Republic of Iraq
Ministry of Higher Education
and Scientific Research
University of Babylon
College of Education for Pure Sciences
Department of Mathematics

Some Conceptions of Chaotic Properties in G- Space

A Thesis

Submitted to the Council of the College of Education for Pure Sciences, University of Babylon as a Partial Fulfillment of the Requirements for the Degree of Master in Education / Mathematics

By

Kadhim Jawad Abdulhussein Alwan Albediri

Supervised by

Asst. Prof. Dr. May A. A. Alyaseen

2022 A.D. 1444 A.H

بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

(فَتَعَالَى اللَّهُ الْمَلِكُ الْحَقُّ وَلَا تَعْجَلْ بِالْقُرْآنِ مِنْ قَبْلِ أَنْ يُقْضَى إِلَيْكَ وَحْيُهُ وَقُلْ رَبِّ زِدْنِي عِلْمًا)

صدق الله العلي العظيم

سورة طه، الآية ١١٤

Supervisor's Certification

I certify that the thesis entitled **" Some Conceptions of Chaotic Properties in G-space "** by**" Kadhim Jawad Abdulhussein Albediri "** has been prepared under my supervision in Babylon University/ College of Education for Pure Sciences as a partial requirements for the degree of Master in Education / Mathematics.

Signature :

Name : Dr. May A. A. Alyaseen

Title: Assistant Professor

Date: / / 2022

In view of the available recommendation, I forward this thesis for debate by the examining committee.

Signature:

Name: Dr. Azal Jaafar Musa

Head of Mathematics Department

Title: Assistant Professor

Date: / / 2022

Certification of linguistic Expert

I certify that I have read this thesis entitled **" Some Conceptions of Chaotic Properties in G-space "** and corrected its grammatical mistakes; therefore, it has qualified for debate.

Signature:

Name: Dr.

Title:

Address:

Date: / / 2022

Certification of Scientific Expert

I certify that I have read the scientific content of this thesis **“ Some Conceptions of Chaotic Properties in G-space "** and I have approved this dissertation is qualified for debate.

Signature:

Name: Dr.

Title:

Date:

Examining Committee Certification

We certify that we have read the thesis entitled **" Some Conceptions of Chaotic Properties in G-space "** by**" Kadhim Jawad Abdulhussein Albediri"** and as a committee examined the student in its contents and, according to our opinion, it is accepted as a thesis for the degree of Master in Education / Mathematics .

Signature:
Name:
Title:
Date: / / 2022
Chairman

Signature:
Name:
Title:
Date: / / 2022
Member

Signature:
Name:
Title:
Date: / / 2022
Member /

Signature:
Name:
Title:
Date: / / 2022
Member / Supervisor

Approved by the Dean of the College.

Signature:
Name: Dr. Bahaa H. Rabee
Scientific grade: Professor
Date: / /2022
Address: **Dean of the College of Education for Pure Sciences**.

Contents

Dedications

To my Sirs and Lords Imam Hussein and his brother Abbas (PBUH)

Researcher

Kadhim Jawad Albediri

Acknowledgments

First of all praise and thanks go to allah almighty . I am most grateful to my supervisor Dr. May Alyaseen for the continuous support and advise.

Thanks and appreciation to anyone who supported me, especially my family and colleagues, for they all the love and appreciation, and also every one which helped me to do this work.

Researcher

Kadhim Jawad Albediri

Abstract

This study investigated some chaotic properties in $G-$ space. Firstly we generalize some definitions and discuss some chaotic concepts in $G-$space which are commonly used in the mathematical literature especially in dynamical system. Through this study, we present chaotic concepts in $G-$space, and investigate new kinds of transitive such as (strongly transitive, strictly orbit transitive, mild mixing, exactly Devany chaotic, totally minimal, scattering, and $F-$system).

Secondly, we satisfied results including ($G-$transitive points, $G-$isolated points, $G-$quasi-isolated points or without $G-$quisi-isolated points, $G-\omega-$transitive, open set $G-$ transitive, orbit $G-$ transitive). We find that if a map is strictly orbit $G-$transitive then it is open set $G-$transitive, and if a map is orbit $G-$transitive then it is strictly orbit $G-$transitive. The relation between $G-$locally everywhere onto and other chaotic notions like ($G-$transitive, totally $G-$transitive, $G-$strongly blending) are studied, also we find that if a map is $G-$transitive and $G-$strongly blending then it is weakly mixing. And we study another new properties.

Finally, we study the product maps on the $G \times G-$ space. We investigate that if the system $(X_1 \times X_2, f_g \times h_q)$ has one of the chaotic properties that introduced in chapter one, then both (X_1, f_g) and (X_2, h_q) have the same chaotic property.

List of Symbols and Abbreviations

Symbols	Description
G	**Topological group**
$\mathbf{G(x)}$	**The orbit of x on G −space**
$\boldsymbol{Orb.(x)}$	**The orbit of f at $\boldsymbol{x}$**
$\mathbb{Z}$	**The set of Integer numbers**
$\mathbb{N}$	**The set of Natural numbers**
$\mathbf{Int(x)}$	**The interior of the point x**
$\boldsymbol{U, V, W}$	**Open sets**
$\boldsymbol{f_g}$	**Self-map on $\boldsymbol{G}$ −space $\boldsymbol{X}$**
$\boldsymbol{\overline{A}}$	**The closure of the set $\boldsymbol{A}$**
G-T	**G-transitive**
$\mathbf{G-O}$	**Orbit G-transitive**
$\mathbf{G-SO}$	**Strictly orbit G-transitive**
$\boldsymbol{G-l.e.o.}$	**G-locally everywhere onto**
$\boldsymbol{G-\omega-T}$	**$\boldsymbol{G-\omega}$ −transitive**
$\mathbf{G-w.b}$	**$\boldsymbol{G}$ − weakly blending**
$\mathbf{G-s.b}$	**$\boldsymbol{G}$ −strongly blending**
Iso. Pt.	**Isolated point**
Qu. iso. pt.	**Quasi isolated point**

Publication

AL-Bediri, Kadhim J., and May AA AL-Yaseen. "Transitivity with quasi-isolated point in G-space." *Journal of Positive School Psychology* (2022): 214-217.

AL-Bediri, Kadhim J., and May AA AL-Yaseen. " Consequence of transitivity in G-space", To Appear

Introduction

In modern mathematical sciences, study of dynamical systems has been interesting field drawing to many mathematicians. A dynamical system is composed of a number of states that are linked by rules or conditions that determine the current state depending on previous states. In topological dynamical system there are classical types, one is discrete and other is continuous. Recently, lots of interesting study has been done for discrete dynamical systems of the form $z_{i+1} = f(z_i)$, $i = 0, 1, 2, \dots$, where f is continuous self-map. A special class of dynamical systems known as chaotic dynamical systems it has been studied in detail in past. The Chaos theory is one of the most active areas in mathematics, it has been interesting field drawing to many mathematicians due to its interesting applications in various fields such as Physics, Economics and Biology. Understanding this theory will simplify and forecast a complicated system. We must be aware of all inputs and keep control over them just as we must when working with any system. There are many Scientists define the chaos theory in own sense as (Devaney, Gualic, Wiggin, ...). But the most popular and stronger definition is Devaney's definition of chaos. Chaos theory offers completely new concepts and properties of maps in topological dynamical systems.

For instance chaos theory is being studied in other setting also, it has been defined and studied for group actions, it extended to cover the action of more general topological group G. A topological group is made up of two basic concepts : group and topological space. Groups in the algebraic sense with continuous group operations are defined as topological groups. This means that a topological group's topology must be compatible with the structure of the group. $G-$space is one of the spaces that we can study the chaotic in dynamical systems for some properties. Many chaotic properties of maps in $G-$space have been investigated by a many of mathematicians, including (transitivity, minimality, blending (strongly and weakly), mixing, etc).

One of the most important properties in topological dynamics is Topological Transitivity which introduced by G. D. Birkhoff in (1920) [5]. It is an important property in topological dynamics and also in the study of chaos it is play an important role. It also defined on G–space as G −transitive.

In [7] (1995), Crannell introduced the definitions of strongly blending and weakly blending. He showed that in low dimension, maps which are blend are not necessarily transitive. He gave the sufficient condition that if the map is strongly blending, it is also transitive.

In [11] (2012), R. Das and T. Das they studied the notion of transitivity on G −space and obtained its characterization. Also gave the sufficient condition for topological G −transitive of the limit function. They determined the necessary conditions for a sequence's limit function of G − transitive maps to be G − transitive.

In [9] (2012), R. Das presented and gave the definition of G −transitive subset for continuous map on a compact metric G −space. Also, she showed under which a G −transitive subset of sequence of continuous maps f_n is also a G −transitive subset of the limit map f.

In [8] (2012), R. Das Studied and discussed the definitions of some chaotic notation for sequence maps in metric G −space in two ways (iterative and successive) like G −periodic point, G −transitive, G −SDIC and G −chaotic. Also gave some examples of maps which are G −chaotic sequence.

In [10] (2013), R. Das she studied and discussed the sufficient conditions when two maps G −chaotic then their product is G −chaotic, also showed the product of two G −mixing maps is G −mixing.

In [1] (2015), M.Abbas and I. ALshara'a they gave some results and generalized the definitions of locally eventually onto, weakly blending, strongly blending and touhey property maps on G −space. They studied the product maps between blending (strongly and weakly) maps in G −space , also studied the relations between strongly blending and Touhey property with another notions in G −space.

In [12] (2016),M. Garge and R.Das they introduced and study some chaotic properties which are stronger than forms of transitivity on G −space like totally G −transitive,weakly G −mixing, strongly G −mixing and G −minimal. They proof some results that showed the relationship between these conceptions.

In [14] (2017), M.Garg and R.Das studied and discussed G −transitivity and G −minimality maps with the notation G −regular periodic decomposition on G −space. Also obtained by using the notion above that a pseudoequivariant G −minimal map on connected G −space is totally G −transitive.

In [2] (2017), M. Abass and I. AL-Shara'a define and studied the convergence for some chaotic properties on G −space ,also proved some of these properties (minimal, blending, and mixing) of a sequence maps and product on G −space.

In [15] (2017), I.J.kadhiml and S.K. Jabur studied some dynamical concepts on G −space and discussed the notation of Devaney's G −chaotic. Also they defined the equicontinuous maps on G −space and proved some results of this notation.

In [13] (2018), M. Garg and R.Das described various types of map transitivity on G −space like (positive G −transitive, infinite G −transitive, orbit G −transitive, $G-\omega$-transitive and G −transitive point). These notions were studied in detail, and also gave the sufficient conditions under which orbit G −transitive implies $G - \omega$-transitive.

In [18] (2021), K. YAN, Q.LIU and F. ZENG studied several topological concepts for group actions. They defined the nation of scattering, mild mixing and another conceptions for group action, also they showed a totally G −transitive with a dense set of G −periodic points is weakly G −mixing.

Our work divided into three chapters. The first one consists of two sections. Several definitions in G −space are introduce in section one. In section two, we present some examples in G −space, and we also presented some theorems and properties that have been proved in G −space by some mathematical researchers and presented as a review of previous studies.

In chapter two, we have three sections, in section one we study some results about the concepts that equivalent to $G -$ transitive like (orbit $G -$ transitive , strictly orbit G −transitive, positive $G -$ transitive and open set G −transitive). In section two, we discuss some results about quasi-isolated point with conditions. The relationships among several chaotic properties in G −space are studied in section three as: (G −locally everywhere onto, totally $G -$ transitive, G −strongly blending and weakly G −mixing).

In chapter three, the product maps are studied. This chapter divide into two sections, in the first section we discuss the product for some properties in

$G-$ space. In section two some chaotic properties studied for product maps in $G-$ space.

Chapter One

Basic Concepts

1.1- General facts about G −space

Let G be a finitely generated topological group. Let X be topological space induced by matric space. In this section we will present some definitions that we use them to prove our results and properties in G −space.

we recall the definitions as suitable to our definitions form and our writing style.

Definition 1.1.1 [4]

Let X be a Topological space, G be a topological group, and $\theta : G \times X \rightarrow X$ be a mapping. The triple (X, G, θ) is called a G −space if the following three conditions satisfied:

i- $\theta(e, x) = x$, for all $x \in X$, where e is the identity of G.

ii- $\theta(g, \theta(k, x)) = \theta(g * k, x)$, for all $x \in X$ and $g, k \in G$, where $*$ is binary operation of the group G.

iii- θ is continuous

Definition 1.1.2 [10]

Let (X, τ) be topological G −space and $f: X \rightarrow X$ be continuous map. Let x be a point in X then the G −orbit of the point x define as follows :

$$G(x) = \{ \theta(g, x): \text{ for some } g \in G\}$$

Let φ is trivial action of G on X then $orb\ (x) = G\ (x)$ and if φ is nontrivial action then $orb(x) \subseteq G(x)$ as the following example:

Example 1.1.3:[3]

Let $X = \{0,1\}$,$\tau = \{\phi, \{0\}, X\}$ and let $G = \{-1,1\}$ under the discrete topology with the action $\varphi: G \times X \to X$ defined by

$$\varphi(-1, x) = 1 - x,\ \ \varphi(1, x) = x\ , x \in X.$$

Define the identity map $g: X \to X$ by $g(0) = 0, g(1) = 1$. Then g is continuous on X . Thus we have

$Orb(x) = orb(0) = \{0\}$, and $orb(x) = orb(1) = \{1\}$

$\varphi(1,0) = 0, \varphi(-1,0) = 1$ implies $G(x) = G(0) = \{0,1\}$

$\varphi(1,1) = 1,\ \varphi(-1,1) = 0$ implies $G(x) = G(1) = \{0,1\}$.

Thus, $orb(x) \subseteq G(x)$

Definition 1.1.4 [11]

Let (X, τ) be topological G −space and $f: X \to X$ be continuous map Then map f is called G −transitive if for every two non−empty open subsets U and V of X , there is $k \in \mathbb{Z}$ and $g \in G$ such that

$$g.f^k(U) \cap V \neq \emptyset,.$$

Definition 1.1.5 [13]

Let (X, τ) be topological G −space and $f: X \to X$ be continuous map. A point $x \in X$ is G −transitive point of the map f if its $G-$ orbit, $G(x)$, is dense in X.

Definition 1.1.6 [13]

Let (X, τ) be topological G −space and $f: X \rightarrow X$ be continuous map. The map f is G −orbit transitive if there is $x \in X$ such that

$$\overline{G\,(\mathrm{x})} = X.$$

So we introduce a new kind of orbit transitive in G −space named strictly orbit G −transitive as we show in the following definition:

Definition 1.1.7

Let (X, τ) be topological G −space and $f: X \rightarrow X$ be continuous map. The map f is strictly orbit $G-$ transitive if there is a point $x \in X$ and such that

$$\overline{G(f\,(x))} = \{\theta_g(f(x))\ , g \in G\} = X$$

Definition 1.1.8 [12]

Let (X, τ) be topological G −space and $f: X \rightarrow X$ be continuous map. The map f is totally G −transitive if f^k is G-transitive for every $k > 1$.

Definition 1.1.9 [13]

Let (X, τ) be topological G −space and $f: X \rightarrow X$ be continuous map. The map f is open –set G −transitive if for every two non-empty open subsets U and V of X there is $k \in \mathbb{N}$ $g \in G$ such that

$$g.f^k(U) \cap V \neq \emptyset.$$

Definition 1.1.10

Let (X, τ) be topological G −space and $f: X \rightarrow X$ be continuous map. A point $y \in X$ is called $G - \omega$ – limit point of x under f if for any $k \in \mathbb{N}$ and any neighborhood U of y there exists integer $n > k$ such that $g.f^n(x) \in U$. the set of all $G - \omega$ −limit points of x under f denoted by $G - \omega(x, f)$, is called the $G - \omega$ −limit set of x.

Definition 1.1.11 [12]

Let (X, τ) be topological G −space and $f: X \rightarrow X$ be continuous map. The map f is $G - \omega$ −transitive if

$$G - \omega(x, f) = \cap \overline{G_f(f^n(x)} = X, \text{ for some } x \in X.$$

Definition 1.1.12 [13]

Let (X, τ) be topological G −space and $f: X \rightarrow X$ be continuous map. The map f is G –infinite transitive if for every two non-empty open subsets U and V of X, such that

$$N_g^+(U, V) = \{k \in \mathbb{N}: g.f^k(U) \cap V \neq \emptyset, g \in G.\}$$

is infinite.

Also, we define strongly transitive in G −space as follows :

Definition $\mathbf{1.1.13}$

Let $(X,\ \tau)$ be topological G −space and $f: X \rightarrow X$ be continuous map. The map f is strongly G −transitive if for each non−empty V open subset of X, there is $r \in \mathbb{N}$ and $g \in G$ such that

$$X = \bigcup_{k=0}^{r} g.f^k(V)$$

Definition $\mathbf{1.1.14}$ [6]

Let $(X,\ \tau)$ be topological G −space and $f: X \rightarrow X$ be continuous map. The map f is G −mixing if for every two non-empty open subsets U and V of X, there is $m \in \mathbb{N}$ and $g \in G$ such that for each $k \geq m$ satisfying

$$g.f^k(U) \cap V \neq \emptyset.$$

Remark 1.1.15

Let X_1 be G_1-space and X_2 be G_2-space, let U_1, U_2 be nonempty open subsets of X_1 and X_2 respectively. If $f: X_1 \rightarrow X_1$ and $h: X_2 \rightarrow X_2$ be maps ,then

$$(g,q)(f \times h)^k (U_1 \times U_2) = g.f^k(U_1) \times q.h^k(U_2)$$

for all $k \in \mathbb{N}$, $g \in G_1$ and $q \in G_2$.

Definition 1.1.16 [12]

Let (X, τ) be topological $G-$space and $f: X \rightarrow X$ be continuous map. The map f_g is said to be weakly $G-$mixing if the Cartesian product $g.f \times g.f$ is $G \times G$-transitive. In other words, for any open sets U, V, W and Y nonempty of X, such that $U \times V$ and $W \times Y$ are open subsets of $X \times X$ then there is $k \in n\ and\ (p, q) \in G \times G$

$$(g, h)(f \times f)^k (U \times V) \cap W \times Y \neq \emptyset .$$

equivalently,

$$(g, h).(f^k(U) \times f^k(V)) \cap W \times Y \neq \emptyset$$

And this equivalently

$$(g.f^k(U) \times h.f^k(V)) \cap W \times Y \neq \emptyset$$

Equivalently

$$g.f^k(U) \cap W \neq \emptyset \text{ and } h.f^k(V) \cap Y \neq \emptyset$$

We generalized the definition of locally everywhere onto to $G-$ space as we see in the following definition:

Definition 1.1.16

Let (X, τ) be topological $G-$space and $f: X \rightarrow X$ be continuous map. The map f is $G-$locally everywhere onto (simple $G - l.e.o$) if for all nonempty subset U of X, there is $k \in N$ and $g \epsilon G$ such that

$$\theta(g.f^k(U)) = X .$$

Definition 1.1.17 [12]

Let (X, τ) be topological G −space and $f: X \to X$ be continuous map. The map f is G −minimal if for every G −orbit of $x \in X$ is dense in X.

We define the backward minimal in G −space as follows:

Definition 1.1.18

Let (X, τ) be topological G −space and $f: X \to X$ be continuous map The map f is Backward G-minimal if the set

$$\{ y \in X: g.f^k(y) = x, \text{ for some } k \in \mathbb{N} \}$$

is dense in X, for all $x \in X$, $g \in G$

Definition 1.1.19 [1]

Let (X, τ) be topological G −space and $f: X \to X$ be continuous map. The map f is G-weakly blending if for any two non−empty open sets U, V of X there is $k \in N$ such that

$$g_1.f^k (U) \cap g_2.f^k (V) \neq \emptyset. \ g_1, g_2 \epsilon G$$

Definition 1.1.20 [1]

Let (X, τ) be topological G −space and $f: X \to X$ be continuous map. The map f is G −Strongly blending if for any two nonempty open sets U, V of X there is $k \in N$ such that

$$g_1.f^k (U) \cap g_2.f^k (V) = W, g_1, g_2 \epsilon G$$

where W is open subset of X.

Definition $\mathbf{1.1.21}$ [16]

Let (X, τ) be topological G −space and $f: X \to X$ be continuous map. A point $x \in X$ is called G −periodic point of f, if there exist $g \in G$ and $k \in \mathbb{N}$ such that

$$g.f^k(x) = x.$$

Example1.1.22 [3]

Let $X = \{0,1\}$,$\tau = \{ \phi, \{0\}, X \}$ and let $G = \{-1,1\}$ under the discrete topology with the action $\varphi: G \times X \to X$ defined by

$$\varphi(-1, x) = 1 - x,\ \varphi(1, x) = x , x \in X.$$

Define a continuous map $g: X \to X$ by $g(x) = 1 - x$. then we have

$$g(0) = 1,\ \ g^2(0) = g\big(g(0)\big) = g(1) = 0$$

$$g(1) = 0,\ \ g^2(1) = g\big(g(1)\big) = g(0) = 1$$

and $\varphi(1,0) = 0,\ \varphi(1,1) = 1.$

$\varphi(-1,1) = 0, \qquad \varphi^2\,(-1,1) = \varphi\big(\varphi(-1,1)\big) = \varphi(-1,0) = 1.$

$\varphi(-1,0) = 1, \qquad \varphi^2\,(-1,0) = \varphi\big(\varphi(-1,0)\big) = \varphi(-1,1) = 0.$

We notice that 1,0 are periodic point of period 2 under g and also G −periodic point.

Definition $1.1.23$ [1]

Let (X, τ) be topological G −space and $f: X \rightarrow X$ be continuous map. The map f is $G -$ Touhey if for all non−empty open subsets U and V of X there is $k \in \mathbb{N}$ $g \in G$,such that for any $x \in U$

$$\theta(g.f^k(x)) \in V$$

where x is G −periodic point.

We define F −system, totally G −minimal and Exactly devaney chaotic in G −space as follows:

Definition $1.1.24$

Let (X, τ) be topological G −space and $f: X \rightarrow X$ be continuous map. Then f is $G - F$-system if f is totally G-transitive and the set of G −periodic point of f is dense in X.

Definition $1.1.25$

Let (X, τ) be topological G −space and $f: X \rightarrow X$ be continuous map. The map f is G −Exactly Devaney chaotic if f is $G - l.e.o$ and the set of G −periodic point of f is dense in X.

1.2- Transitive, minimal and mixing in G −space

In this section, we introduce some theorems and propositions that were generalized in $G -$ space.

Now, we will describe some implication that have been proven in the literature that some conditions implies G −transitivity in G −space. The authors found that these relations between them as the following propositions :

Proposition 1.2.1:

i. Dense G −orbit implies G −transitive. [16]
ii. G −strongly blending with dense G −periodic points implies G −transitive. [16]
iii. $G -$ minimal implies G −transitive. [14]

In our study, we proof that if a map f is $G - l.e.o.$, then it is G −transitive.

Now the $G -$ transitivity implies some properties as following propositions:

Proposition 1.2.2:

i. G −transitive implies $G(x)$ dense in X with its condition for some x in X. [12]
ii. A map f is $G -$ chaotic iff G −transitive and the set of G −periodic point dense. [16]

iii. A map f is G −touhey if and only if G − transitive and G– periodic points dense. [1]

in [13], The notions like (orbit G −transitive, positive G −transitive, $G - \omega$ −transitive, infinite G −transitive) which are equivalently to G −transitivity have been studied in the following propositions :

Proposition 1.2.3:

i. G −orbit transitive implies $G - \omega$ − transitive.
ii. positive G − transitive implies G − ω − transitive.
iii. Infinite G −transitive implies $G - \omega$ −transitive.
iv. $G - \omega$ − transitive implies infinite G −transitive.
v. $G - \omega$ − transitive implies positive G − transitive.
vi. G −orbit transitive implies positive G −transitive.
vii. positive G −transitive implies G −orbit transitive.

In chapter two, we discuss these above chaotic notions with another conditions.

in [12], authors studied some notations as weakly G − mixing , strongly G − mixing and totally G − transitive. They found the following proposition :

Proposition 1.2.4:

i. G −transitive implies strongly G − mixing with its conditions.
ii. Strongly G − mixing implies weakly G − mixing.
iii. Totally G − transitive implies weakly G − mixing with its conditions.

The relation between totally G –transitive and other notations was studied in the following propositions :

Proposition 1.2.5:

i. $G -$ minimal implies totally $G -$ transitive with its conditions. [14]
ii. Strongly $G -$ mixing implies totally $G -$ transitive. [12]

In [1] authors find that $G -$ strongly blending implies $G -$touhey property with its conditions.

Also some chaotic concepts studied for product maps. In [10] Authors proved the following propositions :

Proposition 1.2.6:

i. product of two $G -$ mixing maps is $G \times G -$ mixing.
ii. If two maps are Devaney's $G -$chaotic and $G -$mixing then the product of them is Devaney's $G \times G -$chaotic.

In [1], The notations ($G -$ strongly blending and $G -$ weakly blending) studied for product maps in the following propositions:

Proposition 1.2.7:

i. Two maps are $G -$weakly blending if and only if their product are $G \times G -$weakly blending.

ii. Two maps are G −srongly blending if and only if their product are $G \times G$ −strongly blending.

iii. The product of two G − strongly blending maps implies $G \times G$ −chaotic.

iv. the product of two G −strongly blending maps with G − periodic points satisfies $G \times G$ −touhey property.[16]

Chapter Two

Essential Properties Of Transitivity in G −Space

2.1- Consequence of Transitivity in G −Space

In this section, we will study some concepts of transitivity in $G-$ space like : orbit G −transitivity, open−set G −transitivity, strictly orbit G −transitivity, $\omega - G$ −transitivity and, we show the relations between them in this section.

Proposition 2.1.1:

Let X be a T_2 perfect G-space. If a map f is G −transitive , then for any open subset U of X the set $N_g^+(U,U) \neq \emptyset$.

Proof:

Let f is G −transitive , then by Definition 1.1.4, for any open subset U of X there is an element $k \in \mathbb{N}$ such that

$$N_g^+(U,U) = \{g.f^k(U) \cap U \neq \emptyset, g \in G\}$$

so by induction we will find a decreasing sequence of open sets $U_k \subset U$ and strictly increasing sequences $n_k \in \mathbb{N}$ such that

$$g.f^{n_k}(U_k) \subset U, g \in G.$$

When $k = 1$, and since X is perfect then U contains two distinct points. Since X is T_2, these have two disjoint neighborhoods $V_1, W_1 \subset U$. There for by G −transitive, there is $n_1 \in \mathbb{N}$ such that

$$g.f^{n_1}(V_1) \cap W_1 \neq \emptyset, g \in G$$

and

$$g.f^{n_1}(W_1) \cap V_1 \neq \emptyset, g \in G$$

Thus,

$$n_1 \in (g.f^{n_1}(V_1) \cap W_1 \neq \emptyset) \cup (g.f^{n_1} (W_1) \cap V_1 \neq \emptyset)$$

so,

$$U_1 = U \cap g.f^{-n_1}(U) , g \in G$$

is open , thus

$$g.f^{n_1}(U_1) \subset U , n_1 > 0$$

because V_1 and W_1 are disjoint. Now , for $n = k$, so by induction we have $U_k \subset U$ and $n_k > 0$ such that

$$g.f^{n_k} (U_k) \subset U.$$

And then when $n = k + 1$, we have $U_{k+1} \subset U$, that means k is infinite.

■

Proposition 2.1.2:

Let X be a perfect G-space. If a map f_g is G −transitive then for every non empty open subsets U, V of X the set $N_g^+(U, V) \neq \emptyset$

Proof:

Let U, V be open subsets of X and non-empty, since it is G −transitive then there is an element $k \in \mathbb{N}$ such that

$$N_g^+(U, V) = \{g.f^k (V) \cap U \neq \emptyset, g \in G\}$$

so we have ,

$$g.f^k (U) \cap V = \{ g.f^k (U) \cap V \cup g.f^{-k} (U) \cap V \}$$

for all $k \in \mathbb{N}$. We must find $k \in \mathbb{N}$, to prove the first part it will be enough to show that for $k \in \mathbb{N}$,

$$g.f^k(U) \cap V \neq \emptyset \text{ if and only if } g.f^k(V) \cap U \neq \emptyset.$$

$\Longrightarrow$ Assume there is $k \in \mathbb{N}$ such that

$$g.f^k(U) \cap V \neq \emptyset,$$

we will find natural number k such that

$$g.f^k(V) \cap U \neq \emptyset.$$

Now, let

$$W = U \cap g.f^{-k}(V) \neq \emptyset$$

is open, so by Proposition 2.1.1, there is elements $k \in \mathbb{N}$ such that

$$g.f^k(W) \cap W \neq \emptyset, g \in G.$$

and thus there is $r > k$ such that

$$g.f^r(W) \cap W \neq \emptyset, g \in G.$$

since

$$g.f^r(W) = g.f^r(U \cap g.f^{-k}(V)) \subset g.f^r(g.f^{-k}(\mathrm{V})) = g.f^{r-k}(V).$$

Thus,

$$g.f^r(W) \subset g.f^{r-k}(V),$$

and $W \subset U$ it follows that

$$g.f^{r-k}(V) \cap U \neq \emptyset,$$

so that $r - k \in \mathbb{N}$.

$\Longleftarrow$ By the same way one can get that $g.f^k(U) \cap V \neq \emptyset$.

To prove the second part, suppose that there exist $k \in \mathbb{N}$ such that

$$g.f^k(U) \cap V \neq \emptyset, g \in G .$$

So,

$$W = U \cap g.f^{-k}(V), g \in G \quad (2.1)$$

is open . For any $r \in \mathbb{N}$,

$$g.f^r(W) \cap W \neq \emptyset,$$

and by (2.1)

$$g.f^k(W) = g.f^k(U) \cap V,$$

then since $g.f^k(W) \subset V$, we have

$$g.f^{r+k}(W) \cap g.f^k(W) \subset g.f^{r+k}(W) \cap V \neq \emptyset.$$

Since $W \subset U$ there exists $r + k \in \mathbb{N}$ such that

$$g.f^{r+k}(U) \cap V \neq \emptyset.$$

Thus,

$$g.f^k(W) \cap (W) \subset g.f^k(U) \cap V \neq \emptyset ,$$

$k \in \mathbb{N}$. And by Proposition 2.1.1, there is element $k \in \mathbb{N}$ such that

$$g.f^k(W) \cap W \neq \emptyset,$$

then there is element $k \in \mathbb{N}$ such that

$$g.f^k(U) \cap V \neq \emptyset. \ g \in G. \quad \blacksquare$$

Proposition 2.1.3 :

Let X be a G-space. If a map f is strictly orbit $G-$ transitive then it is $\omega - G -$transitive

Proof:

Let f be a strictly orbit $G-$ transitive, assume a point $x \in X$, by Definition 1.1.7 , then

$$\overline{G\big(g.f(x)\big)} = X, g \in G \tag{2.2}$$

since f is continuous, then f^n is continuous for any $n \in \mathbb{N}$, $g \in G$. Let U be an open neighborhood of x, by (2.2)

$$G\left(g.f(x)\right) \cap U \neq \emptyset$$

Then there is $y \in X$, such that

$$y \in G\left(g.f(x)\right) \text{ and } y \in U$$

Implies that

$$y \in \{ g.f^k(x): k \in \mathbb{N}, for\ some\ g \in G \} \text{ and } y \in U$$

So, there is $k \in \mathbb{N}$ and $g \in G$, such that

$$y = g.f^k(x) \text{ and } y \in U$$

This mean,

$$g.f^k(x) \in U, g \in G$$

and there is an open neighborhood $U_1$of x such that

$$g.f^k\ (U_1) \subset U,$$

by (2.2) there is $n_1 \in \mathbb{N}$ such that $g.f^{n_1}\ (x) \in U_1$. Let k_1=$k + n_1$, then we have

$$\{g.f^k(x), g.f^{k_1}(x)\} \subset U.$$

Then , we can get a sequence of integers

$$k < k_1 < k_2 \ldots,$$

such that

$$\{g.f^k(x), g.f^{k_1}(x), g.f^{k_2}(x), \ldots\} \subset U,$$

$g \in G$, this means $x \in \omega(x, f)$. By Definition of topological group, and since f is continuous, thus

$$g.f\ (G - \omega\ (x, f)) \subset G - \omega\ (x, f)$$

and

$$G(x) \subset G - \omega\ (x, f),$$

since $G - \omega\ (x, f)$ is closed subset of X. We get that

$$G - \omega\ (x, f) \supset \overline{G(x)} \supset \overline{G\big(f(x)\big)} = X.$$

so f is $G - \omega -$ Transitive .

■

Proposition 2.1.4:

Let X be a G-space, if a map f is $G - \omega -$ transitive then it is open-set $G -$ transitive.

Proof:

Let f be a $G - \omega -$ transitive , and let $x \in X$ then by Definition 1.1.11, $G - \omega(x, f) = X$ for all $g \in G$. Now for any two open subsets U and V non-empty in X, there is $m \in \mathbb{N}$ and

$$n_1 < n_2 < n_3 < \ldots$$

is sequence of positive integers, such that

$$g.f^m(x) \in U \text{ and } \{g.f^{ni}(x), i \in \mathbb{N}\} \subset V,$$

for all $g \in G$. Take $i \in \mathbb{N}$ such that $n_i > m$, then

$$g.f^{n_i}(U) \cap V \supset \{ g.f^{n_i}(x) \} \neq \emptyset.$$

This means

$$g.f^{n_i - m}(U) \cap V \neq \emptyset,$$

by Definition 1.1.9, f is open set $G-$ transitive.

■

Corollary 2.1.5 :

Let X be a G-space. If a map f is strictly orbit $G-$ transitive then it is open set $G-$ transitive

Proof :

Let f be a strictly orbit $G-$ transitive then by Definition 1.1.7, there is a point $x \in X$ such that

$$\overline{G\,(f(x))} = X, g \in G.$$

So, by Proposition 2.1.3, f is $G - \omega -$ transitive, and by Proposition 2.1.4, then for any two open sets U and V non-empty in X, there is a positive integer $m \in \mathbb{N}$ such that

$$g.f^m(U) \cap V \neq \emptyset$$

for all $g \in G$, and hence f is open set $G-$ transitive.

■

Proposition 2.1.6

Let X be a G −space and $f: X \rightarrow X$ be a map. If there exists a point x such that $G - \omega(x, f) = X$, then f is G −transitive.

Proof:

Assume that $G - \omega(x, f) = X$, for some point $x \in X$ and $g \in G$. This means that every nonempty open set contains some points $g. f^r(x)$ with $r \in \mathbb{N}$ and $g \in G$. Let U , V be two non−empty open subset of X, then by definition 1.1.10, there exist integers $r_2 > r_1 > 0$ such that

$$g. f^{r_1}(x) \in U \text{ and } g. f^{r_2}(x) \in V .$$

then $r_2 - r_1 > 0$ and $g. f^{r_2 - r_1}(U)$ contains the point

$$g. f^{r_2 - r_1}(g. f^{r_1}(U)) = g. f^{r_2}(x).$$

Thus,

$$g. f^{r_2 - r_1}(U) \cap V \neq \emptyset ,$$

such that $r_2 - r_1 \in \mathbb{Z}$ and $g \in G$. This implies that f is G −transitive

■

2.2- Transitivity with quasi-isolated point in G −space

In this section, Some chaos characterizations on G −space X are investigate, like : (open-set, orbit, strictly orbit,) G −transitive. We find some results related between them in G −space X.

Let A be a subset of X. Recall that a point $x \in X$ is called an isolated point of A if there is a neighborhood $U$of x in X such that$U \cap A = \{x\}$.[17]

Definition 2.2.1: [17]

A point x in topological space X is called a quasi- isolated point of X if there exists a dense subset A of X such that $x \in A$ and x is isolated in A.

Proposition 2.2.2 : [17]

Let X be a topological space, then X is has no quasi isolated points if and only if for all subset U which are non-empty and open in X, and all dense subset of U has at least two points.

Proposition 2.2.3 :[17]

Let X be a T_1 −space. Then a point $x \in X$ is an iso.pt. of X if and only if x is a qu.iso.pt. of X.

Proposition 2.2.4 :

Let X be G −space without quasi-isolated point, then $f(x)$ is strictly orbit $G -$ transitive if and only if $g.f^n(x)$ is $G -$ transitive point of f.

Proof :

$\Longrightarrow$ Since $f(x)$ is strictly orbit $G-$ transitive, consider $n=1$, prove it by induction, by Definition 1.1.5, there exist $x \in X$ such that

$$G(x) = \{g.f^r(x) : g \in G,\ r \geq 0\} \text{ is dense in } X.$$

This means $f\ (x)$ is G −transitive point. Suppose it is true for $n = r-1$, this means that

$$\overline{G(f_g^{r-1}(x))} = X.$$

Thus, $f^{r-1}\ (x)$ is $G-$ transitive point.

Now, for $n = r$, since

$$g.f^r\ (x) = g.f\ (g.f^{r-1}\ (x))$$

and since $f_g^{r-1}\ (x)$ is $G-$ transitive point. Then by Definition 1.1.5,

$$\overline{G(f(f^{r-1}(x)))} = X,$$

thus $f^r\ (x)$ is $G-$ transitive point of f.

$\Longleftarrow$ Suppose that for any given $n \in \mathbb{N}$, $f^n\ (x)$ is a $G-$ transitive point of f , by Definition 1.1.5, is dense that means $\overline{G(f^n\ (x))} = X$, for some $x \in X$. when $n = 1$, then $\overline{G(f\ (x))} = X$. So, by Definition 1.1.7, $f\ (x)$ is strictly orbit $G-$ transitive of f.

■

Corollary 2.2.5:

Let X be a G −space without qu.iso.pts.. Then f is orbit $G-$ transitive if and only if f is strictly orbit $G-$ transitive.

Proof:

$\Longrightarrow$ Suppose that f is orbit $G-$transitive , then by Definition1.1.6, there exist $x \in X$ such that $\overline{G(x)} = X$. This means the orbit of x is dense, hence x is point transitive. By Proposition 2.2.4, we have that for $n = 1$, f is also $G-$transitive point, this means

$$\overline{G(f\,(x))} = X,$$

so by Definition 1.1.7, f is strictly orbit $G-$transitive , $g \in G$.

$\Longleftarrow$ Suppose that f is strictly orbit $G-$transitive, then by Definition 1.1.7, there exist $x \in X$ such that

$$\overline{G(f\,(x))} = X$$

So, by Proposition 2.2.4, when $n = 1$, f is $G-$transitive point, hence f is $orbit G-$transitive .

■

Proposition 2.2.6 :

Let X be a $G-$space without qu.iso.pts.. If f strictly orbit $G-$transitive, then it is $G-\omega-$transitive

Proof:

Let X be without qu.iso.pts., then by Proposition 2.2.4, all points in the orbit $G\,(x)$ are $G-$transitive points of f and for any $y \in X$, any neighborhood U of y in X, any $n \in \mathbb{N}$, there exists $k \geq n$ such that

$$g.f^k\,(x) \in U, g \in G.$$

So, $y \in G-\omega(x\ ,f)$. That means

$$G - \omega(x, f) = X.$$

Therefore, f is $G - \omega -$ transitive .

■

Proposition 2.2.7 :

Let X be G −space without qu.iso.pts.. Every orbit $G -$ transitive map f is open set−$G -$ transitive.

Proof :

Let x be a $G -$ transitive point of f , so by Definition 1.1.5, $G(x)$ is dense in X for every $g \in G$. For any non−empty open sets U and V in X, there exists n ∈ ℕ such that $g.f^n(x) \in U$, for all $g \in G$. By Proposition 2.2.4, $f^{n+1}(x)$ is also $G -$ transitive point of f. Thus there exists $k \in \mathbb{N}$ such that

$$g.f^k\big(g.f^{n+1}(x)\big) \in \mathrm{V}, g \in G.$$

implies that,

$$g.f^{k+n+1}(x) = g.f^n\,(g.f^{k+1}(x)) \in \mathrm{V}, g \in G.$$

this means that $g.f^{k+1}(U) \cap \mathrm{V} \neq \emptyset$, such that $k + 1 \in \mathbb{N}$. Hence f is open set−$G -$ transitive.

■

2.3- Locally Everywhere Onto and Blending in G −space

In this section we discuss some results that showed the relation among $G-$ locally everywhere onto and ($G-$transitivity, totally $G-$transitive and $G-$strongly blending) and another proposition.

Proposition 2.3.1:

Let $f: X \rightarrow X$ be a map on $G-$space X. If f is $G-l.e.o.$, then it is $G-$ transitive.

Proof :

Let $U \; and \; V$ be two non-empty open sets in X and $g \in G$. Since f is $G-l.e.o,$ then by Definition 1.1.16, there is $n \in N$ and $g \in G$ such that

$$g.f^n(U) = X \tag{2.4}$$

since $V \subset X$ thus, $X \cap V \neq \emptyset$, then by (2.4)

$$g.f^n(U) \cap V \neq \emptyset ,$$

for all $g \in G$, so f is $G-$ transitive.

■

Proposition 2.3.2:

Let $f_g: X \rightarrow X$ be a map on $G-$space X. If $f$is $G-l.e.o.$ then it is totally $G-$ transitive.

Proof :

Let U and V be two non−empty open sets in X, since f is $G-l.e.o,$ then by Definition 1.1.16, there is $n \in N$ such that

$$g.f^n(U) = X,$$

for all $g \in G$. Let $r \geq 1$ be an integer and $g \in G$, then

$$(g.f^r)^n(U) = g.f^r(g.f^n(U)) = g.f^r(X) = X.$$

Hence,

$$(g.f^r)^n(U) \cap V = X \cap V = V \neq \emptyset, \; g \in G$$

Thus

$$(g.f^r)^n(U) \cap V \neq \emptyset, g \in G$$

this mean f^r is $G - T$, for all integer $r \geq 1$. So f is totally $G - T$.

■

Proposition $2.3.3$:

Let $f : X \rightarrow X$ be a map on G −space X. If f is $G - l.e.o.$, then it is G −strongly blending

Proof :

Let U and V be two non-empty open sets in X, since f is $G - l.e.o,$ then by Definition 1.1.16, there is $n \in N$ such that

$$g.f^n(U) = X, \text{ for all } g \in G$$

So, we can find an integer $m > 0$ and $g_1, g_2 \in G$, such that

$$g_1.f^m(U) = X \text{ and } g_2.f^m(V) = X,$$

thus

$$g_1.f^m(U) = g_2.f^m(V) = X.$$

Then

$$g_1 . f^m (U) \cap g_2 . f^m (V) = X,$$

which is open. It means that f is G −strongly blending

■

Remark 2. 3. 4 :

By Proposition 2.3.3 above and since G −strongly blending implies G-weakly blending then f is also G −weakly blending

Theorem 2.3.5:

Let $f : X \to X$ be a map on G −space X. If f is $G-$ transitive and G −strongly blending then f_g is weakly $G-$ mixing.

Proof :

Let U, V, M and W be two nonempty open sets of X. since f_g is G −strongle blending then by Definition 1.1.22, there is $k \in N$ and $g , h \in G$,such that

$$g . f^k (U) \cap h . f^k (V) = M,$$

where M is a nonempty open set of X. Since f is $G-$ transitive, by Definition 1.1.4, then there exist integer r and $g \in G$ such that

$$g . f^r (M) \cap V \neq \emptyset, \qquad (2.5)$$

Now, the sets $U \times V$ and $M \times W$ are non-empty open in $X \times X$, thus there exists $(g, h) \in G \times G$ and $k \in \mathbb{N}$ such that

$$(g,h).(f\times f)^k\,(U\times V)\cap(M\times W)\neq\emptyset$$

By Remark,1.115, we have

$$(g,h).(f^k\times f^k)\,(U\times V)\cap(M\times W)\neq\emptyset$$

Which is equivalent to

$$g.f^k(U)\cap W\neq\emptyset \text{ and } h.f^k(V)\cap M\neq\emptyset,$$

Thus, f is weakly G −mixing.

■

Chapter Three

The Product in $G-$Space

3.1- Some Properties in Product G −space

Let $X_{1,} X_2$ be G_1-space , G_2-space respectively, let $f: X_1 \to X_1$and $h: X_2 \longrightarrow X_2$ be maps. In this section, we present the relationships that exist between the maps $(f \times h)$, f and h ,when any of them are: (G −transitive point, $G -$ isolated point, $G -$ periodic point, $G - \omega$ −limit set, orbit G −transitive, strictly orbit G −transitive, $G - \omega$ – transitive, G −locally everywhere onto, G −backward minimal, G −minimal, totally G −minimal, or G −scatring).

Theorem 3.1.2

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \to X_1$ and $h: X_2 \longrightarrow X_2$ be maps, and let $(x_1, x_2) \in X_1 \times X_2$. If (x_1, x_2) is $G_1 \times G_2 -$ transitive point of $f \times h$, then x_1 is G_1– transitive point of f and x_2 is $G_2 -$ transitive point of h .

Proof :

Suppose that (x_1, x_2) is $G_1 \times G_2$–transitive point of $f \times h$, thus by Definition 1.1.5,

$$\overline{G_1 \times G_2((x_1, x_2), f \times h)} = X_1 \times X_2 \ .$$

It means that (x_1, x_2) is dense. Let $U_{,} V$ be a nonempty open subset of $X_{1,}$ X_2 respectively, thus $(U \times V)$ is nonempty open subset of $X_1 \times X_2$. Since (x_1, x_2) is dense, then

$$G_1 \times G_2((x_1, x_2), f \times h) \cap (U \times V) \neq \phi$$

Thus there exists $k \in \mathbb{N}$ and $(p, q) \in G_1 \times G_2$such that

$$(p, q).(f \times h)^{k}((x_1, x_2)) \in (U \times V),$$

By Remark 1.1.15

$$(p,q)\,(f\times h)^{k}((x_1,x_2)) = (p.f^{k}(x_1), q.h^{k}(x_2)),$$

for all p ∈ G_1 and q ∈ G_2. Hence

$$(p.f^{k}(x_1), q.h^{k}(x_2)) \in (U\times V).$$

Thus $p.f^{k}(x_1) \in U$ and $q.h^{k}(x_2) \in V$. Therefore,

$$U \cap G_1(x_1, f) \neq \phi \text{ and } V \cap G_2(x_2, h) \neq \phi,$$

for all p ∈ G_1 and q∈ G_2. Then by Definition 1.1.2,

$$\overline{G_1(x_1, f)} = X_1 \text{ and } \overline{G_2(x_2, h)} = X_2.$$

Hence x_1 is G_1– transitive point of f and x_2 is G_2- transitive point of h .

■

Theorem 3.1.3:

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \to X_1$ and $h: X_2 \to X_2$ be maps. If $f\times h$ is $G_1\times G_2-\omega-$limit set, then f is $G_1-\omega-$limit set and h is $G_2-\omega-$ limit set**.**

Proof :

Suppose that $f\times h$ is $G_1\times G_2-\omega-$limit set then

$$G_1\times G_2-\omega\big((x_1,x_2), f\times h\big) = X_1\times X_2,$$

Let $y_1 \in X_1$ and $y_2 \in X_2$, $k \in \mathbb{N}$, and U_1 , U_2 be nonempty open subsets of X_1 and X_2 respectively, such that $y_1 \in U_1$ and $y_2 \in U_2$. Thus , there exists $r \in \mathbb{N}$ with $r \geq k$ such that

$$(p,q).(f\times h)^{r}\big((x_1,x_2)\big) \in (U_1\times U_2),$$

By Remark 1.1.15 , we have that for all $p \in G_1$ and $q \in G_2$

$$(p.f^r \times q.h^r)\left((x_1, x_2)\right) = (p.f^r(x_1), qh^r(x_2)) \in (U_1 \times U_2).$$

Thus, $p.f^r(x_1) \in U_1$ and $q.h^r(x_2) \in U_2$, therefore

$$y_1 \in G_1 - \omega(x_1, f) \text{ and } y_2 \in G_2 - \omega(x_2, h).$$

So,

$$G_1 - \omega(x_1, f) = X_1 \text{ and } G_2 - \omega(x_2, h) = X_2,$$

Hence, f is $G_1 - \omega$ −limit set and h is $G_2 - \omega -$ limit set.

■

Theorem 3.1.4:

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \rightarrow X_1$ and $h: X_2 \rightarrow X_2$ be maps. If $f \times h: X_1 \times X_2 \rightarrow X_1 \times X_2$ is $\omega{-}G_1 \times G_2$–transitive, then f is $\omega{-}G_1$–transitive and h is $\omega{-}G_2$–transitive.

Proof:

Suppose that $f \times h$ is $\omega{-}G_1 \times G_2$–transitive, then by Definition 1.1.11, there exists $(x_1, x_2) \in X_1 \times X_2$ such that

$$G_1 \times G_2 - \omega((x_1, x_2), f \times h) = X_1 \times X_2,$$

Thus by Theorem 3.1.3,

$$\omega{-}G_1(x_1, f) = X_1 \text{ and } \omega{-}G_2(x_2, h) = X_2.$$

Hence f is $\omega{-}G_1$–transitive and h is $\omega{-}G_2$–transitive.

■

Theorem 3.1.5:

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \rightarrow X_1$ and $h: X_2 \rightarrow X_2$ be maps. Then x_1 is G_1 −periodic point of f and x_2 is G_2 −periodic point of h if and only if (x_1, x_2) is $G_1 \times G_2$− periodic point of $f \times h$.

Proof:

$\Longrightarrow$ suppose that x_1 be G_1 −periodic point of f and x_2 is G_2 −periodic point of h, thus by Definition 1.1.23, there exists $k_1, k_2 \in \mathbb{N}$ such that

$$p.f^{k_1}(x_1) = x_1 \text{ and } q.h^{k_2}(x_2) = x_2,$$

for all $p \in G_1$ and $q \in G_2$. Let $k = k_1.k_2$, it follows that

$$p.f^k(x_1) = x_1 \text{ and } q.h^k(x_2) = x_2.$$

Hence

$$(p.f^k(x_1), q.h^k(x_2)) = (x_1, x_2),$$

for all $p \in G_1$ and $q \in G_2$. By Remark 1.1.15,

$$(p,q).(f \times h)^k((x_1, x_2)) = (x_1, x_2).$$

Therefore, (x_1, x_2) is periodic point of $f \times h$.

$\Longleftarrow$ Now suppose that (x_1, x_2) is $G_1 \times G_2$− periodic point of $f \times h$. Then by Definition 1.1.21, there exists $(p,q) \in G_1 \times G_2$ and $k \in \mathbb{N}$ such that

$$(p,q).(f \times h)^k((x_1, x_2)) = (x_1, x_2), \text{ for all } p \in G_1 \text{ and } q \in G_2.$$

Thus by Remark 1.1.15,

$$(p,q).(f \times h)^k((x_1, x_2)) = (p.f^k \times q.h^k)((x_1, x_2)) = (x_1, x_2).$$

So, we have that

$$p.f^{k}(x_1) = x_1 \text{ and } q.h^{k}(x_2) = x_2.$$

Therefore x_1 is G_1 −periodic point of f and x_2 is G_2 −periodic point of h for all p ∈ G_1 and q∈ G_2.

■

Theorem 3.1.6

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \to X_1$ and $h: X_2 \longrightarrow X_2$ be maps. If $f \times h: X_1 \times X_2 \to X_1 \times X_2$ is orbit $G_1 \times G_2$–transitive, then f orbit G_1–transitive and h orbit G_2–transitive.

Proof:

Suppose that $f \times h$ is orbit $G_1 \times G_2$–transitive. Then, there exists $(x_1, x_2) \in X_1 \times X_2$ such that,

$$\overline{G_1 \times G_2\big((x_1, x_2), f \times h\big)} = X_1 \times X_2,$$

So, by Theorem 3.1.2, we have

$$\overline{G_1(x_1, f)} = X_1 \text{ and } \overline{G_2(x_2, h)} = X_2.$$

Thus f orbit G_1–transitive and h orbit G_2–transitive.

■

Theorem 3.1.7:

Let X_1 be G_1-space and X_2 be G_2-space, $f : X_1 \to X_1$ and $h : X_2 \to X_2$ be maps. If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is strictly orbit $G_1 \times G_2$–transitive. Then f strictly orbit G_1–transitive and h strictly orbit G_2–transitive.

Proof:

Suppose that $f \times h$ is strictly orbit $G_1 \times G_2$–transitive. Then by Definition1.1.7, there exists $(x_1, x_2) \in X_1 \times X_2$ such that

$$\overline{G_1 \times G_2\Big((f \times h)\big((x_1, x_2)\big)\Big)} = X_1 \times X_2 ,$$

Therefore, by Theorem 3.1.2,

$$\overline{G_1\big(f(x_1)\big)} = X_1 \text{ and } \overline{G_2(h(x_2))} = X_2,$$

and we hence that f strictly orbit G_1–transitive and h strictly orbit G_2–transitive.

■

Theorem 3.1.8 :

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \to X_1$ and $h : X_2 \to X_2$ be maps. If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$ –backward minimal, then f is G_1–backward minimal and h is G_2–backward minimal.

Proof:

Suppose that $f \times h$ is $G_1 \times G_2$ −backward minimal. Let $x_1 \in X_1$ and $x_2 \in X_2$, and U_1 , U_2 be nonempty open sets of X_1 and X_2 respectively. Then $U_1 \times U_2$ is nonempty open subset of $X_1 \times X_2$. By Definition 1.1.18 , we have

$\{(y_1, y_2) \in X_1 \times X_2 : (p, q).(f \times h)^r(y_1, y_2) = (x_1, x_2),\ r \in \mathbb{N}$ and $p \in G_1, q \in G_2\} \cap (U_1 \times U_2) \neq \phi$.

Let $(u_1, u_2) \in U_1 \times U_2$ and $r \in \mathbb{N}$ such that

$$(p, q).(f \times h)^r (u_1, u_2) = (x_1, x_2),$$

for all $p \in G_1$ and $q \in G_2$. It follows that,

$$u_1 \in \{y_1 \in X_1 : p. f^{\,r}(y_1) = x_1,\ \text{for } r \in \mathbb{N} \text{ and } p \in G_1\} \cap U_1 \neq \phi,$$

and

$$u_2 \in \{y_2 \in X_2 : q. f^{\,r}(y_2) = x_2, \text{for } r \in \mathbb{N} \text{ and } q \in G_2\} \cap U_2 \neq \phi.$$

Thus, the set

$$\{y_1 \in X_1 : p. f^{\,r}(y_1) = x_1, \text{for } r \in \mathbb{N} \text{ and } p \in G_1\}$$

is dense in X_1, and also the set

$$\{y_2 \in X_2 : q. f^{\,r}(y_2) = x_2, \text{for } r \in \mathbb{N} \text{ and } q \in G_2\}$$

is dense in X_2. So, we have that f is G_1–backward minimal and h is G_2–backward minimal.

■

Theorem 3.1.9 :

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \to X_1$ and $h: X_2 \to X_2$ be maps. Then f is $G_1 - l.e.o$ and h is $G_2 - l.e.o.$ if and only if $f \times h: X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2 - l.e.o.$

proof :

Let $f: X_1 \to X_1$ be $G_1 - l.e.o$ and $h: X_2 \to X_2$ be $G_2 - l.e.o.$, U, V open nonempty subsets of X_1 and X_2 respectively. Let $U \times V$ subset of $X_1 \times X_2$, so by assumption there exist a positive integers n_1 and n_2 such that

$$p.f^{n_1}(U) = X_1 \text{ and } q.h^{n_2}(V) = X_2,$$

for all $p \in G_1$ and $q \in G_2$. Let $n = \max\{n_1, n_2\}$, so by Remark 1.1.15 we get ,

$$(p,q).(f \times h)^n (U \times V) = (p.f^n \times q.h^n)(U \times V)$$
$$= p.f^n(U) \times q.h^n(V)$$
$$= X_1 \times X_2$$

for all $p \in G_1$ and $q \in G_2$. Which mean that $f \times h$ is $G_1 \times G_2 - l.e.o.$

Conversely, assume that $f \times h$ is $G_1 \times G_2 - l.e.o.$ To show that f is $G_1 - l.e.o$ and h is $G_2 - l.e.o.$ Let U_1 nonempty open subset of X_1 and let V_1 be a nonempty open subset of X_2, for all $U = U_1 \times V_1$ are nonempty open subsets of $X_1 \times X_2$. Since $p.f \times q.h$ is $G_1 \times G_2 - l.e.o.$, then there exists a positive integer r such that

$$X_1 \times X_2 = (p,q).(f \times h)^r (U)$$
$$= (p,q).(f \times h)^r (U_1 \times V_1)$$
$$= (p.f^r \times q.h^r)(U_1 \times V_1)$$
$$= p.f^r(U_1) \times q.h^r(V_1)$$

for all $p \in G_1$ and $q \in G_2$. Thus

$$p.f^r(U_1) = X_1 \text{ and } q.h^r(V_1) = X_2,$$

Hence f is $G_1 - l.e.o$ and h is $G_2 - l.e.o.$ ■

Theorem 3.1.10

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \to X_1$ and $h: X_2 \to X_2$ be maps. If $f \times h: X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–minimal, then f_p is G_1– minimal and h_q is G_2–minimal.

Proof:

Suppose that $f \times h$ is $G_1 \times G_2$–minimal, then by Definition 1.1.17 , for each $x_1 \in X_1$ and $x_2 \in X_2$, such that $(x_1, x_2) \in X_1 \times X_2$ and

$$\overline{G_1 \times G_2\big((x_1, x_2), f \times h\big)} = X_1 \times X_2,$$

Thus, (x_1, x_2) is $G_1 \times G_2$–transitive point, so by Theorem 3.1.2,

$$\overline{G_1(x_1, f)} = X_1 \text{ and } \overline{G_2(x_2, h)} = X_2,$$

Thus, x_1 is G_1–transitive point and x_2 is G_2–transitive point, hence f is G_1– minimal and h is G_2–minimal.

■

We will generalized the definition of mild mixing and Scattering which given in [18]

Definition 3.1.11:

Let (X, τ) be topological G –space and $f: X \to X$ be continuous map. The map f is G –mild mixing if for any G – transitive map $h: Y \to Y$, the map $f \times h$ is $G \times G$ –transitive.

Definition 3.1.12

Let (X, τ) be topological $G-$space and $f: X \to X$ be continuous map. The map f is $G-$Scattering if for any $G-$minimal map $h: Y \to Y$, the map $f \times h$ is $G \times G$-transitive

Theorem 3.1.13:

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \to X_1$ and $h: X_2 \to X_2$ be maps. If $f \times h: X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–mild mixing, then f is G_1–mild mixing and h is G_2–mild mixing.

Proof:

Suppose that $f \times h: X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–mild mixing. Let Y be $G-$space and $g: Y \longrightarrow Y$ be $G-$ transitive map, thus by Definition 3.1.11, we have $(f \times h) \times g$ is $(G_1 \times G_2) \times \mathrm{G}-$ transitive. Let U, V be non−empty open subsets of $(X_1 \times X_2) \times \mathrm{Y}$, then there exists nonempty open subsets $U_{1,} U_2$ of X_1, U_3, U_4 of X_2 and $V_{1,} V_2$ of Y, such that

$$(U_1 \times U_2) \times V_1 \subseteq U \text{ and } (U_3 \times U_4) \times V_2 \subseteq V.$$

Thus, $(U_1 \times U_2)$ and $(U_3 \times U_4)$ are non−empty open subsets of$(X_1 \times X_2)$. By hypothesis, there exists $((u_1, u_2), v_2) \in (U_1 \times U_2) \times V_1$ and $k \in \mathbb{N}$ such that

$$\big((p, q), s\big).((f \times h) \times g)^k \big((u_1, u_2), v_1\big) \in (U_3 \times U_4) \times V_2.$$

By Remark 1.1.15,

$$\big((p, q), s\big).((f^k \times h^k) \times g^k) \big((u_1, u_2), v_1\big) \in (U_3 \times U_4) \times V_2,$$

for all $p \in G_1$, $q \in G_2$ and $s \in G$. Now , for $(u_1, v_1) \in (U_1 \times V_1)$ and $(u_2, v_1) \in (U_2 \times V_1)$,thus

$$(p, s).(f \times g)^k(u_1, v_1) \in (U_3 \times V_2)$$

and

$$(q, s).(h \times g)^k(u_2, v_1) \in (U_4 \times V_2).$$

Therefore,

$$(q, s).(f \times g)^k (U_1 \times U_2) \cap (V_1 \times V_2) \neq \emptyset$$

and

$$(q, s).(h \times g)^k (U_3 \times U_4) \cap (V_1 \times V_2) \neq \emptyset.$$

Thus, $f \times g$ is $G_1 \times G$ –transitive and $h \times g$ are $G_2 \times G$ –transitive . This means, f is G_1–mild mixing and h is G_2–mild mixing.

■

Theorem 3.1.14 :

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \to X_1$ and $h: X_2 \to X_2$ be maps. If $f \times h: X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–scattering , then f is G_1–scattering and h is G_2–scattering.

Proof:

Suppose that $f \times h: X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–scattering. Let Y be G –space and $g: Y \longrightarrow Y$ be G – minimal map, thus by Definition 1.1.12, we have $(f \times h) \times g$ is $(G_1 \times G_2) \times G$ – transitive . Let U, V be nonempty open subsets of $(X_1 \times X_2) \times Y$, then there exists nonempty open subsets U_1, U_2 of X_1, U_3, U_4 of X_2 and V_1, V_2 of Y, such that

$$(U_1 \times U_2) \times V_1 \subseteq U \text{ and } (U_3 \times U_4) \times V_2 \subseteq V.$$

Thus, $(U_1 \times U_2)$ and $(U_3 \times U_4)$ are nonempty open subsets of $(X_1 \times X_2)$. By hypothesis, there exists

$$((u_1, u_2), v_2) \in (U_1 \times U_2) \times V_1 \text{ and } k \in \mathbb{N}$$

such that

$$\big((p, q), s\big).((f \times h) \times g)^k \big((u_1, u_2), v_1\big) \in (U_3 \times U_4) \times V_2,$$

by Remark 1.1.15,

$$\big((p, q), s\big).((f^k \times h^k) \times g^k) \big((u_1, u_2), v_1\big) \in (U_3 \times U_4) \times V_2,$$

for all $p \in G_1$, $q \in G_2$ and $s \in G$. Now , for

$$(u_1, v_1) \in (U_1 \times V_1) \quad \text{and} \quad (u_2, v_1) \in (U_2 \times V_1),$$

thus

$$(p, s).(f \times g)^k (u_1, v_1) \in (U_3 \times V_2)$$

and

$$(q, s).(h \times g)^k (u_2, v_1) \in (U_4 \times V_2).$$

Therefore,

$$(p, s).(f \times g)^k (U_1 \times U_2) \cap (V_1 \times V_2) \neq \emptyset$$

and

$$(q, s).(h \times g)^k (U_3 \times U_4) \cap (V_1 \times V_2) \neq \emptyset.$$

Thus, $f \times g$ is $G_1 \times G$ –transitive and $h \times g$ are $G_2 \times G$ –transitive . This means, f is G_1–scattering and h is G_2–scattering.

■

3.2- Chaotic in product G −space

Let $X_{1,} X_2$ be G_1-space , G_2-space respectively**,** let $f: X_1 \rightarrow X_1$and $h: X_2 \rightarrow X_2$ be maps. In this section, we present the relationships that exist between the maps $(f \times h)$, f and h,when any of them are: G −transitive , totally G −transitive, G −mixing, strongly G −transitive, G −weakly mixing, G −chaotic, G −F-system, G −Exactly Devaney chaotic, G −mild mixing, or G −touhey.

Theorem 3.2.1 :

Let X_1 be G_1-space and X_2 be G_2-space, $f_p: X_1 \rightarrow X_1$ and $h_q: X_2 \rightarrow X_2$be maps. If $f \times h: X_1 \times X_2 \rightarrow X_1 \times X_2$ is $G_1 \times G_2$–transitive then f is G_1–transitive and h is G_2–transitive.

Proof:

Let $f \times h$ is $G_1 \times G_2$–transitive, U_1, U_2 be nonempty open subsets of X_1 and V_1 , V_2 be nonempty open subsets of X_2. Then the sets $U = U_1 \times V_1$ and $V = U_2 \times V_2$ are open in $X_1 \times X_2$. Since $f \times h$ is $G_1 \times G_2$–transitive, then there exists $k \in \mathbb{N}$ such that

$$(p, q).(f \times h)^k (U) \cap V \neq \phi,$$

for all $p \in G_1$ and $q \in G_2$. By Remark 1.1.15, this imply

$$(p, q).(f \times h)^k (U) \cap V = (p, q).(f \times h)^k (U_1 \times V_1) \cap (U_2 \times V_2)$$

$$= (p.f^k(U_1) \times h^k (V_1)) \cap (U_2 \times V_2)$$

$$= (p.f^k(U_1) \cap U_2) \times (q.h^k (V_1) \cap V_2)$$

Since , $(p,q).(f_p \times h_q)^k (U) \cap V \neq \phi$, then it follows that

$$(p.f^k(U_1) \cap U_2) \times (q.h^k(V_1) \cap V_2) \neq \phi,$$

for all $p \in G_1$ and $q \in G_2$. So,

$$p.f^k(U_1) \cap U_2 \neq \phi, \quad (3.1)$$

thus, from (3.1) f is G_1–transitive, and

$$q.h^k(V_1) \cap V_2 \neq \phi, \quad (3.2)$$

thus, from(3.2) h is G_2–transitivity.

■

Theorem 3.2.2 :

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \to X_1$ and $h: X_2 \to X_2$ be maps. If $f \times h: X_1 \times X_2 \to X_1 \times X_2$ is weakly $G_1 \times G_2 -$ mixing then f is weakly $G_1 -$ mixing and h is weakly $G_2 -$ mixing.

Proof:

Suppose that $f \times h$ is weakly $G_1 \times G_2-$ mixing, by Definition 1.1.16, it means that $(p,q).((f \times h) \times (f \times h))$ is $(G_1 \times G_2) \times (G_1 \times G_2) -$ transitive.

Let U_1, U_2, U_3 and U_4 be nonempty open subsets of X_1 and V_1, V_2, V_3 and V_4 be a nonempty open subsets of X_2, then for every pairs $(U_1 \times V_1), (U_2 \times V_2), (U_3 \times V_3)$ and $(U_4 \times V_4)$ of nonempty open subsets of $((X_1 \times X_2) \times (X_1 \times X_2))$ there exists $k \in \mathbb{N}$ such that

$(p,q)\big((f \times h) \times (f \times h)\big)^k ((U_1 \times V_1) \times (U_2 \times V_2)) \cap ((U_3 \times V_3) \times (U_4 \times V_4)) \neq \phi.$

By Remark 1.1.15, we have

$((p,q) \times (p,q)).(f \times h)^k \times (f \times h)^k)((U_1 \times V_1) \times (U_2 \times V_2)) \cap ((U_3 \times V_3) \times (U_4 \times V_4)) \neq \phi.$

Equivalently,

$$(p,q).(f \times h)^k ((U_1 \times V_1)) \cap (U_3 \times V_3) \neq \phi$$

$$(p,q).(f \times h)^k ((U_2 \times V_2)) \cap (U_4 \times V_4) \neq \phi$$

By Remark 1.1.5,

$$(p.f^k (U_1) \times q.h^k (V_1) \cap (U_3 \times V_3) \neq \phi$$

and

$$(p.f^k (U_2) \times q.h^k (V_2) \cap (U_4 \times V_4) \neq \phi$$

Thus, it is equivalent to

$$p.f^k (U_1) \cap U_3 \neq \phi , \qquad (3.3)$$

$$qh^k(V_1) \cap V_3 \neq \phi , \qquad (3.4)$$

$$p.f^k (U_2) \cap U_4 \neq \phi , \qquad (3.5)$$

$$q.h^k(V_2) \cap V_4 \neq \phi , \qquad (3.6)$$

Now, we have that by (3.3) and (3.5)

$$(p.f \times p.f)^k(U_1 \times U_2) \cap (U_3 \times U_4) \neq \phi$$

also, by (3.4) and (3.5)

$$(q.f \times q.h)^k(V_1 \times V_2) \cap (V_3 \times V_4) \neq \phi.$$

For all $p \in G_1$ and $q \in G_2$.

Thus, $f \times f$ is $G_1 \times G_1$ −transitive and $h \times h$ is $G_2 \times G_2$–transitive. Hence, f is weakly $G_1 -$ mixing and h is weakly $G_2 -$ mixing.

■

Theorem 3.2.3 :

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \rightarrow X_1$ and $h_q: X_2 \longrightarrow X_2$ be maps. If $f \times h : X_1 \times X_2 \rightarrow X_1 \times X_2$ is totally $G_1 \times G_2$–transitive, then f is totally G_1–transitive and h is totally G_2–transitive.

Proof:

Suppose that $f \times h$ is totally $G_1 \times G_2$–transitive then by Definition 1.1.8, $(p, q)(f \times h)^s$ is $G_1 \times G_2$–transitive for $s \in \mathbb{N}$. Thus by Remark 1.1.14, $f^s \times h^s$ is $G_1 \times G_2$–transitive. Hence, by Theorem 3.2.1, f^s is G_1–transitive and h^s is G_2–transitive therefore f is totally G_1–transitive and h is totally G_2–transitive.

■

Theorem 3.2.4 :

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \rightarrow X_1$ and $h : X_2 \longrightarrow X_2$be maps. If $f \times h : X_1 \times X_2 \rightarrow X_1 \times X_2$ is strongly$G_1 \times G_2$–transitive, then f is strongly G_1–transitive and h is strongly G_2–transitive.

Proof :

Let U and V be non−empty open subsets of X_1 and X_2 respectively, such that $U \times V$ is nonempty open subset of $X_1 \times X_2$ .Suppose that $f \times h$ is strongly $G_1 \times G_2$–transitive, then by Definition 1.1.13, there exists s ∈ ℕ such that

$$X_1 \times X_2 = \bigcup_{k=0}^{s} (p,q).(f \times h)^k (U \times V),$$

for all $p \in G_1$ and $q \in G_2$. Let $x_1 \in X_1$ and $x_2 \in X_2$. Then there exists $k_1 \in \{0,1,\dots,s\}$ such that

$$(x_1, x_2) \in (p,q).(f \times h)^{k_1} (U \times V),$$

for all $p \in G_1$ and $q \in G_2$. By Remark 1.1.15, we have

$$(x_1, x_2) \in (p,h)(f^{k_1} \times h^{k_1})(U \times V).$$

Thus,

$$(x_1, x_2) \in (p,h).\left(f^{k_1}(U) \times h^{k_1}(V)\right),$$

this means,

$$x_1 \in p.f^{k_1}(U) \text{ and } x_2 \in q.h_q^{k_1}(V)$$

for all $p \in G_1$ and $q \in G_2$. Therefore,

$$x_1 \in \bigcup_{k_1=0}^{s} p.f^{k_1}(U) \text{ and } x_2 \in \bigcup_{k_1=0}^{s} q.h^{k_1}(V).$$

So,

$$X_1 = \bigcup_{k_1=0}^{s} p.f^{k_1}(U) \text{ and } X_2 = \bigcup_{k_1=0}^{s} q.h_q^{k_1}(V)$$

Hence, f is strongly G_1–transitive and h is strongly G_2–transitive.

■

Theorem 3.2.5 :

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \rightarrow X_1$ and $h: X_2 \rightarrow X_2$ be maps. If $f \times h: X_1 \times X_2 \rightarrow X_1 \times X_2$ is $G_1 \times G_2$– Touhey, then f is G_1– Touhey and h is G_2–Touhey.

Proof:

Suppose that $f \times h$ is $G_1 \times G_2$– Touhey. Let U_1, U_2 be nonempty open subsets of X_1 and V_1, V_2 be non−empty open subsets of X_2, then $(U_1 \times U_2)$ and $(V_1 \times V_2)$ are non−empty open set of $(X_1 \times X_2)$. By Definition 1.1.23, there exists a periodic point $(x_1, x_2) \in (U_1 \times U_2)$ and $r \in \mathbb{N}$ such that

$$(p, q).(f \times h)^r (x_1, x_2) \in (V_1 \times V_2),$$

for all $p \in G_1$ and $q \in G_2$. By Remark 1.1.15,

$$(p.f^r (x_1), q.h^r (x_2)) \in (V_1 \times V_2)$$

By Theorem 3.1.5, x_1 and x_2 are periodic points of f and h respectively. Thus

$$p.f^r(x_1) \in (V_1).$$

and

$$q.h^r(x_2) \in (V_2).$$

Therefore, f is G_1– Touhey and h is G_2–Touhey. ■

Lemma 3.2.6 : [1]

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \rightarrow X_1$ and $h: X_2 \rightarrow X_2$ be maps. The set of $G_1 \times G_2$-periodic points are dense of $f \times h$ if and only if for f the sets of G_1-periodic points in X_1 are dense and for h the sets of G_2-periodic points in X_2 are dense.

Theorem 3.2.7 :

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \rightarrow X_1$ and $h: X_2 \rightarrow X_2$ be maps. If $f \times h: X_1 \times X_2 \rightarrow X_1 \times X_2$ is $G_1 \times G_2$–F −system, then f is G_1–F −system and h is G_2–F −system.

Proof:

Suppose that $f \times h$ is $G_1 \times G_2$–F −system, thus by Definition 1.1.24, $f \times h$ is totally $G_1 \times G_2$–transitive and $G_1 \times G_2$–periodic point of $f \times h$ is dense in $X_1 \times X_2$. By Theorem 3.2.3, we have that f and h is totally $G_1 \times G_2$–transitive respectively, and by Lemma 3.2.6, the G_1–periodic point of f is dense in X_1 and the G_2–periodic point of h is dense in X_2. Therefore, we have that f is G_1–F −system and h is G_2–F −system.

■

Theorem 3.2.8 :

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \rightarrow X_1$ and $h: X_2 \rightarrow X_2$ be maps. If $f \times h: X_1 \times X_2 \rightarrow X_1 \times X_2$ is $G_1 \times G_2$–chaotic, then f is G_1–chaotic and h is G_2–chaotic.

Proof:

Suppose that $f \times h$ is $G_1 \times G_2$–chaotic. Then by Theorem 3.2.1, we have that f is G_1–transitive and h is G_2–transitive. Hence by Lemma 3.2.6 for f the sets of G_1-periodic points in X_1 are dense and for h the sets of G_2-periodic points in X_2 are dense. So, f is G_1–chaotic and h is G_2–chaotic.

■

Theorem $3.2.9$:

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \to X_1$ and $h: X_2 \to X_2$ be a maps. If $f \times h: X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–exactly Devany chaotic, then f is G_1- exactly Devany chaotic and h is G_2- exactly Devany chaotic.

Proof:

Let $f \times h: X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–exactly Devany chaotic, then by Definition 1.1.25 , $f \times h$ is $G_1 \times G_2 - l.e.o.$ and the $G_1 \times G_2$–periodic points are dense. So, by Theorem 3.1.9, f is $G_1 - l.e.o.$ and h is $G_2 - l.e.o.$ and by Lemma 3.2.6, the set of G_1–periodic points are dense in X_1 and the set of G_2–periodic points are dense in X_2. Hence f is G_1- exactly Devany chaotic and h is G_2- exactly Devany chaotic.

■

Theorem $3.2.10$:

Let X_1 be G_1-space and X_2 be G_2-space, $f: X_1 \to X_1$ and $h: X_2 \to X_2$ be maps. If $f \times h: X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2 -$ mixing then, f is $G_1 -$ mixing and h is $G_2 -$ mixing.

Proof:

Suppose that $f \times h: X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2-$ mixing. Let U_1, U_2 be nonempty open subset of X_1 and V_1, V_2 be nonempty open subsets of X_2, such that $U_1 \times U_2$ and $V_1 \times V_2$ are non−empty open subsets of $X_1 \times X_2$. Since $f \times h$ is $G_1 \times G_2$–mixing , there is $k \in \mathbb{N}$ such that

$$(p,q).(f \times h)^m(U_1 \times U_2) \cap (V_1 \times V_2) \neq \emptyset$$

for each $m \geq k$, and $p \in G_1$ and $q \in G_2$. Let

$$(x_1, x_2) \in (p,q).(f \times h)^m(U_1 \times U_2) \cap (V_1 \times V_2)$$

Then there exists $(u_1, u_2) \in (U_1 \times U_2)$ such that

$$(p,q).(f \times h)^m(u_1, u_2) = (x_1, x_2).$$

for each $m \geq k$, and $p \in G_1$ and $q \in G_2$. By Remark 1.1.15,

$$(p.f^m(u_1), q.h^m(u_2)) = (x_1, x_2).$$

Hence,

$$p.f^m(u_1) = x_1 \text{ and } q.h^m(u_2) = x_2.$$

Thus,

$$x_1 \in p.f^m(U_1) \cap V_1 \text{ and } x_2 \in q.h^m(U_2) \cap V_2.$$

Hence ,

$$p.f^m(U_1) \cap V_1 \neq \emptyset \text{ and } \in q.h^m(U_2) \cap V_2 \neq \emptyset.$$

for each $m \geq k$, and $p \in G_1$ and $q \in G_2$. Therefore f is $G_1 -$ mixing and h is $G_2 -$ mixing.

■

Conclusions and Future Works

Conclusions

Let X be a $G-$space and $f: X \to X$ be a map, in the first chapter, we presented some definitions of some of the chaotic concepts that circulated in the $G-$space X, and we also generalized some other concepts in the $G-$space as follows :

- Strictly orbit $G-$ transitive
- Strongly $G-$ transitive
- $G-$ locally everywhere onto
- Backward $G-$ minimal
- Totally $G-$ minimal
- $G-F-$system
- $G-$ Exactly Devaney chaotic

Also, we presented some theories and properties that were circulated in the $G-$space as a review.

In the second chapter, we studied some relations of the chaotic concepts as follows:

- If a map f is G-transitive, then there is infinite element $k \in \mathbb{N}$ such that $g.f^k(U) \cap U \neq \emptyset$, where U open nonempty subset of X.
- If a map f is $G-T$, then for every open subsets U, V of X there exist infinite element $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$.
- If a map f is $G-SO$ then it is $G-\omega-T$.

- If a map f is $G - \omega -$T then it is open- set $G - T$.
- If a map f is $G - SO$ then it is open set $G - T$.
- If there is a point x such that $G - \omega(x, f) = X$, then f is $G -$T.
- For any given $x \in X$, $g.f(x)$ is $G - SO$ of f if and only if $g.f^n(x)$ is $G - T$ point of f, for all $g \in G$, where X without qu.iso.pt..
- Then f is $G - OT$ if and only if f is $G - SO$, where X without qu.iso.pt..
- If f $G - SO$, then it is $G - \omega - T$, where X without qu.iso.pt..
- Every $G - OT$ map f is open set$-G - T$, , where X without qu.iso.pt..
- If f is $G - l.e.o.$, then it is $G - T$.
- If f is $G - l.e.o.$ then it is totally $G -$T.
- If f is $G - l.e.o.$, then it is $G - s.b.$
- If f is $G -$T and $G - s.b.$, then f is weakly $G -$ mixing.

Now, let $f: X_1 \to X_1$ and $h: X_2 \to X_2$ be maps. We investigate that in chapter three if the product $f \times h: X_1 \times X_2 \to X_1 \times X_2$ satisfied one of the chaotic properties which were introduced in chapter one, then both f and h has the same property as follows:

- If (x_1, x_2) is $G_1 \times G_2 -$ transitive point of $f \times h$, then x_1 is $G_1 -$ transitive point of f and x_2 is $G_2 -$ transitive point of h .
- If $f \times h: X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2 - \omega -$transitive, then f is $G_1 - \omega -$transitive and h is $G_2 - \omega -$transitive.

- x_1, x_2 are isolated points in X_1, X_2 respectively if and only if (x_1, x_2) is an isolated point in $X_1 \times X_2$.
- x_1 is G_1 −periodic point of f and x_2 is G_2 −periodic point of h if and only if (x_1, x_2) is $G_1 \times G_2$– periodic point of $f \times h$.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is orbit $G_1 \times G_2$–transitive, then f orbit G_1–transitive and h orbit G_2–transitive.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is strictly orbit $G_1 \times G_2$–transitive. Then f strictly orbit G_1–transitive and h strictly orbit G_2–transitive.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$ −backward minimal, then f is G_1–backward minimal and h is G_2–backward minimal.
- f is $G_1 - l.e.o$ and h is $G_2 - l.e.o.$ if and only if $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2 - l.e.o.$
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–minimal, then f is G_1–minimal and h is G_2–minimal.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is totally $G_1 \times G_2$–minimal, then f is totally G_1– minimal and h is totally G_2–minimal.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–scattering , then f is G_1–scattering and h is G_2–scattering.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–transitive then f is G_1–transitive and h is G_2–transitive.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is weakly $G_1 \times G_2$ − mixing then f is weakly G_1 − mixing and h is weakly G_2 − mixing.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is totally $G_1 \times G_2$–transitive, then f is totally G_1–transitive and h is totally G_2–transitive.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is strongly $G_1 \times G_2$–transitive, then f is strongly G_1–transitive and h is strongly G_2–transitive.

- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$– Touhey, then f is G_1–Touhey and h is G_2–Touhey.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–F −system, then f is G_1–F −system and h is G_2–F −system.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–chaotic, then f is G_1–chaotic and h is G_2–chaotic.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–exactly Devany chaotic, then f is G_1– exactly Devany chaotic and h is G_2– exactly Devany chaotic.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2 -$ mixing then, f is $G_1 -$ mixing and h is $G_2 -$ mixing.
- If $f \times h : X_1 \times X_2 \to X_1 \times X_2$ is $G_1 \times G_2$–mild mixing, then f is G_1–mild mixing and h is G_2–mild mixing.

Future Works

- Study the chaotic concepts for sets in G −space.
- Are these chaotic properties satisfy for sequence maps in G −space?
- Are these chaotic properties satisfy for uniformly convergent in G −space?
- Are these chaotic properties satisfy for composition maps in G −space?
- Is there any relation between metric properties and our topological properties in G −space?

References

[1] Abbas, Marwa, and Ifticha r Al-shara'a. "Some Chaotic Properties on-space." *Albahir journal* 2.3, 4 (2015).68-75.

[2] Abbas, Marwa, and Iftichar Al-shara'a."On Chaotic Properties of Convergent Sequences Maps on-space." *Journal of University of Babylon* 25.2 (2017). 357-363.

[3] Abid, Salah H., and Ihsan J. Kadhim. "On expansive chaotic maps in G-spaces." *International Journal of Applied Mathematics Research* 3.3 (2014): 225-232.

[4] Ji, Zhan-Jiang, Cong Zhai, and Chun-ru Zhao. "Point Sets and Periodic Shadowing Property of Topological G-Conjugacy on Metric G-Space." *2018 10th International Conference on Measuring Technology and Mechatronics Automation (ICMTMA)*. IEEE, 2018.

[5] Birkhoff, George David. *Dynamical systems*. Vol. 9. American Mathematical Soc., 1927.

[6] Choi, Taeyoung, and Junhui Kim. "Decomposition theorem on $ G $-spaces." *Osaka Journal of Mathematics* 46.1 (2009): 87-104.

[7] Crannell, Annalisa. "The role of transitivity in Devaney's definition of chaos." The American mathematical monthly 102.9 (1995): 788-793.

[8] Das, Ruchi. "Chaos of a sequence of maps in a metric G-space." *Applied Mathematical Sciences* 6.136 (2012): 6769-6775.

[9] Das, Ruchi. "Transitive Subsets on G-spaces." *Applied Mathematical Sciences* 6.92 (2012): 4561-4568.

[10] Das, Ruchi, and Tarun Das. "Chaos of Product Map on G-Spaces." *International Mathematical Forum*. Vol. 8. No. 13. 2013.

[11] Das, Ruchi, and Tarun Das. "Topological transitivity of uniform limit functions on G-spaces." *Int. Journal of Math. Analysis* 6.30 (2012): 1491-1499.

[12] Garg, Mukta, and Ruchi Das. "Exploring stronger forms of transitivity on G-spaces." *Mathematicki vesnik 69.3(2017), 164-175*.

[13] Garg, Mukta, and Ruchi Das. "Transitivities of maps on G-spaces." *Advances in Pure and Applied Mathematics* 9.2 (2018): 75-83.

[14] Garg, Mukta, and Ruchi Das. "When a Minimal Map Is Totally Transitive on a G-Space." *Journal of Dynamical and Control Systems* 23.3 (2017): 499-508.

[15] Kadhim, Ihsan J., and Sema K. Jebur. "Study of Some Dynamical Concepts in General Topological spaces." *Journal of AL-Qadisiyah for computer science and mathematics* 9.1 (2017): 12-22.

[16] Marwa Abbas. "Convergence and chaoticity on G-space." MSC. Thesis, University of Babylon, College of Education for Pure Sciences, Department of Mathematics, 2015.

[17] Mai, Jie-Hua, and Wei-Hua Sun. "Transitivities of maps of general topological spaces." *Topology and its Applications* 157.5 (2010): 946-953.

[18] Yan, Kesong, Qian Liu, and Fanping Zeng. "Classification of transitive group actions." *Discrete & Continuous Dynamical Systems* 41.12 (2021): 5579.

الخلاصة

هذه الرسالة تدرس بعض الخصائص الفوضوية في فضاء $G -$. بدايةً عممنا بعض التعاريف و ناقشنا بعض المفاهيم في فضاء $G -$ والتي لها استخدام شائع في الرياضيات خصوصا في النظم الديناميكية. خلال هذه الدراسة قدمنا بعض المفاهيم الفوضوية وتحققنا من انواع جديدة من التعدي مثل (Strongly transitive، Strictly orbit transitive، weakly mixing، Exactly Devaney chaotic، Scattering، totally minimal) في فضاء $G -$.

ثانياً، اثبتنا بعض النتائج التي تتضمن (isolated point, quasi-isolated point، Open set G- transitive، Orbit G -transitive). وجدنا انه اذا الدالة Strictly Orbit G- transitive فأنه مجموعة Open set G– transitive، و كذلك اذا كانت الدالة transitive Orbit $G -$ فأنه Strictly Orbit G- transitive–. العلاقة بين G-locally everywhere onto و مفاهيم فوضوية مثل (G– transitive، totally $G -$ transitive، Strongly G –blending) دُرست. كذلك وجدنا اذا التطبيق G-transitive و Strongly G –blending في فضاء $G -$ فأنه يحقق خاصة Weakly G-mixing في فضاء $G -$، كذلك درسنا خصائص جديدة اخرى.

اخيراً، درسنا مفهوم المضروب للدوال في الفضاء– ($G \times G$) . حققنا اذا كان النظام $(X_1 \times X_2 , f_p \times h_q)$ يمتلك واحدة من الخصائص الفوضوية التي قدمت في الفصل الاول فأنه كلاً من (X_1, f_p) و (X_2, h_q) يمتلك نفس الخاصية.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

بعض مفاهيم الخواص الفوضوية في فضاء- جي

رسالة

مقدمة الى مجلس كلية التربية للعلوم الصرفة / جامعة بابل كجزء من متطلبات نيل

درجة الماجستير في التربية / الرياضيات

من قبل

كاظم جواد عبد الحسين علوان البديري

بإشراف

أ. م. د. مي علاء عبد الخالق الياسين

١٤٤٤هـ ٢٠٢٢