Republic of Iraq

Ministry of Higher Education

and Scientific Research

University of Babylon

College of Education for Pure Sciences

Generalized Prime System Involving Beurling Zeta function with some enhancing approximation

A Thesis

Submitted to the Council of the College of Education for Pure Sciences, in University of Babylon as a Partial Fulfillment of the Requirements for the Degree of Master in Education / Mathematics

By

Huda Salih Hamza Jasim

Supervised by

Prof.Dr. Faez A. AL-Maamori

2022 A.D. 1443 A.H

بسم الله الرحمن الرحيم

وَآتَاكُمْ مِنْ كُلِّ مَا سَأَلْتُمُوهُ وَإِنْ تَعُدُّوا نِعْمَةَ اللَّهِ
لَا تُحْصُوهَا إِنَّ الْإِنْسَانَ لَظَلُومٌ كَفَّارٌ

صدق الله العلي العظيم

سورة ابراهيم الآية 34

Supervisor's Certification

I certify that the thesis entitled **“Generalised Prime Systems involving Beurling Zeta function with some enhancing approximation** "by"**Huda Salih Hamza Jasim"** has been prepared under my supervision in Babylon University/ College of Education for Pure Sciences as a partial requirement for the degree of Master in Education / Mathematics.

Signature :

Name : Dr. Faez Ali AL-Maamori

Title: Assistant Professor

Date: / / 2022

In view of the available recommendation, I forward this thesis for debate by the examining committee.

Signature:

Name:Dr. Azal Jaafar Musa

Head of Mathematics Department

Title: Assistant Professor

Date: / / 2022

Certification of linguistic Expert

I certify that I have read this thesis entitled **"Generalised Prime Systems involving Beurling Zeta function with some enhancing approximation"** and corrected its grammatical mistakes; therefore, it has qualified for debate.

Signature:

Name: Pof. Dr.Qassim Abbas

Title: Teacher

Date: / / 2022

Certification of Scientific Expert

I certify that I have read the scientific content of this thesis **"Generalised Prime Systems involving Beurling Zeta function with somenhancingapproximation"** and I have approved this dissertation is qualified for debate

Signature:

Name: Dr. Ihsan Jabbar Kadhim

Title: ***Assistant Professor***

Date: / /2022

Certification of Scientific Expert

I certify that I have read the scientific content of this thesis **" Generalised Prime Systems involving Beurling Zeta function with some enhancing approximation"** and I have approved this dissertation is qualified for debate

Signature:

Name: Dr.Amerra N.

Title: ***Professor*** .Dr.

Date: / /2022

Examining Committee Certification

We certify that we have read the thesis entitled " **Generalized Prime Systems involving Beurling Zeta function with some enhancing approximation** " by" "**Huda Salih Hamza Jasim**" and as a committee examined the student in its contents and, according to our opinion, it is accepted as a thesis for the degree of Master in Education / Mathematics .

Signature:
Name:Dr.Ahmed Abed Ali Omer
Title: Assistant. prof
Date: / / 2022
Chairman

Signature:
Name: Eman F.Mohommed
Title: Assistant. prof
Date: / / 2022
Member

Signature:
Name:Ali Youinis shakir
Title: Assistant. prof
Date: / / 2022
Member /

Signature:
Name: ***Dr. Faez Ali AL-Maamori***
Title: Assistant. prof
Date: / / 2022
Member / Advisor

Approved by the Dean of the College.

Address:Dean of the college of Education for pure sciences.

Signature :

Name:Bahaa Hussien Salih Rabee

Scientific grade :Prof.Dr.

Date : / /2022

Dedications

To my Sir and lord Imam Ali (peace be Upon him)

Contents

Acknowledgments

To those with whom I saw the path of my life and derived from them my strength and self-esteem to the ceaseless struggle To those who taught me the meaning of persistence and that nothing is impossible in life with the strength of faith and proper planning My beloved sons (Mortaza, Amir, Azal), whose time I have taken a lot and have always shortened towards in order to complete my studies, I dedicate the fruit of my efforts to all of you.

Abstract

Analytic Number Theory is the main part of the pure mathematics , which regards the core several topics such as mathematical algebra and complex analysis . The distribution of numbers and the related counting functions of primes and integers are studied by several authors. Morever , The distribution of generalized primes, generalized integers and their generalized counting functions (called beurling instead of generalized sometimes) are studied during the last fifty years. There are many results published about the relation of counting function of g- prime $\psi_g(x)$ and counting function of g- integer $N_g(x) involving$ the generalized Riemann Zeta function $\xi_g(\sigma + it)$. for ϵ (0,1) . The aim of this work is to obtain a new system, which is an arithmetic function for prime numbers with an arithmetic function for generalized prime numbers for example (Π,N)

List of symbols

PNT	Prime Number Theorem
$\boldsymbol{\pi}(\mathbf{a})$	Counting function of primes
N(a)	Counting function of integers
d(a)	Counting function of devisors
$\boldsymbol{\sigma}_{\boldsymbol{\alpha}}(\mathbf{a})$	Divisors function
$\mu(\mathbf{a})$	The Mobius function
$\boldsymbol{\phi}(\mathbf{a})$	Euler's function
$\boldsymbol{\Lambda}(\mathbf{a})$	Von Mangoldt function
$\boldsymbol{\theta}(\mathbf{a})$	Chebysheve's θ-function
$\boldsymbol{\Pi}(\mathbf{a})$	Π-function
$\boldsymbol{\psi}(\mathbf{a})$	Chebysheve's ψ-function
O(q(a))	Big oh notation
o(q(a))	Small oh notation
$\sim$	Asymptotic equality
$\boldsymbol{\Omega}$	Ω- notation and Ω- function
$\boldsymbol{\zeta}(\mathbf{s})$	Zeta function
$\boldsymbol{\Re}(\mathbf{s})$	Real part of the complex number s
$\Delta(\mathbf{a})$	Abundance of a
$\mathbb{P}_{\mathbf{p}}$	The set of Beurling's prime numbers
$\mathbb{N}_{\mathbf{p}}$	The set of Beurling's natural numbers
$\boldsymbol{\pi}_{\mathbf{p}}(\mathbf{x})$	Counting function of generalized primes
$\mathbf{N}_{\mathbf{p}}(\mathbf{x})$	Counting function of generalized integers
$\boldsymbol{\zeta}_{\mathbf{p}}(\mathbf{s})$	Beurling's zeta function
$\mathbf{li}(\mathbf{x})$	$\int_2^x \frac{dt}{\log(t)}$
$\mathbf{F}_{\mathbf{1}}^{+}$	The set of all increasing functions with $f(1) = 1$
$\mathbf{F}_{\mathbf{0}}^{+}$	The set of all increasing functions with $f(1) = 0$
$\boldsymbol{\Pi}_{\mathbf{p}}$	Generalized Π-function
$\boldsymbol{\psi}_{\mathbf{p}}(\mathbf{x})$	Generalized Chebysheve's ψ-function
s(a)	The sum of proper devisors of a
$\boldsymbol{\psi}_{\mathbf{0}}(\mathbf{x})$	Integer part of x minus 1
Mod	Modulo
Iff	If and only if

Introduction

During the seventh defined by Bukowski weird numbers and as an update , learners of mathematics at Central Washington University broken Kravitz in 2013 and discovered the largest primitive weird number which is : 1,304,478,802,221,037,336,898,806,955,880,590,950,108,213,611,

184, 211,428,152,436,309,358, 012, ,494,920, 476,023,972,998, 095,015,247,872, Comprising of 127 Digits .

Searched in weird numbers and the weird number n is w positive integer, for which the sum of n proper divisors is higher than the number n itself, and there is no subset of proper divisors that sums up to n.Simply put , wei-rd numbers are numbers that are abundant but not pseudoperfec.

As well as in 19^{th} century the scientist Beurling state that any sequence of

Reals $B = \{p_{1,}p_2, \dots \dots\}$ satisfying : $1 < p_1, p_n \leq p_{n+1}$ and that the sequence of the actual primes {2,3,5…} aould be a case of an called Beurling primes or generalized primes .This can form the sequence of generalized (or Beurling)integers $N_g$formed by the products of the form $\prod_{.=k=0}^{n} p_k^{a_k}$

where k be a natural numbers and $a_k \in N \cup \{0\}$Beurling counting function of primes and of integers are stated as :

$$\pi_g(x) = \sum_{p \leq x\ ,p \in g} 1 \qquad and \quad N_g(x) = \sum_{n \leq x, n \in N} 1\ .$$

All these generalization have been investigated by Beurling (in 1937) and then several authors studied it such as Diamond and Hilberdink and so on . This subject now one word regarded it as a main of the attention of several authors and a phD student in variouse areas around the world which includes many open problems and different branches .Additionally , authors study the generalization of primes in two main entries : Either in the analytic style or in algebraic style where in both cases still the Riemann – Zeta function as a main open problem .

In this thesis , Beurling prime systems are followed (i.e. the generalization of primes).Moreover, an idea about the link between the counting function of primes and of integers involving the generalization of Riemann – Zeta function is given . We will improve the known upper bounds of the generalized integer function by using a sequence of weird numbers .

The aim of thesis is suppose that we have a sequence of weird numbers .

Apply this sequence as a special case of x . By the way , this sequence satisfies the rules of Beurling prime .

In fact , the first chapter it examines the prime numbers as well as the primes and integers arithmetic functions and also the definition of the Riemann Zeta function

The second chapter Developing Landau's idea, Arne Beurling in 1930s introduced generalised (or Beurling) primes. In his definition, from any sequence of reals

P = {p1, p2, p3...} satisfying 1 < p1 ≤ p2 ≤ ... ≤ pn ≤ ..., and pn → ∞

as n → ∞
called 'generalised primes', can be formed the sequence of generalised (or Beurl-ing) integers N formed by the products of the form $\prod_{i=1}^{k} p_i^{a_i}$ .where k ∈ N and a_i ∈ N ∪ {0}

In this sense, Beurling generalises the notion of prime numbers and the natural numbers obtained from them. The generalised primes need not be actual primes,nor even integers and the generalised integers need not to be a uniquely factoris-able. Therefore, P and N are not sets in general, but multisets where elements can occur with a certain multiplicity. Beurling defined π_p (x) to be the counting function of g-primes less than or equal to x and NP (x) to be the counting func-tion of g-integers less than or equal to x (counting multiplicities). Beurling was interested to see under which conditions on N and the multiplicative structure, a Prime Number Theorem holds.

The third chapter we give background to Beurling (or generalised) prime systems and the associated Beurling zeta function and put the theory in its historical context. First, we introduce discrete g-prime systems with some examples, while in the second section we will define the ‘continuous’ version of g-prime systems with some examples. Here π_p and N_p are still increasing functions but need not be step functions. Now ξ_p is defined as the Mellin transform of N_p . That is

$$\xi_p\,(s) = \int_{1-}^{\infty} x^{-s}\ dN_P(x)$$

In the last part, we list relevant known results which are necessary for our work

In four chapter , For the reader, writing $N_p, \pi_p, \xi_p, \ldots$ to inform the reader that these are the generalized (or Beurling) counting function (of integers and primes) and the Beurling Zeta function.

Beurling in 1937 stated that any infinite and positive real sequence such that the first element must precisely>1 called a generalized primes. Further, the sequence of Beurling integers can be build by using the fundamental theorem of arithmetic’s. This article fos on addressing examples of discrete and continues Beurling’s prime systems. .The difficulty of this work is when you have a discrete prime system , then its possible to build a continuous prime system, while the converse is impossible mostly. Further, this work shows the relation between Beurling counting functions and Beurling Zeta function as well.

Chapter one

Prime and Natural Numbers with Arithmatical Function

Introduction

Number theory is a core of the pure science . So its beneficid to address the prime numbers and the natural numbers creating from those actual primes. Moreover, talk about some counting functions of primes $\pi(x), \psi(x)$)andcounting function of integers N(x) as well as the Riemann – Zeta function. Whereas , the second branch study the generalization of what mention above , such as the generalized (or Burling) primes , natural , counting functions and generalization of the Riemann – zeta function.

1.1. Prime Numbers

Definition 1.1.1 [41]. The natural number P is prime if it is greater than 1 and divisibe only by P and 1 .The first few primes few primes are 2,3,5,7,11,13,17,19,23,29,31,……..

The function that counts the prime numbers less than $x \in R$ is

$$\pi(x) = \sum_{p<x} 1$$

Euclide ,the Greek mathematician proved that every natural number is either a prime number or a uniquely product of different powers prime numbers . thise is called The Fundamental Theorem of Arithmetic , and the function that counts the naturals less than the integer x is

$$N(x) = \sum_{\substack{n<x \\ n\in N}} 1$$

Lemma 1.1.2 [41]

An integer $n \geq 2$ is composite if and only if has factors a and b such that.

$$1 < a < n \textbf{ and } 1 < b < n. \quad Such\ that\ n = ab$$

Proof . Let n≥ 2 . The if $direction\ is\ obvious$. For only if, $assume\ that$

$n\ is\ composite$. Then it has a positive integer factor a such that $a \neq 1, a \neq n. This\ means\ that\ there\ is\ ab\ with$

$n = ab$. $Since\ n\ and\ a\ are\ positive\ , So\ is\ b\ Hence\ 1 \leq a\ and\ 1 \leq b$

$a \leq n, and\ b \leq n$. Since $a \neq 1\ and\ a \neq n\ we\ have$

$1 < a < n. If\ b = 1\ then\ a = n\ , Which\ is$

$also$ not possible .So $1 < b < n$, Finishing this half of the argument .

Lemma 1.1.3 [41]

If$n > 1\ then\ there\ is\ a\ prime\ p\ such\ that\ p|n.$

Proof : Let F denote the set of all integers greater than 1 that have no primeDivisor . We must show that F is empty .I f F is not empty then by the Well -Ordering Property it has a smallest member call it m Now $m > 1$and has no prime divisor .Then m cannot be prime

(as every number is a divisor of itself).

Hence, m is composite . Thearefore by lemma 2.2, (m=ab) where

$1 < a < m\ and\ 1 < b < m. since\ 1 < a < m$ $the\ factor\ a\ is\ not\ a\ mem$ber of F . so a must have a prime divisor p|a and a|m So, by Theorem p|m . Thise contradicts the assumption that m has no prime divisor . Therefore , the set F must be empty .

Theorem 1.1.4 (Euclids Theorem) [41]. "There are infinitely many primes

Proof. Assume to get a contradiction that there are only a finitely many primes $p_1 = 2, p_2 = 3$,...,p_n, Consider the number

Since $p_1 \geq 2, Clearly\ N \geq 2$. So N has a prime divisor p . that prime must be one of $p_1, \dots\dots, p_n$ since that list was assumed exhaustive . However observe that the equation

$$N = p_i \quad (p_1\, p_2 \ldots\ldots p_{i-1}\ \ p_{i+1}\ \ \ldots\ldots p_n\ ) + 1$$

Along with $0 \leq 1 < p_i$ shows by lemma 2.3 , that n is not divisible by $p_i{}'$ .this is a contradiction it follows that assumption that there are only finitely many primes is not true .

Theorem 1.1.5 [41].

If $n > 1$ is composite then n has a prime divisor $p \leq \sqrt{n}$

Proof: Let $n > 1$ *be composite Then* $n = ab$ *where* $1 < a < n$, *and* $1 < b < n$

We claim that at least one of a or b is less than or equal to $\sqrt{n}$.'for if not then $a > \sqrt{n}$ and $b > \sqrt{n}$, and hence $n = ab\ \sqrt{n}\ .\sqrt{n} = n$ which is impossible . Suppose , without loss of generality , that $a \leq \sqrt{n}$. since $1 <$ a, by lemma 2.3 there are is a prime p such that p|a. Hence , by Transitivity

in Theorem , sine a|n we have p|n . By Comparison in theorem , since p|a we have $p \leq a \leq \sqrt{n}$.

"We can use theorem to help compute whether an integer is prime ". given

$n > 1$, we need only try to divide it by all primes $p \leq \sqrt{n}$,if none of these

divides ; n then n must be prime .

Example : consider the number" 97". Note that $\sqrt{97} < \sqrt{100} = 10$, The primes

less than (10) are 2,3,5, and 7. None of these divides 97 ; and so 97 is prime

1.2. The Prime Counting Function

Definition 1.2.1[41]:

Let x be a positive real number >1,then $\pi(x)$;is defined as follows :

$\pi(x)=\sum_{\substack{p\leq x \\ p\ prime}} 1$ That is $\pi(x)$is the number of primes less than or equal to x it is also called the prime counting function or the prime distribution function .

Example: The prime numbers up to 50 are :

2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,

Thuse ; we have

$$\pi(1)=0, \pi(2)=1, \pi(3)=2, \pi(10)=4. \pi(20)=8, \pi(30)=10, \ldots$$

Definition 1. 2.2 [41] : $\boldsymbol{\theta(x)}$

If $X>0$ we define cheby shev's $\theta-$ function by the equation

$$\theta(x)=\sum_{p\leqslant x}\log p$$

Where (p) runs over all primes $\leqslant x$.

Definition 1.2.3[41] : for every integer $n\geqslant 1$ we define :

$$\Lambda(n)=\begin{cases} \log p \text{ if } n=p^m \text{ for some prime p and some } m\geqslant 1 \\ 0 \quad \text{other wise} \end{cases}$$

Here is a short table of values $\Lambda(n)$,

(n)	1	2	3	4	5	6	7	8	9	10
$\Lambda(x)=$	0	log2	$\log 3$	$\log 2$	$\log 5$	0	$\log 7$	$\log 2$	$\log 3$	0

The proof the next theorem shows how this function arises naturally from the fundamental theorem of arithmetic .

Theorem 1.2.4[41] : if $n \geqslant 1$ we have ,

$$\log(n) = \sum_{d|n} \wedge (d)$$

Theorem 1.2.5[41] : if $n \geqslant 1$ we have

$$\wedge (n) = \sum_{d|n} M(d) \log\frac{n}{d} = -\sum_{d|n} M(d) \quad \log(d)$$

Definition 1.2.6 [41] : $\psi(x)$

For x $>$ 0 we define che by sher's ψ -function by formula ;

$$\psi(x) = \sum_{n \leqslant x} x$$

Thus the asymptotic formula in (1) states that :

$$\lim_{x \to 1} \frac{\Psi(x)}{x} = 1$$

since $\Lambda(n) = 0$ unlessn is a prime Power ; we can write the definition of $\Psi(x)$; as follows :

$\psi(x) = \sum_{n \leqslant x} \Lambda(n) = \sum_{m=1}^{x} \sum_{p(p)} \Lambda\, m = \sum_{m=1}^{x} \sum_{p \leqslant x} \log p$

Definition 1.2. 7 [41] : **Euller's totient function**

For $n \in N$; define ϕ(n) to be the number of Positive integers less than or equal to n and comprime to m

$$\phi(n) = \sum_{\substack{m=1 \\ (n,m)=1}}^{n} 1$$

Definition 1.2.8 [41]: The divisor function d(n)

For $n \in N$:let d (n) denote the number of divisors of n . we can write this as :

$$d(n) = \sum_{d|n} 1$$

Definition 1.2.9 [41] : The function $\boldsymbol{\sigma(n)}$

for $n \in N$, let $\sigma(n)$ denote the sum of the divisors of n . we can this function as :

$$\sigma(n) = \sum_{d|n} n$$

Definition 1.2.10 [41]: The mobius function M is defined as follows

$$M(a) = \begin{cases} 1 \text{ if } a = 1 \\ 0 \text{ if } a > 1 \ and\ containts\ a\ sequare\ prime\ factor \\ (-1)^n \quad a > 1\ and\ a = p_1 p_2 \ldots . p_n\ (i.e\ sequarefree) \end{cases}$$

For example , $M\ (2) = -1$, $M(4) = 0$

Theorem 1.2.11 [41] : If $n \geqslant 1$ we have:

$$\sum_{d|n} M(d) = \left[\frac{1}{n}\right] = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n > 1 \end{cases}$$

liouville's function $\boldsymbol{\lambda(n)}$ 1.2.12 [26]: An important example of a completely multiplicative function is liouvillés function λ , wich is defined as follows .

Definition 1.2.13 [26]: we define $\lambda(1) = 1$, and if $n = P_k^{a1} \ldots P_k^{ak}$ we define:

$\lambda(n) = (-1)^{a_1 + \cdots + ak}$ The definition shows at once that λ is completely multiplicative .The next theorem describes the divisor sum of λ .

Theorem 1.2.14[26] : for every $n \geqslant 1$ we have :

$$\sum_{d/n} \lambda(d) = \begin{cases} 1 & \text{if n is aperft square} \\ 0 & \text{otherwise} \end{cases}$$

Also ; $\lambda^{-1}(n) = |M(n)|$ for all n .

Proof : let

$g(n) = \sum_{d|n} \lambda(d)$;" then g is

Multiplication so to determine g(n) we need only compute g(p^a) for prime powers we have :

$g(p)^a = \sum_{d|P^a} \lambda(d) = 1 + \lambda(p) + \lambda(p) + \cdots + \lambda(p)^a$

$$= 1 - 1 + 1 \cdots + (-1)^a = \begin{cases} \textit{0 if a is odd} \\ \text{1 if a is even} \end{cases}$$

Hence if

$$n = \prod_{i=1}^{k} p_i^{ai}$$

We have

$$g(n) = \prod_{i=1}^{k} g\left(p_i^{\alpha i}\right)$$

if any exponent ai is and then $g\left(P_i^{a_i}\right) = 0$ so $g(n) = 0$

if all the exponents ai are even then $g\left(p_i^{ai}\right) = 1$ for all I and $g(n) = 1$ this shows that $g(n) = 1$ if n is a square , and $g(n) = 0$ other wise also .

$$\lambda^{-1}(n) = M(n)\lambda(n) = M^2(n) = |M(n)|$$

Difinition 1.2.15[26] : An arithmetical function is real or complex valued function defined on the set of natural numbers.

The following arithmetical functions are needed for our work. For all natural number α one can have σ_α (a) is summing the divisors of a with power α and defined as

$$\sigma_\alpha(n) = \sum_{d^\alpha | n} d^\alpha$$

For example , $\sigma_2(4500) = 38$

Definition 1.2.16 [27]:

The function N(x) is the counting function which count the number of integer numbers less than or equal to x i.e N(x)=$\sum_{\substack{n\le x \\ n\in N}} 1$.

Note that here we mean by say n is integer for $x \in z^+$

1.3 Prime Number Theorem and the prime counting function [28]

The question 'How many primes are there?' and How big is the nth prime are Closely related. If we could answer either of them exactly with a simple formula Then we could also answer the other but we cannot . The best mathematicians Have managed is to get more or less close to the exact answers .The number of Primesless than or equal to n is denoted by $\pi(n)$. Thise should not be misleading There is no connection between the π in $\pi(n)$ *and the constant* π that is related to *the circle* .Since all primes except 2 and 3 are of the form 6n $\pm$ 1,we can say that at most one – third of numbers are prime , but this is a gross over – estimate Noticing that they are also of the theform30n$\pm$ 1,7,11, *or* 13 *means that* at most eight out of thity , or 26.66%, can be prime , but this is also a feeble figure , not least because the number of primes decreases the further we go but this proportion stays the same.The prime number theorem asserts that as n increases ,$\pi(n)$ asymptotically Approaches $\frac{n}{logn'}$ meaning that $\frac{\pi(n)}{\frac{n}{\log n}}$ tends to 1 as n tends to infinity .In notation :$\pi(n) \sim \frac{n}{\log n}$. This is a brilliantly simple estimate , and a pretty good one ,though not the best : it also turned out to be very difficult to prove. The approximation $\frac{n}{\log n}$ to $\pi(n)$ also means that of the numbers 1 to n, roughly $\frac{1}{\log n}$ are prime .we could also say that the proba – bility of a random number between 1 and n being prime is approxi -mately $\frac{1}{\log n}$. It also means that the average gap between two consecutive primes near the number x is close to log x . Thus ,when x is round about 100 , log x is approximately , so roughly every fifth number should be prime

1.4 Historical view of the prime Number theorem number theorem (P.N.T) [28]

During ninetieth century,Legendre was the first to put a version of the prime

number theorem into print .In 1798 he claimed in his book Essai sur Ia theorie des.

The youthful causs had also been studying prime numbers , and in 1792 at the age of fifteen , he proposed the estimate,

$\pi(x) is\ approximately\ L_i$ (x)=$\int_2^x \frac{1}{\log t}\ dt$

Gauss ‘s conjecture is equivalent to the prime number theorem #

There was then a long pause until Tchebycheff proved in (1851) that if ,

$\frac{\pi(x)}{x} / \log x$ does have a limit then the limit must be 1;through he was unable to take the final step and show that the limit existed A year he proved that if n is large enough;then,

$\frac{(0.92\ldots)}{\log x} < \pi(x) < \frac{(1.105\ldots)x}{\log x}$

The final step was taken entirely independently by de la vallee poussin

(1866-1962) and Jacques Hadamard (1865-1963), coinciden-tally two of the longest – lived mathematicians ever .they proved that indeed

$$\pi(x) = L_i\,(x) + an\ error\ term\ of\ the\ order\ of\ x^{\frac{1 \log x}{2}}$$

De la valle'e poussin also showed that gauss's estimate $L_i(x)$ is a better

approximation to $\pi(x)$than x/(logx – a) no matter what value is assigned to the constant a (and also that the best value for a is 1

1.5 Arithmetical Functions

Definition 1.5.1[41] For any two arithmetical functions $\mathrm{w}, \mathrm{q} \in \mathrm{S}$, the convolution is defined by

$$(\mathrm{w} * \mathrm{q})(\mathrm{a}) = \int_{-1}^{\mathrm{a}} \mathrm{w}\left(\frac{\mathrm{a}}{\mathrm{t}}\right) \mathrm{dq}(\mathrm{t})$$

And in term of summation convolution can be defined as

$$(\mathrm{w} * \mathrm{q})(\mathrm{a}) = \sum_{\mathrm{b} \leq \mathrm{a}} \mathrm{w}(\mathrm{b})\mathrm{q}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)$$

Definition 1.5.2 [41] If w and q are arithmetical function then Dirichlet convolution for w and q is known as

$$\mathrm{h}(\mathrm{a}) = \sum_{\mathrm{b}|\mathrm{a}} \mathrm{w}(\mathrm{b})\mathrm{q}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)$$

Theorem 1.5.3 [41] Multiplication of Dirichlet series

Let W and Q be two Dirichlet series defined as

$$W(s) = \sum_{n=1}^{\infty} \frac{w(n)}{n^s} \qquad \text{for } u > a$$

$$Q(s) = \sum_{n=1}^{\infty} \frac{q(n)}{n^s} \qquad \text{for } u > b$$

Then in the half plane that both series are converges absolutely we have

$$W(s)Q(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$$

Where h(s) is Dirichlet convolution of w(s) and q(s)

Theorem 1.5.4 [41] Uniqueness Theorem

For any two absolutely convergent Dirichlet series for $u > u_\alpha$

$$W(s) = \sum_{n=1}^{\infty} \frac{w(n)}{n^s} \quad and \quad Q(s) = \sum_{n=1}^{\infty} \frac{q(n)}{n^s}$$

If W(s) = Q(s) for any s in an infinite sequence $\{S_K\}$ such that $S_K \to \infty$ $as\ K \to \infty$ then w(n) = q(n) for any n .

Theorem 1.5.5 [41] Mobius inversion formula

The following equation

$$f(a) = \sum_{b|a} g(b)$$

Indicates that

$$g(a) = \sum_{b|a} f(b)\mu\left(\frac{a}{b}\right)$$

besides the converse is true .

Theorem 2 which is determine the connection between Euler's function and Mobius function is already one of applications of Mobius inversion formula .

1.6 Zeta function

zeta function ζ(s) is a function defined as a series by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. It was studied firstly by Euler where in 1737 , he provided a linking between zeta function and prime numbers by proving the identity

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ is prime}} \frac{1}{1-p^{-s}}$. That is sometimes , zeta function named by Euler zeta function .

After then , Riemann entered the complex numbers in the study of zeta function by taking complex values of s and now , zeta function is widespread as Riemann zeta function . Zeta function is analytically continues and converges if $\Re(s) > 1$. Thus for $s = u + it$, u and t are reals , we have

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-(u+it)} = \sum_{n=1}^{\infty} e^{-(u+it)\log(n)}$$

$$= \sum_{n=1}^{\infty} n^{-u}\, e^{-it\log(n)}, \text{hence } |n^s| = n^u$$

Actually , zeta function $\zeta(s)$ is a Mellin transform of $f(x) = [x]$ (see[41]) . In fact , Riemann zeta function is analytic at all the points of the complex plane except at the simple pole s = 1 with residue 1 and this proved in [41] . In particular , if s is near 1 then

$$\lim_{s\to 1}\left|\zeta(s)-\frac{1}{s-1}\right| = \gamma$$

Where γ is Euler-Mascheroni constant and equal approximately to 0.57721665 (see[21]) . If t = 0 then zeta function is divergent for u = 1 and absolutely convergent for u > 1 . This was proved in elementary calculus .

Many mathematicians tried to dilate the concept of zeta function , in 1901 Ernest Barnes defined zeta function and named Shintani zeta function as :

$$\zeta_N(s, w|a_1, \dots, a_N) = \sum_{n_1,\dots,n_N} \frac{1}{(w + a_1 n_1 + \dots + a_N a_N)^s}$$

As a generalization , Adolf Hurwits defined zeta function with respect to two complex numbers s and r with real parts graeter than one and zero respectively as

$$\zeta(s, r) = \sum_{n=0}^{\infty} \frac{1}{(n + r)^s}$$

This series is absolutely convergent for all s and r in the half plane

u > 1 and analytic in the same half plane where Riemann zeta function is $\zeta(s, 1)$.

In 1996 Crandall defined zeta function with respect zeros of Airy function as

$$\zeta_{A_i}(s) = \sum_{i=1}^{\infty} \frac{1}{|a_i|^s}$$

where $\Re(s) > \frac{3}{2}$ and A_i is Airy function which is defined as

$$A_i(x) = \frac{1}{\pi}\int_0^{\infty} \cos(\frac{1}{3}t^3 + xt)dt$$

and a_i are the zeros of Airy function such that $|a_1| < |a_2| <\dots$

One of the most important usage of zeta function is when Euler proved in 1737 Euclid's theorem (which is talking about the infiniteness of prime numbers) by showing that the series $\sum \frac{1}{p}$ is divergent (where p run through all prime numbers) . He inferred this result from the fact that zeta function tend to infinity when s = 1 .

Another formula of zeta function can be written if $s \in (0,1)$ as

$$\zeta(s) = \lim_{a \to \infty} \left(\sum_{n \le a} \frac{1}{n^s} - \frac{a^{1-s}}{1-s} \right)$$

And by this formula one can get the following theorem

Theorem 1.6.1[41]

If $a \geq 1 \; and \; s > 0 \, , s \neq 1$ then

$$\sum_{n \le a} \frac{1}{n^s} = \frac{a^{1-s}}{1-s} + \zeta(s) + O\left(\frac{1}{a^s}\right)$$

Chapter two
Generalized prime numbers

2.1 Introduction

The Swedish mathematician Arne Carl-August Beurling (1905 – 1986) produce in 1937 a new concept in Number Theory . He defined the generalized prime numbers by positive increasing real sequence all of its terms greater than one and tend to infinity and mathematically represented as $\mathbb{P}_p = \{r_i\}_{i=1}^{\infty}$, $1 < r_1 \leq r_2 \leq \cdots$. The sequence

$\mathbb{N}_p = \{n_i\}$ that consists of all numbers of the form $r_1^{a_1}.r_2^{a_2} \ldots r_m^{a_m}$ is called generalized integer numbers where $a_1, a_2, a_m \in \mathbb{N} \cup \{0\}$. All r_i is called Beurling primes and n_i are Beurling's integers. Clearly, the elements of $\mathbb{P}_p$ and $\mathbb{N}_p$ may occur more than one time with specific multiplicity . $(\mathbb{P}_p, \mathbb{N}_p)$ is called generalized primes system and briefly ,g-primes system .Naturally , the generalized counting functions of primes and integers will be defined as

$$\pi_p(x) = \sum_{p \leq x, p \in \mathbb{P}_p} 1 \quad \text{and so} \quad N_p(x) = \sum_{n \leq x, n \in \mathbb{N}_p} 1$$

And as all the arithmetic and counting functions mentioned earlier , the analogue zeta function will be named Beurling zeta function and defined by Dirichlet series as $\zeta_p(s) = \sum_{n \in \mathbb{N}_p} n^{-s}$ and by Euler product as

$$\zeta_p(s) = \prod_{p \in \mathbb{P}_p} \frac{1}{1 - p^{-s}}$$

As a Mellin transformation of $N_p(x)$ zeta function can be defined as

$$\zeta_p(s) = \int_{-1}^{\infty} x^{-s} d\, N_p(x)$$

The generalization of the PNT which was by Beurling is as the following

Theorem 2.1.1 [1]. (Beurling's Prime Number Theorem)

If $N_p(x) = Kx + O\left(\frac{x}{\log^{\gamma}(x)}\right)$ for some $K > 0$ and $\gamma > \frac{3}{2}$ then

$$\pi_p(x) \sim \frac{x}{\log(x)}.$$

Beurling also proved that it is necessary that γ be greater than $\frac{3}{2}$ to ensure the Beurling PNT .

Many mathematicians tried to give equivalent theorems to Beurling's generalized prime number theorem . Nyman , for example, and not as a limitation in [5] gave a development of Beurling's generalized prime Number Theory but un fortunately there were some mistakes in the proof . After then , G.Debruyne and J.Vindas in [8] fixed the mistakes and gave the following theorem

Theorem 2.1.2 [8]

For a generalized number system and for complex number $s = u + it$, the following four statements are equivalent

1. For some $K > 0$ the generalized counting function of integers satisfies $N_p(x) = Kx + O\left(\frac{x}{\log^{\gamma}(x)}\right)$, for all $\gamma \in \mathbb{N}$

2. For some $K > 0$, the function $F(s) = \zeta_p(s) - \frac{K}{1-s}$ has C^{∞}-extension to $u \geq 1$ there is $\varkappa > 0$

   such that $F^{(n)}(1 + it) = O(|t|^{\varkappa})$, for all $n \in \mathbb{N}$

3. For some $K > 0$ and each $\varkappa > 0$ the function

   $$F(s) = \zeta_p(s) - \frac{K}{1 - s} \text{ satisfies}$$

   $F^{(n)}(u + it) = O(1 + |t|^{\varkappa})$, for $u > 1$ and all $n \in \mathbb{N}$ with global $O_{\epsilon,n}$-constants

4. $\Pi_p = \text{li}(x) + O\left(\frac{x}{\log^{\gamma}(x)}\right)$ for any $\gamma \in \mathbb{N}$ where $\text{li}(x) = \int_2^x \frac{dt}{\log(t)}$

Nyman in [5] said that the conditions 1 and 4 is equivalent also to :

For any $\varkappa > 0$ and $n \in \mathbb{N}$,

$$\zeta_p^{(n)}(u + it) = O(|t|^{\varkappa}) \text{ and } \frac{1}{\zeta_p(u + it)} = O(|t|^{\varkappa})$$

uniformly on the region $u > 1$ and $|t| \geq \varkappa$. Actually , the equivalence is almost imposable because it does not tell us any information about ζ_p neer $s = 1$ and that was be noticed by Ingham in [3]

2. 2 . The Generaliesed (Beurling counting function):

$\pi_p(x)$,N_p(x), defined respectively ;by:

$\pi_p(x) = \sum_{p \in P, p \leq x} 1$ and $N_p(x) = \sum_{n \in N, n \in x} 1$.

The methods invariably involve the associated

2.3 Beurling Zeta function

$$\xi_p(s) = \prod_{p \in P} \frac{1}{1 - P^{-S}} = \sum_{n \in N} \frac{1}{n^s} .$$

$\psi_p(x) = \sum_{p^k \leq x, p \in P, k \in N} log\ p = \sum_{n \leq x, n \in N} \Lambda_p(n),$

Wher Λ_p denotes the (generalized)von mangoldt function ,defined by:

$\Lambda_p(n) = log\ p\ if\ n = p^m$ for $p \in P\ and\ m \in N, and\ \Lambda_p(n) = 0$ otherwise .

This is because these functions are directly related to $\xi_p(s)\ via$

$\xi_p\ (s) = s\ \int_1^{\infty} \frac{N_p(x)}{x^{s+1}}$ dx and $-\frac{\xi'(s)}{\xi\ (s)}$ = s $\int_1^{\infty} \frac{\psi_p(x)}{x^{s+1}}$ dx .

Chapter Three

Beurling Prime Systems

Introduction

we give background to Beurling (or generalised) prime systems and the associated Beurling zeta function and put the theory in its historical context. First we introduce discrete g-prime systems with some examples, while in the second section we will define the continuous version of g-prime systems with some examples. Here π_p and N_p are still increasing functions but need not
be step functions. Now ξ_p is defined as the Mellin transform of N_p
That is

$$\xi_p(s) = \int_{1-}^{\infty} x^{-s} \, dN_P(x)$$

In the last part we list relevant known results which are necessary for our work Let F denote the space of all functions f : R→C such that f is right- bounded variation with f(x) = 0, ∀x ∈ (−∞,1) continuous and of local. Let$F^+ \subseteq F$ such that for any $f \in F^+$, f is an increasing function. For a ∈R, let
$F_a = \{f \in F: f(1) = a\} and \;\; F_a^+ = F_a \cap F^+$

3.1 Types of g-prime systems

There are two types of g-prime systems but first it needs to move our attention to understand the **outer g-prime system** $(\mathrm{N_p}, \Pi_\mathrm{p})$. Here for $\mathrm{N_p} \in F_1^+ = \{\text{all increasing functions with } \mathrm{f}(1) = 1\}$

and $\Pi_\mathrm{p} \in \mathrm{F}_0^+ = \{\text{all increasing functions with } \mathrm{f}(1) = 0\}$

$$\text{with } \mathrm{N_p} = \exp * \Pi_\mathrm{p}.$$

We refer to the function $\mathrm{f} = \exp_* \mathrm{g}$ if $\mathrm{f} * \mathrm{g_L} = \mathrm{f_L}$ where Clearly if $\Pi_\mathrm{p} \in \mathrm{F}_0^+$ then $\exp * \Pi_\mathrm{p} \in \mathrm{F}_1^+$ hence any $\Pi_\mathrm{p} \in \mathrm{F}_0^+$ generates an outer g-prime with $\mathrm{N_p} = \exp * \Pi_\mathrm{p}$.

Definition 3.1.1 [12]. A g-prime system is an outer g-prime system if there exist $\pi_p \in F_0^+$ and $\Pi_p(x) = \sum_{a=1}^{\infty} \frac{1}{a} \pi_p(x^{1/a})$. π_p can be defined by Mobius function as

$$\pi_p(x) = \sum_{a=1}^{\infty} \frac{\mu(a)}{a} \Pi_p\left(x^{1/a}\right)$$

The generation of Chebyshev function is denoted by

$$\psi_p(x) = \int_1^x \log(t)\, d\Pi_p(t)\,.\,\text{Note that } \psi_p(x) \in F_0^+$$

Definition 3.1.2[12]. The discrete g-prime system is a generalized prime system which π_p is an increasing step function with integer jumps. In this case $\psi_p(x)$ will equal to

$$\sum_{\substack{p^k \le x,\ k \in \mathbb{N} \\ p \in P}} \log(p)$$

Definition 3.1.3 [12]. The continuous g-prime system is a generalized prime system which π_p is increasing and need not to be step function.

Example 3.1.3.1. 1[12]

At first , if one can take out the prime 2 from the sequence of the actual primes $P = \{2,3,5,7,11, \ldots\}$ and see by the fundamental theorem of arithmetic that sequence of Beurling's integers will contain all odd integers and

$\pi_p(x) = \pi(x) - 1$ and thus $N_p(x) = \left[\frac{x+1}{2}\right] = \frac{x}{2} + O(1)$

Example 3.1.3. 2[12]

Let W=$\{w_n\}_{n=0}^{\infty}$ be the sequence or weird numbers and consider the sequence $P_p = \{w_i\}_{i \ge n}$ such that each w_n appears n+1 times
P_p={70,836,836,4030,4030,4030,5830,5830,5830,5830,7192,7192,7192,7192,7192,…}

Then $\mathbb{N}_p$ will contain 1 , all weird numbers and their multiplicities and all the powers of all weird numbers and their multiplicities

Example3.1.3. 3 Let $\amalg_p(x) = \int_1^x \frac{1-t^{-1}}{log\ t}\ dt$,
x≥ 1 . $this\ means\ that\ clearly$ $\amalg_p \in\ F_1^+$ and

$$\psi_p(x) = \int_1^x log\ t\ \ \amalg_p'(x)\ dt = \int_1^x log\ t\ \frac{1-t^{-1}}{log\ t}$$

$$= \int_1^x (1-t^{-1})\ dt \quad = x - 1 - logx \qquad \in\ F_0^+$$

$$\psi_p(x) = x - 1 - logx \qquad ,x > 0 \quad ,\psi_p(x) \in\ F_0^+$$

Now

$$N_p(\mathrm{x}) = \int_1^x log\ t\ d\ N_p(t) \quad = \int_1^x N_p(\frac{x}{t})\ \ d\ \psi_p(t)$$

$$\text{Let u} = \frac{x}{t} \quad \to t = \frac{x}{u} \ \ \to dt = -\frac{x}{u^2}\ du$$

$$= \mathrm{x} \int_1^x N_p(u)(1 - \frac{u}{x})$$

$$\int_1^x log\ t\ dN_p(t) = x\ \int_1^x N_P(u) \quad (1 - \frac{u}{x})\ \frac{du}{u^2}$$

$$N_p(x)log\ x - \int_1^x \frac{N_p(t)}{t} \quad dt = x\ \int_1^x \frac{N_p(u)}{u^2}\ du - \int_1^x \frac{N(u)}{u} \quad du$$

$$\frac{N_p(x)log\ x}{x} \quad = \int_1^x \frac{N_p(u)}{u^2}\ du$$

$$(\frac{N_p(x)log\ x}{x})' \ = \ \frac{N_p(x)}{x^2}$$

$$\frac{x\ (N_p\left(x)'\ log\ x + \frac{N_p(x)}{x}\right) \ - N_p(x)l\ xog}{x^2} = \frac{N_p(x)}{x^2}$$

$$\frac{x\,N_p(x)'\,log\,x}{x^2} + \frac{N_p(x)}{x^2} - \frac{N_p(x)log\,x}{x^2} = \frac{N_p(x)}{x^2}$$

$$\frac{N_p(x)'\;log\,x}{x} = \frac{N_p(x)log\,x}{x^2}$$

$$= 1N_p(x)' = \frac{N_p(x)}{x}$$

$$N_p(x)' = \frac{N_p(x)}{x} \quad = \quad \rightarrow \frac{N_p(x)'}{N_p(x)} = \frac{1}{x}$$

So $\log N_p(x) = log\,x + c$

$$N_p(x) = cx \quad but \quad N_p(1) = 1 \quad , c = 1$$

$$N_p(x) = x \quad , x \ \geq 1$$

3.2 Some known results from literature

1. Beurling in 1937 [1] proved that if $\mathrm{N_p}(\mathrm{x})$ and $\Pi_\mathrm{p}(\mathrm{x})$ are increasing functions in $[1, \infty]$ and related by the relation

   $$\int_1^\infty \mathrm{x}^{-\mathrm{s}}\,\mathrm{dN_p}(\mathrm{x}) = e^{\int_1^\infty x^{-s}\,d\Pi_p(x)}$$

   Where s is complex with $\Re(\mathrm{s}) > 1$ and if $\mathrm{N_p}(1) = 1$ and

   $$\mathrm{N_p}(\mathrm{x}) = \mathrm{kx} + \mathrm{O}(\mathrm{xlog}^{-\lambda}(\mathrm{x}))$$

   for some positive integer k and ג then the Burling Prime Number Theorem is satisfied if ג > 3/2 and he gave an example in which if ג = 3/2 but Burling Prime Number Theorem fails to hold.

2. Nyman , in 1947 in [5] showed that for x tend to infinity , if there is a positive real number K then for any j > 0

   $$\mathrm{N_p}(\mathrm{x}) = \mathrm{Kx} + \mathrm{O}(\mathrm{x}(\log(\mathrm{x}))^{-\mathrm{j}}$$

Iff

$$\pi_p(x) = li(x) + O(x(\log(x))^{-j}$$

3. Dimond in 1969 in [18] showed that if
   $N_p(x) = kx + O(x(\log(x))^{-\frac{3}{2}}(x) \times e^{-(\log\log(x))^{\alpha}})$

with $k > 0$ and $\alpha > \frac{1}{3}$ then

   a. $\zeta(s) = \int_1^{\infty} x^{-s}\, dN_p(x)$ has no zeros on the line $u = 1$
   b. $\Pi_p(x) \sim \frac{x}{\log(x)}$

4. Mallaivin , in 1969 in [27] proved that for $0 < \alpha < 1$ and

   a. $b, c > 0$ $N_p(x) = cx + O\left(xe^{-b(\log(x))^{\beta}}\right)$ implying
   b. $\Pi_p(x) = li(x) + O\left(xe^{-a(\log(x))^{\alpha}}\right)$ where $\alpha = 10\beta$ for some a .
5. Diamond again in 1970 [17] , enhanced Mallavin's result conversely by showing that

$$N_p(x) = \rho x + O\left(xe^{-b(\log x \log\log x)^{\beta}}\right)$$

   for some $b > 0$ where $\beta = \frac{\alpha}{\alpha+1}$ If

$$\Pi_p(x) = li(x) + O\left(xe^{-a(\log(x))^{\alpha}}\right)$$

   holds for $0 < \alpha < 1$ and for some posative constant a .

6. Diamond , in 1977 [19] showed as a converse of Beurling's PNT, that there exist a positive constant c such that

$$N_p(x) \sim cx \text{ as } x \longrightarrow \infty \text{ if } \int_2^{\infty} t^{-2} \left|\Pi_p(t) - \frac{t}{\log(t)}\right| dt < \infty.$$

7. Kahane , in 1997 [24] presented a great extension to the Beurling's PNT. he proved that

$$\int_{1}^{\infty} \left|\frac{(N(t) - at)\log(t)}{t}\right|^2 \frac{dt}{t} < \infty$$

for some a > 0 implies the Beurlings PNT.

8. Balanzario , in 1998 in [ep] proved that

$$\Pi_p(x) = li(x) + O(xe^{-\alpha(\log(x))^{\frac{1}{2}}})$$

and

$$N_p(x) = cx + \Omega_{\pm}(xe^{-b(\log(x))^{\frac{1}{2}}})$$

For some positive a , b and c.

9. Hilberdink , in 2006 in [41] improved Diamond result in [hda] as : Let $\psi_p(x) = x + O(x^a)$ for som $a \in (0,1)$ then

$$N_p(x) = bx + O\left(xe^{-c\left(\frac{1}{\log(x)\log\log(x)}\right)^{-\frac{1}{2}}} \right)$$

Chapter Four

Some examples of Discrete and continues Beurling prime systems

Introduction

For the reader, writing $N_p, \pi_p, \xi_p, \ldots$ to inform the reader that these are the generalized (or Beurling) counting function(of integers and primes) and the Beurling Zeta function. **Beurling in 1937 stated that any infinite** and positive real sequence such that the first element must precisely>1 called a generalized primes. Further, the sequence of Beurling integers can be build by using the fundamental theorem of arithmetic's. This article focus on addressing an examples of discrete and continues beurling's prime systems. .The difficulty of this work is when you have a discrete prime system then its possible to build a continuous prime system, while the converse is impossible mostly. Further, this work shows the relation between Beurling counting functions and Beurling Zeta function as well.

The aim of the paper

Is to show there are a discrete and counting g-prime systems with some difference of the define of $\pi(x)$

The well know counting functions of prime and integers defined as

$$\psi\,(x) = \sum_{n\leq x} 1\ ,\ \theta(x) = \sum_{p\leq x} \log p\ ,\quad \pi(x) = \sum_{p\leq 1} 1$$

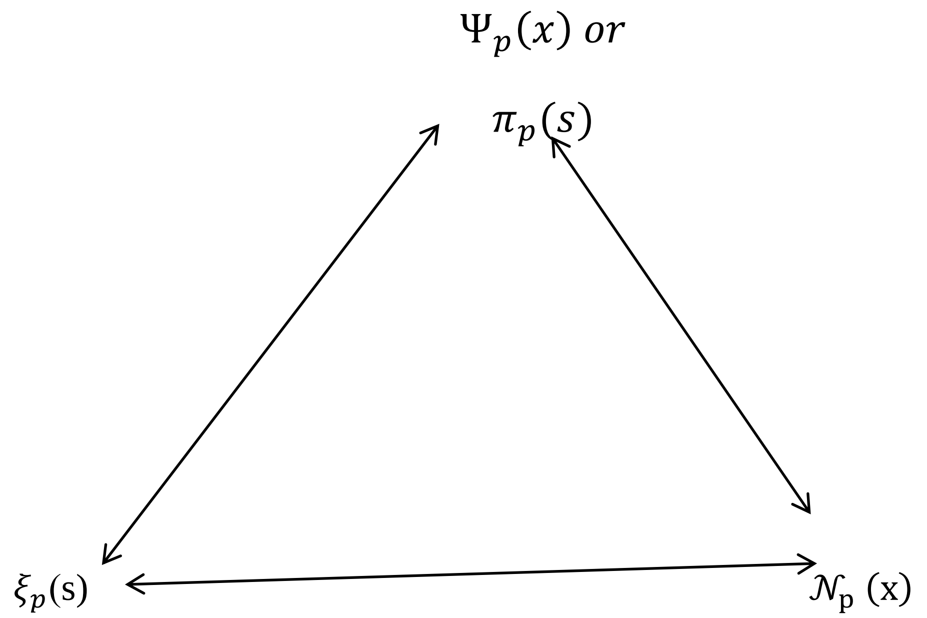
4.1. The link between the counting function $\psi(x), N(x)$ and Beurling Zeta function

(1) $\zeta(s)=\int_1^\infty x^{-s} \quad dN(x)$

(2) $-\frac{\zeta(s)}{\zeta(s)} = s\int_1^\infty x^{-s-1}\ \psi(x)\ d(x)$

(3) $-\frac{d}{ds}\log\zeta(s) = s\ \int_1^\infty x^{-s-1}\ \psi(x)\ dx$

So, any information a bout $\pi(x) or\ N(x)$ give information a bout $\xi(s)$ and ricervers



Figure(1) Shows the triangular relation

We able to introduce F the space of all function $f: R \to ¢$ such that f is right – continuous and of local bounded variation with f(x)=0 , $\forall\, x \in (-\infty, 1). let\ F^+ \subseteq Fsuch\ that\ for\ any$
$F \in F^+\ , f\ is\ an\ increasing\ function\ . For\ a \in R$.

4.2. Types of g -prime systems

There are two types of g – prime systems as shown in the follows examples

Example 4.5.1 :∗ [On Chebyshev's for beurling's generalized primes]

Suppose that $\pi(x) = \int_1^x \frac{1-t^{-\rho}}{\log t}\ G\ (\log t)dt$ exist with

G(logt)=1+ $2\sum_{j=1}^{n-1} \alpha_j \cos(jx) so\ , as\ mention\ in\ [*] after\ deep$ details

ξ(s)=$[\frac{s+\rho-1}{s-1}\ \prod_{0\ <|j|<n(} 1 - \frac{\rho}{s+ij-\rho})^{-\alpha j}]0 < |j|\ < n$

$N\ (X) = c\,x + o\ (x\ (log\,x)^{-\boldsymbol{\alpha}})$

THIS show there counting g -prime($\amalg(x), N(x)$) $\boldsymbol{there\ is\ a\ counting\ g-prime\ system}$

$$\Pi(x) = li(x) + E_1(x)$$
$$N(x) = li(x) + E_2(x)$$

Example2 *: using the definition of* $\Pi(\mathrm{x}) = \int_1^{\mathrm{x}} \frac{1-\mathrm{t}^{\frac{-1}{2}}}{\log \mathrm{t}} \mathrm{G}\,(\log \mathrm{t})\mathrm{dt}$ *with fixed g(logt) to be constan and* $\rho \in (0, \frac{1}{2})$ *then*

$$\amalg_1(x) = \int_1^x \frac{1 - t^{-\frac{1}{2}}}{logt} dt$$
$$\psi(x) = \int_1^x \amalg_1(\acute{x}) dt$$
$$\psi(x) = \int_1^x \left(1 - t^{-\frac{1}{2}}\right) dt$$
$$\psi(x) = (x-1) - \sqrt{x} + 2\,, \psi(x) = x - \sqrt{x} + 1$$
$$\int_1^x logt\, d\, N(t) = \int_1^x N(\frac{1}{\sqrt{t}}) d\, \psi(t) dt$$
$$let\, u = \frac{x}{t} \rightarrow t = \frac{x}{u}, \frac{dt}{du} = -\frac{x}{u^2}, dt = \frac{-x}{u^2} du$$
$$\int_1^x logt\, d\, N(t) = \int_1^x N(\frac{1}{t}) \left(1 - t^{\frac{-1}{2}}\right) dt$$
$$\int_1^x logt\, dN(t) = \int_1^x N(u)(1 - \frac{x}{u})^{\frac{-1}{2}} \left(\frac{x}{u^2}\right) du$$
$$L.H.S \rightarrow \int_1^X logt\, dN(t) \qquad , let\ u = logt$$
$$du = \frac{1}{t} dt\,, dv = dN(t), v = N(t)$$
$$\int_1^x udv = uv - \int_1^x vdu = \int_1^x logt\, dN(t)$$
$$= logx\, N_1(x)|_1^x - \int_1^x \frac{N(t)}{t} dt = logxN(x) - \int_1^x \frac{N(t)}{t} dt$$
$$\text{R.H.S} \rightarrow \int_1^X N(u)(1 - (\frac{X}{U})^{\frac{-1}{2}}) \left(\frac{x}{u^2}\right) du$$
$$= logx\, N_1(x) - \int_1^x \frac{N(t)}{t} dt = \int_1^x N(u)(1 - (\frac{x}{u})^{-\frac{1}{2}}) \left(\frac{x}{u^2}\right) du$$

$$=\int_1^x N(u)(\frac{x}{u^2}-\frac{x^{\frac{1}{2}}}{(u^2)^{\frac{3}{2}}})du=\int_1^x N(u)(\frac{x}{u^2}-\frac{\sqrt{x}}{u^{\frac{3}{2}}})du$$

$$logxN_1(x)=\int_1^x[(\frac{N(u)}{u}+N(u)(\frac{x}{u^2}-\frac{\sqrt{x}}{u^{\frac{3}{2}}})]du$$

$$\frac{d}{dx}\left(logxN_1(x)\right)=N\left(\frac{x}{u}\right)+N_1(x)(\frac{x}{x^2}-\frac{\sqrt{x}}{x^{\frac{3}{2}}})$$

$$\frac{d}{dx}\left(logx\,N_1(x)\right)=\frac{N(x)}{x}+N(x)\left(\frac{1}{x}-\frac{1}{x^2}\right)$$

$$=\frac{N_1(x)}{x}(1+\left(1-\frac{1}{x}\right))$$

$$\frac{d}{dx}\left(logxN(x)\right)=\frac{N_1(x)}{x}(2-\frac{1}{x})$$

$$\acute{N}(x).logx+\frac{N_1(x)}{x}=\frac{N(x)}{x}(2-\frac{1}{x})$$

$$\acute{N}\,(x).logx=\frac{N_1(x)}{x}(2-\frac{1}{x}-1)$$

$$\acute{N}(x).logx=\frac{N_1(x)}{x}(1-\frac{1}{x})$$

$$\frac{\acute{N}(X)}{N_1(x)}=\frac{1}{xlogx}(1-\frac{1}{x})$$

$$\frac{d}{dx}logN_1(x)=\frac{d}{dx}(\frac{1}{xlogx}\left(1-\frac{1}{x}\right))$$

$$logN_1(x)=\int\frac{1}{xlogx}(1-\frac{1}{x}))$$

$$e^{\mathrm{LOG}\,N(x)}=e^{\int\frac{1}{xlogx}(1-\frac{1}{x})}$$

$$N_1(1)=1$$

The above shows that there is adiscreat G-P systems(ψ(X), $\boldsymbol{N_1}(x)$)

$$\psi(x) = x - \sqrt{x} + 1, N_1(x) = \frac{1}{xlogx}(1 - \frac{1}{x})$$

so there is a system such that ($\mathbb{III}_{\boldsymbol{1}}(\boldsymbol{x})$,$\boldsymbol{N_1}(\boldsymbol{x})$) *exist*

$$\Pi_1(x) = li(x) + E_1^0(x)$$
$$N_1(x) = li(x) + E_2^0(x)$$

Chapter Five

Conclusion and Future work

Conclusion

Chapter three shows the restricting of the gap between the error term of $|N_0(x) - \tau x|$ and itself in Riemann hypothesis . The important question here is : Can anyone get restrict the error term furthermore? This question leads to a big future work . For more precise , anyone can restrict the gap by using other sequences but , a big challenge will be forced because of the difficulty of manipulation that should use to get closer from Riemann hypothesis .

Future work

- This work opens window to the reader for enhancing the results in chapter three and the guide line is Riemann hypothesis .
- Finding a sequence of odd primitive weird numbers by applying the given algorithm in chapter three .
- Decreasing key generation time of proposed RSA algorithm in chapter four by enhancing the keys generation algorithm .
- Mix the proposed RSA algorithm in chapter four with other cryptography algorithms

References

[1] A. Beurling, *Analyse de la loi asymptotique de la distribution des nombres premiers generalizes*, Acta Arithmetica, 68(1937), 255-291.

[2] A. Goel, *Encryption algorithm using dual modulus,* International Conference on Computational Intelligence & Communication Technology, Ghaziabad, 2017, 1-4.

[3] A. Ingham, *The distribution of prime numbers*, Cambridge University Press,1990, 2–5.

[4] A. Ivic, *The Riemann zeta function theory and applications*, NewYork:Wiley, 1985.

[5] B. Nyman, *A general prime number theorem*, Acta Math. 81 (1949), 299-307.

[6] C. N. Friedman, *Sums of divisors and Egyptian Fractions*, Journal of Number Theory, 44 (1993), 328–339.

[7] D. E. Iannucci, *On primitive weird numbers of the form* $2^k pq$, (2015), arXiv:1504.02761v1

[8] D, Gregory, J. Vindas, *On general prime number theorems with remainder*, In: Oberguggenberger M., Toft J., Vindas J., Wahlberg P. (eds) Generalized Functions and Fourier Analysis. Operator Theory: Advances and Applications, Springer, 2017.

[9] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, Oxford Clarendon Press, 1986.x.

[10] E. P. Balanzario, *An example in Beurling's theory of primes*, Acta Arithmetica, 87 (1998), 121-139.

[11] F. Al-Maamori, T. Hilberdink, *An example in Beurling's theory of generalised primes*, Acta Arithmetica, 4 (2015), 383-395.

[12] F. Almaamori, *Theory and examples of generalised prime Systems*, PhD thesis, University of Reading, 2013.

[13] F. Chowdhury, M. Chowdhury, *Essentials of number theory*, Pi Publications, Dhaka Bangladesh, 2005.

[14] G. Amato, M. Hasler, G. Melfi, M. Parton, *Primitive weird numbers having more than three distinct prime factors*, Riv. Mat. Univ. Parma, 7(1)(2016), 153-163.

[15] G. H. Hardy, *Sur les zeros de la function*, Comptes rendus de l'Académie des Sciences, 158(1914), 1012–1014.

[16] G. Melfi, *On the conditional infiniteness of primitive weird numbers, Journal of Number Theory,* 147 (2015), 508-514.

[17] H. Diamond, *Asymptotic distribution of Beurling's generalized integers*, Illinois J. Math., 14 (1970), 12-28.

[18] H. Diamond, *The prime number theorem for Beurling's generalized numbers*, J. Number Theory, 1 (1969), 200-207.

[19] H. Diamond, *When do Beurling's generalized integers have density?*, J. Reine Angew. Math, 295 (1977), 22-39.

[20] H. Eves, *An introduction to the history of mathematics*, 6th Edition, Thomson, 1990.

[21] H. M. Srivastava, E. A. Miller, *A reducible case of double hypergeometric series involving the Riemann ζ-function*, Bulletin of the Korean Mathematical Society, 33 (1) (1996), 107-110.

[22] H. Sara, *Beurling prime systems and the Riemann hypothesis*, PhD thesis, University of Baghdad, 2019.

[23] I. Jahan, M. Asif, L. Rozario, *Improved RSA cryptosystem based on the study of number theory and public key cryptosystems*, American Journal of Engineering Research, 1 (2015), 143-146.

[24] J. P. Kahane, *Sur les nombres premiers generalises de Beurling. Preuve d'une conjecture de Bateman et Diamond, Journal de theorie des nombres de Bordeaux*, 9 (1997), 251-266.

[25] M. Thangavel, *An enhanced and secured RSA key generation scheme*, Journal of Information Security and Applications (2014).

[26] N. Iswari, *Key generation algorithm design combination of RSA and ElGamal algorithm,* In: International Conference on Information Technology and Electrical Engineering,8^{th} edition, Yogyakarta, 2016.

[27] P. Malliavin, *Sur le reste de la loi asymptotique de repartition des nombres premiers generalises de Beurling*, Acta Math, 106 (1961), 281-298.

[28] R. Crandall, E. Richard, *On the quantum zeta function*, Journal of Physics, **29** (21) (1996), 6795–6816.

[29] R. K. Guy, *Unsolved Problems in Number Theory*, 3rd Edition, Springer, 2004.
[30] R. Patidar, R. Bhartiya, *Modified RSA cryptosystem based on offline storage and prime number*, International Conference on Computational Intelligence and Computing Research, Enathi, 2013,1-6.
[31] R. Rivest, L. Ronald, A. Shamir, L. Adleman. *A method for obtaining digital signatures and public-key cryptosystems*, Communications of the ACM, 1978, 120-126.

[32] S. A. Jaju, S. S. Chowhan, "*A Modified RSA algorithm to enhance security for digital signature*," 2015. International Conference and Workshop on Computing and Communication, Vancouver, 2015, 1-5.
[33] S. Al-Kaabi, S. Belhaouari, *Methods toward enhancing RSA algorithm*, International Journal of Network Security & Its Applications, 2019.
[34] S. Gupta, J. Sharma, *A hybrid encryption algorithm based on RSA and Diffie-Hellman*, 2012, International Conference on Computational Intelligence and Computing Research, Coimbatore, 2012, 1-4.
[35] S. J. Benkoski, P. Erdos, *On weird and pseudo perfect numbers*, Mathematics of computation, 126(1974), 617-623.
[36] Sh. Sarah, *Introduce a generalized public keys using a sequence of ω-numbers built from Beurling's function of reals*, Journal of Global Scientific Research, 5 (2020), 596-601.
[37] S. Kravitz, *A Search for Large Weird Numbers*, J. Recr. Math, 9 (1976), 82-85.
[38] S. Pajunen, *On primitive weird numbers*, In: A Collection of Manuscripts Related to the Fibonacci Sequence (18th Anniversary Volume), V. Hoggat, ed., Fibonacci Association, 1980, 162-166.
[39] T. Elgamal, *A public key cryptosystem and a signature scheme based on discrete logarithms*, In: Transactions on Information Theory, vol. 31, no. 4 (1985), 469-472.
[40] T. M. Apostol, *Mathematical Analysis* ,Second Edition, Addison-Wesley, 1974.

[41] T. M. Apostol, *Introduction to analytic number theory*, Springer, 1976.

[42] T. Hilberdink, L. Lapidus, *Beurling zeta function , generalized primes, and fractal membrane*, Acta Appl. Math, 94 (2006), 21-48.

[43] W. Scallings, *Cryptography and network security*, 7th Edition, Pearson, 2017.

[44] W. Zhang, Beurling primes with RH, Beurling primes with large oscillation, Math. Ann., 337 (2007), 671{704.

الخلاصة

نظرية الاعداد التحليلية هي الجزء الرئيسي من الرياضيات الصرفة والتي تعتبر اساس العديد من مواضيع الرياضيات مثل الجبر والتحليل المعقد وغيرها يتم دراسة توزيع الاعداد ووضائف العد ذات الصلة بلاعداد الاولية والاعداد الصحيحة من قبل العديد من المؤلفين بالاضافة الى دراسة توزيع الاعداد الاولية المعممهالتي تسمى اعداد بيرلنك بعض الاحيان خلال الخمسين عام الماضية برزت العديد من النتائج المنشورة حول الاعداد الاولية المعممة والاعداد الصحيحة المعممة وريمان زيتا المعممة الهدف من هذا العمل هو الحصول على نظام جديد والذي هو عبارة عن دالة حسابية لاعداد اولية مع دالة حسابية لاعداد اولية معممة

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بابل
كلية التربية للعلوم الصرفة
قسم الرياضيات

تعميم الانظمة الأولية لدالة بيرلنك زيتا مع بعض التقريب المحسن

رسالة

مقدمة الى مجلس كلية التربية للعلوم الصرفة / جامعة بابل كجزء من متطلبات نيل درجة الماجستير في التربية / الرياضيات

من قبل

هدى صالح حمزة جاسم

بإشراف

أ. د. فائز علي المعموري

1443هـ 2022م