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New Results of The Algebraic Properties Using Geometric Function Theory

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بسم الله الرحمن الرحيم

۞ نرفعُ درجاتٍ مَن نشاءُ وفَوقَ كُلِ ذِي عِلمٍ عَليم ۞

صدق الله العلي العظيم

من سورة يوسف (76)

Declaration

Aware of legal liability I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

Intissar Abdulhur Kadum

DEDICATION

To my dear father's soul

To the source of kindness

My mother

To my support in life

My brothers and sisters

I dedicate the fruit of my humble effort

Intissar Abdulhur

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Praise be to God, Lord of the worlds, and prayers and peace be upon the most honorable of the prophets and messengers, our Master Muhammad, his family, his companions, and those who followed them with kindness until the Day of Judgment, and after...

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Abstract

This study aims to build new formulas and connection between geometric function theory (especially sandwich theorems) and Viete’s theorem in elementary algebra through Newton's identities, and all the algebraic concepts that can be related with it. The relationships between those theorems within Hardy space, where some basic concepts of Viete’s theorem are reviewed. The algebraic properties of symmetric group with new formula that defined as ordered cyclic subgroups, this represents in the first stage of this work.

The second stage: We introduced concepts of companion matrix in the Hardy spaces by studying algebraic properties as well as the concepts of composition of matrices. Indeed the effect of operators on the powers' traces of companion matrix, which effects on the complete homogeneous symmetric functions class that involving an ordered cyclic operator. Several operators have been identified in many details such as:

- Newton's identities by companion matrix.
- A new formula of complete homogeneous symmetric functions.
- Derivation the ordered cyclic operator.
- Derivation the inverse of the ordered cyclic operator.

The third stage: The concept of differential subordination and superordination for type of meromorphic complete symmetric functions that defined by some operators were introduced (ordered cyclic subgroups, fundamental differential for ordered cyclic operator).

The fourth stage: The concepts of (third and fourth)- order differential subordination, superordination and sandwich theorems for class of complete homogeneous symmetric functions of three - order and four - order and some classifications of that concept are introduced (linear differential for cyclic operator of this functions, composition of inverse for an ordered cyclic operator).

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List of symbols

Symbols	Description
$\mathbb{C}$	The omplex plane
$\mathbb{R}$	The set of all real numbers
$\mathbb{Z}$	The Integer numbers
$\mathbb{N}$	The set of natural numbers
$\mathbb{U}$	Open unit disk $\{z:z\in\mathbb{C} ; \lvert z\rvert<1\}$
$\partial\mathbb{U}$	Boundary of unit disk $\mathbb{U}$,$\{z\in\mathbb{C} ; \lvert z\rvert=1\}$
$\overline{\mathbb{U}}$	$\mathbb{U} \cup\partial\mathbb{U} =\{z:z\in\mathbb{C} ; \lvert z\rvert\leq 1\}$.
Q	The set of all functions ₫
$f \prec g$	f is subordinate to g
S_n	Symmetric group
$\circ$	The composition of functions
$e_m(x_1, x_2 , \dots , x_n)$	The m-th symmetric polynomial in $x_1, x_2 , \dots , x_n$
C_φ	Composition operator
$\bar{x}_i$	The complex conjugate of x_i
$tr(A)$	The trace of matrix A
$C(p)$	The companion matrix
$det\ (A)$	The determinant of a matrix A

The publication of the following papers from this dissertation has been accepted.

1. Sandwich Theorems for a New Class of Complete Homogeneous Symmetric Functions by Using Cyclic Operator, Symmetry, MPDI (ISSN: 2073-8994), symmetry-1920224(2022).
2. The Differential Subordination and Superordination of Order Three for Complete homogeneous Elementary Functions Class, Journal of Discrete Mathematical Sciences & Cryptography. Taru publication Ref. No . JDMSC-T-1525 (2022).

Introduction

The geometric function theory is the branch of complex analysis which deals with the geometric properties of analytic functions. It was founded around the turn of the 20th century and has remained one of the active fields of the current research. Moreover, in spite of the famous coefficient problem, "Bieberbach Conjecture" that was solved by Louis de Branges in 1985‹it suggests various approaches and directions for the study of geometric function theory [20].

The first aim of the study presented of this thesis is the use of the concept of symmetric polynomials, this type of identities was identified by Albert Girard (1629) [21], It was rediscovered and axiomatized by Isaac Newton around 1666 [34]. Newton's identities, also known as the Girard–Newton formulae, gave relations between two types of symmetric polynomials, namely between power sums and elementary symmetric polynomials. Evaluated at the roots of a monic polynomial $P(x)$ in one variable, that allow expressing the sums of the k-th powers of all roots of $P(x)$ (counted with their multiplicity) in terms of the coefficients of $P(x)$, without actually finding those roots.

Where the idea of the concept of convergence crystallized through Hardy space, we may define it as a relationship as follows:

Let f be analytic at all points within a circle C with center at z_0 and radius r_0. Then at each point z inside C, the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, converges uniformly to $f(z)$ in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$.

The second aim of our work is the derivation some cyclic operator. A frobenius matrix is a special kind of square matrix from numerical mathematics. Some researchers use the transpose of this matrix, which (dually) cycles coordinates, and is more convenient for some purposes. There was a suggestion method by the scientist Barakat[11] to calculate the sum of the powers for the eigenvalues of a diagonal matrix and works iteratively through Newton's identities to get the elementary symmetric functions for these eigenvalues, which represent the coefficients of the characteristic polynomial. Each homomorphic function φ that takes $\mathbb{U}$ into itself we associate the composition operator C_φ defined by:

$$C_\varphi f = f \circ \varphi \ (f \in \mathcal{H}^2)$$

The Littlewood subordination theorem tells us that the operator C_φ takes the Hardy space $\mathcal{H}^2$ into itself. Littlewood's principle also supplies an estimate which shows that C_φ is a bounded operator on $\mathcal{H}^2$, see [48,14] for more details.

Let $\mathcal{H}(\mathbb{U})$ denotes the analytic function class in the open unit disk $\mathbb{U}$, the resolvent det $(\mathrm{I} - \mathrm{A})^{-1}$ of a complex matrix A is naturally an analytic function of eigenvalues $\lambda \in \mathbb{C}$ and this eigenvalues are isolated singularities. In general any matrix has finitely eigenvalues. The resolvent set of A is defined as follows:

$$\varrho(A) = \{\lambda \in \mathbb{C}: \lambda I - A \text{ is invertible}\}$$

and the spectrum of A is expressed by $\vartheta(\mathrm{A}) = \mathbb{C}/\varrho(\mathrm{A})$. Macdonald [34] defined the characteristic polynomial for finite distinct eigenvalues and this class represent the subclasses of analytical functions $\mathcal{H}[a, n]$ which denoted by $\mathbb{H}$ such that $\mathcal{H}[a, 1] = \mathbb{H}$ and has coefficients of the form

$$h_n = \sum_{\substack{j_1 + j_2 + \cdots + j_n = k \\ j_1 + 2j + 3j_3 + \cdots + nj_n = n}} \frac{(-1)^{n-k} \, \mathbb{s}_1^{j_1} \mathbb{s}_2^{j_2} \ldots \mathbb{s}_n^{j_n}}{j_1! \ j_2! \ldots j_n! \ 1^{j_1} \, 2^{j_2} \ldots n^{j_n}} .$$

Indeed we can be reduces a class $\mathbb{H}$ to the class $\mathcal{H}$ of normalized univalent analytical functions and composed of functions of the following form:

$$\frac{H(z)-h_0}{h_1} = z + \sum_{n=2}^{\infty} a_n\, z^n; a_n = \frac{h_i}{h_1}, (z \in \mathbb{U}).$$

This thesis is devoted for studying differential subordination and superordination results for types of meromorphic univalent functions that are defined by some operators. A differential subordination in the complex plane is the generalization of the differential inequality on the real line. The concept of differential subordination plays a very important role in functions of real variable. In the theory of complex-valued function, there are several differential applications in which a characterization of a function is determined by a differential condition. Scientific research have contributed on the differential subordination and superordination with sandwich theorems as [20,29,36,42].

It is very important for us to find new observational and theoretical results in this field with various applications. The cornerstone of geometric function theory is the theory of univalent functions, but new related topics appeared and developed with many interesting results and applications.

This thesis consists of five chapters:

In **chapter one**, the most important basics of our work are given, by presenting necessary theorems and results. The first section view the concept of symmetric group and its most important characteristics, theories and examples. The second section shows with the concept of Newton's identities and its algebraic properties. The third section discuss with the concept of Hardy space and cyclic operator. The last section concerned with important definitions and theorems related with sandwich theorem.

Chapter two consists of three sections. In section one, a new construction will describe in detail in application to cyclic groups, though it could have been introduced at once for symmetric group. In second section, a new formula has been derived with theorems of abstract algebra that is deeper than the elementary properties of symmetric groups. Namely, proceeding from the description this group by using given of cyclic groups, it's called ordered cyclic subgroups. We introduce a new formula of symmetric group as direct product of ordered cyclic subgroup. Third section shows Lagrange's theorem in terms of ordered cyclic subgroup.

Chapter three consists of four sections. Section one is introduced the composition of complete homogeneous symmetric functions as an analytic function which defined in open unit disk. In section two, we recall and founded newton's identities for characteristic polynomials by companion matrix. In section three, we derived a new formula for coefficients of complete homogeneous symmetric in terms of powers of traces' matrix. In section four, we derived an ordered cyclic operator. In section five, we deals with the inverse for an ordered cyclic operator.

Chapter four consists of three sections. In section one; we introduced several results on second-order differential subordination and superordination for class of complete elementary functions. Second section is represents with sandwich theorems for certain analytic completely homogeneous symmetric functions class defined by a new cyclic operator. Here, we derive some results certain complete homogeneous symmetric functions by using differential subordination and superordination defined by an ordered cyclic operator. In the third section deals with the sandwich results of complete homogeneous symmetric functions defined by a differential for ordered cyclic operator.

Chapter five consists of three sections. In section one; we introduced several results on (third and fourth)-order differential subordination and superordination for class of complete elementary functions. The second section concerned with third-order differential subordination results for analytic functions associated with a certain diffrrential of an ordered cyclic operator. The third section deals with the fourth-order differential subordination and superordination results of meromorphic multivalent functions defined by the inverse for ordered cyclic operator. Here, we obtain some applications of fourth-order differential subordination and superordination results involving traces function for powers of matrix transformation $\mathbb{I}^{\mathbb{i}}\left(H(z)\right)$ for meromorphic multivalent functions.

Chapter One: Definitions and Fundamental Concepts

1.1 Introduction

In this chapter, we deal with the most important special mathematical tools in the process of building a thesis, as they are the mainstay of our work. The main aspect of the proximity relationship includes its definition, its most important characteristics and the construction processes that were the basic building block for building the concept of ordered cyclic subgroup, and which was the starting point for the process of building modern concepts that was adopted in this thesis. The first chapter contains four parts:

The first section deals with the concept of symmetric group and its most important characteristics, theorems and examples. The second section dealt with the concept of Newton's identities and its algebraic properties. The third section dealt with the concept of Hardy space and cyclic operator. The last topic concerned important definition and theorems about sandwich theorem.

1.2 Symmetric Group

In this section, we review the concept of symmetric group and its basic properties, also give some examples and basic theorems that we need in our work.

Definition 1.2.1 [15]: Let X be a set with n elements, which label as $1, 2, \dots, n$, a permutation of X is a bijective function $\sigma: X \to X$.

A permutation σ can describe by the following form:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

Because σ is bijective, each element of X occurs exactly once on the second line. The set of all permutations with composition of functions is called a symmetric group and denoted by S_n .

Definition1.2.2[19]: A permutation is called a *k-cycle* if it fixes all elements except k elements, let it be $a_0, \ldots, a_{k-1}$ and acts on those according to:

$$\sigma(a_0) = a_1, \sigma(a_1) = a_2, \ldots, \sigma(a_{k-2}) = a_{k-1}, \sigma(a_{k-1}) = a_0.$$

Such a cycle is denoted by listing the elements it moves between parentheses in such a manner that the image of each element is listed immediately after it:

$$\sigma = (a_0\ a_1\ \ldots\ a_{k-1}),$$

we may start with any element in the cycle:

$$(a_0\ a_1\ \ldots\ a_{k-2}\ a_{k-1}) = (a_1\ a_2\ \ldots\ a_{k-1}\ a_0) = \cdots = (a_{k-1} a_0\ a_1 \ldots\ a_{k-2}).$$

A cycle of length 1 is the identity permutation. A cycle of length 2 is called a transposition. Two cycles are said to be disjoint if they have no elements in common .

Definition 1.2.3[15]: Let φ be a permutation of a set X. An element $x \in X$ is called a fixed point of φ if $\varphi(x) = x$. That is, the fixed points of a permutation are the points not moved by the permutation. For example, $\varphi = \begin{pmatrix} 1\ 2\ 3\ 4\ 5\ 6 \\ 1\ 3\ 2\ 6\ 5\ 4 \end{pmatrix}$ has fixed points {1, 5}, since $\varphi(1) = 1$ and $\varphi(5) = 5$ and everything else is sent to something different.

Definition 1.2.4[19]: Let $(G,*)$ and $(G',*')$ be two groups. We say that G and G' are isomorphic if there is a bijective map $\varphi: G \rightarrow G'$. That is to say, for every g and g' in G, $\varphi(g * g') = \varphi(g) *' \varphi(g')$, and the mapping φ is called isomorphism.

Theorem 1.2.5(Cayley's Theorem)[19]: Let G be a group, then G is isomorphic to a subgroup of a permutation group. Moreover, if G is a finite, then so is the permutation group, so that every finite group is a subgroup of S_n, for some n.

Definition 1.2.6 [32]: The order of an element a in a group is the least positive integer m such that $a^m = e$, if no positive powers of a equals the identity e, a has order infinity. The group G cyclic if it contains some elements x whose powers primary to power of G ,this element is said to generate the group.

Proposition 1.2.7[19]: A cyclic permutation of n elements has order n.

Theorem 1.2.8[15]: If G and G' are two cyclic groups of the same order, then G and G' are isomorphic.

Lemma 1.2.9[15]: Let σ be a permutation of X. If k is the smallest (strictly) positive integer such that $\sigma^k(x) = x$, then the elements $\{x, \sigma(x), \sigma^2\,(x), \ldots, \sigma^{k-1}\,(x)\}$ are all distinct.

Theorem 1.2.10[19]: Any permutation ϕ can be written as a product of disjoint cycles.

Theorem 1.2.11(Lagrange's Theorem)[13]: If G is a finite group of order n and H is a subgroup of G of order k, then $k|n$ and $\frac{n}{k}$ is the number of distinct cosets of H in G.

Definition 1.2.12[28]: A permutation matrix is a square matrix obtained from the same size identity matrix by a permutation of rows. Every row and column has a single 1 with 0's everywhere else. There are $n!$ permutation matrices of size n.

Example 1.2.13[28]: The permutation matrices of order two are given by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and of order three are given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

A permutation matrix is nonsingular, and the determinant is always ∓ 1. In addition, a permutation matrix A satisfies

$$AA^T = I,$$

where A^T is a transpose of a matrix A and I is the identity matrix.

Definition1.2.14[28]: A square matrix is called a Frobenius matrix if it has the following three properties:

1) All entries on the main diagonal are ones,
2) The entries below the main diagonal of at most one column are arbitrary,
3) Every other entry is zero.

Frobenius matrices are named after Ferdinand Georg Frobenius. An alternative name for this class of matrices is Gauss transformation.

Example 1.2.15[37]: A Moore matrix is a matrix of the form:

$$M = \begin{bmatrix} x_1 & x_1^q & \dots & x_1^{q^{n-1}} \\ x_2 & x_2^q & \dots & x_2^{q^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x_m & x_m^q & \dots & x_m^{q^{n-1}} \end{bmatrix}$$

It is a matrix defined over a finite field. The Moore matrix has successive powers of the Frobenius automorphism applied to the first column. Therefore, it is an $m \times n$ n matrix.

Definition 1.2.16[46]: The Stirling numbers of the first kind $s(n,m)$ are defined such that the number of permutation of n elements which contain exactly m permutation cycles is the nonnegative number

$$|s(n,m)| = (-1)^{n-m} s(n,m)$$

This means that $s(n,m) = 0$ for m $>$ n and $s(n,n) = 1$.

The usual stirling numbers of the first kind is a special case of a general function $d_r(n,k)$ which is related to the number of cycles in a permutation.

Definition1.2.17 [46]: The number of ways of partitioning a set of n elements into m nonempty sets (i.e., m set blocks), is called a Stirling set number of the second kind. The Stirling numbers of the second kind are variously and denoted by $S(n,m)$. Since a set of n elements can only be partitioned in a single way into 1 or n subsets, $S(n,1) = S(n,n) = 1$.

Example 1.2.18 [46]: The set $\{1,2,3\}$ can be partitioned into three subsets in one way: $\{\{1\},\{2\},\{3\}\}$, into two subsets in three ways: $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, and $\{\{2,3\},\{1\}\}$, and into one subset in one way: $\{\{1,2,3\}\}$.

1.3 Newton's Identities

Newton's identities give relations between two types of symmetric polynomials, namely between power sums and elementary symmetric polynomials. Evaluated at the roots of a monic polynomial $P(x)$ in one variable, they allow expressing the sums of the k-th powers of all roots of polynomial in terms of the coefficients of $P(x)$, without actually finding those roots. These identities were found by Isaac Newton around 1666, apparently in ignorance of earlier work (1629) by Albert Girard. They have applications in many areas of mathematics, including Galois theory, invariant theory, group theory, combinatorics, as well as further applications outside mathematics, including general relativity. In this section we introduce some definition and theorems of newton's identities.

Definition 1.3.1[34]: A polynomial $p(x_1, x_2, \dots, x_n)$ is called a symmetric polynomial if it satisfies:

$$p(x_{\varphi(1)}, x_{\varphi(2)}, \dots, x_{\varphi(n)}) = p(x_1, x_2, \dots, x_n)$$

for all permutations φ of $\{1, \dots, n\}$. The space of all symmetric polynomials in $x_1, x_2, \dots, x_n$ denoted by Λ_n.

Definition 1.3.2[34]: Suppose $x_1, x_2, \dots, x_n$ are the n roots of a polynomial

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n.$$

Then

$$e_0 = 1,$$

$$e_1(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} x_i = -a_1,$$

$$e_2(x_1, x_2, \dots, x_n) = \sum_{1 \le i_1 < i_2 \le n} x_{i_1} x_{i_2} = a_2,$$

…

$$e_m(x_1, x_2, \dots, x_n) = \sum_{1 \le i_1 < \cdots < i_m \le n}^{n} x_{i_1} \dots x_{i_m} = (-1)^m a_m,$$

…

$$e_n(x_1, x_2, \dots, x_n) = x_1 x_2 \dots x_n.$$

The polynomial $e_m(x_1, x_2, \dots, x_n)$ is called the m-th symmetric polynomial in $x_1, x_2, \dots, x_n$.

It has the following property:

$$e_m(x_{\varphi(1)}, x_{\varphi(2)}, \dots, x_{\varphi(n)}) = e_m(x_1, x_2, \dots, x_n),$$

for all permutations φ of {1, . . . , n}. Recall a permutation of $\{1, \dots, n\}$ is a one-to-one correspondence:

$$\varphi: \{1, \dots, n\} \to \{1, \dots, n\}.$$

Definition 1.3.3 [34]: For each $k \ge 0$, the complete symmetric polynomial is the sum of all monomials of degree k:

$$h_k(x_1, x_2, \dots, x_n) = \sum_{d_1 + \cdots + d_n = k} x_1^{d_1} \dots x_n^{d_n}.$$

In particular $h_0(x_1, x_2, \dots, x_n) = 1$.

It is not hard to see that

$$h_k(x_1, x_2, \dots, x_n) = \sum_{\lambda \in \mathrm{P(k,n)}} m_\lambda(x_1, x_2, \dots, x_n),$$

$s$uch that m_λ is the partition of k.

Thus, for each nonnegative integer k, there exists exactly one complete homogeneous symmetric polynomial of degree k in n variables.

Another way of rewriting the definition is to take summation over all sequences i_k, without condition of ordering $i_p < i_{p+1}$:

$$h_k(x_1, x_2, \dots, x_n) = \sum_{1 \le i_1, \dots, i_k \le n} \frac{m_1!\, m_2! \dots m_n!}{k!} x_{i_{12}} x_{i_2} \dots x_{i_n} .$$

Define the generating function for h_k by:

$$H_n(t) = \sum_{k \ge 0} h_k(x_1, x_2, \dots, x_n)\, t^k .$$

Then we have

$$H_n(t) = \sum_{d_1, \dots, d_n \ge 0} {x_1}^{d_1} \dots {x_n}^{d_n}\, t^{d_1 + \dots + d_n = k} = \frac{1}{\prod_{i=1}^{n} (1 - t x_i)}.$$

Theorem 1.3.4[34]: Consider the generating function of elementary symmetric polynomials:

$$E_n(t) = \sum_{i=1}^{n} e_i(x_1, x_2, \dots, x_n)\, t^i = \prod_{i=1}^{n} (1 - t x_i).$$

Clearly we have $H(t)E(-t) = 1$,

or equivalently,

$$\sum_{r=0}^{n} e_r h_{n-r} = 0 \qquad \text{for all } k \ge 1 .$$

Here we have set

$$e_r(x_1, x_2, \dots, x_n) = 0 ,$$

for $r > n$.

Proposition 1.3.5[34]: The formula $\sum_{r=0}^{n} e_r h_{n-r}$ can be inductively.

For $k = 1$,

$$h_1 - e_1 = 0,$$

$h$ence

$$h_1 = e_1 .$$

For $k = 2$,

$$h_2 - e_1 h_1 - e_2 = 0$$

hence

$$h_2 = e_1^2 - e_2 = \begin{vmatrix} e_1 & e_2 \\ 1 & e_1 \end{vmatrix} .$$

Inductively one finds:

$$h_k = \det\left(e_{1-i+j}\right)_{1 \le i,j \le n} .$$

By symmetry between h and e in the above formula, one also get

$$e_k = \det\left(h_{1-i+j}\right)_{1 \le i,j \le n} .$$

Here we have used the convention that

$e_i(x_1, x_2, \dots, x_n) = 0$ for $i < 0$ or $i > n$.

Proposition 1.3.6[34]: The symmetry between h and e suggests the introduction of the following map $\omega: \Lambda_n \to \Lambda_n$

$$\omega\left(\sum_{m_1,\dots,m_n} a_{m_1,\dots,m_n} e_1^{m_1} + \dots + e_n^{m_n}\right) = \sum_{m_1,\dots,m_n} a_{m_1,\dots,m_n} h_1^{m_1} + \dots + h_n^{m_n} , \qquad (1.2.1)$$

and

$$\omega\left(\sum_{m_1,\dots,m_n} a_{m_1,\dots,m_n} h_1^{m_1} + \dots + h_n^{m_n}\right) = \sum_{m_1,\dots,m_n} a_{m_1,\dots,m_n} e_1^{m_1} + \dots + e_n^{m_n} . \qquad (1.2.2)$$

It has the following properties:

1) ω is a ring isomorphism, i.e. $\omega(p + q) = \omega(p) + \omega(q), \omega(p \cdot q) = \omega(p) \cdot \omega(q)$, for $p, q \in n$.
2) $\omega(e_i) = h_i \ and \ \omega(h_i) = e_i$.
3) $\omega^2 = id$.

In other words, if we define for $\lambda' = (\lambda'_1, \dots, \lambda'_k) \in p'(k, n)$,

$$h_{\lambda'}(x_1, x_2, \dots, x_n) = h_{\lambda'_1}(x_1, x_2, \dots, x_n) \dots h_{\lambda'_k}(x_1, x_2, \dots, x_n)$$

$\{h_{\lambda'}\}_{\lambda' \in p'(k,n)}$ is a basis of Λ_k^n.

Theorem 1.3.7[34]: For $r \geq 1$, the r-th Newton polynomial (power sum) in $x_1, x_2, \dots, x_n$ is $p_r(x_1, x_2, \dots, x_n) = x_1^r + \cdots + x_n^r$.

The generating function for them is

$$P_n = \sum_{r\geq 1} p_r(x_1, x_2, \dots, x_n)\, t^{r-1} = \sum_{i=1}^{n}\sum_{r\geq 1} x_i^r\, t^{r-1} = \sum_{i=1}^{n} \frac{x_i}{1 - x_i t}$$

$$= \frac{d}{dt} log \frac{1}{\prod_{i=1}^{n} 1 - x_i t} .$$

It's represented Newton formulas. By comparing, one gets:

$$P_n(t) = \frac{H_n'(t)}{H_n(t)} = \frac{E_n'(-t)}{E_n(-t)} .$$

By applying ω, one gets:

$\omega(p_n)(t) = p_n(-t)$ or equivalently,

$\omega(p_r) = (-1)^{r-1} p_r$ One also has

$H_n'(t) = p_n(t) H_n(t)$, $E_n'(t) = p_n(t) E_n(-t)$.

Equivalently

$$k\, h_k = \sum_{r=1}^{k} p_r h_{k-r} \quad ,$$

$$k\, e_k = \sum_{r=1}^{k} (-1)^{r-1} p_r e_{k-r} .$$

These are called the Newton formulas

Proposition 1.3.8[34]: The formula $\sum_{r=1}^{k} p_r h_{k-r}$ can be inductively.

For $k = 1$,

$$e_1 = p_1.$$

For $k = 2$,

$$2e_2 = p_1\, e_1 - p_2 = \begin{vmatrix} p_1 & p_2 \\ 1 & p_1 \end{vmatrix}.$$

For $k = 3$,

$$3!\,e_3 = 2p_1\,e_2 - p_1\,e_1 + p_2 = \begin{vmatrix} p_1 & p_2 & p_3 \\ 2 & p_1 & p_2 \\ 0 & 1 & p_1 \end{vmatrix}.$$

By induction, one finds

$$k!\,e_k = \begin{vmatrix} p_1 & p_2 & p_3 & \dots & p_{k-1} & p_k \\ k-1 & p_1 & p_2 & \dots & p_{k-2} & p_{k-1} \\ 0 & k-2 & p_1 & \dots & p_{k-3} & p_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p_1 & p_2 \\ 0 & 0 & 0 & \dots & 1 & p_1 \end{vmatrix}.$$

One can also rewrite $k\,e_k$ as:

$$p_k = \sum_{r=1}^{k-1} (-1)^{k-r-1}\ e_{k-r}\ p_r + (-1)^{k-1}\ ke_k.$$

For $k = 1$,

$$p_1 = e_1.$$

For $k = 2$,

$$p_2 = e_1\,p_1 - 2e_2 = \begin{vmatrix} e_1 & 2e_2 \\ 1 & e_1 \end{vmatrix}.$$

For $k = 3$,

$$p_3 = e_1\,p_2 - e_2\,p_1 + 3e_3 = \begin{vmatrix} e_1 & e_2 & 3e_3 \\ 2 & e_1 & 2e_2 \\ 0 & 1 & e_1 \end{vmatrix}.$$

By induction, one finds

$$p_k = \begin{vmatrix} e_1 & e_2 & e_3 & \dots & e_{k-1} & ke_k \\ 1 & e_1 & e_2 & \dots & e_{k-2} & (k-1)e_{k-1} \\ 0 & 1 & e_1 & \dots & e_{k-3} & (k-2)e_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & e_1 & 2e_2 \\ 0 & 0 & 0 & \dots & 1 & e_1 \end{vmatrix}$$

By applying ω ,one gets:

$$k!\,h_k = \begin{vmatrix} p_1 & p_2 & p_3 & \dots & p_{k-1} & p_k \\ -(k-1) & p_1 & p_2 & \dots & p_{k-2} & p_{k-1} \\ 0 & -(k-2) & p_1 & \dots & p_{k-3} & p_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p_1 & p_2 \\ 0 & 0 & 0 & \dots & -1 & p_1 \end{vmatrix},$$

and

$$(-1)^{k-1}p_k = \begin{vmatrix} h_1 & h_2 & h_3 & \dots & h_{k-1} & kh_k \\ 1 & h_1 & h_2 & \dots & h_{k-2} & (k-1)h_{k-1} \\ 0 & 1 & h_1 & \dots & h_{k-3} & (k-2)h_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & h_1 & 2h_2 \\ 0 & 0 & 0 & \dots & 1 & h_1 \end{vmatrix}$$

As a corollary, we have $\Lambda_n = \mathbb{C}[h_1, \dots, h_n]$.

In other words, if we define for $\lambda' = (\lambda'_1, \dots, \lambda'_k) \in P'(k,n)$

$$p_{\lambda'}(x_1, x_2, \dots, x_n) = p_{\lambda'_1}(x_1, x_2, \dots, x_n) \dots p_{\lambda'_k}(x_1, x_2, \dots, x_n),$$

$\{p_{\lambda'}\}_{\lambda' \in P'(k,n)}$ is a basis of Λ_k^n.

1.4 Hardy Space

In this section, we define the Hardy space $\mathcal{H}^2$ and prove some basic results. We refer the reader to Duren's book [8] and J. H. shapiro [18] for more details about Hardy space. Let $\mathbb{U}$ be the unit disk in the complex plane $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$, and let $\mathcal{H}(\mathbb{U})$ be the set of all complex valued functions which are analytic on $\mathbb{U}$. Since pointwise sums and products of analytic functions are again analytic, then $\mathcal{H}(\mathbb{U})$ is a vector space over the field of the complex numbers. Before we give the definition of the Hardy space $\mathcal{H}^2$, we recall Taylor theorem without proof.

We review some links between function theory and operator theory that are created by Littlewood's subordination principle. To each analytic function φ that takes the unit disk $\mathbb{U}$ of the complex plane $\mathbb{C}$ into itself, we associated the composition operator C_φ defined by:

$$C_\varphi f = f \circ \varphi \text{ for all } f \in \mathcal{H}^2.$$

Theorem 1.4.1(Taylor) [17]: Let f be analytic at all points within a circle C with center at z_0 and radius r_0. Then at each point z inside C, the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, converges uniformly to $f(z)$, i.e.,

$f(z) = \sum_{n=0}^{\infty} a_i(z-z_0)^n$ for all z inside C, where $a_n = \frac{f^{(n)}(z_0)}{n!}$ is said to be the n-th Taylor coefficient of the function f.

Remark 1.4.2[17]: The function f belongs to $\mathcal{H}(\mathbb{U})$,then by Taylor theorem:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Now, we give the definition of Hardy space $\mathcal{H}^2$.

Definition 1.4.3[48]: The Hardy space $\mathcal{H}^2$ is the set of all functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{U}),$$

such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, i.e., $\mathcal{H}^2 = \{f \in \mathcal{H}(\mathbb{U}): \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$.

We can define an inner product on $\mathcal{H}^2$ as follows:

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, are any functions in $\mathcal{H}^2$, then the inner product of f and g is:

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \bar{b}_n.$$

Remark 1.4.4 [14]: If f is any function in $\mathcal{H}^2$, then we define the norm of the function f as follows:

$$\|f\|^2 = \langle f, f \rangle = \sum_{n=0}^{\infty} |a_n|^2$$

i.e., $\mathcal{H}^2$ is a normed space.

Definition 1.4.5 [47]:

1) The function φ is said to be self-map of $\mathbb{U}$ if it takes the unit disk $\mathbb{U}$ of the complex plane $\mathbb{C}$ into itself.

2) A one-to-one analytic map is called univalent.

3) Let φ be an analytic self-map of $\mathbb{U}$. If φ is univalent and onto $\mathbb{U}$, then φ is said to be a conformal automorphism of $\mathbb{U}$ or just automorphism of $\mathbb{U}$.

Now, we give two examples of analytic functions:

Example 1.4.6 [47]: For each $p \in \mathbb{U}$, define the special automorphism function:

$$\propto_p (z) = \frac{p-z}{1-\bar{p}z}, \text{ for all } z \in \hat{\mathbb{C}}$$

This function interchanges p with the origin, i.e., $\propto_p (p) = 0$ and $\propto_p (0) = p$.

Littlewood's Subordination Principle 1.4.7 [47]: Suppose φ is an analytic self-map of $\mathbb{U}$, with $\varphi(0) = 0$. Then for each $f \in \mathcal{H}^2, C_\varphi f \in \mathcal{H}^2$ and $\|C_\varphi f\| \leq \|f\|$.

The following theorem gives the general case for the map φ (φ does not necessarily fix the origin).

Theorem 1.4.8[48]: Let φ be an analytic self-map of $\mathbb{U}$, then $f \circ \varphi \in \mathcal{H}^2$,

$$\|f \circ \varphi\| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}} \text{ for all } f \in \mathcal{H}^2.$$

Recall that if T is bounded operator on a Hilbert space $\mathcal{H}$, then the norm of such an operator is defined by:

$$\|T\| = \sup\{\|Tf\|: f \in \mathcal{H}, \|f\| = 1\}$$

if $\|T\| \leq 1$, then T is said to be a contraction on $\mathcal{H}$ [48].

Corollary 1.4.9 [47]: Let φ be an analytic self-map of $\mathbb{U}$, then C_φ is bounded operator on $\mathcal{H}^2$ and $\|C_\varphi\| \le \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}$.

Remarks 1.4.10 [47]:

1) If φ is analytic self-map of $\mathbb{U}$ that fixes the origin, then from corollary (1.3.9), C_φ is a contraction on $\mathcal{H}^2$.

2) One can easily show that $C_\varphi \circ C_\psi = C_{\varphi \circ \psi} = C_{\psi \circ \varphi}$ and hence

$C_\varphi^n = C_{\varphi \circ \varphi \circ \ldots \circ \varphi} = C_{\varphi_n}$.

3) If φ is a conformal automorphism, then the composition operator C_φ is invertible operator and $C_\varphi^{-1} = C_{\varphi^{-1}}$.

We recall that if $\mathcal{H}$ is a Hilbert space and T_1, T_2 are two operators on $\mathcal{H}$, then T_{12}, T_2 are similar if there is an invertible operator S, such that $T_2 = S^{-1}T_1S$.

Definition 1.4.11 [48]: The composition of operators C_φ and C_ψ are said to be compositionally similar if there is a conformal automorphism mapping α of the unit ball $\mathbb{U}$, such that:

$$\varphi = \alpha^{-1} \circ \psi \circ \alpha.$$

Proposition 1.4.12[48]: Every compositionally similar composition operators are similar.

Remark 1.3.13 [48]: If φ is univalent self-map of $\mathbb{U}$, then σ in the previous theorem is also univalent.

Definitions 1.4.14 [48]: Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$, then T is cyclic if there exists a vector $x \in \mathcal{H}$, such that the set span $\{T^n x: n = 0, 1, \ldots\}$ is dense in $\mathcal{H}$. The vector x is called a cyclic vector for the operator T.

Theorem 1.4.15 [31]: Suppose that S, T, X are bounded operators on

a Hilbert space $\mathcal{H}$, such that $SX = XT$, if T is cyclic and X has a dense range, then S is also cyclic.

Proposition 1.4.16 [30]: Let T be an operator on a Hilbert space $\mathcal{H}$ that has diagonal matrix $\mathrm{A} = \mathrm{diag}(\lambda_1, \lambda_2, \dots)$ with respect to some orthonormal basis $\{e_n\}$, then T is cyclic if and only if the diagonal entries $\{\lambda_j\}$ are distinct.

Lemma 1.4.17 [30]: If T is a cyclic operator on H has a matrix $A = (a_{ij})$ with cyclic vector $x = (x_1, x_2, \dots)$, then the operator $\bar{T}$ is cyclic with cyclic vector $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots)$, where $\bar{x}_i$ is the complex conjugate of x_i, for all i and $\bar{T}$ is the operator that has the matrix $\bar{A} = (\bar{a}_{ij})$, $(\bar{a}_{ij})$ is the complex conjugate of (a_{ij}).

Notation: Let $\varphi(z) = \sum_n a_n z^n$, $a_n \in \mathbb{C}$, $\forall\, n$; be an analytic self-map of $\mathbb{U}$. We denote by $\overline{\varphi}(z)$ to the map $\sum_n a_n z^n$, where $\bar{a}_n$ is the complex conjugate of a_n.

Proposition 1.4.18 [30]: Let $\varphi(z)$ be an analytic self-map of $\mathbb{U}$. If C_φ is a cyclic operator with cyclic vector f, then $C_{\overline{\varphi}}$ is a cyclic operator with cyclic vector $\bar{f}$.

Corollary 1.4.19 [30]: Suppose that $\varphi(z) = \frac{P_n(z)}{q_m(z)}$ is an analytic self-map of $\mathbb{U}$, where $P_n(z)$ and $q_m(z)$ are polynomials of degree $n \; and \; m$, respectively. If C_φ is a cyclic operator, then C_ψ is a cyclic operator, where $\psi(z) = \frac{\bar{P}_n(z)}{\bar{q}_m(z)}$.

Lemma 1.4.20 [12]: Let T be an operator that has the matrix $A = (a_{ij})$ with respect to the orthonormal basis $\{e_n\}$, then the matrix of T^*(the adjoint of T) with respect to the same orthonormal basis is $\bar{A}^t = \bar{a}_{ij} = (a_{ij})$, where $\bar{a}_{ij}$ is the complex conjugate of a_{ij}.

Theorem 1.4.21 [3]: Let φ be a conformal automorphism of $\mathbb{U}$ and has an interior fixed point p, then C_φ is cyclic if and only if $(\varphi'(p))^n \neq 1$, for all $n = 1, 2, \dots$.

Recall that, the operator T on $\mathcal{H}$ is called a normal operator if $TT^* = T^*T$ and called isometric if $T^*T = I$ [3].

Theorem 1.4.22[14]: Let φ be an analytic self-map of $\mathbb{U}$. If C_φ is cyclic, then φ is univalent on $\mathbb{U}$.

Theorem 1.4.23 [14]: Let φ be an analytic self-map of $\mathbb{U}$. If C_φ is cyclic, then its range is dense in $\mathcal{H}^2$.

Corollary 1.4.24 [14]: If C_φ is cyclic, then the set of polynomials in φ is dense in $\mathcal{H}^2$. Equivalently, the set of polynomials in z is dense in $\mathcal{H}^2(\varphi(\mathbb{U}))$.

Let us say that a function $f \in \mathcal{H}^2$ is univalent almost everywhere on $\partial\mathbb{U}$ provided that there is a set E $\subset \partial\mathbb{U}$ having zero Lebesgue measure, such that f is univalent on $\partial\mathbb{U}\backslash$E.

Theorem 1.4.25 [14]: If C_φ is cyclic, then φ is univalent almost everywhere on $\partial\mathbb{U}$.

Proposition 1.4.25[14]: Let φ be an analytic self-map of $\mathbb{U}$. If $\varphi'(0) = 0$, then C_φ is not cyclic.

1.5 The Differential Subordination

In this section, we presented definitions, theorems, and the most important results on differential subordination and sandwich theorem of subclasses in geometric function theory.

Definition 1.5.1 [29]: A set $E \subseteq \mathbb{C}$ is said to be starlike with respect to $w_0 \in E$ if the line segment joining w_0 to every other point $w \in E$ lies entirely in E. In a more picturesque language, the requirement is that every point of E be visible from w_0 . The set E is said to be convex if it is starlike with respect to each of its points, that is, if the linear segment joining any two points of E lies entirely in E.

Definition 1.5.2 [20]: The class of all analytic functions denoted by S.

A function $f \in S$ is said to be starlike function of order β if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \text{ where } z \in \mathbb{U}; 0 \leq \beta < 1.$$

Denotes the class of all starlike functions of order β in $\mathbb{U}$ by $S^*(\beta)$ and S^* the class of all starlike functions of order 0, $S^*(0) = S^*$. Geometrically, we can say that a starlike function is conformal mapping of the unit disk onto a starlike domain with respect to the origin. For example, the function $f(z) = \frac{z}{(1-z)^{2(1-\beta)}}$, is starlike function of order β.

Definition 1.5.3 [20]: A function $f \in S$ is said to be convex function of order β if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta, \text{ where } 0 \leq \beta < 1, z \in \mathbb{U}.$$

Denotes the class of all convex functions of order β in U by $C(\beta)$ and C for the convex function of order 0 by $C(0) = C$. We note that $C(\beta) \subset S^*(\beta)$.

Definition 1.5.4 [20]: A function f analytic in the unit disk $\mathbb{U}$ is said to be close-to-convex of order α; $(0 \leq \alpha < 1)$ if there is a convex function g such that

$$Re\left(\frac{f'(z)}{g'(z)}\right) > \alpha, \forall z \in \mathbb{U}.$$

We denote by $K(\alpha)$, the class of close - to - convex functions of order α *in* $\mathbb{U}$. We note that $C(\alpha) \subset S^*(\alpha) \subset K(\alpha)$.

Definition 1.5.5 [29]: Let f be an analytic function in the unit disk. If the equation $w = f(z)$ has never more than p-solutions in $\mathbb{U}$, then f is said to be p-valent in $\mathbb{U}$. The class of all p-valent analytic functions is denoted by A_p and expressed in one of the following forms:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, z \in \mathbb{U}; p \in \mathbb{N},$$

and let f be a function analytic in the punctured unit disk $\mathbb{U}$ If the equation w=$f(z)$ has never more than p-solutions in $\mathbb{U}$, then f is said to be p-valent in

$\mathbb{U}^*$. The class of all p-valent meromorphic functions is denoted by $A^*(p)$ and expressed in one of the following forms:

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} a_k\, z^k, z \in \mathbb{U}; p \in \mathbb{N},$$

Definition 1.5.6 [20]: Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a function. We call f a Schwarz function, if for all $c \in \mathbb{R}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then $f^{(n)}(x) = 0(|z|^c)$. where "capital O " is defined as follows:

Let $\{a_n\}$ and $\{b_n\}$ be any two sequences and $b_n \geq 0$ for all n. If there exists a constant number $\tau > 0$ such that $a_n \leq \tau b_n$ (for all n), then we write

$a_n = O(b_n)$.

Definition 1.5.7[35]: Let f_1 and f_2 are analytic in $\mathbb{U}$, we say that the function f_1 is subordinate to f_2 or, f_2 is said to be superordinate to f_1 if there exists a Schwarz function w in $\mathbb{U}$ $with$ $w(0) = 0$, and $\left|w(z)\right| < 1$ $(z \in \mathbb{U})$ where $f_1(z) = f_2\big(w(z)\big)$. In such a case, we write $f_1 \prec f_2$ or $f_1(z) \prec f_2(z)$ $(z \in \mathbb{U})$.

Particularly, if the function f_2 is univalent in $\mathbb{U}$, then $f_1 \prec f_2$ if and only if $f_1(0) = f_2(0)$ and $f_1(\mathbb{U}) \subset f_2(\mathbb{U})$.

Definition 1.5.8 [35]: The set of all functions q that are analytic and injective on $\overline{\mathbb{U}} \setminus E(\mathbb{q})$, denote by Q, where $\overline{\mathbb{U}} = \mathbb{U} \cup \{z \in \partial\mathbb{U}\}$, and

$E(\mathbb{q}) = \left\{\xi \in \partial\mathbb{U}: \lim_{\xi \to 0} \mathbb{q}(z) = \infty\right\}$, such that $\mathbb{q}'(\xi) \neq 0$ for $\xi \in \partial\mathbb{U} \setminus E(\mathbb{q})$.

Further, let the subclass of Q for which $f(0) = a$ be denoted by $Q(a)$,

$Q(0) = Q_0$ and $Q(1) \equiv Q_1 = \{f \in Q: f(0) = 1\}$.

Lemma 1.5.9 [35]: Let the function $\mathbb{q}(z)$ be univalent in the open unit disc $\mathbb{U}$ and let $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ be analytic in a domain D containing $q(\mathbb{U})$ with $\boldsymbol{\varphi}(w) \neq 0$ when $w \in \mathbb{q}(\mathbb{U})$. put $Q(z) = z\mathbb{q}'(z)\boldsymbol{\varphi}\big(\mathbb{q}(z)\big)$ and $h(z) = \boldsymbol{\theta}(\mathbb{q}(z) + Q(z)$.

Suppose that

1) $Q(z)$is starlike univalent in $\mathbb{U}$,

2) $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0, z \in \mathbb{U}$.

If h is analytic in $\mathbb{U}$ with $h(0) = \mathbb{q}(0), h(\mathbb{U}) \subseteq D$ and

$$\boldsymbol{\theta}\big(h(z)\big) + zh'(z)\boldsymbol{\varphi}\big(h(z)\big) \prec \boldsymbol{\theta}\big(\mathbb{q}(z)\big) + z\mathbb{q}'(z)\boldsymbol{\varphi}\big(\mathbb{q}(z)\big), \qquad (1.5.1)$$

then $h(z) \prec \mathbb{q}(z), and\ \mathbb{q}(z)$ is the best dominant.

Lemma 1.5.10[2]: Let $\mathbb{q}(z)$ be convex univalent function in open unit disk $\mathbb{U}$, let $\psi \in \mathbb{C},\ \gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with

$$Re\left(1 + \frac{z\mathbb{q}''(z)}{\mathbb{q}'(z)}\right) > max\left\{0, -Re\left(\frac{\psi}{\gamma}\right)\right\}. \qquad (1.5.2)$$

If $h(z)$ is analytic in $\mathbb{U}$ with $h(0) = \mathbb{q}(0)$ and

$$\psi h(z) + \gamma zh'(z) \prec \psi\mathbb{q}(z) + \gamma z\mathbb{q}'(z), \qquad (1.5.3)$$

then $h(z) \prec \mathbb{q}(z); z \in \mathbb{U}$ and $\mathbb{q}$ is the best dominant.

Lemma 1.5.11[14]: Let $\mathbb{q}(z)$ be convex univalent in the unit disk $\mathbb{U}$ and let $\boldsymbol{\theta}\ and\ \boldsymbol{\varphi}$ be analytic in a domain D containing $\mathbb{q}(\mathbb{U})$. Suppose that

1) $Re\left\{\frac{\boldsymbol{\theta}'(\mathbb{q}(z))}{\boldsymbol{\varphi}(\mathbb{q}(z))}\right\} > 0\ \ for\ \ z \in \mathbb{U}$,
2) $z\mathbb{q}'(z)\boldsymbol{\varphi}(\mathbb{q}(z))$ is starlike univalent in $z \in \mathbb{U}$.

If $h(z) \in \mathcal{H}[\mathbb{q}(0), 1] \cap Q$, with $h(\mathbb{U}) \subseteq D,\ and\ \boldsymbol{\theta}(h(z) + zh'(z)\boldsymbol{\varphi}(h(z))$ is univalent in $\mathbb{U}$, and

$$\boldsymbol{\theta}\big(\mathbb{q}(z)\big) + z\mathbb{q}'(z)\boldsymbol{\varphi}\big(\mathbb{q}(z)\big) \prec \boldsymbol{\theta}\big(h(z)\big) + zh'(z)\boldsymbol{\varphi}\big(h(z)\big), \qquad (1.5.4)$$

then $\mathbb{q}(z) \prec h(z),\ and\ \mathbb{q}(z); z \in \mathbb{U}$ is the best subordinant.

Lemma 1.5.12[14]: Let $\mathbb{q}(z)$ be convex univalent in $\mathbb{U}$ and $\mathbb{q}(0) = 1$. Let $\gamma \in \mathbb{C},\ that\ Re\{\gamma\} > 0$. I$f$ $h(z) \in \mathcal{H}[\mathbb{q}(0), 1] \cap Q\ and\ h(z) + \gamma zh'(z)$ is univalent in $\mathbb{U}$, then

$$\mathbb{q}(z) + \gamma z \mathbb{q}'(z) \prec h(z) + \gamma z h'(z), \qquad (1.5.5)$$

which implies that $\mathbb{q}(z) \prec h(z)$ *and* $\mathbb{q}(z)$ is the best subordinant.

Definition 1.5.13 [35]: Let $\boldsymbol{\psi}: \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $\mathbb{p}(z)$ is analytic in $\mathbb{U}$ and satisfies the second - order differential subordination:

$$\boldsymbol{\psi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z); z) \prec h(z), \qquad (1.5.6)$$

then $\mathbb{p}(z)$ is called a solution of the differential subordination (1.5.6). The univalent function $\mathbb{q}(z)$ is called a dominant of the solutions of the differential subordination (1.5.6), moreover simply dominant, if $\mathbb{p}(z) \prec \mathbb{q}(z)$ for all $\mathbb{p}\,(z)$ satisfying (1.5.6). A univalent dominant $\widehat{\mathbb{q}}(z)$ that satisfies $\widehat{\mathbb{q}}(z) \prec \mathbb{q}(z)$ for all dominant $\mathbb{q}(z)$ of (1.5.6) is said to be the best dominant of (1.5.6), is unique up to a relation of $\mathbb{U}$.

Definition 1.5.14[36]: Let h and k are two analytic functions in $\mathbb{U}$ and $\boldsymbol{\phi}(r, s, t;\, z): \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. If h and $\boldsymbol{\phi}(h(z), zh'(z), z^2h''(z);\, z)$ are univalent functions in $\mathbb{U}$ and if h satisfies the second-order superordination

$$k(z) \prec \boldsymbol{\phi}(h(z), zh'(z), z^2h''(z);\, z), \qquad (1.5.7)$$

then h is called a solution of the differential superordination (1.5.7). A function $\mathbb{q} \in H(\mathbb{U})$ is called a subordinant of (1.5.7), if $h(z) \prec \mathbb{q}(z)$ for all the functions h satisfying (1.5.7). A univalent subordinant $\widehat{\mathbb{q}}$ that satisfies $\mathbb{q}(z) \prec \widehat{\mathbb{q}}(z)$ for all the subordinants $\mathbb{q}$ of (1.5.7), is said to be the best subordinant.

Definition 1.5.15 [35]: Let $\boldsymbol{\phi} = \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and $K(z)$ be univalent in $\mathbb{U}$ .If $h(z)$ is analytic in $\mathbb{U}$ and satisfies the second-order differential subordination :

$$\boldsymbol{\phi}(h(z), zh'(z), z^2h''(z);\, z) \prec k(z), \qquad (1.5.8)$$

then h is called a solution of the differential subordination (1.5.8), and the univalent function $\mathbb{q}(z)$ is called a dominant of the solution of the differential subordination(1.5.8), or more simply dominant if $h(z) \prec \mathbb{q}(z)$ for all $\mathbb{p}(z)$ satisfying (1.5.8). A univalent dominant $\widehat{\mathbb{q}}(z)$ that satisfies $\widehat{\mathbb{q}}(z) \prec \mathbb{q}(z)$

for all dominant $\mathbb{q}(z)$ of (1.5.8), is said to be the best dominant is unique up to a relation of $\mathbb{U}$.

Definition 1.5.16 [36]: Let $\boldsymbol{\psi}: \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be analytic in $\mathbb{U}$. If $\mathbb{p}(z)$ and $\boldsymbol{\psi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z);\, z)$ are univalent in $\mathbb{U}$ and if $\mathbb{p}(z)$satisfies the second - order differential superordination:

$$h(z) \prec \boldsymbol{\psi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z);\, z), \tag{1.5.9}$$

then $\mathbb{p}(z)$ is called a solution of the differential superordination (1.5.9).

An analytic function $\mathbb{q}(z)$is called a subordinant of the solutions of the differential superordination (1.5.9) or more simply a subordinant, if

$\mathbb{q}(z) \prec \mathbb{p}(z)$ for all $\mathbb{p}(z)$satisfying (1.5.9). A univalent subordinant $\widehat{\mathbb{q}}(z)$that satisfies $\widehat{\mathbb{q}}(z) \prec \mathbb{q}(z)$ for all dominant $\mathbb{q}(z)$ of (1.5.9) is said to be the best subordinant.

Definition 1.5.17 [2]: Let $\boldsymbol{\Pi}: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and suppose that the function $h(z)$ is univalent in $\mathbb{U}$. If the function $\mathbb{p}(z)$ is analytic in $\mathbb{U}$ and satisfies the following third-order differential subordination:

$$\boldsymbol{\Pi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z), z^3\mathbb{p}'''(z);\, z) \prec h(z), \tag{1.5.10}$$

then $p(z)$ is called a solution of the differential subordination (1.5.10).

Furthermore, a given univalent function $\mathbb{q}(z)$ is called a dominant of the solutions of (1.5.10) or more simply, a dominant if $\mathbb{p}(z) \prec \mathbb{q}(z)$ for all $p(z)$ satisfying (1.5.10). A dominant $\widehat{\mathbb{q}}(z)$that satisfies $\widehat{\mathbb{q}}(z) \prec \mathbb{q}(z) \prec q(z)$ for all dominants $\mathbb{q}(z)$of (1.5.10) is said to be the best dominant.

Lemma 1.5.18 [2]: Let $\mathbb{p} \in \mathcal{H}[a, n]$ with $n \in \mathbb{N}\backslash\{1\}$ and $\mathbb{p} \in Q(a)$ satisfying the following conditions: $Re\left\{\frac{\xi^2\mathbb{q}''(\xi)}{\mathbb{q}'(\xi)}\right\} \geqq 0$ and $\left|\frac{z\mathbb{p}'(z)}{\mathbb{q}'(\xi)}\right| \leqq \mathrm{k}$,

where $z \in \mathbb{U}, \xi \in \partial\mathbb{U}\backslash E(\mathbb{q})$ and $k \geqq n$. If Ω is a set in $\mathbb{C}$, $\boldsymbol{\Pi} \in \Psi_n[\Omega, \mathbb{q}]$, and $\boldsymbol{\Pi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z), z^3\mathbb{p}'''(z);\, z) \subset \Omega$,

then $\mathbb{p}(z) \prec \mathbb{q}(z), (z \in \mathbb{U})$.

Definition 1.5.19 [2]: Let Ω be a set in $\mathbb{C}$, $\mathbb{q} \in Q$ and $n \in \mathbb{N}\backslash\{1\}$. The class of admissible functions $\Psi_n[\Omega, \mathbb{q}]$ consists of those functions $\boldsymbol{\psi}: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition : $\boldsymbol{\psi}(r, s, t, u; z) \notin \Omega$ whenever

$$r = \mathbb{q}(\zeta), s = k\zeta\mathbb{q}'(\zeta), Re\left\{\frac{t}{s} + 1\right\} \geq kRe\left\{1 + \frac{\zeta\mathbb{q}''(\zeta)}{\mathbb{q}'(\zeta)}\right\},$$

$$Re\left\{\frac{u}{s}\right\} \geq k^2 Re\left\{1 + \frac{\zeta^2\mathbb{q}'''(\zeta)}{\mathbb{q}'(\zeta)}\right\},$$

$where\ z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus E(\mathbb{q})$ and $k \geq n$.

Definition 1.5.20[5,9]: Let $\boldsymbol{\phi}: \mathbb{C}^5 \times \mathbb{U} \to \mathbb{C}$ and suppose $h(z)$ be univalent function in $\mathbb{U}$. If $\mathbb{p}(z)$ is analytic function in $\mathbb{U}$ and satisfies the following fourth-order differential subordination:

$$\boldsymbol{\phi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z), z^3\mathbb{p}'''(z), z^4\mathbb{p}''''(z); z) \prec h(z), \quad (1.5.11)$$

then $\mathbb{p}(z)$ is called a solution of the differential subordination (1.5.11).

A univalent function $\mathbb{q}(z)$ is called a dominant of the solutions of (1.5.11), or, more simply, a dominant $\mathbb{q}(z)$ if $\mathbb{p}(z) \prec \mathbb{q}(z)$) for all $\mathbb{p}(z)$satisfying (1.5.11).

A dominant $\hat{\mathbb{q}}(z)$ which satisfies $\hat{\mathbb{q}}(z) \prec \mathbb{q}(z)$ for all dominants $\mathbb{q}(z)$ of (1.5.11) is said to be the best dominant.

Definition 1.5.21 [5,9]: $Let\ \boldsymbol{\phi}: \mathbb{C}^5 \times \mathbb{U} \to \mathbb{C}$ and suppose that $h(z)$ be analytic function in $\mathbb{U}$. If $\mathbb{p}(z)$ and $(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z), z^3\mathbb{p}'''(z), z^4\mathbb{p}''''(z); z)$, are univalent functions in $\mathbb{U}$ and satisfies the following fourth-order differential superordination

$$h(z) \prec \boldsymbol{\phi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z), z^3\mathbb{p}'''(z), z^4\mathbb{p}''''(z); z), \quad (1.5.12)$$

then $\mathbb{p}(z)$ is called a solution of the differential superordination (1.5.12).

An analytic function $\mathbb{p}(z)$ is called a subordinant of the solution of (1.5.12), or, more simply, a subordinant, if $\mathbb{q}(z) \prec \mathbb{p}(z)$for all $\mathbb{p}(z)$ satisfying (1.5.12).

A univalent subordinant $\hat{\mathbb{q}}(z)$ which satisfies $\mathbb{q}(z) \prec \hat{\mathbb{q}}(z)$ for all subordinants of (1.5.12) is said to be the best subordinant. We note that the best subordinant is unique up to rotation of $\mathbb{U}$.

Lemma 1.5.22[9]: Let $\mathbb{p} \in H[a,n]$ with $\boldsymbol{\phi} \in \Psi'_n[\Omega, \mathbb{q}]$. If

$$\boldsymbol{\phi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z), z^3\mathbb{p}'''(z), z^4\mathbb{p}''''(z); z),$$

is a univalent in $\mathbb{U}$ and $\mathbb{p}(z) \in \mathrm{Q(a)}$ satisfy the following admissibility conditions:

$Re\left\{\frac{z^2\mathbb{q}'''(\xi)}{\mathbb{q}'(\xi)}\right\} \geqq 0$,*and and* $\left|\frac{z^2\mathbb{p}''(z)}{\mathbb{q}'(\xi)}\right| \leqq \frac{1}{m}$, where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}$ *and*

$m \geqq n \geqq 3$, then

$$\Omega \subset \{\boldsymbol{\phi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z), z^3\mathbb{p}'''(z), z^4\mathbb{p}''''(z); z), : \mathrm{z} \in \mathbb{U}\},$$

implies that $\mathbb{q}(z) \prec \mathbb{p}(z), (z \in \mathbb{U})$.

Definition 1.5.23 [5,9]: Let Ω be a set in $\mathbb{C}$, $\mathbb{q}(z) \in \mathcal{Q}$ *and* $n \in \mathbb{N} \setminus \{2\}$. The class $\Psi_n[\Omega, \mathbb{q}]$ of admissible functions consists of those functions

$\boldsymbol{\phi}: \mathbb{C}^5 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility conditions:

$\boldsymbol{\phi}(\mathbb{x}, \mathbb{y}, \mathbb{z}, \mathbb{w}, \mathbb{v}; \zeta) \notin \Omega$,

$$\mathbb{x} = \mathbb{q}'(\zeta), \mathbb{y} = kz\mathbb{q}'(\zeta), \left\{Re\frac{\mathbb{z}}{\mathbb{y}} + 1\right\} \leqq kRe\left\{\frac{\zeta\mathbb{q}''(\zeta)}{\mathbb{q}'(\zeta)} + 1\right\},$$

and

$$Re\left\{\frac{\mathbb{w}}{\mathbb{y}}\right\} \leqq k^2 Re\left\{\frac{\zeta^2\mathbb{q}'''(\zeta)}{\mathbb{q}'(\zeta)}\right\}, Re\left\{\frac{\mathbb{w}}{\mathbb{y}}\right\} \leqq k^3 Re\left\{\frac{z^3\mathbb{q}''''(\zeta)}{\mathbb{q}'(\zeta)}\right\},$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}$ *and* $m \geqq n \geqq 3$.

Definition 1.5.24 [5,9]: Let Ω be a set in $\mathbb{C}$,$\mathbb{q}(z) \in \mathcal{H}[a,n]$ *with* $\mathbb{q}'(z) \neq 0$. The class $\Psi'_n[\Omega, \mathbb{q}]$ of admissible functions consists of those functions

$\boldsymbol{\phi}: \mathbb{C}^5 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility conditions:

$\boldsymbol{\phi}(\mathbb{x}, \mathbb{y}, \mathbb{z}, \mathbb{u}, \mathbb{w}; \zeta) \in \Omega$, whenever

$$\mathbb{x} = \mathbb{q}'(z), \mathbb{y} = \frac{z\mathbb{q}'(z)}{m}, \left\{Re\frac{\mathbb{z}}{\mathbb{y}} + 1\right\} \leqq \frac{1}{m} Re\left\{\frac{z\mathbb{q}''(z)}{\mathbb{q}'(z)} + 1\right\},$$

and

$$Re\left\{\frac{\mathbb{w}}{\mathbb{y}}\right\} \leqq \frac{1}{m^2} Re\left\{\frac{z^2\mathbb{q}'''(z)}{\mathbb{q}'(z)}\right\}, Re\left\{\frac{\mathbb{w}}{\mathbb{y}}\right\} \leqq \frac{1}{m^3} Re\left\{\frac{z^3\mathbb{q}''''(z)}{\mathbb{q}'(z)}\right\},$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}$ *and* $m \geqq n \geqq 3$.

Chapter Two: New Formula for Symmetric Group

2.1 Introduction

The symmetric group S_n plays a fundamental role in mathematics. It arises in all sorts of different contexts, so its importance can hardly be overstated. We have already seen from Cayley's theorem that every finite group can be treated as a subgroup of S_n for some n. Also S_n plays a role in describing the structure of the general linear group, as well as, certain other linear groups[19]. The purpose of this chapter is to derive new ways of representing its elements of S_n and some of the elementary properties of S_n. It is customary in the theory of symmetric groups, we use the composition notation for the group operation, we say the composition of permutations f_1 and f_2 denoted by $f_1 \circ f_2$, the identity subgroup $I = (1)$, the multiples f^k for some integer number k.

This construction will described in detail in cyclic groups, though it could have been introduced at once for symmetric group. In second section, a new formula has been derived with theorems of abstract algebra that is deeper than the elementary properties of symmetric groups. Proceeding from the description this group by using cyclic groups, it's called ordered cyclic subgroups. This formula will give us a simple description to express the group of symmetries. We introduce a new formula of symmetric group as direct product of ordered cyclic subgroup. In third section we review Lagrange's theorem in terms of ordered cyclic subgroup.

2.2 Symmetric Group as Direct Product of an Ordered Cyclic subgroups

A permutations regarded as a two composition is a permutation group with respect to the composition of cycles. If C_1 and C_2 are two distinct order of this kind, then any element in the cycle that issues from the origin is uniquely represented by the composition of it's projections on the cyclic subgroups C_1 and C_2. Similarly, any element of three composition permutation can be uniquely written as the composition of three elements belonging to three given ordered cyclic subgroups C_1, C_2 and C_3, provided the cycles do not lie in the same cyclic an so on.

Definition 2.2.1: For $j \in \mathbb{N}$ we define an ordered cycle permutation of order j,

$$c_j = (\ 1\ 2\ \ 3\ \ \dots\ j)$$

which it generated an ordered cyclic subgroup of order j,

$$C_j = \{c_j\,, c_j^2, \dots, c_j^j\},$$

and can be rewriting as cyclic permutations set as follows:

$C_j = \{(\ 1\ 2\ \ 3\ \ \dots\ j), (\ 1\ 2\ \ 3\ \ \dots\ \ j)^2, \dots, (\ 1\ 2\ \ 3\ \ \dots\ \ j)^j = (1)\}.$

Example 2.2.2: If $j = 4$, the ordered cycle permutation of order 4,

$c_4 = (\ 1\ 2\ \ 3\ 4)$ which it generated an ordered cyclic subgroup of order 4, $C_4 = \{c_4\,, c_4^2, c_4^3, c_4^4\}$ and can be rewriting as cyclic permutations set as follows:

$$C_4 = \{(1\ 2\ 3\ 4), (13)(24), (1432), (1)\}.$$

Remark 2.2.3: We considered the elements within the cycle that order lower than k are fixed element, for instance that of $c_j = (1\ 2 \dots j)$, the elements $j+1, j+2, \dots, k$ are fixed elements.

Proposition 2.2.4: The inverse of a permutation $c_1^{k_1} \circ c_2^{k_2} \circ \dots \circ c_k^{k_l}$ for all

$j = 1,2, \dots, l,$ is a permutation $c_k^{l-k_l} \circ \dots \circ c_2^{2-k_{2;}}$; $k_l = 1, \dots, l$ and $\left(c_1^{k_1}\right)^{-1} = c_1^{k_1} = 1.$

Proof:

Let C_j be an ordered cyclic subgroup for all $j = 1,2,\dots,k$, the inverse of cycle $c_j^k \in C_j$ is the cycle $c_j^{j-k} \in C_j$; $k = 1,2,\dots,j$.

From properties of composition of function $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$, using the generalization of this property and the relation $\left(c_j^k\right)^{-1} = c_j^{j-k}$ and

$$\left(c_1^{k_1} \circ c_2^{k_2} \circ \dots \circ c_l^{k_l}\right)^{-1} = c_l^{l-k_l} \circ \dots \circ c_2^{2-k_2};\ \left(c_1^{k_1}\right)^{-1} = c_1^{k_1} = 1\ ;\ k_l = 1,\dots,l.$$

■

<u>Example 2.2.5</u>: To find he inverse of a permutation (123) with respect to above proposition can be written as:

$(123) = (1) \circ (12)^1 \circ (123)^1 \circ (1234)^1$ is a permutation

$$\begin{aligned}(123)^{-1} &= (1234)^{4-1} \circ (123)^{3-1} \circ (12)^{2-1} \circ (12)\\ &= (1234)^3 \circ (123)^2 \circ (12)^1 \circ (1)\\ &= (1432) \circ (132) \circ (12) \circ (1) = (132)\end{aligned}$$

<u>Corollary 2.2.6</u>: Let C_j be ordered cyclic subgroup of order j, then $\left(\{C_j\}_{j=1,2,\dots,k}, \circ\right)$ is a group.

Proof:

It follows directly from theorems closure and associative of symmetric group, every subgroup of symmetric group contains identity element, and proposition (2.2.4). ■

<u>Theorem 2.2.7</u>: A symmetric group S_k can be represented as composition of ordered cyclic subgroups $\{C_j\}_{j=1,2,\dots,k}$.

Proof:

Assume that $\mathrm{X} = \{1, 2, \dots, \mathrm{n}\}$, its known the symmetric set is contain all permutation $\varphi: \mathrm{X} \to X$.

Let $\{C_j\}$ be ordered cyclic subgroups of S_k have order j respectively, every $\{C_j\}; j = 1,2, \dots, k$ has j cycles which it represented cyclic permutation.

To prove the computations' set $C_1 \circ C_2 \circ \dots \circ C_k$ is equivalent to the set S_k.

For any element $x = 1,2, \dots, k$ in C_k, the set $\{x, c_k^1(x), c_k^2(x), \dots, c_k^{k-1}(x)\}$ distinct elements for all x, by Lemma (1.1.9), i.e. the element x will be linked with k distinct elements in C_k, for any element $x = 1,2, \dots, k$.

After this, that k distinct elements will be linked by composition operation with k-1 distinct elements in C_{k-1} for any element $y = 1,2, \dots, k-1$ with observe the element k is a fixed element in the set $\{y, c_k^1(y), c_k^2(y), \dots, c_k^{k-1}(y)\}$, which contain k-1 distinct element for any elements $y = 1,2, \dots, k-1$. And so we get 4 distinct elements will be associated with 3 distinct elements in C_3 for any element $z = 1,2,3$ with observe the element 4 is associated with himself. i.e., $k, k-1, \dots$,4 are fixed elements in the set $\{z, c_3^1(z), c_3^2(z)\}$, which contain 3 distinct elements for any element $y = 1,2,3$.

Also $C_2 = \{(1), (1\ 2)\}$,the element 1 is associated with 1 and 2, the element 2 is linked with 1 and 2, and 3 is a fixed element in C_2. By Writing the corresponding cycle after the one previously written, with continue choosing previously unused elements and writing out the cycles they traverse until every element of X has been named, where:

1) x_{j+1} is an element of X that does not occur among the $\sigma^j(x_s)$ with $s \leq j$.
2) For each element of X appears exactly once among the $\sigma^l(x_j)$.
3) $\sigma^j(x_j) = x_j$.
4) For each $k_j \geq 1$ when $k_j = 1$, then the cycle containing x_j is (x_j).

The total number of permutation are conclusion from distinct k elements in C_k which composition with distinct $k-1$ elements in C_{k-1} and so on until we obtain $k!$ distinct permutation. ■

Example 2.2.8: Let $S_3 = \{(1),(12),(23),(13),(123),(132)\}$ is a symmetric group of order 3 with respect to above theorem can be written as

$S_3 = C_1 \circ C_2 \circ C_3$ such that $C_1 = \{(1)\}, C_2 = \{(1),(12)\}$, and

$C_3 = \{(1),(123),(132)\}$ are ordered cyclic subgroups of order 1,2, and 3 respectively as follows:

$$(1) \circ (1) \circ (1) = (1),$$

$$(1) \circ (1) \circ (123) = (123),$$

$$(1) \circ (1) \circ (132) = (132),$$

$$(1) \circ (12) \circ (1) = (12),$$

$$(1) \circ (12) \circ (123) = (23),$$

$$(1) \circ (12) \circ (132) = (13).$$

All of this, show the composition of first three ordered cyclic subgroups generated the symmetric group of order 3.

Remark 2. 2.9: Let C_j be ordered cyclic subgroup of order j, the cycles which are generated of this subgroup are cycles have order k divided the order of ordered cyclic group. i.e.,$k|\, j$.

Example 2.2.10: Let C_4 be an ordered cyclic subgroup of order 4

$$\left((1234)\right) = \{(1234),(13)(24),(1432),(1)\} = C_4,$$

$$\left((1432)\right) = \{(1432),(13)(24),(1234),(1)\} = C_4.$$

The generators of C_4 are $(1234), (1432)$ since o(1234) =1, o(1432) =3.

Theorem 2.2.11: A symmetric group S_k is called the direct product of its ordered cyclic subgroups $C_1, C_2,\ldots, C_k$

$$S_k = C_1 \circ C_2 \circ \ldots \circ C_k, \tag{2.2.1}$$

if every permutation s of S_k is uniquely written as the composition of powers of the cycles $c_1, c_2,\ldots, c_k$ respectively, taken in the ordered cyclic subgroups $C_1, C_2,\ldots, C_k$

$$s = c_{j1} \circ c_{j2} \circ \ldots \circ c_{jk} . \tag{2.2.2}$$

Proof :

The notation (2.2.1) is called the direct decomposition of the group S_k, the ordered cyclic subgroups $C_j; j = 1,2, \ldots, k$ are direct product of this decomposition, and the cycles c_{jl} in (2.2.2) is a component of the permutation in the direct product C_j of the decomposition (1), $j = 1,2, \ldots, k$.

If we are given a direct decomposition (2.2.1) of a symmetric group S_k and if the direct product C_j; of this decomposition (all or some of them), are themselves decomposed into a direct product ,

$$C_j = C_{j1} \circ C_{j2} \circ \ldots \circ C_{jk_j} ; k_j \geq 1, \tag{2.2.3}$$

then the group S_k is the direct product of all its cyclic subgroups

C_{jl};, $i = 1,2, \ldots, k$, $l = 1,2, \ldots, k_j$.

Indeed, for an arbitrary permutation s of S_k we have the notation (2.2.2) relative to the direct decomposition (2.2.1) and for each component c_{jl};

$j = 1,2, \ldots, k,\ l = 1,2, \ldots, k_j$ we have the notation

$$s = c_{j1} \circ c_{j2} \circ \dots \circ c_{jk} \,, \tag{2.2.4}$$

relative to the direct decomposition (2.2.3) of the cyclic group C_j.

It is clear that s is the composition of all cycle $c_{jl}; j = 1,2, \dots, k,\ l = 1,2, \dots, k_j$,

the uniqueness of this notation follows from that we must obtain precisely equality (2.2.2) by taking any notation of the permutation s as composition of the cycles, taken one each in the cyclic group in the subgroups C_{jl}, and by adding the composition belonging to the same subgroup C_{jl} ; $j = 1,2, \dots, k$.

On the other hand, each cycle c_{jl} only has one notation of the type (2.2.4).

The definition of a direct product may be restated. First let us introduce a new concept, if it is given that a symmetric group S_k has certain ordered cyclic subgroup $C_1, C_2, \dots, C_m$, then we denote by $\{C_1, C_2, \dots, C_m\}$ the set of permutation s_α of S_k which can in at least one way be written as a composition of cycles $c_{\alpha 1}, c_{\alpha 2}, \dots, c_{\alpha m}$ taken in the cyclic subgroups $C_1, C_2, \dots, C_m$ respectively,

$$s_\alpha = c_{\alpha 1} \circ c_{\alpha 2} \circ \dots \circ\ c_{\alpha m} \,. \tag{2.2.5}$$

The set $\{C_1, C_2, \dots, C_m\}$ will be ordered cyclic subgroups of S_k.

Take in $\{C_1, C_2, \dots, C_m\}$ a permutation s_β with notation (2.2.5), and also cycles $c_{\beta j}$ with a similar notation

$$s_\beta = c_{\beta 1} \circ c_{\beta 2} \circ \dots \circ\ c_{\beta m},$$

where $c_{\beta j}$ is a cycle in C_j ; $j = 1, 2, \dots, m$. Then

$$s_\alpha \circ s_\beta = (c_{\alpha 1} \circ c_{\alpha 2} \circ \dots \circ\ c_{\alpha m}) \circ \left(c_{\beta 1} \circ c_{\beta 2} \circ \dots \circ\ c_{\beta m}\right)$$

$s_\alpha^{-1} = c_{\alpha m}^{-1} \circ \dots \circ\ c_{\alpha 2}^{-1} \circ\ c_{\alpha 1}^{-1}$ which is to say that the elements $s_\alpha \circ s_\beta$ and s_α^{-1} also have at least one notation of the type (2.2.5) hence, belong to the set $\{C_1, C_2, \dots, C_m\}$.

The ordered cyclic subgroup $\{C_1, C_2, \dots, C_m\}$ contains each of the C_{jm} ;

$j = 1,2, \dots, m$. Indeed, every subgroup of the group S_k contains the identity permutation of this group, for instance in the ordered cyclic subgroup C_2, any cycle c_2, the identity permutation (1) of ordered cyclic subgroup $\{C_3, \dots, C_m\}$, we obtain the following notation of type (2.2.5) for element c_α:

$$c_2 = (1) \circ c_2 \dots \circ (1).$$

Thus

$$S_k = \circ \{C_1, C_2, \dots, C_k\}, \tag{2.2.6}$$

and if the intersection of each ordered cyclic subgroups $\{C_j\}$; $j = 2, \dots, k$ with the subgroup generated by all preceding ordered cyclic subgroups $\{C_1, C_2, \dots, C_{j-1}\}$ contains identity alone :

$$\{C_1, C_2, \dots, C_{j-1}\} \cap C_j = \{(1)\}; j = 2, \dots, k \ . \tag{2.2.7}$$

Indeed, if the symmetric group S_k has the direct decomposition (1), then for any permutation s of S_k the notation (2.2.2) exist, therefore it have notation (2.2.6) follows uniqueness of the notation (2.2.2) for any cycle c_j, for some i, the intersection $\{C_1, C_2, \dots, C_{j-1}\} \cap C_j = \{(1)\}$ contained a non-identity permutation s, then on the one hand, s could be written as a cycle c_j in C_j ,i.e., $s = c_{\alpha j}$ and so

$$s = (1) \circ \dots \circ c_{\alpha j} \circ \dots \circ (1), \tag{2.2.8}$$

and s as a permutation of the subgroup $\{C_1, C_2, \dots, C_{j-1}\}$ would have a notation of the form $s = c_{\alpha 1} \circ c_{\alpha 2} \circ \dots \circ c_{\alpha j-1}$ which is to say that

$$s = c_{\alpha 1} \circ c_{\alpha 2} \circ \dots \circ c_{\alpha j-1} \circ (1) \circ \dots \circ (1). \tag{2.2.9}$$

It evident that (2.2.8) and (2.2.9) are two distinct notation of type (2.2.2) for the element s .

Conversely, we assume that every permutation s of S_k is uniquely written as the composition of powers of the cycle $c_1, c_2, \ldots, c_k$. i.e., the notations (2.2.6) and (2.2.7) hold.

From (2.2.6) it follows that any permutation s of S_k has at least one notation of type (2.2.2). However, let there be two distinct notations of type (2.2.2) for some permutation s:

$$s = c_{\alpha 1} \circ c_{\alpha 2} \circ \ldots \circ c_{\alpha k} = c_{\beta 1} \circ c_{\beta 2} \circ \ldots \circ c_{\beta k} \,. \qquad (2.2.10)$$

Then an $j; j \leq k$ such that

$$c_{\alpha k} = c_{\beta k}, c_{\alpha(k-1)} = c_{\beta(k-1)}, \ldots, c_{\alpha(j+1)} = c_{\beta(j+1)}, \qquad (2.2.11)$$

but $c_{\alpha j} \neq c_{\beta j}$,

that is $$c_{\alpha j} \circ \left(c_{\beta j}\right)^{-1} \neq (1). \qquad (2.2.12)$$

From (2.2.10) and (2.2.11) follows however, the equality

$$c_{\alpha j} \circ \left(c_{\beta j}\right)^{-1} = \left(c_{\alpha 1} \circ \left(c_{\beta 1}\right)^{-1}\right) \circ \left(c_{\alpha 2} \circ \left(c_{\beta 2}\right)^{-1}\right) \circ \ldots \circ \left(c_{\alpha(j-1)} \circ \left(c_{\beta(j-1)}\right)^{-1}\right)$$

Which contradicts (2.2.7) due to (2.2.12). The theorem is proved. ■

The concept direct product may be regarded from quite a different angle. Suppose k arbitrary a symmetric group among cyclic subgroups $C_1, C_2, \ldots, C_k$ which there may be isomorphic, denoted by S_k to the set of all possible systems of the form

$$(c_{\alpha 1}, c_{\alpha 2}, \ldots, c_{\alpha k}) \qquad (2.2.13)$$

composed of cycles taken one at a time in each of the ordered cyclic subgroups $\{C_1, C_2, \dots, C_k\}$.

Indeed, the associativity of this composition follows from the validity of these properties in each of the specified ordered cyclic subgroups ;the role of identity is played by the system $((1)_1, (1)_2, \dots, (1)_k)$ where $(1)_i$ denotes the identity permutation of the ordered cyclic subgroups $C_j; j = 1,2, \dots, k$.

The inverse of (2.2.13) is the system $\left(\left(c_{\alpha(k-1)}\right)^{-1},\ (c_{\alpha k})^{-1}, \dots,\ (c_{\alpha 1})^{-1}\right)$.

Thus the symmetric group S_k can be constructed as the direct product of the ordered cyclic subgroups

$$S_k = C_1 \circ C_2 \circ \dots \circ C_k$$

This expression is justified by the fact the symmetric group S_k, which is the direct product of the ordered cyclic subgroups $C_1, C_2, \dots, C_k$ in the sense just defined, can be decomposed into direct product of the ordered cyclic subgroups $C'_1, C'_2, \dots, C'_k$ which are isomorphic, respectively, to the groups $C_1, C_2, \dots, C_k$.

Namely, denote by C'_j ; $j = 1,2, \dots, k$ the set of permutations of S_k,that is systems of type (2.2.13), with an arbitrary permutation $c_{\alpha j}$ of S_k in the j-th ordered cyclic subgroups, all other cycle being occupied by identity of the corresponding ordered cyclic : these will thus be systems of the form

$$((1), (1), \dots, (1), c_{\alpha i}, (1), \dots, (1)) \qquad (2.2.14)$$

The notation (2.2.14) of composition shows that the set C'_j is a subgroup of the symmetric group S_k.

We obtain the isomorphisnm of this subgroup and subgroup C_j, by associating to each system (2.2.14) a permutation $c_{\alpha j}$ of the j-th ordered cyclic subgroups.

It remains to prove that the symmetric group S_k is the direct product of the ordered cyclic subgroups $C_1', C_2', \dots, C_k'$,

Indeed, any permutation (2.2.13) of S_k may be represented as a composition of cycles of the indicated ordered cyclic subgroups:

$$(c_{\alpha 1}, c_{\alpha 2}, \dots, c_{\alpha k}) = (c_{\alpha 1}, (1), \dots, (1)) \circ \big((1), c_{\alpha 1}, \dots, (1)\big) \dots \circ \big((1), (1), \dots, c_{\alpha k}\big) \qquad \dots$$

The uniqueness of this representation follows from the fact that distinct systems of type (2.2.13) are distinct permutation of the symmetric group S_k. If we have two Systems of ordered cyclic subgroups, $C_1, C_2, \dots, C_k$ and $C_1{}', C_2{}', \dots, C_k{}'$ and the subgroups C_j and $C_1{}'$, are isomorphic. $j = 1, 2, \dots, k$

$$S_k = C_1 \circ C_2 \circ \dots \circ \ C_k$$

$$S_k{}' = C_1' \circ C_2' \circ \dots \circ \ C_k',$$

then the symmetric groups are also isomorphic.

Also if for $j = 1, 2, \dots, k$ then there exist isomorphism φ_j between subgroups C_j and $C_j{}'$, which associates with each cycle $c_{\alpha j}$ of C_j a cycle $\varphi(c_{\alpha i})$ of $C_i{}'$, thus the mapping φ which associate with every permutation $(c_{\alpha 1}, c_{\alpha 2}, \dots, c_{\alpha k})$ of S_k there exist a permutation of $S_k{}'$defined by the equation

$$(c_{\alpha 1}, c_{\alpha 2}, \dots, c_{\alpha k}) = (\varphi_i(c_{\alpha 1}), \varphi_2(c_{\alpha 2}), \dots, \varphi_k(c_{\alpha k})),$$

will obviously be an isomorphic mapping of the symmetric group S_k onto the symmetric group $S_k{}'$.

■

2.3 Lagrange Theorem in Terms of Ordered Cyclic Subgroups

Lagrange's theorem is one of the essential theorems for abstract algebra and it's uses in many important theorems. Before proving Lagrange's theorem, we state some proposition. It is easy to see a relationship between symmetric groups as follows:

$$S_m \subseteq S_n \leftrightarrow m \leq n,$$

such that, fix the final $n - m$ elements of $\{1, \ldots, n\}$ so that

$$S_m = \{\sigma \in S_n : \sigma(j) = j, \forall j > m\}.$$

In fact S_m is a subgroup of S_n in precisely $\binom{n}{m}$ different ways: each copy of S_m arises by fixing $n - m$ elements of the set $\{1, \ldots, n\}$ there are precisely $\binom{n}{n-m} = \binom{n}{m}$ ways of choosing these fixed elements.

<u>Proposition 2.3.1</u>: Let C_k be an ordered cyclic subgroups of a symmetric group S_k of order k, then S_k can be represented as composition $S_{k-1} \circ C_k$

Proof:

Let $\{C_j\}_{j=1,2,\ldots,k}$ be ordered cyclic subgroups of S_k of order j.

From theorem (2.2.7), $S_k = \circ \{C_j\}_{j=1,2,\ldots,k}$ and $S_{k-1} = \circ \{C_j\}_{j=1,2,\ldots,k-1}$,

thus $S_k = \{S_{k-1} \circ C_k\}$. ■

<u>Example 2.3.2</u>: Let $S_3 = \{S_2 \circ C_3\}$

It's known $S_2 = \{(1), (12)\}$, $C_3 = \{(1), (123), (132)\}$,

$$S_2 \circ C_3 = \{(1) \circ (1), (12) \circ (1), (1) \circ (123), (12) \circ (123), (1) \circ (132), (12) \circ (132)\} = \{(1), (12), (123), (23), (132), (13)\} = S_3$$

<u>Proposition 2.3.3</u>: Let C_k^{-1} be an inverse ordered cyclic subgroups of a symmetric group S_k of order k ,then S_{k-1} can be represented as composition of $S_{k-1} = \{S_k \circ C_k^{-1}\}$.

Proof:

Let $\{C_k^{-1}\}$ be an inverse for an ordered cyclic subgroup of S_k of order k.

From proposition (2.3.1) , $S_k = \{S_{k-1} \circ C_k\}$ and $S_k = \circ \{C_j\}_{j=1,2,\dots,k}$, so by composition two sides with C_k^{-1},we obtain

$\{S_k \circ C_k^{-1}\} = \{(S_{k-1} \circ C_k) \circ C_k^{-1}\} = \{S_{k-1} \circ (C_k \circ C_k^{-1})\} = S_{k-1}$,

then $S_{k-1} = \{S_k \circ C_k^{-1}\}$. ■

Example 2.3.4: Let $S_2 = \{S_3 \circ C_3^{-1}\}$, such that

$S_3 = \{(1),(12),(123),(23),(132),(13)\}$, $C_3 = \{(1),(132),(123)\}$.

Thus

$$\{S_3 \circ C_3^{-1}\} = \{(1) \circ (1), (12) \circ (1), (123) \circ (1), (23) \circ (1), (132) \circ (1), (13) \circ (1), (1) \circ (1), (12) \circ (132), (123) \circ (132), (23) \circ (132), (132) \circ (132), (13) \circ (132), (1) \circ (123), (12) \circ (123), (123) \circ (123), (23) \circ (123), (132) \circ (123), (13) \circ (123)\}$$

$$= \{\{(1),(12)\}\} = S_2.$$

Lemma 2.3.5: If S_k is a symmetric group with subgroup S_{k-1}, then there is a one to one correspondence between S_{k-1} and any coset of S_{k-1}.

Proof:

Let C be a left coset of S_{k-1} in S_k, then there is a $c_k^j \in C_k; j = 1, \dots, k$ such that $C = c_k^j \circ S_{k-1}$ Define $f: S_{k-1} \to C$ by $f(s) = c_k^j \circ s, s \in S_{k-1}$.

1) f is one to one.

If $s_1 \neq s_2; s_1, s_2 \in S_{k-1}$, then since by Lemma 1.1.9, $s_1 \circ c_k^j \neq s_2 \circ c_k^j$.

Hence $f(s_1) \neq f(s_2)$

2) f is onto.

If $c \in C$, then since $C = c_k^j \circ S_{k-1}$, there is an $s \in S_{k-1}$ such that $c_k^j \circ s = c$.

It follows that $f(s) = c$ and as c was arbitrary, f is onto. ■

Lemma 2.3.6: If S_k is a symmetric group with subgroup S_{k-1}, then there is a one to one correspondence between S_k and any coset of S_k.

Proof:

Let C^{-1} be a left coset of S_k in S_{k-1}, then there is a $c_k^{-j} \in C_k^{-1}; j = 1, \dots, k$ such that $C^{-1} = c_k^{-j} \circ S_k$. Define $g: S_k \rightarrow C^{-1}$ by $g(s) = c_k^{-j} \circ s, s \in S_k$.

1) g is one to one.

If $s_1 \neq s_2; s_1, s_2 \in S_k$, then since by Lemma 1.1.9, $s_1 \circ c_k^{-j} \neq s_2 \circ c_k^{-j}$.

Hence $g(s_1) \neq g(s_2)$

2) g is onto.

If $c^{-1} \in C^{-1}$, then since $C = \{c_k^j \circ S_k\}$, there is an $s \in S_k$ such that $c_k^j \circ s = c$. It follows that $f(s) = c^{-1}$
and as c^{-1} was arbitrary, g is onto. ■

<u>Definition 2.3.7</u>: Let S_{k-1} be a symmetric group of size $n-1$ and C_k is an ordered cyclic subgroup of S_k ,the relation $\sim$ is defined as $c_k^{j_1} \sim c_k^{j_2}$ if and only if the left cosets $\{c_k^{j_1} \circ S_{k-1}\} = \{c_k^{j_2} \circ S_{k-1}\} \forall c_k^{j_1}, c_k^{j_2} \in C_k$.

<u>Lemma 2.3.8</u>: If S_k is a group with subgroup S_{k-1}, then the relation $\sim$ is an equivalence relation.

Proof :

The essence of this proof is that $\sim$ is an equivalence relation because it is defined in terms of set equality and equality for sets is an equivalence relation. The details are below.

1) $\sim$ is reflexive.

Let $s_\alpha \in S_k$ be given. Then, $s_\alpha \circ S_{k-1} = \{s_\alpha \circ s : s \in S_{k-1}\}$ and as this set is well defined, $s \circ S_{k-1} = s \circ S_{k-1}$.

2) $\sim$ is symmetric.

Let $s_\alpha, s_\beta \in S_k$ with $s_\alpha \sim s_\beta$, then by the definition of $\sim$, $\{s_\alpha \circ S_{k-1}\} = \{s_\beta \circ S_{k-1}\}$.

That is, $\{s_\alpha \circ s : s \in S_{k-1}\} = \{s_\beta \circ s : s \in S_{k-1}\}$ and as set equality is symmetric, $\{s_\beta \circ s : s \in S_{k-1}\} = \{s_\alpha \circ s : s \in S_{k-1}\}$.

Hence, $s_\beta \sim s_\alpha$ and as s_α and s_β were arbitrary, $\sim$ is symmetric.

3) $\sim$ is transitive.

Let $s_\alpha, s_\beta, s_\partial \in S_k$ with $s_\alpha \sim s_\beta$ and $s_\beta \sim s_\partial$. Then,

$\{s_\alpha \circ S_{k-1}\} = \{s_\alpha \circ s: s \in S_{k-1}\} = \{s_\beta \circ s: s \in S_{k-1}\} = s_\beta \circ S_{k-1}$ and $\{s_\beta \circ S_{k-1}\} = \{s_\beta \circ s: s \in S_{k-1}\} = \{s_\partial \circ s: s \in S_{k-1}\} = s_\partial \circ S_{k-1}$ as set equality is transitive, it follows that

$$s_\alpha \circ S_{k-1} = \{s_\alpha \circ s: s \in S_{k-1}\} = \{s_\partial \circ s: s \in S_{k-1}\} = s_\partial \circ S_{k-1}.$$

That is, $s_\alpha \sim s_\partial$, and as $s_\alpha, s_\beta, s_\partial \in S_k$ are arbitrary, $\sim$ is transitive.

■

Lemma 2.3.9: Let S_n be a symmetric group and $\sim$ be an equivalence relation on S_n. If $\{s_\alpha \circ S_{k-1}\}$ and $\{s_\beta \circ S_{k-1}\}$ are two equivalence classes with $\{s_\alpha \circ S_{k-1}\} \cap \{s_\beta \circ S_{k-1}\} \neq \emptyset$, then $\{s_\alpha \circ S_{k-1}\} = \{s_\beta \circ S_{k-1}\}$.

Proof:

To prove the lemma, we must show that $\{s_\alpha \circ S_{k-1}\} \subset \{s_\beta \circ S_{k-1}\}$ and $\{s_\beta \circ S_{k-1}\} \subset \{s_\alpha \circ S_{k-1}\}$. As $\{s_\alpha \circ S_{k-1}\}$ and $\{s_\beta \circ S_{k-1}\}$ are arbitrarily labeled, it suffices to show the former.

Let $s_{\alpha k} \in \{s_\alpha \circ S_{k-1}\}$ and suppose $\{s_\alpha \circ S_{k-1}\} \cap \{s_\beta \circ S_{k-1}\} \neq \emptyset$, there is $s_{\alpha\beta} \in \{s_\alpha \circ S_{k-1}\} \cap \{s_\beta \circ S_{k-1}\}$. As $\{s_\alpha \circ S_{k-1}\}$ is an equivalence class of $\sim$ and both a and c are in $\{s_\alpha \circ S_{k-1}\}$, it follows that $\{s_{\alpha k} \sim s_{\alpha\beta}\}$.

Now since $\{s_{\alpha k} \sim s_{\alpha\beta}\}$, $s_{\alpha\beta} \in \{s_\beta \circ S_{k-1}\}$ and $\{s_\beta \circ S_{k-1}\}$ is an equivalence class of $\sim$, it follows that $s_{\alpha k} \in \{s_\beta \circ S_{k-1}\}$ and then $\{s_\alpha \circ S_{k-1}\} = \{s_\beta \circ S_{k-1}\}$.

■

Theorem 2.3.10: If S_k is a finite group of order $k!$ and S_{k-1} is a subgroup of S_{k-1} of order $(k-1)!$, then $(k-1)!/k!$ and k is the number of distinct cosets of S_{k-1} in S_k.

Proof :

Let $\sim$ be the left coset equivalence relation defined in definition 2.3.7. It follows from lemma 2.3.8 that $\sim$ is an equivalence relation and by Lemma 2.3.4 any two distinct cosets of $\sim$ are disjoint. Hence,we can write

$S_k = \{(1) \circ S_{k-1}\} \cup \{c_k^1 \circ S_{k-1}\} \cup \ldots \{c_k^{k-1} \circ S_{k-1}\}$

where the $\{c_k^j \circ S_{k-1}\}, j = 1,2, \ldots, k$ are the disjoint left cosets of S_{k-1} guaranteed by Lemma 2.3.9.

By theorem 1.2.11, the cardinality of each of these cosets is the same as the order of S_{k-1}, and so

$|S_k| = |\{(1) \circ S_{k-1}\}| + |\{c_k^1 \circ S_{k-1}\}| + \cdots + |\{c_k^{k-1} \circ S_{k-1}\}| = k(k-1)! = k!$

■

<u>Example 2.3.11</u>: If $S_3 = \{(1) \circ (1), (12) \circ (1), (1) \circ (123), (12) \circ (123), (1) \circ (132), (12) \circ (132)\} = \{(1), (12), (123), (23), (132), (13)\}$

and

$$C_4 = \{(1), (1234), (13)(24), (1432)\}$$

Then there are four distinct cosets of S_3 in S_4:

$$S_3 \circ (1) = \{(1), (12), (123), (23), (132), (13)\} = S_3,$$

$$S_3 \circ (1234) = \{(1234), (234), (1342), (134), (34), (12)(34)\},$$

$$S_3 \circ (13)(24) = \{(13)(24), (1324), (243), (1243), (124), (24)\},$$

$$S_3 \circ (1432) = \{(1432), (143), (14), (142), (1423), (14)(23)\}.$$

<u>Definition 2.3.12</u>: let S_k be a symmetric group of size n and C_k^{-1} is an ordered cyclic subgroup of S_k ,the relation $\sim$ is defined as $c_k^{-j_1} \sim c_k^{-j_2}$ if and only if the left cosets $c_k^{-j_1} \circ S_k = c_k^{-j_2} \circ S_k \forall c_k^{-j_1}, c_k^{-j_2} \in C_k^{-1}$.

<u>Lemma 2.3.13</u>: If S_k is a group with subgroup S_{k-1}, then the relation $\sim$ is an equivalence relation.

Proof:

The essence of this proof is that $\sim$ is an equivalence relation because it is defined in terms of set equality and equality for sets is an equivalence relation. The details are below.

1)$\sim$ is reflexive.

Let $s_\alpha^{-1} \in S_k$ be given. Then, $s_\alpha^{-1} \circ S_k = \{s_\alpha^{-1} \circ s : s \in S_k\}$ and as this set is well defined, $s \circ S_k = s \circ S_k$.

2)$\sim$ is symmetric.

Let $s_\alpha^{-1}, s_\beta^{-1} \in S_k$ with $s_\alpha^{-1} \sim s_\beta^{-1}$, then by the definition of $\sim$,

$\{s_\alpha^{-1} \circ S_k\} = \{s_\beta^{-1} \circ S_k\}$. That is, $\{s_\alpha^{-1} \circ s : s \in S_k\} = \{s_\beta^{-1} \circ s : s \in S_k\}$ and as set equality is symmetric, $\{s_\beta^{-1} \circ s : s \in S_k\} = \{s_\beta^{-1} \circ s : s \in S_k\}$.

Hence, $s_\beta^{-1} \sim s_\alpha^{-1}$ and as $s_\alpha^{-1}$and s_β^{-1} are arbitrary, $\sim$ is symmetric.

3)$\sim$ is transitive.

Let $s_\alpha^{-1}, s_\beta^{-1}, s_\partial^{-1} \in S_k$ with $s_\alpha^{-1} \sim s_\beta^{-1}$ and $s_\beta^{-1} \sim s_\partial^{-1}$.

Then, $s_\alpha^{-1} \circ S_k = \{s_\alpha^{-1} \circ s : s \in S_k\} = \{s_\beta^{-1} \circ s : s \in S_k\} = s_\beta^{-1} \circ S_k$ and

$s_\beta^{-1} \circ S_k = \{s_\beta^{-1} \circ s : s \in S_k\} = \{s_\partial^{-1} \circ s : s \in S_k\} = s_\partial^{-1} \circ S_k$ As set equality is transitive, it follows that

$s_\alpha^{-1} \circ S_k = \{s_\alpha^{-1} \circ s : s \in S_k\} = \{s_\partial^{-1} \circ s : s \in S_k\} = s_\partial^{-1} \circ S_k$ or $s_\alpha^{-1} \circ S_k = s_\partial^{-1} \circ S_k$. That is, $s_\alpha^{-1} \sim s_\partial^{-1}$, and as $s_\alpha^{-1}, s_\beta^{-1}, s_\partial^{-1} \in S_k$ are arbitrary, $\sim$ is transitive. ■

Lemma 2.3.14: Let S_n be a symmetric group and $\sim$ be an equivalence relation on S_n. If $\{s_\alpha^{-1} \circ S_k\}$ and $\{s_\beta^{-1} \circ S_k\}$ are two equivalence classes with

$\{s_\alpha^{-1} \circ S_k\} \cap \{s_\beta^{-1} \circ S_k\} \neq \emptyset$, then $\{s_\alpha^{-1} \circ S_k\} = \{s_\beta^{-1} \circ S_k\}$.

Proof:

To prove the lemma, we must show that $(s_\alpha^{-1} \circ S_k) \subset (s_\beta^{-1} \circ S_k)$ and $(s_\beta^{-1} \circ S_k) \subset (s_\alpha^{-1} \circ S_k)$. As $s_\alpha^{-1} \circ S_k$ and $s_\beta^{-1} \circ S_k$ are arbitrarily labeled, it suffices to show the former.

Let $s_{\alpha k} \in s_\alpha^{-1} \circ S_k$ and suppose $(s_\alpha^{-1} \circ S_k) \cap \left(s_\beta^{-1} \circ S_k\right) \neq \emptyset$, there is $s_{\alpha\beta} \in (s_\alpha^{-1} \circ S_k) \cap \left(s_\beta^{-1} \circ S_k\right)$. As $\{s_\alpha^{-1} \circ S_k\}$ is an equivalence class of $\sim$ and both $s_{\alpha k}$ and $s_{\alpha\beta}$ are in $s_\alpha^{-1} \circ S_k$, it follows that $s_{\alpha k} \sim s_{\alpha\beta}$.

Now since $s_{\alpha k} \sim s_{\alpha\beta}$, $s_{\alpha\beta} \in \left\{s_\beta^{-1} \circ S_k\right\}$ and $\left\{s_\beta^{-1} \circ S_k\right\}$ is an equivalence class of $\sim$, it follows that $s_{\alpha k} \in \left\{s_\beta^{-1} \circ S_k\right\}$ and then $\{s_\alpha^{-1} \circ S_k\} = \left\{s_\beta^{-1} \circ S_k\right\}$.

■

Example 2.3.15:

If $S_4 = \left\{\begin{matrix} (1),(12),(123),(23),(132),(13), \\ (1234),(234),(1342),(134),(34),(12)(34), \\ (13)(24),(1324),(243),(1243),(124),(24), \\ (1432),(143),(14),(142),(1423),(14)(23) \end{matrix}\right\}$.

From example 2.3.4

$$S_4 = \left\{\begin{matrix} S_3 \circ (1), \\ S_3 \circ (1234), \\ S_3 \circ (13)(24), \\ S_3 \circ (1432) \end{matrix}\right\},$$

and $C_4^{-1} = \{(1),(1432),(13)(24),(1234)\}$

$$C_4^{-1} \circ S_4 = \left\{\begin{matrix} S_3 \circ (1) \circ (1), \\ S_3 \circ (1234) \circ (1432), \\ S_3 \circ (13)(24) \circ (13)(24), \\ S_3 \circ (1432) \circ (1234) \end{matrix}\right\} = S_3 .$$

Chapter Three: Some Operators For Complete Homogeneous Symmetric Functions

3.1 Introduction

A Frobenius matrix is a special kind of square matrix from numerical mathematics. Some researchers use the transpose of this matrix, which (dually) cycles coordinates, and is more convenient for some purposes. There is suggestion method by the scientist Randic [45] to calculate the sum of the powers for the eigenvalues and works iteratively through Newton's identities to get the elementary symmetric functions for the eigenvalues, which represent the coefficients of the characteristic polynomial.

For every analytic function φ that takes $\mathbb{U}$ into itself we associate the composition operator C_φ defined by:

$$C_\varphi f = f \circ \varphi \ (f \in H^2)$$

The Littlewoods a subordination theorem (1.3.7) tells us that the operator C_φ takes the Hardy space H^2 into itself. Littlewood's principle also supplies an estimate which shows that C_φ is a bounded operator on H^2, see [14, 48] for more details.

If φ be an analytic self-map of $\mathbb{E}$, then $\varphi_n = \varphi \circ \varphi \circ \ldots \circ \varphi$ (n-times).

This chapter consists of five sections. The section one is introduction. In section two, we recall and derived newton's identities for characteristic polynomials by companion matrix. In section three, we derived a new formula for coefficients of complete homogeneous symmetric in terms of powers of traces' matrix. In section four, we derived an ordered cyclic operator. In section five, we derived an inverse for an ordered cyclic operator.

3.2 Derivation of Newton's Identities for Characteristic Polynomials by Companion Matrix

If A is an $m \times m$ matrix with entries from some field K such that A is similar to the companion matrix over K of its characteristic polynomial $p(t)$ and there exists a cyclic vector v in $V = K^m$ for A, meaning that $\{v, Av, A^2v, \dots, A^{m-1}v\}$ is a basis of V. Equivalently, such that V is cyclic as a $K[A]$-$module$ and $V = K[A]/(p(A))$.

Definition 3.2.1: Let $\{a_n\}$ be an iterative sequence, the Frobenius companion matrix of the monic polynomial $p(t) = t^n + a_1t^{n-1} + \cdots + a_{n-1}t + a_n; t \in K$ is defined as:

$$C(p) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix},$$

and the transpose's matrix

$$C^T(p) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}.$$

The transpose generates the sequence, in the sense that:

$$C^T(p) \begin{bmatrix} a_k \\ a_{k+1} \\ \vdots \\ a_{k+m-1} \end{bmatrix} = \begin{bmatrix} a_{k+1} \\ a_{k+2} \\ \vdots \\ a_{k+m} \end{bmatrix},$$

increments the series by 1. The vector $(1, t, t^2, \dots, t^{m-1})$ is an eigenvector of this matrix for eigenvalue t, when t is a root of the characteristic polynomial $p(t)$. If $p(t)$ has distinct roots $\lambda_1, \lambda_2, \dots, \lambda_m$ (the eigenvalues of $C^T(p)$), then $C(p)$ is diagonalizable as follows:

$$\mathbb{V}C^T(p)\mathbb{V}^{-1} = diag\,\{\lambda_1,\lambda_2,\dots,\lambda_m\}\,;\mathbb{V} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{m-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_m & \lambda_m^2 & \cdots & \lambda_m^{m-1} \end{bmatrix}.$$

where $\mathbb{V}$ is the Vandermonde matrix corresponding to the $\lambda's$. In that case,[2] traces of powers j of $C^T(p)$ readily yield sums of the same powers m of all roots of $p(t), Tr\big((C^T(p))^j\big) = \sum \lambda_k^j\,; k = 1,2,\dots,m$.

If $p(t)$ has a non-simple root, then $C(p)$ isn't diagonalizable (its Jordan canonical form contains one block for each distinct root). The scientist Radink explained [37] that:

$$C^T(p) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}; tr\big(C^T(p)\big) = -a_1.$$

$$\big(C^T(p)\big)^2 = \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \\ a_n a_1 & -a_n + a_{n-1}a_1 & -a_{n-1} + a_{n-2}a_1 & \cdots & -a_2 + a_1^2 \end{bmatrix}; \; tr\left(\big(C^T(p)\big)^2\right) = a_1^2 - 2a_2.$$

$$\big(C^T(p)\big)^3 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \\ a_1 a_n & -a_n + a_1 a_{n-1} & -a_{n-1} + a_1 a_{n-2} & \cdots & -a_2 + a_1^2 \\ -a_1^2 a_n + a_2 a_n & a_1 a_n - a_1^2 a_{n-1} + a_2 a_{n-1} & a_1 a_{n-1} - a_1^2 a_{n-2} + a_2 a_{n-2} & \cdots & -a_3 + 2a_1 a_2 - a_1^3 \end{bmatrix};$$

$$;tr\left(\big(C^t(p)\big)^3\right) = -a_1^3 - 3a_3 + 3a_1 a_2,$$

and so on by taking $tr\left(\left(C^T(p)\right)^j\right); j = 1,2,\dots,n$,that yields to :

$$tr\left(\left(C^T(p)\right)^1\right) = -a_1 = \mathbb{s}_1; a_1 = -\mathbb{s}_1,$$

$$tr\left(\left(C^T(p)\right)^2\right) = a_1^2 - 2a_2 = \mathbb{s}_2; a_2 = \frac{1}{2}((\mathbb{s}_1)^2 - \mathbb{s}_2),$$

$$tr\left(\left(C^T(p)\right)^3\right) = -a_1^3 - 3a_3 + 3a_1a_2 = \mathbb{s}_3 ; a_3 = \frac{1}{3}\left(\frac{-(\mathbb{s}_1)^3}{2} + 3\mathbb{s}_1\,\mathbb{s}_2 - \mathbb{s}_3\right),$$

$\vdots$

<u>Example 3.2.1</u>: Let $p(t) = t^4 + a_1t^3 + a_2t^2 + a_3t + a_4$ be a characteristic polynomial which has four distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, the Frobenius companion matrix of this polynomial is defined as: $C(p) = \begin{bmatrix} 0 & 0 & 0 & -a_4 \\ 1 & 0 & 0 & -a_3 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_1 \end{bmatrix}$,

$$C^T(p) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix}; tr\left(C^T(p)\right) = -a_1 .$$

$$\left(C^T(p)\right)^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \\ a_1a_4 & -a_4 + a_3a_1 & -a_3 + a_2a_1 & a_1^2 - a_2 \end{bmatrix}; tr\left(\left(C^T(p)\right)^2\right) = a_1^2 - 2a_2 .$$

$$\left(C^T(p)\right)^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \\ a_1 a_4 & -a_4 + a_3 a_1 & -a_3 + a_2 a_1 & a_1^2 - a_2 \\ -a_1^2 a_4 + a_2 a_4 & a_1 a_4 - a_1^2 a_3 + a_3 a_2 & -a_4 + a_3 a_1 + a_2^2 - a_1^2 a_2 & -a_1^3 - a_3 + 2a_2 a_1 \end{bmatrix};$$

$$tr\left(\left(C^T(p)\right)^3\right) = -a_1^3 + 3a_2 a_1 - 3a_3.$$

$$\left(C^T(p)\right)^4 = \begin{bmatrix} -a_4 & -a_3 & -a_2 & -a_1 \\ a_1 a_4 & -a_4 + a_3 a_1 & -a_3 + a_2 a_1 & a_1^2 - a_2 \\ -a_1^2 a_4 + a_2 a_4 & a_1 a_4 - a_1^2 a_3 + a_3 a_2 & -a_4 + a_3 a_1 + a_2^2 - a_1^2 a_2 & -a_1^3 - a_3 + 2a_2 a_1 \\ a_1^3 a_4 + a_3 a_4 - 2a_1 a_2 a_4 & -a_1^2 a_4 + a_3^2 + a_1^3 a_3 + a_2 a_4 - 2a_1 a_2 a_3 & a_1 a_4 - a_1^2 a_3 - 2a_1 a_2^2 + a_1^3 a_2 + 2a_3 a_2 & a_1^4 - a_4 + 2a_1 a_3 + a_2^2 - 3a_1^2 a_2 \end{bmatrix};$$

$$tr\left(\left(C^T(p)\right)^4\right) = a_1^4 + 4a_1 a_3 + 2a_2^2 - 3a_1^2 a_2 - 4a_4.$$

3.3 Derivation of the Formula for h_k in Terms of $\mathbb{s}_k$

If $A = (a_{ij})_{m\times m}, a_{ij} \in \mathbb{C}$, be a diagonal complex matrix, the rational polynomial $\frac{1}{det\,(I - A)}$ which have factors into $\frac{1}{\prod_{l=1}^{m}(1 - z\lambda_l)}$ where λ_l are the eigenvalues of the matrix. The coefficients $h_1, h_2, \cdots, h_n, \ldots$ of this polynomial

$$H(z) = h_0 + h_1 z + h_2 z^2 + \cdots + h_k z^k + \cdots; z \in \mathbb{E},$$

are given in terms of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ by

$$h_k(\lambda_1, \lambda_2, \ldots, \lambda_m) = \sum_{j_1 + \cdots + j_m = k}^{n} {\lambda_1}^{j_1} \ldots {\lambda_m}^{j_m}.$$

$$h_k(\lambda_1, \lambda_2, \ldots, \lambda_m) = \sum_{1 \le j_1, \ldots, j_k \le m} \frac{m_1!\, m_2! \ldots m_m!}{k!} \lambda_{j_1} \lambda_{j_2} \ldots \lambda_{j_m}.$$

The symmetric powers of the eigenvalues λ_l are defined by

$\mathbb{s}_n = \text{trace}\,(A^n) = \sum_{l=1}^{m} \lambda_l^n$. The summation here is over all the eigenvalues of A. We start with identities

$$\mathbb{s}_1 - h_1 = 0,$$

$$\mathbb{s}_2 - h_1\mathbb{s}_1 + 2h_2 = 0,$$
$$\mathbb{s}_3 - h_1\mathbb{s}_2 + h_2\mathbb{s}_1 - 3h_3 = 0,$$
$$\mathbb{s}_4 - h_1\mathbb{s}_3 + h_2\mathbb{s}_2 - h_3\mathbb{s}_1 + 4h_4 = 0,$$
$$\vdots$$

Consider the formal power series

$$\mathbb{S}(z) = \sum_{n=0}^{\infty} \mathbb{s}_n\, z^n \text{ and } H(z) = \sum_{n=0}^{\infty} (-1)^{n-1} h_n\, z^n.$$

It is convenient to take $\mathbb{s}_0 = 0\ \ and\ \ a_0 = 1$.Then, by newton's identities and applying ω, one gets:

$\omega(h_n)(t) = a_n(-t)$ or equivalently, $\omega(h_n) = (-1)^{n-1}a_n$ we obtain identities similar to newton's identities and are equivalent to the formal differential equation:

$$\mathbb{S}(z)H(z) + z\acute{H}(z) = 0,$$

$$\begin{aligned}
&\mathbb{S}(z)H(z) + z\acute{H}(z) \\
&\qquad = \left(\sum_{n=0}^{\infty} \mathbb{s}_n\, z^n\right)\left(\sum_{n=0}^{\infty} (-1)^{n-1} h_n\, z^n\right) \\
&\qquad + \sum_{n=0}^{\infty} (-1)^{n-1} n h_n\, z^n \\
&\qquad = (-1 + h_0) + (\mathbb{s}_1 - h_1)z + (\mathbb{s}_2 - h_1\mathbb{s}_1 + 2h_2)z^2 \\
&\qquad + (\mathbb{s}_3 - h_1\mathbb{s}_2 + h_2\mathbb{s}_1 - 3h_3)z^3 \\
&\qquad + (\mathbb{s}_4 - h_1\mathbb{s}_3 + h_2\mathbb{s}_2 - h_3\mathbb{s}_1 + 4h_4)z^4 + \cdots = 0 + 0 + 0 + \cdots \\
&\qquad = 0\,.
\end{aligned}$$

This can be solved by separating the variables:

$$\mathbb{S}(z) = -\frac{z\acute{H}(z)}{H(z)},$$

and

$$\int \mathbb{S}(z)dz = -\int \frac{z\acute{H}(z)}{H(z)} dz = -\ln H(z) + c\,.$$

We can integrate the left side term by term to get

$$\int \sum_{n=1}^{\infty} \mathbb{s}_n \, z^{n-1} \, dz = \sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n} = -\ln H(z) + c \, .$$

When $z = 0$, the left side is 0 and the right side is c. So, $c = 0$ and we have two power series whose coefficients involve h_n and $\mathbb{s}_n$.

Since $\ln H(z) = -\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}$, and that yields $H(z) = e^{-\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}}$. Expanding using the power series for the exponential function,

$$H(z) = 1 - \frac{1}{1!}\left(\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}\right) + \frac{1}{2!}\left(\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}\right)^2 - \frac{1}{3!}\left(\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}\right)^3 + \cdots \quad .$$

Therefore, collecting coefficients of z^nin this series as before,

$$(-1)^{n-1} h_n = \frac{1}{1!}\frac{\mathbb{s}_n}{n} + \sum_{\substack{j_1 + j_2 = 2 \\ j_1, j_2 \geq 1}} \frac{1}{2!}\frac{\mathbb{s}_{j_1}\, \mathbb{s}_{j_2}}{j_1\, j_2} - \sum_{\substack{j_1 + j_2 + j_3 = 3 \\ j_1, j_2, j_3 \geq 1}} \frac{1}{3!}\frac{\mathbb{s}_{j_1}\, \mathbb{s}_{j_2}\, \mathbb{s}_{j_3}}{j_1\, j_2 j_3} + \cdots$$

$$= \sum_{\substack{j_1 + j_2 + \cdots + j_n = k \\ j_1 + 2j_2 + 3j_3 + \cdots + nj_n = n}} \frac{(-1)^{k-1}\, \mathbb{s}_1^{j_1} \mathbb{s}_2^{j_2} \ldots\, \mathbb{s}_n^{j_n}}{j_1!\;\, j_2! \ldots\, j_n!\;\, 1^{j_1}\, 2^{j_2} \ldots n^{j_n}} \, .$$

One also has

$$h_n = \sum_{\substack{j_1 + j_2 + \cdots + j_n = k \\ j_1 + 2j_2 + 3j_3 + \cdots + nj_n = n}} \frac{(-1)^{k-1}\, \mathbb{s}_1^{j_1} \mathbb{s}_2^{j_2} \ldots\, \mathbb{s}_n^{j_n}}{j_1!\;\, j_2! \ldots\, j_n!\;\, 1^{j_1}\, 2^{j_2} \ldots n^{j_n}} \, ,$$

and in fact the magnitude of the coefficient

$$\frac{n!}{j_1!\, 1^{j_1}\;\, j_2!\, 2^{j_2} \ldots\;\, j_n!\;\, n^{j_n}} \, ,$$

is the number of permutations of n symbols composed of j_l) cycles of length l for $l = 1,2, \ldots n.$.,

$$h_n = \sum_{\substack{j_1+j_2+\cdots+j_n=k \\ j_1+2j_2+3j_3+\cdots+nj_n=n}} \frac{n!}{n!} . \frac{(-1)^{k-1}\, \mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2} \dots\, \mathbb{s}_n^{j_n}}{j_1!\ \ j_2! \dots j_n!\ 1^{j_1}\ 2^{j_2} \dots n^{j_n}} \ .$$

It also provides a check on computations, viz

$$\sum_{\substack{j_1+j_2+\cdots+j_n=k \\ j_1+2j_2+3j_3+\cdots+nj_n=n}} \frac{n!}{j_1!\, 1^{j_1}\ \ j_2!\, 2^{j_2} \dots\ \ j_n!\ n^{j_n}},$$

it equals to $|s(n,k)| = (-1)^{n-k}s(n,k)$ where $s(n,k)$ are the well-known Sterling numbers of the first kind.

$$h_n = \sum_{\substack{j_1+j_2+\cdots+j_n=k \\ j_1+2j_2+3j_3+\cdots+nj_n=n}} \frac{s(n,k)\ \mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2} \dots\ \mathbb{s}_n^{j_n}}{n!} .$$

<u>Example 3.3.1</u>: Let $A = [1]$ be a diagonal matrix with eigenvalue $\lambda_1 = 1$. The powers of trace of A

$$\mathbb{s}_1 = 1, \mathbb{s}_2 = 1, \mathbb{s}_3 = 1, \mathbb{s}_4 = 1, \dots \quad ,$$

$$\mathbb{s}_1^2 = 1, \mathbb{s}_1^3 = 1, \mathbb{s}_1^4 = 1, \dots \quad ,$$

$$h_n = \sum_{\substack{j_1+j_2+\cdots+j_n=k \\ j_1+2j_2+3j_3+\cdots+nj_n=n}} \frac{|s(n,k)|\ \mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2} \dots\ \mathbb{s}_n^{j_n}}{n!} .$$

$$s(1,1) = 1,$$

$$s(2,1) = -1, s(2,2) = 1,$$

$$s(3,1) = 2, s(3,2) = -3, s(3,3) = 1,$$

$$s(4,1) = -6, s(4,2) = 11, s(4,3) = -6, s(4,4) = 1,$$

$$h_0 = 1,$$

$$h_1 = \sum_{i_1=1} \frac{|s(1,1)|\mathbb{s}_1^1}{1!} = 1(1) = 1,$$

$$h_2 = \sum_{\substack{j_1+j_2=1,2 \\ j_1+2j_2=2}} \frac{|s(2,k=1,2)|\,\mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2}}{2!} = \frac{s(2,1)\mathbb{s}_1^0 S\mathbb{s}_2^1 - s(2,2)\mathbb{s}_1^2\mathbb{s}_2^0}{2!} = \frac{1+1}{2}$$

$$= 1,$$

$$h_3 = \sum_{\substack{j_1+j_2+j_3=1,2,3 \\ j_1+2j_2+3j_3=3}} \frac{|s(3,k=1,2,3)|\,\mathbb{s}_1^{j_1}\mathbb{s}_2^{j}\,\mathbb{s}_3^{j_3}}{3!}$$

$$= \frac{|s(3,1)|\mathbb{s}_1^0\mathbb{s}_2^0\,\mathbb{s}_3^1 + |s(3,2)|\mathbb{s}_1^1\mathbb{s}_2^1\mathbb{s}_3^0 + |s(3,3)|\,\mathbb{s}_1^0\mathbb{s}_2^0\,\mathbb{s}_1^3}{3!}$$

$$= \frac{2\,(1) + 3(1) + (1)}{3!} = 1,$$

$$h_4 = \sum_{\substack{j_1+j_2+j_3+j_4=1,2,3,4 \\ j_1+2j_2+3j_3+4j_4=4}} \frac{|s(4,k=1,2,3,4)|\,\mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2}\,\mathbb{s}_3^{j_3}\mathbb{s}_4^{j_4}}{4!}$$

$$= \frac{|s(4,1)|\,\mathbb{s}_1^0\mathbb{s}_2^0\,\mathbb{s}_3^0\mathbb{s}_4^1 + |s(4,2)|\,\{\mathbb{s}_1^1\mathbb{s}_2^0\,\mathbb{s}_3^1\mathbb{s}_4^0 + \mathbb{s}_1^0\mathbb{s}_2^2\,\mathbb{s}_3^0\mathbb{s}_4^0\} + |s(4,3)|\,\mathbb{s}_1^2\mathbb{s}_2^1\,\mathbb{s}_3^0 S_4^0 + |s(4,4)|\,\mathbb{s}_1^4\mathbb{s}_2^0\,\mathbb{s}_3^0\mathbb{s}_4^0}{4!}$$

$$= \frac{6(1) + 8(1) + 3(1) + 6\,(1) + 1(1)}{4!} = 1\,.$$

$$H(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \cdots$$

<u>Example 3.3.2</u>: Let $A = \begin{bmatrix} i & 0 \\ 0 & 1+i \end{bmatrix}$ be a diagonal matrix with eigenvalues $\lambda_1 = i, and\ \lambda_2 = 1 + i$.

The powers of trace of A

$$\mathbb{s}_1 = 1 + 2i, \mathbb{s}_2 = -1 + 2i, \mathbb{s}_3 = -2 + i, \mathbb{s}_4 = -3, \ldots$$

$$\mathbb{s}_1^2 = -3 + 4i, \mathbb{s}_1^3 = -11 - 2i, \mathbb{s}_1^4 = -7 - 24i, \mathbb{s}_2^2 = -3 - 4i\,.$$

$$\sum_{\substack{j_1+j_2+\cdots+j_n=k \\ j_1+2j_2+3j_3+\cdots+nj_n=n}} \frac{|s(n,k)|\ \mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2}\ \dots\ \mathbb{s}_n^{j_n}}{n!}\ .$$

$$s(1,1) = 1,$$

$$s(2,1) = -1, s(2,2) = 1,$$

$$s(3,1) = 2, s(3,2) = -3, s(3,3) = 1,$$

$$s(4,1) = -6, s(4,2) = 11, s(4,3) = -6, s(4,4) = 1,$$

$$s(5,1) = 24, s(5,2) = -50, s(5,3) = 35, s(5,4) = -10, s(5,5) = 1,$$

$$h_0 = 1,$$

$$h_1 = \sum_{i_1=1} \frac{|s(1,1)|\mathbb{s}_1^1}{1!} = \frac{1(1+2i)}{1!} = 1+2i,$$

$$h_2 = \sum_{\substack{j+j_2=1,2 \\ j_1+2j_2=2}} \frac{|s(2,k=1,2)|\ \mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2}}{2!} = \frac{s(2,1)\mathbb{s}_1^0\mathbb{s}_2^1 +\ s(2,2)\mathbb{s}_1^2\mathbb{s}_2^0}{2!}$$

$$= \frac{(1)(-1+2i)+(1)(-3+4i)}{2} = -2+3i\ ,$$

$$h_3 = \sum_{\substack{j_1+j_2+j_3=1,2,3 \\ j_1+2j_2+3j_3=3}} \frac{|s(3,k=1,2,3)|\ \mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2}\ \mathbb{s}_3^{j_3}}{3!}$$

$$= \frac{|s(3,1)|\mathbb{s}_1^0\mathbb{s}_2^0\ \mathbb{s}_3^1 +\ |s(3,2)|\mathbb{s}_1^1\mathbb{s}_2^1\mathbb{s}_3^0 +\ |s(3,3)|\ \mathbb{s}_1^0\mathbb{s}_2^0\ \mathbb{s}_1^3}{3!}$$

$$= \frac{(2)\ (-2+i)+(3)(1+2i)(-1+2i)+(11-2i)}{3!} = -5,$$

$$h_4 = \sum_{\substack{j_1+j_2+j_3+j_4=1,2,3,4 \\ j_1+2j_2+3j_3+4j_4=4}} \frac{|s(4,k=1,2,3,4)|\ \mathbb{S}_1^{j_1}\mathbb{S}_2^{j_2}\ \mathbb{S}_3^{j_3}\mathbb{S}_4^{j_4}}{4!}$$

$$= \frac{|s(4,1)|\ \mathbb{S}_1^0\mathbb{S}_2^0\ \mathbb{S}_3^0\mathbb{S}_4^1 +\ |s(4,2)|\ \{\mathbb{S}_1^1\mathbb{S}_2^0\ \mathbb{S}_3^1\mathbb{S}_4^0 + \mathbb{S}_1^0\mathbb{S}_2^2\ \mathbb{S}_3^0\mathbb{S}_4^0\} +\ |s(4,3)|\ \mathbb{S}_1^2\mathbb{S}_2^1\ \mathbb{S}_3^0\mathbb{S}_4^0 + |\ s(4,4)|\ \mathbb{S}_1^4\mathbb{S}_2^0\ \mathbb{S}_3^0\mathbb{S}_4^0}{4!}$$

$$= \frac{(6)(-3) + 8(-4-3i) + (3)(-3-4i) + (6)(-5-10i) + (-7-24i)}{4!} = -4-5i\,.$$

$$H(z) = \frac{1}{(1-iz)(1-(1+i)z)}$$
$$= 1 + (1+2i)z + (-2+3i)z^2 + (-5)z^3 + (-4-5i)z^4 + \cdots \quad .$$

3.4 Derivation of an Ordered Cyclic Operator

From definition 2.2.1, can be defined an ordered cyclic operator of complete homogeneous symmetric function and we conclude the action of an ordered cyclic operator $\mathcal{T}^{\mathbb{i}}$ of order $\mathbb{i}$ on each coefficient h_n of complete homogeneous symmetric function ;i.e. $\mathcal{T}^{\mathbb{i}}(h_n) = h_{n+\mathbb{i}}$ is similar to an action of ordered cyclic subgroups $C_n^{\mathbb{i}}$ of order $\mathbb{i}$ on S_n ;i.e.$C_n^{\mathbb{i}} \circ S_n = S_{n+\mathbb{i}}$.

Here $\mathcal{T}^{\mathbb{i}}\big(H(z)\big)$ is represented univalent function because the coefficient $h_{\mathbb{i}}$ is coefficient of z^0 and so on .

<u>Definition 3.4.1</u>: Let C_m be an ordered cyclic subgroup of symmetric group S_m and $\mathcal{M}_m$is an ordered cyclic matrices for symmetric matrices of size m as follows:

$$\mathcal{m}_m^0 = I_m = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix},$$

$$m_m^1 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \dots , m_m^{m-1} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

We define the isomorphism $\mathcal{T}: (C_m, \circ) \to (\mathcal{M}_m, .)$ such that $\mathcal{T}\left(c_m^{\mathbb{i}}\right) = m_m^{\mathbb{i}}$ for all $\mathbb{i}$.

Example 3.4.2: Let C_4 be an ordered cyclic subgroup of S_4 and $\mathcal{M}_4$ is a binary matrix of size 4 as follows:

$$C_4 = \{c_4^0 = (1), c_4^1 = (1234), c_4^2 = (13)(24), c_4^3 = (1432)\}$$

$$\mathcal{M}_4 = \{m_4^0, m_4^1, m_4^2, m_4^3\}; m_4^0 = I_4, m_4^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$m_4^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, m_4^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Such that the $\mathcal{T}\left(c_4^{\mathbb{i}}\right) = m_4^{\mathbb{i}}$ for all $\mathbb{i} = 0,1,2,3$.

Also these matrices act on the sequence $\{a_n\}$ as follows:

$$m_m^{\mathbb{i}} \begin{bmatrix} a_k \\ a_{k+1} \\ \vdots \\ a_{k+n-1} \end{bmatrix} = \begin{bmatrix} a_{k+\mathbb{i}} \\ a_{k+\mathbb{i}+1} \\ \vdots \\ a_{k+\mathbb{i}+n-1} \end{bmatrix}, \text{ it follows from that :}$$

$$m_m^{\mathbb{i}} \left(C^T(p) \begin{bmatrix} a_k \\ a_{k+1} \\ \vdots \\ a_{k+n-1} \end{bmatrix} \right) = m_m^{\mathbb{i}} \left(\begin{bmatrix} a_{k+1} \\ a_{k+2} \\ \vdots \\ a_{k+n} \end{bmatrix} \right) = \begin{bmatrix} a_{k+\mathbb{i}+1} \\ a_{k+\mathbb{i}+2} \\ \vdots \\ a_{k+\mathbb{i}+n} \end{bmatrix}.$$

It's satisfies for each powers of companion matrix and thus follow action of ordered cyclic operator of elementary symmetric polynomials

$$tr\left(m_m^{\mathbb{i}}\left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}\right)\right) = -a_{1+\mathbb{i}},$$

$$tr\left(m_m^{\mathbb{i}}\left(\begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \\ a_n a_1 & -a_n + a_{n-1}a_1 & -a_{n-1} + a_{n-2}a_1 & \cdots & -a_2 + a_1^2 \end{bmatrix}\right)\right)$$

$$= a_{1+\mathbb{i}}^2 - 2a_{2+\mathbb{i}} \,,$$

$$tr\left(m_m^{\mathbb{i}}\left(\left(C^T(p)\right)^3\right)\right) =$$

$$tr\left(m_m^{\mathbb{i}}\left(\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \\ a_1 a_n & -a_n + a_1 a_{n-1} & -a_{n-1} + a_1 a_{n-2} & \cdots & -a_2 + a_1^2 \\ -a_1^2 a_n + a_2 a_n & a_1 a_n - a_1^2 a_{n-1} + a_2 a_{n-1} & a_1 a_{n-1} - a_1^2 a_{n-2} + a_2 a_{n-2} & \cdots & 2a_1 a_2 - a_1^3 \end{bmatrix}\right)\right) =$$

$$-a_{1+\mathbb{i}}^3 - a_{3+\mathbb{i}} + 3a_{1+\mathbb{i}}a_{2+\mathbb{i}} \,,$$

$$\vdots$$

and so on by taking $tr\left(m_m^{\mathbb{i}}\left(C^T(p)\right)^k\right); k = 1,2,\ldots,n$,that yields to :

$$tr\left(m_m^{\mathbb{i}}\left(C^T(p)\right)^1\right) = -a_{1+\mathbb{i}} = \mathbb{s}_1,$$

$$tr\left(m_m^{\mathbb{i}}\left(C^T(p)\right)^2\right) = a_{1+\mathbb{i}}^2 - 2a_{2+\mathbb{i}} = \mathbb{s}_2; a_{2+\mathbb{i}} = \frac{1}{2}((\mathbb{s}_1)^2 - \mathbb{s}_2),$$

$$tr\left(\mathscr{m}_m^{\mathbb{i}}\left(C^T(p)\right)^3\right) = -a_{1+\mathbb{i}}^3 - a_{3+\mathbb{i}} + 3a_{1+\mathbb{i}}a_{2+\mathbb{i}}a_{3+\mathbb{i}} = \mathbb{s}_3; a_{3+\mathbb{i}}$$

$$= \frac{1}{3}\left(\frac{-(\mathbb{s}_1)^3}{2} + 3\mathbb{s}_1\,\mathbb{s}_2 - \mathbb{s}_3\right).$$

⋮

Example 3.4.3: Let $p(t) = z^4 + a_1z^3 + a_2z^2 + a_3z + a_4$ be a characteristic polynomial

$\mathscr{m}_4^{\mathbb{i}}\left(C^T(p)\right) =$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{\mathbb{i}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix}; tr\left(\mathscr{m}_4^{\mathbb{i}}\left(C^T(p)\right)\right) = -a_{1+\mathbb{i}}; \mathbb{i} = 0,1,2,3.$$

$\mathscr{m}_4^{\mathbb{i}}\left(C^T(p)\right)^2 =$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{\mathbb{i}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_4 & -a_3 & -a_2 & -a_1 \\ a_1a_4 & -a_4 + a_3a_1 & -a_3 + a_2a_1 & a_1^2 - a_2 \end{bmatrix}; tr\left(\mathscr{m}_4^{\mathbb{i}}\left(C^T(p)\right)^2\right) = a_{1+\mathbb{i}}^2 - 2a_{2+\mathbb{i}}; \mathbb{i} = 0,1,2,3.$$

$\mathscr{m}_4^{\mathbb{i}}\left(C^T(p)\right)^3$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{\mathbb{i}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -a_4 & -a_3 & -a_2 & -a_1 \\ a_1a_4 & -a_4 + a_3a_1 & -a_3 + a_2a_1 & a_1^2 - a_2 \\ -a_1^2a_4 + a_2a_4 & a_1a_4 - a_1^2a_3 + a_3a_2 & -a_4 + a_3a_1 + a_2^2 - a_1^2a_2 & -a_1^3 - a_3 + 2a_2a_1 \end{bmatrix};$$

$$tr\left(\mathscr{m}_4^{\mathbb{i}}\left(C^T(p)\right)^3\right) = -a_{1+\mathbb{i}}^3 + 3a_{2+i}a_{1+\mathbb{i}} - 3a_{3+\mathbb{i}}; \mathbb{i} = 0,1,2,3.$$

$$\mathscr{m}_4^{\mathbb{i}}\left(C^T(p)\right)^4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{i}$$

$$\begin{bmatrix} -a_4 & -a_3 & -a_2 & -a_1 \\ a_1a_4 & -a_4 + a_3a_1 & -a_3 + a_2a_1 & a_1^2 - a_2 \\ -a_1^2a_4 + a_2a_4 & a_1a_4 - a_1^2a_3 + a_3a_2 & -a_4 + a_3a_1 + a_2^2 - a_1^2a_2 & -a_1^3 - a_3 + 2a_2a_1 \\ a_1^3a_4 + a_3a_4 - 2a_1a_2a_4 & -a_1^2a_4 + a_3^2 + a_1^3a_3 + a_2a_4 - 2a_1a_2a_3 & a_1a_4 - a_1^2a_3 - 2a_1a_2^2 + a_1^3a_2 + 2a_3a_2 & a_1^4 - a_4 + 2a_1a_3 + a_2^2 - 3a_1^2a_2 \end{bmatrix};$$

$$tr\left(\mathscr{m}_4^{\mathbb{i}}\left(C^T(p)\right)^4\right) = a_{1+\mathbb{i}}^4 + 4a_{1+\mathbb{i}}a_{3+\mathbb{i}} + 2a_{2+\mathbb{i}}^2 - 3a_{1+\mathbb{i}}^2a_{2+\mathbb{i}} - 4a_{4+\mathbb{i}}; \mathbb{i} = 0,1,2,3\,.$$

It follows from acting of ordered cyclic operator on characteristic polynomial

$$\mathcal{T}^{\mathbb{i}}(p(z)) = \mathcal{T}^{\mathbb{i}}(a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n)$$
$$= a_{\mathbb{i}} z^n + a_{1+\mathbb{i}} z^{n-1} + \cdots + a_{n-1+\mathbb{i}} z + a_{n+\mathbb{i}}; n \; mod \; \mathbb{i} \, .$$

By applying ω, one gets:

$\omega(a_n)(t) = h_n(-t)$or equivalently,

$\omega(a_n) = (-1)^{n-1} h_n$ One also has

$$\mathcal{T}^{\mathbb{i}}\big(H(z)\big) = \mathcal{T}^{\mathbb{i}}(h_0 + h_1 z + h_2 z^2 + \cdots + h_k z^k + \cdots)$$
$$= h_{\mathbb{i}} + h_{1+\mathbb{i}} z + h_{2+\mathbb{i}} z^2 + \cdots + h_{k+\mathbb{i}} z^k + \cdots \; .$$

With start with identities

$$\mathbb{s}_1 - h_{1+\mathbb{i}} = 0,$$
$$\mathbb{s}_2 - h_{1+\mathbb{i}}\mathbb{s}_1 + 2h_{2+\mathbb{i}} = 0,$$
$$\mathbb{s}_3 - h_{1+\mathbb{i}}\mathbb{s}_2 + h_{2+\mathbb{i}}\mathbb{s}_1 - 3h_{3+\mathbb{i}} = 0,$$
$$\mathbb{s}_4 - h_{1+\mathbb{i}}\mathbb{s}_3 + h_{2+\mathbb{i}}\mathbb{s}_2 - h_{3+\mathbb{i}}\mathbb{s}_1 + 4h_{4+\mathbb{i}} = 0,$$
$$\vdots$$

Consider the formal power series $\mathbb{S}(z) = \sum_{n=0}^{\infty} \mathbb{s}_n \, z^n$ and $\mathcal{T}^{\mathbb{i}}\big(H(z)\big) = \sum_{n=0}^{\infty} (-1)^{n-1} h_{n+\mathbb{i}} \, z^n$.It is convenient to take $\mathbb{s}_0 = 0 \;\; and \;\; a_0 = 1$.

Then, by newton's identities and applying ω, one gets:

$\omega(h_n)(t) = a_n(-t)$ or equivalently, $\omega(h_n) = (-1)^{n-1} a_n$ we obtain identities similar to newton's identities and are equivalent to the formal differential equation:

$$\mathbb{S}(z)\mathcal{T}^{\mathbb{i}}\big(H(z)\big) + \; z\mathcal{T}^{\mathbb{i}}\left(\acute{H}(z)\right) = \; 0,$$

$$\mathbb{S}(z)\mathcal{T}^{\mathbb{i}}\big(H(z)\big) + z\mathcal{T}^{\mathbb{i}}\left(\acute{H}(z)\right)$$
$$= \left(\sum_{n=0}^{\infty} \mathbb{s}_n\, z^n\right)\left(\sum_{n=0}^{\infty}(-1)^{n-1}h_{n+\mathbb{i}}\, z^n\right)$$
$$+ \sum_{n=0}^{\infty}(-1)^{n-1}nh_{n+\mathbb{i}}\, z^n$$
$$= (-1 + h_{\mathbb{i}}) + (\mathbb{s}_1 - h_{1+\mathbb{i}})z + (\mathbb{s}_2 - h_{1+\mathbb{i}}\mathbb{s}_1 + 2h_{2+\mathbb{i}})z^2$$
$$+ (\mathbb{s}_3 - h_{1+\mathbb{i}}\mathbb{s}_2 + h_{2+\mathbb{i}}\mathbb{s}_1 - 3h_{3+\mathbb{i}})z^3$$
$$+ (\mathbb{s}_4 - h_{1+\mathbb{i}}\mathbb{s}_3 + h_{2+\mathbb{i}}\mathbb{s}_2 - h_{3+\mathbb{i}}\mathbb{s}_1 + 4h_{4+\mathbb{i}})z^4 + \cdots$$
$$= 0 + 0 + 0 + \cdots = 0\,.$$

This can be solved by separating the variables:

$$\mathbb{S}(z) = -\frac{z\mathcal{T}^{\mathbb{i}}\left(\acute{H}(z)\right)}{\mathcal{T}^{\mathbb{i}}\big(H(z)\big)}; \mathbb{i} = 0,1,2, \ldots$$

and

$$\int \mathbb{S}(z)dz = -\int \frac{z\mathcal{T}^{\mathbb{i}}\left(\acute{H}(z)\right)}{\mathcal{T}^{\mathbb{i}}\big(H(z)\big)}dz = -\ln \mathcal{T}^{\mathbb{i}}\big(H(z)\big) + c\,.$$

We can integrate the left side term by term to get

$$\int \sum_{n=1}^{\infty} \mathbb{s}_n\, z^{n-1}\, dz = \sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n} = -\ln \mathcal{T}^{\mathbb{i}}\big(H(z)\big) + c.$$

When $z = 0$, the left side is 0 and the right side is c. So, $c = 0$ and we have two power series whose coefficients involve h_n and $\mathbb{s}_n$.

Since $\ln \mathcal{T}^{\mathbb{i}}\big(H(z)\big) = -\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}$, and that yields $\mathcal{T}^{\mathbb{i}}\big(H(z)\big) = e^{-\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}}$.

Expanding using the power series for the exponential function,

$$\mathcal{T}^{\mathbb{i}}\big(H(z)\big) = 1 - \frac{1}{1!}\left(\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}\right) + \frac{1}{2!}\left(\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}\right)^2 - \frac{1}{3!}\left(\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}\right)^3 + \cdots$$

Therefore, collecting coefficients of z^nin this series as before,

$$(-1)^{n-1} h_{n+\mathbb{i}} = \frac{1}{1!}\frac{\mathbb{s}_{(n+\mathbb{i})}}{(n+\mathbb{i})} + \sum_{\substack{j_1+j_2=2\\ j_1,j_2\geq 1}} \frac{1}{2!}\frac{\mathbb{s}_{j_1}\ \mathbb{s}_{j_2}}{j_1 j_2} - \sum_{\substack{j_1+j_2+j_3=3\\ j_1,j_2,j_3\geq 1}} \frac{1}{3!}\frac{\mathbb{s}_{j_1}\ \mathbb{s}_{j_2}\ \mathbb{s}_{j_3}}{j_1\ j_2\ j_3} + \cdots$$

$$= \sum_{\substack{j_1+j_2+\cdots+j_{(n+\mathbb{i})}=k\\ j+2j_2+3j_3+\cdots+(n+\mathbb{i})j_{(n+\mathbb{i})}=(n+\mathbb{i})}} \frac{(-1)^{k-1}\ \mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2}\ \ldots\ \mathbb{s}_{(n+\mathbb{i})}^{j_{(n+\mathbb{i})}}}{j_1!\ \ j_2!\ \ldots\ j_{(n+\mathbb{i})}!\ 1^{j_1}\ 2^{j_2}\ \ldots (n+\mathbb{i})^{j_{(n+\mathbb{i})}}}.$$

One also has

$$h_{n+\mathbb{i}} = \sum_{\substack{j_1+j_2+\cdots+j_{(n+\mathbb{i})}=k\\ j+2j_2+3j_3+\cdots+(n+\mathbb{i})j_{(n+\mathbb{i})}=(n+\mathbb{i})}} \frac{(-1)^{n-k}\ \mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2}\ \ldots\ \mathbb{s}_{(n+\mathbb{i})}^{j_{(n+\mathbb{i})}}}{j_1!\ \ j_2!\ \ldots\ j_{(n+\mathbb{i})}!\ 1^{j_1}\ 2^{j_2}\ \ldots (n+\mathbb{i})^{j_{(n+\mathbb{i})}}},$$

and in fact the magnitude of the coefficient

$$\frac{(n+\mathbb{i})!}{j_1!\,1^{j_1}\ \ j_2!\,2^{j_2}\ \ldots\ \ j_{(n+\mathbb{i})}!\ \ (n+\mathbb{i})^{j_{(n+\mathbb{i})}}},$$

is the number of permutations of $(n+\mathbb{i})$ symbols composed of j_l- cycles of length l for $l = 1,2,\ldots(n+\mathbb{i})$,

$h_{n+\mathbb{i}}$

$$= \sum_{\substack{j_1+j_2+\cdots+j_{(n+\mathbb{i})}=k\\ j+2j_2+3j_3+\cdots+(n+\mathbb{i})j_{(n+\mathbb{i})}=(n+\mathbb{i})}} \frac{(n+\mathbb{i})!}{(n+\mathbb{i})!}.\frac{(-1)^{n-k}\ \mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2}\ \ldots\ \mathbb{s}_{(n+\mathbb{i})}^{j_{(n+\mathbb{i})}}}{j_1!\ \ j_2!\ \ldots\ j_{(n+\mathbb{i})}!\ 1^{j_1}\ 2^{j_2}\ \ldots (n+\mathbb{i})^{j_{(n+\mathbb{i})}}}.$$

It also provides a check on computations, viz

$$\sum_{\substack{j_1+j_2+\cdots+j_{(n+\mathbb{i})}=k\\ j+2j_2+3j_3+\cdots+(n+\mathbb{i})j_{(n+\mathbb{i})}=(n+\mathbb{i})}} \frac{n!}{j_1!\,1^{j_1}\ \ j_2!\,2^{j_2}\ \ldots\ \ j_n!\ n^{j_n}},$$

it equals to $|s(n+\mathbb{i},k)| = (-1)^{n-k}s(n+\mathbb{i},k)$where $s(n+\mathbb{i},k)$ are the well-known Sterling numbers of the first kind.

$$h_{n+\mathbb{i}} = \sum_{\substack{j_1+j_2+\cdots+j_{(n+\mathbb{i})}=k \\ j+2j_2+3j_3+\cdots+(n+\mathbb{i})j_{(n+\mathbb{i})}=(n+\mathbb{i})}} \frac{s(n+\mathbb{i},k)\ \mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2}\ \ldots\ \mathbb{s}_{(n+\mathbb{i})}^{j_{(n+\mathbb{i})}}}{(n+\mathbb{i})!}\ .$$

<u>Example 3.4.4</u>: From example 3.3.2 , the polynomial

$$H(z) = \frac{1}{(1-iz)(1-(1+i)z)}$$
$$= 1 + (1+2i)z + (-2+3i)z^2 + (-5)z^3 + (-4-5i)z^4 + \cdots\ .$$

Then $\mathcal{T}^1\big(H(z)\big) = (1+2i) + (-2+3i)z + (-5)z^2 + (-4-5i)z^3 + \cdots$.

3.5 Derivation The Inverse of an Ordered Cyclic Operator

From proposition 2.2.4 can be defined an inverse for an ordered cyclic operator and we conclude the action of inverse for ordered cyclic operator $\mathcal{T}^{-\mathbb{i}}$ of order $\mathbb{i}$ on each coefficient h_n of complete homogeneous symmetric function ;i.e.$\mathcal{T}^{-\mathbb{i}}(h_n) = h_{n-\mathbb{i}}$ is similar to the action of inverse of ordered cyclic subgroups $C_n^{-\mathbb{i}}$ of order $\mathbb{i}$ on S_n ;i.e.$C_n^{-\mathbb{i}} \circ S_n = S_{n-\mathbb{i}}$.

Here $\mathcal{T}^{-\mathbb{i}}\big(H(z)\big)$ is represented multivalent function because the coefficient h_0 is coefficient of $z^{\mathbb{i}}$ when the coefficients of $z^{0,1,\ldots,\mathbb{i}-1}$is equal to zero .

<u>Definition 3.5.1</u>: Let C_m^{-1} be an inverse for an ordered cyclic subgroup of symmetric group S_m and $\mathcal{M}_m^{-1}$is an inverse for an ordered cyclic matrices for symmetric matrices of size m as follows:

$$\mathcal{m}_m^{-m} = \mathcal{m}_m^0 = I_m = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix},$$

$$m_m^{-1} = m_m^{m-1} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \dots, m_m^{-(m-1)}$$

$$= m_m^{1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

We define the isomorphism $\mathcal{T}: (C_m^{-1}, \circ) \to (\mathcal{M}_m^{-1}, .)$ such that $\mathcal{T}\left(c_m^{-\mathbb{i}}\right) = m_m^{-\mathbb{i}}$ for all $\mathbb{i}$.

<u>Example 3.5.2</u>: Let C_4^{-1} be an inverse for an ordered cyclic subgroup of S_4 and $\mathcal{M}_4^{-1}$is a binary matrix of size 4 as follows:

$$C_4^{-1} = c_4^{-4} = c_4^{0} = (1), c_4^{-1} = (1432), c_4^{-2} = (13)(24), c_4^{-3} = (1234)\}$$

$$\mathcal{M}_4 = \{m_4^0, m_4^{-1}, m_4^{-2}, m_4^{-3}\}; m_4^0 = I_4, m_4^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, m_4^{-2}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, m_4^{-3} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Such that the $\mathcal{T}\left(c_4^{-\mathbb{i}}\right) = m_4^{-\mathbb{i}}$ for all $i = 0,1,2,3$.

Also, these matrices act on the sequence $\{a_n\}$ as follows:

$$\mathcal{m}_m^{-\mathbb{i}}\begin{bmatrix} a_k \\ a_{k+1} \\ \vdots \\ a_{k+n-1}\end{bmatrix} = \begin{bmatrix} a_{k-\mathbb{i}} \\ a_{k+\mathbb{i}-1} \\ \vdots \\ a_{k-\mathbb{i}+n-1}\end{bmatrix}$$ it follows from that :

$$\mathcal{m}_m^{-\mathbb{i}}\left(C^T(p)\begin{bmatrix} a_k \\ a_{k+1} \\ \vdots \\ a_{k+n-1}\end{bmatrix}\right) = \mathcal{m}_m^{-\mathbb{i}}\left(\begin{bmatrix} a_{k+1} \\ a_{k+2} \\ \vdots \\ a_{k+n}\end{bmatrix}\right) = \begin{bmatrix} a_{k-\mathbb{i}+1} \\ a_{k-\mathbb{i}+2} \\ \vdots \\ a_{k-\mathbb{i}+n}\end{bmatrix}.$$

It's satisfies for each powers of companion matrix, thus the action of ordered cyclic operator of elementary symmetric polynomials

$$tr\left(\mathcal{m}_m^{-\mathbb{i}}\left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}\right)\right) = -a_{1-\mathbb{i}},$$

$$tr\left(\mathcal{m}_m^{-\mathbb{i}}\left(\begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \\ a_n a_1 & -a_n + a_{n-1}a_1 & -a_{n-1} + a_{n-2}a_1 & \cdots & -a_2 + a_1^2 \end{bmatrix}\right)\right)$$

$$= a_{1-i}^2 - 2a_{2-i},$$

$$tr\left(\mathcal{m}_m^{-\mathbb{i}}\left(\left(C^T(p)\right)^3\right)\right) =$$

$$tr\left(\mathcal{m}_m^{-\mathbb{i}}\left(\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \\ a_1 a_n & -a_n + a_1 a_{n-1} & -a_{n-1} + a_1 a_{n-2} & \cdots & -a_2 + a_1^2 \\ -a_1^2 a_n + a_2 a_n & a_1 a_n - a_1^2 a_{n-1} + a_2 a_{n-1} & a_1 a_{n-1} - a_1^2 a_{n-2} + a_2 a_{n-2} & \cdots & 2a_1 a_2 - a_1^3 \end{bmatrix}\right)\right) =$$

$$-a_{1-\mathbb{i}}^3 - a_{3-\mathbb{i}} + 3a_{1-\mathbb{i}}a_{2-\mathbb{i}},$$

$$\vdots$$

and so on by taking $tr\left(m_m^{-i}\left(C^T(p)\right)^k\right); k = 1,2,\dots,n$,that yields to :

$$tr\left(m_m^{-i}\left(C^T(p)\right)^1\right) = -a_{1-i} = \mathbb{s}_1,$$

$$tr\left(m_m^{-i}\left(C^T(p)\right)^2\right) = a_{1-i}^2 - 2a_{2-i} = \mathbb{s}_2; a_{2-i} = \frac{1}{2}((\mathbb{s}_1)^2 - \mathbb{s}_2),$$

$$tr\left(m_m^{-i}\left(C^T(p)\right)^3\right) = -a_{1-i}^3 - 3a_{3-i} + 3a_{1-i}a_{2-i}a_{3-i} = \mathbb{s}_3; a_{3-i} = \frac{1}{3}\left(\frac{-(\mathbb{s}_1)^3}{2} + 3\mathbb{s}_1\mathbb{s}_2 - \mathbb{s}_3\right),$$

$$\vdots$$

<u>Example 3.5.3</u>: Let $p(t) = z^4 + a_1z^3 + a_2z^2 + a_3z + a_4$ be a characteristic polynomial

$$m_4^{-i}\left(C^T(p)\right) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{i} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix}; tr\left(m_4^{-i}\left(C^T(p)\right)\right) = -a_{1-i}; i = 0,1,2,3.$$

$$m_4^{-i}\left(C^T(p)\right)^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{i} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_4 & -a_3 & -a_2 & -a_1 \\ a_1a_4 & -a_4 + a_3a_1 & -a_3 + a_2a_1 & a_1^2 - a_2 \end{bmatrix}; tr\left(m_4^{-i}\left(C^T(p)\right)^2\right) = a_{1-i}^2 - 2a_{2-i}; i = 0,1,2,3.$$

$$m_4^{-i}\left(C^T(p)\right)^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{i} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -a_4 & -a_3 & -a_2 & -a_1 \\ a_1a_4 & -a_4 + a_3a_1 & -a_3 + a_2a_1 & a_1^2 - a_2 \\ -a_1^2a_4 + a_2a_4 & a_1a_4 - a_1^2a_3 + a_3a_2 & -a_4 + a_3a_1 + a_2^2 - a_1^2a_2 & -a_1^3 - a_3 + 2a_2a_1 \end{bmatrix};$$

$$tr\left(m_4^{-i}\left(C^T(p)\right)^3\right) = -a_{1-i}^3 + 3a_{2-i}a_{1-i} - 3a_{3-i}; i = 0,1,2,3.$$

$$m_4^{-\mathbb{i}}\left(C^T(p)\right)^4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{\mathbb{i}}$$

$$\begin{bmatrix} -a_4 & -a_3 & -a_2 & -a_1 \\ a_1a_4 & -a_4+a_3a_1 & -a_3+a_2a_1 & a_1^2-a_2 \\ -a_1^2a_4+a_2a_4 & a_1a_4-a_1^2a_3+a_3a_2 & -a_4+a_3a_1+a_2^2-a_1^2a_2 & -a_1^3-a_3+2a_2a_1 \\ a_1^3a_4+a_3a_4-2a_1a_2a_4 & -a_1^2a_4+a_3^2+a_1^3a_3+a_2a_4-2a_1a_2a_3 & a_1a_4-a_1^2a_3-2a_1a_2^2+a_1^3a_2+2a_3a_2 & a_1^4-a_4+2a_1a_3+a_2^2-3a_1^2a_2 \end{bmatrix};$$

$$tr\left(m_4^{-\mathbb{i}}\left(C^T(p)\right)^4\right) = a_{1-\mathbb{i}}^4 + 4a_{1-\mathbb{i}}a_{3-\mathbb{i}} + 2a_{2-\mathbb{i}}^2 - 3a_{1-\mathbb{i}}^2a_{2-\mathbb{i}} - 4a_{4-\mathbb{i}};\ \mathbb{i} = 0,1,2,3.$$

It follows from acting of ordered cyclic operator on characteristic polynomial $\mathcal{T}^{-\mathbb{i}}(p(z)) = \mathcal{T}^{-\mathbb{i}}(a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n) = a_{-\mathbb{i}}z^n + a_{1-\mathbb{i}}z^{n-1} + \cdots + a_{n-1-\mathbb{i}}z + a_{n-\mathbb{i}};\ n\ mod - \mathbb{i}$.

By applying ω, one gets:

$\omega(a_n)(t) = h_n(-t)$or equivalently,

$\omega(a_n) = (-1)^{n-1}h_n$ One also has

$$\mathcal{T}^{-\mathbb{i}}\left(H(z)\right) = \mathcal{T}^{-\mathbb{i}}(h_0 + h_1z + h_2z^2 + \cdots + h_kz^k + \cdots)$$
$$= h_{-\mathbb{i}} + h_{1-\mathbb{i}}z + h_{2-\mathbb{i}}z^2 + \cdots + h_{k-\mathbb{i}}z^k + \cdots \quad .$$

With start with identities

$$\mathbb{s}_1 - h_{1-\mathbb{i}} = 0,$$
$$\mathbb{s}_2 - h_{1-\mathbb{i}}\mathbb{s}_1 + 2h_{2-\mathbb{i}} = 0,$$
$$\mathbb{s}_3 - h_{1-\mathbb{i}}\mathbb{s}_2 + h_{2-\mathbb{i}}\mathbb{s}_1 - 3h_{3-\mathbb{i}} = 0,$$
$$\mathbb{s}_4 - h_{1-\mathbb{i}}\mathbb{s}_3 + h_{2-\mathbb{i}}\mathbb{s}_2 - h_{3-\mathbb{i}}\mathbb{s}_1 + 4h_{4-\mathbb{i}} = 0,$$
$$\vdots$$

Consider the formal power series $\mathbb{S}(z) = \sum_{n=0}^{\infty} \mathbb{s}_n\, z^n$ and $\mathcal{T}^{-\mathbb{i}}\left(H(z)\right) = \sum_{n=0}^{\infty}(-1)^{n-1}h_{n+\mathbb{i}}\, z^n$.It is convenient to take $\mathbb{s}_0 = 0$ and $a_0 = 1$.

Then, by newton's identities and applying ω, one gets:

$\omega(h_n)(t) = a_n(-t)$or equivalently, $\omega(h_n) = (-1)^{n-1}a_n$ we obtain identities similar to newton's identities and are equivalent to the formal differential equation:

$$\mathbb{S}(z)\mathcal{T}^{-\mathbb{i}}\big(H(z)\big) + z\mathcal{T}^{-\mathbb{i}}\left(\acute{H}(z)\right) = 0,$$

$$\begin{aligned}\mathbb{S}(z)\mathcal{T}^{-\mathbb{i}}\big(H(z)\big) &+ z\mathcal{T}^{-\mathbb{i}}\left(\acute{H}(z)\right)\\ &= \left(\sum_{n=0}^{\infty} \mathbb{s}_n\, z^n\right)\left(\sum_{n=0}^{\infty}(-1)^{n-1}h_{n-\mathbb{i}}\, z^n\right)\\ &+ \sum_{n=0}^{\infty}(-1)^{n-1}nh_{n-\mathbb{i}}\, z^n\\ &= (-1 + h_{-\mathbb{i}}) + (\mathbb{s}_1 - h_{1-\mathbb{i}})z + (\mathbb{s}_2 - h_{1-\mathbb{i}}\mathbb{s}_1 + 2h_{2-\mathbb{i}})z^2\\ &+ (\mathbb{s}_3 - h_{1-\mathbb{i}}\mathbb{s}_2 + h_{2-\mathbb{i}}\mathbb{s}_1 - 3h_{3-\mathbb{i}})z^3\\ &+ (\mathbb{s}_4 - h_{1-\mathbb{i}}\mathbb{s}_3 + h_{2-\mathbb{i}}\mathbb{s}_2 - h_{3-\mathbb{i}}\mathbb{s}_1 + 4h_{4-\mathbb{i}})z^4 + \cdots\\ &= 0 + 0 + 0 + \cdots = 0\,.\end{aligned}$$

This can be solved by separating the variables:

$$\mathbb{S}(z) = -\frac{z\mathcal{T}^{-\mathbb{i}}\left(\acute{H}(z)\right)}{\mathcal{T}^{-\mathbb{i}}\big(H(z)\big)}; \mathbb{i} = 0,1,2, \dots \;,$$

and

$$\int \mathbb{S}(z)dz = -\int \frac{z\mathcal{T}^{-\mathbb{i}}\left(\acute{H}(z)\right)}{\mathcal{T}^{-\mathbb{i}}\big(H(z)\big)}dz = -\ln \mathcal{T}^{-\mathbb{i}}\big(H(z)\big) + c\,.$$

We can integrate the left side term by term to get

$$\int \sum_{n=1}^{\infty} \mathbb{s}_n\, z^{n-1}\, dz = \sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n} = -\ln \mathcal{T}^{-\mathbb{i}}\big(H(z)\big) + c\,.$$

When $z = 0$, the left side is 0 and the right side is c. So, $c = 0$ and we have two power series whose coefficients involve h_n and $\mathbb{s}_n$.

Since $\ln \mathcal{T}^{-\mathbb{i}}\big(H(z)\big) = -\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}$, and that yields $\mathcal{T}^{-\mathbb{i}}\big(H(z)\big) = e^{-\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}}$.

Expanding using the power series for the exponential function,

$$\mathcal{T}^{-\mathbb{i}}\big(H(z)\big) = 1 - \frac{1}{1!}\left(\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}\right) + \frac{1}{2!}\left(\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}\right)^2 - \frac{1}{3!}\left(\sum_{n=1}^{\infty} \mathbb{s}_n \frac{z^n}{n}\right)^3 + \cdots \ .$$

Therefore, collecting coefficients of z^nin this series as before,

$$(-1)^{n-1} h_{n-\mathbb{i}} = \frac{1}{1!}\frac{\mathbb{s}_{n-\mathbb{i}}}{n-\mathbb{i}} + \sum_{\substack{j_1+j_2=2\\ j_1,j_2\ge 1}} \frac{1}{2!}\frac{\mathbb{s}_{j_1}\,\mathbb{s}_{j_2}}{j_1\,j_2} - \sum_{\substack{j_1+j_2+j_3=3\\ j_1,j_2,j_3\ge 1}} \frac{1}{3!}\frac{\mathbb{s}_{j_1}\,\mathbb{s}_{j_2}\,\mathbb{s}_{j_3}}{j_1\,j_2\,j_3} + \cdots$$

$$= \sum_{\substack{j_1+j_2+\cdots+j_{(n-\mathbb{i})}=k\\ j_1+2j_2+3j_3+\cdots+(n-\mathbb{i})j_{(n-\mathbb{i})}=(n-\mathbb{i})}} \frac{(-1)^{k-1}\,\mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2}\ldots\,\mathbb{s}_{(n-\mathbb{i})}^{j_{(n-\mathbb{i})}}}{j_1!\ \ j_2!\ldots\,j_{(n-\mathbb{i})}!\ 1^{j_1}\,2^{j_2}\ldots(n-\mathbb{i})^{j_{(n-\mathbb{i})}}}\,.$$

One also has

$$h_{n-\mathbb{i}} = \sum_{\substack{j_1+j_2+\cdots+j_{(n-\mathbb{i})}=k\\ j_1+2j_2+3j_3+\cdots+(n-\mathbb{i})j_{(n-\mathbb{i})}=(n-\mathbb{i})}} \frac{(-1)^{n-k}\,\mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2}\ldots\,\mathbb{s}_{(n-\mathbb{i})}^{j_{(n-\mathbb{i})}}}{j_1!\ \ j_2!\ldots\,j_{(n-\mathbb{i})}!\ 1^{j_1}\,2^{j_2}\ldots(n-\mathbb{i})^{j_{(n-\mathbb{i})}}}\,,$$

and in fact the magnitude of the coefficient

$$\frac{(n-\mathbb{i})!}{j_1!\,1^{j_1}\ \ j_2!\,2^{j_2}\ldots\ j_{(n-\mathbb{i})}!\ (n-\mathbb{i})^{j_{(n-\mathbb{i})}}}\,,$$

is the number of permutations of $n-\mathbb{i}$ symbols composed of j_l- cycles of length l for $l = 1,2,\ldots(n-\mathbb{i})$,

$$h_{n-\mathbb{i}} = \sum_{\substack{j_1+j_2+\cdots+j_{(n-\mathbb{i})}=k\\ j_1+2j_2+3j_3+\cdots+(n-\mathbb{i})j_{(n-\mathbb{i})}=(n-\mathbb{i})}} \frac{(n-\mathbb{i})!}{(n-\mathbb{i})!}.\frac{(-1)^{n-k}\,\mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2}\ldots\,\mathbb{s}_{(n-\mathbb{i})}^{j_{(n-\mathbb{i})}}}{j_1!\ \ j_2!\ldots\,j_{(n-\mathbb{i})}!\ 1^{j_1}\,2^{j_2}\ldots(n-\mathbb{i})^{j_{(n-\mathbb{i})}}}\,.$$

It also provides a check on computations, viz

$$\sum_{\substack{j_1+j_2+\cdots+j_{(n-\mathbb{i})}=k \\ j_1+2j_2+3j_3+\cdots+(n-\mathbb{i})j_{(n-\mathbb{i})}=(n-\mathbb{i})}} \frac{(n-\mathbb{i})!}{j_1!\,1^{j_1}\ j_2!\,2^{j_2}\ \ldots\ j_{(n-\mathbb{i})}!\ (n-\mathbb{i})^{j_{(n-\mathbb{i})}}},$$

it equals to $|s(n-\mathbb{i},k)| = (-1)^{n-k}s(n-\mathbb{i},k)$where $s(n-\mathbb{i},k)$ are the well-known Sterling numbers of the first kind.

$$h_{n-\mathbb{i}} = \sum_{\substack{j_1+j_2+\cdots+j_{(n-\mathbb{i})}=k \\ j_1+2j_2+3j_3+\cdots+(n-\mathbb{i})j_{(n-\mathbb{i})}=(n-\mathbb{i})}} \frac{s(n-\mathbb{i},k)\ \mathbb{s}_1^{j_1}\mathbb{s}_2^{j_2}\ \ldots\ \mathbb{s}_{(n-\mathbb{i})}^{j_{(n-\mathbb{i})}}}{(n-\mathbb{i})!}.$$

Example 3.5.4: From example 3.3.2 , the polynomial

$$H(z) = \frac{1}{(1-iz)(1-(1+i)z)}$$
$$= 1 + (1+2i)z + (-2+3i)z^2 + (-5)z^3 + (-4-5i)z^4 + \cdots \ .$$

Then $\mathcal{T}^{-1}\big(H(z)\big) = z + (1+2i)z^2 + (-2+3i)z^3 + (-5)z^4 + (-4-5i)z^5 + \cdots$.

Chapter Four: Differential Subordination and Superordination Results for Completely Homogeneous Symmetric Functions Class

4.1 Introduction

This chapter is devoted for studying differential subordination and superordination results for type of special meromorphic univalent functions defined by some operators. A differential subordination in the complex plane is the generalization of the differential inequality on the real line. The concept of differential subordination plays a very important role in functions of real variable. In the theory of complex-valued function, there are several differential applications in which a characterization of a function is determined by a differential condition. Miller and Mocanu [36] have contributed number of papers on differential subordinations and superordinations. Many textbooks studied the differential subordination and superordination with sandwich theorems such as Duren [20], Goodman[29] and Pommerenke [42].This chapter is divided into three sections. The first section is introduction. In the second section is concerned with sandwich theorems for certain analytic completely homogeneous symmetric functions class defined by a new cyclic operator. Here, we derive some results certain of completely homogeneous symmetric functions by using differential subordination and superordination defined by an ordered cyclic operator such that $\mathcal{H}(\mathbb{U})$ denotes the analytic function class in the open unit disk $\mathbb{U} = \{|z| < 1; z \in \mathbb{C}\}$, and let $\mathcal{H}[a,p]$ denotes the subclass of functions $f \in \mathcal{H}(\mathbb{U})$ as:

$$\mathcal{H}[a,p] = \left\{f \in \mathcal{H}: f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots\right\}; a \in \mathbb{C}, p \in N = \{1,2,\ldots\}.$$

The resolvent $det\ (I - A)^{-1}$ of a complex matrix A is naturally an analytic function of eigenvalues $\lambda \in \mathbb{C}$ and this eigenvalues are isolated singularities. In general, any matrix has finitely eigenvalues. The resolvent set of A is defined as follows :

$$\varrho(A) = \{\lambda \in \mathbb{C}: \lambda I - A \text{ is invertible}\}$$

and the spectrum of A is expressed by $\vartheta(\mathrm{A}) = \mathbb{C}/\varrho(\mathrm{A})$.

For distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$, the polynomial

$$H(z) = h_0 + h_1 z + h_2 z^2 + \cdots + h_n z^n + \cdots$$

are given in terms of the eigenvalues$\lambda_1, \lambda_2, \dots, \lambda_m$ by

$$H(z) = \sum_{n=0}^{\infty} h_n\, z^n, h_n = \sum_{1 \le j_1, \dots, j_m \le m} \frac{m_1!\, m_2! \dots m_m!}{n!} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_m}\ ; z \in \mathbb{C}. \tag{4.1.1}$$

This formula can also be written in terms of the distinct powers of traces of matrix as follows:

$$H(z) = \sum_{n=0}^{\infty} h_n\, z^n ; h_n = \sum_{\substack{j_1 + j_2 + \cdots + j_n = k \\ j_1 + 2j + 3j_3 + \cdots + nj_n = n}} \frac{(-1)^{n-k}\, \mathbb{s}_1^{j_1} \mathbb{s}_2^{j_2} \dots \mathbb{s}_n^{j_n}}{j_1!\ j_2! \dots j_n!\ 1^{j_1}\, 2^{j_2} \dots n^{j_n}}. \tag{4.1.2}$$

This class represent the subclasses of analytical functions $\mathcal{H}[a, n]$ and denoted by $\mathbb{H}$ such that $\mathcal{H}[a, 1] = \mathbb{H}$ and has coefficients of the form (4.1.2), i.e. when the value of n is equal to one, and can be reduces a class $\mathbb{H}$ to the class $\mathcal{H}$ of normalized univalent analytical functions and composed of functions of the following form:

$$\frac{H(z) - h_0}{h_1} = z + \sum_{n=2}^{\infty} a_n\, z^n; a_n = \frac{h_i}{h_1}, (z \in \mathbb{U}).$$

To each analytic function φ in open unit disk $\mathbb{U}$ into itself, we associated the composition operator C_φ defined by:

$C_\varphi H = H \circ \varphi$ for all $H \in \mathbb{H}$.

Then, we define ordered cyclic operator $\mathbb{T}^{\mathbb{i}}$ of f as follows:

$$\mathcal{T}^{\mathbb{i}}\big(H(z)\big) = \big(C_{\mathcal{T}^{\mathbb{i}}}H\big)(z) = \big(\mathcal{T}^{\mathbb{i}} \circ H\big)(\mathrm{z}) = \mathcal{T}^{\mathbb{i}}\big(H(z)\big) = \mathcal{T}^{\mathbb{i}}\left(h_0 + \sum_{n=1}^{\infty} h_n z^n\right)$$

$$= h_{\mathbb{i}} + \sum_{n=1}^{\infty} h_{n+\mathbb{i}} z^n \;\; h_{n+\mathbb{i}} \in \mathbb{C}.$$

Let $\mathbb{q}(z)$ be convex univalent in $\mathbb{U}$ $with$ $\mathbb{q}(0) = 1$and $\eta \in \mathbb{C} \setminus \{0\}$.
Suppose that

$$Re\left(1 + \frac{z\mathbb{q}''(z)}{\mathbb{q}'(z)}\right) > max\left(0, -Re\left(\frac{1}{\eta}\right)\right).$$

If

$$e_1(z) = (1+\eta)\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) - \eta\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\left(\frac{z\left(\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big)\right)'}{\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big)}\right),$$

$$e_1(z) \prec \mathbb{q}(z) + \eta z \mathbb{q}'(z),$$

then $\mathcal{T}^{\mathbb{i}}H(z) \prec \mathbb{q}(z)$,and $\mathbb{q}(z)$ is the best dominant.

The third section deals with the sandwich results of completely homogeneous symmetric functions defined by an differential for ordered cyclic operator. Here, we establish subordination and superordination results for completely homogeneous symmetric functions defined by the cyclic operator in open unit disk. We get a number of sandwich-type results, such as, let q_j be two univalent convex functions in $\mathbb{U}$ in condition for $\mathbb{q}_j(0) = 1, \mathbb{q}_j(z) \neq 0, (j = 1,2)$. Assume that $\mathbb{q}_1$ and $\mathbb{q}_2$ satisfy the conditions

$$e_2(z) \prec \mathbb{x} + \mathbb{y}\mathbb{q}(z) + \mathbb{z}\big(\mathbb{q}(z)\big)^2 + \eta\left(\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}\right)$$

and

$$\mathbb{x} + \mathbb{y}\mathbb{q}(z) + \mathbb{z}\big(\mathbb{q}(z)\big)^2 + \eta\left(\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}\right) \prec e_2(z),$$

respectively. If $H \in \mathbb{H}$, and suppose that H satisfies the next condition:

$\big(z\acute{D}\big)^{n+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) \in \mathcal{H}[\mathbb{q}(0),1] \cap \mathrm{Q}$,

and $\big(z\acute{D}\big)^{n+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) \neq 0$ *and* $e_2(z)$ is univalent in $\mathbb{U}$, then

$$\mathbb{x} + \mathbb{y}\mathbb{q}(z) + \mathbb{z}\big(\mathbb{q}(z)\big)^2 + \eta\left(\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}\right) \prec e_2(z)$$

$$\prec \mathbb{x} + \mathbb{y}\mathbb{q}(z) + \mathbb{z}\big(\mathbb{q}(z)\big)^2 + \eta\left(\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}\right),$$

Implies that:

$$\mathbb{q}_1(z)) \prec t\big(z\acute{D}\big)^{n+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) + (1-t)\big(z\acute{D}\big)^{n}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) \prec \mathbb{q}_2(z),$$

where $\mathbb{q}_1(z)$ *and* $\mathbb{q}_2(z)$are the best subordination and best dominant respectively.

4.2 Sandwich Theorems for Certain Completely Homogeneous Symmetric Functions Defined by an Ordered Cyclic Operator Cyclic Operator

Let $\mathbb{H} = \mathcal{H}[a,1]$ denote the class of functions H of the form:

$$H(z) = \sum_{n=0}^{\infty} h_n z^n, h_n \in \mathbb{C},$$

and

$$\mathcal{T}^{\mathbb{i}}\big(H(z)\big) = H(z) = \sum_{n=0}^{\infty} h_{n+\mathbb{i}} z^n, h_{n+\mathbb{i}} \in \mathbb{C},$$

which are analytic univalent in the open unit disk $\mathbb{U} = \{z: z \in \mathbb{C}, |z| < 1\}$. Miller and Mocanu [36] consider the problem *of* determining conditions an admissible functions Ψ such that (1.5.6) is satisfied, then it implies that

$\mathbb{p}(z) \prec \mathbb{q}(z)$, for all functions $\mathbb{p}(z) \in \mathcal{H}$. Moreover they found conditions so that $\mathbb{q}$ is the smallest function with this property called the best domainant of the subordination (1.5.6). Also, they studied the dual problem and determined conditions on $\boldsymbol{\Psi}$ such that (1.5.9) is satisfied, implies $\mathbb{q}(z) \prec \mathbb{p}(z)$, for all functions $\mathbb{q} \in Q$, that satisfy the superordination (1.5.9). They also found conditions so that the function q is the largest function with this property, called the best subordinant of the superordination (1.5.9). See also [6,8,10,22].

For the function $\mathcal{T}^{\mathbb{i}}(H(z)) \in \mathbb{H}$, we using the results (see [2,10]) to obtain sufficient conditions for normalized analytic functions to satisfy :

$$\mathbb{q}_1(z) \prec \frac{z\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)'}{\mathcal{T}^{\mathbb{i}}(H(z))} \prec \mathbb{q}_2(z),$$

where $\mathbb{q}_1$ and $\mathbb{q}_2$ are given univalent functions in $\mathbb{U}$ with $\mathbb{q}_1(0) = \mathbb{q}_2(0) = 1$. Also, Al-Ameedee et al. [1] and El-Ashwah and Aouf [22] derived some differential subordination and superordination results for analytic functions in $\mathbb{U}$. Recently, several authors obtained some sandwich results for subclasses of analytic functions (see [2,6,7,8,10]).

<u>Lemma (4.2.1)</u>: Let $H \in \mathbb{H}$, an ordered cyclic operator $\mathcal{T}^{\mathbb{i}}$ for an analytic function H is given as :

$$\mathcal{T}^{\mathbb{i}}(H(z)) = \sum_{k=0}^{n} h_{n+i} z^n, z \in \mathbb{U},$$

we define the ordered cyclic operator of degree 1 as follows:

$$\mathcal{T}^1 : \mathbb{H} \to \mathbb{H}, \quad by$$

$$\mathcal{T}^1(H(z)) = \sum_{k=0}^{n} h_{n+1} z^n, z \in \mathbb{U},$$

and

$$\mathcal{T}^{\mathbb{i}}\big(H(z)\big) = \sum_{k=0}^{n} h_{n+\mathbb{i}} z^n \,, z \in \mathbb{U},$$

where

$$\mathcal{T}^{0}\big(H(z)\big) = H(z).$$

Proof:

Let $H(z) \in \mathbb{H},$ then

$$\mathcal{T}^{1}\big(H(z)\big) = C_{\mathcal{T}^1} H(z) = H \circ \mathcal{T}^{1}(z) = \mathcal{T}^{1}\left(\sum_{k=1}^{\infty} h_k\, z^k\right) = \sum_{n=1}^{\infty} h_{n+1}\, z^n,$$

and

$$\mathcal{T}^{2}\big(H(z)\big) = C_{\mathcal{T}^2} H(z) = (H \circ \mathcal{T}^{2})(z) = (H \circ \mathcal{T} \circ \mathcal{T})(z) = \mathcal{T}\left(\mathcal{T}\left(\sum_{k=1}^{\infty} h_k\, z^k\right)\right)$$

$$= \mathcal{T}\left(\sum_{n=1}^{\infty} h_{n+1}\, z^n\right) = \sum_{n=1}^{\infty} h_{n+2}\, z^n,$$

… so on

$$\mathcal{T}^{\mathbb{i}}\big(H(z)\big) = \big(C_{\mathcal{T}^{\mathbb{i}}} H\big)(z) = \big(H \circ \mathcal{T}^{\mathbb{i}}\big)(z) = \Big(H \circ \underbrace{\mathcal{T} \dots \circ \mathcal{T}}_{\mathbb{i}-times}\Big)(z)$$

$$= \mathcal{T}\left(\mathcal{T}\left(\dots \mathcal{T}\left(\sum_{n=0}^{\infty} h_n\, z^n\right) \dots\right)\right) =$$

$$= \sum_{n=0}^{\infty} h_{n+\mathbb{i}}\, z^n \quad . \tag{4.2.1}$$

This, complete the proof. ■

By the simple calculations, and using

$$z\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)' = -\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\mathbb{S}(z). \tag{4.2.2}$$

That is the main purpose of this idea to find sufficient conditions for certain normalized analytic functions $\mathcal{T}^{\mathbb{i}}\big(H(z)\big)$ to satisfy:

$$\mathbb{q}_1(z) \prec tz\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big) + (1-t)z\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \prec \mathbb{q}_2(z),$$

$$\mathbb{q}_1(z) \prec z\big(z\acute{D}\big)^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \prec \mathbb{q}_2(z),$$

the univalent functions $\mathbb{q}_1(z)\ and\ \mathbb{q}_2(z)$ in $\mathbb{U}$ with $\mathbb{q}_1(0) = \mathbb{q}_2(0) = 1$.

<u>Theorem (4.2.2)</u>: Let $\mathbb{q}(z)\ be$ convex univalent in $\mathbb{U}\ with\ \mathbb{q}(0) = 1\ \ and\ \ \eta \in \mathbb{C} \setminus \{0\}$.

Assume that

$$Re\left\{1 + \frac{z\mathbb{q}''(z)}{\mathbb{q}'(z)}\right\} > max\left\{0, -Re\left(\frac{1}{\eta}\right)\right\}, \tag{4.2.3}$$

if

$$e_1(z) = (1+\eta)\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) - \eta\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\left(\frac{z\left(\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big)\right)'}{\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big)}\right), \tag{4.2.4}$$

$$e_1(z) \prec \mathbb{q}(z) + \eta z\mathbb{q}'(z), \tag{4.2.5}$$

then

$$\mathcal{T}^{\mathbb{i}}H(z) \prec \mathbb{q}(z), \tag{4.2.6}$$

and $\mathbb{q}(z)$ is the best dominant.

Proof:

If we consider the analytic function

$$\mathbb{p}(z) = \mathcal{T}^{\mathbb{i}}H(z);\ \ \mathbb{i} = 0,1,2,\dots, z \in \mathbb{U}, \tag{4.2.7}$$

differentiating (4.2.7) with respect to z, we have $\mathbb{p}'(z) = \left(\mathcal{T}^{\mathbb{i}} H(z)\right)'$.

Now, using the identity (4.2.2), we obtain

$$\frac{z\mathbb{p}'(z)}{\mathbb{p}(z)} = \frac{z\left(\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big)\right)'}{\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big)},$$

therefore

$$z\mathbb{p}'(z) = \frac{\mathrm{z}\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\left(\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big)\right)'}{\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big)}. \tag{4.2.8}$$

Since $e_1(z) = (1+\eta)\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) - \eta\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\left(\frac{z\left(\mathcal{T}^{\mathbb{i}+1}(H(z))\right)'}{\mathcal{T}^{\mathbb{i}+1}(H(z))}\right)$,

$$e_1(z) \prec \mathbb{q}(z) + \eta z\mathbb{q}'(z).$$

thus, the subordination (4.2.5) is equivalent to

$$\mathbb{p}(z) + \eta z\mathbb{p}'(z) \prec \mathbb{q}(z) + \eta z\mathbb{q}'(z). \tag{4.2.9}$$

Application of Lemma (1.5.10) with $\gamma = \eta$, $\psi = 1$, we obtain (4.2.5). ■

Taking $\mathbb{q}(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, in Theorem (4.2.2), we get the following result.

<u>Corollary (4.2.3)</u>: Let $\eta \in \mathbb{C} \setminus \{0\}$, and suppose that

$$Re\left(\frac{1+z}{1-z}\right) > max\left\{0, -Re\left(\frac{1}{\eta}\right)\right\}.$$

If $H \in \mathbb{H}$ is satisfy the following subordination condition:

$$\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) + \left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\left(\frac{z\left(\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big)\right)'}{\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big)}\right) \prec \frac{1+z}{1-z} + \frac{2z}{(1-z)^2},$$

then

$$z\left(\mathcal{J}^{\mathbb{i}+1}\big(H(z)\big)\right)' \prec \frac{1+z}{1-z} \ ,$$

and $\frac{1+z}{1-z}$ is the best dominant.

<u>Theorem (4.2.4)</u>: Let $\mathbb{q}(z)$ be a convex univalent in unit disk $\mathbb{U}$ with $\mathbb{q}(0)=1$,

$\mathbb{q}(z) \neq 0$ and $\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}$ is starlike univalent function in $\mathbb{U}$,

$\eta \in \mathbb{C}\setminus\{0\}, a, \lambda, \mu, \varrho, \mathbb{x}, \mathbb{y}, \mathbb{z} \in \mathbb{C},\ H \in \mathbb{H}$, and suppose that H and $\mathbb{q}$ satisfy the next two conditions

$$tz\left(\mathcal{J}^{\mathbb{i}+1}\big(H(z)\big)\right) + (1-t)z\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right) \neq 0, z \in \mathbb{U}, 0 \leq t \leq 1, \qquad (4.2.10)$$

and

$$Re\left\{1 + \frac{\mathbb{y}}{\eta}\mathbb{q}(z) + \frac{2\mathbb{z}}{\eta}\big(\mathbb{q}(z)\big)^2 - z\frac{\mathbb{q}'(z)}{\mathbb{q}(z)} + z\frac{\mathbb{q}''(z)}{\mathbb{q}'(z)}\right\} > 0.$$

If

$$e_2(z) = a + \lambda\mathbb{q}(z) + \mu\mathbb{z}\mathbb{q}^2(z) + \varrho\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}, \qquad (4.2.11)$$

such that

$$\begin{aligned} &e_2(z) \\ &= \mathbb{x} + \mathbb{y}\left(tz\left(\mathcal{J}^{\mathbb{i}+1}\big(H(z)\big)\right) + (1-t)z\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right)\right) \\ &+ \mathbb{z}\left(tz\left(\mathcal{J}^{\mathbb{i}+1}\big(H(z)\big)\right) + (1-t)z\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right)\right) \\ &+ \eta\left[\frac{tz\left(\mathcal{J}^{\mathbb{i}+1}\big(H(z)\big)\right)' + (1-t)z\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right)'}{t\mathcal{J}^{\mathbb{i}+1}\big(H(z)\big) + (1-t)\mathcal{J}^{\mathbb{i}}\big(H(z)\big)}\right], \end{aligned} \qquad (4.2.12)$$

and

$$e_2(z) \prec \mathbb{x} + \mathbb{y}\mathbb{q}(z) + \mathbb{z}\big(\mathbb{q}(z)\big)^2 + \eta\ \frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}, \tag{4.2.13}$$

then $tz\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big) + (1-t)z\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \prec \mathbb{q}(z),$ (4.2.14)

and $\mathbb{q}(z)$ is the best dominant.

Proof:

Define analytic function $\mathbb{p}(z)$ by

$$\mathbb{p}(z) = t\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big) + (1-t)\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \tag{4.2.15}$$

Then the function $\mathbb{p}(z)$ is analytic in $\mathbb{U}$ and $\mathbb{p}(0) = 1$, differentiating (4.2.15) with respect to z, we get

$$\frac{z\mathbb{p}'(z)}{\mathbb{p}(z)} = \frac{tz\left(\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big)\right)' + (1-t)z\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)'}{t\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big) + (1-t)\mathcal{T}^{\mathbb{i}}\big(H(z)\big)}. \tag{4.2.16}$$

By setting $\boldsymbol{\theta}(w) = \mathbb{x} + \mathbb{y}w + \mathbb{z}w^2$ *and* $\boldsymbol{\phi}(w) = \frac{\eta}{w}$, it can be easily observed that $\boldsymbol{\theta}(w)$ is analytic in $\mathbb{C}$, $\boldsymbol{\phi}(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ *and* $\boldsymbol{\phi}(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Also we get

$$Q(z) = z\mathbb{q}'(z)\boldsymbol{\phi}(z) = \eta\ \frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}\ , z \in \mathbb{U}\ and\ \ h(z) = \boldsymbol{\theta}(\mathbb{p}(z)) + Q(z)$$

$$= \mathbb{x} + \mathbb{y}\mathbb{q}'(z) + \mathbb{z}\big(\mathbb{q}(z)\big)^2 + \eta\ \frac{z\mathbb{q}'(z)}{\mathbb{q}(z)},$$

It is clear that $Q(z)$ is starlike univalent in $\mathbb{U}$, and that

$$Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(1 + \frac{\mathbb{y}}{\eta}\mathbb{q}(z) + \frac{2\mathbb{z}}{\eta}\big(\mathbb{q}(z)\big)^2 - z\frac{\mathbb{q}'(z)}{\mathbb{q}(z)} + z\frac{\mathbb{q}''(z)}{\mathbb{q}'(z)}\right) > 0; z \in \mathbb{U}.$$

Using (4.2.16), the hypothesis (4.2.13) can be equivalently written as

$$\theta\big(\mathbb{p}(z)\big) + z\mathbb{p}'(z)\phi\big(\mathbb{p}(z)\big) \prec \phi\big(\mathbb{q}(z)\big) + z\mathbb{q}'(z)\phi\big(\mathbb{q}(z)\big),$$

thus, by applying Lemma (1.5.9), and the function $\mathbb{q}(z)$ is the best dominant. ■

Theorem (4.2.5): Let $\mathbb{q}(z)$ be a convex univalent function in $\mathbb{U}$ with $\mathbb{q}(0) = 1$, $Re\{\eta\} > 0$. Let $H \in \mathbb{H}$,satisfies

$$\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \in \mathcal{H}[\mathbb{q}(0), 1] \cap Q.$$

If the function $e_1(z)$ defined by (4.2.4) is univalent in $\mathbb{U}$ and

$$\mathbb{q}(z) + \eta z\mathbb{q}'(z) \prec e_1(z), \tag{4.2.17}$$

then

$$\mathbb{q}(z) \prec \mathcal{T}^{\mathbb{i}}\big(H(z)\big), \tag{4.2.18}$$

and $\mathbb{q}(z)$ is the best subordinant.

Proof:

Define the analytic function $\mathbb{p}(z)$ by

$$\mathbb{p}(z) = \mathcal{T}^{\mathbb{i}}\big(f(z)\big) \tag{4.2.19}$$

Differentiating (4.2.19) with respect to z, we have

$$\frac{z\mathbb{p}'(z)}{\mathbb{p}(z)} = \frac{z\left(\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big)\right)'}{\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big)}. \tag{4.2.20}$$

After some computation and using (4.2.2), from (4.2.20), we get

$$e_1(z) = \mathbb{p}(z) + \eta z\mathbb{p}'(z),$$

now, by using Lemma (1.5.10), we get the desired result.

Taking $\mathbb{q}(z) = \frac{1+Az}{1+Bz}$, $(-1 \leq B < A \leq 1)$, in Theorem (4.2.5), we get the following corollary. ■

Corollary (4.2.6): Let $-1 \leq B < A \leq 1$, $\eta \in \mathbb{C} \setminus \{0\}$ with $Re\{\eta\} > 0$, also let

$$\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \in \mathcal{H}[\mathbb{q}(0), 1] \cap Q.$$

If the function $e_1(z)$ given by (4.2.4) is univalent in $\mathbb{U}$ and $H \in \mathbb{H}$ satisfies the following superordination condition

$$\frac{1 + Az}{1 + Bz} + \eta \frac{(A - B)z}{(1 + Bz)^2} \prec e_1(z),$$

then

$$\frac{1 + Az}{1 + Bz} \prec \mathcal{T}^{\mathbb{i}}\big(H(z)\big),$$

and the function $\frac{1+Az}{1+Bz}$ is the best subordinant. ■

Theorem (4.2.7): Let $\mathbb{q}(z)$ be convex univalent in unit disk $\mathbb{U}$, with $\mathbb{q}(0) = 1$,

$\mathbb{q}(z) \neq 0,\ and\ \frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}$, is starlike in $\mathbb{U}$, let $\eta \in \mathbb{C} \setminus \{0\}\ and\ \mathbb{x}, \mathbb{y} \in \mathbb{C}$. Further assume that $\mathbb{q}$ satisfies

$$Re\left\{(\mathbb{y} + 2\mathbb{z}\mathbb{q}(z)) \frac{\mathbb{q}(z)\mathbb{q}'(z)}{\eta}\right\} > 0, (z \in \mathbb{U}). \tag{4.2.21}$$

Let $H(z) \in, \mathbb{H}$, and suppose that $H(z)$ satisfies the next condition

$$tz\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big) + (1 - t)z\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \neq 0; z \in \mathbb{U}, (0 \leq t \leq 1), \tag{4.2.22}$$

and

$$tz\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big) + (1 - t)z\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \in \mathcal{H}[\mathbb{q}(0), 1] \cap Q. \tag{4.2.23}$$

If the function $e_2(z)$, given by (4.2.11) is univalent in $\mathbb{U}$,*and*

$$\mathbb{x} + \mathbb{y}\mathbb{q}(z) + \mathbb{z}\big(\mathbb{q}(z)\big)^2 + \eta\, \frac{z\mathbb{q}'(z)}{\mathbb{q}(z)} \prec e_2(z), \tag{4.2.24}$$

then

$$\mathbb{q}(z) \prec tz\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big) + (1-t)z\mathcal{T}^{\mathbb{i}}\big(H(z)\big), \tag{4.2.25}$$

and $\mathbb{q}(z)$ is the best subordinant.

Proof :

Let the function $\mathbb{p}(z)$ defined on $\mathbb{U}$ by (4.2.14).

Then a computation show that

$$\frac{z\mathbb{p}'(z)}{\mathbb{p}(z)} = \frac{tz\mathcal{T}^{\mathbb{i}+1}\big(f(z)\big)' + (1-t)z\mathcal{T}^{\mathbb{i}}\big(f(z)\big)'}{t\mathcal{T}^{\mathbb{i}+1}\big(f(z)\big) + (1-t)\mathbb{T}\mathcal{T}^{\mathbb{i}}\big(f(z)\big)}, \tag{4.2.26}$$

by setting

$$\boldsymbol{\Theta}(w) = \mathbb{x} + \mathbb{y}w + \mathbb{z}w^2, and\ \ \boldsymbol{\phi}(w) = \frac{\eta}{w}\ \ ,(w \in \mathbb{C} \setminus \{0\}).$$

We see that $\boldsymbol{\Theta}(w)$ is analytic in $\mathbb{C}$, $\boldsymbol{\phi}(w)$ is analytic in $\mathbb{C} \setminus \{0\}$, and that $\boldsymbol{\phi}(w) \neq 0$ $(w \in \mathbb{C} \setminus \{0\})$. Also , we get

$$Q(z) = z\mathbb{q}'(z)\boldsymbol{\phi}\big(\mathbb{q}(z)\big) = \eta\,\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}\ , z \in \mathbb{U}$$

it observed $that\ Q(z)$ is starlike univalent $in\ \mathbb{U},\ and\ that$

$$Re\left(\frac{z\boldsymbol{\Theta}'\big(\mathbb{q}(z)\big)}{\boldsymbol{\phi}\big(\mathbb{q}(z)\big)}\right) = Re\left\{\mathbb{y} + 2\mathbb{z}\mathbb{q}(z)\frac{\mathbb{q}(z)\mathbb{q}'(z)}{\eta}\right\} > 0\,.$$

By making use (4.2.25) the hypothesis (4.2.23) can by equivalently written as

$$\boldsymbol{\Theta}\big(\mathbb{q}(z)\big) + z\mathbb{q}'(z)\boldsymbol{\phi}\big(\mathbb{q}(z)\big) \prec \boldsymbol{\Theta}\big(\mathbb{p}(z)\big) + z\mathbb{p}'(z)\boldsymbol{\phi}\big(\mathbb{p}(z)\big),$$

thus, by applying Lemma (1.5.11), the proof is complete. ■

Combination Theorem (4.2.1) with Theorem (4.2.7) , we obtain the following sandwich Theorem.

<u>Theorem (4.2.8)</u>: Let $\mathbb{q}_1$ and $\mathbb{q}_2$ be convex univalent functions in $\mathbb{U}$ with $\mathbb{q}_1(0) = \mathbb{q}_2(0) = 1$ and $\mathbb{q}_2$ satisfies (4.2.3). Suppose that $Re\{\eta\} > 0$. If $H \in \mathbb{H}$, such that

$$\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \in \mathcal{H}[\mathbb{q}(0), 1] \cap Q,$$

and the function $e_1(z)$, is univalent in $\mathbb{U}$ and satisfies

$$\mathbb{q}_1(z) + \eta z \mathbb{q}_1{}'(z) \prec e_1(z) \prec \mathbb{q}_2(z) + \eta z \mathbb{q}_2{}'(z), \tag{4.2.27}$$

where $e_1(z)$ is given by (4.2.4), then

$$\mathbb{q}_1(z) \prec \mathcal{T}^{\mathbb{i}}\big(H(z)\big) \prec \mathbb{q}_2(z),$$

where $\mathbb{q}_1$ and $\mathbb{q}_2$, are respectively the best subordinant and best dominant of (4.2.27) . ■

Combining Theorem (4.2.4) with Theorem (4.2.8), we obtain the following sandwich theorem:

<u>Theorem (4.2.9)</u>: Let $\mathbb{q}_j$, be two convex univalent functions in $\mathbb{U}$, such that $\mathbb{q}_j(0) = 1, \mathbb{q}_j(z) \neq 0,$ and $\frac{z\mathbb{q}_j'(z)}{\mathbb{q}_j(z)}$ $(j = 1,2)$ is starlike univalent in $\mathbb{U}$, let $\eta \in \mathbb{C} \setminus \{0\}$ and $\mathbb{x}, \mathbb{y}, \mathbb{z} \in \mathbb{C}$. Suppose that $\mathbb{q}_1$ and $\mathbb{q}_2$ satisfies (4.2.14), and (4.2.25), respectively.

If $H \in \mathbb{H}$, and suppose that H satisfies the next condition

$$tz\,\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big) + (1-t)z\,\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \neq 0, (z \in \mathbb{U}, 0 \le t \le 1),$$

and

$$tz\,\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big) + (1-t)z\,\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \in \mathcal{H}[\mathbb{q}(0), 1] \cap Q.$$

If the function $e_2(z)$ given by (4.2.11) is univalent in $\mathbb{U}$, and

$$\mathbb{x} + \mathbb{y}\mathbb{q}_1(z) + \mathbb{z}\big(\mathbb{q}_1(z)\big)^2 + \eta\,\frac{z\mathbb{q}_1{}'(z)}{\mathbb{q}_1(z)} \prec e_2(z)$$

$$\prec \mathbb{x} + \mathbb{y}\mathbb{q}_2(z) + \mathbb{z}\big(\mathbb{q}_2(z)\big)^2 + \eta\,\frac{z\mathbb{q}_2{}'(z)}{\mathbb{q}_2(z)}, \qquad (4.2.28)$$

then

$$\mathbb{q}_1(z) \prec tz\mathcal{T}^{\mathbb{i}+1}\big(H(z)\big) + (1-t)z\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \prec \mathbb{q}_2(z),$$

and $\mathbb{q}_1 and\ \mathbb{q}_2$ are the best subodinant and best dominant respectively of (4.2.28).

4.3 Sandwich Results of Complete Homogeneous Symmetric Functions Defined by Differential of Ordered Cyclic Operator

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ be the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. For positive integer p and $a \in \mathbb{C}$. Let $\mathcal{H}[a,p]$ be the subclass of $\mathcal{H}$ and

$$\mathcal{H}[a,p] = \{f \in \mathcal{H}: f(z) = a + \sum_{n=p}^{\infty} a_n z^n.$$

Let $\mathbb{H} = \mathcal{H}[a,1]$ denote the class of functions H of the form:

$$H(z) = \sum_{n=0}^{\infty} h_n\, z^n, h_n \in \mathbb{C}\,,$$

and

$$\mathcal{T}^{\mathbb{i}}\big(H(z)\big) = H(z) = \sum_{n=0}^{\infty} h_{n+\mathbb{i}}\, z^n, h_{n+\mathbb{i}} \in \mathbb{C}\,,$$

which are analytic univalent in the open unit disk $\mathbb{U} = \{z: z \in \mathbb{C}, \left| z \right| < 1\}$.

For the function $\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \in \mathbb{H}$, The Differential operator representing the computation of a derivative, $D' \equiv \frac{d}{dz}$ sometimes also called the Newton-Leibniz operator. The second derivative is then denoted ${D'}^2$,the third ${D'}^3$, etc. The integral is denoted ${D'}^{-1}$. A fundamental identity for this operator is given by

$$(zD')^n = \sum_{k=0}^{n} S(n,k) z^k {D'}^k$$

where $S(n,k)$ is a Stirling number of the second kind [46] giving:

$$(zD')^1 = zD'$$

$$(zD')^2 = zD' + z^2 {D'}^2$$

$$(zD')^3 = zD' + 3z^2 {D'}^2 + z^3 {D'}^3$$

$$(zD')^4 = zD' + 7z^2 {D'}^2 + 6z^3 {D'}^3 + z^4 {D'}^4$$

and so on .

then fundamental differential operator of H is defined as:

$$(zD')^n \left(\mathcal{T}^{\mathbb{i}}(H(z))\right) = \sum_{k=0}^{\infty} k^n h_{k+\mathbb{i}} z^k, h_{n+\mathbb{i}} \in \mathbb{C} ,$$

Miller and Mocanu [35,36] consider the problem of determining conditions an admissible functions $\boldsymbol{\Psi}$ such that (1.5.6) is satisfied and implies $\mathbb{p}(z) \prec \mathbb{q}(z)$, for all functions $\mathbb{p}(z) \in \mathcal{H}$. Moreover they found conditions so that $\mathbb{q}$ is the smallest function with this property called the best domainant of the subordination (1.5.6). Also, they studied the dual problem and determined conditions on Ψ such that (1.5.9) is satisfied, implies $\mathbb{q}(z) \prec \mathbb{p}(z)$, for all functions $\mathbb{q} \in Q$, that satisfy the superordination (1.5.9). They also found conditions so that the function $\mathbb{q}$ is the largest function with this property, called the best subordinant of the superordination (1.5.9). See also [6,8,10,22].

Using the results (see [1]) to obtain sufficient conditions for normalized analytic functions to satisfy :

$$\mathbb{q}_1(z) \prec \frac{z\left((zD')^n\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\right)'}{(zD')^n\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)} \prec \mathbb{q}_2(z),$$

where $\mathbb{q}_1$ and $\mathbb{q}_2$ are given univalent functions in $\mathbb{U}$ $with$ $\mathbb{q}_1(0) = \mathbb{q}_2(0) = 1.$

Lemma (4.3.1): Let $H \in \mathbb{H},$ a fundamental identity for operator is given as :

$$(zD')^n = \sum_{k=0}^{n} S(n,k) z^n D'^k,$$

we define the differential operator of degree 1

$$zD' : \mathbb{H} \to \mathbb{H}, \quad by$$

$$zD'\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) = z\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)'_z ; z \in \mathbb{U}$$

then
$$(zD')^2\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) = \left(zD' + z^2 D'^2\right)\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right);$$

$$(zD')^n\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) = \sum_{l=0}^{n} S(n,k) z^n D'^l\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right);$$

where

$$(xD')^0\big(H(z)\big) = H(z).$$

Proof:

Let $\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \in \mathbb{H},$ then

$$(zD')\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) = zH(z)' = \sum_{k=1}^{\infty} k h_k z^k,$$

and $(zD')^2\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) = \left(zD' + z^2 D'^2\right)\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) = z\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)' +$ $z^2\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)'' = h_{1+\mathbb{i}} z + 2h_{2+\mathbb{i}} z^2 + 3h_{3+\mathbb{i}} z^3 + 4h_{4+\mathbb{i}} z^4 + \cdots + kh_{k+\mathbb{i}} z^n +$

$\cdots + 2h_{2+\mathbb{i}}z^2 + 6h_{3+\mathbb{i}}z^3 + 12h_{4+\mathbb{i}}z^4 + \cdots + n(n-1)h_{k+\mathbb{i}}z^k + \cdots = h_{1+\mathbb{i}}z + 4h_{2+\mathbb{i}}z^2 + 9h_{3+\mathbb{i}}z^3 + 16h_{4+\mathbb{i}}z^4 + \cdots + n^2h_{k+\mathbb{i}}z^k + \cdots = \sum_{k=0}^{\infty} k^2 h_{k+\mathbb{i}}\, z^k,$

… so on

$$(zD')^n H(z) = \sum_{l=0}^{n} S(n,l) z^l \acute{D}^l \left(H(z)\right) = h_{1+\mathbb{i}}z + 2h_{2+\mathbb{i}}z^2 + 3h_{3+\mathbb{i}}z^3 + 4h_{4+\mathbb{i}}z^4 + \cdots + kh_{k+\mathbb{i}}z^n + \cdots + 6h_{2+\mathbb{i}}z^2 + 18h_{3+\mathbb{i}}z^3 + 36h_{4+\mathbb{i}}z^4 + \cdots + 3k(k-1)h_{k+\mathbb{i}} + \cdots + 6h_{3+\mathbb{i}}z^3 + 24h_{4+\mathbb{i}}z^4 + \cdots + k(k-1)(k-2)h_{k+\mathbb{i}}z^k + \cdots + \cdots + k(k-1)(k-2)\ldots(k-n+1)h_{k+\mathbb{i}}z^k + \cdots = \sum_{k=0}^{\infty} k^n h_{k+\mathbb{i}}\, z^k. \tag{4.3.1}$$

This, complete the proof. ■

By the simple calculations, and using

$$z\left((zD')^n \mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)' = z\left(\sum_{k=1}^{\infty} k^n h_{k+\mathbb{i}}\, z^k\right)' = \sum_{k=1}^{\infty} k^{n+1} h_{k+\mathbb{i}}\, z^k = (zD')^{n+1}\mathcal{T}^{\mathbb{i}}\left(H(z)\right), \tag{4.3.2}$$

That is the main purpose of this idea to find sufficient conditions for certain normalized analytic functions $\mathcal{T}^{\mathbb{i}}\left(H(z)\right)$ to satisfy:

$$\mathbb{q}_1(z) \prec tz(zD')^{n+1}\mathcal{T}^{\mathbb{i}}\left(H(z)\right) + (1-t)z(zD')^n\mathcal{T}^{\mathbb{i}}\left(H(z)\right) \prec \mathbb{q}_2(z),$$

$$\mathbb{q}_1(z) \prec z(zD')^n\mathcal{T}^{\mathbb{i}}\left(H(z)\right) \prec \mathbb{q}_2(z),$$

the univalent functions $\mathbb{q}_1(z)$ and $\mathbb{q}_2(z)$ in $\mathbb{U}$ with $\mathbb{q}_1(0) = \mathbb{q}_2(0) = 1$.

Theorem 4.3.2: Suppose that $\mathbb{q}(z)$ is a convex univalent in $\mathbb{U}$ with $\mathbb{q}(0) = 1$, and

$$Re\left\{1 + \frac{z\mathbb{q}''(z)}{\mathbb{q}'(z)}\right\} > max\left\{0, -Re\left(\frac{1}{\eta}\right)\right\};\ \eta \in \mathbb{C} \setminus \{0\},. \tag{4.3.3}$$

If

$$e_1(z) = (1+\eta)(zD')^n\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) - \eta z\left(\frac{(zD')^{n+1}\mathcal{T}^{\mathbb{i}}\big(H(z)\big)}{(zD')^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big)} - 1\right), \quad (4.3.4)$$

$$e_1(z) \prec \mathbb{q}(z) + \eta z\mathbb{q}'(z), \quad (4.3.5)$$

then

$$(zD')^n\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) \prec \mathbb{q}(z), \quad (3.3.6)$$

and $\mathbb{q}(z)$ is the best dominant.

Proof :

If we consider the analytic function

$$\mathbb{p}(z) = \frac{z(zD')^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big)}{z};\ z \in \mathbb{U}, \quad (4.3.7)$$

differentiating (4.3.7) with respect to z , we have

$$\mathbb{S}(z)\mathcal{T}^{\mathbb{i}}H(z) + z\mathcal{T}^{\mathbb{i}}\acute{H}(z) = 0$$

$$\mathbb{p}(z)' = \left(\frac{z(zD')^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big)}{z}\right)' = \frac{z(zD')^{n+1}\mathcal{T}^{\mathbb{i}}\big(H(z)\big) - z(zD')^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big)}{z^2}$$

$$z\mathbb{p}(z)' = (zD')^{n+1}\mathcal{T}^{\mathbb{i}}\big(H(z)\big) - (zD')^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big)$$

Now, using the identity (4.3.2), we obtain

$$\frac{z\mathbb{p}'(z)}{\mathbb{p}(z)} = \frac{(zD')^{n+1}\mathcal{T}^{\mathbb{i}}\big(H(z)\big) - (zD')^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big)}{(zD')^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big)},$$

therefore

$$z\mathbb{p}'(z) = \frac{(zD')^{n+1}\mathcal{T}^{\mathbb{i}}\big(H(z)\big)}{(zD')^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big)} - 1, \quad (4.3.8)$$

Since $e_1(z) = (1+\eta)\left(\frac{z(zD')^n\mathcal{T}^{\mathbb{i}}(H(z))}{z}\right) - \eta\left(\frac{(zD')^{n+1}\mathcal{T}^{\mathbb{i}}(H(z))}{(zD')^n\mathcal{T}^{\mathbb{i}}(H(z))} - 1\right)$,

$$e_1(z) \prec \mathbb{q}(z) + \eta\big(z\mathbb{q}'(z)\big)$$

thus, the subordination (4.3.3) is equivalent to

$$\mathbb{p}(z) + \eta z\mathbb{p}'(z) \prec \mathbb{q}(z) + \eta z\mathbb{q}'(z). \qquad (4.3.9)$$

Application of Lemma (1.5.10) with $\gamma = \eta$, $\psi = 1$we obtain (4.3.6). ■

Taking $\mathbb{q}(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, in Theorem (4.3.2), we get the following result.

<u>Corollary 4.3.3</u>: Let $\eta \in \mathbb{C}$ and $(-1 \leq B < A \leq 1)$. Suppose $that$

$$Re\left(\frac{1 - Bz}{1 + Bz}\right) > max\left\{0, -Re\left(\frac{1}{\eta}\right)\right\}.$$

If $H \in \mathbb{H}$ is satisfies the following subordination condition:

$$(zD')^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big) + \frac{(zD')^{n+1}\mathcal{T}^{\mathbb{i}}\big(H(z)\big)}{(zD')^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big)} - 1 \prec \frac{1 + Az}{1 + Bz} + \frac{(A - B)z}{(1 + Bz)^2},$$

then

$$\big(z\acute{D}\big)^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big) \prec \frac{1 + Az}{1 + Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant. ■

Taking $A = 1$ and $B = -1$, in Theorem (4.3.2), we get the following result.

<u>Corollary (4.3.4)</u>: Let $\eta \in \mathbb{C}$, and suppose that

$$Re\left(\frac{1 + z}{1 - z}\right) max > \left\{0, -Re\left(\frac{1}{\eta}\right)\right\}.$$

If $H \in \mathbb{H}$, the following subordination condition:

$$(zD')^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big) + \frac{(zD')^{n+1}\mathcal{T}^{\mathbb{i}}\big(H(z)\big)}{(zD')^n\mathcal{T}^{\mathbb{i}}\big(H(z)\big)} - 1 \prec \frac{1 + z}{1 + z} + \frac{2z}{(1 + z)^2},$$

then

$$(zD')^n \mathcal{T}^{\mathbb{i}}\big(H(z)\big) \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant. ■

Theorem 4.3.5: Let $\mathbb{q}(z)$ be convex univalent in unit disk $\mathbb{U}$ $with$ $q(0)=1$, $q(z) \neq 0$ and $\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}$ is starlike univalent in $\mathbb{U}$, $\eta \in \mathbb{C} \setminus \{0\}$, $a, \lambda, \mu, \varrho, \mathbb{x}, \mathbb{y}, \mathbb{z} \in \mathbb{C}$, $H \in \mathbb{H}$, suppose that $(zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)$ and $(zD')^{n}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)$ satisfy the following two conditions

$$\frac{tz(zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)+(1-t)z(zD')^{n}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)}{z} \neq 0, z \in \mathbb{U}, 0 \leq t \leq 1, \tag{4.3.10}$$

and

$$Re\left\{1+\frac{\mathbb{y}}{\eta}\mathbb{q}(z)+\frac{2\mathbb{z}}{\eta}\big(\mathbb{q}(z)\big)^2 - z\frac{\mathbb{q}'(z)}{\mathbb{q}(z)} + z\frac{\mathbb{q}''(z)}{\mathbb{q}'(z)}\right\} > 0. \tag{4.3.11}$$

Then if

$$e_2(z) = a + \lambda\mathbb{q}(z) + \mu\mathbb{z}\mathbb{q}^2(z) + \varrho\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}, \tag{4.3.12}$$

if

$$\begin{aligned} e_2(z) = \mathbb{x} &+ \mathbb{y}\left(t(zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)+(1-t)(zD')^{n}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\right) \\ &+ \mathbb{z}\left(t(zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)+(1-t)(zD')^{n}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\right)^2 \\ &+ \eta\left(\frac{tz\left((zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\right)'+(1-t)z\left((zD')^{n}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\right)'}{t(zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)+(1-t)(zD')^{n}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)} \right. \\ &\left. - 1\right), \end{aligned} \tag{4.3.13}$$

and

$$e_2(z) \prec \mathbb{x} + \mathbb{y}\mathbb{q}(z) + \mathbb{z}\big(\mathbb{q}(z)\big)^2 + \eta\, \frac{z\mathbb{q}'(z)}{\mathbb{q}(z)} \ , \qquad (4.3.14)$$

Then

$$tz(zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) + (1-t)(zD')^{n}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) \prec \mathbb{q}(z), \quad (4.3.15)$$

and $\mathbb{q}(z)$ is the best dominant.

Proof :

Define analytic function $\mathbb{p}(z)$ by

$$\mathbb{p}(z) = \frac{tz(zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) + (1-t)z(zD')^{n}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)}{z}. \qquad (4.3.16)$$

Then the function $\mathbb{p}(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$, differentiating (4.3.16) with respect to z, we get

$$\frac{z\mathbb{p}'(z)}{\mathbb{p}(z)} = \frac{tz\left((zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\right)' + (1-t)z\left((zD')^{n}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\right)'}{t(zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) + (1-t)(zD')^{n}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)}. \qquad (4.3.17)$$

By setting $\boldsymbol{\theta}(w) = \mathbb{x} + \mathbb{y}w + \mathbb{z}w^2$ *and* $\boldsymbol{\phi}(w) = \frac{\eta}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\boldsymbol{\phi}(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and

$\boldsymbol{\phi}(w) \neq 0\,, w \in \mathbb{C} \setminus \{0\}$.

Also we get

$$\boldsymbol{Q}(z) = z\mathbb{q}'(z)\boldsymbol{\phi}(z) = \eta\, \frac{z\mathbb{q}'(z)}{\mathbb{q}(z)} \ , z \in \mathbb{U} \quad and$$

$$h(z) = \boldsymbol{\theta}(\mathbb{q}(z)) + \boldsymbol{Q}(z) = \mathbb{x} + \mathbb{y}\mathbb{q}'(z) + \mathbb{z}\big(\mathbb{q}(z)\big)^2 + \eta\, \frac{z\mathbb{q}'(z)}{\mathbb{q}(z)},$$

It is clear that $\boldsymbol{Q}(z)$ is starlike univalent in $\mathbb{U}$, and that

$$Re\left(\frac{zh\prime(z)}{Q(z)}\right) = Re\left(1 + \frac{\mathbb{y}}{\eta}\mathbb{q}(z) + \frac{2\mathbb{z}}{\eta}\left(\mathbb{q}(z)\right)^2 - z\frac{\mathbb{q}\prime(z)}{\mathbb{q}(z)} + z\frac{\mathbb{q}\prime\prime(z)}{\mathbb{q}\prime(z)}\right) > 0, z \in \mathbb{U}.$$

By using (4.3.15), the hypothesis (4.3.12) can be equivalently written as

$$\boldsymbol{\theta}\big(\mathbb{p}(z)\big) + z\mathbb{p}'(z)\boldsymbol{\phi}\big(\mathbb{p}(z)\big) \prec \boldsymbol{\phi}\big(\mathbb{q}(z)\big) + z\mathbb{q}'(z)\boldsymbol{\phi}\big(\mathbb{q}(z)\big),$$

thus, by applying Lemma (1.5.9), and the function $\mathbb{q}(z)$ is the best dominant. ■

Theorem (4.3.6): Let $\mathbb{q}(z)$ be a convex univalent function in $\mathbb{U}$ with $\mathbb{q}(0) = 1$, $Re\{\eta\} > 0$. Let $H \in \mathbb{H}$,satisfies

$$\frac{z(zD')^n\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)}{z} \in \mathcal{H}[\mathbb{q}(0), 1] \cap Q.$$

If the function $e_1(z)$ defined by (4.3.2) is univalent in $\mathbb{U}$ and

$$\mathbb{q}(z) + \eta z\mathbb{q}'(z) \prec e_1(z), \tag{4.3.18}$$

then

$$\mathbb{q}(z) \prec \frac{z(zD')^n\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)}{z}, \tag{4.3.19}$$

and $\mathbb{q}(z)$ is the best subordinant.

Proof:

Define the analytic function $\mathbb{p}(z)$ by

$$\mathbb{p}(z) = \frac{z(zD')^n\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)}{\mathrm{z}}. \tag{4.3.20}$$

Differentiating (4.3.20) with respect to z, we have

$$\frac{z\mathbb{p}'(z)}{\mathbb{p}(z)} = \frac{z\left((zD')^n\left(\mathcal{J}^{\mathbb{i}}\left(H(z)\right)\right)\right)'}{(zD')^n\left(\mathcal{J}^{\mathbb{i}}\left(H(z)\right)\right)} - 1. \qquad (4.3.21)$$

After some computation and using ((4.3.2)from (4.3.22)we get

$$e_1(z) = \mathbb{p}(z) + \eta z\mathbb{p}'(z),$$

now, by using Lemma (1.4.10), we get the desired result.

Taking $\mathbb{q}(z) = \frac{1+Az}{1+Bz}$, $(-1 \le B < A \le 1)$, in Theorem (4.3.5), we get the following corollary. ■

<u>**Corollary (4.3.7)**</u>**:** Let $-1 \le B < A \le 1$, $\eta \in \mathbb{C} \setminus \{0\}$ $with\ Re\{\eta\} > 0$, also let

$$\frac{z(zD')^n\left(\mathcal{J}^{\mathbb{i}}\left(H(z)\right)\right)}{z^2} \in \mathcal{H}[\mathbb{q}(0), 1] \cap Q.$$

If the function $e_1(z)$ given by (4.3.4) is univalent in $\mathbb{U}$ and $H \in \mathbb{H}$ satisfies the following superordination condition

$$\frac{1 + Az}{1 + Bz} + \eta\frac{(A - B)z}{(1 + Bz)^2} \prec e_1(z),$$

then

$$\frac{1 + Az}{1 + Bz} \prec \frac{z\left(z\acute{D}\right)^n\left(\mathcal{J}^{\mathbb{i}}\left(H(z)\right)\right)}{z^2},$$

and the function $\frac{1+Az}{1+Bz}$ is the best subordinant. ■

<u>**Theorem (4.3.8)**</u>**:** Let $\mathbb{q}(z)$ be convex univalent in unit disk $\mathbb{U}$, with $\mathbb{q}(0) = 1$,

$\mathbb{q}(z) \neq 0,\ and\ \frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}$, is starlike in $\mathbb{U}$, let $\eta \in \mathbb{C} \setminus \{0\}$ and $\sigma, \alpha \in \mathbb{C}$. Further assume that $\mathbb{q}$ satisfies

$$Re\left\{(\mathbb{y}+2\mathbb{z}\mathbb{q}(z))\frac{\mathbb{q}(z)\mathbb{q}'(z)}{\eta}\right\}>0; z\in\mathbb{U}. \qquad (4.3.22)$$

Let $H(z)\in\mathbb{H}$, and suppose that $H(z)$ satisfies the next condition

$$t(zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)+(1-t)(zD')^{n}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)\neq 0, (z\in\mathbb{U}),$$
$$(0\leq t\leq 1)\,, \qquad (4.3.23)$$

and

$$t(zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)+(1-t)(zD')^{n}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)$$
$$\in\mathcal{H}[\mathbb{q}(0),1]\cap Q. \qquad (4.3.24)$$

If the function $e_2(z)$, given by (4.3.11) is univalent in $\mathbb{U}$,and

$$\mathbb{x}+\mathbb{y}\mathbb{q}(z)+\mathbb{z}\left(\mathbb{q}(z)\right)^2+\eta\,\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}\prec e_2(z), \qquad (4.3.25)$$

then

$$\mathbb{q}(z)\prec t(zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)$$
$$+(1-t)(zD')^{n}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right), \qquad (4.3.26)$$

and $\mathbb{q}(z)$ is the best subordinant.

Proof:

Let the function $\mathbb{p}(z)$ defined on $\mathbb{U}$ by (4.3.16).

Then a computation show that

$$\frac{z\mathbb{p}'(z)}{\mathbb{p}(z)}=\frac{tz\left((zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)\right)'+(1-t)z\left((zD')^{n}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)\right)'}{t(zD')^{n+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)+(1-t)(zD')^{n}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}$$
$$-1, \qquad (4.3.27)$$

by setting

$$\boldsymbol{\Theta}(w) = \mathbb{x} + \mathbb{y}w + \mathbb{z}w^2, and \ \ \boldsymbol{\phi}(w) = \frac{\eta}{w} \ \ ,(w \in \mathbb{C} \setminus \{0\}).$$

We see that $\boldsymbol{\Theta}(w)$ is analytic in $\mathbb{C}$, $\boldsymbol{\phi}(w)$ is analytic in $\mathbb{C} \setminus \{0\}$, and that $\boldsymbol{\phi}(w) \neq 0$ $(w \in \mathbb{C} \setminus \{0\})$. Also, we get

$$Q(z) = z\mathbb{q}'(z)\phi\big(\mathbb{q}(z)\big) = \eta\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)} \ ,z \in \mathbb{U},$$

it observed $that\ Q(z)$ is starlike univalent $in\ \mathbb{U}$, $and\ that$

$$Re\left(\frac{z\boldsymbol{\Theta}'\big(\mathbb{q}(z)\big)}{\boldsymbol{\phi}\big(\mathbb{q}(z)\big)}\right) = Re\left\{\mathbb{y} + 2\mathbb{y}\mathbb{q}(z)\frac{\mathbb{q}(z)\mathbb{q}'(z)}{\eta}\right\} > 0\ .$$

By making use (4.3.25) the hypothesis (4.3.23) can by equivalently written as

$$\boldsymbol{\Theta}\big(\mathbb{q}(z)\big) + z\mathbb{q}'(z)\boldsymbol{\phi}\big(\mathbb{q}(z)\big) \prec \boldsymbol{\Theta}\big(\mathbb{p}(z)\big) + z\mathbb{p}'(z)\boldsymbol{\phi}\big(\mathbb{p}(z)\big),$$

thus, by applying Lemma (1.4.11), the proof is complete. ■

Combination Theorem (4.3.2) with Theorem (4.3.8) , we obtain the following sandwich Theorem.

<u>Theorem (4.3.9)</u>: Let $\mathbb{q}_1\ and\ \ \mathbb{q}_2$ be convex univalent functions $in\ \mathbb{U}$ with $\mathbb{q}_1(0) = \mathbb{q}_2(0) = 1\ \ \ and\ \mathbb{q}_2$ satisfies (4.3.1). Suppose that $Re\{y\} > 0$, $H \in\ \mathbb{H}$ such that

$$(zD')^n\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) \in \mathcal{H}[\mathbb{q}(0), 1] \cap Q,$$

and the function $e_1(z)$, is univalent in $\mathbb{U}$ and satisfies

$$\mathbb{q}_1(z) + \eta z\mathbb{q}_1{}'(z) \prec\ e_1(z) \prec \mathbb{q}_2(z) + \eta z\mathbb{q}_2{}'(z), \tag{4.3.28}$$

where $e_1(z)$ is given by (4.3.2), then

$$\mathbb{q}_1(z) \prec (zD')^n\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right) \prec \mathbb{q}_2(z),$$

where $\mathbb{q}_1$ and $\mathbb{q}_2$, are respectively the best subordinant and best dominant of (4.3.26). ■

Combining Theorem (4.3.5) with Theorem (4.3.9), we obtain the following sandwich theorem:

Theorem (4.3.10): Let $\mathbb{q}_j;\ j = 1,2$ be two convex univalent functions in $\mathbb{U}$, such that $\mathbb{q}_j(0) = 1, \mathbb{q}_j(z) \neq 0,\ and\ \frac{z\mathbb{q}_j'(z)}{\mathbb{q}_j(z)}$ is starlike univalent in $\mathbb{U}$, let $\eta \in \mathbb{C} \setminus \{0\}$ and $\mathbb{x}, \mathbb{y}, \mathbb{z} \in \mathbb{C}$. Suppose that $\mathbb{q}_1$ and $\mathbb{q}_2$ satisfies (4.3.10) and (4.3.20), respectively.

If $H \in \mathbb{H}$, and suppose that H satisfies the next condition

$$t(zD')^{n+1}\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right) + (1-t)(zD')^n\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right) \neq 0, (z \in \mathbb{U},\ 0 \le t \le 1),$$

and

$$\frac{tz(zD')^{n+1}\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right) + (1-t)z(zD')^n\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right)}{z^2} \in \mathcal{H}[\mathbb{q}(0),1] \cap Q.$$

If the function $e_2(z)$ given by (4.3.12) is univalent in $\mathbb{U}$, and

$$\mathbb{x} + \mathbb{y}\mathbb{q}_1(z) + \mathbb{z}\big(\mathbb{q}_1(z)\big)^2 + \eta\,\frac{z\mathbb{q}_1{}'(z)}{\mathbb{q}_1(z)} \prec e_2(z) \prec \mathbb{x} + \mathbb{y}\mathbb{q}_2(z) + \mathbb{z}\big(\mathbb{q}_2(z)\big)^2 + \eta\,\frac{z\mathbb{q}_2{}'(z)}{\mathbb{q}_2(z)}, \qquad (4.3.29)$$

then $\mathbb{q}_1(z) \prec t(zD')^{n+1}\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right) + (1-t)(zD')^n\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right) \prec \mathbb{q}_2(z)$,

and $\mathbb{q}_1 and$ $\mathbb{q}_2$ are the best subodinant and best dominant respectively of (4.3.29). ■

Chapter Five: (Third and Fourth) - Order Differential Subordination and superordination for Completely Homogeneous Symmetric Functions Class

5.1 Introduction

This chapter is totally dedicated for study of several results on (third and fourth)-order differential subordination and superordination for class of complete elementary functions. In fact the concept of differential subordination considered important in functions of real variable and this concept enables us to study the range of original function. In the theory of complex-valued function, there are various differential applications in which a characterization of a function is determined from a differential condition. Many of papers on differential subordinations contributed by Miller and Mocaun [36]. The differential subordination was presented by text books by Duren [20], Goodman [29] and Pommerenke [42].

This chapter is divided into two sections. The first section is concerned with third-order differential subordination results for analytic functions associated with a certain diffrrential of an ordered cyclic operator, we study suitable classes of admissible functions and establish the properties of third-order differential subordination by making use a certain differential operator of analytic functions in $\mathbb{U}$ and have the normalized Taylor-Maclaurin series of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \, ; z \in \mathbb{U}.$$

Some new conclusions on differential subordination with some corollaries are obtained, such as, let $\boldsymbol{\phi} \in J_j[\Omega, \mathbb{q}]$. If the functions $H \in \mathbb{H}$ and $\mathbb{q} \in Q_0$, satisfy the following conditions:

$$Re\left(\frac{\xi \mathbb{q}''(\xi)}{\mathbb{q}'(\xi)}\right) \geqq 0 \, , \left|\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{\mathbb{q}'(\xi)}\right| \leqq k, \text{ and}$$

$\left\{\boldsymbol{\phi}\left(\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right),\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right),\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right),\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right);z\right),z \in \mathbb{U}\right\} \subset \Omega,$

then

$$\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{J}^{\mathbb{i}}\big(H(z)\big)\right) \prec \mathbb{q}(z), z \in \mathbb{U}.$$

The second section deals with the fourth-order differential subordination and superordination results of meromorphic multivalent functions defined by the inverse of an ordered cyclic operator. We obtain some applications of fourth-order differential subordination and superordination results involving trace' powers of companion matrix for the inverse of ordered cyclic operator $\mathbb{I}^{\mathbb{i}}\big(H(z)\big)$ of complete homogenous symmetric polynomial.

Also we obtain several sandwich-type results, such as, let $\boldsymbol{\phi} \in \mathcal{A}\mathscr{j}[\Omega, \mathbb{q}]$.

If $H \in \mathbb{H}$ and $\mathbb{q} \in Q_0$ and satisfy the following conditions:

$$Re\left(\frac{\zeta^2\mathbb{q}'''(\zeta)}{\mathbb{q}'(\zeta)}\right) \geqq 0, \left|\frac{\mathbb{T}^{\mathbb{i}+2}\big(H(z)\big)}{\mathbb{q}'(\zeta)}\right| \leqq k^2,$$

and

$$\left\{\boldsymbol{\phi}\left(\mathbb{T}^{\mathbb{i}}\big(H(z)\big),\mathbb{T}^{\mathbb{i}+1}\big(H(z)\big),\mathbb{T}^{\mathbb{i}+2}\big(H(z)\big),\mathbb{T}^{\mathbb{i}+3}\big(H(z)\big),\mathbb{T}^{\mathbb{i}+4}\big(H(z)\big)\right)\right\} \subset \Omega,$$

then

$$\mathbb{T}^{\mathbb{i}}\big(H(z)\big) \prec \mathbb{q}(z)\ (z \in \mathbb{U}).$$

5.2 Third-Order Differential Subordination Results for Complete Homogeneous Symmetric Functions Associated with a Certain Differential of Cyclic Operator

let $\mathbb{H} = \mathcal{H}(a, 1)$ be the subclass of $\mathcal{H}$ in which the functions satisfy the following form:

$$H(z) = \sum_{n=0}^{\infty} h_n z^n ; z \in \mathbb{U}. \quad (5.2.1)$$

Let S be a subclass of $\mathcal{H}$ which are analytic in $\mathbb{U}$ and have the normalized Taylor-Maclaurin series of the form:

$$f(z) = \frac{H(z) - 1}{h_1}; z \in \mathbb{U}. \qquad (5.2..2)$$

For a function $f(z) \in \mathbb{H}$ given by (5.2.1), the cyclic composition of $H(z)$ denoted by $\mathcal{T}^{\mathbb{i}}$ is defined by

$$\mathcal{T}^{\mathbb{i}}\big(H(z)\big) = \mathcal{T}^{\mathbb{i}}\left(\sum_{n=0}^{\infty} h_n z^n\right) = \sum_{n=0}^{\infty} h_{n+\mathbb{i}} z^n \qquad (5.2.3)$$

The geometric function theory relies heavily on the study of operators. Now, we introduce new operator by using the cyclic composition in this study.

<u>Definition 5.2.1</u>: Let $H \in \mathbb{H}$, $\alpha, \beta \in N_0$ and $\alpha + \beta \neq 0$,we define the operator

$$\mathbb{D}^1_{\alpha,\beta}: \mathbb{H} \to \mathbb{H},$$

by

$$\mathbb{D}^1_{\alpha,\beta}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) = \frac{\alpha}{\alpha+\beta}\mathbb{D}^0_{\alpha,\beta}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) + \frac{\beta}{\alpha+\beta} z\left(\mathbb{D}^0_{\alpha,\beta}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\right)'$$

$$= \sum_{n=0}^{\infty} \frac{\alpha + n\beta}{\alpha+\beta} h_n z^n.$$

Where $\mathbb{D}^1_{\alpha,0}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) = \mathcal{T}^{\mathbb{i}}\big(H(z)\big)$, and $\mathbb{D}^1_{\alpha,0}\left(\mathcal{T}^{\mathbb{i}=0}\big(H(z)\big)\right) = H(z)$.

Suppose that $H \in \mathbb{H}, \alpha, \beta \in N_0$ and $\alpha + \beta \neq 0$

$$\mathbb{D}^1_{\alpha,\beta}\big(H(z)\big) = \frac{\alpha}{\alpha+\beta}\mathbb{D}^0_{\alpha,\beta}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) + \frac{\beta}{\alpha+\beta} z\left(\mathbb{D}^0_{\alpha,\beta}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\right)'$$

$$= \frac{\alpha}{\alpha+\beta}\sum_{n=0}^{\infty} h_{n+\mathbb{i}} z^n + \frac{\beta}{\alpha+\beta} z\left(\sum_{n=0}^{\infty} n h_{n+\mathbb{i}} z^{n-1}\right)$$

$$= \frac{\alpha}{\alpha+\beta}\sum_{n=0}^{\infty} h_{n+\mathbb{i}} z^n + \frac{\beta}{\alpha+\beta}\sum_{n=0}^{\infty} n h_{n+\mathbb{i}} z^n = \sum_{n=0}^{\infty} \frac{\alpha + n\beta}{\alpha+\beta} h_{n+\mathbb{i}} z^n.$$

In general

$$\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}(H(z)) = \frac{\alpha}{\alpha+\beta}\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) + \frac{\beta}{\alpha+\beta}z\left(\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\right)'$$

$$= \sum_{n=0}^{\infty}\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{\mathbb{j}} h_{n+\mathbb{i}}\, z^{n}, \qquad (5.2..4)$$

where $\mathbb{D}_{\alpha,0}^{\mathbb{i}+1}(H(z)) = H(z)$.

By simple calculation, we obtain

$$\beta z\left(\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\right)' = (\alpha+\beta)\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}(H(z)) - \alpha\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right), \quad (5.2.5)$$

The notion of the third-order differential subordination can be found in the work of Ponnusamy and Juneja [43]. The recent work by several authors (see [7,26,49,50]), on the differential subordination attracted many researchers in this field. For example, see [1,6,8,10,16,18,27,33,38,44,51].

In this research, we investigate suitable classes of admissible functions associated with the new differential operator $\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)$ and establish the properties of third-order differential subordination by making use a certain new differential cyclic operator of analytic functions in $\mathbb{U}$ and have the Taylor-Maclaurin series of the form:

$$H(z) = \sum_{n=0}^{\infty} h_{n+\mathbb{i}}\, z^{n}; z \in \mathbb{U}.$$

Some new results on differential subordinations with some corollaries are obtained. Here, we obtain the symmetry of the differential superordination results.

We establish a set of acceptable functions so that (1.5.10) holds. We construct the following new class of admissible functions for this purpose, which will be needed to establish the key third-order differential subordination theorems for the operator $\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)$ described by (5.2.4).

Definition 5.2.2: Let Ω be a set in $\mathbb{C}$ and $\mathbb{q} \in Q_0 \cap H_0$. The class $J_j[\Omega, \mathbb{q}]$ of admissible functions consists of those functions $\boldsymbol{\phi}: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$, that satisfy the following admissibility conditions:

$\boldsymbol{\vartheta}(\mathbb{x}, \mathbb{y}, \mathbb{z}, \mathbb{w}; z) \notin \Omega$,

$$\mathbb{x} = \mathbb{q}(\xi), \mathbb{y} = \frac{k\xi\mathbb{q}'(\xi) + \alpha\mathbb{q}(\xi)}{\alpha + \beta},$$

$$Re\left(\frac{2\alpha(\alpha\mathbb{x}+\beta\mathbb{y})+\alpha\beta(\mathbb{x}+\mathbb{y})+\beta^2(\mathbb{y}+\mathbb{z})}{(\alpha+\beta)(\alpha\mathbb{x}+\beta\mathbb{y})}\right) \geqq kRe\left(\frac{\xi\mathbb{q}''(\xi)}{\mathbb{q}'(\xi)} + 1\right),$$

and

$$Re\left(\frac{\alpha^3\mathbb{x} + 3\alpha^2\beta\mathbb{y} + 3\alpha\beta^2\mathbb{z} + \beta^3\mathbb{w}}{(\alpha + \beta)^2(\alpha\mathbb{x} + \beta\mathbb{y})}\right) \geqq k^2 Re\left(\frac{\xi^2\mathbb{q}'''(\xi)}{\mathbb{q}'(\xi)}\right),$$

where

$z \in \mathbb{U}, \xi \in \partial\mathbb{U} \setminus E(\mathbb{q})$ and $k \geqq 2$.

Theorem 5.2.3: Let $\boldsymbol{\phi} \in J_j[\Omega, \mathbb{q}]$ If the functions $H \in \mathbb{H}$ and $\mathbb{q} \in Q_0$, satisfy the following conditions:

$$Re\left(\frac{\xi\mathbb{q}''(\xi)}{\mathbb{q}'(\xi)}\right) \geqq 0\,, \left|\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)}{\mathbb{q}'(\xi)}\right| \leqq k\,, \tag{5.2.6}$$

and

$$\left\{\boldsymbol{\phi}\left(\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right); z\right) : z \in \mathbb{U}\right\} \subset \Omega, \tag{5.2.7}$$

then

$$\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) \prec \mathbb{q}(z), (z \in \mathbb{U}).$$

Proof:

Define the analytic function $\mathbb{p}(z)$ in $\mathbb{U}$ by

$$\mathbb{p}(z) = \mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right). \tag{5.2.8}$$

Form equations (5.2.5) and (5.2.8), we have

$$\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) = \frac{\beta z\mathbb{p}'(z) + \alpha\mathbb{p}(z)}{\alpha+\beta}. \tag{5.2.9}$$

By a similar argument, we get

$$\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) = \frac{\beta^2 z^2\mathbb{p}''(z) + 2\alpha\beta z\mathbb{p}'(z) + \alpha^2\mathbb{p}(z)}{(\alpha+\beta)^2}, \tag{5.2.10}$$

and

$$\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) = \frac{\beta^3 z^3\mathbb{p}'''(z) + 3\alpha\beta^2 z^2\mathbb{p}''(z) + 3\alpha^2\beta z\mathbb{p}'(z) + \alpha^3\mathbb{p}(z)}{(\alpha+\beta)^3}. \tag{5.2.11}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\mathbb{x}(r,s,t,u) = r, \mathbb{y}(r,s,t,u) = \frac{\beta s + \alpha r}{\alpha+\beta},$$

$$\mathbb{z}(r,s,t,u) = \frac{\beta^2 t + 2\alpha\beta s + \alpha^2 r}{(\alpha+\beta)^2}, \tag{5.2.12}$$

$$\mathbb{w}(r,s,t,u) = \frac{\beta^3 u + 3\alpha\beta^2 t + 3\alpha^2\beta s + \alpha^3 r}{(\alpha+\beta)^3}. \tag{5.2.13}$$

Let

$$\boldsymbol{\Pi}(r,s,t,u) = \phi(\mathbb{x},\mathbb{y},\mathbb{z},\mathbb{w};z)$$

$$= \boldsymbol{\phi}\left(r, \frac{\beta s + \alpha r}{\alpha+\beta}, \frac{\beta^2 t + 2\alpha\beta s + \alpha^2 r}{(\alpha+\beta)^2}, \frac{\beta^3 u + 3\alpha\beta^2 t + 3\alpha^2\beta s + \alpha^3 r}{(\alpha+\beta)^3}\right). \tag{5.2.14}$$

Using the equations (5.2.8) to (5.2.11), and from the equation (5.2.14), we have

$$\boldsymbol{\Pi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z), z^3\mathbb{p}'''(z); z)$$
$$= \boldsymbol{\phi}\left(\mathbb{D}^{\mathbb{j}}_{\alpha,\beta}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right), \mathbb{D}^{\mathbb{j}+1}_{\alpha,\beta}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right), \mathbb{D}^{\mathbb{j}+2}_{\alpha,\beta}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right), \mathbb{D}^{\mathbb{j}+3}_{\alpha,\beta}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right); z\right) : z \in \mathbb{U}. \quad (5.2.15)$$

Hence, clearly (5.2.14) becomes

$$\boldsymbol{\Pi}\left(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z), z^3\mathbb{p}'''(z); z\right) \in \Omega,$$

we note that

$$1 + \frac{t}{s} = \frac{2\alpha(\alpha\mathbb{x} + \beta\mathbb{y}) + \alpha\beta(\mathbb{x} + \mathbb{y}) + \beta^2(\mathbb{y} + \mathbb{z})}{(\alpha + \beta)(\alpha\mathbb{x} + \beta\mathbb{y})}$$

and

$$\frac{u}{s} = \frac{\alpha^3\mathbb{x} + 3\alpha^2\beta\mathbb{y} + 3\alpha\beta^2\mathbb{z} + \beta^3\mathbb{w}}{(\alpha + \beta)^2(\alpha\mathbb{x} + \beta\mathbb{y})}$$

Thus clearly, the admissibility condition for $\boldsymbol{\phi} \in J_j[\Omega, \mathbb{q}]$ in Definition 5.1.1, is equivalent to admissibility condition $\boldsymbol{\Pi} \in \Psi_2[\Omega, \mathbb{q}]$ as given in Definition 1.5.19 with $n = 2$.

Therefore, by using (5.2.6) and Lemma 1.5.18, we have $\mathbb{D}^{\mathbb{j}}_{\alpha,\beta}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right) \prec \mathbb{q}(z)$.

■

Our next result is consequence of Theorem 5.2.3, when the behavior of $\mathbb{q}(z)$ on $\partial\mathbb{U}$ is not known.

Corollary 5.2.4: Let $\Omega \subset \mathbb{C}$ and let the function $\mathbb{q}$ be univalent in $\mathbb{U}$ with $\mathbb{q}(0) = 1$. Let $\boldsymbol{\phi} \in J_j[\Omega, \mathbb{q}_p]$ for some $p \in (0,1)$, where $\mathbb{q}_p(z) = \mathbb{q}(pz)$. If the function $H \in \mathbb{H}$ and $\mathbb{q}_p$ satisfies the following conditions:

$$Re\left(\frac{\xi \mathbb{q}_p''(\xi)}{\mathbb{q}_p'(\xi)}\right) \geqq 0\,, \left(\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)}{\mathbb{q}_p'(\xi)}\right) \leqq k, (z \in \mathbb{U}; k \geqq 2; \xi \in \partial\mathbb{U} \setminus E\big(\mathbb{q}_p\big),$$

and

$\left(\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right); z\right) \in \Omega,$

then

$$\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) \prec \mathbb{q}(z), (z \in \mathbb{U}).$$

Proof :

By applying theorem 5.2.3, we get

$$\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) \prec \mathbb{q}_p(z), z \in \mathbb{U}.$$

The result asserted by corollary 5.2.4 is now deduced from following subordination property $\mathbb{q}_p(z) \prec \mathbb{q}(z), z \in \mathbb{U}$.

If $\Omega \neq \mathbb{C}$, is a simply connected domain, the $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ on to Ω. In this case the class $J_j[h(\mathbb{U})\,, \mathbb{q}]$ is written as $J_j[h\,, \mathbb{q}]$. This leads to the following immediate consequence of Theorem 5.2.3. ■

<u>Theorem 5.2.5</u>: Let $\boldsymbol{\phi} \in J_j[h, \mathbb{q}]$. If the function $H \in \mathbb{H}$ and $\mathbb{q} \in Q_0$, satisfy the following conditions:

$$Re\left(\frac{\xi \mathbb{q}_p''(\xi)}{\mathbb{q}_p'(\xi)}\right) \geqq 0\,, \left|\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)}{\mathbb{q}_{\mathbb{p}}'(\xi)}\right| \leqq k \qquad (5.2\,.16)$$

and

$\boldsymbol{\phi}\left(\mathbb{D}_{\alpha,\beta}^{\mathrm{j}}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathrm{j}+1}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathrm{j}+2}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathrm{j}+3}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right); z\right) \prec h(z),$ (5.2.17)

then

$$\mathbb{D}_{\alpha,\beta}^{\mathrm{j}+1}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right) \prec \mathrm{q}(z), z \in \mathbb{U}.$$

The next result is an immediate consequence of corollary 5.2.4.

<u>Corollary 5.2.6</u>: Let $\Omega \subset \mathbb{C}$ and let the function q be univalent in $\mathbb{U}$ with $\mathrm{q}(0) = 1$. Also let $\phi \in J_j[\Omega, \mathrm{q}_p]$ for some $p \in (0,1)$, where $\mathrm{q}_p(z) = \mathrm{q}(pz)$. If the function $H \in \mathbb{H}$ and $\mathrm{q}_{\mathbb{p}}$ satisfies the following conditions:

$$Re\left(\frac{\xi \mathrm{q}_p''(\xi)}{\mathrm{q}_p'(\xi)}\right) \geqq 0\,, \left|\frac{\mathbb{D}_{\alpha,\beta}^{\mathrm{j}+1}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)}{\mathrm{q}_p'(\xi)}\right| \leqq k\,, z \in \mathbb{U}; k \geqq 2; \xi \in \partial\mathbb{U} \setminus E(\mathrm{q}_p),$$

and

$\boldsymbol{\phi}\left(\mathbb{D}_{\alpha,\beta}^{\mathrm{j}}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathrm{j}+1}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathrm{j}+2}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathrm{j}+3}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right); z\right) \prec h(z),$

then

$$\mathbb{D}_{\alpha,\beta}^{\mathrm{j}}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right) \prec \mathcal{T}^{\mathrm{i}}(H(z)), z \in \mathbb{U}.$$

The following result yield the best dominant of differential subordination (5.2.17). ■

<u>Theorem 5.2.7</u>: Let the function h be univalent in $\mathbb{U}$. Also let $\boldsymbol{\phi}: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$

and $\boldsymbol{\Pi}$ given by (5.2.14). Suppose that following differential equation

$$\boldsymbol{\Pi}(\mathrm{q}(z), z\mathrm{q}'(z), z^2\mathrm{q}''(z), z^3\mathrm{q}'''(z); z) = h(z). \quad (5.2.18)$$

Then

$$\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right) \prec \mathcal{T}^{\mathbb{i}}\big(H(z)\big), z \in \mathbb{U},$$

and $\mathbb{q}(z)$ is the best dominant.

Proof:

From theorem 5.2.3, we see that $\mathbb{q}$ is a dominant of (5.2.17) since $\mathbb{q}$ satisfies (5.2.16), it is also a solution of (5.2.17). Therefore, $\mathbb{q}$ will be dominated by all dominants. Hence $\mathbb{q}$ is the best dominant. This completes the proof of theorem 5.2.7. ■

In view of definition 5.2.2, and in special case when $\mathbb{q}(z) = Mz\ (M > 0)$, the class $J_j[\Omega, \mathbb{q}]$, of admissible functions, denoted by $J_j[\Omega, M]$ is expressed follows.

Definition 5.2.8: Let Ω be set in $\mathbb{C}$ and $M > 0$. The class $J_j[\Omega, M]$ of admissible functions consists of those functions $\boldsymbol{\phi}: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$, such that

$$\boldsymbol{\phi}\begin{pmatrix} Me^{i\theta}, \\ \left(\dfrac{\beta K + \alpha M}{(\alpha+\beta)}\right)e^{i\theta}, \\ \left(\dfrac{\beta^2 L + 2\alpha\beta z K + \alpha^2 M}{(\alpha+\beta)^2}\right)e^{i\theta}, \\ \left(\dfrac{\beta^3 N + 3\alpha\beta^2 L + 3\alpha^2\beta z K + \alpha^3 M}{(\alpha+\beta)^3}\right)e^{i\theta} \end{pmatrix}, \notin \Omega, \qquad (5.2.19)$$

where $z \in \mathbb{U}, Re(Le^{-i\theta}) \geqq (K-1)KM$ and $Re(Ne^{-i\theta}) \geqq 0, \forall\ \theta \in \mathbb{R}; k \geqq 0.$

Corollary 5.2.9: Let $\boldsymbol{\phi} \in J_j[\Omega, M]$. If the function $H \in \mathbb{H}$ satisfies the following conditions $\left|\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)\right| \leqq kM, z \in \mathbb{U}; k \geqq 2;\ M > 0,$

and

$\left(\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right); z \in \Omega\right.$,

then $\left|\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)\right| < M$.

In special case, when $\Omega = q(\mathbb{U}) = \{\omega: |\omega| < M\}$, the class $J_j[\Omega, M]$. It is simple denoted by $J_j[M]$. ■

Corollary 5.2.9 can now be rewritten in the following .

<u>Corollary 3.2.10</u>: Let $\boldsymbol{\phi} \in J_j[M]$. If the function $H \in \mathbb{H}$ satisfies the following conditions $\left|\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)\right| \leqq kM$, $z \in \mathbb{U}; k \geqq 2;\ M > 0$,

and

$\left|\boldsymbol{\phi}\left(\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right); z\right)\right| < M$,

then

$\left|\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)\right| < M$. ■

<u>Corollary 5.2.11</u>: Let $k \geqq 2, 0 \neq \mathbb{q} \in \mathbb{C}$ *and* $M > 0$. If the function $H \in \mathbb{H}$ satisfies the following conditions:

$$\left|\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)\right| \leqq kM,$$

and $\left|\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}\right| \leqq \frac{Mz}{|\alpha+\beta|}$, then $\left|\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)\right| \leqq M$.

Proof:

Let $\boldsymbol{\phi}(\mathbb{x}, \mathbb{y}, \mathbb{z}, \mathbb{w}; z) = \frac{\mathbb{y}}{\mathbb{x}}$ and $\Omega = h(\mathbb{U})$, where

$$h(z) = \frac{Mz}{|\alpha+\beta|}, M > 0,$$

use Corollary 5.2.9, we need to show that $\boldsymbol{\phi} \in J_j[\Omega, M]$, that is the admissibility condition (5.2.19), is satisfied.

This follows readily, since it is seen that $|\boldsymbol{\phi}(\mathbb{x}, \mathbb{y}, \mathbb{z}, \mathbb{w}; z)| = \left|\frac{(k-1)Me^{i\theta}}{\alpha+\beta}\right| \geqq \frac{M}{|\alpha+\beta|}$,

where $z \in \mathbb{U}, \theta \in \mathbb{R}$ *and* $k \geqq 2$. The required result now follows from Corollary 5.2.9. This completes the proof . ∎

<u>Definition 5.2.12</u>: Let Ω be a set in $\mathbb{C}$ *and* $\mathbb{q} \in Q_1 \cap S_1$, the class $J_{j,1}[\Omega, \mathbb{q}]$ of admissible functions consists of those functions $\boldsymbol{\phi}: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$,which satisfy the following admissibility conditions $\boldsymbol{\phi}(\mathbb{x}, \mathbb{y}, \mathbb{z}, \mathbb{w}; z) \notin \Omega$ whenever

$$\mathbb{x} = \mathbb{q}(\xi), \mathbb{y} = \frac{\beta K \xi \mathbb{q}'(\xi) + \alpha \mathbb{q}(\xi)}{(\alpha+\beta)},$$

$$Re\left(\frac{2\alpha(\alpha\mathbb{x}+\beta\mathbb{y}) + \alpha\beta(\mathbb{x}+\mathbb{y}) + \beta^2(\mathbb{y}+\mathbb{z})}{(\alpha+\beta)(\alpha\mathbb{x}+\beta\mathbb{y})}\right) \geqq kRe\left(\frac{\xi\mathbb{q}''(\xi)}{\mathbb{q}'(\xi)} + 1\right)$$

and

$$Re\left(\frac{\alpha^3\mathbb{x} + 3\alpha^2\beta\mathbb{y} + 3\alpha\beta^2\mathbb{z} + \beta^3\mathbb{w}}{(\alpha+\beta)^2(\alpha\mathbb{x}+\beta\mathbb{y})}\right) \geqq k^2 Re\left(\frac{\xi^2\mathbb{q}'''(\xi)}{\mathbb{q}'(\xi)}\right),$$

$$\mathbb{x}(r,s,t,u) = r, \mathbb{y}(r,s,t,u) = \frac{\beta s + \alpha r}{\alpha+\beta},$$

$$\mathbb{z}(r,s,t,u) = \frac{\beta^2 t + 2\alpha\beta s + \alpha^2 r}{(\alpha+\beta)^2}, \quad (5.2.20)$$

$$\mathbb{w}(r,s,t,u) = \frac{\beta^3 u + 3\alpha\beta^2 t + 3\alpha^2\beta s + \alpha^3 r}{(\alpha+\beta)^3},$$

$$Re\left(\frac{\alpha^3\mathbb{x} + 3\alpha^2\beta\mathbb{y} + 3\alpha\beta^2\mathbb{z} + \beta^3\mathbb{w}}{(\alpha+\beta)^2(\alpha\mathbb{x}+\beta\mathbb{y})}\right) \geqq k^2 Re\left(\frac{\xi^2\mathbb{q}'''(\xi)}{\mathbb{q}'(\xi)}\right),$$

where $z \in \mathbb{U}, \xi \in \partial\mathbb{U} \setminus E(\mathbb{q})\ and\ k \geqq n.$

<u>Theorem 5.2.13</u>: Let $\boldsymbol{\phi} \in J_{j,1}[\Omega, \mathbb{q}]$. If the function $H \in \mathbb{H}\ and\ \mathbb{q} \in Q_1$, satisfy the following conditions:

$$Re\left(\frac{\xi \mathbb{q}''(\xi)}{\mathbb{q}'(\xi)}\right) \geqq 0,$$

$$\left|\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{z\mathbb{q}'(\xi)}\right| \leqq k \qquad (5.2.21)$$

and

$$\left\{\boldsymbol{\phi}\left(\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{z}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{z}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{z}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{z}; z\right), z \in \mathbb{U}\right\} \subset \Omega, \qquad (5.2.22)$$

then

$$\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{z} \prec \mathbb{q}(z); z \in \mathbb{U}.$$

Proof:

Define $\mathbb{p}(z)\ in\ \mathbb{U}$, by

$$\mathbb{p}(z) = \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{z}, \qquad (5.2.23)$$

from equation (5.2.5) and (5.2.8,22), we have

$$\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{z} = \frac{(\alpha+\beta)\mathbb{p}(z) + \beta z\mathbb{p}'(z)}{\alpha+\beta}. \qquad (5.2.24)$$

By a similar argument, we get

$$\frac{\mathbb{D}_{\alpha,\beta}^{\mathfrak{j}+2}\left(\mathcal{T}^{\mathfrak{i}}\big(H(z)\big)\right)}{z} = \frac{(\alpha+\beta)^2\mathbb{p}(z) + 2\beta(\alpha+\beta)z\mathbb{p}'(z) + \beta^2 z^2 \mathbb{p}''(z)}{(\alpha+\beta)^2}, \qquad (5.2.25)$$

and

$$\frac{\mathbb{D}_{\alpha,\beta}^{\mathfrak{j}+3}\left(\mathcal{T}^{\mathfrak{i}}\big(H(z)\big)\right)}{z} = \frac{(\alpha+\beta)^3\mathbb{p}(z) + (\alpha+\beta)(5\beta^2+3\alpha\beta)\, z\mathbb{p}'(z) + 3\beta^2(3\alpha+4\beta)z^2\mathbb{p}''(z) + \beta^3 z^3 \mathbb{p}'''(z)}{(\alpha+\beta)^3}. \qquad (5.2.26)$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\mathbb{x}(r,s,t,u) = r, \mathbb{y}(r,s,t,u) = \frac{(\alpha+\beta)r + \beta s}{\alpha+\beta}, \mathbb{z}(r,s,t,u) = \frac{(\alpha+\beta)^2 r + 2\beta(\alpha+\beta)s + \beta^2 t}{(\alpha+\beta)^2} \qquad (5.2.27)$$

and

$$\mathbb{w}(r,s,t,u) = \frac{(\alpha+\beta)^3 r + (\alpha+\beta)(5\beta^2+3\alpha\beta)\, s + 3\beta^2(3\alpha+4\beta)t + \beta^3 z^2 u}{(\alpha+\beta)^3}. \qquad (5.2.28)$$

Let

$$\boldsymbol{\Pi}(r,s,t,u) = \boldsymbol{\phi}(\mathbb{x},\mathbb{y},\mathbb{z},\mathbb{w};z) = \boldsymbol{\phi}\left(\begin{array}{c} r, \dfrac{(\alpha+\beta)r+\beta s}{\alpha+\beta}, \\ \dfrac{(\alpha+\beta)^2 r + 2\beta(\alpha+\beta)s + \beta^2 t}{(\alpha+\beta)^2}, \\ \dfrac{(\alpha+\beta)^3 r + (\alpha+\beta)(5\beta^2+3\alpha\beta)\, s + 3\beta^2(3\alpha+4\beta)t + \beta^3 z^2 u}{(\alpha+\beta)^3} \end{array}\right). \qquad (5.2.29)$$

The proof will make use of lemma (1.5.18). Using the equations (5.2.22) to (5.2.25) and from the equation (5.2.29), we have

$$\boldsymbol{\Pi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z), z^3\mathbb{p}'''(z); z) =$$
$$\phi\left(\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right), \mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right); z\right). \tag{5.2.29}$$

Hence, clearly (5.2.21) becomes

$$\boldsymbol{\Pi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z), z^3\mathbb{p}'''(z); z) \in \Omega.$$

We note that

$$\frac{t}{s} + 1 = \frac{2(\alpha+\beta)^2\mathbb{x} + 3\beta(\alpha+\beta)\mathbb{y} + \beta^2\mathbb{z}}{(\alpha+\beta)\big((\alpha+\beta)\mathbb{x} + \beta\mathbb{y}\big)},$$

and

$$\frac{u}{s} = \frac{(\alpha+\beta)^3\mathbb{x} + (\alpha+\beta)(5\beta^2 + 3\alpha\beta)\,\mathbb{y} + 3\beta^2(3\alpha+4\beta)\mathbb{z} + \beta^3\mathbb{w}}{(\alpha+\beta)^2\big((\alpha+\beta)\mathbb{x} + \beta\mathbb{y}\big)}.$$

Thus clearly, the admissibility condition for $\boldsymbol{\phi} \in J_{j,1}[\Omega, \mathbb{q}]$ in definition 5.2.4 is equivalent to admissibility condition for $\boldsymbol{\Pi} \in \Psi_2[\Omega, \mathbb{q}]$, as given in definition 1.4.20 with $n \geqq 2$.

Therefore, by using (5.2.20) and lemma (1.4.18), we have

$$\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right) z \prec \mathbb{q}(z).$$

If $\Omega \neq \mathbb{C}$, is a simply connected domain, the $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case the class $J_{j,1}[h(\mathbb{U}), \mathbb{q}\,]$ is written as $J_{j,1}[h, \mathbb{q}\,]$. ■

This leads to the following immediate consequence of theorem 5.2.13 is stated below.

<u>Theorem 5.2.14</u>: Let $\boldsymbol{\phi} \in J_{j,1}[\Omega, \mathbb{q}]$. If $H \in \mathbb{H}$ and $\mathbb{q} \in Q_1$, satisfy the following conditions:

$$Re\left(\frac{\xi \mathbb{q}''(\xi)}{\mathbb{q}'(\xi)}\right) \geqq 0, \left|\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z\mathbb{q}'(\xi)}\right| \leqq k, \tag{5.2.30}$$

and

$$\boldsymbol{\phi}\left(\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}. \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}; z\right)$$
$$\prec h(z), \tag{5.2.31}$$

then $\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z} \prec \mathbb{q}(z), z \in \mathbb{U}$. ■

In view of definition 5.2.11, and in special case when $\mathbb{q}(z) = Mz, (M > 0)$, the class $J_{j,1}[\Omega, \mathbb{q}]$, of admissible functions, denoted by $J_{j,1}[\Omega, M]$ is expressed follows.

<u>Definition 5.2.15</u>: Let Ω be set in $\mathbb{C}$ and $M > 0$. The class $J_{j,1}[\Omega, M]$ of admissible functions consists of those function $\boldsymbol{\phi}: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$, $such\ that$

$$\boldsymbol{\phi}\begin{pmatrix} Me^{i\theta}, \\ \left(\frac{(\alpha+\beta)M+\beta K}{\alpha+\beta}\right)e^{i\theta}, \\ \left(\frac{(\alpha+\beta)^2 M+2\beta(\alpha+\beta)K+\beta^2 L}{(\alpha+\beta)^2}\right)e^{i\theta}, \\ \left(\frac{(\alpha+\beta)^3 M+(\alpha+\beta)(5\beta^2+3\alpha\beta)\ K+3\beta^2(3\alpha+4\beta)L+\beta^3 z^2 N}{(\alpha+\beta)^3}\right)e^{i\theta} \end{pmatrix}, \notin \Omega, \tag{5.2.32}$$

whenever, $z \in \mathbb{U}, Re(Le^{-i\theta}) \geqq (K-1)KM$

and $Re(Ne^{-i\theta}) \geqq 0, \forall\ \theta \in \mathbb{R}; k \geqq 0$.

<u>Corollary 5.2.16</u>: Let $\boldsymbol{\phi} \in J_{j,1}[\Omega, M]$. If the function $H \in \mathbb{H}$ satisfies the following conditions:

$\left|\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}\right| \leqq kM, (z \in U; k \geqq 2;\ M > 0),$

and $\boldsymbol{\phi}\left(\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}. \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}; z\right) \in \Omega,$

then $\left|\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}\right| < M.$

When $\Omega = \mathbb{q}(\mathbb{U}) = \{\omega: |\omega| < M\}$, the class $J_{j,1}[\Omega, M]$ is simple denoted by $J_{j,1}[\Omega, M]$. ■

Corollary 5.2.16 can now be rewritten in the following from.

<u>Corollary 5.2.17</u>: Let $\boldsymbol{\phi} \in J_{j,1}[\Omega, M]$. If the function $H \in \mathbb{H}$ satisfies the following conditions $\left|\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}\right| \leqq kM, (z \in \mathbb{U}; k \geqq 2;\ M > 0)$

And $\left|\boldsymbol{\phi}\left(\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}. \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)}{z}; z\right)\right| < M,$

then $\left|\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)\right| < M.$

<u>Definition 5.2.18</u>: Let Ω be a set in $\mathbb{C}$ $.Also$ $\mathbb{q} \in Q_1 \cap S_1$, the class $J_{j,2}[\Omega, \mathbb{q}]$ of admissible functions consists of those functions $\boldsymbol{\phi}: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$,which satisfy the following admissibility conditions $\boldsymbol{\phi}(\mathbb{x}, \mathbb{y}, \mathbb{z}, \mathbb{w}; z) \notin \Omega$

whenever $\mathbb{x} = \mathbb{q}(\xi), \mathbb{y} = \frac{(\alpha+\beta)r+\beta s}{\alpha+\beta},$

$$Re\left(\frac{2(\alpha+\beta)^2\mathbb{x} + 3\beta(\alpha+\beta)\mathbb{y} + \beta^2\mathbb{z}}{(\alpha+\beta)\left((\alpha+\beta)\mathbb{x} + \beta\mathbb{y}\right)}\right) \geqq k\ Re\left(\frac{\xi\mathbb{q}''(\xi)}{\mathbb{q}'(\xi)} + 1\right),$$

and

$$Re\left(\frac{(\alpha+\beta)^3\mathbb{x}+(\alpha+\beta)(5\beta^2+3\alpha\beta)\,\mathbb{y}+3\beta^2(3\alpha+4\beta)\mathbb{z}+\beta^3\mathbb{w}}{(\alpha+\beta)^2\big((\alpha+\beta)\mathbb{x}+\beta\mathbb{y}\big)}\right)$$

$$\geqq k^2 Re\left(\frac{\xi^2\mathbb{q}'''(\xi)}{\mathbb{q}'(\xi)}\right),$$

where $z \in \mathbb{U}, \xi \in \partial\mathbb{U} \setminus E(\mathbb{q})\ and\ k \geqq n.$

<u>Theorem 5.2.19</u>: Let $\boldsymbol{\phi} \in J_{\mathrm{j},2}[\Omega,\mathbb{q}]$. If the function $H \in \mathbb{H}\ and\ \mathbb{q} \in Q_1$, satisfy the following conditions: $Re\left(\frac{\xi\mathbb{q}''(\xi)}{\mathbb{q}'(\xi)}\right) \geqq 0,$

$$\left|\frac{\mathbb{D}^{\mathrm{j}+1}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}\big(H(z)\big)\right)}{\mathbb{D}^{\mathrm{j}}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}\big(H(z)\big)\right)}\right| \leqq k, \tag{5.2.33}$$

and $\left\{\boldsymbol{\phi}\left(\frac{\mathbb{D}^{\mathrm{j}+1}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)}{\mathbb{D}^{\mathrm{j}}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)},\frac{\mathbb{D}^{\mathrm{j}+2}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)}{\mathbb{D}^{\mathrm{j}+1}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)},\frac{\mathbb{D}^{\mathrm{j}+3}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)}{\mathbb{D}^{\mathrm{j}+2}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)},\frac{\mathbb{D}^{\mathrm{j}+4}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)}{\mathbb{D}^{\mathrm{j}+3}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)};z\right);\ z \in U\right\} \subset \Omega,$ (5.2.34)

then $\frac{\mathbb{D}^{\mathrm{j}+1}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)}{\mathbb{D}^{\mathrm{j}}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)} \prec \mathbb{q}(z), (z \in \mathbb{U}).$

Proof:

Define the analytic function $\mathbb{p}(z)\ in\ \mathbb{U}$ by

$$\mathbb{p}(z) = \frac{\mathbb{D}^{\mathrm{j}+1}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)}{\mathbb{D}^{\mathrm{j}}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)}. \tag{5.2.35}$$

From equations (5.2.5) and (5.2.35), we have

$$\frac{\mathbb{D}^{\mathrm{j}+2}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}\big(H(z)\big)\right)}{\mathbb{D}^{\mathrm{j}+1}_{\alpha,\beta}\left(\mathcal{T}^{\mathrm{i}}\big(H(z)\big)\right)} = \frac{(\alpha+\beta)\mathbb{p}^2(z)+\beta z\mathbb{p}'(z)}{(\alpha+\beta)\mathbb{p}(z)}. \tag{5.2.36}$$

By a similar argument, we get

$$\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)}{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)} = \frac{(\alpha+\beta)^2\mathbb{p}^3(z)+3(\alpha+\beta)\beta z\mathbb{p}(z)\mathbb{p}'(z)+2\beta^2 z\mathbb{p}'(z)+2\beta^2 z^2\mathbb{p}''(z)}{(\alpha+\beta)^2\mathbb{p}^2(z)+\beta(\alpha+\beta)z\mathbb{p}'(z)}, \quad (5.2.37)$$

and

$$\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+4}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)}{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}\big(H(z)\big)\right)} = \frac{d\beta^2 z^3\mathbb{p}'''(z)+ab+c+de}{a^2}, \quad (5.2.38)$$

where

$$a(z) = \big((\alpha+\beta)^2\mathbb{p}^3(z)+3(\alpha+\beta)\beta z\mathbb{p}(z)\mathbb{p}'(z)+\beta^2 z\mathbb{p}'(z)+\beta^2 z^2\mathbb{p}''(z)\big),$$

$$b(z) = \big((\alpha+\beta)^2\mathbb{p}^3(z)+(\alpha+\beta)\beta z\mathbb{p}(z)\mathbb{p}'(z)-\beta^2 z\mathbb{p}'(z)-\beta^2 z^2\mathbb{p}''(z)\big),$$

$$c(z) = \big(\beta^2 z\mathbb{p}'(z)+\beta^2 z^2\mathbb{p}''(z)\big)^2,$$

$$d(z) = \big((\alpha+\beta)\beta\mathbb{p}^2(z)+\beta^2 z\mathbb{p}'(z)\big),$$

$$e(z) = \big((\alpha+\beta)^2\mathbb{p}^3(z)+3(\alpha+\beta)\beta\mathbb{p}(z)z^2\mathbb{p}''(z)+3(\alpha+\beta)\beta z^2\mathbb{p}'(z)\mathbb{p}'(z) + 3(\alpha+\beta)\beta z\mathbb{p}(z)\mathbb{p}'(z)+3\beta^2 z^2\mathbb{p}''(z)+z\mathbb{p}'(z)\big).$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\mathbb{x}(r,s,t,u) = r, \mathbb{y}(r,s,t,u) = \frac{(\alpha+\beta)r^2+\beta s}{(\alpha+\beta)r},$$

$$\mathbb{z}(r,s,t,u) = \frac{(\alpha+\beta)^2 r^3+3(\alpha+\beta)\beta rs+\beta^2 s+\beta^2 t}{(\alpha+\beta)^2 r^2+\beta(\alpha+\beta)s}, \quad (5.2.39)$$

and

$$\mathbb{w}(r,s,t,u) = \frac{d(r,s,t,u)\beta^2 u + a(r,s,t,u)b(r,s,t,u) + c(r,s,t,u) + d(r,s,t,u)e(r,s,t,u)}{\left(a(r,s,t,u)\right)^2}, \tag{5.2.40}$$

where

$$a(r,s,t,u) = ((\alpha+\beta)^2 r^3 + 3(\alpha+\beta)\beta rs + \beta^2 s + \beta^2 t),$$

$$b(r,s,t,u) = ((\alpha+\beta)^2 r^3 + (\alpha+\beta)\beta rs - \beta^2 s - \beta^2 t),$$

$$c(r,s,t,u) = (\beta^2 s + \beta^2 t)^2,$$

$$d(r,s,t,u) = \left((\alpha+\beta)\beta r^2 + \beta^2 s\right),$$

$$e(r,s,t,u) = ((\alpha+\beta)^2 r^3 + 3(\alpha+\beta)\beta rt + 3(\alpha+\beta)\beta s^2 + 3(\alpha+\beta)\beta rs + 3\beta^2 t + s).$$

Let

$$\boldsymbol{\Pi}(r,s,t,u) = \boldsymbol{\phi}(\mathbb{x},\mathbb{y},\mathbb{z},\mathbb{w};z) = \boldsymbol{\phi}\left(\begin{array}{c} r, \\ \dfrac{(\alpha+\beta)r^2+\beta s}{(\alpha+\beta)r}, \\ \dfrac{(\alpha+\beta)^2 r^3 + 3(\alpha+\beta)\beta rs + \beta^2 s + \beta^2 t}{(\alpha+\beta)^2 r^2 + \beta(\alpha+\beta)s}, \\ \dfrac{d(r,s,t,u)\beta^2 u + a(r,s,t,u)b(r,s,t,u) + c(r,s,t,u) + d(r,s,t,u)e(r,s,t,u)}{\left(a(r,s,t,u)\right)^2} \end{array}\right). \tag{5.2.41}$$

The proof will make use of lemma 1.4.17. Using the equations (5.2.35) to (5.2.38), and from the equation (5.2.41), we have

$$\boldsymbol{\Pi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z), z^3\mathbb{p}'''(z); z)$$

$$= \boldsymbol{\phi}\left(\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+4}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}; z\right).$$

Hence, clearly (5.2.34) becomes

$$\boldsymbol{\Pi}(\mathbb{p}(z), z\mathbb{p}'(z), z^2\mathbb{p}''(z), z^3\mathbb{p}'''(z); z) \in \Omega.$$

We note that

$$1 + \frac{t}{s} = \frac{2(\alpha+\beta)^2\mathbb{x}^4 + 5(\alpha+\beta)\beta\mathbb{x}^2\mathbb{y} + 2\beta^2\mathbb{x}\mathbb{y} + \beta^2\mathbb{y}^2 + 2\beta^2\mathbb{x}\mathbb{z}}{\left((\alpha+\beta)\mathbb{x}^2 + \beta\mathbb{y}\right)^2},$$

and

$$\frac{u}{s} = \frac{(\alpha+\beta)\mathbb{x}\big(d(\mathbb{x},\mathbb{y},\mathbb{z},\mathbb{u})\beta^2 u + a(\mathbb{x},\mathbb{y},\mathbb{z},\mathbb{u})b(\mathbb{x},\mathbb{y},\mathbb{z},\mathbb{u}) + c(\mathbb{x},\mathbb{y},\mathbb{z},\mathbb{u}) + d(\mathbb{x},\mathbb{y},\mathbb{z},\mathbb{u})e(\mathbb{x},\mathbb{y},\mathbb{z},\mathbb{u})\big)}{\left((\alpha+\beta)\mathbb{x}^2 + \beta\mathbb{y}\right)\left(a\left((\mathbb{x},\mathbb{y},\mathbb{z},\mathbb{u})\right)\right)^2}.$$

Thus clearly, the admissibility condition for $\boldsymbol{\phi} \in J_{j,2}[\Omega, \mathbb{q}]$, in definition 5.2.6 is equivalent to admissibility condition for $\boldsymbol{\Pi} \in \Psi_2[\Omega, \mathbb{q}]$, as given in definition 1.4.19 with n=2.

Therefore, by using (5.2.33) and lemma 1.4.17, we have

$$\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)} \prec \mathbb{q}(z). \qquad (5.2.43)$$

This completes the proof of Theorem 5.2.19. ■

If $\Omega \neq \mathbb{C}$, is simply-connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ on to Ω. In this case, the class $J_{j,2}[h(\mathbb{U}), \mathbb{q}]$ is written as $J_{j,2}[h, \mathbb{q}]$. An immediate consequence of Theorem (5.2.11) is now stated below without proof.

<u>Theorem 5.2.20</u>: Let $\boldsymbol{\phi} \in J_{j,2}[\Omega, \mathbb{q}]$. If $H \in \mathbb{H}$ *and* $\mathbb{q} \in Q_1$, satisfy the following conditions (5.2.34), and

$$\boldsymbol{\phi}\left(\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+2}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}, \frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+4}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+3}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}; z\right)$$

$$\prec h(z), \qquad (5.2.44)$$

then

$$\frac{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}+1}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)}{\mathbb{D}_{\alpha,\beta}^{\mathbb{j}}\left(\mathcal{T}^{\mathbb{i}}\left(H(z)\right)\right)} \prec \mathbb{q}(z), z \in \mathbb{U}.$$

■

5.3 Fourth-Order Differential Subordination and Superordination Results for Completely Homogeneous Symmetric Defined by the Inverse of an Ordered Cyclic Operator

Assume that $\mathbb{H}$ denotes the class of functions of the form

$$H(z) = \sum_{n=0}^{\infty} h_n z^n, z \in \mathbb{U}, \qquad (5.3.1)$$

let $H \in \mathbb{H}$ given by (5.3.1) the inverse $\mathbb{I}^{\mathbb{i}}$ of $H(z)$ is defined as follows:

$$\mathbb{I}^{\mathbb{i}}\left(H(z)\right) = \sum_{n=0}^{\infty} h_{n-\mathbb{i}} z^n, z \in \mathbb{U} \text{ and } \mathbb{i} = 0,1,2,\ldots,$$

it is easily verified from (5.3.2), that

$$\frac{z\left(\mathbb{I}^{\mathbb{i}}\left(H(z)\right)\right)'}{\mathbb{I}^{\mathbb{i}}\left(H(z)\right)} = -\mathbb{S}(\mathrm{z}) \text{ and } z\left(\mathbb{I}^{\mathbb{i}}\left(H(z)\right)\right)' = -\mathbb{I}^{\mathbb{i}}\left(H(z)\right)\mathbb{S}(\mathrm{z}). \qquad (5.3.3)$$

In recent years, there are many authors presented and dealing with the theory of second-order differential subordination and superordination see ([1,10,16,36]). Also, many authors discussed the theory of the third-order differential subordination and superordination see ([2,5]). In 2011, Antonino and

Miller [2] presented basic concepts and extended the theory of the second-order differential subordination in the open unit disk introduced by Miller and Mocanu [36] to the third case. Atshan et al. ([5,9]) extended the third-order case to fourth-order differential subordination and determined properties of functions $\mathbb{q}$ that satisfy the following fourth-order differential subordination:

$$\phi(\mathbb{q}(z), z\mathbb{q}'(z), z^2\mathbb{q}''(z), z^3\mathbb{q}'''(z), z^4\mathbb{q}''''(z); z) \prec h(z),$$

where h be analytic univalent function in $\mathbb{U}$, $\mathbb{q}$ is analytic function and $\boldsymbol{\phi}: \mathbb{C}^5 \times \mathbb{U} \to \mathbb{C}$. Now, we extended the third-order case to fourth-order differential superordination and determined properties of the function $\mathbb{q}$ that satisfy the following fourth-order differential superordination

$$h(z) \prec \boldsymbol{\phi}(\mathbb{q}(z), z\mathbb{q}'(z), z^2\mathbb{q}''(z), z^3\mathbb{q}'''(z), z^4\mathbb{q}''''(z); z),$$

where $h(z)$ be analytic univalent function in $\mathbb{U}$, $\mathbb{p}$ is analytic function $\boldsymbol{\phi}: \mathbb{C}^5 \times \mathbb{U} \to \mathbb{C}$. To prove our main result, we need the basic concepts in the theory of fourth-order.

We present some results for differential subordination and superordination for analytic functions defined by the inverse of an cyclic operator $\mathbb{I}^{\mathbb{i}}$ of $H(z)$.

Second, we define the following class of admissible functions, which are needed in proving the differential theorems associated with $\mathbb{I}^{\mathbb{i}}$ defined by (5.3.2).

<u>Definition 5.3.1</u>: Let Ω be a set in $\mathbb{C}$ and $\mathbb{q} \in Q_0$. The class $\mathcal{A}_j[\Omega, \mathbb{q}]$ of admissible functions consists of those functions $\boldsymbol{\phi}: \mathbb{C}^5 \times \mathbb{U} \to \mathbb{C}$, that satisfy the following admissibility conditions:

$$\boldsymbol{\phi}(\mathbb{x}, \mathbb{y}, \mathbb{z}, \mathbb{u}, \mathbb{w}; z) \notin \Omega,$$

whenever $\mathbb{x} = \mathbb{q}(\xi), \mathbb{y} = \mathbb{I}\big(H(z)\big)\mathbb{p}(z) - \frac{\mathbb{S}(z)\mathbb{p}^2(z)}{z\mathbb{p}'(z)}$,

$Re\left(\frac{\mathbb{I}^3(\mathbb{p}(z))}{\mathbb{I}(\mathbb{p}(z))}\right) \geqq k^2 Re\left(\frac{\zeta^2\mathbb{q}'''(\zeta)}{\mathbb{q}'(\zeta)}\right)$ and $Re\left(\frac{\mathbb{I}^4(\mathbb{p}(z))}{\mathbb{I}(\mathbb{p}(z))}\right) \geqq k^3 Re\left(\frac{\zeta^3\mathbb{q}''''(\zeta)}{\mathbb{q}'(\zeta)}\right)$,

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus E(\mathbb{q})$ and $k \geqq 3$.

Theorem 5.3.2: Let $\phi \in \mathcal{A}_j[\Omega, \mathbb{q}]$. If $H(z) \in \mathbb{H}\ and\ \mathbb{q} \in Q_0$ and satisfy the following conditions

$$Re\left(\frac{\zeta^2\mathbb{q}'''(\zeta)}{\mathbb{q}'(\zeta)}\right) \geqq 0, \left|\frac{\zeta^2\left(\mathbb{I}^{\mathbb{i}+2}(H(z))\right)}{\mathbb{q}'(\zeta)}\right| \leqq k^2, \qquad (5.3.4)$$

and

$$\left\{\phi\left(\mathbb{I}^{\mathbb{i}}(H(z)), \mathbb{I}^{\mathbb{i}+1}(H(z)), \mathbb{I}^{\mathbb{i}+2}(H(z)), \mathbb{I}^{\mathbb{i}+3}(H(z)), \mathbb{I}^{\mathbb{i}+4}(H(z))\right)\right\} \subset \Omega, \qquad (5.3.5)$$

then $\mathbb{I}^{\mathbb{i}}(H(z)) \prec \mathbb{q}(z)\ (z \in \mathbb{U})$.

Proof:

Define the analytic function $\mathbb{p}\ in\ \mathbb{U}$ by

$$\mathbb{p}(z) = \mathbb{I}^{\mathbb{i}}(H(z)); z \in \mathbb{U}. \qquad (5.3.6)$$

Then, differentiating (5.3.6) with respect to z and using (4.3.3), we have

$$\mathbb{p}'(z) = \left(\mathbb{I}^{\mathbb{i}}(H(z))\right)' \text{and} \frac{z\mathbb{p}'(z)}{\mathbb{p}(z)} = -\mathbb{S}(z). \qquad (5.3.7)$$

$$\mathbb{I}^{\mathbb{i}+1}(H(z)) = -\frac{z\left(\mathbb{I}^{\mathbb{i}+1}(H(z))\right)'}{\mathbb{S}(z)} = -\frac{z\left(\mathbb{I} \circ \mathbb{I}^{\mathbb{i}}(H(z))\right)'}{\mathbb{S}(z)}$$

$$= -\left(\frac{z\mathbb{I}(H(z))\left(\mathbb{I}^{\mathbb{i}}(H(z))\right)' + z\mathbb{I}^{\mathbb{i}}(H(z))\left(\mathbb{I}(H(z))\right)'}{\mathbb{S}(z)}\right) =$$

$$= \mathbb{I}(H(z))\mathbb{p}(z) - \frac{\mathbb{S}(z)\mathbb{p}^2(z)}{z\mathbb{p}'(z)}$$

Further computations show that

$$\mathbb{I}^{\mathbb{i}+2}(H(z)) = -\frac{z\left(\mathbb{I}^{\mathbb{i}+2}(H(z))\right)'}{\mathbb{S}(z)} = -\frac{z\left(\mathbb{I} \circ \mathbb{I}^{\mathbb{i}+1}(H(z))\right)'}{\mathbb{S}(z)}$$

$$= \mathbb{I}(H(z))\mathbb{p}(z) - \frac{\mathbb{I}(H(z))z\mathbb{p}'(z)}{\mathbb{S}(z)} + 2\mathbb{S}(z)\mathbb{p}(z)$$

$$+ \frac{\mathbb{I}(H(z))z^2\mathbb{p}'(z)\mathbb{S}'(z)\left(\left(\mathbb{p}(z)\right)^2\right) - \mathbb{S}(z)\left(\mathbb{p}(z)\right)^2\mathbb{I}(H(z))\left(z\mathbb{p}'(z) + z^2\mathbb{p}''(z)\right)}{\mathbb{S}(z)\left(z\mathbb{p}'(z)\right)^2}$$

$$\mathbb{I}^{\mathbb{i}+3}\big(H(z)\big) =$$

$$= \left(\frac{\left(2z^2\mathbb{p}''^{(z)}\mathbb{I}\big(H(z)\big)\mathbb{p}^2(z) + z\mathbb{p}'(z)\mathbb{I}\big(H(z)\big)\right)\left(2z\mathbb{p}'(z)\big(2z^2\mathbb{p}''(z) + z\mathbb{p}'(z)\big)\right)}{\mathbb{S}^3(z)\big(z\mathbb{p}'(z)\big)^4}\right)$$

$$+ \left(\frac{z^5\mathbb{p}'(z)\mathbb{p}'(z)\mathbb{p}'''(z)}{\mathbb{S}^3(z)\big(z\mathbb{p}'(z)\big)^4}\right) \tag{5.3.9}$$

and

$$\mathbb{I}^{\mathbb{i}+4}\big(H(z)\big) = \left(\frac{z^5\mathbb{p}'(z)\mathbb{p}''(z)\mathbb{p}''''(z)}{\mathbb{S}^5(z)\big(z\mathbb{p}'(z)\big)^6}\right)$$

$$+ \Big(\mathbb{I}\big(H(z)\big)\Big)\Big(\mathbb{I}\big(H(z)\big)\mathbb{p}(z) - 2\mathbb{S}(z)\mathbb{I}\big(H(z)\big)z\mathbb{p}'(z)$$

$$+ \frac{\mathbb{S}(z)\mathbb{I}\big(H(z)\big)z\mathbb{p}'(z) - \mathbb{I}\big(H(z)\big)z\mathbb{S}'(z)z\mathbb{p}'(z) + \mathbb{S}(z)\mathbb{I}\big(H(z)\big)z^2\mathbb{p}''(z)}{\mathbb{S}^2(z)}\Big)$$

$$- \frac{z\Big(\mathbb{I}\big(H(z)\big)\Big)\Big(\mathbb{I}\big(H(z)\big)\mathbb{S}^2(z)\mathbb{p}(z) - 2\mathbb{S}^3(z)\mathbb{I}\big(H(z)\big)z\mathbb{p}''(z) + \mathbb{S}(z)\mathbb{I}\big(H(z)\big)\mathbb{p}'(z)\Big)'}{\mathbb{S}^5(z)\big(z\mathbb{p}'(z)\big)^6}$$

$$- \frac{-\mathbb{I}\big(H(z)\big)z\mathbb{S}'(z)z\mathbb{p}'(z) + \mathbb{S}(z)\mathbb{I}\big(H(z)\big)z^2\mathbb{p}''(z)}{\mathbb{S}^5(z)\big(z\mathbb{p}'(z)\big)^6}\,. \tag{5.3.10}$$

We, now define the transformation $\mathbb{C}^5$ to $\mathbb{C}$,

$$\mathbb{x}(r,s,t,u,w) = r,$$

$$\mathbb{y}(r,s,t,u,w) = \mathbb{I}\big(H(z)\big)r - \frac{\mathbb{S}(z)r^2}{s},$$

$$\mathbb{z}(r,s,t,u,w)$$

$$= \mathbb{I}\big(H(z)\big)r - \frac{\mathbb{I}\big(H(z)\big)s}{\mathbb{S}(z)} + 2\mathbb{S}(z)\mathbb{p}(z)$$

$$+ \frac{\mathbb{I}\big(H(z)\big)z\mathbb{S}'(z)sr^2 - \mathbb{S}(z)r^2\mathbb{I}\big(H(z)\big)(s+u)}{\mathbb{S}(z)s^2},$$

$$\mathbb{u}(r,s,t,u,w) = \left(\frac{\Big(2t\mathbb{I}\big(H(z)\big)\mathbb{p}^2(z) + s\mathbb{I}\big(H(z)\big)\Big)\big(2s(2t+s)\big)}{\mathbb{S}^3(z)\big(z\mathbb{p}'(z)\big)^4}\right) + \left(\frac{stu}{\mathbb{S}^3(z)(s)^4}\right) \tag{5.3.11}$$

and

$$\mathbb{w}(r,s,t,u,w)=\left(\frac{z^5\mathbb{p}'(z)\mathbb{p}''(z)\mathbb{p}''''(z)}{\mathbb{S}^5(z)\big(z\mathbb{p}'(z)\big)^6}\right)$$

$$+\Big(\mathbb{I}\big(H(z)\big)\Big)\Bigg(\mathbb{I}\big(H(z)\big)\mathbb{p}(z)-2\mathbb{S}(z)\mathbb{I}\big(H(z)\big)z\mathbb{p}'(z)$$

$$+\frac{\mathbb{S}(z)\mathbb{I}\big(H(z)\big)z\mathbb{p}'(z)-\mathbb{I}\big(H(z)\big)z\mathbb{S}'(z)z\mathbb{p}'(z)+\mathbb{S}(z)\mathbb{I}\big(H(z)\big)z^2\mathbb{p}''(z)}{\mathbb{S}^2(z)}\Bigg)$$

$$-\frac{z\Big(\mathbb{I}\big(H(z)\big)\Big)\Big(\mathbb{I}\big(H(z)\big)\mathbb{S}^2(z)\mathbb{p}(z)-2\mathbb{S}^3(z)\mathbb{I}\big(H(z)\big)z\mathbb{p}''(z)+\mathbb{S}(z)\mathbb{I}\big(H(z)\big)\mathbb{p}'(z)\Big)'}{\mathbb{S}^5(z)\big(z\mathbb{p}'(z)\big)^6}$$

$$-\frac{-\mathbb{I}\big(H(z)\big)z\mathbb{S}'(z)z\mathbb{p}'(z)+\mathbb{S}(z)\mathbb{I}\big(H(z)\big)z^2\mathbb{p}''(z)}{\mathbb{S}^5(z)\big(z\mathbb{p}'(z)\big)^6}. \qquad (5.3.12)$$

Let

$$\boldsymbol{\chi}(r,s,t,u,w;z)=\boldsymbol{\phi}(\mathbb{x},\mathbb{y},\mathbb{z},\mathbb{u},\mathbb{w};z). \qquad (5.3.13)$$

The proof will make use Lemma 1.5.22. Using the equations (5.3.7) to (5.3.10), we have from (5.3.11) that

$$\boldsymbol{\chi}\big(\mathbb{p}(z),\mathbb{I}\big(\mathbb{p}(z)\big),\mathbb{I}^2\big(\mathbb{p}(z)\big),\mathbb{I}^3\big(\mathbb{p}(z)\big),\mathbb{I}^4\big(\mathbb{p}(z)\big);z\big)=$$

$$\boldsymbol{\phi}\Big(\mathbb{I}^{\mathbb{i}}\big(H(z)\big),\mathbb{I}^{\mathbb{i}+1}\big(H(z)\big),\mathbb{I}^{\mathbb{i}+2}\big(H(z)\big),\mathbb{I}^{\mathbb{i}+3}\big(H(z)\big),\mathbb{I}^{\mathbb{i}+4}\big(H(z)\big)\Big). \qquad (5.3.14)$$

Hence, clearly (5.3.5) becomes

$\boldsymbol{\chi}\big(\mathbb{p}(z),\mathbb{I}\big(\mathbb{p}(z)\big),\mathbb{I}^2\big(\mathbb{p}(z)\big),\mathbb{I}^3\big(\mathbb{p}(z)\big),\mathbb{I}^4\big(\mathbb{p}(z)\big);z\big)\in\ \Omega,$

we note that $\frac{t}{s}+1=\frac{\mathbb{I}^2(\mathbb{p}(z))}{\mathbb{I}(\mathbb{p}(z))}+1,\ \frac{u}{s}=\frac{\mathbb{I}^3(\mathbb{p}(z))}{\mathbb{I}(\mathbb{p}(z))}$

and $\frac{w}{s}=\frac{\mathbb{I}^4(\mathbb{p}(z))}{\mathbb{I}(\mathbb{p}(z))}$.

Therefore, the admissibility condition for $\boldsymbol{\phi}\in\mathcal{A}_j[\Omega,\mathbb{q}]$, in Definition 5.3.1 is equivalent to admissibility condition for $\boldsymbol{\chi}\in\Psi_3[\Omega,\mathbb{q}]$, as given in Definition

1.5.23 with $n = 3$. Therefore, by using (5.3.4) and Lemma 1.5.22, we obtain $\mathbb{p}(z) = \mathbb{I}^{\mathbb{i}}\big(H(z)\big) \prec \mathbb{q}(z)$.

This completes the proof of Theorem 5.3.2. ■

Our next corollary, is an extension of Theorem 5.3.2 to the case when the behavior of $\mathbb{q}(z) \in \partial\mathbb{U}$, is not known.

Corollary 5.3.3: Let $\Omega \subset \mathbb{C}$, and let the function $\mathbb{q}(z)$be univalent in $\mathbb{U}$ $with$ $\mathbb{q}(0) = 1$. Let $\boldsymbol{\phi} \in \mathcal{A}_j[\Omega, \mathbb{q}]$ for some $p \in (0,1)$, where $\mathbb{q}_p(z) = \mathbb{q}(pz)$. If $H \in \mathbb{H}$ and $\mathbb{q}_p$ satisfies the following conditions:

$$Re\left(\frac{\mathbb{I}^3(\mathbb{q}(\zeta))}{\zeta^2\mathbb{I}(\mathbb{q}(\zeta))}\right) \geqq 0,$$

$$\left|\frac{\mathbb{I}^{\mathbb{i}+2}(H(z))}{\zeta^{\mathbb{i}+2}\mathbb{I}(\mathbb{q}(\zeta))}\right| \leqq \mathrm{k}^2, \big(z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus \mathrm{E}(\mathbb{q}_p)\text{and } \mathrm{k} \geqq 3\big), \tag{5.3.15}$$

and

$$\boldsymbol{\phi}\left(\mathbb{I}^{\mathbb{i}}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+1}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+2}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+3}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+4}\big(H(z)\big)\right) \prec h(z), \tag{5.3.16}$$

then $\mathbb{I}^{\mathbb{i}}\big(H(z)\big) \prec \mathbb{q}(z), z \in \mathbb{U}$.

Proof:

By using theorem 5.3.2, yield $\mathbb{I}^{\mathbb{i}}\big(H(z)\big) \prec \mathbb{q}_p(z), (z \in \mathbb{U})$,

then we obtain the result from $\mathbb{q}_p(\mathrm{z}) \prec \mathbb{q}(z), (z \in \mathbb{U})$.

This completes the proof of Corollary 5.3.1.

If $\Omega \neq \mathbb{C}$ is simply-connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\mathcal{A}_j[h(\mathbb{U}), \mathbb{q}]$ is written an $\mathcal{A}_j[h, \mathbb{q}]$. The following two results are immediate consequence of theorem 5.3.2 and Corollary 5.3.3. ■

Theorem (5.3.4): Let $\boldsymbol{\phi} \in \mathcal{A}_j[h(\mathbb{U}), \mathbb{q}]$ If $H \in \mathbb{H}$ and $\mathbb{q} \in \mathcal{Q}_0$ satisfy the following conditions (5.3.4), and

$$\boldsymbol{\phi}\left(\mathbb{I}^{\mathbb{i}}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+1}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+2}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+3}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+4}\big(H(z)\big)\right) \prec h(z), \tag{5.3.17}$$

then $\mathbb{I}^{\mathbb{i}}\big(H(z)\big) \prec \mathbb{q}(z) (z \in \mathbb{U})$. ■

Corollary 5.3.5: Let $\Omega \subset \mathbb{C}$, and let the function $\mathbb{q}$ be univalent in $\mathbb{U}$ with $\mathbb{q}(0) = 1$. Let $\boldsymbol{\phi} \in \mathcal{A}j[\Omega, \mathbb{q}]$ for some $p \in (0,1)$, where $\mathbb{q}_p(z) = \mathbb{q}(pz)$. If $H \in \mathbb{H}$ *and* $\mathbb{q}_p$ satisfies the conditions (5.3.15) and

$$\boldsymbol{\phi}\Big(\mathbb{I}^{\mathbb{i}}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+1}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+2}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+3}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+4}\big(H(z)\big)\Big) \prec h(z), \tag{5.3.18}$$

then $\mathbb{I}^{\mathbb{i}}\big(H(z)\big) \prec \mathbb{q}(z) (z \in \mathbb{U})$. ■

The following result yield the best dominant of differential subordination (5.3.15).

Theorem 5.3.6: Let the function h be univalent in $\mathbb{U}$. Also let

$$\boldsymbol{\phi}\left(\mathbb{q}(z), \mathbb{I}\big(H(z)\big)\mathbb{q}(z) - \frac{\mathbb{S}(z)\mathbb{q}^2(z)}{z\mathbb{q}'(z)}, \left(\frac{\big(2z^2\mathbb{q}''(z)\mathbb{I}\big(H(z)\big)\mathbb{q}^2(z) + z\mathbb{q}'(z)\mathbb{I}\big(H(z)\big)\big)\big(2z\mathbb{q}'(z)\big(2z^2\mathbb{q}''(z) + z\mathbb{q}(z)\big)\big)}{\mathbb{S}^3(z)\big(z\mathbb{p}'(z)\big)^4}\right) + \left(\frac{z^5\mathbb{q}'(z)\mathbb{q}'(z)\mathbb{q}'''(z)}{\mathbb{S}^3(z)\big(z\mathbb{q}'(z)\big)^4}\right), A, B\right) = h(z), \tag{5.3.19}$$

$$A = \left(\frac{\big(2z^2\mathbb{q}''(z)\mathbb{I}\big(H(z)\big)\mathbb{q}^2(z) + z\mathbb{q}'(z)\mathbb{I}\big(H(z)\big)\big)\big(2z\mathbb{q}'(z)\big(2z^2\mathbb{q}''(z) + z\mathbb{q}'(z)\big)\big)}{\mathbb{S}^3(z)\big(z\mathbb{q}'(z)\big)^4}\right) + \left(\frac{z^5\mathbb{q}'(z)\mathbb{q}'(z)\mathbb{q}'''(z)}{\mathbb{S}^3(z)\big(z\mathbb{q}'(z)\big)^4}\right)$$

$$B = \left(\frac{z^5\mathbb{q}'(z)\mathbb{p}''(z)\mathbb{q}''''(z)}{\mathbb{S}^5(z)\big(z\mathbb{q}'(z)\big)^6}\right)$$

$$+\left(\mathbb{I}\big(H(z)\big)\right)\Bigg(\mathbb{I}\big(H(z)\big)\mathbb{p}(z) - 2\mathbb{S}(z)\mathbb{I}\big(H(z)\big)z\mathbb{p}'(z)$$

$$+\frac{\mathbb{S}(z)\mathbb{I}\big(H(z)\big)z\mathbb{q}'(z) - \mathbb{I}\big(H(z)\big)z\mathbb{S}'(z)z\mathbb{q}'(z) + \mathbb{S}(z)\mathbb{I}\big(H(z)\big)z^2\mathbb{q}''(z)}{\mathbb{S}^2(z)}\Bigg)$$

$$-\frac{z\left(\mathbb{I}\big(H(z)\big)\right)\left(\mathbb{I}\big(H(z)\big)\mathbb{S}^2(z)\mathbb{q}(z) - 2\mathbb{S}^3(z)\mathbb{I}\big(H(z)\big)z\mathbb{q}''(z) + \mathbb{S}(z)\mathbb{I}\big(H(z)\big)\mathbb{p}\mathbb{q}'(z)\right)'}{\mathbb{S}^5(z)\big(z\mathbb{q}'(z)\big)^6}$$

$$-\frac{-\mathbb{I}\big(H(z)\big)z\mathbb{S}'(z)z\mathbb{q}'(z) + \mathbb{S}(z)\mathbb{I}\big(H(z)\big)z^2\mathbb{q}''(z)}{\mathbb{S}^5(z)\big(z\mathbb{q}'(z)\big)^6}$$

has a solution $\mathbb{q}(z)$ with $\mathbb{q}(0) = 1$, which satisfies the condition (4.3.4).

If $H \in \mathbb{H}$, satisfies the condition (5.3.15) and if

$\boldsymbol{\phi}\left(\mathbb{I}^{\mathbb{i}}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+1}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+2}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+3}\big(H(z)\big), \mathbb{I}^{\mathbb{i}+4}\big(H(z)\big)\right)$ is analytic in $\mathbb{U}$

then $\mathbb{I}^{\mathbb{i}}\big(H(z)\big) \prec \mathbb{q}(z)(z \in \mathbb{U})$ and $\mathbb{q}(z)$ is the best dominant.

Proof:

Using Theorem 5.3.2, that $\mathbb{q}(z)$is a dominant of (5.3.15). Since $\mathbb{q}(z)$satisfies (5.3.17), it is also a solution of (5.3.15). Therefore, $\mathbb{q}(z)$will be dominated by all dominant. ■

Conclusion and Future Work

Through this study related to the Geometric theory, which was limited to the complete homogenous symmetric function with the some operator which defined by the effect of composition relation of analytic function on Hardy space. We found types of an ordered cyclic operator using the cyclic subgroups and the effect was obvious by studying the transition behavior of items and sets for each of these types of functions and obtained significant results, In addition to the possibility of applying these results to some of the research presented before In light of the new definitions presented for compactness and some of its types in this study and the results presented therein. Also, we can add several studies, including:

1. With the concept of the ordered cyclic subgroups present in the symmetric polynomials, it is possible to further study the concept of an ordered cyclic subgroups and an ordered cyclic subgroups.
2. Expanding the concept of composition and its types and studying the relationship between them and the algebrical concepts an ordered cyclic subgroups and an ordered cyclic subgroups.

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المستخلص

تهدف هذه الدراسة إلى بناء صيغ جديدة وربط بين نظرية الدالة الهندسية (خاصة نظرية الشطائر) ونظرية فيت في الجبر من خلال متطابقات نيوتن، ودراسة جميع المفاهيم الجبرية التي يمكن دراستها فيها. كذلك العلاقات بين تلك النظريات داخل فضاء هاردي. حيث تمت مراجعة بعض المفاهيم الأساسية لنظرية فيت، بالإضافة إلى مفهوم مجموعة التبديلات والخصائص الجبرية لتلك المجموعة واشتقاق صيغة جديدة للزمرة التناظرية بدلالة نوع جديد من الزمر الجزئية الدائرية المرتبة ، والتي تمثل المرحلة الأولى من هذا العمل.

المرحلة الثانية: تم تقديم مفهوم المصفوفة المصاحبة في فضاء هاردي من خلال دراسة خصائصها الجبرية وكذلك تطرقنا الى مفهوم تركيب المصفوفات وفعل المؤثرات على القوى المختلفة لأثر المصفوفة القطرية وبالتالي تأثيرها على الدوال المتماثلة المتجانسة الكاملة التي تتضمن الصيغ الدائرية.حيث تم تعريف عدة صيغ ومؤثرات منها :

- متطابقات نيوتن باستخدام المصفوفة المرافقة.
- صيغة جديدة للدوال المتناظرة المتجانسة المتكاملة.
- اشتقاق المؤثر الدائري المرتب.
- اشتقاق المؤثر النظير للمؤثر الدائري المرتب.

المرحلة الثالثة : تم تقديم مفهومي (subordination , superordination) للدوال المتناظرة المتجانسة المتكاملة وبالشكل الذي حددته بعض المؤثرات (الزمر الجزئية الدورية المرتبة ، الاشتقاق التفاضلي الاساسي للمؤثر الدوري المرتب).

المرحلة الرابعة: تم تقديم مفهوم (subordination , superordination) من الرتبة الثالثة والرابعة لصنف الدوال المتناظرة المتجانسة المتكاملة وبعض المؤثرات في فضاء هاردي (مؤثر تفاضلي للمؤثر الخطي بالصيغة الدورية للدالة المتماثلة المتجانسة المتكاملة و التركيب لمعكوس المؤثر الدائري المرتب).

من الرتب الثانية والثالثة الرابعة وبصيغ متعددة.

جمهورية العراق

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قسم الرياضيات

نتائج جديدة للخواص الجبرية باستخدام نظرية الدالة الهندسية

اطروحة مقدمة الى

مجلس كلية التربية للعلوم الصرفة في جامعة بابل كجزء من متطلبات نيل درجة الدكتوراه فلسفة في التربية / الرياضيات

من قبل

انتصار عبدالحر كاظم زغير

بأشراف

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