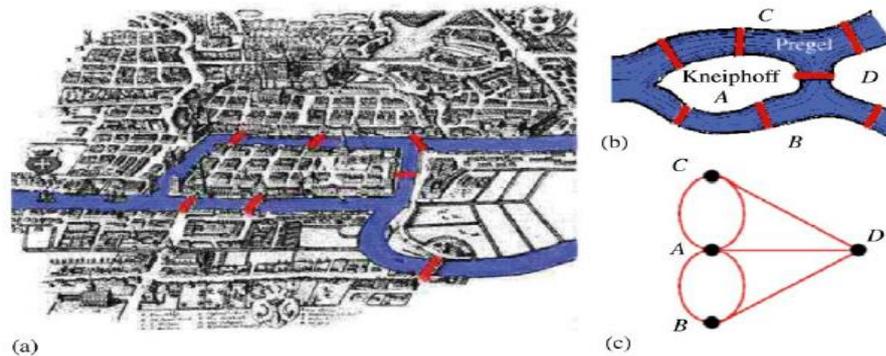


Figure 1.1: Seven bridges in Kaliningrad, Russia

In view of this, there is a need to understand the basic terminology concepts for graph and some theories. Some important references may be found in [1, 3, 7, 11, 24, and 29].

1.3 Basic Definitions

Definition 1.3.1. [1] A **graph** $G = (V(G), E(G))$ or $G = (V, E)$ consists of two finite set $V(G)$ or V , the vertex set of the graph, which is a non-empty set of elements called vertices (or points) and $E(G)$ or E the edge set of the graph, which is a possibly empty set of elements called edges (or lines), such that each edge e in E is assigned as an unordered pair of vertices called the end vertices of e . A graph G with n vertices and m edges is called a (n, m) –graph.

Example 1.3.2. A graph G of 6 vertices and 7 edges such that $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ a set of vertices and $E = \{v_1v_2, v_1v_5, v_1v_6, v_2v_3, v_2v_4, v_3v_4, v_5v_6\}$ a set of edge of a graph G or $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$.

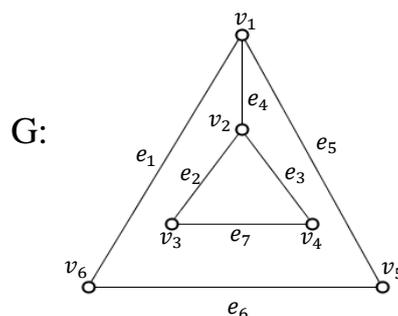


Figure 1.2: (6,7) –graph

Definition 1.3.3 [1] The **order** of G is $n = |V|$ and the **size** of G is $m = |E|$. In Fig 1.2 , order and size of G is 6 and 7 respectively.

Definition 1.3.4 [6] A graph is **trivial** if its vertex set is a singleton and it contains no edges.

Definition 1.3.5 [1] A graph with a finite number of vertices as well as finite number of edges is called a **finite graph**, otherwise it is an **infinite graph**.

Definition 1.3.6 [3] If two vertices of a graph are joined by an edge then these vertices are called **adjacent vertices**.

Definition 1.3.7 [3] If two or more edges of a graph have a common vertex then these edges are called **adjacent edges** (or **incident edges**). Note that if $e = vu$ is an edge of G , then e is **incident** to u and v . we also say that u and v are the endpoints of e .

Definition 1.3.8 [8] The **open neighborhood** $N(v)$ of the vertex v consists of the set vertices adjacent to v , that is, $N(v) = \{u \in V : vu \in E\}$, and the **closed neighborhood** of v is $N[v] = N(v) \cup \{v\}$.

In Fig.1.2, $N(v_1) = \{v_2, v_5, v_6\}$ and $N[v_1] = \{v_1, v_2, v_5, v_6\}$.

Definition 1.3.9 If $X \subseteq V(G)$ and $u \in X$ then the **private neighbor** of u with respect to X , is defined by $pn[u, X] = \{v : N[v] \cap X = \{u\}\}$.

Definition 1.3.10 [3] The **degree** of a vertex v in a graph G , denoted by $d(v)$ or $deg(v)$ is the number of edges incident with v . For example, in Fig.1.2, $deg(v_1) = 3$ i.e. $deg(v) = |N(v)|$.

Definition 1.3.11 [5] The **maximum degree** of a graph G , denoted by $\Delta(G)$ is the maximum value among the degrees of all the vertices of G , i.e., $\Delta(G) = \max_{v \in V(G)} deg(v)$. Similarly, we define the **minimum degree** of a

graph G and denote it by $\delta(G)$, i.e., $\delta(G) = \min_{v \in V(G)} \deg(v)$. In Fig.1.2, $\Delta(G) = 3$ and $\delta(G) = 2$.

Definition 1.3.12 [6] A vertex of degree 0 is an **isolated vertex** of G , A vertex of degree 1 is called a **pendant vertex** of G (or end-vertex or leaf), and the unique edge of G incident to such a vertex of G is a **pendant edge** of G .

Definition 1.3.13 [1] If for some positive integer k , $\deg(v) = k$ for every vertex v of the graph G , then G is called **k -regular**.

Definition 1.3.14 [5] A **loop** is an edge whose end-vertices are the same. (see Fig.1.3 (e_1))

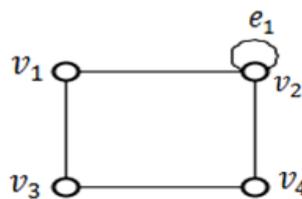


Figure 1.3: Graph with loop

Definition 1.3.15 [5] **Multiple edges** are edges with the same pair of end-vertices.

Example 1.3.16 (Fig. 1.4) show loops and multiple edge where the edges e_1 and e_2 are loops, so the edges e_3 and e_4 are multiple edges.

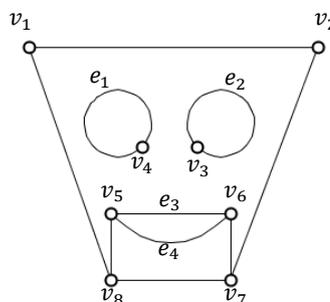


Figure 1.4: A graph has loops and multiple edges.

Definition 1.3.17 [4] *Simple graph* is undirected graph without loops and multiple edges. In other words simple graph is a pair $G = (V, E)$ where V is an arbitrary set and E is a set of unordered pairs of distinct elements from V .

Definition 1.3.18 [7] A $u - v$ *walk* W in G is a sequence of vertices in G , beginning with u and ending at v such that consecutive vertices in the sequence are adjacent.

Definition 1.3.19 [7] A $u - v$ walk in a graph in which no vertices are repeated is a $u - v$ *path*.

Definition 1.3.20 The thorn path $P_{n,r,s}$ is obtained from the path P_n by adding r neighbors to each of its nonterminal vertices and s neighbors to each of its terminal vertices. (Gutman, 1998). In addition to if $r = s$, then the thorn graph is called a homogenous caterpillar.

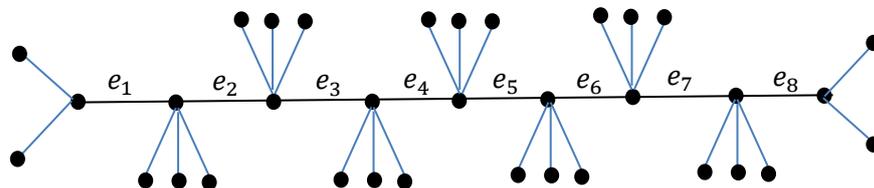


Figure 1.5. The thorn graph $P_{9,3,2}$

Definition 1.3.21 A thorn rod is a graph, $P_{n,m}$ which includes a linear chain of n vertices and degree- m terminal vertices at each of the two rod ends. (Gutman, 1998).

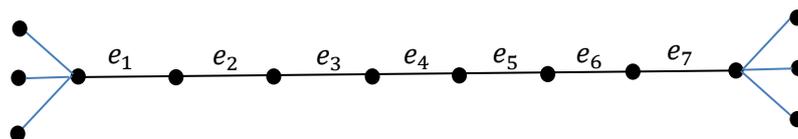


Figure 1.6. The thorn rod graph $P_{8,3}$

Definition 1.3.22 The graph obtained by joining cycle C_m to a path P_n with a bridge called tadpole graph denoted by $T_{m,n}$. It is also known as dragon graph. (see Fig.21)

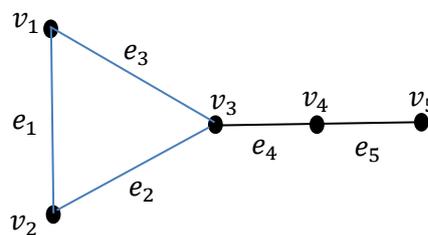


Figure 1.7 : Tadpole graph $T_{3,2}$

Definition 1.3.20 [5] A **cycle graph** is one that is obtained by joining the two end-vertices of a path graph. Thus, the degree of each vertex of a cycle graph is two. A cycle graph with n vertices is often denoted by C_n .

Definition 1.3.21 [7] A **wheel** W_n is the graph $C_n + K_1$. (see Fig. 1.5)

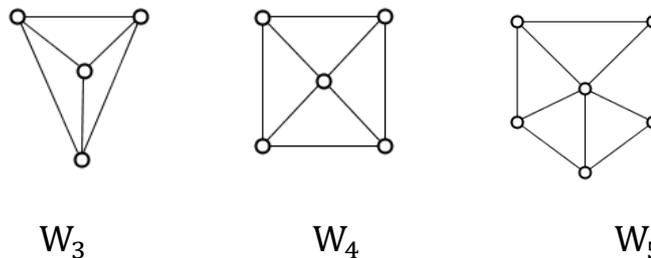


Figure 1.8: wheel graph W_3, W_4 and W_5 .

Definition 1.3.22 A *Cartesian product* denoted by $G = G_1 \times G_2$ of two graphs G_1 and G_2 has $V(G) = V(G_1) \times V(G_2)$ such that the vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if

(a) $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or

(b) $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$

Definition 1.3.23. The *corona* $(G_1 \odot G_2)$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 , where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2 .

Definition 1.3.24. A *grid graph* is obtained by a Cartesian product of two paths $P_{n_1} \times P_{n_2}$, $n_1, n_2 \geq 2$.

Definition 1.3.25. A *web graph* (W_m) is given by the Cartesian product graphs $C_m \times P_n$ where, C_m is a cycle with m vertices and P_n is a path with n vertices.

Definition 1.3.26. A *helm graph* (H_n) is obtained by attaching a single edge and vertex to each vertex of the outer circuit of a wheel graph W_n , the number of vertices of H_n is $2n + 1$ and the number of edges is $3n$.

Definition 1.3.27. A *tadpole graph* is also called *dragon graph* $(T_{m,n})$ which is obtained by joining a cycle graph C_m to a path graph P_n with a bridge.

Definition 1.3.28. A *lollipop graph* $(L_{m,n})$ is a graph obtained by joining a complete graph K_m to a path graph P_n with a bridge.

Definition 1.3.29 [49] Let C_n be a graph of order n , the vertex common $(n - 3)$ chords is called the *apex vertex*.

Definition 1.3.30 [49] A subgraph induced by apex vertex with end vertices of chords is called a *shell graph*.

Definition 1.3.31 [49] A *bow graph* is a double shell (comprises of two edge separate shells with a common apex) in which apiece shell has any order.

Definition 1.3.32 [49] A *butterfly graph* is a bow graph with precisely two pendant edges at the apex vertex and denoted that $BF(m,n)$.(as an example, see Figure1.6).

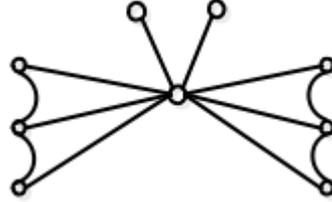


Figure 1.9: Butterfly graph $BF(3,3)$

Definition 1.3.33. A *barbell graph* ($B_{n,n}$) is obtained by connecting two copies of a complete graph K_n by a bridge.

Definition 1.3.34. A *dumbbell graph* ($D_b(n_1, n_2, k)$) is the graph of order $n = n_1 + n_2 + k$ obtained from the cycles C_{n_1} and C_{n_2} by joining a vertex of C_{n_1} to a vertex of C_{n_2} and subdividing the resulting edge k times.

Definition 1.3.35. A *windmill graph* (W_n^k) can be constructed by joining n copies of the complete graph K_k with a common vertex. It has $(k - 1)n + 1$ vertices and $nk(k - 1)/2$ edges.

Definition 1.3.36. A *daisies graph* $D(n_1, n_2, \dots, n_k)$ is k cycles have lengths n_1, n_2, \dots, n_k we denote the daisy graph.

Further, if $n_1 = n_2 = \dots = n_k$ then, $D(n_1, n_2, \dots, n_k)$ is simply as $D_k(n_1)$ and it is also known as, *Dutch windmill graph*.

Definition 1.3.37. A *Dutch windmill graph* (D_n^m) is a graph obtained by taking m copies of the cycle C_n with a vertex in common. Dutch windmill graph D_n^m contains $(n - 1)m + 1$ vertices and mn edges.

The Dutch windmill graph is also called as *friendship graph* if $n = 3$, i.e. friendship graph is the graph obtained by taking m copies of the cycle C_3 with a vertex in common.

All these types of definitions and more can be found in recent references [10, 14, and 16].

Definition 1.3.38 [8] A graph G is **connected** if for every pair u, v of vertices there exist a $u - v$ path, otherwise, G is **disconnected**.

Definition 1.3.39 [1] Let H be a graph with vertex set $V(H)$ and edge set $E(H)$, and similarly let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Then, we say that H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 1.3.40 [3] A **spanning subgraph** is a subgraph containing all the vertices of G .

Definition 1.3.41 [6] A subgraph H of G is said to be an **induced subgraph** of G if each edge of G having its ends in $V(H)$ is also an edge of H . Such H is denoted by $G[H]$ also denoted by $\langle H \rangle_G$ or simply by $\langle H \rangle$.

Example 1.3.42 : In (Fig. 1.7) we show types of subgraph of a graph G such that G_1 and G_2 are induced subgraph, so G_3 and G_4 are spanning subgraph and G_5 is subgraph but neither induced nor spanning subgraph.

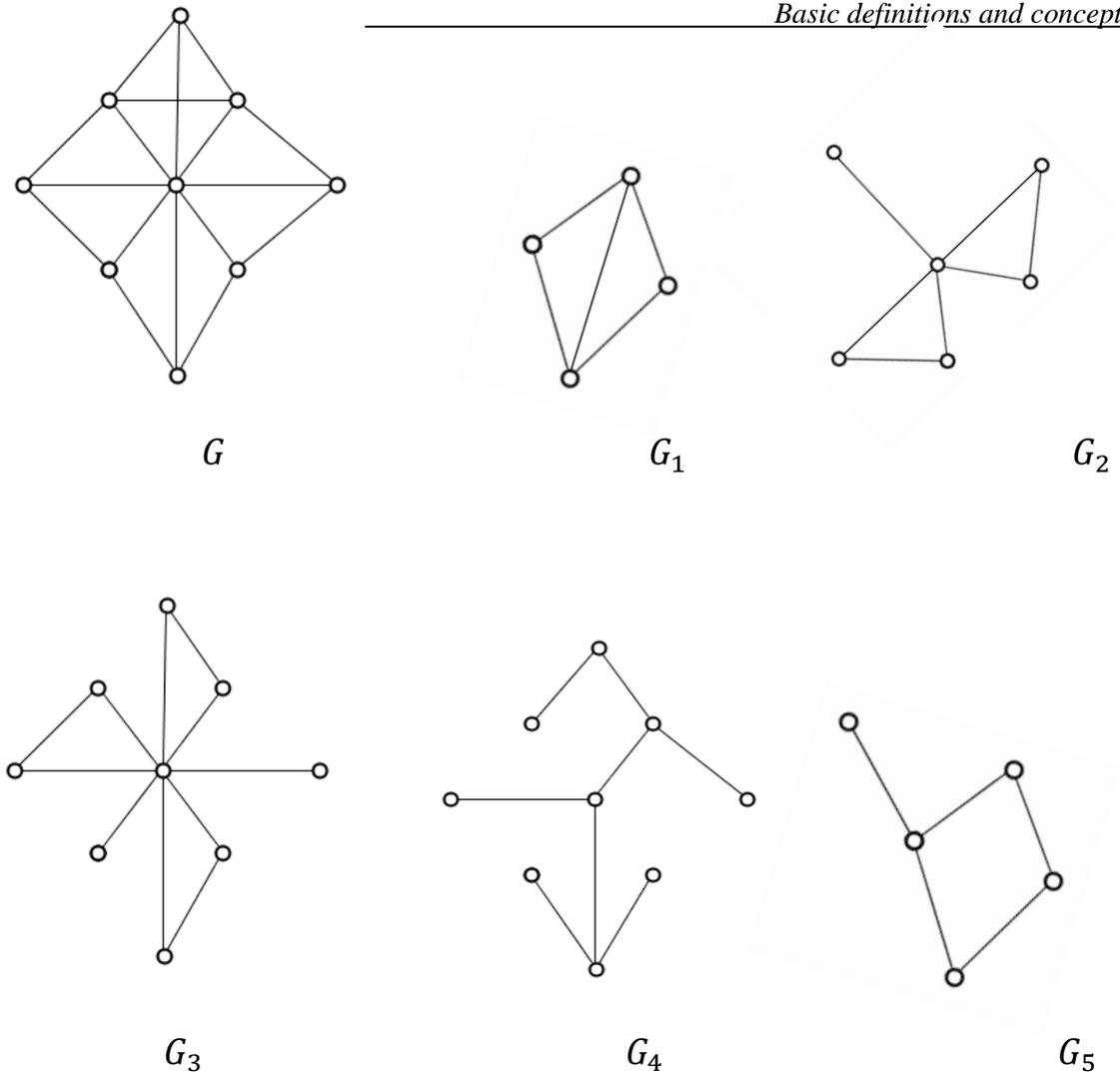


Figure 1.10: subgraph, induced subgraph and spanning subgraph.

Definition 1.3.43 [3] *The **complement** \bar{G} of a graph G also has $V(G)$ as its vertex set, but two vertices are adjacent in \bar{G} if and only if they are not adjacent in G . (for example, see fig. 1.8)*

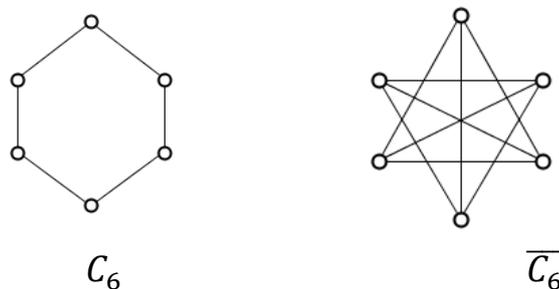


Figure 1.11: A cycle and complement of cycle.

Definition 1.3.44 [7] A graph G is **complete** if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n . (Fig. 1.9 show complete graph where $n = 1, \dots, 4$)

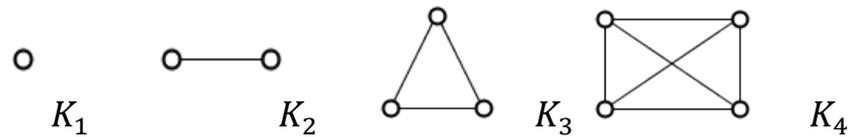


Figure 1.12: complete graph K_1, K_2, K_3 and K_4 .

Definition 1.3.45 [1] If $E = \emptyset$, in a graph $G(V, E)$ then such a graph without any edges is called a **null graph** and denoted by \overline{K}_n with n vertices. (An example can be seen in Fig. 1.10)

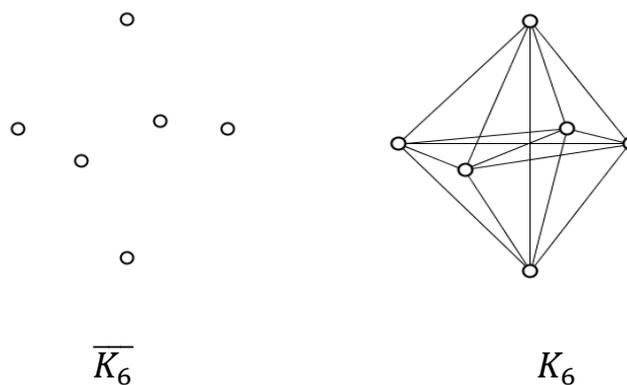


Figure 1.13: Null graph of complete.

Definition 1.3.46 [6] A connected graph without cycles is defined as a **tree**.

Definition 1.3.47 [3] A **bipartite** graph G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2 .

Definition 1.3.48 [5] A bipartite graph G is called **complete bipartite** graph if for each vertex $u \in V_1$ and each vertex $v \in V_2$, there is an edge (u, v) in G . Such denoted by $K_{m,n}$.

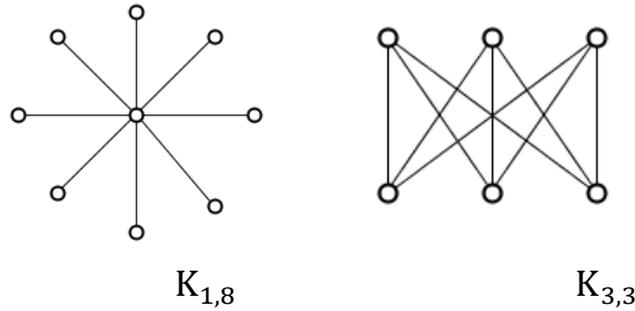


Figure 1.14: complete bipartite graph $K_{1,8}$ and $K_{3,3}$.

Definition 1.3.49 [2] A complete bipartite $K_{m,n}$ is called **star** if $K_{1,n-1}$, $n \geq 2$. In (Fig. 1.11), $K_{1,8}$ is star.

Definition 1.3.50 [4] Let $e = (a, b)$ be an edge of a graph. The operation called **contraction of the edge e** removes the edge from the graph and merges the vertices a, b to one new vertex. All the other edges incident to a or to b become incident to the new vertex.

Definition 1.3.51 [8] For a graph $G = (V, E)$, a set $S \subseteq V$ is **independent** if no two vertices in S are adjacent.

Definition 1.3.52 [7] A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is a **component** of G .

Definition 1.3.53 [5] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. **The union** of G_1 and G_2 , denoted by $G_1 \cup G_2$, is another graph $G = (V, E)$ whose vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. (see Fig. 1.12).

Definition 1.3.54 [8] The disjoint graphs G_1 and G_2 , **the join** $G = G_1 + G_2$ is a graph which has a vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv: u \in V(G_1), v \in V(G_2)\}$. (see Fig. 1.12).

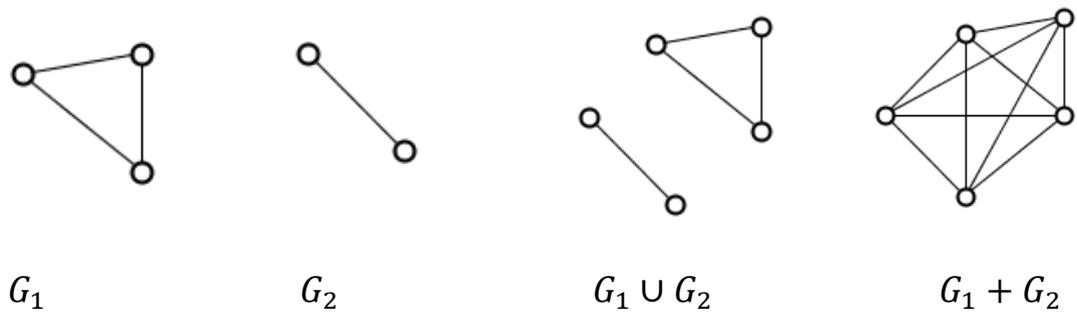


Figure 1.15: The union and join two graph G_1 and G_2 .

Definition 1.3.55 [1] A graph $G_1 = (V_1, E_1)$ is said to be **isomorphic** to the graph $G_2 = (V_2, E_2)$ if there is a one-to-one correspondence between the vertex sets V_1 and V_2 and a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1 is an edge with end vertices u_1 and u_2 in G_1 then the corresponding edge e_2 in G_2 has its end vertices v_1 and v_2 in G_2 which corresponds to u_1 and u_2 , respectively. Such a pair of correspondence is called a graph isomorphism and denoted by $(G_1 \cong G_2)$. (see Fig. 1.13).

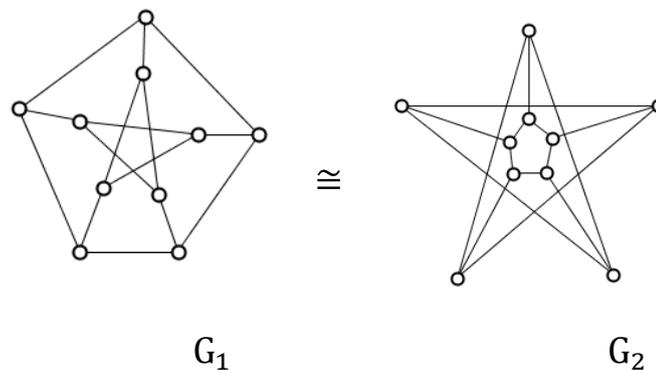


Figure 1.16: Two isomorphic graphs.

Definition 1.3.56 [5] Let e be an edge of G . We denote by $G - e$ the graph obtained by **deleting the edge e** from G . (see Fig. 1.14).

Definition 1.3.57 [6] Let u and v be vertices of a graph G and the edge $e = uv$ is an edge of complement graph \bar{G} . By $G + e$, we mean the graph obtained by **adding a new edge e** to G . (see Fig. 1.14).

Definition 1.3.58 [6] The subgraph $G - v$ is the graph obtained from G by deleting a vertex v . (see Fig. 1.14).

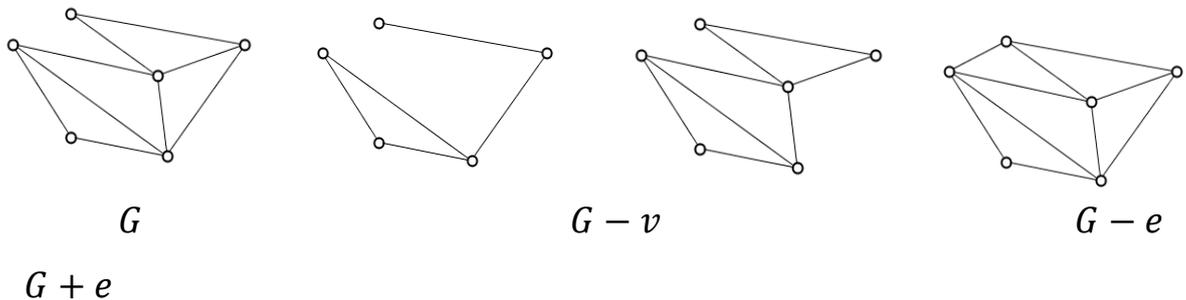


Figure 1.17: Deleting vertex and edge and adding edge for a graph G .

Definition 1.3.59. The edge removal from a graph is known as a **bridge** if it increases the number of components for the graph.

Definition 1.3.60 [7] The **distance** between u and v is the smallest length of any $u - v$ path in G and is denoted by $d_G(u, v)$ or simply $d(u, v)$.

Definition 1.3.61. The **diameter** of G denoted (**diam** (G)), is the maximum distance among all pairs of vertices in G .

Definition 1.3.62 [49] A subgraph induced by apex vertex with end vertices of chords is called a **shell** graph.

Definition 1.3.63 [49] A **jellyfish** graph is obtained from a cycle of order 4 with vertices $v_1, v_2, v_3,$ and v_4 by joining v_1 and v_3 with an edge and appending m pendant edges to v_2 and n pendant edges to v_4 and denoted that $J(m, n)$. (as an example, see Figure 1.15).

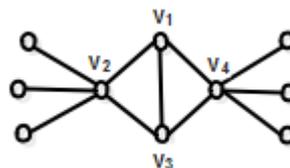


Figure 1.18: Jellyfish graph $J(3,3)$

Definition 1.3.64 [22] A **Ladder** graph of $2n$ vertices and $3n-2$ edges which is obtained by Cartesian product of P_2 and P_n and that is denoted that L_n . (as an example, see Figure 1.16).

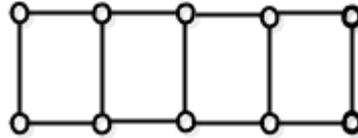


Figure 1.19: Ladder graph L_5

Definition 1.3.65 [47] A **Fan** graph is obtained from the join graphs $P_n + K_1$ and denoted that F_n . (as an example, see Figure 1.17).

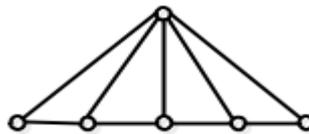


Figure 1.20: Fan graph F_5

Definition 1.3.66 [47] A **Double Fan** graph is obtained from the join graphs $P_n + \overline{K_2}$ and denoted that DF_n . (as an example, see Figure 1.18).

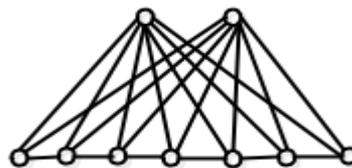


Figure 1.21: Double fan graph DF_7

Definition 1.3.67 [49] The **floor function** of a real number x is the greatest integer less than or equal to x and denoted that $\lfloor x \rfloor$.

Definition 1.3.68 [49] The **ceiling function** of a real number x is the least integer greater than or equal to x and denoted that $\lceil x \rceil$.

1.4 Hamiltonian graph

Definition 1.4.1 [8] A graph G is said to be **Hamiltonian graph** if it has a cycle containing all vertices of G .

Definition 1.4.2 [8] A cycle graph that containing all vertices in a graph G is called **cycle Hamiltonian**.

1.5 Domination in graph theory

Definition 1.5.1 [8] A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a **dominating set** if every vertex $v \in V$ is either an element of S or is adjacent to an element of S .

Definition 1.5.2 [8] A dominating set D is called a **minimal dominating set** if no proper subset $S \subset D$ is dominating set.

Definition 1.5.3 [8] The **domination number** $\gamma(G)$ of a graph G equals the minimum cardinality of a set of minimal dominating set of G . Such a set is called the γ -set of G .

Proposition 1.5.4

- (i) $\gamma(K_n) = \gamma(W_n) = 1$
- (ii) $\gamma(P_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$

1.6 Connected dominating set

Definition 1.6.1 [8] A dominating set D is called **connected dominating set** if induced subgraph $\langle D \rangle$ is connected.

Proposition 1.6.2

- (i) $\gamma_c(K_n) = 1$

$$(ii) \gamma_c(P_n) = \begin{cases} n - 2, & n \geq 3 \\ 1, & \text{other wise} \end{cases}$$

$$(iii) \gamma_c(C_n) = n - 2$$

$$(iv) \gamma_c(W_n) = 1$$

1.7 Total dominating set

Definition 1.7.1 [6] A subset D of the vertex set of a graph is called a **total dominating set** of G if any vertex v of G has a neighbor in D . (In other words, D dominates not only vertices outside D but also vertices in D).

Definition 1.7.2 [8] The **total domination number** of G equals the minimum cardinality of a total dominating set of G and denoted by $\gamma_t(G)$.

Proposition 1.7.3

$$(i) \gamma_t(K_n) = 2$$

$$(ii) \gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & n \equiv 0,3(\text{mod } 4) \\ \left\lceil \frac{n}{2} + 1 \right\rceil, & \text{other wise} \end{cases}$$

$$(iii) \gamma_t(W_n) = 2$$

1.8 Independent dominating set

Definition 1.8.1 [2] A set D is an **independent dominating set** if it is both independent and dominating set.

Definition 1.8.2 [2] The **independent domination number** $i(G)$ of G is the minimum cardinality among all independent dominating set of G .

Proposition 1.8.3

$$(i) \gamma_i(K_n) = 1$$

$$(ii) \gamma_i(P_n) = \gamma_i(C_n) = \left\lceil \frac{n}{3} \right\rceil$$

$$(iii) \gamma_i(W_n) = 1$$

1.9 Restrained domination number

Definition 1.9.1. A dominating set $D \subseteq V(G)$ is a **restrained dominating set** of G , where every vertex in $V - D$ is adjacent to a vertex in D as well as another vertex in $V - D$. The minimum cardinality of a restrained dominating set in G is called the **restrained domination number** of G and is denoted by $\gamma_r(G)$.

This concept was introduced by Domke, et al. in 1999 [8]. It is clear that $\gamma(G) \leq \gamma_r(G)$ (for examples, see Fig. 1.10).

The restrained domination number is given for some graphs as follow.

- (i) $\gamma_r(P_n) = n - 2 \left\lfloor \frac{n-1}{3} \right\rfloor, n \geq 4$
- (ii) $\gamma_r(C_n) = n - 2 \left\lfloor \frac{n}{3} \right\rfloor, n \geq 3$
- (iii) $\gamma_r(K_n) = 1$
- (iv) $\gamma_r(K_{n_1, n_2}) = 2, n_1, n_2 \geq 2$

1.10 Complementary tree domination number

Definition 1.10.1. A dominating set $D \subset V(G)$ is a **complementary tree dominating set** of G if the induced subgraph $G[V - D]$ is a tree. The minimum cardinality of a complementary tree dominating set in G is called the **complementary tree domination number** of G and is denoted by $\gamma_{ctd}(G)$.

This definition was displayed by Muthammai, et al. in 2011 [22]. It is clear that $\gamma(G) \leq \gamma_{ctd}(G)$ (see Fig. 1.10 (e)).

The complementary tree domination number for some graphs is given by

- (i) $\gamma_{ctd}(C_n) = n - 2, n \geq 3$
- (ii) $\gamma_{ctd}(K_n) = n - 2$
- (iii) $\gamma_{ctd}(K_{1, n}) = n, n \geq 2$

$$(iv) \gamma_{ctd}(W_n) = 2$$

1.11 Inverse dominating set

Definition 1.11.1 [10] *The set $V - D$ contains a dominating set in G with respect D . This set is called an inverse dominating set of D in G and denoted by D^{-1} . The **inverse domination number** of G , denoted by $\gamma^{-1}(G)$ is the minimum cardinality over all inverse bi-dominating sets in G .*

The inverse domination number is given for some graphs by

$$(i) \quad \gamma^{-1}(C_n) = \left\lceil \frac{n}{3} \right\rceil$$

$$(ii) \quad \gamma^{-1}(P_n) = \left\lceil \frac{(n+1)}{3} \right\rceil$$

$$(iii) \quad \gamma^{-1}(K_n) = 1$$

$$(iv) \quad \gamma^{-1}(W_n) = \left\lceil \frac{n-1}{3} \right\rceil$$

In Fig.1.19 some types of domination are illustrated in a graph.

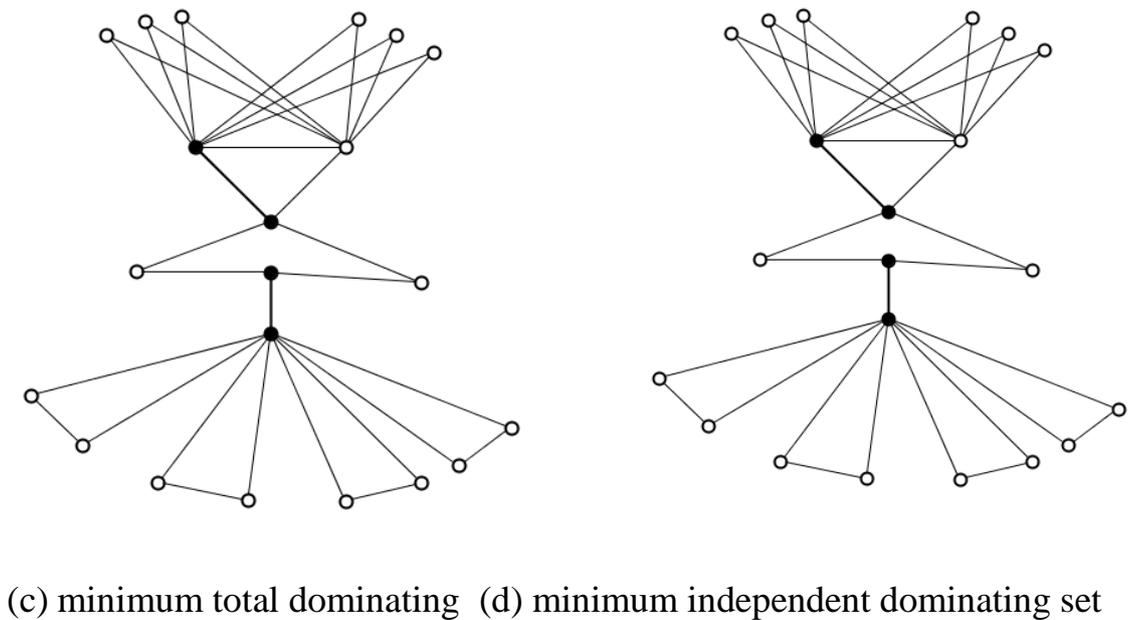
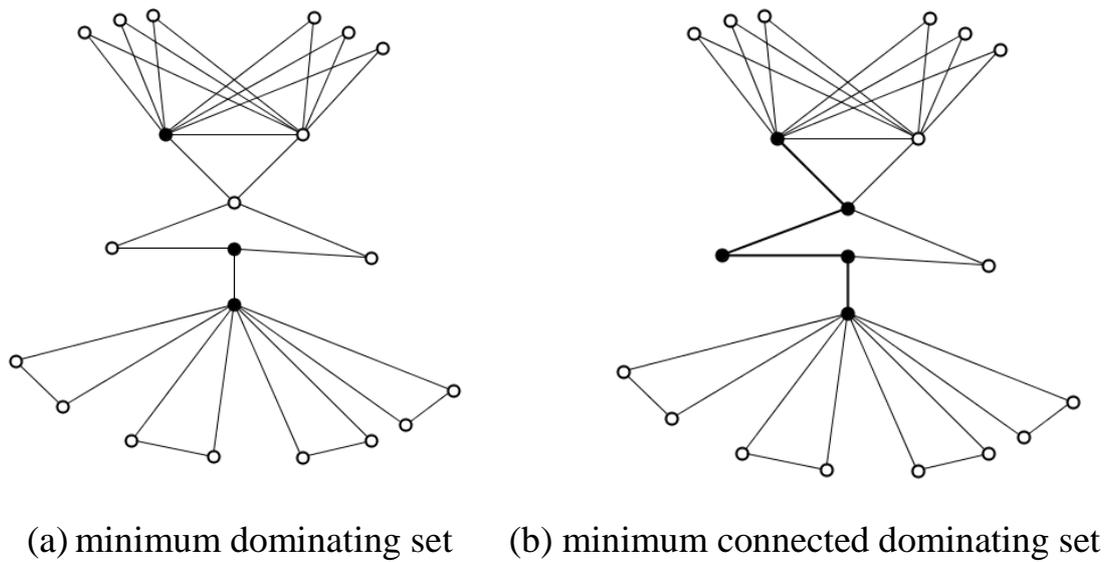


Figure 1.22: Some types of domination in a graph

1.12 Edge dominating set

Definition 1.12.1 [11] A set D_e of edges in a graph G is an *edge dominating set*, if every edge not in D_e is adjacent to at least one edge in D_e .

Definition 1.12.2 [11] An edge dominating set D_e is called a **minimal edge dominating set**, if for every edge e , $D_e - \{e\}$ is not an edge dominating set.

Definition 1.12.3 [11] The number of edges in a minimum edge dominating set is defined as the **edge domination number** of a graph G , and it is denoted by $\gamma_e(G)$.

- (i) For a complete graph K_n , $\gamma_e(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$.
- (ii) For a path P_n , $\gamma_e(P_n) = \left\lfloor \frac{n+1}{3} \right\rfloor$.
- (iii) For a cycle C_n and wheel W_n , $\gamma_e(C_n) = \left\lfloor \frac{n}{2} \right\rfloor = \gamma_e(W_n)$.
- (iv) For a complete bipartite graph $K_{m,n}$, $\gamma_e(K_{m,n}) = \min(m, n)$.

1.13 Inverse edge dominating set

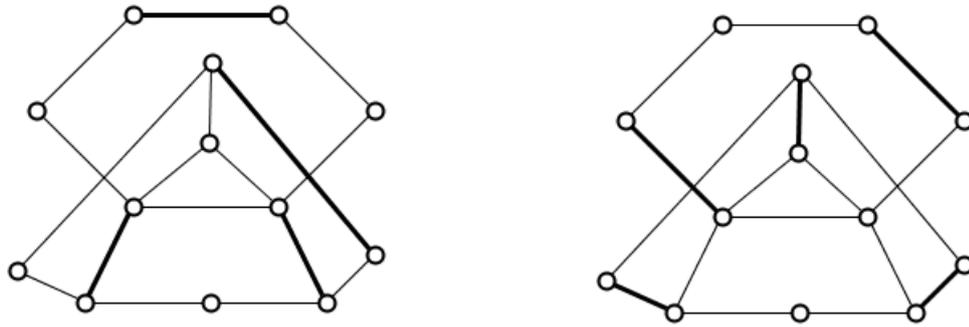
Definition 1.13.1 [11] Let D_e be a minimum edge dominating set in a graph G . If $E - D_e$ contains an edge dominating set D_e^{-1} of G , then D_e^{-1} is called an **inverse edge dominating set** of G with respect to D_e .

Definition 1.13.2 [11] The number of edges in a minimum inverse edge dominating set is defined as the **inverse edge domination number** of a graph G , and it is denoted by $\gamma_e^{-1}(G)$.

Observation 1.13.3 [9]

- (i) For a complete graph K_n , $\gamma_e^{-1}(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$ if $n \geq 3$.
- (ii) For a path P_n , $\gamma_e^{-1}(P_n) = \left\lfloor \frac{n}{3} \right\rfloor$ if $n \geq 3$.
- (iii) For a cycle C_n , $\gamma_e^{-1}(C_n) = \left\lfloor \frac{n}{3} \right\rfloor$
- (iv) For a wheel W_n , $\gamma_e^{-1}(W_n) = \left\lfloor \frac{n}{3} \right\rfloor$ if $n \geq 4$.

In (Fig. 1.16) we show edge dominating set and inverse edge dominating set.



(a) minimum edge dominating set (b) minimum inverse edge dominating set

Figure 1.23: Edge dominating set and its inverse.

1.14 Algorithms

Informally, an algorithm is any well-defined computational procedure that takes some value, or set of values, as input and produces some value, or set of values, as output. An algorithm is thus a sequence of computational steps that transform the input into the output.

We can also view an algorithm as a tool for solving a well-specified computational problem. The statement of the problem specifies in general terms the desired input/output relationship. The algorithm describes a specific computational procedure for achieving that input/output relationship.

1.14.1 Algorithm example

One of the simplest algorithms is to find the largest number in a set (list) of numbers of random order. Finding the solution requires looking at every number in the list. From this follows a simple algorithm, which can be stated in a high-level description, as:

High-level description:

1. If there are no numbers in the set then there is no largest number.
2. Assume the first number in the set is the largest number in the list.

3. For each remaining number in the set: if this number is larger than the current largest number, set this number to be the largest number in the list.
4. When there are no numbers left in the list to iterate over, set the current largest number to be the largest number of the list.

Algorithm **LargestNumber**

Input: A list of numbers D

1. if $D.size = 0$ return null;
2. $LN := D[0]$;
3. for each item in D , do;
4. if item $>$ LN , then;
5. $LN := item$;

Output: LN

Definition 1.14.1 [53] *An **instance** is input sequence of a problem. In general, an **instance of a problem** consists of the input (satisfying whatever constraints are imposed in the problem statement) needed to compute a solution to the problem.*

1.14.1 Minimum Spanning Trees

Electronic circuit designs often need to make the pins of several components electrically equivalent by wiring them together. To interconnect a set of n pins, we can use an arrangement of $n - 1$ wires, each connecting two pins. Of all such arrangements, the one that uses the least amount of wire is usually the most desirable.

We can model this wiring problem with a connected, undirected graph $G = (V, E)$ where V is the set of pins, E is the set of possible interconnections between pairs of pins, and for each edge $(u, v) \in E$, we have a weight $w(u, v)$ specifying the cost (amount of wire needed) to

connect u and v . We then wish to find an acyclic subset $T \subseteq E$ that connects all of the vertices and whose total weight

$$w(T) = \sum_{(u,v) \in T} w(u,v)$$

is minimized. Since T is acyclic and connects all of the vertices, it must form a tree, which we call a spanning tree since it “spans” the graph G . The problem of determining the tree T called the minimum-spanning-tree problem.

1.15 Hard Problems

Most of this thesis is about efficient algorithms. The usual measure of efficiency is speed, i.e., how long an algorithm takes to produce its result. There are some problems, however, for which no efficient solution is known, which are known as NP-complete. Like CIRCUIT-SAT, SAT, 3-CNF-SAT, CLIQUE, VERTEX-COVER, HAM-CYCLE, TSP problems.

All the algorithms we study are polynomial-time algorithms: on inputs of size n , their worst-case running time is $O(n^k)$ for some constant k .

Chapter Two

Even Sum Domination in Graphs and some Algorithms

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Even Sum Domination in Graphs and some Algorithms

2.1 Introduction

Domination is one of the modern international standards that have entered all sciences and research fields. The first to demonstrate this concept is Claude Berge in 1962 [6]. After that, this concept began to appear in multiple forms and formats to treat many life problems. In mathematics, this concept appeared in many fields including fuzzy graph [17-18], topological graph [11-12], labeled graph [3-4], and others. Moreover, recently, many general definitions of this concept have appeared, which depend on certain conditions placed on the dominating set or outside the dominating set or both as in [1-2], [5], and [15-16]. In this work the new parameter of domination is initiated it is called even sum domination. This definition depends on the life problems in which it is possible to classify the elements of the set into two classes so that each class is dominated by the same type of that class. In this work, the important Theorems, properties, observation, and proposition are been discussed with giving algorithms. Moreover, this number was calculated for certain graphs as a path, cycle, complete, complete bipartite, null, wheel, and star. Furthermore, this number was calculated for the complement of certain graphs mentioned above. This concept of inverse domination was first introduced by Kulli and Sigarkanti [13], after this appeared many papers deal with a new parameter of this concept as in [14]. Again, in this work, the new concept called the inverse even sum domination is introduced and determined this concept to certain graphs too. Finally, It has been proved that the vertex cover problem is NP-complete [8]. Thus, in this paper, the NP-complete for ES domination

problem is proved by reduction from the vertex cover problem to the ES domination problem. In terms of the complexity of the ES domination problem in graphs, also, a linear-time algorithm for the ES domination problem in wheels is been written. For more details see [7], and [9-10].

2.2 BASIC CONCEPTS

Definition 2.2.1 Let G be a graph and D is a dominating set, the set D is called even sum dominating set (ESDS) if, for each $u \in V - D$ there is a vertex $v \in D$ adjacent to u such that $\deg(v) + \deg(u)$ is even.

Definition 2.2.2 Let D be a ESDS in a graph G and has no proper even sum dominating set, then D is called a minimal ESDS. The minimum cardinality of all minimal ESDSs is called the even sum domination number and denoted by $\gamma_{es}(G)$.

If D is even sum dominating set with minimum cardinality, then D is called γ_{es} -set.

Definition 2.2.3. Let D be a γ_{es} -set. in a graph G , then if there is a ESDS D_1 such that $D \cap D_1 = \emptyset$, then the set is an inverse ESDS with respect to the set D , denoted by D^{-1} . The minimum cardinality of these sets is called the inverse even sum domination number and denoted by $\gamma_{es}^{-1}(G)$.

Example

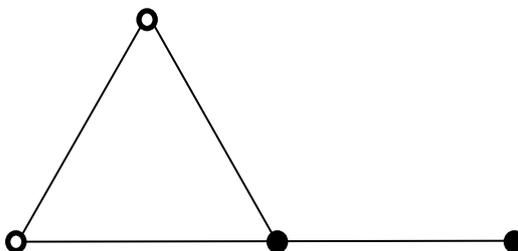


Figure 2.1: A graph has ESDS the black vertices.

Observation 2.2.4. Let G be a graph of order n and D is a γ_{es} -set, then

- 1) Every isolated vertex in graph G belongs to every even sum dominating set.
- 2) If there is a vertex u in the set $V - D$ has an even (odd) degree, then there is at least one vertex v in D has an even (odd) degree adjacent to u .
- 3) Every vertex has an odd (even) degree not adjacent to every vertex has an odd (even) degree belongs to every ESDS.
- 4) $\gamma_{es}(G) \geq \gamma(G)$.

Proposition 2.2.5. Let G be a regular graph, then $\gamma_{es}(G) = \gamma(G)$.

Proof. Let G be a regular graph, then all vertices of this graph have the same degree say r . Thus, the summation of the degree of each two vertices is even, so the result is obtained.

Corollary 2.2.6.

- 1) $\gamma_{es}(K_n) = \gamma(K_n) = 1$
- 2) $\gamma_{es}(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$
- 3) $\gamma_{es}(N_n) = \gamma(N_n) = n$

Proposition 2.2.7. Let G be a path of order n , so

$$\gamma_{es}(G) = \begin{cases} 1, & \text{if } n = 2 \\ 2 + \left\lceil \frac{n-2}{3} \right\rceil, & \text{if } n \geq 3 \end{cases}$$

Proof. Let the vertices of the path are $\{v_1, v_2, \dots, v_n\}$. There are three cases that depend on the order of the path as follows.

Case 1. If $n = 2$, then the result is clear.

Case 2. There are two subcases as follows.

Subcase 1. If $n = 3$, then all vertices belong to the even sum dominating set since there is no vertex of degree even(odd) adjacent to the other vertex of degree even (odd).

Subcase 2. If $n \geq 4$, then the by the same manner in case 2, the two vertices $\{v_1, v_n\}$ belong to every ESDS and these vertices not dominate the adjacent vertices $\{v_2, v_{n-1}\}$, since the two vertices $\{v_1, v_n\}$ have the odd degree and the vertices $\{v_2, v_{n-1}\}$ have even degree. Thus, the remain vertices that not dominated by the vertices $\{v_1, v_n\}$ are $\{v_2, v_3, \dots, v_{n-1}\}$. Therefore there are two subcases in this step as follows.

I. If $n \equiv 0 \pmod{3}$, then let $D_1 = \{v_{3i+3}, i = 0, 1, \dots, \lfloor \frac{n-2}{3} \rfloor - 2\} \cup \{v_{n-1}\}$, it is clear that the set D_1 is minimum ESDS to the subgraph induced by the vertices $\{v_2, v_3, \dots, v_{n-1}\}$. Thus, the set $D = D_1 \cup \{v_1, v_n\}$ is γ_{es} -set.

II. If $n \equiv 1, 2 \pmod{3}$, then let $D_1 = \{v_{3i+3}, i = 0, 1, \dots, \lfloor \frac{n-2}{3} \rfloor - 1\}$, by the same manner, the set $D = D_1 \cup \{v_1, v_n\}$ is γ_{es} -set.

From all cases above, the proof is done.

Algorithm 2.2.7. ESDS-PATH (G is a path of order n)

Input: A set of vertices $V = \{v_1, v_2, \dots, v_n\}$

1. ESDS: = \emptyset ;
2. Case 1: $n = 2$;
3. ESDS: = $\{v_1\}$;
4. Case 2: $n = 3$;
5. ESDS: = $\{v_1, v_2, v_3\}$;
6. Case 2: $n > 3$;

7. $ESDS := \{v_1, v_n\};$
8. For $i := 3$ to $n - 1$ step 2, do;
9. $ESDS := ESDS \cup \{v_i\};$
10. end for;

Output: $ESDS$

Proposition 2.2.8. *If G is a complete bipartite graph $K_{m,n}$, then*

$$\gamma_{es}(G) = \left\{ \begin{array}{l} 1, \quad \text{if } n = 1 \text{ and } m \text{ is odd or vice versa} \\ 2, \quad \text{if } n \text{ and } m \text{ are both even or odd, } n, m \geq 2 \\ n + m, \quad \text{if } n \text{ is odd and } m \text{ is even or vice versa} \end{array} \right\}.$$

Proof. Suppose that $V_1 = \{u_i, i = 1, 2, \dots, n\}$ and $V_2 = \{v_j, j = 1, 2, \dots, m\}$ are the bipartite sets of the graph G . Then three different cases are obtained as follows.

Case 1. If $n = 1$ and m is odd, then one can easily conclude that $D = \{u_1\}$ is γ_{es} -set.

Thus, $\gamma_{es}(G) = 1$.

Case 2. If n and m are both even, then it is obvious that let $D = \{u_1, v_1\}$ is γ_{es} -set. Similarly, if n and m are both odd. Thus, $\gamma_{es}(G) = 2$.

Case 3. If n is odd and m is even, then each vertex in the graph G belongs to the $ESDS$, since if that vertex belongs to the set V_1 , then this vertex has an odd degree and it is not adjacent to any other vertex of odd degree. Again, if that vertex belongs to the set V_2 , then this vertex has an even degree and it is not adjacent to any other vertex of even degree. Similar proof if n is even and m is odd. Thus, $\gamma_{es}(G) = n + m$.

From all cases above, the proof is done. □

Algorithm 2.2.8. *ESDS- complete bipartite (G is a complete bipartite $K_{m,n}$)*

Input: A graph $G = (V, E)$

1. $ESDS := \emptyset$;
2. $X := \{v_1, v_2, \dots, v_n\}$;
3. $Y := \{u_1, u_2, \dots, u_m\}$;
4. $V := X \cup Y; X \cap Y = \emptyset$;
5. case 1: $n = 1$ and m is odd
6. $ESDS := \{v_1\}$;
7. case 2: n and m both even or both odd
8. $ESDS := \{v_1, u_1\}$;
9. case 3: n even and m odd or m even and n odd
10. $ESDS := V$;

Output: $ESDS$

Proof. According to the proof of proposition 2.2.8 the obtained set is an ESDS. \square

Corollary 2.2.9. *If G is a star graph $S_n \equiv K_{1,n-1}$, then $\gamma_{es}(S_n) = 1$, if n is even.*

Proposition 2.2.10. *If G is a wheel graph $W_n \equiv K_1 + C_{n-1}$, then*

$$\gamma_{es}(G) = \begin{cases} 1, & \text{if } n \text{ is even} \\ 1 + \left\lceil \frac{n-1}{3} \right\rceil, & \text{if } n \text{ is odd} \end{cases}.$$

Proof. Let the vertex set of the wheel graph is $\{v_1, v_2, \dots, v_n\}$ where the vertex v_1 is the center of the wheel ($K_1 = \{v_1\}$), then two cases are obtained as follows.

Case 1. If n is even, then the center vertex has an odd degree and all other vertices have degree equal to three. Thus, the center vertex dominates all other vertices, therefore $\gamma_{es}(G) = 1$.

Case 2. If n is odd, then the center vertex has even degree and all other vertices have degree equal to three. So, the center vertex belongs to every ESDS, since it is not adjacent to any vertex has an even degree, according to Observation 2.3(3). The induced subgraph of remained vertices constitutes a cycle of order $n-1$. Therefore, $\gamma_{es}(G) = 1 + \left\lfloor \frac{n-1}{3} \right\rfloor$ according to the Corollary 2.5(2). From all cases above, the proof is done . \square

Algorithm 2.2.10. ESDS-WHEEL (G is a wheel of order n , v_1 is the center vertex)

Input: A graph $G = (V, E)$

1. $ESDS := \emptyset$;
2. case 1: n is even
3. $ESDS := \{v_1\}$;
4. case 2: n is odd
5. For $i := 2$ to $n - 1$ step 3
6. $ESDS := ESDS \cup \{v_i\}$;
7. end for;
8. $ESDS := ESDS \cup \{v_1\}$;

Output: $ESDS$.

Proof. According to the proof of proposition 2.13 the obtained set is an ESDS. \square

2.3 THE COMPLEMENT OF CERTAIN GRAPH

Observation 2.3.1

$$1) \gamma_{es}(\overline{K_n}) = \gamma_{es}(N_n) = n.$$

$$2) \gamma_{es}(\overline{N_n}) = \gamma_{es}(K_n) = 1.$$

Proposition 2.3.2. *If G is a cycle of order n , then*

$$\gamma_{es}(\overline{C_n}) = \begin{cases} 3, & \text{if } n = 3 \\ 2, & \text{otherwise} \end{cases}$$

Proof. *There are two different cases as follows.*

Case 1. *If $n = 3$, then $\gamma_{es}(\overline{C_3}) = \gamma_{es}(N_3) = 3$ according to Corollary 2.5 (3).*

Case 2. *If $n \geq 4$, then three different subcases are obtained.*

Subcase 1. *If $n = 4$, then $\overline{C_4} \equiv K_2 \cup K_2$, then $\gamma_{es}(\overline{C_4}) = 2$, according to Corollary 2.5 (2).*

Subcase 2. *If $n = 5$, then $\overline{C_5} \equiv C_5$ that means C_5 is self-complementary. Again, $\gamma_{es}(\overline{C_5}) = \gamma_{es}(C_5) = 2$, according to Corollary 2.5 (2).*

Subcase 3. *If $n \geq 6$, then the graph $\overline{C_n}$ is $(n-3)$ -regular graph and each vertex v_i in this graph is adjacent to all vertices except the two vertices that adjacent to it in the graph C_n . Let $D = \{v_i, v_j\}$ where the vertex v_i is adjacent to the vertex v_j in the graph C_n (as an example, see Figure 2.2). One can easily conclude that the set D is γ_{es} -set, therefore $\gamma_{es}(\overline{C_n}) = 2$. From each cases above, the proof is done. \square*

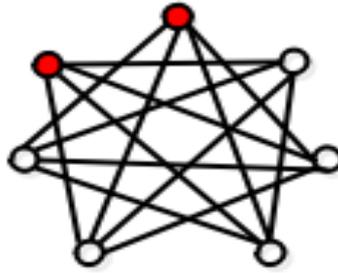


Figure 2.2. Even sum domination of the complement of C_7 .

Algorithm 2.3.2. *ESDS- complement cycle of order n (G is a complete cycle $\overline{C_n}$)*

Input: A graph $G = (V, E)$

1. $ESDS := \emptyset$;
2. case 1: $n = 3$
3. $ESDS := \{v_1, v_2, v_3\}$;
4. case 2: $n > 3$
5. $ESDS := \{v_1, v_2\}$;

Output: $ESDS$

Proposition 2.3.3. *If G is a complete bipartite graph $K_{m,n}$, then $\gamma_{es}(\overline{K_{m,n}}) = 2$.*

Proof. *It is clear that $\overline{K_{m,n}} \equiv K_m \cup K_n$, then $\gamma_{es}(\overline{K_{m,n}}) = 2$, according to Corollary 2.5 (1). \square*

Note that $\gamma_{es}(\overline{S_n}) = \gamma_{es}(\overline{K_{1,n-1}}) = 2$.

Proposition 2.3.4. *If G is a wheel graph $W_n \equiv K_1 + C_{n-1}$, then*

$$\gamma_{es}(\overline{W_n}) = \begin{cases} 4, & \text{if } n = 4 \\ 3, & \text{otherwise} \end{cases}$$

Proof. The graph $\overline{W}_n \equiv K_1 \cup \overline{C}_{n-1}$, therefore the result is obtained according to the to Corollary 2.5 (1) and Proposition 3.2.

□

Proposition 2.3.5. If G is a path graph P_n , then $\gamma_{es}(\overline{P}_n) = \begin{cases} 2, & \text{if } n = 2, 3 \\ 3, & \text{if } n \geq 4 \end{cases}$.

Proof. There are two different cases as follows.

Case 1. There are two subcases.

Subcase 1. If $n = 2$, then $\overline{P}_2 \equiv K_1 \cup K_1$, so $\gamma_{es}(\overline{P}_2) = 2$.

Subcase 2. If $n = 3$, then $\overline{P}_3 \equiv P_2 \cup K_1$, so $\gamma_{es}(\overline{P}_3) = 2$.

Case 2. There are three different cases as follows.

Subcase 1. If $n = 4$, then $\overline{P}_4 \equiv P_4$, so $\gamma_{es}(\overline{P}_4) = 3$, according to Theorem 2.10.

Subcase 2. If $n = 5$, then the two vertices $\{v_1, v_5\}$ have odd degrees and they are adjacent, so one of them is be chosen in the ESDS. The other vertices have even degree, two of them are adjacent that are $\{v_2, v_4\}$, again choose one of them to add to the ESDS. The vertex v_3 has even degree but not adjacent to any other vertex of an even degree, so add this vertex to the ESDS. Thus, $\gamma_{es}(\overline{P}_5) = 3$.

Subcase 4. If n is even and $n \geq 6$, then in \overline{P}_n the vertices $\{v_1, v_n\}$ have odd degrees and adjacent, so one of them adds to the ESDS. The other vertices have even degrees and each vertex of them is adjacent to all other vertices except the vertices that adjacent to it in P_n . Therefore, two adjacent vertices of even degree are be chosen in the ESDS to dominate all vertices of even degree. Thus, $\gamma_{es}(\overline{P}_n) = 3$.

Subcase 5. If n is odd and $n \geq 7$, then by the similar technique in Subcase 4, the result is obtained. From all cases above, the proof is done. □

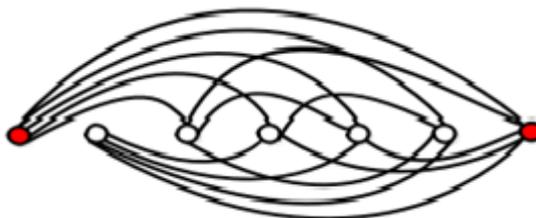


Figure 2.3. Even sum domination of the complement of P_7 .

2.4 THE INVERSE OF CERTAIN GRAPH

Proposition 2.4.1 *If G is a graph, then G has no inverse ESDS if G contains a vertex of degree zero or an odd (even) degree and not adjacent to each vertex has an odd (even) degree.*

Proof. *It is straightforward from Observation 2.3.*

□

Observation 2.4.2.

- 1) *The path graph has no inverse ESDS.*
- 2) *The null graph has no inverse ESDS.*

Observation 2.4.3.

- 1) $\gamma^{-1}(K_n) = 1, \quad n \geq 2.$
- 2) $\gamma^{-1}(C_n) = \left\lfloor \frac{n}{3} \right\rfloor.$

Proposition 2.4.4. *If G is a complete bipartite graph $K_{m,n}$, then*

$$\gamma_{es}^{-1}(K_{m,n}) = \left\{ \begin{array}{ll} m, & \text{if } n = 1 \text{ and } m \text{ is odd} \\ 2, & \text{if } n \text{ and } m \text{ are both even or odd, } n, m \geq 2 \end{array} \right\} \text{ and}$$

G has no inverse ESDS if n is odd and m is even or vice versa.

Proof. Suppose that $V_1 = \{u_i, i = 1, 2, \dots, n\}$ and $V_2 = \{v_j, j = 1, 2, \dots, m\}$ are the bipartite sets of the graph G . Then three different cases are obtained as follows.

Case 1. If $n = 1$ and m is odd, then $D = \{u_1\}$ is γ_{es} -set according to Proposition 2.11. Thus, the other ES DS is the set V_2 . Therefore, $\gamma_{es}^{-1}(K_{m,1}) = m$.

Case 2. If n and m are both even or odd, then let $D = \{u_1, v_1\}$ is γ_{es} -set by Proposition 2.11. Therefore let $D^{-1} = \{u_2, v_2\}$. Thus, $\gamma_{es}(G) = 2$.

Case 3. If n is odd (even) and m is even (odd), then the ES DS contains all vertices of the two sets V_1 and V_2 . Thus, it is clear that there is no disjoint ES DS and the result is obtained.

From all cases above, the proof is done. □

Not that $\gamma_{es}^{-1}(S_n) = \gamma^{-1}(K_{1,n-1}) = n - 1$.

Proposition 2.4.5. If G is a wheel graph $W_n \equiv K_1 + C_{n-1}$, then

$\gamma_{es}^{-1}(W_n) = \left\lceil \frac{n-1}{3} \right\rceil$, if n is even and has no inverse ES DS if n is odd.

Proof. Let the vertex set of the wheel graph is $\{v_1, v_2, \dots, v_n\}$ where the vertex v_1 is the center of the wheel ($K_1 = \{v_1\}$), then two cases are obtained as follows.

Case 1. If n is even, then $D = \{v_1\}$, so any disjoint ES DS does not contains the vertex v_1 . Thus the remaining vertices constitute a cycle of order $n-1$, so the vertices that dominate this cycle is already dominates the center vertex v_1 . Thus, $\gamma_{es}^{-1}(W_n) = \left\lceil \frac{n-1}{3} \right\rceil$ by using Corollary 2.5.

Case 2. If n is odd, then the center vertex have even degree and all other vertices has degree equal to three. Thus, the center vertex belongs to each

ESDS. Thus, there are no disjoint ES DS, and the result is obtained. From all cases above, the proof is done.

2.5 NP-COMPLETENESS OF ES DOMINATION PROBLEM

We need a well-known NP-complete problem, such as the vertex cover problem, which is defined as follows.

VERTEX COVER

INSTANCE: A nontrivial graph $G = (V, E)$ and a positive integer k

QUESTION: Does G have a vertex subset of cardinal at most k such that each edge of the graph has at least one end vertex in this set?

Proposition 2.5.1 [8] *The vertex cover problem is NP-complete.*

We defined the ES DOMINATION problem as follows.

ES DOMINATION

INSTANCE: A nontrivial graph $G = (V, E)$ and a positive integer k

QUESTION: Does G have an ES dominating set of cardinal at most k ?

Proposition 2.5.2. *The ES domination problem is NP-complete.*

Proof. We must do two things. First, we must show that ES DOMINATION \in NP. This is easy to do since for $G = (V, E)$, a positive integer k , and an arbitrary subset $S \subseteq V$ with $|S| \leq k$, easy to verify in polynomial time whether S is an ES DS of G . Hence the ES domination problem is in NP.

Second, we must construct a reduction from VERTEX COVER to ES DOMINATION. Given a nontrivial not regular graph $G = (V, E)$, (for regular graphs the two problems are isomorphic, according to proposition 4.2) where $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$, suppose that G has exactly t even vertex and exactly s odd vertex ($t + s = n$). Construct a graph $G' = (V', E')$ with vertex set $V' = V \cup V_t$ and edge set $E' = E \cup E_t$,

where $V_t = \{v'_1, v'_2, \dots, v'_t\}$ (additional t pendant vertices) and $E' = \{e'_1, e'_2, \dots, e'_t\}$ such that e'_i ($i = 1, 2, \dots, t$) incident to v'_i and the end vertex of even degree in G . Clear that this construction has polynomial time. All vertices of G' of odd degree so, G has a vertex cover of size at most k if and only if G' has an ES DS of size at most k . By Proposition 5.1, we conclude that the ES domination problem is NP-complete.

Chapter Three

Corona of the Cycle Graph with Some Certain Graphs

Chapter Three

Corona of the Cycle Graph with Some Certain Graphs

3.1 INTRODUCTION

Many life problems, scientific applications, engineering project management, and the distribution of power all over the world need control sites that regulate the work of all of the above. So when domination in graphs appeared, it took a wide field in all applications. The first person who introduced the domination in graphs is Claude Berge in 1962 [4]. This concept took an important place in all types of sciences such as chemistry, physics, biology, engineering, and others. Therefore, a lot of research has appeared recently on this concept that addresses the problems of these sciences.

Especially in mathematics, this concept has entered many fields of mathematics as topological graph [8], fuzzy graph [15, 16], and others. Also, many new formulas have appeared for this concept that are consistent with the terms of the problem as in [1-3] and [10-14]. In the research, the researchers Rasheed and Omran presented a new definition of this concept, which finds optimal solutions for a society that consists of two classes, where each class is controlled from within the same class which is called even sum domination, the dominating set $D \subseteq V$ is called even sum dominating set (ESDS) if, $\forall u \in V - D$ there is a vertex $v \in D$ such that $\deg(v) + \deg(u)$ is even. In this research, the authors pay attention to the operation of two graphs, which is the corona, whereby it takes one copy of the first graph and copies of the second graph as many as the number of vertices of the first graph and then join each vertex of the first graph with all the vertices of one copy from the second graph. Moreover, this number was determined to find the corona operation of a

cycle graph and one of the certain graphs as a cycle, path, null, complete, complete bipartite, star, and wheel. For more details see [5, 6, 7], and [9].

3.2 THE MAIN RESULTS

Theorem 3.2.1. *If G_1 and G_2 are two cycle graphs of order n and m respectively, then*

$$\gamma_{es}(C_n \odot C_m) = \begin{cases} n, & \text{if } m \text{ is odd} \\ n \lfloor \frac{m}{3} \rfloor + \lfloor \frac{n}{3} \rfloor, & \text{if } m \text{ is even} \end{cases}$$

Proof. *To obtain all cases in the graph $C_n \odot C_m$, there are two different cases.*

Case 1) *If m is odd, then the vertices of the cycle C_n have an odd degree in the graph $C_n \odot C_m$ since each vertex in it say v , $\deg(v)$ adjacent to two vertices the vertices of the cycle C_n and with it adjacent to all vertices in a copy of cycle C_m (for an example, see Fig.3.1 (a)). On another hand, all vertices in each copy of cycle C_m have an odd degree. Thus, the vertices of the cycle C_n dominate all vertices in the graph $C_n \odot C_m$, so, $\gamma_{es}(C_n \odot C_m) = n$.*

Case 2) *If m is even, then all vertices in the vertices of the cycle C_n have even degree, one can do that with the same technique that was followed in the previous case (for an example, see Fig.3.1 (b)). Again, on another hand, all vertices in each copy of cycle C_m have odd degrees. Thus, the vertices of the cycle C_n do not dominate the vertices of each copy of cycle C_m . Therefore, each copy of cycle C_m is dominated themselves by its vertices which equal to $\lfloor \frac{m}{3} \rfloor$. Also, the vertices of the cycle C_n is dominated by $\lfloor \frac{n}{3} \rfloor$ from its vertices. Thus, $\gamma_{es}(C_n \odot C_m) = n \lfloor \frac{m}{3} \rfloor + \lfloor \frac{n}{3} \rfloor$.*

Based to the above, the required is obtained. \square

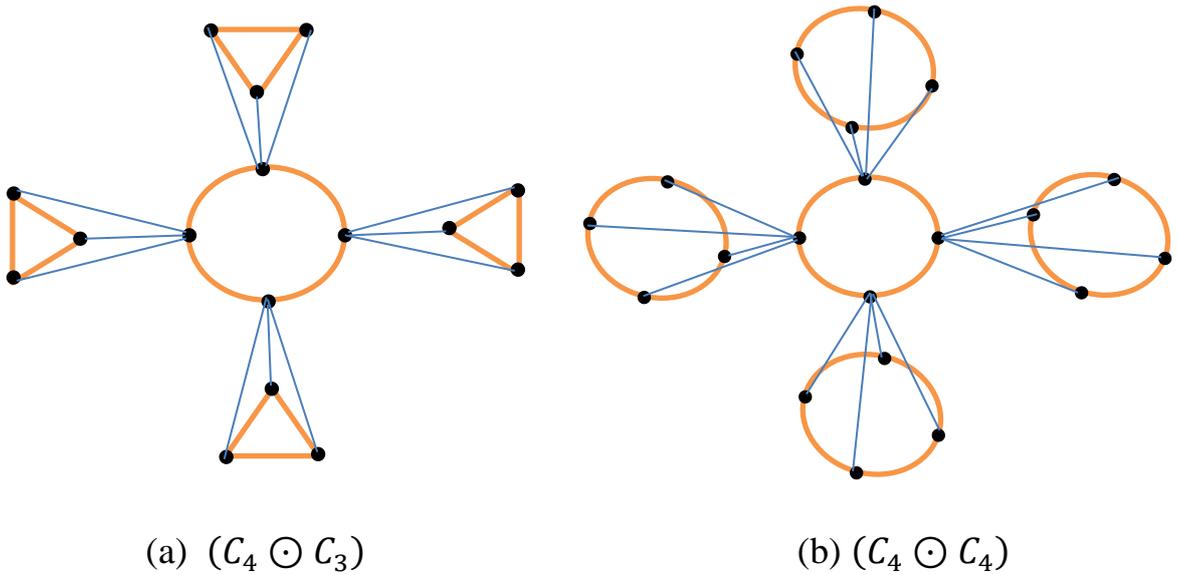


FIGURE 3.1: Corona of cycle graph with cycle graph

Algorithm 3.2.1. ESDS- corona of two cycles of order n and m (G_1 is a cycle of order n , G_2 is a cycle of order m , corona of two cycles $G = C_n \odot C_m$)

Input: A graph $G_1 = C_n$, a graph $G_2 = C_m$

1. $ESDS := \emptyset$;
2. $V_1 := \{v_1, v_2, \dots, v_n\}$;
3. $V_2 := \{u_1, u_2, \dots, u_m\}$;
- 4.

$V :=$

$$\left\{ v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_m, u_{2,1}, u_{2,2}, \dots, u_{2,m}, u_{3,1}, u_{3,2}, \dots, u_{3,m}, \dots, \right. \\ \left. u_{n,1}, u_{n,2}, \dots, u_{n,m} \right\};$$

5. case 1: m is odd

6. $ESDS := V_1$;

7. case 2: m is even

8. if n is odd then

9. For $i := 1$ to $n - 2$ step 2, do;

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10.           $ESDS := ESDS \cup \{v_i\};$ 
11.          end for;
12.        end if;
13.      if  $n$  is even then
14.          For  $i := 1$  to  $n - 1$  step 2, do;
15.               $ESDS := ESDS \cup \{v_i, u_i\};$ 
16.          end for;
17.          For  $i := 2$  to  $n - 1$  step 2, do;
18.              For  $j := 1$  to  $n - 1$  step 2, do;
19.                   $ESDS := ESDS \cup \{u_{i,j}\};$ 
20.              end for;
21.          end for;
22.      end if;

```

Output: $ESDS$

Theorem 3.2.2 *If G_1 is a cycle graph of order n and G_2 is a path graph of order m , then*

$$\gamma_{es}(C_n \odot P_m) = \begin{cases} 3n, & \text{if } m \text{ is odd} \\ n + n \left\lfloor \frac{m-2}{3} \right\rfloor, & \text{if } m \text{ is even} \end{cases}$$

Proof. *To obtain all cases in the graph $C_n \odot P_m$, there are two different cases.*

Case 1) *If m is odd, then the vertices of the cycle C_n have odd degree, one can do that with the same technique that was followed in the case 1 from Theorem 2.1. On another hand, the pendants two vertices of each copy of*

an induced subgraph isomorphic to the path P_m have an even degrees and the other vertices of each copy of the path P_m have an odd degrees (for an example, see Fig. 3.2(a)). Thus, each vertex in the cycle C_n dominates of all vertices in a copy of the path P_m that adjacent to it except the two pendants of this copy. Therefore, the two pendants of each copy must added to minimum dominating set. Thus, $\gamma_{es}(C_n \odot P_m) = n + 2n = 3n$.

Case 2) If m is even, then the vertices of the cycle C_n have even degree, one can do that with the same technique that was followed in the case 1 from Theorem 2.1. On another hand, the pendants two vertices of each copy of the path P_m have an even degrees and the other vertices of each copy of the path P_m have an odd degrees (for an example, see Fig. 3.2(b)). Thus, each vertex in the cycle C_n dominates the two pendants vertices of each copy of the path P_m in the graph $C_n \odot P_m$. Therefore, the remained vertices which are not dominated by vertices of the cycle C_n are the vertices of each copy of the path P_m except the pendants vertices and these vertices dominated by $\lceil \frac{m-2}{3} \rceil$ vertices. Thus, $\gamma_{es}(C_n \odot P_m) = n \lceil \frac{m}{3} \rceil + \lceil \frac{n}{3} \rceil$.

Based to the above, the required is obtained. \square

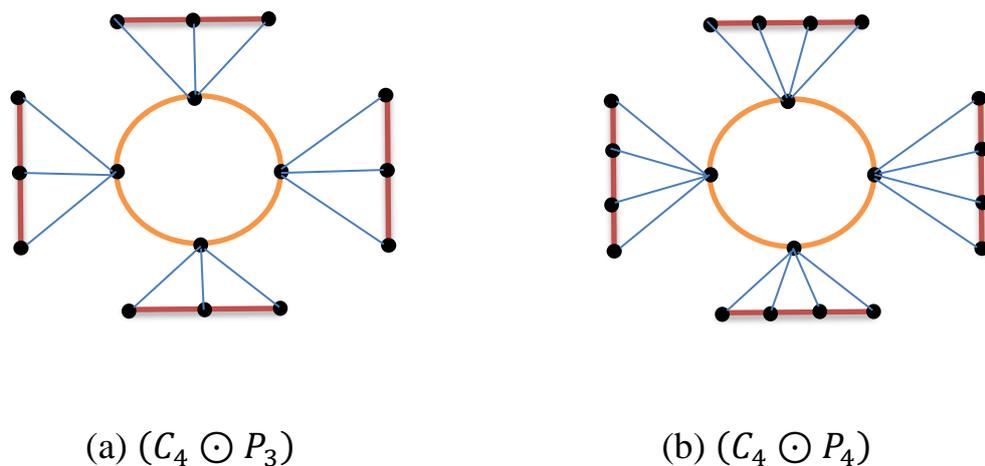


FIGURE 3.2: Corona of cycle graph with path graph

Theorem 3.2.3. If G_1 is a cycle graph of order n and G_2 is a complete graph of order m , then

$$\gamma_{es}(C_n \odot K_m) = n.$$

Proof. To obtain all cases in the graph $C_n \odot K_m$, there are two different cases, the technique of the two cases as same. If m is odd (even), then the vertices of the cycle C_n

have odd (even) degree, one can do that with the same technique that was followed in the case 1 from Theorem 2.1. On another hand, the vertices of each copy of the complete K_m have an odd (even) degrees in the graph $C_n \odot K_m$ (for an example, see Fig.3.1 (a) (Fig.3.3)). Thus, each vertex in the cycle C_n dominates of all vertices in a copy of the complete K_m that adjacent to it in the graph $C_n \odot K_m$. Thus, $\gamma_{es}(C_n \odot K_m) = n$.

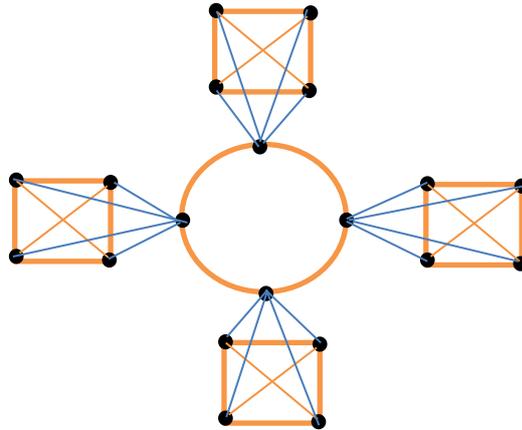


FIGURE 3.3: Corona of cycle graph with complete graph ($C_4 \odot K_4$)

Theorem 3.2.4. If G_1 is a cycle graph of order n and G_2 is a null graph of order m , then

$$\gamma_{es}(C_n \odot N_m) = \begin{cases} n, & \text{if } m \text{ is odd} \\ nm + \left\lfloor \frac{n}{3} \right\rfloor, & \text{if } m \text{ is even} \end{cases}$$

Proof. To obtain all cases in the graph $C_n \odot N_m$, there are two different cases.

Theorem 3.2.5. If G_1 is a cycle graph of order n and G_2 is a complete bipartite graph $K_{r,t}$ of order rt , then

$$\gamma_{es}(C_n \odot K_{r,t}) = \begin{cases} n, & \text{if } r \text{ and } t \text{ are odd} \\ 2n + \left\lceil \frac{n}{3} \right\rceil, & \text{if } r \text{ and } t \text{ are even} \\ n + nt, & \text{if } r \text{ is even and } t \text{ is odd} \\ n + nr, & \text{if } r \text{ is odd and } t \text{ is even} \end{cases}$$

Proof. It is know that the complete bipartite graph contains two sets say X and Y such that each set is isomorphic to null graph of same order of this set and $|X| = r$ and $|Y| = t$. To obtain all cases in the graph $C_n \odot P_m$, there are different four steps.

Case 1) If r and t are odd, then the vertices in the cycle C_n have even degree, one can do that with the same technique that was followed in the case 1 from Theorem 2.1. On another hand, the vertices of each copy of the complete bipartite graph $K_{r,t}$ have an even degrees in the graph $C_n \odot K_{r,t}$. Thus, each vertex in the cycle C_n dominates of all vertices in a copy of the complete bipartite graph $K_{r,t}$. Thus, $\gamma(C_n \odot K_{r,t}) = n$.

Case 2) If r and t are even, then the vertices in the cycle C_n have even degree, one can do that with the same technique that was followed in the case 1 from Theorem 2.1. On another hand, the vertices of each copy of the complete bipartite graph $K_{r,t}$ have an odd degrees in the graph $C_n \odot K_{r,t}$. Thus, the vertices of the cycle C_n cannot dominate the vertices in each copy of the complete bipartite graph $K_{r,t}$. Again, vertices in the cycle C_n can be dominated by $\left\lceil \frac{n}{3} \right\rceil$ vertices from the same set. Also, since each vertex in the complete bipartite graph $K_{r,t}$ has odd degree in the graph $C_n \odot K_{r,t}$, then two vertices one of them from the set X and the other from the set Y can be dominate all vertices in the complete bipartite graph $K_{r,t}$ and it is clear that these two vertices represent the minimum dominating set. Thus, $\gamma_{es}(C_n \odot K_{r,t}) = 2n + \left\lceil \frac{n}{3} \right\rceil$.

Case 3) If r is even and t is odd, then the vertices in the cycle C_n have odd degree, one can do that with the same technique that was followed in the case 1 from Theorem 2.1. On another hand, the vertices of each copy of the complete bipartite graph $K_{r,t}$ have an odd degrees in the graph $C_n \odot K_{r,t}$ (for an example, see Fig.3.5). Thus, the vertices of the induced subgraph generated by the vertices X have odd degree and the vertices of the induced subgraph generated by the vertices Y have even degree. Thus, each vertex in the cycle C_n dominates of all vertices in a copy of an induced subgraph generated by the vertices X and cannot dominates of all vertices in a copy of an induced subgraph generated by the vertices Y . So, the vertices in a copy of an induced subgraph generated by the vertices Y dominate themselves. Thus, $\gamma_{es}(C_n \odot K_{r,t}) = n + nt$.

Case 4) If r is odd and t is even, then in the same manner in previous case, one can concluded that $\gamma_{es}(C_n \odot K_{r,t}) = n + nr$.

Based to the above, the required is obtained. \square

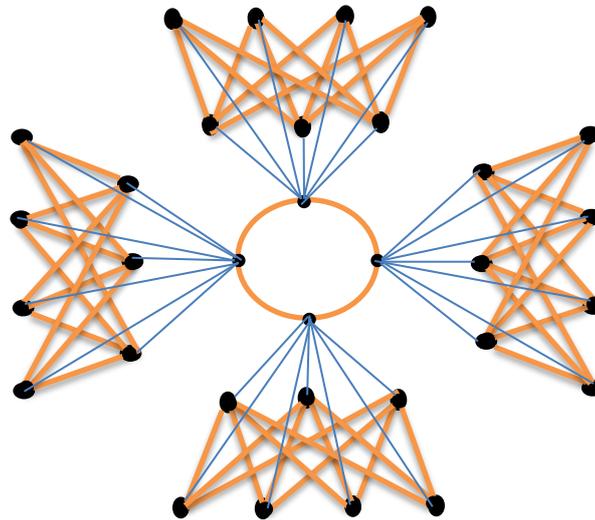


FIGURE 3.5: Corona of cycle graph with complete bipartite graph
 $(C_4 \odot K_{3,4})$

Corollary 3.2.6. If G_1 is a cycle graph of order n and G_2 is a complete bipartite graph $S_m \equiv K_{1,m-1}$ of order m , then

$$\gamma_{es}(C_n \odot S_m) = \begin{cases} 2n, & \text{if } m \text{ is odd} \\ n, & \text{if } m \text{ is even} \end{cases}$$

Proof. It is straightforward from proof of last theorem.

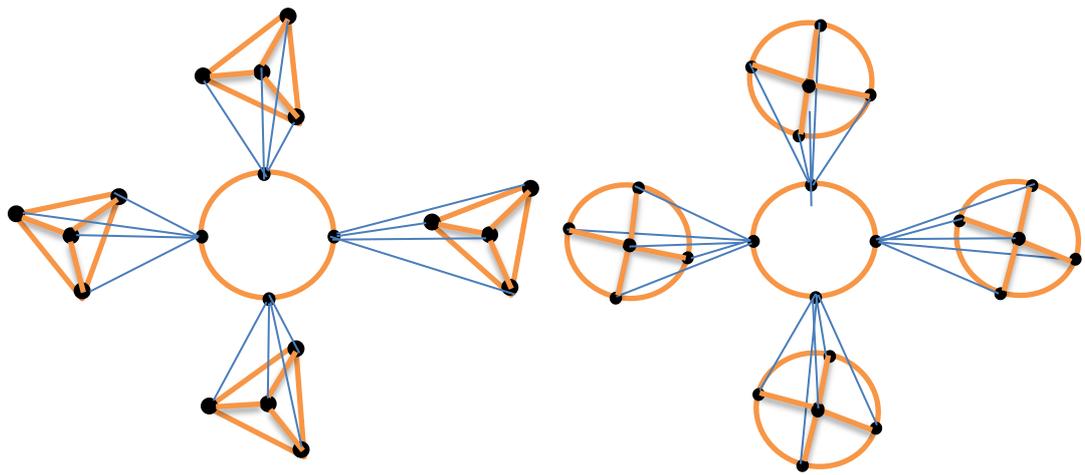
Theorem 3.2.7. If G_1 is a cycle graph of order n and G_2 is a wheel graph W_m of order m , where $W_m \equiv K_1 + C_{n-1}$, then

$$\gamma_{es}(C_n \odot W_m) = \begin{cases} n + n \left\lfloor \frac{m-1}{3} \right\rfloor, & \text{if } m \text{ is odd} \\ n, & \text{if } m \text{ is even} \end{cases}$$

Proof. To obtain all cases in the graph $C_n \odot W_m$, there are different two steps.

Case 1) If m is odd, then the vertices in the cycle C_n have odd degree, one can do that with the same technique that was followed in the case 1 from Theorem 2.1. On another hand, the vertices of each copy of the wheel graph W_m have an even degrees except the center vertex of that wheel graph (for an example, see Fig.3.6 (a)). Thus, each vertex in the cycle C_n dominates only center vertex of each copy of the wheel graph W_m . Therefore, the remained vertices which are not dominated by vertices of the cycle C_n are the vertices of each copy of an induced subgraph generated by the vertices of each copy of C_{m-1} these vertices dominated by $\left\lfloor \frac{m-1}{3} \right\rfloor$ vertices. Thus, $\gamma_{es}(C_n \odot W_m) = n + n \left\lfloor \frac{m-1}{3} \right\rfloor$.

Case 2) If m is even, then the vertices in the cycle C_n have odd degree, one can do that with the same technique that was followed in the case 1 from Theorem 2.1. On another hand, the vertices of each copy of the wheel graph W_m have an even degrees too in the graph $C_n \odot W_m$ (for an example, see Fig.3.6 (b)). Thus, each vertex the cycle C_n dominates all vertices of each copy of an induced subgraph isomorphic to the wheel graph W_m . Thus, $\gamma_{es}(C_n \odot W_m) = n$.

(a) $(C_4 \odot W_4)$ (b) $(C_4 \odot W_5)$ **FIGURE 3.6:** Corona of cycle graph with Wheel graph

Chapter Four

Edge Even Sum Domination in Graphs

Chapter Four

Edge Even Sum Domination in Graphs

4.1 INTRODUCTION

Similar to the work we did on the vertices we define the edge even sum dominating set.

4.2 Basic Definitions

Definition 4.2.1. The edge degree $d(e)$ of the edge $e = uv$ is defined as the number of neighbors (a common vertex with the edge e) of e , i.e., $|N(u) + N(v) - 2|$.

Definition 4.2.2. Let (V,E) be a graph that has no isolated vertex and F is a set of edges, the set F is called *an edge dominating set* if, $\forall e_i \in E - F$ there is an edge $e_j \in F$ adjacent to e_i .

Definition 4.2.3. Let $G(V,E)$ be a graph that has no isolated vertex and F is an edge dominating set, the set F is called *even sum edge dominating set* (ESEDS) if, $\forall e_i \in E - F$ there is an edge $e_j \in F$ adjacent to e_i such that $deg(e_i) + deg(e_j)$ is even.

Definition 4.2.4. Let D be an ESEDS in a graph G and has no proper even sum edge dominating set, then D is called a minimal ESEDS. The minimum cardinality of all minimal ESEDSs is called the even sum edge domination number and denoted by $\gamma'_{es}(G)$.

If D is an ESEDS with minimum cardinality, then D is called γ'_{es} -set.

Observation 4.2.5. Let G be a graph of order n and D is a γ'_{es} -set, then

- 1) Every K_2 component in graph G belongs to every ESEDS.
- 2) If there is an edge e_i in the set $V - D$ has an even(odd) degree, then there is at least one edge e_j in D that has an even(odd) degree adjacent to e_i .

3) Every edge has an odd (even) degree not adjacent to every edge has an odd (even) degree belongs to every ESEDS.

$$4) \gamma'_{es}(G) \leq \gamma'(G).$$

4.3 THE MAIN RESULTS

Proposition 4.3.1. Let G be a regular graph, then $\gamma'_{es}(G) = \gamma'(G)$.

Proof. Let G be a regular graph, then all vertices of this graph have the same degree say r . Thus, the degree of all edges is $(2r - 2)$, so there are two cases one of them $(2r - 2 = 0)$, therefore $G \equiv K_2$ otherwise $(2r - 2)$ is even.

Thus, $\gamma'_{es}(G) = \gamma'(G)$. \square

Remark 4.3.2. The converse of the previous proposition is not necessarily true for example take the star graph (S_6) where $S_6 \equiv K_{1,5}$.

Corollary 2.6.

$$1) \gamma'_{es}(K_n) = \gamma'(K_n) = \left\lfloor \frac{n}{2} \right\rfloor.$$

$$2) \gamma'_{es}(C_n) = \gamma'(C_n) = \left\lfloor \frac{n}{3} \right\rfloor.$$

Theorem 4.3.3. Let G be a path of order n ; $n \geq 2$, so

$$\gamma'_{es}(P_n) = \begin{cases} 1, & \text{if } n = 2, 3 \\ 2 + \left\lfloor \frac{n-3}{3} \right\rfloor, & \text{if } n \geq 4 \end{cases}.$$

Proof. The proof is treated separately for two cases.

Case 1. If $n = 2$, then $P_2 \equiv K_2$, thus $\gamma'_{es}(P_2) = 1$. Also, if $n = 3$, then there are only two disjoint edges each of them of degree 3, so $\gamma'_{es}(P_3) = 1$.

Case 2. If $n \geq 4$, then the degree of each pendant edge is 1 and not adjacent to an edge of odd degree, so by Observation 4.2.5(3) these edges belong to every ESEDS. These two edges do not dominate to adjacent edges, since these it have an odd degree and the adjacent have an even degree. All remained edges have even degree, so all three consecutive edges can be

dominated by one edge. It is clear that the three consecutive edges is covering four vertices thus, $\gamma_{es}(P_n) = 2 + \left\lceil \frac{n-3}{3} \right\rceil$.

From the cases above, the result is obtained. \square

Algorithm 4.3.3. ESEDS-PATH (G is a path of order n)

Input: the vertex set $V = \{v_1, v_2, \dots, v_n\}$, v_1 and v_n are the pendant vertices

1. $ESEDS := \emptyset$;
2. Case 1: $n = 2$ or 3 ;
3. $ESEDS := \{e_1\}$;
4. Case 2: $n \geq 4$;
5. $ESEDS := \{v_1, v_n\}$;
6. For $i := 3$ to $n - 1$ step 2, do;
7. $ESEDS := ESEDS \cup \{e_i\}$;
8. end for;

Output: $ESDS$

Theorem 4.3.4. Let G be a thorn path $P_{n,r,s}$. Then

$$\gamma'_{es}(P_{n,r,s}) = \left. \begin{array}{l} n + \left\lceil \frac{n}{3} \right\rceil, \quad \text{if } s \text{ is odd and } r \text{ is even} \\ 2 + \left\lceil \frac{n-4}{2} \right\rceil, \text{ if } s \text{ and } r \text{ are odd or } s \text{ is even and } r \text{ is odd} \\ n + \left\lceil \frac{n-2}{3} \right\rceil, \quad \text{if } s \text{ and } r \text{ are even} \end{array} \right\}.$$

Proof. The proof is treated separately for three cases.

Case 1. If s is odd and r is even, then all terminal edges in the thorn graph have an odd degree and all edges of the path graph (non-terminal edges in the thorn graph) have an even degree (as an example, see Figure 1). Thus, the terminal edges do not dominate the non-terminal edges, so take one edge from each terminal that is adjacent to a vertex (say v) in the

path of order n . This edge dominates all edges which are adjacent to the vertex v and do not belong to the path graph. Therefore, the number of these edges is n . The remained edges not adjacent by these edges are the edges of the path graph. Now, all edges in the thorn graph that belong in the path graph have an even degree, so all three consecutive edges can be dominated by one edges. Thus, the minimum number of edges that dominate the edges of the path graph is $\lceil \frac{n}{3} \rceil$. Therefore, $\gamma'_{es}(P_{n,r,s}) = n + \lceil \frac{n}{3} \rceil$.

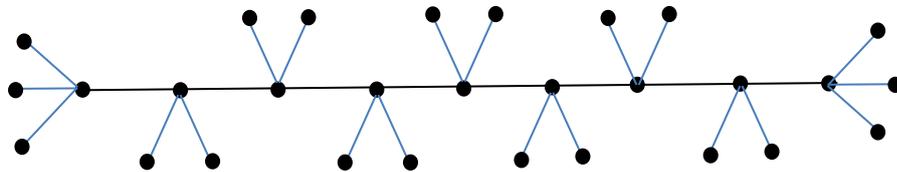


Figure 4.1. The thorn graph $P_{9,2,3}$

Case 2. If s and r are odd, then all terminal edges in the thorn graph that are adjacent to the pendants vertices of the path graph have odd degree also the pendants vertices of path. Thus, the pendant vertices must belong to the minimum ESEDS. These two vertices dominate the terminal edges in the thorn graph that adjacent to it (as an example, see Figure 2).

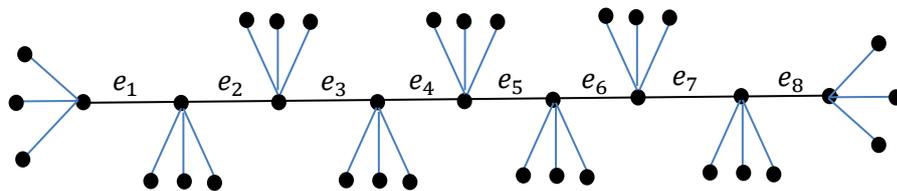


Figure 4.2. The thorn graph $P_{9,3,3}$

All remained edges of the thorn graph have even degree, so the set $D_1 = \{e_{3+2k}, k = 0, 1, \dots, \lceil \frac{n-4}{2} \rceil\}$ where all these edges belong to the path graph (as an example, see Figure 2). The set D_1 is the minimum ESEDS of

the remain vertex that not dominate by the edges e_1 and e_n and the number vertices of the set D_1 is $\left\lceil \frac{n-2}{2} \right\rceil$. Thus, $\gamma'_{es}(P_{n,r,s}) = 2 + \left\lceil \frac{n-4}{2} \right\rceil$.

Case 3. If s is even and r is odd, then all edges in the thorn graph have even degree (as an example, see Figure 1).

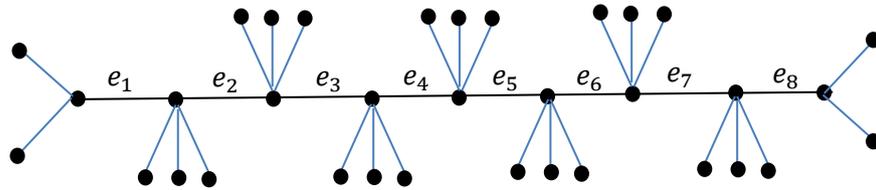


Figure 4.3. The thorn graph $P_{9,3,2}$

The pendant edges of the path graph (as an example, e_1 and e_8 in the Figure 3) must belong to the minimum ESEDS to dominate the terminal edges that adjacent to it. In same technique in previous case the set D_1 is minimum ESEDS to other edges in the thorn graph. Thus, $\gamma'_{es}(P_{n,r,s}) = 2 + \left\lceil \frac{n-4}{2} \right\rceil$.

Case 4. If s and r are even, the terminal edges in the thorn graph that adjacent to the pendant vertices in the path graph have even degree and the pendant vertices of the path have odd degree, thus one edge from the terminal edges must be taken in the minimum ESEDS (as an example, see the Figure 4).

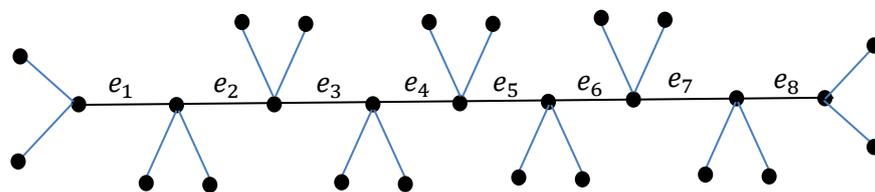


Figure 4.4. The thorn graph $P_{9,2,2}$

This edge dominates all other terminal edges that adjacent to it. The same technique used in the other terminal, sine for each vertex in the path graph, all terminal edges that adjacent to it have odd degree. Also, the edges of

path graph except the pendant have even graph. Thus, again one edge from the terminal edges of a vertex in the path graph must be taken in the minimum ESEDS. The remained vertices which no dominate by the select previous edges are the vertices of path graph except the terminal edges. Therefore, the minimum number of edges that dominate the remained edges is $\lceil \frac{n-2}{3} \rceil$. Thus, $\gamma'_{es}(P_{n,r,s}) = n + \lceil \frac{n-2}{3} \rceil$.

From the cases above, the result is obtained. \square

Theorem 4.3.5. Let G be a thorn rod path $P_{n,m}$. Then

$$\gamma'_{es}(P_{n,m}) = \begin{cases} 1, & \text{if } n = 2 \text{ and } m \text{ is even} \\ 3, & \text{if } n = 2 \text{ and } m \text{ is odd} \\ 4 + \lceil \frac{n-3}{3} \rceil, & \text{if } n > 2 \text{ and } m \text{ is even} \\ 2 + \lceil \frac{n-1}{3} \rceil, & \text{if } n > 2 \text{ and } m \text{ is even} \end{cases}.$$

Proof. The proof is treated separately for four cases.

Case 1. If $n = 2$ and m is even, then all edges of the thorn rod graph are even degree, so the edge of path dominates all other edges. Thus, $\gamma'_{es}(P_{2,m}) = 1$.

Case 2. If $n = 2$ and m is odd, then all terminal edges of the thorn rod graph have odd degree and the edge of path has even degree. Thus, one of terminal edges must be taken in the minimum ESEDS from each side of the path. Also, the edge of P_2 must be taken according to observation 2.4(3). Thus, $\gamma'_{es}(P_{2,m}) = 3$.

Case 3. If $n > 2$ and m is even, then all terminal edge of the thorn rod path graph have even degree. In addition, the pendant edges of the path graph have odd degree. Finally, the other edges of path graph have even degree (as an example, see the Figure 5).

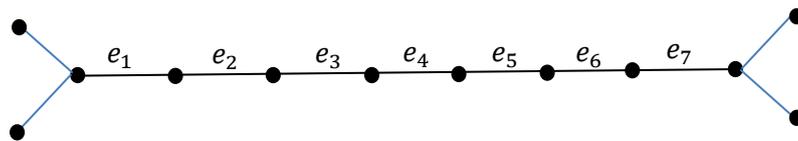


Figure 4.5. The thorn rod graph $P_{n,2}$

In each side of the path, one each must be taken to add to the minimum ESEDS to dominate all terminal edges in the thorn rod path graph. Also, the terminal edges of the path graph (as an example, e_1 and e_8 in the Figure 5) must be taken in the minimum ESEDS, according to Observation 2.4(3). The remained edges not dominated by the four vertices are $\{e_2, e_3, \dots, e_{n-1}\}$ of the path graph. All these edges have even degree, so all edges in thorn graph that belong in the path graph have even degree. Therefore, all three consecutive edges can be dominated by one edges and $\gamma'_{es}(P_{2,m}) = 4 + \left\lceil \frac{n-3}{3} \right\rceil$.

Case 4. If $n > 2$ and m is odd, then all terminal edge of the thorn rod path graph have odd degree. In addition, all the edges of the path graph have even degree (as an example, see the Figure 6).

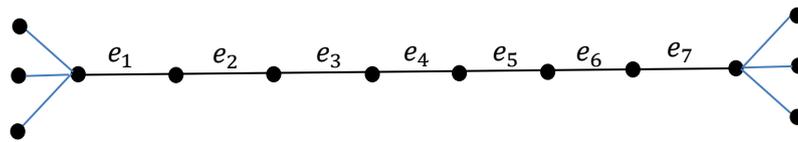


Figure 4.6. The thorn rod graph $P_{n,3}$

In each side of the path, one each must be taken to add to the minimum ESEDS to dominate all terminal edges in the thorn rod path graph. The remained edges not dominated by the two vertices are $\{e_1, e_2, \dots, e_n\}$ of the path graph. All these edges have even degree, so all edges in thorn graph that belong in the path graph have even degree. Therefore, all three consecutive edges can be dominated by one edges and $\gamma'_{es}(P_{2,m}) = 2 + \left\lceil \frac{n-1}{3} \right\rceil$.

From the cases above, the result is obtained. □

Theorem 4.3.6. *If G is a ladder graph (Cartesian product of P_2 and P_n ; $n \geq 2$) denoted by*

$$G \equiv P_2 \times P_n, \text{ then } \gamma'_{es}(G) = \left\{ \begin{array}{ll} 2, & \text{if } n = 2 \\ 6, & \text{if } n = 3 \\ 8, & \text{if } n = 4,5 \\ 7 + \frac{2(n-3)}{3}, & \text{if } n \equiv 0; n \neq 3 \\ 8 + \frac{2(n-4)}{3}, & \text{if } n \equiv 1; n \neq 4 \\ 7 + \frac{2(n-2)}{3}, & \text{if } n \equiv 2; n \neq 2,5 \end{array} \right\}.$$

Proof. It is possible to partition the set of edges of the ladder graph to three part as follows: $S_1 = \{e_1^1, e_2^1, \dots, e_{n-1}^1\}$ this is the set of edges of upper path of the ladder graph, $S_2 = \{e_1^2, e_2^2, \dots, e_{n-1}^2\}$ the set of edges of lower path of the ladder graph, and $S_3 = \{e_1, e_2, \dots, e_n\}$ the set of edges that joining the vertex v_i from upper path with the corresponding vertex from lower path (as an example, see Figure 6).

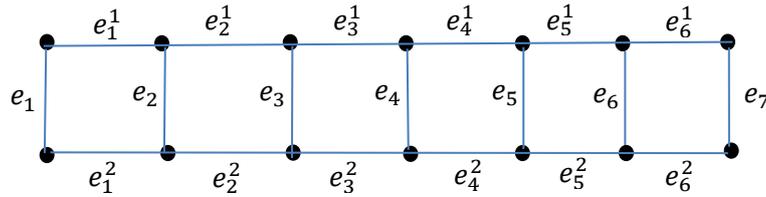


Figure 4.7. The ladder graph $P_2 \times P_7$.

The complete proof is treated separately for four cases.

Case 1. If $n = 2$, then $G \equiv C_4$ and $\gamma'_{es}(P_2 \times P_2) = 2$, according to corollary 2.6.

Case 2. If $n = 3$, then the edges $\{e_1, e_2, e_3\}$ have even degree, so all these edges belong to the minimum ESEDS according to Observation 2.4(3). The remained edges not dominate by these edges are $e_1^1, e_2^1, e_1^2, e_2^2$, all these vertices have odd degree so the two vertices e_1^1 and e_1^2 dominate the vertices $e_1^1, e_2^1, e_1^2, e_2^2$. Thus, $\gamma'_{es}(P_2 \times P_3) = 6$.

Case 3. If $n = 4,5$, then there are two cases:

I) If $n = 4$, then the vertices $e_1, e_1^1, e_1^2, e_3^1, e_3^2, e_4$ must be belong to the minimum ESEDS according to Observation 2.4(3). The remain edges not dominated by these edges make as induced subgraph isomorphic to a cycle of order four. Thus, by using corollary 2.6 (2) $\gamma'_{es}(P_2 \times P_4) = 8$.

II) If $n = 5$, then the set $\{e_1, e_1^1, e_2^1, e_4^1, e_1^2, e_3^2, e_4^2, e_5\}$ is the minimum ESEDS. Thus, $\gamma'_{es}(P_2 \times P_5) = 8$.

Now, there is a general formula for $n \geq 6$ depending on the modulo of three as the follows:

let $D_1 = \{e_1, e_1^1, e_1^2, e_{n-1}^1, e_{n-1}^2, e_n\}$ this set must be belong to the minimum ESEDS according to Observation 2.4(3).

Case 4. If $n \equiv 0(mod 3)$, then Let $D_2 = \{e_{3+3k}^1, k = 0, 1, \dots, \frac{n-3}{3} - 1\}$ and $D_3 = \{e_{4+3k}^2, k = 0, 1, \dots, \frac{n-3}{3} - 1\}$, then let $D = D_1 \cup D_2 \cup D_3 \cup \{e_2\}$. It is clear that the set D is ESEDS and it is minimum, since if we delete any edge (e) from the set D, then $D - e$ not ESEDS (as an example, see Figure 7). Thus, $\gamma'_{es}(G) = |D_1| + |D_2| + |D_3| + 1 = 6 + \frac{n-3}{3} + \frac{n-3}{3} + 1 = 7 + \frac{2(n-3)}{3}$.

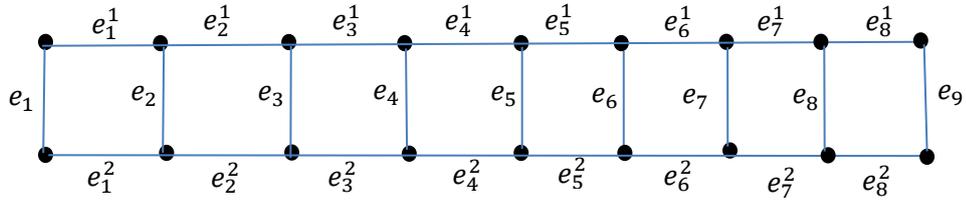


Figure 4.8. The ladder graph $P_2 \times P_9$.

Case 5. If $n \equiv 1(mod 3)$, then Let $D_2 = \{e_{3+3k}^1, k = 0, 1, \dots, \frac{n-4}{3} - 1\} \cup \{e_{n-1}^1\}$ and $D_3 = \{e_{4+3k}^2, k = 0, 1, \dots, \frac{n-4}{3} - 1\}$, then let $D = D_1 \cup D_2 \cup D_3 \cup \{e_2\}$. Again it is clear that the set D is ESEDS and it is minimum, since if we delete any edge (e) from the set D, then $D - e$ not ESEDS (as an example, see Figure 8). Thus, $\gamma'_{es}(G) = |D_1| + |D_2| + |D_3| + 1 = 6 + \frac{n-4}{3} + 1 + \frac{n-4}{3} + 1 = 8 + \frac{2(n-4)}{3}$.

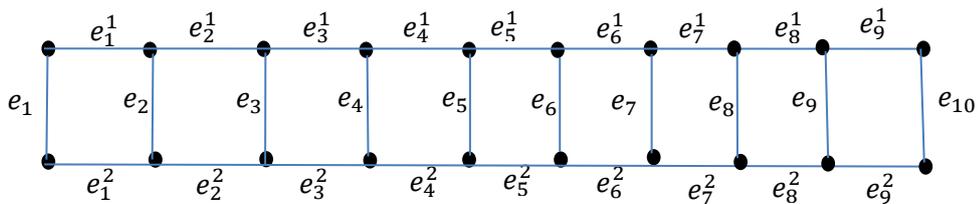


Figure 4.8. The ladder graph $P_2 \times P_{10}$.

Case 6. If $n \equiv 2 \pmod{3}$, then Let $D_2 = \{e_{3+3k}^1, k = 0, 1, \dots, \frac{n-2}{3} - 1\}$ and $D_3 = \{e_{4+3k}^2, k = 0, 1, \dots, \frac{n-2}{3} - 2\} \cup \{e_{n-1}^2\}$, then let $D = D_1 \cup D_2 \cup D_3 \cup \{e_2\}$. Again it is clear that the set D is ESEDS and it is minimum, since if we delete any edge (e) from the set D , then $D - e$ not ESEDS (as an example, see Figure 4.9). Thus, $\gamma'_{es}(G) = |D_1| + |D_2| + |D_3| + 1 = 6 + \frac{n-2}{3} + \frac{n-2}{3} - 1 + 1 + 1 = 7 + \frac{2(n-2)}{3}$.

From the cases above, the result is obtained. □

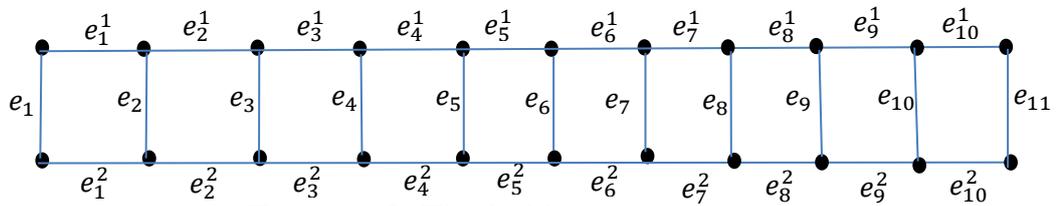


Figure 4.9. The ladder graph $P_2 \times P_{11}$.

Theorem 4.3.7. Let G be tadpole graph denoted by $T_{m,n}$, then

$$\gamma'_{es}(T_{m,n}) = \begin{cases} 2 + \left\lfloor \frac{m-2}{3} \right\rfloor, & \text{if } n = 1 \\ 2 + \left\lfloor \frac{m-2}{3} \right\rfloor, & \text{if } n = 2 \\ 2 + \left\lfloor \frac{m-2}{3} \right\rfloor + \left\lfloor \frac{n-2}{3} \right\rfloor, & \text{if } n \geq 3 \end{cases}$$

Proof. The proof is treated separately for three cases depend on the number of vertices of path graph.

Let $S_c = \{e_1^c, e_2^c, \dots, e_m^c\}$ and $S_p = \{e_1^p, e_2^p, \dots, e_n^p\}$ be a vertex set of a cycle and path respectively and e is the bridge (as an example, see Figure)

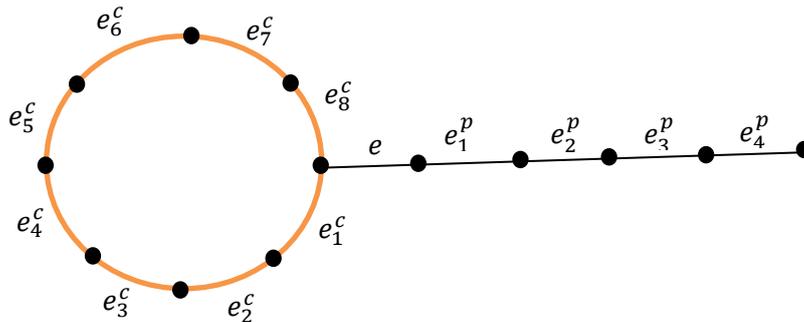


Figure 4.10. The tadpole graph $T_{9,5}$.

Case 1. If $n = 1$, then $S_p = \emptyset$, the bridge edge e has even degree and the two edges e_1^c and e_m^c adjacent to it have odd degree so, e belong to the minimum ESEDS according to Observation 2.4(3). Also, one of the two vertices e_1^c and e_m^c (say e_1^c) belong to the minimum ESEDS. The remain vertices not dominated by the two vertices e and e_1^c are $e_2^c, e_3^c, \dots, e_{m-1}^c$, all these edges have even degree so all three consecutive edges can be dominated by one edges. Thus, $\gamma'_{es}(T_{m,1}) = 2 + \left\lfloor \frac{m-2}{3} \right\rfloor$.

Case 2. . If $n = 2$, then $S_p = \{e_1^p\}$, this vertex and three edges e , e_1^c , and e_m^c have odd degree so the bride edge e dominate all these edges. Again, the remain vertices not dominated by the vertex e are $e_2^c, e_3^c, \dots, e_{m-1}^c$, all these edges have even degree so all three consecutive edges can be dominated by one edges. Thus, $\gamma'_{es}(T_{m,1}) = 1 + \left\lfloor \frac{m-2}{3} \right\rfloor$.

Case 3. If $n \geq 3$, then the terminal edge e_{n-1}^p in the tadpole graph has odd degree and the edge adjacent to it has even degree, so this vertex belong to the minimum ESEDS. The three edges e , e_1^c , and e_m^c have odd degree so the bride edge e dominate all these edges. The remain edges not dominated by the two edges divided into set, one of them the edge set of path graph except the edge e_{n-1}^p and the other is the edge set of cycle graph except two vertices e_1^c , and e_m^c . All remained edge have even degree, so by use the same technique one can be concluded that, $\gamma'_{es}(T_{m,1}) = 2 + \left\lfloor \frac{m-2}{3} \right\rfloor + \left\lfloor \frac{n-2}{3} \right\rfloor$.

From the cases above, the result is obtained. \square

Theorem 4.3.8. Let G be a wheel graph denoted by $W_n \equiv C_n + K_1$, then

$$\gamma'_{es}(W_n) = \begin{cases} 1 + \left\lfloor \frac{n-2}{3} \right\rfloor, & \text{if } n \text{ is even} \\ 1 + \left\lfloor \frac{n}{3} \right\rfloor, & \text{if } n \text{ is odd} \end{cases}$$

Proof. The proof is treated separately for two cases depend on the number of vertices of wheel graph.

Case 1. If n is even, then the center vertex (K_1) has odd degree and all vertices of the cycle graph have odd degree too. Thus, all edges in the wheel graph have even degree, so, to get the minimum ESEDS, take one

edge say e_i from inside edges (the edges joining the center vertex with the vertices of the cycle graph). This edge dominates all inside edges and two edges from cycle graph. The remain edges are the edges of cycle except two edges that adjacent to the edge e_i , then by use the same technique in previous theorem, one can be concluded that $\gamma'_{es}(W_n) = 1 + \left\lceil \frac{n-2}{3} \right\rceil$.

Case 2. If n is odd, then the center vertex has even degree and all vertices of the cycle graph have odd degree. Thus, all inside edges in the wheel graph have odd degree and all edges in the cycle graph have even degree. So, to get the minimum ESEDS, take one edge say e_i from inside edges, this edge dominates all inside edges. The remain edges are the edges of cycle, then by use the same technique in previous theorem, one can be concluded that $\gamma'_{es}(W_n) = 1 + \left\lceil \frac{n}{3} \right\rceil$.

From the cases above, the result is obtained. □

Algorithm 4.3.8. ESEDS-WHEEL (G is a wheel of order n , v_o is the center vertex, the edge $e_{o,j} = v_o v_j$)

Input: A graph $G = (V, E)$

1. $ESEDS := \emptyset$;
2. case 1: n is odd
3. $ESEDS := \{e_{o,1}\}$;
4. For $i := 3$ to $n - 1$ step 3
5. $ESEDS := ESEDS \cup \{e_i\}$;
6. end for;
7. case 2: n is even
8. For $i := 1$ to $n - 1$ step 3
9. $ESEDS := ESEDS \cup \{e_i\}$;
10. end for;

Output: *ESEDS*.

Theorem 4.3.9. Let G be a complete bipartite graph $K_{m,n}$ then

$$\gamma'_{es}(K_{m,n}) = \min\{m, n\}.$$

Proof. With loss of generality suppose that $m \leq n$, now take a vertex (say v_1) from the set with cardinal m (say X) and take an edge joining the vertex v_1 with a vertex (say u_1) in the set of cardinal n (say Y). This edge dominates all edges incident with v_1 and u_1 . Again take another vertex from the set X say v_2 and take an edge joining the vertex v_2 with a vertex in the set Y different from u_1 (say u_2), again this edge dominates all incident with v_2 and u_2 . Continue with this procedure until reach the last vertex in the set X (say v_m). Thus, there are separately cases depend on the relation between n and m as follows.

I) If $n = m$, then it is obvious that the edges selected above make a minimum *ESEDS*, since they dominate all edges in the graph $K_{m,n}$ and it is impossible contains a proper *ESEDS*.

II) If $n > m$, then there is at least one vertex (say u_i) in the set Y not incident to all selected edges in case I. Now, for each edge incident with the vertex u_i must be incident with all vertices in the set Y , then it is adjacent with the selected edges in the case I. Thus, the selected edges dominate all edges in the graph $K_{m,n}$.

From the cases above, $\gamma'_{es}(K_{m,n}) = \min\{m, n\}$. □

Chapter Five

Fuzzy even sum domination in strong fuzzy graph

Chapter Five***Fuzzy even sum domination in strong fuzzy graph*****5.1 Introduction**

The study of domination in fuzzy graphs worth to care. The concept of domination in fuzzy graphs was introduced by A. Somasundram, S. Somasundram [14-15] . There is more than one definition of domination in a fuzzy graph where Mahioub et al. [10] defined it by taking the minimum fuzzy cardinality to an all-dominating set. Another definition is presented by Xavior et al. [16] by determined the minimum dominating set and taking the sum of all its vertices. In this chapter, we followed the used the definition that has been put by Xavior et al. with some additions which seems to be the best adapted to our work.

Consider $G(V,E)$ be a simple, finite, and undirected graph. A map $\sigma : V \rightarrow [0, 1]$ where V has at least one element (set of vertices) called a fuzzy subset, and $G = (\sigma, \mu)$ where $\mu: V \times V \rightarrow [0,1]$ and $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$, (the min of $\sigma(u)$ and $\sigma(v)$), is said to be a fuzzy graph. An edge $e = (u,v)$ is said to be an effective if it satisfy $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ and the fuzzy graph is called strong if each edge belongs to it is effective. The open effective neighborhood is $N_E(v) = \{u; (u,v) \text{ is an effective edge}\}$ and $N_E[v] = N_E(v) \cup \{v\}$. Degree of a vertex v is the number of effective edges in $N_E(v)$ and denoted by $deg_E(v)$. In this chapter, we deal with the strong fuzzy graph, so every edge is an effective edge. Moreover, any null graph is strong fuzzy null, and a strong fuzzy star graph denoted by S_n ; $S_n \equiv K_{1,n-1}$.

Definition 5.1.1 *Let $G = (\sigma, \mu)$ be a fuzzy graph of $G(V, E)$ and D is a fuzzy dominating set, the set D is called fuzzy even sum dominating set*

(FESDS) if, $\forall u \in V - D$ there is a vertex $v \in D$ adjacent to u such that $deg_E(v) + deg_E(u)$ is even.

Definition 5.1.2 Let D be a FESDS in a fuzzy graph $G = (\sigma, \mu)$ and has no proper fuzzy even sum dominating set, then D is called a minimal MFESDS. The minimum cardinality of all MFESDS is called the minimum fuzzy even sum dominating set (M^m FESDS).

Definition 5.1.3 Consider

$W(D_i) = \{\sum \sigma(v); \forall v \in D_i; D_i \text{ is a } M^m \text{FESDS}\}$, then the fuzzy even sum domination number of a fuzzy graph is

$$\gamma_{fes}(G) = \min \{W(D_i); D_i \text{ is a minimum even sum dominating set}\}.$$

Example 5.1.4 In the following figure, there are four M^m FESDS that are $D_1 = \{v_1, v_3\}$, $D_2 = \{v_1, v_4\}$, $D_3 = \{v_2, v_3\}$, $D_4 = \{v_2, v_4\}$.

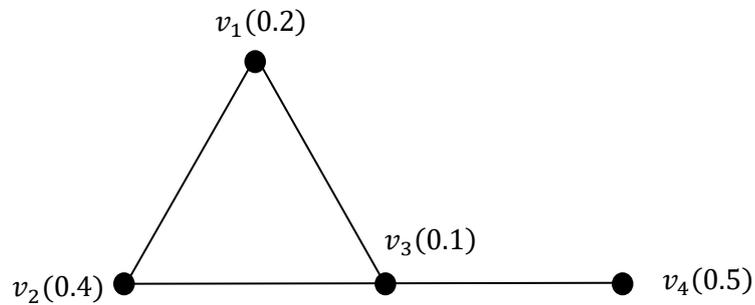


Figure 5.1.

Then $|D_1| = 0.3$, $|D_2| = 0.7$, $|D_3| = 0.5$, and $|D_4| = 0.9$. Thus, $\gamma_{fes}(G) = 0.3$.

5.2 The main results

Proposition 5.2.1.

- 1) If G has m isolated vertices then $\gamma_{fes}(G) \geq \sum \sigma(v)$, where v has an isolated vertex.
- 2) $\min(\sigma(v_j)) \leq \gamma_{fes}(G) \leq \sum_{i=1}^n \sigma(v_i)$, where v_j the vertex that dominates all vertices in G .
- 3) If a vertex v has an odd degree and all neighbors have an even degree or vice versa, then this vertex belonging to every $M^m FESDS$.

Proof.

- 1) By the definition of FESDS, each isolated vertex must belong to every $M^m FESDS$, so the required is obtained.
- 2) The lower bound obtained where there is a vertex adjacent to all other vertices the same kind of degree that means if this vertex has an odd degree, then all neighbors have an odd degree too and vice versa. The upper bound is obtained if the fuzzy graph is null or each vertex in the fuzzy graph has a degree differ kind of all neighbors that means if this vertex has an odd degree, then all neighbors have an even degree or vice versa.
- 3) If a vertex v has an odd degree and all neighbors have an even degree, then this vertex dominates itself only and there is no vertex dominates the vertex v . Thus, the vertex v must belonging to every $M^m FESDS$, so the result is obtained.

Proposition 5.2.2 If $G = (\sigma, \mu)$ be a strong path and has n vertices, then

$$\gamma_{fes}(G) = \left\{ \begin{array}{ll} \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} \sigma(v_{3+3i}), & \text{if } n \equiv 2 \pmod{3} \\ \min\{\sum \sigma(v_i), v_i \in D_1, D_2, \text{ or } D_k\}, & \text{if } n \equiv 1 \pmod{3} \end{array} \right\},$$

where $D_1 = \{v_{2+3i}, i = 0, \dots, \lfloor \frac{n-2}{3} \rfloor\}$, $D_2 = \{v_{3+3i}, i = 0, \dots, \lfloor \frac{n-2}{3} \rfloor\}$, and

$$D_k = \left\{ v_{3+3i}, i = 0, \dots, \left\lfloor \frac{k-1}{3} \right\rfloor \right\} \cup \left\{ v_{\left\lfloor \frac{k-1}{3} \right\rfloor + 2} \right\} \cup \left\{ v_{\left\lfloor \frac{k-1}{3} \right\rfloor + 2 + 3j}, j = 1, \dots, \left\lfloor \frac{n-2}{3} \right\rfloor - \left(\left\lfloor \frac{k-1}{3} \right\rfloor + 1 \right) \right\}; k = 3, \dots, n-4.$$

Proof.

There are two cases that depend on the modulo of n as follows.

Case 1. If $n \equiv 0 \pmod{3}$, then the pendant vertices $\{v_1, v_n\}$ have an odd degree and are not adjacent, since $n \equiv 0 \pmod{3}$. The other vertices have an even degree, so the pendant vertices belong to every $M^m FESDS$. Each vertex, not pendant dominates three vertices at most one of them is itself the vertex. Therefore, there is a unique $M^m FESDS$ that is $D = \left\{ v_{3+3i}, i = 0, \dots, \left\lfloor \frac{n-2}{3} \right\rfloor \right\}$. Thus, $\gamma_{fes}(G) = \sum_{i=0}^{\frac{n-2}{3}} \sigma(v_{3+3i})$, if $n \equiv 2 \pmod{3}$.

Case 2. If $n \equiv 1 \pmod{3}$, then there are many $M^m FESDS$ as the following

I) If v_2 belongs to the $M^m FESDS$, then there is a unique $M^m FESDS$ that is $D_1 = \left\{ v_{2+3i}, i = 0, \dots, \left\lfloor \frac{n-2}{3} \right\rfloor \right\}$. The set D_1 is $ESDS$ and it has no $ESDS$ as proper, then the set D_1 is $M^m FESDS$.

II) If v_2 not belonging to the $M^m FESDS$, then the vertex v_3 must belong to the $M^m FESDS$ to dominate the vertex v_2 , so again there are two ways to choose the vertices of the $M^m FESDS$ to depend on the distinct of these vertices as the following:

1) If $d(v_i, v_j) = 3 \forall i, j \neq 1, n$, then in this case there are unique $M^m FESDS$ as the following $D_2 = \left\{ v_{3+3i}, i = 0, \dots, \left\lfloor \frac{n-2}{3} \right\rfloor \right\}$.

2) If $d(v_i, v_j) = 2$ at only two consecutive vertices and $d(v_i, v_j) = 3$ at the other vertices. We cannot choose more than two vertices such that

$d(v_i, v_j) = 2$, since the M^mFESDS is been loosen. There are $\lfloor \frac{n-2}{3} \rfloor$ different M^mFESDS in this case as the following $D_k = \{v_{3+3i}, i = 0, \dots, \lfloor \frac{k-1}{3} \rfloor\} \cup \{v_{\lfloor \frac{k-1}{3} \rfloor + 2}\} \cup \{v_{\lfloor \frac{k-1}{3} \rfloor + 2 + 3j}, j = 1, \dots, \lfloor \frac{n-2}{3} \rfloor - (\lfloor \frac{k-1}{3} \rfloor + 1)\}; k = 3, \dots, n - 4$. Now, the first part of the set is M^mFESDS to all vertices of the set $\{v_2, v_2, \dots, v_{\lfloor \frac{k-1}{3} \rfloor + 1}\}$, the vertex dominates itself and two vertices $\{v_{\lfloor \frac{k-1}{3} \rfloor + 1}, v_{\lfloor \frac{k-1}{3} \rfloor + 3}\}$, and the set $\{v_{\lfloor \frac{k-1}{3} \rfloor + 2 + 3j}, j = 0, \dots, \lfloor \frac{n-2}{3} \rfloor - (\lfloor \frac{k-1}{3} \rfloor + 1)\}$ dominates all the vertices $\{v_{\lfloor \frac{k-1}{3} \rfloor + 4}, \dots, n - 1\}$. Thus, the set D_2 is M^mFESDS of path in this case.

Observation 5.2.3. $\gamma_{fes}(K_n) = \min\{\sigma(v_i); v_i \in V(K_n)\}$.

Proposition 5.2.4. *If G is a strong fuzzy cycle has n vertices (C_n); $n \equiv 0(mod 3)$, then*

$$\gamma_{fes}(C_n) = \left\{ \begin{array}{l} \min\{\sum_{i=1}^{n/3} \sigma(v_{i+j}); j = 0, 1, 2\}; \text{ if } n \equiv 0(mod 3) \\ \min\{\sum \sigma(v_i), v_i \in D_1, D_2, \text{ or } D_k\}, \text{ if } n \equiv 2(mod 3) \end{array} \right\} \text{ where}$$

$$D_1 = \{v_{2+3i}, i = 0, \dots, \lfloor \frac{n-2}{3} \rfloor\}, D_2 = \{v_{3+3i}, i = 0, \dots, \lfloor \frac{n-2}{3} \rfloor\}, \text{ and}$$

$$D_k = \{v_{3+3i}, i = 0, \dots, \lfloor \frac{k-1}{3} \rfloor\} \cup \{v_{\lfloor \frac{k-1}{3} \rfloor + 2}\} \cup \{v_{\lfloor \frac{k-1}{3} \rfloor + 2 + 3j}, j = 1, \dots, \lfloor \frac{n-2}{3} \rfloor - (\lfloor \frac{k-1}{3} \rfloor + 1)\}; k = 3, \dots, n - 4.$$

Proof. There are two cases as follows.

Case 1 . If $n \equiv 0(mod 3)$, then there are exactly three M^mFESDS sets that are $D_1 = \{v_{1+3i}, i = 0, \dots, \frac{n}{3} - 1\}$, $D_2 = \{v_{2+3i}, i = 0, \dots, \frac{n}{3} - 1\}$, and $D_3 = \{v_{3+3i}, i = 0, \dots, \frac{n}{3} - 1\}$. It is obvious that the sets are M^mFESDS , since each vertex has maximum neighborhood and all vertices have even

degree. Also, these sets have the same cardinality where $|D_1| = |D_2| = |D_3| = \frac{n}{3}$. Therefore,

$$\gamma_{fes}(C_n) = \min \left\{ \sum_{i=1}^{n/3} \sigma(v_{i+j}); j = 0, 1, 2 \right\}.$$

Case 2. If $n \equiv 2 \pmod{3}$, in the same technique in the case 2 of **Theorem 2.2.**, the result is obtained.

Proposition 5.2.5 *Let $G = (\sigma, \mu)$ be a fuzzy wheel graph $W_n \equiv K_1 + C_{n-1}$, where $V(K_1) = \{v_0\}$ and $V(C_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$, then*

$$\gamma_{fes}(W_n) = \begin{cases} \sigma(v_0), & \text{if } n \text{ is even} \\ 1 + \gamma_{fes}(C_{n-1}), & \text{if } n \text{ is odd} \end{cases}.$$

Proof. There are two cases depend on the number of vertices of the cycle (C_{n-1}):

Case 1. If the number of vertices of the induced subgraph of the wheel graph isomorphic to the cycle (C_{n-1}) is odd that means n is an even number, then all vertices of the wheel graph have an odd degree. Thus, the set $D = \{v_0\}$ is the unique $M^m FESDS$, so $\gamma_{fes}(W_n) = \sigma(v_0)$.

Case 2. If n is odd, then the number of vertices of the cycle (C_{n-1}) that means the degree of the vertex v_0 is even and all other vertices have an odd degree, then the vertex v_0 belonging to every $M^m FESDS$ and this vertex dominates only itself. The remained vertices which are not dominated by the vertex v_0 constitute the cycle graph (C_{n-1}), so by the same manner in Proposition 2.7. can be getting the required.

Proposition 5.2.6 *If G be a fuzzy complete bipartite graph $K_{n,m}$ contains two partite sets V_1 and V_2 such that V_1 has n vertices $\{u_1, u_2, \dots, u_n\}$ and V_2 has m vertices $\{v_1, v_2, \dots, v_m\}$; $n \leq m$, then*

$$\gamma_{fes}(K_{n,m}) = \left\{ \begin{array}{l} \sigma(u_1), \quad \text{if } n = 1 \text{ and } m \text{ is odd} \\ \min(\sigma(u_i), i = 1, \dots, n) + \min(\sigma(v_j), j = 1, \dots, m), \text{ if } n > 1 \text{ and} \\ \quad n \text{ and } m \text{ are even or odd together} \\ \sum_{i=1}^n \sigma(u_i) + \sum_{j=1}^m \sigma(v_j), \text{ if } n > 1 \text{ odd is and } m \text{ is even or vice versa} \end{array} \right\}$$

Proof. There are three cases:

Case 1. If $n = 1$ and m is odd, then the vertex u_1 that dominates all vertices in the graph $K_{n,m}$ and there is no other vertex dominates all vertices. Thus, $\gamma_{fes}(K_{n,m}) = \sigma(u_1)$.

Case 2. If $n > 1$ and n and m are even or odd together, then each vertex from any partite sets dominates all other vertices in other partite sets. Therefore, a vertex with the least value will be the chosen one from the set V_1 and the second vertex chosen from the set V_2 . These two vertices make a $M^m FESDS$. Thus,

$$\gamma_{fes}(K_{n,m}) = \min(\sigma(u_i), i = 1, \dots, n) + \min(\sigma(v_j), j = 1, \dots, m).$$

Case 3. If $n > 1$ is odd and m is even or n is even and m is odd, then each vertex dominates itself only, since if a vertex in the set V_1 has an odd degree, then all vertices in the set V_2 have an even degree and vice versa. Thus, $\gamma_{fes}(K_{n,m}) = \sum_{i=1}^n \sigma(u_i) + \sum_{j=1}^m \sigma(v_j)$.

Proposition 5.2.7 *If G is a strong fuzzy star graph has n vertices ($S_n \equiv K_{1,n-1}$)*

$$\gamma_{fes}(S_n) = \left\{ \begin{array}{l} \sigma(u), \text{ if } n \text{ is even and } u \text{ is the root vertex} \\ \sigma(u) + \sum_{j=1}^{n-1} \sigma(v_j), \text{ if } n \text{ is odd} \end{array} \right\}.$$

Proof. It is straightforward from previous proposition.

Conclusion and Future Work

Conclusions

Throughout this dissertation a new definition of domination in graphs is introduced which is called even sum domination. The results we obtained depended on the set we are dealing with, as there were two sets, namely the vertex set and the edge set. In the second, third and fifth chapters, the results depended on the vertex set, while the fourth chapter relied on the edge set.

For the vertex set many properties and bounded with their algorithms are determined . For the certain graphs such as path, cycle, wheel, complete, and complete partite, this number is calculated with giving an algorithm for each case. Moreover, this number is determined to complement and an inverse of the certain graphs mentioned above with give an algorithm for some of theirs. Also, determined this number to corona operation of two graphs, especially to the cycle graph with one of the certain graphs mentioned above is calculated. Additionally, in the fuzzy graph especially, in strong fuzzy graph which every edge is effective this number is determined to certain graphs mentioned above with some properties and bounded are presented with an algorithm for some of theirs. Finally, for the edge set this number is determined to certain graphs mentioned above and some properties and bounden are introduced. In addition, this number is determined for the thorn, thorn rod, tadpole, and ladder graphs.

Future Works

There are many ideas related to work that can be worked on in the future, which can be summarized in the following points.

- 1) Add a new condition to the set of dominating or to the set of out dominating or together to obtained a new definition on even sum as total even sum, independent even sum, paired even sum, restrained even sum, and others.
- 2) Study other operations on this number as join, Cartesian product, and others.
- 3) Study the fuzzy graph on the edge set.
- 4) Study the topological indices to this number on the vertex set or edge set.

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المستخلص

أدت التطبيقات المختلفة للهيمنة والعمل الحديث على أشكال مختلفة من الهيمنة في نظرية البيان إلى اهتمامنا باستكشاف أنواع خاصة جديدة من الهيمنة في نظرية البيان في هذه الدراسة.

حيث أن معظم العمل الذي تمت مناقشته في موضوع الهيمنة يركز أكثر على تهيئة الظروف للمجموعة المهيمنة للتوصل إلى مفهوم جديد للهيمنة. في هذه الأطروحة، تم تقديم المعاملين الجديدين للهيمنة في نظرية البيان وهما هيمنة المجموع الزوجي والهيمنة العكسية ذات المجموع الزوجي. وكذلك مفهوم هيمنة الأضلاع ذات المجموع الزوجي، وتمت دراسة ومناقشة العديد من خصائص هذه المفاهيم ووضع النظريات لها بشكل خاص على الرسوم البيانية المحددة كالدورة، المسار، الكامل، ثنائي الكامل، النجم، الفارغ، والعجلة والشوكة. علاوة على ذلك، تمت كتابة خوارزمية الوقت الخطي لمشكلة الهيمنة ES في العجلة والمسار وتم إثبات أن مسألة الهيمنة ES هي مسألة NP تامة. بالإضافة إلى ذلك، عملية الإكليل للرسم البياني الدائري وواحد من الرسوم البيانية المعينة مثل رسم البيان للعجلة والمسار والدورة والكامل والنجمة والثنائي الجزء الكامل والرسم الفارغ لهيمنة المجموع الزوجي. أيضًا، لكل حالة هناك بعض الأرقام التي تم تقديمها لتوضيح هذه الحالات. كما تمت دراسة الرسوم البيانية الضبابية وتطبيق المفاهيم الجديدة عليها كتعريف مجموعة المهيمنة ذات المجموع الزوجي الضبابية M^mFESDS وكتابة بعض الخوارزميات لها.



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خوارزميات لهيئات متنوعة في نظرية البيان

أطروحة مقدمة إلى مجلس كلية التربية للعلوم الصرفة في جامعة بابل
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