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ON TRUNCATED DISTRIBUTIONS

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1444 A.H

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

"مَا أَنْزَلْنَا عَلَيْكَ الْقُرْآنَ لِتَشْقَى * إِلَّا تَذِكْرَةً لِّمَن يَخْشَى *
تَنْزِيلًا مِّمَّنْ خَلَقَ الْأَرْضَ وَالسَّمَاوَاتِ الْعُلَى * الرَّحْمَنِ عَلَى
الْعَرْشِ اسْتَوَى * لَهُ مَا فِي السَّمَاوَاتِ وَمَا فِي الْأَرْضِ وَمَا
بَيْنَهُمَا وَمَا تَحْتَ الثَّرَى * وَإِن تَجَهَّرْ بِالْقَوْلِ فَإِنَّهُ يَعْلَمُ السِّرَّ -
وَأَخْفَى "

صدق الله العلي العظيم

سورة طه (٧-١)

Dedications

to my God,

to my father's soul,

To my family ... husband and children,

for my mother and my brothers,

*To everyone who helped me, even in one
word, we dedicate this modest work.*

Researcher

Kawther Fawzi Hamza Abbas

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Researcher

Kawther Fawzi Hamza Abbas

Abstract

It is complex and difficult to model the natural phenomena easily. For this reason the statistical distributions are considered very significant in describing these phenomena experienced by researchers.

In this study, we concentrated on truncated distributions that have multiple truncations since it suitable for representing natural phenomena that can not be studied or tracked in all its growth and development stages. For example, phenomena that are related to agriculture, airplanes, health, and the environment.

This study introduces multi- truncated distributions depending on double and left truncated. In this study, we constructed the formula to find multi-truncated continuous distribution, with double truncation type and obtained its statistical properties. multi-double truncated Weibull distribution, and multi-double truncated Rayleigh distribution, have been employed. Some examples of multi- double truncated, such as one double-truncated Weibull distribution, one double truncated Rayleigh distribution, and two double truncated Weibull distributions have been introduced, in addition to the properties statistical for each one.

Besides, we utilized the left truncated to introduce a formula that describes multi-truncation of the left truncated, with statistical properties. We explained this construction through two examples, three subintervals being cut off.

The triple truncated Rayleigh distribution, and triple truncated exponential distribution, are presented with, The r^{th} moment about origin and mean, the moment generating and characteristic functions, and order statistics. further, all the above distributions are presented graphically.

We used the maximum likelihood method to estimate the parameter of distributions. Also, the performance of each distribution was tested by calculating some statistical criteria(AIC,CAIC,BIC,and HQIC) and testing the goodness of fit for each distribution, with comparisons between the distributions and testing them on real data for patients infected with COVID-19 in Babylon city.

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List of Symbols and Abbreviations

Symbol	Description
x_T	Random variable after truncated
\mathbb{C}	The set of complex numbers.
$L(\Phi)$	Likelihood function
Φ	The vector of parameters
Ω	The parameter space
skew	The Coefficient of Skewness
kurt	The Coefficient of Kurtosis
CV	Coefficient of variance
erf	Error function
$f(x)$	Probability distribution function
$\gamma(\alpha, x)$	The lower incomplete Gamma
$\Gamma(\alpha, x)$	The upper incomplete Gamma
AIC	Akaike Information Criterion
CAIC	corrected Akaike Information Criterion
BIC	Bayesian Information Criterion
HQIC	Hannan-Quinn information criterion
SSE	Sum of squares due to error
SST	Sum of squares about the mean
RMSE	Root mean squared error
MLE	Maximum likelihood estimator
$g_n(x_T)$	The probability density function of the multi-truncated distribution
$S(T)$	survival function

$\mathbb{S}_n(x_T)$	survival function of multi-truncated
$\mathbb{G}_n(x_T)$	The cumulative function of multi-truncated
$\mathbb{H}(t)$	The hazard function
$\mathbb{H}_n(x_T)$	The hazard function of multi-truncated
$\tilde{\mathbb{H}}(t)$	The reversed hazard function
$\tilde{\mathbb{H}}_n(x_T)$	The reversed hazard function of multi-truncated
$\mathcal{M}(t)$	The moment generating function
$\mathcal{M}_{X_T}(t)$	The moment generating function of the truncated distribution
$\Phi_{X_T}(t)$	The characteristic function of the truncated distribution
MDTCD	the multi-double truncated continuous distribution
$\mathbb{g}_n(x_T)_{[j]}$	The pdf of j^{th} Order Statistics of the multi-truncated continuous distribution
ODTWD	One-Double Truncated Weibull Distribution
TDTWD	Two-Double Truncated Weibull Distribution
ODTRD	One-Double Truncated Rayleigh Distribution
TDTRD	Two-Double Truncated Rayleigh Distribution
MDTRD	Multi-Double Truncated Rayleigh Distribution
DTRD	Double Truncated Rayleigh Distribution
MLTCD	Multi-Left Truncation Continuous Distribution
MLTD	multi-left truncated distribution
LTRD	Left Truncation of Rayleigh Distribution
TLTRD	Triple Left Truncation of Rayleigh Distribution
MLTRD	Multi- Left Truncation of Rayleigh Distribution

LTED	Left Truncated of Exponential Distribution
MLETD	Multi-Left Truncated for Exponential Distribution
MAWE	Maxwell distribution
CH-SQ	Chi-square distribution
GARA	Gamma Rayleigh distribution
MOWE	Modified Weibull distribution
RATO	Rayleigh with two parameters
GERA	generalized Rayleigh

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Publications

- 1.** Kawther F. H., Kareema Abad Al-Kadima, "Estimation of multi-truncated Rayleigh and Weibull distributions with an application", IJNA Publishing, Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 3131-3140.
- 2.** Kawther F. H., Kareema Abad Al-Kadima, "A New Study: Multi Left Truncation of Continuous Distribution" Accepted in Swiss of Economics and Statistics journal our in Volume(158),(2022).
- 3.** Kawther F. H., Kareema Abad Al-Kadima," Multi-Double Truncated of Continuous Distribution ", Accepted in Journal of interdisciplinary mathematics, Taylor & Francis, 31/January/2022.
- 4.** Kawther F. H., Kareema Abad Al-Kadima," Formulation Multi-Double Truncated of Rayleigh Distribution ", Accepted in Journal of AIP conference proceeding journal, 4NICST2022, 13/7/2022.

Introduction

Our life contains many scientific, health, and social problems, which are complex and difficult to easy modeling problems. Therefore, most researchers resorted to studying distributions in different forms to be able to represent these phenomena (problems of life)in several fields. These new forms of distributions are contagious (mixture, compound), and truncated distributions. We focused in this study on truncation. Truncation is a statistical phenomenon that has been shown to occur in a wide range of applications, including survival analysis, epidemiology, and economics. Some mathematical researchers may arise difficulty in using samples whose measurements are outside specific ranges. For example, samples of households with income below a certain level: in this case, the households with income above that specified level are systematically excluded. In many studies, samples may be cut off from the data due to their measurements.

Efron and Petrosian 1999[12] give an example of double-truncation, where an astronomers which cannot determine the luminosity of quasars if they are very dim, or very bright, that is, (left-truncation or right-truncation), respectively. The problem of double-truncation arises especially when the studies are dependent on the statistical inference of the data that relate to phenomena.

The truncated distribution excludes an unimportant part of the data, at the same time they offer more flexibility for modeling lifetime data.

This essential problem in truncated statistics has a long history return back to:

In (1898)[14] when Galton analyzed truncated samples corresponding to the speeds of American trotting horses. In the early 1950s interest in

truncated distributions appeared revived and there begins a record of research into estimation problems regarding truncated distributions.

For discrete distributions, David and Johnson (1952)[10] implemented a truncated Poisson distribution to examine the number of accidents per worker. Sampford (1955)[33] discussed the doubly truncated binomial and negative binomial distributions with examples in biology. In (1980)[22], the mean and variance of partially-truncated distributions, introduced by Lee and Krutchkoff, in this work commercial fish populations are examined for the existence of certain contaminants. after Part of the contaminant distribution above a certain value, that called action level', is eliminated, that is mean contaminant is decreased, although perhaps not optimally decreased. In(1984)[26], Mittal obtained estimating the parameters of some truncated distributions.

Hegde and Dahiya (1989)[16] derived the necessary and sufficient conditions for the existence of the ML estimators in the doubly truncated normal. Mittal and Dahiya (1989)[27] showed that the ML estimator of the scale parameter of the Weibull distribution truncated at T.

Paul G. Staneski obtain an estimation of parameters for truncated Cauchy distribution and it applied on to stock returns, in (1990)[41].

Patrick and Dahiya (1999)[36], considered the right truncated, such that the truncation point is unknown, and obtained the maximum likelihood is a unique solution over parameter space for the exponential distribution.

L.Zaninetti (2002)[23], M.Ali, and Nadarajah in 2006[31] studied the truncated Pareto distribution with applications.

Kotz and Nadarajah (2008)[21] worked on the truncated T and F distributions. the truncated exponential distribution, has been derived by Mathias, (2012)[37]. L.Zaninetti (2013)[24] studied right and left

truncated gamma distribution with application to the stars. Moreia introduced kernel density double truncated data (2012)[35]. On mid-truncated distributions and their applications in order statistics, an attempt by Hassan et al (2011)[18] and in (2013), Mohie et al studied on mid-truncated distributions and their applications [25].

Chattopadhyay et al (2014)[40], obtained fit truncated in the real-world networks. In (2017)[44] Y.Zhany and Nadarajah introduced a paper on partially truncated. In (2018), M.tokmachev provided work in modeling the truncated probability distribution[32], and T.Nick's studied the probability distribution with truncated log and bivariate extensions[43].

In (2019), analysis of double truncated data obtained Achmi and Takeshi [8], Mohamed introduced generalized truncated distributions with Interval [29], truncated Fréchet Pareto, has been obtained by E.mohamed [11], and efficient truncated statistics was an attempt by Vasilis et al[42]. Truncated Rayleigh Pareto distribution is studied by Reyah and Kareema 2020[1]. Amal and Mohamed introduced a new family study of upper truncated, (2020)[15]. E.Khalifa's et al worked in statistical inference truncated of Weibull Rayleigh,(2021)[13], Also, in the same year, a new left truncated Gamble distribution, has been worked by M.Neamah[30].

The main aim of the study: is proposing general formula to describe the multi- truncation distribution and study some the statistical properties.

In this study, three main contributions had been made:

The first contribution, new multiple truncations of a continuous distribution, using the double type with statistical properties. The applications on it, one double truncated Weibull distribution(ODTWD), two double truncated Weibull distribution (TDTWD), one double truncated Rayleigh distribution(ODTRD), and two double truncated Rayleigh distribution(TDTRD).

The second contribution aims to create, a new formula for a multi-left truncated continuous distribution, and some statistical properties of the multi-left truncated continuous distribution derived. Moreover, multi-left truncated Rayleigh distribution and multi-left truncated exponential distribution are introduced with statistical properties as applications to MLTCD. Also, graphs for PDF, CDF, and survival function for both models, have been utilized.

The flexibility of one double truncated Weibull distribution, one double truncated Rayleigh distribution, two double truncated Weibull distribution, triple-left truncated Rayleigh distribution(TLTRD), and triple-left truncated exponential distribution(TLTED) are illustrated throughout comparison with other distributions via applications to real data sets. Moreover, the goodness of fit curve of these models is considered.

This dissertation is organized as follows:

Chapter one contains some basic definitions and concepts that are used. Furthermore, the literature background of the preceding researchers is reviewed.

Chapter two consists of the general formula of double truncated for $(n-1)$ gabs, which have occurred on the original distribution. also, examples are introduced as the one-double truncated Weibull, and two-double truncated Weibull with some statistical properties and introduced some functions graphically.

In chapter three, Rayleigh distribution with multi-double truncated is derived and applied to (one- double truncated Rayleigh) and (two- double truncated Rayleigh). Some statistical properties, survival function, cumulative distribution function, the r^{th} moment about origin, and about mean, the order statistic,..., etc, are found.

Introduction

In chapter four, a new study with multi-left truncated of a continuous distribution is derived and applied to (three-left truncated Rayleigh) and (three-left truncated Exponential). The survival function and cumulative distribution function for these models for different parameters are obtained graphically. Also, the r^{th} moment about origin and mean, moment generating function, and order statistic are introduced.

Chapter five formulates a theory for calculating the maximum likelihood function of all the above models for estimating their parameters and parameters of comparison distributions. Using many criteria to compare our models with others are introduced, moreover, a good fit of data is obtained for different sets. We clarify the utility of the models proposed by application of real data (data of COVID-19) in Babylon city," Babylon health directorate /department of public health / transmitted diseases control section/ epidemiology monitoring unit"

Finally in chapter six, the conclusions and future works, have been presented

CHAPTER ONE

SOME NEEDED DEFINITIONS AND CONCEPTS

This chapter involves some basic concepts and definitions used during our dissertation. the distributions that were utilized and also statistical measurements for data fit have been presented.

1.1 Definitions and Basic Concepts

This section includes some important definitions and concepts

1.1.1 Truncation [4]

Truncation is defined as a phenomenon that some individuals in the population have a minimum chance to be selected because their measurement is very short or very long. That is, the probability that an individual is chosen from a population depends on its measurement.

For example, in many studies of a lifetime, it may be too costly or time-needing to get hold of simple random sampling from a population under study, such as, one cannot identify an individual whose life is shorter or longer than a specific value (*truncation limit*).

Suppose we wish to know the probability density function of the random variable after restricting the observations of x to (a,b) .

Mathematically, let X be a random variable, X distributed according to some probability density function, $f(X)$ define of $(-\infty,\infty)$, and has the cumulative distribution function $F(X)$. then, the probability density of the random variable after restricting the observations of x to (a, b) , is:

$$\mathfrak{g}(x_T) = \frac{f(x_T)}{P(a < x_T < b)} \quad x_T \in (a, b). \quad (1.1)$$

We call $\mathfrak{g}(x_T)$ the truncated from $f(x_T)$. Then truncation means deleting some values of the domain of the random variables.

1.1.2. Left Truncation [4]

Truncated from below, low values of X are cut off so your range is from some minimum value of X to positive infinity $\{X \min, \infty\}$

Now, If we wish to find p.d.f to X define on the subset of Ω , is:

$$\Omega^* = \{x: a < x < \infty\}, a \in \Omega.$$

$$g(x_T) = \frac{f(x_T)}{P(a < x_T < \infty)} = \frac{f(x_T)}{1-F(a)}, x_T \in (a, \infty) \quad (1.2)$$

1.1.3. Statistical Models [34]

A mathematical model is classified through a statistical model, which incorporated a set of statistical hypotheses related to the pointing of a statistical sample taken from a statistical community. The assumptions included in the statistical model describe a collection of probability distributions. That is, the statistical model is a statistical characterization of the construction of a system that aims to correspond to the real phenomena as much as possible. The population model fits with the sample by estimating the special parameters of the model, then possible to attain hypothesis testing, build confidence intervals, and get conclusions about the phenomenon.

1.1.4. Maximum Likelihood Methods [4]

This method is very important for finding estimators of parameters which was proposed by statistician Fisher in 1922.

It is a method generally applicable to the construction of statistical estimators that have “good” properties. The performance of the maximum likelihood estimator is optimal for large enough data and also for the many models of practical interest, The M.L.E is one of the more applicable methods for fitting parametric statistical models to data.

Let $(X_1, X_2, X_3, \dots, X_m)$ are random variables continuous iid, with observations $(x_1, x_2, x_3, \dots, x_m)$ from the density function $f(x; \Phi)$, where Φ is a vector of unknown parameters, $\Phi = (\theta_1, \theta_2, \theta_3, \dots, \theta_k)$, then the likelihood function $L(\Phi) = \prod_{i=1}^m f(x_i; \Phi)$, for all $\Phi \in \Omega$. (1.3)

The M.L.E are those values of the parameters that maximize the likelihood function defined in (1-3). By taking the partial derivative of the log to equation (1-3) for each parameter and setting it equal to origin:

$$\frac{\partial \ell(\Phi)}{\partial \theta_j} = 0, \quad j = 1, 2, \dots, k. \quad (1.4)$$

and then get to values for the parameters that outcome in the highest value for this function. this follows in some equations with an equal number of unknowns, which can be solved at the same time. This can be a relatively simple solution for the partial derivatives, and in others, numerical techniques need to be used.

1.1.5. Order Statistic [18]

Order statistics utilize in statistical theory and play an important role in it. Therefore, the failure times which are observed from a life-testing experiment come in increasing order naturally. Therefore uses the theory of order statistics to lifetime data analyzing the data widely.

Let $(X_1, X_2, X_3, \dots, X_m)$ are random sample continuous iid, each with density function $f(x)$ and cumulative function $F(x)$, then $X_{[1]} \leq X_{[2]} \leq X_{[3]} \leq X_{[4]} \leq \dots \leq X_{[m]}$, be the order statistic, so that the $X_{[j]}$, is the j^{th} smallest order statistics. In particular,

$$X_{[1]} = y_1 = \min(X_1, X_2, X_3, \dots, X_m)$$

$$X_{[m]} = y_m = \max(X_1, X_2, X_3, \dots, X_m)$$

The probability of density function of the j^{th} smallest order statistics is defined as:

$$f_{y_j}(y) = \frac{1}{B(j, m-j+1)} (\mathbb{F}(y))^{j-1} (1 - \mathbb{F}(y))^{m-j} f(y), \quad -\infty < y_i < \infty, \quad (1.5)$$

where $B(j, m - j + 1)$ is the Beta function.

The p.d.f of the minimum order statistics is:

$$f_{y_1}(y) = m(1 - \mathbb{F}(y))^{m-1} f(y), \quad -\infty < y_i < \infty \quad (1.6)$$

The pdf of the maximum order statistics is:

$$f_{y_m}(y) = m(\mathbb{F}(y))^{m-1} f(y), \quad -\infty < y_i < \infty \quad (1.7)$$

1.1.6. Error Function[5]

The error function is the area under the Gaussian curve in the function $y = e^{-x^2}$, and defined as:

$$\text{erf}: \mathbb{C} \rightarrow [0,1]$$

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt, \quad z \in \mathbb{C}, \quad \mathbb{C} \text{ is a complex set} \quad (1.8)$$

This function occurs in probability and distribution. In statistics, for a random variable Y that is normally distributed with mean 0 and standard deviation $(2)^{-1/2}$, $\text{erf}(x)$ is the probability that Y falls in the range $[-x, x]$, for non-negative values of x .

The first who introduced Normal distribution was de Moivre in 1733. In 1812, Laplace used normal distribution in statistical distributions for the purpose analysis the errors. Defined normal distribution function as:

$$f(x; \sigma) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty, \quad \sigma > 0. \quad (1.9)$$

When $\mu = 0$, $\sigma^2 = 1$, the standard normal distribution is:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Gauss studied integral $I = \frac{1}{\sqrt{\pi}} \int e^{-t^2} dt$ in the range $[-x, x]$, and found this integral is the error of Normal distribution,

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

So, if $I = \frac{1}{\sqrt{\pi}} \int e^{-t^2} dt$ defined in the range $[-\infty, \infty]$,

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

That is, $\int_0^{\infty} e^{-t^2} dt = \frac{1}{2}\sqrt{\pi}$,

$$\frac{2}{\sqrt{\pi}} \left(\int_0^x e^{-t^2} dt + \int_x^{\infty} e^{-t^2} dt \right) = 1$$

From the above equation, we can define the complementary error function, denoted by

$$\begin{aligned} \operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt \\ &= 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \\ \operatorname{erfc}(x) &= 1 - \operatorname{erf}(x). \end{aligned} \tag{1.10}$$

1.1.6.1 Some properties of the error function

1- Deferential error function

$$\frac{d}{dx} \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

2- Integration of error function

$$\begin{aligned} \int \operatorname{erf}(x) dx &= x \operatorname{erf}(x) - \frac{2}{\sqrt{\pi}} \int x e^{-x^2} dx + c \\ &= x \operatorname{erf}(x) - \frac{1}{\sqrt{\pi}} e^{-x^2} + c, \text{ where } c \text{ is the constant integration} \end{aligned}$$

3- $\operatorname{erf}(\infty) = 1$

$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2} dx$$

$$\begin{aligned} \text{Take } I &= \int_0^{\infty} e^{-x^2} dx, I^2 = \left(\int_0^{\infty} e^{-x^2} dx \right)^2 = \int_0^{\infty} e^{-x^2} dx \int_0^{\infty} e^{-y^2} dy \\ &= \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy \end{aligned}$$

Now, translate the above integral to polar coordinates by two equations:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2, \quad r dr d\theta = dx dy$$

$$I^2 = \int_{\theta=0}^{\theta=\frac{\pi}{2}} d\theta \int_{r=0}^{r=\infty} e^{-r^2} r dr = \frac{1}{2} \int_{\theta=0}^{\theta=\frac{\pi}{2}} d\theta = \frac{\pi}{4}.$$

$$I^2 = \frac{\pi}{4}, \quad I = \frac{\sqrt{\pi}}{2}$$

$$\text{Then } \operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = 1.$$

4. $\int_a^b e^{-t^2} dt = \frac{\sqrt{\pi}}{2} (\operatorname{erf}(b) - \operatorname{erf}(a))$, $b > a$. represent the probability of an error lying between a and b .

$$\operatorname{erf}(0) = 0,$$

$$\operatorname{erf}(-\infty) = -1,$$

$$\operatorname{erfc}(\infty) = 0,$$

$$\operatorname{erf}(0) = 1$$

$$\operatorname{erf}(-x) = -\operatorname{erf}(x)$$

$$\operatorname{erfc}(-x) + \operatorname{erfc}(x) = 2$$

1.1.6.2 Relationship between error function and incomplete Gamma function

Gamma function is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0 \quad (1.11)$$

The upper incomplete Gamma function is defined as;

$$\Gamma(\alpha, x) = \int_x^{\infty} t^{\alpha-1} e^{-t} dt, \quad x > 0 \quad (1.12)$$

While the lower incomplete Gamma function

$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t^2} dt$, $x > 0$, where α is the complex number and the real part is positive.

Such that,

$$\gamma\left(\frac{1}{2}, x^2\right) = \int_0^{x^2} t^{-\frac{1}{2}} e^{-t} dt = 2 \int_0^x e^{-y^2} dy = \sqrt{\pi} \operatorname{erf}(x)$$

$$\Gamma\left(\frac{1}{2}, x^2\right) = \int_{x^2}^{\infty} t^{-\frac{1}{2}} e^{-t} dt = 2 \int_x^{\infty} e^{-y^2} dy = \sqrt{\pi} \operatorname{erfc}(x)$$

1.1.6.3 Some Properties of Incomplete Gamma Function

1. $\Gamma(\alpha + 1, x) = \alpha \Gamma(\alpha, x) + x^{\alpha-1} e^{-x^2}$
2. $\gamma(\alpha + 1, x) = \alpha \gamma(\alpha, x) - x^{\alpha-1} e^{-x^2}$
3. $\Gamma(\alpha, x) + \gamma(\alpha, x) = \Gamma(\alpha)$
4. $\int_c^d t^{\alpha-1} e^{-t^2} dt = \gamma(\alpha, d) - \gamma(\alpha, c)$,
5. $\Gamma(1, x) = e^{-x}$, $\gamma(1, x) = 1 - e^{-x}$

For any c, d is a complex number not equal to zero.

1.2 Useful Distributions[7]

1.2.1. Rayleigh distribution

Rayleigh distribution is one of the most important distributions in the analysis of data for many fields related to life: medical, and age of production. The Rayleigh distribution was proposed in acoustics and optics by Lord Rayleigh (12 November 1842 – 30 June 1919), a British scientist who made extensive contributions to both theoretical and experimental physics.

Let X be a random variable, has Rayleigh distribution with probability density function,

$$f(x; \sigma) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, I_{(0,\infty)}(x) . \quad (1.13)$$

Such that, $\sigma > 0$ is the scale parameter, and the cumulative distribution function is

$$F(x; \sigma) = 1 - e^{-\frac{x^2}{2\sigma^2}}, \sigma^2 > 0$$

1.2.2 Weibull Distribution:

Weibull distribution is one of the most essential distributions because it is usually used in reliability theory, such as manufacturing and electrical engineering, in the study distribution of lifetime.

The probability distribution function and cumulative function of Weibull distribution are, respectively

$$f(x) = \frac{\delta}{\omega^\delta} x^{\delta-1} e^{-\left(\frac{x}{\omega}\right)^\delta} I_{(0,\infty)}(x) , \quad (1.14)$$

Such that, $\delta > 0$ is shape parameter, and $\omega > 0$ scale parameter.

$$F(x) = 1 - e^{-\left(\frac{x}{\omega}\right)^\delta}, x > 0 \quad \delta, \omega > 0.$$

1.2.3 Exponential distribution

Exponential distribution considers one of the used continuous distributions widely since It is used to formulate the time terminated between events. mathematically define the exponential distribution as:

$$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x) \quad (1.15)$$

$$F(x) = 1 - e^{-\lambda x}, \lambda > 0$$

And the mean of exponential distribution is $\frac{1}{\lambda}$.

1.3 Some Statistical Measurements

A measure of how well the new distribution 'fits' the observed distribution, underlies the essential notion in cases of the goodness of fit statistics. such that, the data which fits the specified distribution know as the ' Goodness of Fit' problem. If the data are univariate or multivariate, continuous or discrete, researchers are focused on determining whether the observed data differs from the expected data.

1.3.1 Information Criterion[17][38]

Akaike Information Criterion (AIC) was introduced firstly in 1973 by Hirotugu Akaike. it is used in statistical models for a given set of data as an estimator of the relative quality, and it is a criterion for model selection defined according to the following equation:

$$AIC = -2\ell^{\wedge} + 2q , \quad (1.16)$$

In some cases, the AIC could have a significant negative bias for such a case Hurvich and Tsai introduce *the corrected Akaike Information criterion CAIC* where the vantage of the CAIC on the AIC is that the expected difference is estimated with a tendency lower than the AIC. The formula of the CAIC can be written as follows:

$$CAIC = -2\ell^{\wedge} + \frac{2qm}{m-q-1} , \quad (1.17)$$

Bayesian Information Criterion (BIC)

This criterion was introduced by Schwarz (1978) and it is used in Bayesian statistics to choose between two or more alternative models. Schwarz is one of the most widely known tools in choosing the best statistical models that are according to simplicity in computational and effective performance. The BIC can be evaluated by the following:

$$BIC = -2\ell^{\wedge} + q\log(m), \quad (1.18)$$

Hannan-Quinn information criterion (HQIC)

$$HQIC = -2\ell^{\wedge} + 2q\log(\log(m)).$$

Where

m: is the sample size of the data application.

q: is the number of parameters in the statistical distribution.

We used AIC, CAIC, BIC, and HQIC to indicate the better fit among the introduced distributions such that a lower value of the used criteria will indicate the best fit among the tested models.

1.3.2 Curve Fitting [6]

To decide, which model exhibited a better curve fit for our models: ODTWD, ODTRD, TDTWD, TLTRD, and TLTED with other models, we examine four quantities:

1. The sum of squares due to error (SSE): this statistic measures the total deviation of the response values from the fit to the response values. It is also called the summed square of residuals.

$$SSE = \sum_{i=1}^n w_i (f_i - \hat{f}_i)^2$$

Where f_i, \hat{f}_i the response is computed by the smoothing spline method and w_i is the smoothing parameter.

2. R-squared: This statistic measures how successful the fit is in explaining the variation of the data. R-square is defined as the ratio of the sum of squares of the regression (SSR) and the total sum of squares (SST). SSR is defined as:

$$SSR = \sum_{i=1}^n w_i (\hat{f}_i - \bar{f})^2$$

SST is also called the sum of squares about the mean, and is defined as:

$$SST = \sum_{i=1}^n w_i (\bar{f} - f_i)^2$$

$$R\text{-square} = SSR/SST = 1 - SSE/SST$$

R-square can take any value between 0 and 1, with a value closer to 1 indicating that a greater proportion of variance is accounted for by the model.

3. Adjusted R-squared: is generally the best indicator of the fit quality when one compares two models. The adjusted R-square statistic can take on any value less than or equal to 1, with a value closer to 1 indicating a better fit.
4. Root mean squared error (RMSE): mean squared error is the measure of the closeness of a regression line to a set of points. It measures the distance called errors and squares to remove any negative signs. Lower values of RMSE mean that the regression line is close to the data points, indicating a better fit.

CHAPTER TWO

MULTI-DOUBLE TRUNCATED CONTINUOUS DISTRIBUTION (MDTCD)

In this chapter, we will introduce multi- truncated continuous distribution of type double truncation. Presented examples about this, such as one-double truncated Weibull distribution, and two-double truncated Weibull distribution. Furthermore, some statistical properties there are introduced.

2.1 Construction of Multi- Double Truncated for Continuous Distribution

Let X , be a random variable, distributed according to the probability density function, $f_X(x)$, and the cumulative distribution function $F_X(x)$. such that, $x \in (-\infty, \infty)$. In this section, we will derive the probability density function $g_n(x_T)$ with multiple truncated for the double truncation.

We partition the original interval into

$$(a_1, b_1) \cup (a_2, b_2) \cup \dots \cup (a_n, b_n),$$

such that, $X_T \in (a_1, b_1) \cup (a_2, b_2) \cup \dots \cup (a_n, b_n)$.

That is,

$$(-\infty, \infty) = (a_1, b_1) \cup (a_2, b_2) \cup \dots \cup (a_n, b_n),$$

$$\text{or } (a_1, b_n) = (-\infty, b_1) \cup (a_2, b_2) \cup \dots \cup (a_n, \infty), \quad (2.1)$$

such that, $\bigcap_{i=1}^n (a_i, b_i) = \emptyset$

that is, the intervals which cutoff are

$(b_1, a_2) \cup (b_2, a_3) \cup \dots \cup (b_{n-1}, a_n)$, and their number is $n - 1$.

Now, we define the truncated distribution for all partitions in these intervals $(-\infty, b_1) \cup (a_2, b_2) \cup \dots \cup (a_n, \infty)$ as:

$\frac{f(x_T)}{F(b_1) - F(a_1)}$, the truncated distribution on the interval (a_1, b_1) ,

$\frac{f(x_T)}{F(b_2) - F(a_2)}$, the truncated distribution on the interval (a_2, b_2) , ..., and so.

Then, the probability density function of multi- double truncated is:

$$\begin{aligned} g_n(x_T) &= \frac{1}{n} \left(\frac{f(x_T)}{F(b_1) - F(a_1)} + \frac{f(x_T)}{F(b_2) - F(a_2)} + \dots + \frac{f(x_T)}{F(b_n) - F(a_n)} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \frac{f(x_T)}{F(b_i) - F(a_i)} \end{aligned}$$

$$\mathbb{g}_n(x_T) = \begin{cases} \frac{1}{n} \sum_{i=1}^n \frac{f(x_T)}{F(b_i) - F(a_i)} & \text{when } a_i < x_T < b_i \\ 0 & \text{otherwise} \end{cases} \quad (2.2)$$

That is

- 1- Since that $f(x_T)$ is a pdf over $(-\infty, \infty)$, $-\infty < x_T < \infty$, $n, F(b_i) - F(a_i) > 0$, then $\mathbb{g}_n(x_T) > 0$ for all $x_T \in (-\infty, \infty)$.
- 2- $\int_{-\infty}^{\infty} \mathbb{g}_n(x_T) dx_T = 1$.

$$\begin{aligned} \text{Since, } \int_{a_i}^{b_i} \mathbb{g}_n(x_T) dx_T &= \frac{1}{n} \sum_{i=1}^n \left(\frac{\int_{a_i}^{b_i} f(x_T) dx_T}{F(b_i) - F(a_i)} \right) \\ &= \frac{1}{n} \left[\frac{\int_{a_1}^{b_1} f(x_T) dx_T}{F(b_1) - F(a_1)} + \frac{\int_{a_2}^{b_2} f(x_T) dx_T}{F(b_2) - F(a_2)} + \dots + \frac{\int_{a_n}^{b_n} f(x_T) dx_T}{F(b_n) - F(a_n)} \right] \\ &= \frac{1}{n} \left[\frac{F(b_1) - F(a_1)}{F(b_1) - F(a_1)} + \frac{F(b_2) - F(a_2)}{F(b_2) - F(a_2)} + \dots + \frac{F(b_n) - F(a_n)}{F(b_n) - F(a_n)} \right] \\ &= \frac{1}{n} (1 + 1 + 1 + \dots + 1) = \frac{1}{n} (n \text{ terms}) = 1. \end{aligned}$$

So, the cumulative function of multi- double truncated continuous distribution (MDTCD) is:

$$\begin{aligned} \mathbb{G}_n(x_T) &= P(X \leq x), \\ &= \\ &= \frac{1}{n} \left[\frac{F(x_T) - F(a_1)}{F(b_1) - F(a_1)} + \frac{F(x_T) - F(a_2) - F(a_1) + F(b_1)}{F(b_2) - F(a_2)} + \right. \\ & \quad \left. \frac{F(x_T) - F(a_3) - F(a_2) - F(a_1) + F(b_2) + F(b_1)}{F(b_3) - F(a_3)} + \right. \\ & \quad \left. \frac{F(x_T) - F(a_4) - F(a_3) - F(a_2) - F(a_1) + F(b_3) + F(b_2) + F(b_1)}{F(b_4) - F(a_4)} \right] \\ & \quad + \dots + \frac{F(x_T) - F(a_n) - F(a_{n-1}) - \dots - F(a_{n-(n-1)}) + F(b_{n-1}) + F(b_{n-2}) + \dots + F(b_{n-(n-1)})}{F(b_n) - F(a_n)} \\ &= \frac{1}{n} \sum_{i=1}^n \left(\frac{F(x_T) I_{(b_i, a_i)} - \sum_{j=1}^n F(a_j) + \sum_{j=1}^n F(b_{j-1})}{F(b_i) - F(a_i)} \right) \\ \mathbb{G}_n(x_T) &= \frac{1}{n} \sum_{i=1}^n \left(\frac{F(x_T) - \sum_{j=1}^n F(a_j) + \sum_{j=1}^n F(b_{j-1})}{F(b_i) - F(a_i)} \right), \text{ where } b_0 = -\infty \quad (2.3) \end{aligned}$$

2.2 Other Functions of Multi- Double Truncated Continuous Distribution (MDTCD)

2.2.1 The survival function of Multi- Double Truncated continuous distribution

Survival analysis is a branch of the branches of statistics, it's employed for data life analysis and engineering field (time modeling); for example in experiments medical, the study time for the injured from diagnosis of injury with any disease until occurs death.

Define the survival function as:

$$\mathbb{S}(T) = P(T > t) = \int_0^{t_{\max}} f(t)dt = 1 - P(T < t),$$

Such that,

$$0 < \mathbb{S}(T) < 1, \mathbb{S}(T \rightarrow \infty) = 0, \mathbb{S}(0) = 1.$$

Now, we find the survival function of the probability density function for multiple double truncated for any continuous distribution, denoted by, $\mathbb{S}_n(x_T)$.

$$\begin{aligned} \mathbb{S}_n(x_T) &= 1 - \mathbb{G}_n(x_T) \\ &= 1 - \frac{1}{n} \sum_{i=1}^n \left(\frac{\mathbb{F}(x_T) - \sum_{j=1}^n \mathbb{F}(a_j) + \sum_{j=1}^n \mathbb{F}(b_{j-1})}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \right) \end{aligned} \quad (2.4)$$

If $n = 2$

$$\begin{aligned} \mathbb{S}_2(x_T) &= \left\{ 1 - \frac{1}{2} \left[\frac{\mathbb{F}(x_T) - \mathbb{F}(a_1)}{\mathbb{F}(b_1) - \mathbb{F}(a_1)} + \frac{\mathbb{F}(x_T) - \mathbb{F}(a_2) - \mathbb{F}(a_1) + \mathbb{F}(b_1)}{\mathbb{F}(b_2) - \mathbb{F}(a_2)} \right] \right\} \\ &= \\ &= \frac{2\mathbb{F}(b_1)\mathbb{F}(b_2) - \mathbb{F}(b_1)\mathbb{F}(a_2) - \mathbb{F}(b_2)\mathbb{F}(a_1) + \mathbb{F}(b_1)\mathbb{F}(a_1) + \mathbb{F}(x_T)[\mathbb{F}(a_1) + \mathbb{F}(a_2) - \mathbb{F}(b_1) - \mathbb{F}(b_2)] + \mathbb{F}(a_1)^2 + \mathbb{F}(b_1)^2}{2\mathbb{F}(b_1)\mathbb{F}(b_2) - 2\mathbb{F}(b_1)\mathbb{F}(a_2) - 2\mathbb{F}(b_2)\mathbb{F}(a_1) + \mathbb{F}(a_2)\mathbb{F}(a_1)} \end{aligned}$$

2.2.2 Hazard Function of Multi- Double Truncated Continuous Distribution

The hazard function is the failure rate function and it is interpreted as the probability that the object (item) will fail through $(t + \Delta t]$ for small Δt , given, It's still working until t , denoted by $\mathbb{H}(t)$. [34][4]

$$\int_{-\infty}^{\infty} \mathbb{H}(t) dt = 0, \quad \mathbb{H}(t) \geq 0, \text{ for all } t.$$

$$\begin{aligned} \mathbb{H}(t) &= \lim_{\Delta t \rightarrow 0} \frac{P(t < T < t + \Delta t | (T > t))}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{P(t < T < t + \Delta t / P(T > t))}{\Delta t} \\ &= \frac{1}{P(T > t)} \lim_{\Delta t \rightarrow 0} \frac{P(t < T < t + \Delta t)}{\Delta t} \end{aligned}$$

$\mathbb{H}(t) = \frac{f(t)}{S(t)}$, is the ratio of the probability function to the survival function.

Now, we find the hazard function of the probability density function for multiple double truncated for any continuous distribution, denoted by, $\mathbb{H}_n(x_T)$.

$$\mathbb{H}_n(x_T) = \frac{g_n(x_T)}{S_n(x_T)} = \frac{\frac{1}{n} \sum_{i=1}^n \frac{f(x_T)}{F(b_i) - F(a_i)}}{1 - \frac{1}{n} \sum_{i=1}^n \left(\frac{(F(x_T) - \sum_{j=1}^n F(a_j) + \sum_{j=1}^n F(b_{j-1}))}{F(b_i) - F(a_i)} \right)} \quad (2.5)$$

If $n = 2$,

$$\mathbb{H}_2(x_T) = \frac{\frac{f(x_T)}{2(F(b_1) - F(a_1))} + \frac{f(x_T)}{2(F(b_1) - F(a_1))}}{1 - \frac{1}{2} \left[\frac{F(x_T) - F(a_1)}{F(b_1) - F(a_1)} + \frac{F(x_T) - F(a_2) - F(a_1) + F(b_1)}{F(b_2) - F(a_2)} \right]}$$

$\mathbb{H}_2(x_T) =$

$$\frac{\frac{f(x_T)}{2(F(b_1) - F(a_1))} + \frac{f(x_T)}{2(F(b_1) - F(a_1))}}{\frac{2F(b_1)F(b_2) - F(b_1)F(a_2) - F(b_2)F(a_1) + F(b_1)F(a_1) + F(x_T)[F(a_1) + F(a_2) - F(b_1) - F(b_2)] + F(a_1)^2 + F(b_1)^2}{2F(b_1)F(b_2) - 2F(b_1)F(a_2) - 2F(b_2)F(a_1) + F(a_2)F(a_1)}}$$

2.2.3 Reverse Hazard function of Multi- Double Truncated Continuous Distribution

This function is very important, it is used for analyzing data in the presence of left-censored observations and it is used to discuss the age distribution with an inverse timescale, and it defines as the ratio of the PDF to CDF, denoted by $\check{\mathbb{H}}(t)$, [4][34].

$$\check{\mathbb{H}}_n(X_T) = \frac{\mathbb{g}_n(X_T)}{\mathbb{G}_n(X_T)} = \frac{\frac{1}{n} \sum_{i=1}^n \frac{f(x_T)}{\mathbb{F}(b_i) - \mathbb{F}(a_i)}}{\frac{1}{n} \sum_{i=1}^n \left(\frac{\mathbb{F}(x_T) - \sum_{j=1}^n \mathbb{F}(a_j) + \sum_{j=1}^n \mathbb{F}(b_{j-1})}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \right)} \quad (2.6)$$

2.2.4 The r^{th} Moment about of Multi- Double Truncated distribution

$$\begin{aligned} E(X_T^r) &= \int_{a_1}^{b_n} x_T^r (\mathbb{g}_n(x_T)) dx_T = \frac{1}{n} \sum_{i=1}^n \left(\frac{\int_{a_i}^{b_i} x^r f(x_T) dx_T}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \right) \\ &= \frac{1}{n} \left[\frac{\int_{a_1}^{b_1} x^r f(x_T) dx_T}{\mathbb{F}(b_1) - \mathbb{F}(a_1)} + \frac{\int_{a_2}^{b_2} x^r f(x_T) dx_T}{\mathbb{F}(b_2) - \mathbb{F}(a_2)} + \dots + \frac{\int_{a_n}^{b_n} x^r f(x_T) dx_T}{\mathbb{F}(b_n) - \mathbb{F}(a_n)} \right] \\ &= \frac{1}{n} \left[\frac{b_1^r \mathbb{F}(b_1) - a_1^r \mathbb{F}(a_1) - r \int_{a_1}^{b_1} x^{r-1} \mathbb{F}(x_T) dx_T}{\mathbb{F}(b_1) - \mathbb{F}(a_1)} + \dots + \right. \\ &\quad \left. \frac{b_2^r \mathbb{F}(b_2) - a_2^r \mathbb{F}(a_2) - r \int_{a_2}^{b_2} x^{r-1} \mathbb{F}(x_T) dx_T}{\mathbb{F}(b_2) - \mathbb{F}(a_2)} + \dots \right. \\ &\quad \left. \dots \frac{b_n^r \mathbb{F}(b_n) - a_n^r \mathbb{F}(a_n) - r \int_{a_n}^{b_n} x^{r-1} \mathbb{F}(x_T) dx_T}{\mathbb{F}(b_n) - \mathbb{F}(a_n)} \right] \\ E(X_T^r) &= \frac{1}{n} \sum_{i=1}^n \left(\frac{b_i^r \mathbb{F}(b_i) - a_i^r \mathbb{F}(a_i) - r \int_{a_i}^{b_i} x^{r-1} \mathbb{F}(x_T) dx_T}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \right) \quad (2.7) \end{aligned}$$

2.2.5 Some examples of the r^{th} moment about Origin for Multi- Double Truncated Exponential Distributions

If take $X \sim \exp(\lambda)$, then

Case1: when $n=1$, $r=1$ (this special case: meaning of double exponential truncated), first moment.

$$E(X_T) = \frac{e^{-\lambda a_1} \left(a_1 + \frac{1}{\lambda} \right) - e^{-\lambda b_1} \left(b_1 + \frac{1}{\lambda} \right)}{e^{-\lambda a_1} - e^{-\lambda b_1}}$$

Case 2: when $n=1, r=2$, second moment

$$E(X_T^2) = \frac{e^{-\lambda b_1} \left(\frac{2}{\lambda} - \frac{2b_1}{\lambda} - b_1^2 \right) - e^{-\lambda a_1} \left(\frac{2}{\lambda} - \frac{2a_1}{\lambda} - a_1^2 \right)}{e^{-\lambda a_1} - e^{-\lambda b_1}}$$

Case 3: when $n=1, r=3$, the third moment

$$\begin{aligned} E(X_T^3) &= \frac{b_1^3 F(b_1) - a_1^3 F(a_1) - (3) \int_{a_1}^{b_1} x^{3-1} F(x_T) dx_T}{e^{-\lambda a_1} - e^{-\lambda b_1}} \\ &= \frac{1}{e^{-\lambda a_1} - e^{-\lambda b_1}} \left[a_1^3 e^{-\lambda a_1} - b_1^3 e^{-\lambda b_1} \right. \\ &\quad \left. - \frac{3}{\lambda} (b_1^2 e^{-\lambda b_1} - a_1^2 e^{-\lambda a_1}) + \frac{6}{\lambda^2} (b_1 e^{-\lambda b_1} - a_1 e^{-\lambda a_1}) + \right. \\ &\quad \left. \frac{6}{\lambda} (e^{-\lambda b_1} - e^{-\lambda a_1}) \right]. \end{aligned}$$

Case 4: when $n=2, r=2$ (one-double truncated Exponential)

$$\begin{aligned} E(X_T^2) &= \frac{1}{2} \left(\frac{b_1^2 F(b_1) - a_1^2 F(a_1) - 2 \int_{a_1}^{b_1} x^{2-1} F(x_T) dx_T}{F(b_1) - F(a_1)} \right) \\ &\quad + \frac{1}{2} \left(\frac{b_2^2 F(b_2) - a_2^2 F(a_2) - 2 \int_{a_2}^{b_2} x^{2-1} F(x_T) dx_T}{F(b_2) - F(a_2)} \right) \\ E(X_T^2) &= \frac{e^{-\lambda a_1} \left(a_1 + \frac{1}{\lambda} \right) - e^{-\lambda b_1} \left(b_1 + \frac{1}{\lambda} \right)}{2(e^{-\lambda a_1} - e^{-\lambda b_1})} + \frac{e^{-\lambda a_2} \left(a_2 + \frac{1}{\lambda} \right) - e^{-\lambda b_2} \left(b_2 + \frac{1}{\lambda} \right)}{2(e^{-\lambda a_2} - e^{-\lambda b_2})}. \end{aligned}$$

Case 5: when $n=3, r=3$

$$\begin{aligned} E(X_T^3) &= \frac{b_1^3 F(b_1) - a_1^3 F(a_1) - 3 \int_{a_1}^{b_1} x^2 F(x_T) dx_T}{3(e^{-\lambda a_1} - e^{-\lambda b_1})} + \frac{b_2^3 F(b_2) - a_2^3 F(a_2) - 3 \int_{a_2}^{b_2} x^2 F(x_T) dx_T}{3(e^{-\lambda a_2} - e^{-\lambda b_2})} + \\ &\quad \frac{b_3^3 F(b_3) - a_3^3 F(a_3) - 3 \int_{a_3}^{b_3} x^2 F(x_T) dx_T}{3(e^{-\lambda a_3} - e^{-\lambda b_3})} \\ &= \frac{a_1^3 e^{-\lambda a_1} - b_1^3 e^{-\lambda b_1} - \frac{3(b_1^2 e^{-\lambda b_1} - a_1^2 e^{-\lambda a_1})}{\lambda} + \frac{6(b_1 e^{-\lambda b_1} - a_1 e^{-\lambda a_1})}{\lambda^2} + \frac{6(e^{-\lambda b_1} - e^{-\lambda a_1})}{\lambda}}{3(e^{-\lambda a_1} - e^{-\lambda b_1})} + \end{aligned}$$

$$+ \frac{a_2^3 e^{-\lambda a_2} - b_2^3 e^{-\lambda b_2} - \frac{3(b_2^2 e^{-\lambda b_1} - a_2^2 e^{-\lambda a_2})}{\lambda} + \frac{6(b_2 e^{-\lambda b_2} - a_2 e^{-\lambda a_2})}{\lambda^2} + \frac{6(e^{-\lambda b_2} - e^{-\lambda a_2})}{\lambda}}{3(e^{-\lambda a_2} - e^{-\lambda b_2})} +$$

$$+ \frac{a_3^3 e^{-\lambda a_3} - b_3^3 e^{-\lambda b_3} - \frac{3(b_3^2 e^{-\lambda b_3} - a_3^2 e^{-\lambda a_3})}{\lambda} + \frac{6(b_3 e^{-\lambda b_3} - a_3 e^{-\lambda a_3})}{\lambda^2} + \frac{6(e^{-\lambda b_3} - e^{-\lambda a_3})}{\lambda}}{3(e^{-\lambda a_3} - e^{-\lambda b_3})}$$

2.2.6 The r^{th} Moment about a Mean of Multi- Double Truncated Continuous Distribution

$$\begin{aligned} E(X_T - \mu)^r &= \int_{-\infty}^{\infty} (x_T - \mu)^r \mathfrak{G}_n(x_T) dx_T \\ &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} (\mu)^{r-j} \mu'_j \\ &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} (\mu)^{r-j} \int_{-\infty}^{\infty} x_T^j \mathfrak{G}_n(x_T) dx_T \\ &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} (\mu)^{r-j} \left(\frac{1}{n} \sum_{i=1}^n \left(\frac{\int_{a_i}^{b_i} (x_T)^j \mathbb{F}(x_T) dx_T}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \right) \right) \end{aligned}$$

Where $\mu'_j = \frac{1}{n} \sum_{i=1}^n \left(\frac{(b_i)^j \mathbb{F}(b_i) - (a_i)^j \mathbb{F}(a_i) - j \int_{a_i}^{b_i} (x_T)^{j-1} \mathbb{F}(x_T) dx_T}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \right)$ (2.8)

2.2.7 Some Examples of the r^{th} Moment About Origin Multi-Double Truncated Distributions

If $X \sim U(a,b)$, that is, the PDF and CDF of X are:

$$f(x) = \frac{1}{b-a}, \quad \mathbb{F}(x) = \frac{x-a}{b-a}, \text{ respectively.}$$

Case 1: when $n = 1, r = 2$

$$\begin{aligned} E(X_T - \mu)^2 &= \sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} (\mu)^{2-j} \left(\frac{1}{n} \sum_{i=1}^n \left(\frac{(b_i)^j \mathbb{F}(b_i) - (a_i)^j \mathbb{F}(a_i) - j \int_{a_i}^{b_i} (x_T)^{j-1} \mathbb{F}(x_T) dx_T}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \right) \right) \\ &= \frac{1}{\mathbb{F}(b_1) - \mathbb{F}(a_1)} \left[\left(\left(\frac{3(\mu)^2 - 6(\mu)^2 + 2b_1^2 + b_1 a_1 + a_1^2}{3} \right) \right) \right] \\ &= \frac{1}{\mathbb{F}(b_1) - \mathbb{F}(a_1)} \left[\left(\left(\frac{b_1^2 - 2b_1 a_1 + a_1^2}{12} \right) \right) \right] \end{aligned}$$

Case 2: when $n = 2, r = 2$

$$\begin{aligned}
E(X_T - \mu)^2 &= \\
&\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} (\mu)^{2-j} \left(\frac{1}{2} \right) \sum_{i=1}^2 \left(\frac{(b_i)^j \mathbb{F}(b_i) - (a_i)^j \mathbb{F}(a_i) - j \int_{a_i}^{b_i} (x_T)^{j-1} \mathbb{F}(x_T) dx_T}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \right) \\
&= \frac{1}{2(\mathbb{F}(b_1) - \mathbb{F}(a_1))} \left[(\mu)^2 - 2\mu \left((b_1) \mathbb{F}(b_1) - a_1 \mathbb{F}(a_1) - \int_{a_1}^{b_1} \mathbb{F}(x_T) dx_T \right) + \right. \\
&\quad \left. \left((b_1)^2 \mathbb{F}(b_1) - (a_1)^2 \mathbb{F}(a_1) - 2 \int_{a_1}^{b_1} x_T \mathbb{F}(x_T) dx_T \right) \right] \\
&+ \frac{1}{2(\mathbb{F}(b_2) - \mathbb{F}(a_2))} \left[(\mu)^2 - 2\mu \left((b_2) \mathbb{F}(b_2) - a_2 \mathbb{F}(a_2) - \int_{a_2}^{b_2} \mathbb{F}(x_T) dx_T \right) + \right. \\
&\quad \left. \left((b_2)^2 \mathbb{F}(b_2) - (a_2)^2 \mathbb{F}(a_2) - 2 \int_{a_2}^{b_2} x_T \mathbb{F}(x_T) dx_T \right) \right] \\
&= \frac{1}{2(\mathbb{F}(b_1) - \mathbb{F}(a_1))} \left[\left(\frac{b_1^2 - 2b_1 a_1 + a_1^2}{12} \right) \right] = \\
&\frac{1}{2(\mathbb{F}(b_2) - \mathbb{F}(a_2))} \left[\left(\frac{b_2^2 - 2b_2 a_2 + a_2^2}{12} \right) \right]
\end{aligned}$$

2.2.8 Moment Generating and Characteristic Functions of the Multi-Double Truncated Continuous Distribution

Let X be a random variable with a density function $f_x(x)$, and t is another variable. The expected value of e^{tx} is defined to be the moment generating function of X denoted by $\mathcal{M}_X(t)$,

$$\mathcal{M}_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} f(x) e^{tx} dx, \quad -h < t < h, h > 0. \quad (2.9)$$

And according to the above supposition, we define characteristic functions as:

$$\Phi_X(t) = E(e^{itX}) = \int_{-\infty}^{\infty} f(x) e^{itx} dx = \mathcal{M}_X(it), \quad -h < t < h, h > 0.$$

Thus, the moment generating and characteristic functions of (MDTCD) is:

$$\begin{aligned}
\mathcal{M}_{X_T}(t) &= \int_{-\infty}^{\infty} e^{tx_T} g_n(x_T) dx_T \\
&= \frac{e^{b_1 t} \mathbb{F}(b_1) - e^{a_1 t} \mathbb{F}(a_1) - t \int_{a_1}^{b_1} e^{tx_T} \mathbb{F}(x_T) dx_T}{n(\mathbb{F}(b_1) - \mathbb{F}(a_1))} +
\end{aligned}$$

$$\begin{aligned}
& \frac{e^{b_2 t} F(b_2) - e^{a_2 t} F(a_2) - t \int_{a_2}^{b_2} e^{t x_T} F(x_T) dx_T}{n(F(b_2) - F(a_2))} + \\
& \dots + \frac{e^{b_n t} F(b_n) - e^{a_n t} F(a_n) - t \int_{a_n}^{b_n} e^{t x_T} F(x_T) dx_T}{n(F(b_n) - F(a_n))} \\
\mathcal{M}_{X_T}(t) &= \frac{1}{n} \sum_{i=1}^n \left(\frac{e^{b_i t} F(b_i) - e^{a_i t} F(a_i) - t \int_{a_i}^{b_i} e^{t x_T} F(x_T) dx_T}{F(b_i) - F(a_i)} \right) \quad (2-10)
\end{aligned}$$

Similarly, obtaining the characteristic function of the multi-double truncated continuous distribution is as follows:

$$\Phi_{X_T}(t) = E(e^{it X_T})$$

$$\begin{aligned}
\Phi_{X_T}(t) &= \\
& \frac{\Phi^{b_1 t} F(b_1) - \Phi^{a_1 t} F(a_1) - t \int_{a_1}^{b_1} \Phi(x_T) F(x_T) dx_T}{n(F(b_1) - F(a_1))} + \\
& \frac{\Phi^{b_2 t} F(b_2) - \Phi^{a_2 t} F(a_2) - t \int_{a_2}^{b_2} \Phi(x_T) F(x_T) dx_T}{n(F(b_2) - F(a_2))} + \dots + \frac{\Phi^{b_n t} F(b_n) - \Phi^{a_n t} F(a_n) - t \int_{a_n}^{b_n} \Phi(x_T) F(x_T) dx_T}{n(F(b_n) - F(a_n))} \\
& \frac{1}{n} \sum_{j=1}^n \left(\frac{\Phi^{b_j t} F(b_j) - \Phi^{a_j t} F(a_j) - t \int_{a_j}^{b_j} \Phi(x_T) F(x_T) dx_T}{F(b_j) - F(a_j)} \right). \\
\Phi_{X_T}(t) &= \frac{1}{n} \sum_{j=1}^n \left(\frac{e^{ib_j t} F(b_j) - e^{ia_j t} F(a_j) - it \int_{a_j}^{b_j} e^{it x_T} F(x_T) dx_T}{F(b_j) - F(a_j)} \right) \quad (2.11)
\end{aligned}$$

2.2.9 Order Statistics of the Multi-Double Truncated Continuous Distribution

Let $X_{T_1}, X_{T_2}, X_{T_3}, X_{T_4}, \dots, X_{T_m}$ be the independent and identical distribution of random variables with p.d.f and c.d.f. Then $X_{T_{[1]}} \leq X_{T_{[2]}} \leq X_{T_{[3]}} \leq X_{T_{[4]}} \leq \dots \leq X_{T_{[m]}}$, be the order statistic, where $X_{T_{[j]}} = X_{T_j}$ are arranged in order of increasing magnitude.

If the random variable x_T has a multi- double truncated function with equation (2.2), then

$$\mathbb{G}_n(x_T)_{[j]} = \frac{m! \sum_{i=1}^n \frac{f(x_T)}{\mathbb{F}(b_i) - \mathbb{F}(a_i)}}{n(j-1)!(m-j)!} \left(\frac{1}{n} \sum_{i=1}^n \frac{\mathbb{F}(x_T) - \sum_{s=1}^n \mathbb{F}(a_s) + \sum_{s=1}^n \mathbb{F}(b_{s-1})}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \right)^{j-1} \\ \times \left(1 - \frac{1}{n} \sum_{i=1}^n \frac{\mathbb{F}(x_T) - \sum_{s=1}^n \mathbb{F}(a_s) + \sum_{s=1}^n \mathbb{F}(b_{s-1})}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \right)^{m-j}$$

But,

$$\left(\frac{1}{n} \sum_{i=1}^n \frac{\mathbb{F}(x_T) - \sum_{s=1}^n \mathbb{F}(a_s) + \sum_{s=1}^n \mathbb{F}(b_{s-1})}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \right)^{j-1} = \\ \sum_{i_1}^n \sum_{i_2}^n \cdots \sum_{i_{(j-1)}}^n \left(\frac{\mathbb{F}(x_T) - \sum_{s=1}^n \mathbb{F}(a_s) + \sum_{s=1}^n \mathbb{F}(b_{s-1})}{n(\mathbb{F}(b_i) - \mathbb{F}(a_i))} \right)_{i_1 i_2 \dots i_{j-1}} \\ \mathbb{G}_n(x_T)_{[j]} = \\ \frac{m! \sum_{i=1}^n \frac{f(x_T)}{\mathbb{F}(b_i) - \mathbb{F}(a_i)}}{n(j-1)!(m-j)!} \left[\sum_{i_1}^n \sum_{i_2}^n \cdots \sum_{i_{(j-1)}}^n \left(\frac{\mathbb{F}(x_T) - \sum_{s=1}^n \mathbb{F}(a_s) + \sum_{s=1}^n \mathbb{F}(b_{s-1})}{n(\mathbb{F}(b_i) - \mathbb{F}(a_i))} \right)_{i_1 i_2 \dots i_{j-1}} \right] \\ \left[\sum_{k=0}^{m-j} \binom{m-j}{k} (-1)^k \sum_{i_1}^n \cdots \sum_{i_k}^n \left(\frac{\mathbb{F}(x_T) - \sum_{s=1}^n \mathbb{F}(a_s) + \sum_{s=1}^n \mathbb{F}(b_{s-1})}{n(\mathbb{F}(b_i) - \mathbb{F}(a_i))} \right)_{i_1 i_2 \dots i_k} \right] \quad (2.12)$$

The smallest order statistics

$$\mathbb{G}_n(x_T)_{[1]} = \frac{m \sum_{i=1}^n \frac{f(x_T)}{\mathbb{F}(b_i) - \mathbb{F}(a_i)}}{n} \left(1 - \frac{1}{n} \sum_{i=1}^n \frac{\mathbb{F}(x_T) - \sum_{s=1}^n \mathbb{F}(a_s) + \sum_{s=1}^n \mathbb{F}(b_{s-1})}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \right)^{m-1} \\ \left[\sum_{k=0}^{m-1} \frac{m(-1)^k}{n} \binom{m-1}{k} \sum_{i=1}^n \frac{f(x_T)}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n \frac{1}{n} \left(\frac{\mathbb{F}(x_T) - \sum_{s=1}^n \mathbb{F}(a_s) + \sum_{s=1}^n \mathbb{F}(b_{s-1})}{(\mathbb{F}(b_i) - \mathbb{F}(a_i))} \right)_{i_1 i_2 \dots i_k} \right] \quad (2.13)$$

2.3 One-Double Truncated of Weibull Distribution (ODTWD)

In this section, the probability multi-double truncated is applied to Weibull distribution and introduces some statistical results and various illustration shapes for (ODTWD).

2.3.1 The Probability Function of One-Double Truncated Weibull Distribution, (ODTWD)

Weibull distribution is named after Waloddi Weibull (1887–1979) who was Swedish engineer, scientist, and mathematician and the first to found the utility of this distribution to model data sets. In (1951) he modeled data sets from many different disciplines, and its applications.

To find the probability density function of one-double truncated Weibull distribution (ODTWD), by using equation (2.2), is:

$$g_2(x_T) = \begin{cases} \frac{\frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta}}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} & 0 < x_T < b_1 \\ \frac{\frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta}}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} & a_2 < x_T < \infty \end{cases} \quad (2.14)$$

that is, the interval which deleted is (b_1, a_2) .

Such that, $\left(\frac{\int_{a_i}^{b_i} f(x_T) dx_T}{\mathbb{F}(b_i) - \mathbb{F}(a_i)}\right) = \frac{1}{2} \left[\frac{\mathbb{F}(b_1) - \mathbb{F}(a_1)}{\mathbb{F}(b_1) - \mathbb{F}(a_1)} + \frac{\mathbb{F}(b_2) - \mathbb{F}(a_2)}{\mathbb{F}(b_2) - \mathbb{F}(a_2)}\right] = 1, i = 1, 2.$

Figures (2.1), (2.2), and (2.3) show the curve of one double truncated Weibull distribution function for different values of the parameters δ, ω . Note that, if there exist two intervals $(a_1, b_1), (a_2, b_2)$, then the interval which deleted is $(b_1, a_2) = (6, 10)$.

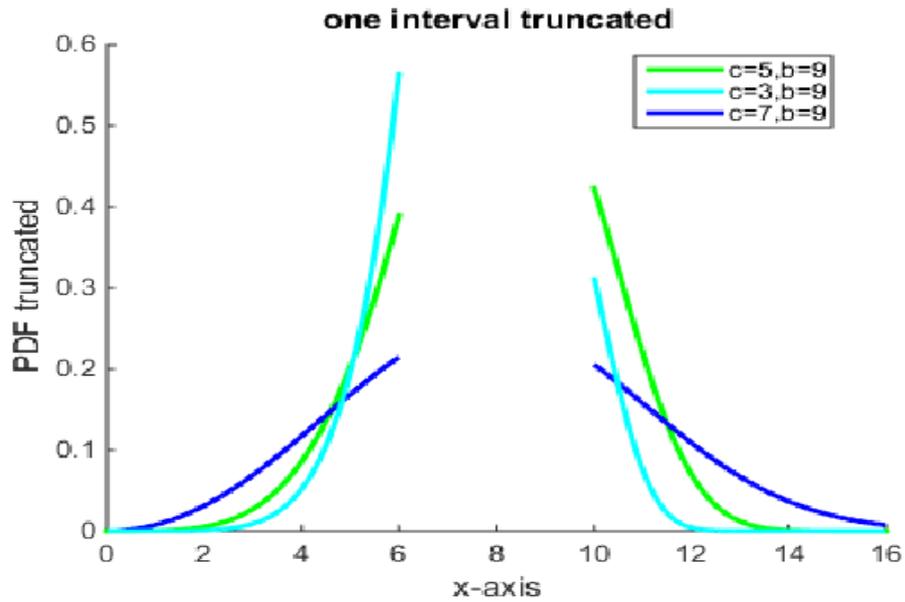


FIGURE 2.1 One interval truncated of PDF Weibull, $\delta = 5, 3, 7$

In this figure, show the PDF of one double truncated Weibull distribution for different values of the parameters δ, ω . Note that, the two intervals are $(a_1, b_1) = (0, 6)$, $(a_2, b_2) = (10, 16)$, then the interval which deleted is $(b_1, a_2) = (6, 10)$.

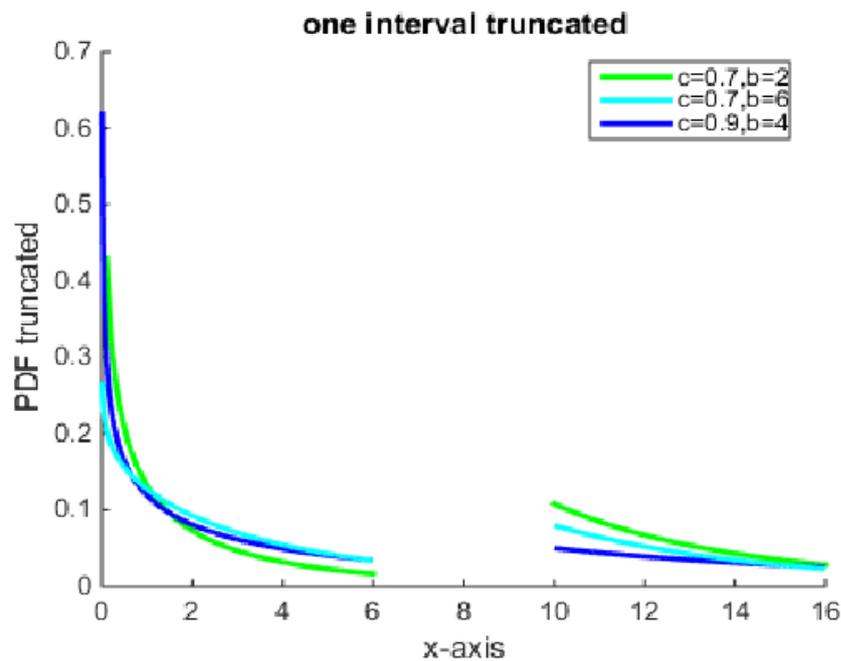


FIGURE 2.2 One interval truncated of PDF Weibull, $\omega = 2, 6, 4$

In this figure, show the PDF of one double truncated Weibull distribution function for different values of the parameters δ, ω . Such that $(a_1, b_1) = (0, 6)$, $(a_2, b_2) = (10, 16)$, so the interval which deleted is $(b_1, a_2) = (6, 10)$.

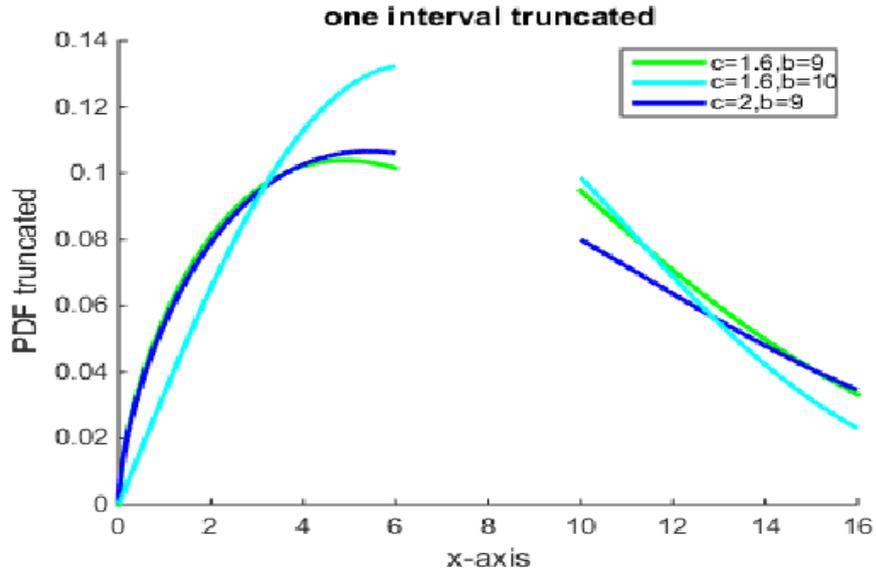


FIGURE 2.3 One interval truncated of PDF Weibull $\delta = 1,1,2$ $\omega = 9,10,9$

While in figure (2.3), we show the PDF of one double truncated Weibull distribution function for $\delta = 1,1,2$, $\omega = 9,10,9$. Such that $(a_1, b_1)=(0,6)$, $(a_2, b_2)=(10,16)$, so the interval which deleted is $(b_1, a_2)=(6,10)$.

2.3.2 Cumulative Distribution Function of One-Double Truncated Weibull

We can obtain the cumulative distribution function of one truncated Weibull distribution, by using equation (2.2), as:

$$\mathbb{G}_2(X_T) = \left(\frac{e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{x_T}{\omega}\right)^\delta}}{2(e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta})} \right) + \left(\frac{e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{x_T}{\omega}\right)^\delta} + e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta}}{2(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta})} \right) \quad (2.15)$$

Figures (2.4),(2.5), and (2.6) presented curves of the cumulative distribution function of one- double truncated Weibull distribution for different values of parameters δ, ω : where $(a_1, b_1) = (0,5)$, $(a_2, b_2) = (10,25)$, that is, the interval which was deleted was $(5,10)$.

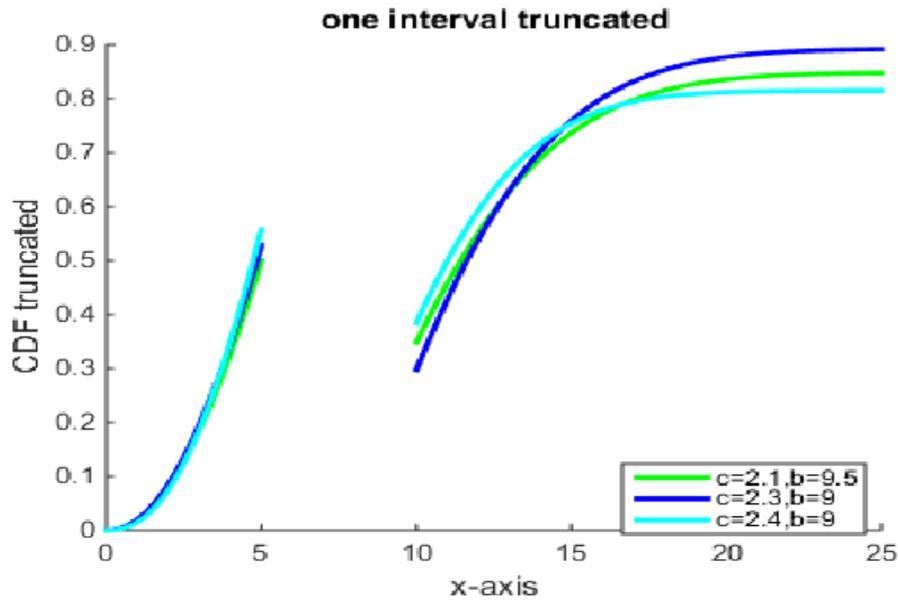


FIGURE 2.4 One interval truncated of CDF Weibull $\delta = 2.1, 2.3, 2.4$

In this figure, we presented curve of the cumulative distribution function of one- double truncated Weibull distribution for different values of parameters $\delta = 2.1, 2.3, 2.4$: where $(a_1, b_1) = (0, 5)$, $(a_2, b_2) = (10, 25)$, that is, the interval which deleted is $(5, 10)$.

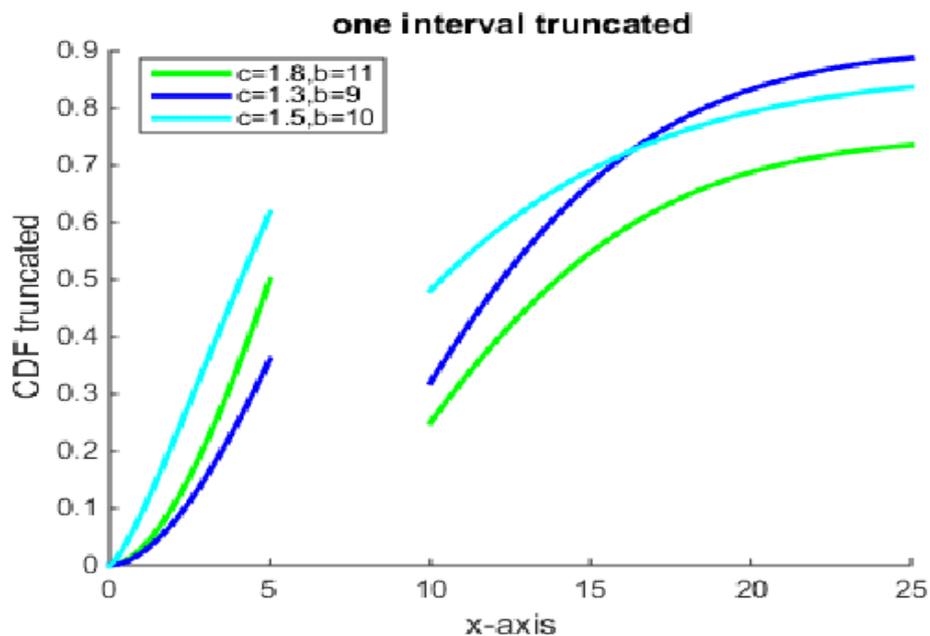


FIGURE 2.5 One interval truncated of CDF Weibull various $\omega = 11, 9, 10$

In this figure, we presented curve of the cumulative distribution function of one- double truncated Weibull distribution for different values of parameters $\omega = 11, 9, 10$. Where $(a_1, b_1) = (0, 5)$, $(a_2, b_2) = (10, 25)$, that is, the interval which deleted is $(5, 10)$.

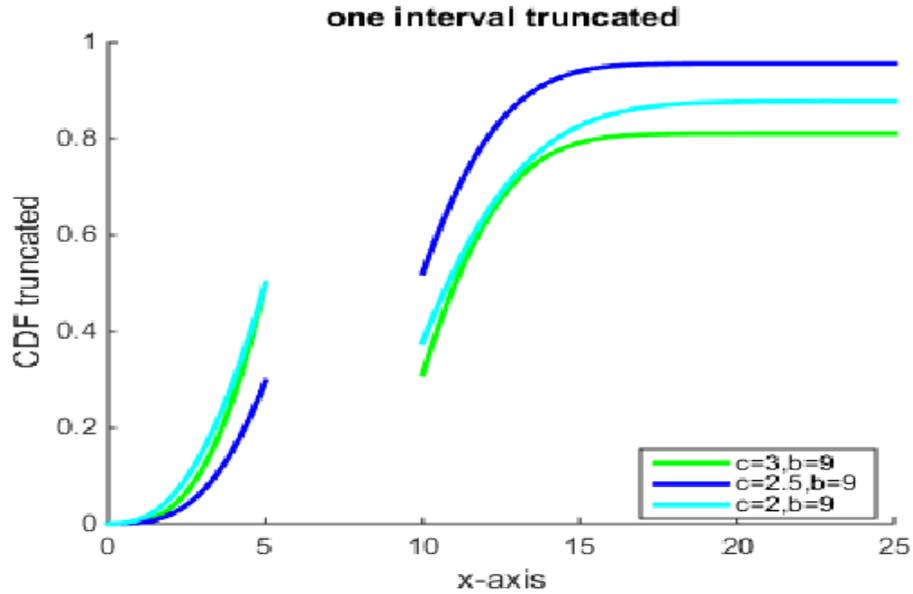


FIGURE 2.6 One interval truncated of CDF Weibull, $\delta = 3,2,2$.

In this figure, we introduced the cumulative distribution function of one-double truncated Weibull distribution for different values of parameters $\delta = 3,2,2$. Where $(a_1, b_1) = (0,5)$, $(a_2, b_2) = (10,25)$, that is, the interval which deleted is $(5,10)$.

Theorem 2.1 Let x_T be a random variable distributed has probability density function in equation (2.14), then the r^{th} Moment of One-Double Truncated Weibull Distribution is:

$$\begin{aligned} E(X_T^r) &= \frac{1}{2} \sum_{i=1}^2 \left(\frac{\int_{a_i}^{b_i} x^r f(x_T) dx_T}{F(b_i) - F(a_i)} \right) = \frac{\frac{r}{\delta} \omega^r \gamma\left(\frac{r}{\delta} + 1, \frac{(b_1)^\delta}{\omega^\delta}\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\frac{r}{\delta} \omega^r \Gamma\left(\frac{r}{\delta} + 1, \frac{(a_2)^\delta}{\omega^\delta}\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \\ &= \frac{1}{2} \sum_{i=1}^2 \left(\frac{b_i^r F(b_i) - a_i^r F(a_i) - r \int_{a_i}^{b_i} x^{r-1} F(x_T) dx_T}{F(b_i) - F(a_i)} \right) \end{aligned}$$

Proof:-

$$\begin{aligned} E(X_T^r) &= \int_0^\infty x^r g_n(x_T) dx_T = \frac{1}{2} \sum_{i=1}^2 \left(\frac{\int_{a_i}^{b_i} x^r f(x_T) dx_T}{F(b_i) - F(a_i)} \right) \\ &= \frac{1}{2} \frac{\int_{a_1}^{b_1} x^r f(x_T) dx_T}{F(b_1) - F(a_1)} + \frac{1}{2} \frac{\int_{a_2}^{b_2} x^r f(x_T) dx_T}{F(b_2) - F(a_2)} \\ &= \frac{\int_0^{b_1} x_T^r \frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta} dx_T}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\int_{a_2}^\infty x_T^r \frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta} dx_T}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \end{aligned}$$

Take: $I_1 = \int_{a_1}^{b_1} x_T^r \frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta} dx_T$

let $\lambda = 1/\omega^\delta$, $x_T^\delta = y \rightarrow dy = \delta x_T^{\delta-1} dx_T$, $dx_T = \frac{1}{\delta} y^{\frac{1-\delta}{\delta}} dy$.

$$\begin{aligned} I_1 &= \int_0^{b_1^\delta} \frac{\delta}{\omega^\delta} y^{\frac{r}{\delta}} y^{\frac{1}{\delta}(\delta-1)} e^{-\lambda y} \frac{1}{\delta} y^{\frac{1-\delta}{\delta}} dy \\ &= \int_0^{b_1^\delta} \lambda \delta y^{\frac{r}{\delta}} y^{\frac{1}{\delta}(\delta-1)} e^{-\lambda y} \frac{1}{\delta} y^{\frac{1-\delta}{\delta}} dy = \int_0^{b_1^\delta} \lambda y^{\frac{r+\delta-1+1-\delta}{\delta}} e^{-\lambda y} dy \\ &= \int_0^{b_1^\delta} \lambda y^{\frac{r}{\delta}} e^{-\lambda y} dy, \text{ let } \lambda y = z, \frac{z}{\lambda} = y \rightarrow dy = \frac{dz}{\lambda} \end{aligned}$$

$$I_1 = \lambda \int_0^{\lambda(b_1)^\delta} \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} z^{\frac{r}{\delta}+1-1} e^{-z} \frac{dz}{\lambda} = \int_0^{\lambda(b_1)^\delta} \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} z^{\frac{r}{\delta}+1-1} e^{-z} dz$$

Then $I_1 = \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} \gamma\left(\frac{r}{\delta} + 1, \lambda(b_1)^\delta\right)$, since $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$

Suppose $I_2 = \int_{a_2}^{\infty} x_T^r \frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta} dx_T$, and use the same above suppositions, we get:

$$I_2 = \lambda \int_{(\lambda a_2)^\delta}^{\infty} \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} z^{\frac{r}{\delta}+1-1} e^{-z} \frac{dz}{\lambda} = \int_{(\lambda a_2)^\delta}^{\infty} \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} z^{\frac{r}{\delta}+1-1} e^{-z} dz$$

Then $I_2 = \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} \Gamma\left(\frac{r}{\delta} + 1, \lambda(a_2)^\delta\right)$, since $[\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt]$

Thus,

$$E(X_T^r) = \frac{(\omega^\delta)^{\frac{r}{\delta}} \gamma\left(\frac{r}{\delta}+1, \lambda(b_1)^\delta\right)}{2\left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{(\omega^\delta)^{\frac{r}{\delta}} \Gamma\left(\frac{r}{\delta}+1, \lambda(a_2)^\delta\right)}{2\left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \quad (2.16)$$

Now, if take $E(X_T^r) = \frac{1}{2} \sum_{i=1}^2 \left(\frac{\int_{a_i}^{b_i} x^r f(x) dx}{F(b_i) - F(a_i)} \right)$

$$= \frac{1}{2} \sum_{i=1}^2 \left(\frac{b_i^r F(b_i) - a_i^r F(a_i) - r \int_{a_i}^{b_i} x^{r-1} F(x) dx}{F(b_i) - F(a_i)} \right)$$

=

$$\frac{1}{2} \frac{b_1^r F(b_1) - a_1^r F(a_1) - r \int_{a_1}^{b_1} x^{r-1} F(x) dx}{F(b_1) - F(a_1)} + \frac{1}{2} \frac{b_2^r F(b_2) - a_2^r F(a_2) - r \int_{a_2}^{b_2} x^{r-1} F(x) dx}{F(b_2) - F(a_2)}$$

$$= \frac{b_1^r (1 - e^{-\lambda b_1^\delta})^{-r} \int_0^{b_1} x^{r-1} dx_T + r \int_0^{b_1} x^{r-1} e^{-\lambda b_1^\delta} dx_T}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta} \right)} +$$

$$\frac{b_2^r (1 - e^{-\lambda b_2^\delta})^{-r} \int_{a_2}^\infty x^{r-1} dx_T + r \int_{a_2}^\infty x^{r-1} e^{-\lambda b_2^\delta} dx_T}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} \right)}$$

$$\text{Take: } I_3 = \int_0^{b_1} x_T^{r-1} e^{-\lambda x_T^\delta} dx_T$$

$$\text{Let } y = x_T^\delta \rightarrow dy = \delta x_T^{\delta-1} dx_T, \text{ and } \lambda = 1/\omega^\delta, \quad dx_T = \frac{1}{\delta} y^{\frac{1-\delta}{\delta}} dy.$$

$$I_3 = \int_0^{b_1^\delta} y^{\frac{r-1}{\delta}} y^{\frac{1-\delta}{\delta}} e^{-\lambda y} \frac{1}{\delta} dy$$

$$= \frac{1}{\delta} \int_0^{b_1^\delta} y^{\frac{r-1}{\delta}} y^{\frac{1-\delta}{\delta}} e^{-\lambda y} dy = \int_0^{b_1^\delta} y^{\frac{r-\delta}{\delta}} e^{-\lambda y} dy$$

$$\text{Let } z = \lambda y, \quad \frac{z}{\lambda} = y \rightarrow dy = \frac{dz}{\lambda}$$

$$I_3 = \frac{1}{\delta} \int_0^{\lambda b_1^\delta} \left(\frac{1}{\lambda}\right)^{\frac{r-\delta}{\delta}} z^{\frac{r-\delta}{\delta}+1-1} e^{-z} \frac{dz}{\lambda} =$$

$$\frac{1}{\delta} \left(\frac{1}{\lambda}\right)^{\frac{r-\delta+\delta}{\delta}} \int_0^{\lambda b_1^\delta} z^{\frac{r-\delta+\delta}{\delta}-1} e^{-z} dz$$

$$= \frac{1}{\delta} \left(\frac{1}{\lambda}\right)^{\frac{r-\delta+\delta}{\delta}} \int_0^{\lambda b_1^\delta} z^{\frac{r}{\delta}-1} e^{-z} dz$$

$$I_3 = \frac{1}{\delta} \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} \gamma\left(\frac{r}{\delta}, \lambda b_1^\delta\right),$$

$$\text{Thus, } -b_1^r e^{-\lambda b_1^\delta} + r \int_0^{b_1} x_T^{r-1} e^{-\lambda x_T^\delta} dx_T = -b_1^r e^{-\lambda b_1^\delta} + r I_3$$

$$-b_1^r e^{-\lambda b_1^\delta} + r \int_0^{b_1} x_T^{r-1} e^{-\lambda x_T^\delta} dx_T = -b_1^r e^{-\lambda b_1^\delta} +$$

$$r \frac{1}{\delta} \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} \gamma\left(\frac{r}{\delta}, \lambda b_1^\delta\right)$$

$$\text{Take: } I_4 = \int_{a_2}^\infty x^{r-1} e^{-\lambda x_T^\delta} dx_T$$

$$\begin{aligned}
I_4 &= \frac{1}{\delta} \int_{a_2 \delta}^{\infty} y^{\frac{r-\delta}{\delta}} e^{-\lambda y} dy \\
&= \frac{1}{\delta} \int_{\lambda a_2 \delta}^{\infty} \left(\frac{z}{\lambda}\right)^{\frac{r-\delta}{\delta}} e^{-z} \frac{dz}{\lambda} \\
&= \frac{1}{\delta \lambda} \left(\frac{1}{\lambda}\right)^{\frac{r-\delta}{\delta}} \int_{\lambda a_2 \delta}^{\infty} (z)^{\frac{r-\delta}{\delta}+1-1} e^{-z} dz = \frac{1}{\delta} \left(\frac{1}{\lambda}\right)^{\frac{r-\delta+1}{\delta}} \int_{\lambda a_2 \delta}^{\infty} z^{\frac{r}{\delta}-1} e^{-z} dz
\end{aligned}$$

$$\text{Then } I_2 = \frac{1}{\delta} \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} \Gamma\left(\frac{r}{\delta}, \lambda a_2 \delta\right)$$

$$\text{Thus, } -a_2^r e^{-\lambda a_2 \delta} + r \int_{a_2}^{\infty} x^{r-1} e^{-\lambda x_T \delta} dx_T = -a_2^r e^{-\lambda b_2 \delta} + r I_2$$

$$a_2^r e^{-\lambda a_2 \delta} + r \int_{a_2}^{\infty} x^{r-1} e^{-\lambda x_T \delta} dx_T = a_2^r e^{-\lambda a_2 \delta} + r \frac{1}{\delta} \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} \Gamma\left(\frac{r}{\delta}, \lambda a_2 \delta\right)$$

$$\begin{aligned}
E(X_T^r) &= \\
&= \frac{-(b_1)^r e^{-\frac{1}{\omega \delta} (b_1) \delta} + \frac{r}{\delta} (\omega \delta)^{\frac{r}{\delta}} \gamma\left(\frac{r}{\delta}, \frac{1}{\omega \delta} b_1 \delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right) \delta}\right)} + \frac{(a_2)^r e^{-\frac{1}{\omega \delta} (a_2) \delta} + \frac{r}{\delta} (\omega \delta)^{\frac{r}{\delta}} \gamma\left(\frac{r}{\delta}, \frac{1}{\omega \delta} a_2 \delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right) \delta}\right)} \quad (2.17)
\end{aligned}$$

From equation (2.16), and properties of incomplete Gamma function, we obtain:

$$\begin{aligned}
&\frac{(\omega \delta)^{\frac{r}{\delta}} \gamma\left(\frac{r}{\delta}+1, \lambda (b_1) \delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right) \delta}\right)} + \frac{(\omega \delta)^{\frac{r}{\delta}} \Gamma\left(\frac{r}{\delta}+1, \lambda (a_2) \delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right) \delta}\right)} \\
&= \\
&\frac{\frac{r}{\delta} (\omega \delta)^{\frac{r}{\delta}} \gamma\left(\frac{r}{\delta}, \lambda b_1 \delta\right) - (\omega \delta)^{\frac{r}{\delta}} (\lambda b_1 \delta)^{\frac{r}{\delta}} e^{-\lambda b_1 \delta}}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right) \delta}\right)} + \\
&\quad \frac{\frac{r}{\delta} (\omega \delta)^{\frac{r}{\delta}} \Gamma\left(\frac{r}{\delta}+1, \lambda (a_2) \delta\right) + (\omega \delta)^{\frac{r}{\delta}} (\lambda a_2 \delta)^{\frac{r}{\delta}} e^{-\lambda a_2 \delta}}{2 \left(e^{-\left(\frac{a_2}{\omega}\right) \delta}\right)}
\end{aligned}$$

$$= \frac{\frac{r}{\delta} \omega^r \gamma\left(\frac{r-1}{\delta}, \frac{1}{\omega\delta} b_1^\delta\right) - (b_1)^r e^{-\frac{(b_1)^\delta}{\omega\delta}}}{2\left(1 - e^{-\frac{(b_1)^\delta}{\omega\delta}}\right)} + \frac{\frac{r}{\delta} \omega^r \Gamma\left(\frac{r-1}{\delta}, \frac{1}{\omega\delta} a_2^\delta\right) + (a_2)^r e^{-\frac{(a_2)^\delta}{\omega\delta}}}{2\left(e^{-\frac{(a_2)^\delta}{\omega\delta}}\right)},$$

and the last formula is equal to equation (2.17).

The mean of (ODTWD) is:

$$E(X_T) = \frac{\omega \gamma\left(\frac{1}{\delta} + 1, \frac{1}{\omega\delta} b_1^\delta\right)}{2\left(1 - e^{-\frac{(b_1)^\delta}{\omega\delta}}\right)} + \frac{\omega \Gamma\left(\frac{1}{\delta} + 1, \frac{1}{\omega\delta} a_2^\delta\right)}{2\left(e^{-\frac{(a_2)^\delta}{\omega\delta}}\right)} \quad (2.18)$$

Theorem 2.2

Let x_T be a random variable distributed has probability density function in equation (2.14), then the r^{th} Moment about a mean of One-Double Truncated Weibull Distribution is:

$$\sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \left[\frac{\omega \frac{1}{\delta} \gamma\left(\frac{1}{\delta}, \frac{1}{\omega\delta} b_1^\delta\right) - b_1 e^{-\frac{(b_1)^\delta}{\omega\delta}}}{2\left(1 - e^{-\frac{(b_1)^\delta}{\omega\delta}}\right)} + \frac{\omega \frac{1}{\delta} \Gamma\left(\frac{1}{\delta}, \frac{1}{\omega\delta} a_2^\delta\right) + a_2 e^{-\frac{(a_2)^\delta}{\omega\delta}}}{2\left(e^{-\frac{(a_2)^\delta}{\omega\delta}}\right)} \right]^{r-j}$$

$$\left[\frac{\frac{j}{\delta} (\omega\delta)^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta}, \frac{1}{\omega\delta} b_1^\delta\right) - (b_1)^j e^{-\frac{(b_1)^\delta}{\omega\delta}}}{2\left(1 - e^{-\frac{(b_1)^\delta}{\omega\delta}}\right)} + \frac{\frac{j}{\delta} (\omega\delta)^{\frac{j}{\delta}} \Gamma\left(\frac{j}{\delta}, \frac{1}{\omega\delta} a_2^\delta\right) + (a_2)^j e^{-\frac{(a_2)^\delta}{\omega\delta}}}{2\left(e^{-\frac{(a_2)^\delta}{\omega\delta}}\right)} \right]$$

Proof:- the r^{th} Moment about the mean is:

$$E(X_T - \mu)^r = \int_0^\infty (X_T - \mu)^r g_{\text{N}}(x_T) dx_T = \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} (\mu)^{r-j} \mu_j'$$

$$= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} (\mu)^{r-j} \int_0^\infty (x_T)^j g_{\text{N}}(x_T) dx_T$$

$$= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} (\mu)^{r-j} \left(\frac{1}{2}\right) \sum_{i=1}^2 \left(\frac{(b_i)^j \mathbb{F}(b_i) - (a_i)^j \mathbb{F}(a_i) - j \int_{a_i}^{b_i} (x_T)^j \mathbb{F}(x_T) dx_T}{\mathbb{F}(b_i) - \mathbb{F}(a_i)} \right)$$

$$E(X_T - \mu)^r =$$

$$\sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \left[\frac{\omega^{\frac{1}{\delta}} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega^{\delta}} b_1^{\delta}\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^{\delta}}\right)} + \frac{\omega^{\frac{1}{\delta}} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega^{\delta}} a_2^{\delta}\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^{\delta}}\right)} \right]^{r-j} \left[\frac{\frac{j}{\delta} (\omega^{\delta})^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega^{\delta}} b_1^{\delta}\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^{\delta}}\right)} + \frac{\frac{j}{\delta} (\omega^{\delta})^{\frac{j}{\delta}} \Gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega^{\delta}} a_2^{\delta}\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^{\delta}}\right)} \right]$$

$$E(X_T - \mu)^r =$$

$$\sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \left[\frac{\omega^{\frac{1}{\delta}} \gamma\left(\frac{1}{\delta}, \frac{1}{\omega^{\delta}} b_1^{\delta}\right) - b_1 e^{-\frac{(b_1)^{\delta}}{\omega^{\delta}}}}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^{\delta}}\right)} + \frac{\omega^{\frac{1}{\delta}} \Gamma\left(\frac{1}{\delta}, \frac{1}{\omega^{\delta}} a_2^{\delta}\right) + a_2 e^{-\frac{(a_2)^{\delta}}{\omega^{\delta}}}}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^{\delta}}\right)} \right]^{r-j} \left[\frac{\frac{j}{\delta} (\omega^{\delta})^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta}, \frac{1}{\omega^{\delta}} b_1^{\delta}\right) - (b_1)^j e^{-\frac{(b_1)^{\delta}}{\omega^{\delta}}}}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^{\delta}}\right)} + \frac{\frac{j}{\delta} (\omega^{\delta})^{\frac{j}{\delta}} \Gamma\left(\frac{j}{\delta}, \frac{1}{\omega^{\delta}} a_2^{\delta}\right) + (a_2)^j e^{-\frac{(a_2)^{\delta}}{\omega^{\delta}}}}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^{\delta}}\right)} \right] \quad (2.19)$$

2.3.3 Skewness And Kurtosis of One-Double Truncated Weibull Distribution

Skewness: The coefficient of skewness is a tool important that measures symmetry, or more precisely, the deficiency of symmetry. distribution is symmetric if it appears the same to the left and right of the center point within its curve.

$$\text{Skew}(x_T) = \frac{E(x_T - \mu)^3}{[(E(x_T - \mu)^2)^{3/2}]}$$

$$\begin{aligned}
& \left[\sum_{j=0}^3 (-1)^{3-j} \binom{3}{j} \left[\frac{\frac{j}{\delta} (\omega \delta)^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega \delta} b_1 \delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\frac{j}{\delta} (\omega \delta)^{\frac{j}{\delta}} \Gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega \delta} a_2 \delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right] \right. \\
& \quad \left. \left[\frac{\omega \frac{1}{\delta} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} b_1 \delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\omega \frac{1}{\delta} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} a_2 \delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right]^{3-j} \right] \\
= & \left[\left(\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left[\frac{\frac{j}{\delta} (\omega \delta)^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega \delta} b_1 \delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\frac{j}{\delta} (\omega \delta)^{\frac{j}{\delta}} \Gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega \delta} a_2 \delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right] \right)^{3/2} \right. \\
& \quad \left. \left[\frac{\omega \frac{1}{\delta} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} b_1 \delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\omega \frac{1}{\delta} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} a_2 \delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right]^{2-j} \right]
\end{aligned}$$

And, according to equation (2.17):

Skew(x_T)=

$$\begin{aligned}
& \left[\sum_{j=0}^3 (-1)^{3-j} \binom{3}{j} \left[\frac{\frac{j}{\delta} \omega^j \gamma\left(\frac{j}{\delta}, \frac{1}{\omega \delta} b_1 \delta\right) - (b_1)^j e^{-\frac{(b_1)^\delta}{\omega \delta}}}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\frac{j}{\delta} \omega^j \Gamma\left(\frac{j}{\delta}, \frac{1}{\omega \delta} a_2 \delta\right) + (a_2)^j e^{-\frac{(a_2)^\delta}{\omega \delta}}}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right] \right. \\
& \quad \left. \left[\frac{\frac{1}{\delta} \omega \gamma\left(\frac{1}{\delta}, \frac{1}{\omega \delta} b_1 \delta\right) - b_1 e^{-\frac{(b_1)^\delta}{\omega \delta}}}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\frac{1}{\delta} \omega \Gamma\left(\frac{1}{\delta}, \frac{1}{\omega \delta} a_2 \delta\right) + a_2 e^{-\frac{(a_2)^\delta}{\omega \delta}}}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right]^{3-j} \right] \\
& \left[\left(\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left[\frac{\frac{j}{\delta} \omega^j \gamma\left(\frac{j}{\delta}, \frac{1}{\omega \delta} b_1 \delta\right) - (b_1)^j e^{-\frac{(b_1)^\delta}{\omega \delta}}}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\frac{j}{\delta} \omega^j \Gamma\left(\frac{j}{\delta}, \frac{1}{\omega \delta} a_2 \delta\right) + (a_2)^j e^{-\frac{(a_2)^\delta}{\omega \delta}}}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right] \right)^{3/2} \right. \\
& \quad \left. \left[\frac{\frac{1}{\delta} \omega \gamma\left(\frac{1}{\delta}, \frac{1}{\omega \delta} b_1 \delta\right) - b_1 e^{-\frac{(b_1)^\delta}{\omega \delta}}}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\frac{1}{\delta} \omega \Gamma\left(\frac{1}{\delta}, \frac{1}{\omega \delta} a_2 \delta\right) + a_2 e^{-\frac{(a_2)^\delta}{\omega \delta}}}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right]^{2-j} \right]
\end{aligned}$$

Now, Suppose

$$\begin{aligned}
A &= \sum_{j=0}^3 (-1)^{3-j} \binom{3}{j} \left[\frac{\frac{j}{\delta} (\omega^\delta)^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega^\delta} b_1^\delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \right. \\
&\quad \left. \frac{\frac{j}{\delta} (\omega^\delta)^{\frac{j}{\delta}} \Gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega^\delta} a_2^\delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right] \left[\frac{\omega \frac{1}{\delta} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega^\delta} b_1^\delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\omega \frac{1}{\delta} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega^\delta} a_2^\delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right]^{3-j} \\
&= (-1) \left[\frac{\gamma\left(1, \frac{1}{\omega^\delta} b_1^\delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\Gamma\left(1, \frac{1}{\omega^\delta} a_2^\delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right] \left[\frac{\omega \frac{1}{\delta} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega^\delta} b_1^\delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\omega \frac{1}{\delta} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega^\delta} a_2^\delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right]^3 \\
&\quad + 3 \left[\frac{\omega \frac{1}{\delta} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega^\delta} b_1^\delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\omega \frac{1}{\delta} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega^\delta} a_2^\delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right] \left[\frac{\omega \frac{1}{\delta} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega^\delta} b_1^\delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\omega \frac{1}{\delta} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega^\delta} a_2^\delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right]^2 \\
&\quad - 3 \left[\frac{\frac{2}{\delta} \omega^2 \gamma\left(\frac{2}{\delta}+1, \frac{1}{\omega^\delta} b_1^\delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\frac{2}{\delta} \omega^2 \Gamma\left(\frac{2}{\delta}+1, \frac{1}{\omega^\delta} a_2^\delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right] \left[\frac{\omega \frac{1}{\delta} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega^\delta} b_1^\delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\omega \frac{1}{\delta} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega^\delta} a_2^\delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right] \\
&\quad + \left[\frac{\frac{3}{\delta} \omega^3 \gamma\left(\frac{3}{\delta}+1, \frac{1}{\omega^\delta} b_1^\delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\frac{3}{\delta} \omega^3 \Gamma\left(\frac{3}{\delta}+1, \frac{1}{\omega^\delta} a_2^\delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right] \\
\text{And } B &= \sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left[\frac{\frac{j}{\delta} (\omega^\delta)^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega^\delta} b_1^\delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\frac{j}{\delta} (\omega^\delta)^{\frac{j}{\delta}} \Gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega^\delta} a_2^\delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right]
\end{aligned}$$

$$\begin{aligned}
& \left[\frac{\omega \frac{1}{\delta} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} b_1 \delta\right)}{2\left(1-e^{-\left(\frac{b_1}{\omega}\right) \delta}\right)} + \frac{\omega \frac{1}{\delta} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} a_2 \delta\right)}{2\left(e^{-\left(\frac{a_2}{\omega}\right) \delta}\right)} \right]^{2-j} \\
& \left[\frac{\gamma\left(1, \frac{1}{\omega \delta} b_1 \delta\right)}{2\left(1-e^{-\left(\frac{b_1}{\omega}\right) \delta}\right)} + \frac{\Gamma\left(1, \frac{1}{\omega \delta} a_2 \delta\right)}{2\left(e^{-\left(\frac{a_2}{\omega}\right) \delta}\right)} \right] \left[\frac{\omega \frac{1}{\delta} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} b_1 \delta\right)}{2\left(1-e^{-\left(\frac{b_1}{\omega}\right) \delta}\right)} + \frac{\omega \frac{1}{\delta} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} a_2 \delta\right)}{2\left(e^{-\left(\frac{a_2}{\omega}\right) \delta}\right)} \right]^2 \\
& -2 \left[\frac{\frac{\omega}{\delta} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} b_1 \delta\right)}{2\left(1-e^{-\left(\frac{b_1}{\omega}\right) \delta}\right)} + \frac{\frac{\omega}{\delta} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} a_2 \delta\right)}{2\left(e^{-\left(\frac{a_2}{\omega}\right) \delta}\right)} \right] \left[\frac{\frac{\omega}{\delta} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} b_1 \delta\right)}{2\left(1-e^{-\left(\frac{b_1}{\omega}\right) \delta}\right)} + \frac{\frac{\omega}{\delta} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} a_2 \delta\right)}{2\left(e^{-\left(\frac{a_2}{\omega}\right) \delta}\right)} \right] \\
& + \left[\frac{\frac{2}{\delta} \omega^2 \gamma\left(\frac{2}{\delta}+1, \frac{1}{\omega \delta} b_1 \delta\right)}{2\left(1-e^{-\left(\frac{b_1}{\omega}\right) \delta}\right)} + \frac{\frac{2}{\delta} \omega^2 \Gamma\left(\frac{2}{\delta}+1, \frac{1}{\omega \delta} a_2 \delta\right)}{2\left(e^{-\left(\frac{a_2}{\omega}\right) \delta}\right)} \right].
\end{aligned}$$

Kurtosis: The coefficient of kurtosis is a tool that identifies if the tails of the specified distribution, have extreme values or not. and it is used as a measure of kurtosis, which is the degree of flatness or peakedness of a density near its center.

$$\text{Kurt}(x_T) = \frac{E(x_T - \mu)^4}{[E(x_T - \mu)^2]^2}$$

$$\begin{aligned}
& \left[\sum_{j=0}^4 (-1)^{4-j} \binom{4}{j} \left[\frac{\frac{j}{\delta} (\omega \delta)^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega \delta} b_1 \delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\frac{j}{\delta} (\omega \delta)^{\frac{j}{\delta}} \Gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega \delta} a_2 \delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right] \right] \\
& \left[\frac{\omega \frac{1}{\delta} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} b_1 \delta\right) + \omega \frac{1}{\delta} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} a_2 \delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right) + 2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right]^{4-j} \\
& \left[\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left[\frac{\frac{j}{\delta} (\omega \delta)^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega \delta} b_1 \delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\frac{j}{\delta} (\omega \delta)^{\frac{j}{\delta}} \Gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega \delta} a_2 \delta\right)}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right] \right]^{2-j} \\
& \left[\frac{\omega \frac{1}{\delta} \gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} b_1 \delta\right) + \omega \frac{1}{\delta} \Gamma\left(\frac{1}{\delta}+1, \frac{1}{\omega \delta} a_2 \delta\right)}{2 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right) + 2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta}\right)} \right]^{2-j}
\end{aligned} \tag{2.21}$$

2.3.4 The Moment Generating Function of One-Double Truncated Weibull Distribution

The moment generating function of (ODTWD) random variable is given by:

$$\begin{aligned}
\mathcal{M}_{x_T}(t) &= \int_0^\infty e^{t x_T} (\mathbb{G}_n(x_T)) dx_T = \frac{1}{2} \sum_{i=1}^2 \left(\frac{\int_{a_i}^{b_i} e^{t x_T} f(x_T) dx_T}{F(b_i) - F(a_i)} \right) \\
&= \frac{\int_0^{b_1} e^{t x_T} f(x_T) dx_T}{F(b_1) - F(a_1)} + \frac{\int_{a_2}^{b_2} e^{t x_T} f(x_T) dx_T}{F(b_2) - F(a_2)}
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int_0^{b_1} e^{t x_T} f(x_T) dx_T = \int_0^{b_1} e^{t x_T} \frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta} dx_T \\
&= \int_0^{b_1} \sum_{j=0}^{\infty} \frac{(t x_T)^j}{j!} e^{-\left(\frac{x_T}{\omega}\right)^\delta} dx_T = \sum_{j=0}^{\infty} \frac{(t)^j}{j!} \frac{\delta}{\omega^\delta} \int_0^{b_1} (x_T)^{j+\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta} dx_T
\end{aligned}$$

$$\text{Let } y = \lambda x_T^\delta, x_T = (y/\lambda)^{1/\delta}, dx_T = \frac{1}{\delta} \left(\frac{1}{\lambda}\right)^{\frac{1}{\delta}} (y)^{\frac{1}{\delta}-1} dy$$

$$= \sum_{j=0}^{\infty} \frac{(t)^j}{j!} \lambda \left(\frac{1}{\lambda}\right)^{\frac{j+\delta}{\delta}} \int_0^{\lambda(b_1)^\delta} (y)^{\frac{j}{\delta}+1-1} e^{-y} dy$$

$$I_1 = \sum_{j=0}^{\infty} \frac{(t)^j}{j!} (\omega^\delta)^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta} + 1, \frac{1}{\omega^\delta} (b_1)^\delta\right)$$

If take: $I_2 = \int_{a_2}^{\infty} e^{t x_T} f(x_T) dx_T$, by using the same above suppositions;

$$\begin{aligned} I_2 &= \sum_{j=0}^{\infty} \frac{(t)^j}{j!} \lambda \left(\frac{1}{\lambda}\right)^{\frac{j+\delta}{\delta}} \int_{\lambda(a_2)^\delta}^{\infty} (y)^{\left(\frac{j}{\delta}\right)+1-1} e^{-y} dy \\ &= \sum_{j=0}^{\infty} \frac{(t)^j}{j!} (\omega^\delta)^{\frac{j}{\delta}} \Gamma\left(\frac{j}{\delta} + 1, \frac{1}{\omega^\delta} (a_2)^\delta\right) \end{aligned}$$

$$\mathcal{M}_{x_T}(t) = \frac{\sum_{j=0}^{\infty} \frac{(t)^j}{j!} (\omega^\delta)^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta} + 1, \frac{1}{\omega^\delta} (b_1)^\delta\right)}{e^{-\left(\frac{a_1}{\omega}\right)^\delta} - 1} + \frac{\sum_{j=0}^{\infty} \frac{(t)^j}{j!} (\omega^\delta)^{\frac{j}{\delta}} \Gamma\left(\frac{j}{\delta} + 1, \frac{1}{\omega^\delta} (a_2)^\delta\right)}{e^{-\left(\frac{a_2}{\omega}\right)^\delta}} \quad (2.22)$$

2.4 Two - Double Truncated Weibull Distribution (TDTWD)

In this subsection we will be presented, a double truncated Weibull distribution with two deleted intervals from the original interval.

2.4.1 The probability function of two-double truncated Weibull distribution

The probability density function of two intervals truncated double truncated for Weibull distribution is:

$$\mathbb{G}_3(x_T) = \begin{cases} \frac{\frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta}}{3 \left(e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta} \right)} & 0 < x_T < b_1 \\ \frac{\frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta}}{3 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta} \right)} & a_2 < x_T < b_2 \\ \frac{\frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta}}{3 \left(e^{-\left(\frac{a_3}{\omega}\right)^\delta} - e^{-\left(\frac{b_3}{\omega}\right)^\delta} \right)} & a_3 < x_T < \infty \end{cases} \quad (2.23)$$

That is, the intervals $(b_1, a_2), (b_2, a_3)$, are deleted from the original interval.

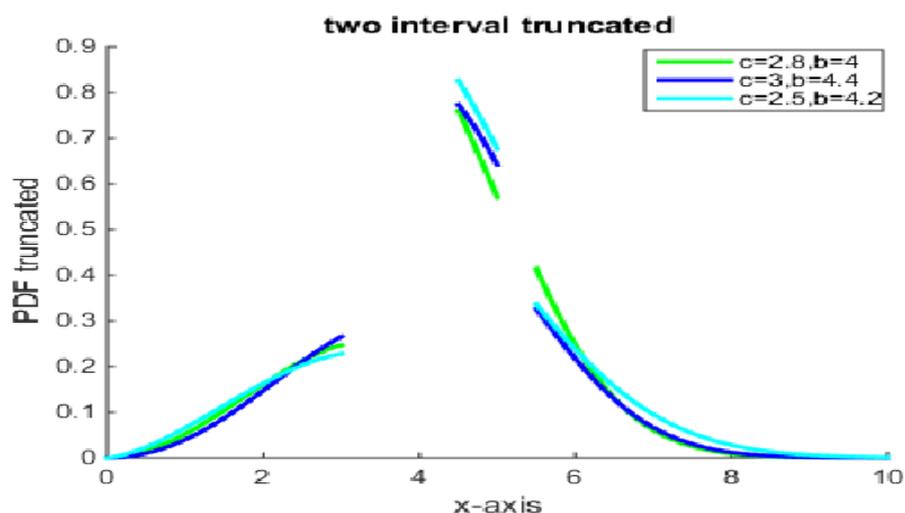


FIGURE 2.7 PDF of (TDTWD), $\delta = 2, 3, 2$, $\omega = 4, 4, 4, 4, 2$

In this figure, two-double truncated of Weibull distribution for different values of parameters δ, ω , when drawing the curves of PDF have been presented. The two intervals which deleted are: $(b_1, a_2) = (3, 4)$, and $(b_2, a_3) = (5, 5.5)$.

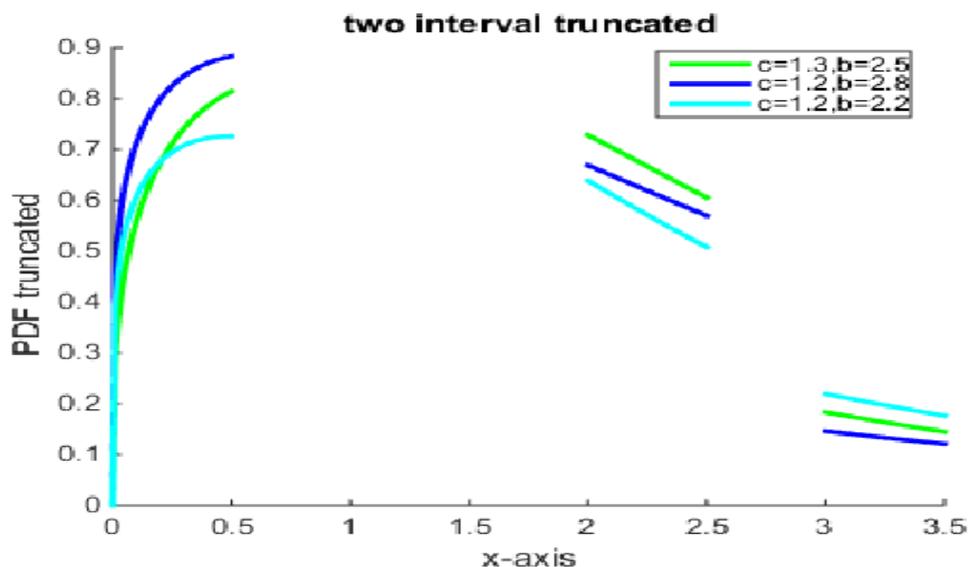


FIGURE 2.8 PDF of (TDTWD) $\omega = 2.5, 2.8, 2.2$

Two-double truncated of Weibull distribution for different values of parameters δ, ω , when drawing the curves of PDF. The two intervals which deleted are: $(b_1, a_2) = (0.5, 2)$, and $(b_2, a_3) = (2.5, 3)$, which are presented in figure (2-8).

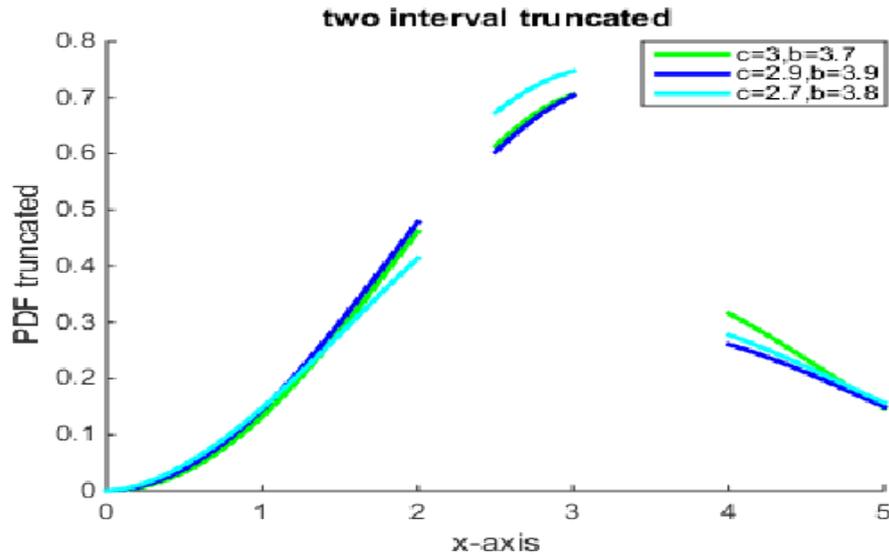


FIGURE 2.9 PDF of (TDTWD) , $\delta = 3, 2.9, 2.7$, $\omega = 3.7, 3.9, 3.8$

Two-double truncated of Weibull distribution for $\delta = 3, 2.9, 2.7$, $\omega = 3.7, 3.9, 3.8$, when drawing the curves of PDF. The two intervals which deleted are: (2,2.5), and (3,4), which are presented in figure (2-9).

2.4.2 Cumulative Distribution Function of Two- Double Truncated Weibull

The cumulative distribution function of two intervals truncated of Weibull distribution is:

$$\begin{aligned}
 \mathbb{G}_3(x_T) = & \left(\frac{e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{x_T}{\omega}\right)^\delta}}{3 \left(e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta} \right)} \right) + \left(\frac{e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{x_T}{\omega}\right)^\delta} + e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta}}{3 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta} \right)} \right) \\
 & + \left(\frac{e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{x_T}{\omega}\right)^\delta} + e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta} + e^{-\left(\frac{a_3}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta}}{3 \left(e^{-\left(\frac{a_3}{\omega}\right)^\delta} - e^{-\left(\frac{b_3}{\omega}\right)^\delta} \right)} \right) \quad (2.25)
 \end{aligned}$$

Figures (2.10),(2.11), and (2.12), presented some examples of the truncated cumulative distribution function for different values of parameters δ, ω : two-double truncated Weibull distribution, where $(a_1, b_1) = (0, 5)$, $(a_2, b_2) = (10, 25)$, that is, the intervals which deleted were (5,10),(15,20), in figure (2.10), while in both figures (2.11),(2.12), the intervals which deleted were: $\{(1,5),(7,9)\}$.

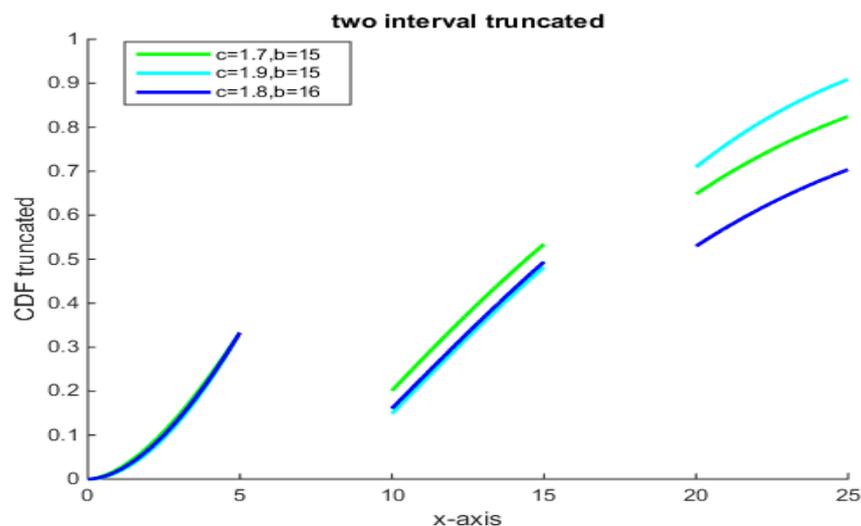


FIGURE 2.10 CDF of (TDTWD), $\delta = 1.7, 1.9, 1.8$

In this figure, two-double truncated Weibull distribution, where $(a_1, b_1) = (0, 5)$, $(a_2, b_2) = (10, 15)$, and $(a_3, b_3) = (20, 25)$ that is, the intervals which deleted are $(5, 10), (15, 20)$.

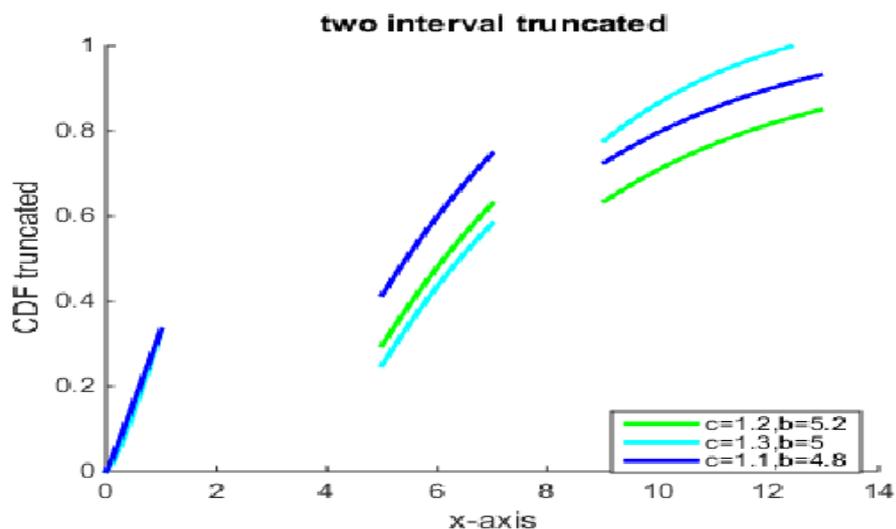


FIGURE 2.11 CDF of (TDTWD) $\delta = 1.2, 1.3, 1.1, \omega = 5.2, 5, 4.8$

In figure (2.11), CDF of two-double truncated Weibull distribution has been introduced, where $(a_1, b_1) = (0, 1)$, $(a_2, b_2) = (5, 7)$, and $(a_3, b_3) = (9, 14)$ that is, the intervals which deleted are $\{(1, 5), (7, 9)\}$

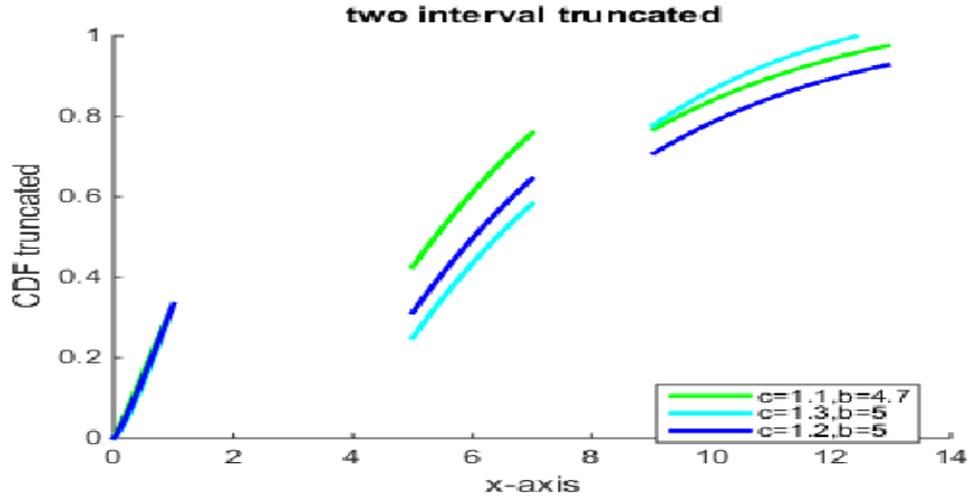


FIGURE 2.12 CDF of (TDTWD) $\delta = 1.1, 1.3, 1.2, \omega = 4.7, 5, 5$

While this figure, presented CDF of two-double truncated Weibull distribution, where $(a_1, b_1) = (0, 1)$, $(a_2, b_2) = (5, 7)$, and $(a_3, b_3) = (9, 14)$ that is, the intervals which deleted are $\{(1, 5), (7, 9)\}$, where $\delta = 1.1, 1.3, 1.2, \omega = 4.7, 5, 5$.

2.4.3 Two-Double Truncated Survival Function of Weibull Distribution

In this subsection, two intervals from the multi-double truncated survival function of Weibull distribution are deleted as:

$$\begin{aligned} \mathbb{S}_3(x_T) &= 1 - \left(\frac{e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{x_T}{\omega}\right)^\delta}}{3 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta} \right)} \right) - \left(\frac{e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{x_T}{\omega}\right)^\delta} + e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta}}{3 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta} \right)} \right) \\ &\quad - \left(\frac{e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{x_T}{\omega}\right)^\delta} + e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta} + e^{-\left(\frac{a_3}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta}}{3 \left(e^{-\left(\frac{a_3}{\omega}\right)^\delta} \right)} \right) \\ \mathbb{S}_3(x_T) &= \left(\frac{3 \left(e^{-\left(\frac{a_1}{\omega}\right)^\delta} \right)^2 - 6 e^{-\left(\frac{a_1}{\omega}\right)^\delta} e^{-\left(\frac{b_1}{\omega}\right)^\delta} + 3 \left(e^{-\left(\frac{b_1}{\omega}\right)^\delta} \right)^2}{3 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta} \right)} \right) - \left(\frac{e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{x_T}{\omega}\right)^\delta} + e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta}}{3 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta} \right)} \right) \end{aligned}$$

$$-\left(\frac{e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{x_T}{\omega}\right)^\delta} + e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta} + e^{-\left(\frac{a_3}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta}}{3\left(e^{-\left(\frac{a_3}{\omega}\right)^\delta}\right)}\right) \quad (2.26)$$

The survival function of two-double truncated for Weibull distribution, such that, the two-interval that deleted are: (1,5),(7,9) and (5,10) , (15,20) are introduced in figures (2.13),(2.14), for different values of parameters respectively.

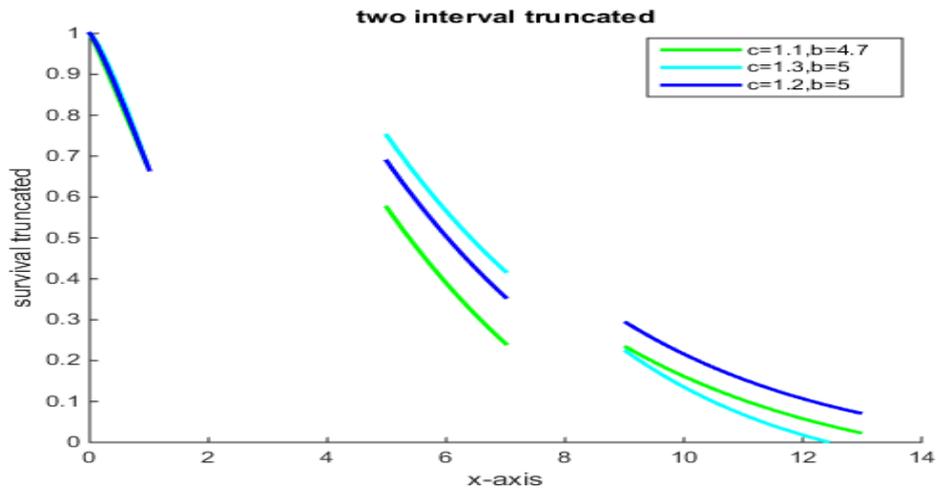


FIGURE 2.13 Survival of (TDTWD) $\delta = 1.1, 1.3, 1.2, \omega = 4.7, 5, 5$

The survival function of (TDTWD), such that the intervals which deleted are (1,5),(7,9).

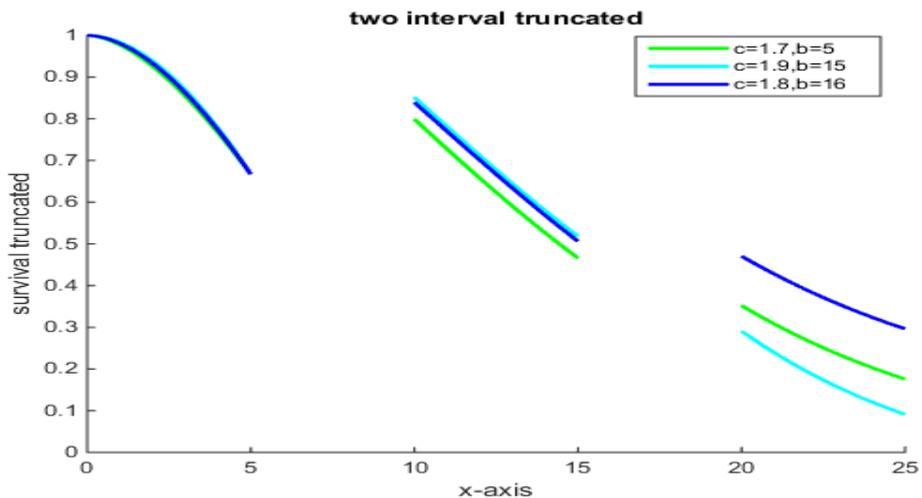


FIGURE 2.14 Survival of (TDTWD) $\delta = 1.7, 1.9, 1.8, \omega = 5, 15, 16$

The survival function of (TDTWD), such that the intervals which deleted are (5,10),(15,20).

2.4.4 The r^{th} Moment About Origin of Two Double Truncated Weibull Distribution

$$\begin{aligned} E(X_T^r) &= \int_{-\infty}^{\infty} x^r \mathfrak{G}_n(x_T) dx_T = \frac{1}{3} \sum_{i=1}^3 \left(\frac{\int_{a_i}^{b_i} x^r f(x_T) dx_T}{F(b_i) - F(a_i)} \right) \\ &= \frac{1}{3} \frac{\int_{a_1}^{b_1} x^r f(x_T) dx_T}{F(b_1) - F(a_1)} + \frac{1}{3} \frac{\int_{a_2}^{b_2} x^r f(x_T) dx_T}{F(b_2) - F(a_2)} + \frac{1}{3} \frac{\int_{a_3}^{b_3} x^r f(x_T) dx_T}{F(b_3) - F(a_3)} \\ &= \frac{\int_0^{b_1} \frac{\delta x_T^r}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta} dx_T}{3 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\int_{a_2}^{b_2} \frac{\delta x_T^r}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta} dx_T}{3 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta}\right)} + \frac{\int_{a_3}^{\infty} \frac{\delta x_T^r}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta} dx_T}{3 \left(e^{-\left(\frac{a_3}{\omega}\right)^\delta}\right)} \end{aligned}$$

$$\text{Take: } I_1 = \int_{a_1}^{b_1} x_T^r \frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta} dx_T = \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} \gamma\left(\frac{r}{\delta} + 1, \lambda b_1^\delta\right)$$

$$\begin{aligned} \text{So, } I_2 &= \int_{a_2}^{b_2} x_T^r \frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta} dx_T \\ &= \lambda \int_{\lambda a_2^\delta}^{\lambda b_2^\delta} \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} z^{\frac{r}{\delta}+1-1} e^{-z} \frac{dz}{\lambda} = \int_{\lambda a_2^\delta}^{\lambda b_2^\delta} \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} z^{\frac{r}{\delta}+1-1} e^{-z} dz \end{aligned}$$

$$I_2 = \left(\frac{1}{\lambda}\right)^{\frac{r}{\delta}} \left(\gamma\left(\frac{r}{\delta} + 1, \lambda b_2^\delta\right) - \gamma\left(\frac{r}{\delta} + 1, \lambda a_2^\delta\right) \right),$$

$$E(X_T^r) = \frac{\omega^r \gamma\left(\frac{r}{\delta} + 1, \lambda b_1^\delta\right)}{3 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\omega^r \left(\gamma\left(\frac{r}{\delta} + 1, \lambda b_2^\delta\right) - \gamma\left(\frac{r}{\delta} + 1, \lambda a_2^\delta\right) \right)}{3 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta} \right)} + \frac{\omega^r \Gamma\left(\frac{r}{\delta} + 1, \lambda a_3^\delta\right)}{3 \left(e^{-\left(\frac{a_3}{\omega}\right)^\delta} \right)} \quad (2.27)$$

$$E(X_T) = \frac{(\omega^\delta)^{\frac{1}{\delta}} \gamma\left(\frac{1}{\delta} + 1, \lambda b_1^\delta\right)}{3 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{(\omega^\delta)^{\frac{1}{\delta}} \left(\gamma\left(\frac{1}{\delta} + 1, \lambda b_2^\delta\right) - \gamma\left(\frac{1}{\delta} + 1, \lambda a_2^\delta\right) \right)}{3 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta} \right)} + \frac{(\omega^\delta)^{\frac{1}{\delta}} \Gamma\left(\frac{1}{\delta} + 1, \lambda a_3^\delta\right)}{3 \left(e^{-\left(\frac{a_3}{\omega}\right)^\delta} \right)} \quad (2.28)$$

2.4.5 The r^{th} Moment About Mean of Two Double Truncated Weibull Distribution

$$E(X_T - \mu)^r = \int_{-\infty}^{\infty} (x_T - \mu)^r \mathfrak{G}_n(x_T) dx_T =$$

$$= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \left[\frac{\omega \gamma\left(\frac{1}{\delta} + 1, \lambda b_1^\delta\right)}{3 \left(1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{\omega \left(\gamma\left(\frac{1}{\delta} + 1, \lambda b_2^\delta\right) - \gamma\left(\frac{1}{\delta} + 1, \lambda a_2^\delta\right) \right)}{3 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta} \right)} + \right.$$

$$\left. \frac{\omega \Gamma\left(\frac{1}{\delta}+1, \lambda a_3 \delta\right)}{3\left(e^{-\left(\frac{a_3}{\omega}\right)^\delta}\right)} \right]^{r-j} \left[\frac{(\omega^\delta)^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta}+1, \lambda b_1 \delta\right)}{3\left(1-e^{-\left(\frac{b_1}{\omega}\right)^\delta}\right)} + \frac{(\omega^\delta)^{\frac{j}{\delta}} \left(\gamma\left(\frac{j}{\delta}+1, \lambda b_2 \delta\right) - \gamma\left(\frac{j}{\delta}+1, \lambda a_2 \delta\right)\right)}{3\left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta}\right)} + \frac{(\omega^\delta)^{\frac{j}{\delta}} \Gamma\left(\frac{j}{\delta}+1, \lambda a_3 \delta\right)}{3\left(e^{-\left(\frac{a_3}{\omega}\right)^\delta}\right)} \right] \quad (2.29)$$

2.4.6 The Moment Generating Function of TDTWD

The moment generating function of the (TDTWD) random variable is given by:

$$\begin{aligned} \mathcal{M}_{x_T}(t) &= \int e^{tx_T} (\mathbb{G}_n(x_T)) dx_T = \frac{1}{2} \sum_{i=1}^2 \left(\frac{\int_{a_i}^{b_i} e^{tx_T} f(x_T) dx_T}{F(b_i) - F(a_i)} \right) \\ &= \frac{\int_{a_1}^{b_1} e^{tx_T} f(x_T) dx_T}{F(b_1) - F(a_1)} + \frac{\int_{a_2}^{b_2} e^{tx_T} f(x_T) dx_T}{F(b_2) - F(a_2)} + \frac{\int_{a_3}^{b_3} e^{tx_T} f(x_T) dx_T}{F(b_3) - F(a_3)} \end{aligned}$$

$$\text{Let } I_1 = \int_0^{b_1} e^{tx_T} f(x_T) dx_T = \sum_{j=0}^{\infty} \frac{(t)^j}{j!} (\omega^\delta)^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta} + 1, \frac{1}{\omega^\delta} (b_1)^\delta\right)$$

$$I_2 = \int_{a_2}^{b_2} e^{tx_T} \frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta} dx_T = \sum_{j=0}^{\infty} \frac{(t)^j}{j!} (\omega^\delta)^{\frac{j}{\delta}} \left\{ \gamma\left(\frac{j}{\delta} + 1, \frac{(b_1)^\delta}{\omega^\delta}\right) - \gamma\left(\frac{j}{\delta} + 1, \frac{(b_2)^\delta}{\omega^\delta}\right) \right\}$$

$$I_3 = \int_{a_3}^{\infty} e^{tx_T} f(x_T) dx_T = \sum_{j=0}^{\infty} \frac{(t)^j}{j!} (\omega^\delta)^{\frac{j}{\delta}} \Gamma\left(\frac{j+\delta}{\delta}, \frac{(a_3)^\delta}{\omega^\delta}\right)$$

$$\mathcal{M}_{x_T}(t) =$$

$$\begin{aligned} & \frac{\sum_{j=0}^{\infty} \frac{(t)^j}{j!} (\omega^\delta)^{\frac{j}{\delta}} \gamma\left(\frac{j}{\delta}+1, \frac{1}{\omega^\delta} (b_1)^\delta\right)}{1 - e^{-\left(\frac{b_1}{\omega}\right)^\delta}} + \\ & \frac{\sum_{j=0}^{\infty} \frac{(t)^j}{j!} (\omega^\delta)^{\frac{j}{\delta}} \left\{ \gamma\left(\frac{j}{\delta}+1, \frac{(b_1)^\delta}{\omega^\delta}\right) - \gamma\left(\frac{j}{\delta}+1, \frac{(b_2)^\delta}{\omega^\delta}\right) \right\}}{e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta}} + \frac{\sum_{j=0}^{\infty} \frac{(t)^j}{j!} (\omega^\delta)^{\frac{j}{\delta}} \Gamma\left(\frac{j+\delta}{\delta}, \frac{(a_3)^\delta}{\omega^\delta}\right)}{e^{-\left(\frac{a_3}{\omega}\right)^\delta}} \end{aligned} \quad (2.30)$$

CHAPTER THREE

MULTI- DOUBLE TRUNCATED OF RAYLEIGH DISTRIBUTION

In this chapter, another multi-truncated distribution has been introduced, that is deletes some (subintervals) from an original range of data values from a domain for the random variables for double truncation type, this distribution is the Rayleigh distribution. Some statistical properties such as cumulative, survival, hazard, reversed hazard, moment generating, and characteristic functions are derived. In addition to obtaining the r^{th} moment of the new distribution, order statistics. Shapes of one and two truncated Rayleigh distributions (PDF, CDF, survival) were represented by the graphs.

3.1 Deriving of Multi-Double Truncated Rayleigh Distribution (MDTRD)

The Rayleigh distribution is one of the most important distributions in the analysis of data for many fields related to life: medical, and age of production. The Rayleigh distribution was proposed in acoustics and optics by Lord Rayleigh and it became widely used in radio signals, oceanography, communication, and for modeling wave propagation, radiation, radar, survival function, etc. hazard of Rayleigh is increasing linearly, therefore the monotone property of the hazard rate Rayleigh distribution helpfully a model for the lifetimes of components that age quickly with time.

Let X , be a random variable, has a probability Rayleigh function, $f_X(x)$, such that $x \in (0, a_1) \cup (b_1, a_2) \cup \dots \dots \dots \cup (b_{n-1}, a_n)$. The truncation is to delete some subintervals from the original interval. In this section, we will derive the probability density $g(x_T)$ of the random variable has Rayleigh distribution, X_T , after multi-double subintervals have been deleted, the observations of X_T to (b_{i-1}, a_i) . To find the formula that represents multi- double intervals truncation of Rayleigh distribution on the original interval, we will partition the original interval as:

$$(-\infty, \infty) = (a_1, b_1) \cup (a_2, b_2) \cup \dots \dots \dots, (a_n, b_n).$$

or, $(-\infty, \infty) = (b_0, a_1) \cup (b_1, a_2) \cup \dots \dots \dots \cup (b_{n-1}, a_n).$

Then the probability density function of multi-intervals double truncated in Rayleigh distribution is:

$$\mathbb{G}_n(x_T) = \left(\frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{\left(\frac{-b_0^2}{e \frac{2\sigma^2}{2\sigma^2}} - e^{-\frac{a_1^2}{2\sigma^2}} \right)} + \frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{\left(\frac{-b_1^2}{e \frac{2\sigma^2}{2\sigma^2}} - e^{-\frac{a_2^2}{2\sigma^2}} \right)} + \dots + \frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{\left(\frac{-b_{n-1}^2}{e \frac{2\sigma^2}{2\sigma^2}} - e^{-\frac{a_n^2}{2\sigma^2}} \right)} \right)$$

$$\mathbb{G}_n(x_T) = \begin{cases} \sum_{i=1}^n \frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{n \left(\frac{-b_{i-1}^2}{e \frac{2\sigma^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}} \right)} & \text{when } b_{i-1} < x_T < a_i \\ 0 & \text{otherwise} \end{cases} \quad (3.1)$$

The probability density function $\mathbb{G}_n(x_T)$ is a pdf over $(0, \infty), 0 < x_T < \infty$, $n, F(a_i) - F(b_{i-1}) > 0$, then $\mathbb{G}_n(x_T) > 0$ for all $x_T \in (0, \infty)$, and $\int_0^\infty \mathbb{G}_n(x_T) dx_T = 1$.

Note: In this chapter we used $a_1 = b_0, a_2 = b_1, \dots, a_i = b_{i-1}$

3.1.1 The Cumulative Distribution Function For Multi-Double Truncated Rayleigh Distribution

$$\mathbb{G}_n(x_T) = P(X_T \leq x_T),$$

$$= \frac{1}{n} \sum_{i=1}^n \left(\frac{F(x_T) I_{(b_{i-1}, a_i]} - \sum_{j=1}^{i-1} F(b_{j-1}) + \sum_{j=1}^n F(a_j)}{F(a_i) - F(b_{i-1})} \right)$$

$$\frac{1}{n} \left[\frac{F(x_T) - F(b_0)}{F(a_1) - F(b_0)} + \frac{F(x_T) - F(b_1) - F(b_0) + F(a_1)}{F(a_2) - F(b_1)} + \right.$$

$$\left. \frac{F(x_T) - F(b_2) - F(b_1) - F(b_0) + F(a_1) + F(a_2)}{F(a_3) - F(b_2)} + \right.$$

$$\left. \frac{F(x_T) - F(b_3) - F(b_2) - F(b_1) - F(b_0) + F(a_3) + F(a_2) + F(a_1)}{F(a_4) - F(b_3)} \right]$$

$$\begin{aligned}
 & + \dots + \left[\frac{\mathbb{F}(x_T) - \mathbb{F}(b_{n-1}) - \mathbb{F}(b_{n-2}) - \dots - \mathbb{F}(b_{n-n}) + \mathbb{F}(a_n) + \mathbb{F}(a_{n-1}) + \dots + \mathbb{F}(a_1)}{\mathbb{F}(a_n) - \mathbb{F}(b_{n-1})} \right] \\
 & = \frac{\frac{-b_0^2}{e^{2\sigma^2}} - \frac{-x_T^2}{e^{2\sigma^2}}}{\left(e^{2\sigma^2} - e^{\frac{-a_1^2}{2\sigma^2}} \right)} + \dots + \frac{1 - e^{\frac{-x_T^2}{2\sigma^2}} - \left(1 - e^{\frac{-b_1^2}{2\sigma^2}} \right) - \left(1 - e^{\frac{-b_0^2}{2\sigma^2}} \right) + \left(1 - e^{\frac{-a_1^2}{2\sigma^2}} \right)}{\left(e^{\frac{-b_1^2}{2\sigma^2}} - e^{\frac{-a_2^2}{2\sigma^2}} \right)} \\
 & + \frac{1 - e^{\frac{-x_T^2}{2\sigma^2}} - \left(1 - e^{\frac{-b_2^2}{2\sigma^2}} \right) - \left(1 - e^{\frac{-b_1^2}{2\sigma^2}} \right) - \left(1 - e^{\frac{-b_0^2}{2\sigma^2}} \right) + \left(1 - e^{\frac{-a_1^2}{2\sigma^2}} \right) + \left(1 - e^{\frac{-a_2^2}{2\sigma^2}} \right)}{\left(e^{\frac{-b_2^2}{2\sigma^2}} - e^{\frac{-a_3^2}{2\sigma^2}} \right)} \\
 & \left[\frac{1 - e^{\frac{-x_T^2}{2\sigma^2}} - \left(1 - e^{\frac{-b_{n-1}^2}{2\sigma^2}} \right) - \left(1 - e^{\frac{-b_{n-2}^2}{2\sigma^2}} \right) \dots}{\left(e^{\frac{-b_{n-1}^2}{2\sigma^2}} - e^{\frac{-a_n^2}{2\sigma^2}} \right)} \right. \\
 & \left. - \left(1 - e^{\frac{-b_{n-n}^2}{2\sigma^2}} \right) + \left(1 - e^{\frac{-a_n^2}{2\sigma^2}} \right) + \left(1 - e^{\frac{-a_{n-1}^2}{2\sigma^2}} \right) + \left(1 - e^{\frac{-a_1^2}{2\sigma^2}} \right) \right] \\
 & = \frac{\frac{-b_0^2}{e^{2\sigma^2}} - \frac{-x_T^2}{e^{2\sigma^2}}}{\left(e^{2\sigma^2} - e^{\frac{-a_1^2}{2\sigma^2}} \right)} + \dots + \frac{\frac{-b_1^2}{e^{2\sigma^2}} - \frac{-x_T^2}{e^{2\sigma^2}} + \frac{-b_0^2}{e^{2\sigma^2}} - \frac{-a_1^2}{e^{2\sigma^2}}}{\left(e^{\frac{-b_1^2}{2\sigma^2}} - e^{\frac{-a_2^2}{2\sigma^2}} \right)} \\
 & + \frac{\frac{-b_2^2}{e^{2\sigma^2}} + \frac{-b_1^2}{e^{2\sigma^2}} - \frac{-x_T^2}{e^{2\sigma^2}} + \frac{-b_0^2}{e^{2\sigma^2}} - \frac{-a_1^2}{e^{2\sigma^2}} - \frac{-a_2^2}{e^{2\sigma^2}}}{\left(e^{2\sigma^2} - e^{\frac{-a_3^2}{2\sigma^2}} \right)} \\
 & + \frac{\frac{-x_T^2}{e^{2\sigma^2}} + \frac{-b_{n-1}^2}{e^{2\sigma^2}} + \frac{-b_{n-2}^2}{e^{2\sigma^2}} + \dots + \frac{-b_{n-n}^2}{e^{2\sigma^2}} - \frac{-a_n^2}{e^{2\sigma^2}} - \frac{-a_{n-1}^2}{e^{2\sigma^2}} - \dots - \frac{-a_1^2}{e^{2\sigma^2}}}{\left(e^{\frac{-b_{n-1}^2}{2\sigma^2}} - e^{\frac{-a_n^2}{2\sigma^2}} \right)}
 \end{aligned}$$

Thus,

$$\mathbb{G}_n(x_T) = \sum_{i=1}^n \left(\frac{e^{\frac{-x_T^2}{2\sigma^2}} - \sum_{j=1}^n \left(e^{\frac{-b_{j-1}^2}{2\sigma^2}} - e^{\frac{-a_j^2}{2\sigma^2}} \right)}{n \left(e^{\frac{-b_{i-1}^2}{2\sigma^2}} - e^{\frac{-a_i^2}{2\sigma^2}} \right)} \right) \tag{3.2}$$

3.1.2 Other Functions of Multi-Double Truncated Rayleigh Distribution

The Survival Function is:

$$S_n(x_T) = 1 - G_n(x_T) = 1 - \frac{1}{n} \sum_{i=1}^n \left(\frac{e^{-\frac{x_T^2}{2\sigma^2}} - \sum_{j=1}^n \left(e^{-\frac{b_{j-1}^2}{2\sigma^2}} - e^{-\frac{a_j^2}{2\sigma^2}} \right)}{n \left(e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}} \right)} \right) \quad (3.3)$$

Hazard Function,

$$H_n(x_T) = \frac{g_n(x_T)}{S_n(x_T)} = \frac{\sum_{i=1}^n \left(\frac{x_T e^{-\frac{x_T^2}{2\sigma^2}}}{n\sigma^2 \left[e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}} \right]} \right)}{1 - \frac{1}{n} \sum_{i=1}^n \left(\frac{e^{-\frac{x_T^2}{2\sigma^2}} - \sum_{j=1}^n \left(e^{-\frac{b_{j-1}^2}{2\sigma^2}} - e^{-\frac{a_j^2}{2\sigma^2}} \right)}{n \left(e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}} \right)} \right)} \quad (3.4)$$

Reversed Hazard Function, $\tilde{H}_n(x_T)$

$$\tilde{H}_n(x_T) = \frac{g_n(x_T)}{G_n(x_T)} = \frac{\sum_{i=1}^n \left(\frac{x_T e^{-\frac{x_T^2}{2\sigma^2}}}{n\sigma^2 \left[e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}} \right]} \right)}{\sum_{i=1}^n \left(\frac{e^{-\frac{x_T^2}{2\sigma^2}} - \sum_{j=1}^n \left(e^{-\frac{b_{j-1}^2}{2\sigma^2}} - e^{-\frac{a_j^2}{2\sigma^2}} \right)}{n \left(e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}} \right)} \right)} \quad (3.5)$$

Theorem 3.1

Let x_T be random variable has probability density function is multi-double truncated Rayleigh distribution, $\mathfrak{G}_n(x_T)$ and cumulative distribution function, $\mathbb{G}_n(x_T)$, then the r^{th} moment about origin of x_T is;

$$E(X_T^r) = \frac{(2\sigma^2)^{\frac{r}{2}} \left[\gamma\left(\frac{r}{2}+1, \frac{a_i^2}{2\sigma^2}\right) - \gamma\left(\frac{r}{2}+1, \frac{b_{i-1}^2}{2\sigma^2}\right) \right]}{\mathbb{F}(a_i) - \mathbb{F}(b_{i-1})}, b_{i-1} < x_T < a_i$$

Proof:-

$$\begin{aligned} E(X_T^r) &= \frac{1}{n} \left\{ \frac{\int_{b_0}^{a_1} x_T^r f(x_T) dx_T}{\mathbb{F}(a_1) - \mathbb{F}(b_0)} + \frac{\int_{b_1}^{a_2} x_T^r f(x_T) dx_T}{\mathbb{F}(a_2) - \mathbb{F}(b_1)} + \dots + \frac{\int_{b_{n-1}}^{a_n} x_T^r f(x_T) dx_T}{\mathbb{F}(a_n) - \mathbb{F}(b_{n-1})} \right\} \\ &= \frac{1}{n} \left\{ \frac{\int_{b_0}^{a_1} x_T^r f(x_T) dx_T}{\mathbb{F}(a_1) - \mathbb{F}(b_0)} + \frac{\int_{b_1}^{a_2} x_T^r f(x_T) dx_T}{\mathbb{F}(a_2) - \mathbb{F}(b_1)} + \dots + \frac{\int_{b_{n-1}}^{a_n} x_T^r f(x_T) dx_T}{\mathbb{F}(a_n) - \mathbb{F}(b_{n-1})} \right\} \\ &= \frac{1}{n} \left\{ \frac{\int_{b_0}^{a_1} x_T^r \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}} dx_T}{\mathbb{F}(a_1) - \mathbb{F}(b_0)} + \frac{\int_{b_1}^{a_2} x_T^r \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}} dx_T}{\mathbb{F}(a_2) - \mathbb{F}(b_1)} + \dots + \frac{\int_{b_{n-1}}^{a_n} x_T^r \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}} dx_T}{\mathbb{F}(a_n) - \mathbb{F}(b_{n-1})} \right\} \end{aligned}$$

$$\text{Take: } I_1 = \int_{b_0}^{a_1} x_T^r \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}} dx_T = \frac{1}{\sigma^2} \int_{b_0}^{a_1} x_T^{r+1} e^{-\frac{x_T^2}{2\sigma^2}} dx_T$$

$$\text{Let } y = \frac{x_T^2}{2\sigma^2}, x_T = (2\sigma^2 y)^{1/2}, dx_T = (2\sigma^2)^{1/2} \frac{1}{2} (y)^{-1/2} dy$$

$$I_1 = \frac{1}{\sigma^2} \int_0^{a_1^2/2\sigma^2} ((2\sigma^2 y)^{1/2})^{r+1} e^{-y} \left(\frac{2\sigma^2}{2}\right)^{1/2} (y)^{-1/2} dy$$

$$= \left((2\sigma^2)^{\frac{1}{2}} \right)^{r+1} \left(\frac{(2\sigma^2)^{\frac{1}{2}}}{2} \right) \frac{1}{\sigma^2} \int_0^{a_1^2/2\sigma^2} \left((y)^{\frac{1}{2}} \right)^{r+1} e^{-y} \left((y)^{-\frac{1}{2}} dy \right)$$

$$= (2\sigma^2)^{\frac{r}{2}} \frac{1}{2\sigma^2} \int_0^{a_1^2/2\sigma^2} ((y))^{\frac{r}{2}+1-1} e^{-y} dy$$

$$I_1 = (2\sigma^2)^{\frac{r}{2}} \gamma\left(\frac{r}{2} + 1, a_1^2/2\sigma^2\right)$$

$$\text{Let } I_2 = \int_{b_1}^{a_2} x_T^r \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}} dx_T$$

$$= \left((2\sigma^2)^{\frac{1}{2}} \right)^{r+1} \left(\frac{(2\sigma^2)^{\frac{1}{2}}}{2} \right) \frac{1}{\sigma^2} \int_{b_1^2/2\sigma^2}^{a_2^2/2\sigma^2} \left((y)^{\frac{1}{2}} \right)^{r+1} e^{-y} \left((y)^{-\frac{1}{2}} dy \right)$$

$$I_2 = (2\sigma^2)^{\frac{r}{2}} \left[\gamma\left(\frac{r}{2} + 1, \frac{a_2^2}{2\sigma^2}\right) - \gamma\left(\frac{r}{2} + 1, \frac{b_1^2}{2\sigma^2}\right) \right]$$

Thus,

$$I_n = \int_{b_{n-1}}^{a_n} x_T^r \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}} dx_T = (2\sigma^2)^{\frac{r}{2}} \left[\gamma\left(\frac{r}{2} + 1, \frac{a_n^2}{2\sigma^2}\right) - \gamma\left(\frac{r}{2} + 1, \frac{b_{n-1}^2}{2\sigma^2}\right) \right]$$

$$E(X_T^r) = \frac{1}{n} \sum_{i=1}^n \frac{\left((2\sigma^2)^{\frac{r}{2}} \left[\gamma\left(\frac{r}{2} + 1, \frac{a_i^2}{2\sigma^2}\right) - \gamma\left(\frac{r}{2} + 1, \frac{b_{i-1}^2}{2\sigma^2}\right) \right] \right)}{e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}}}, \quad (3.6)$$

such that, The mean of multi-double Rayleigh truncated distribution is:

$$E(X_T) = \frac{1}{n} \sum_{i=1}^n \frac{\left((2\sigma^2)^{\frac{1}{2}} \left[\gamma\left(\frac{1}{2} + 1, \frac{a_i^2}{2\sigma^2}\right) - \gamma\left(\frac{1}{2} + 1, \frac{b_{i-1}^2}{2\sigma^2}\right) \right] \right)}{e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}}} \quad (3.7)$$

$$\text{And } E(X_T^2) = \frac{1}{n} \sum_{i=1}^n \frac{(2\sigma^2)^{\frac{2}{2}} \left[\gamma\left(2, \frac{a_i^2}{2\sigma^2}\right) - \gamma\left(2, \frac{b_{i-1}^2}{2\sigma^2}\right) \right]}{F(a_i) - F(b_{i-1})} \quad (3.8)$$

We can find the other formula for $E(X_T^r)$, see the next theorem:

Theorem 3.2

Let x_T be random variable has probability density function is multi-double truncated Rayleigh distribution, $\mathfrak{g}_n(x_T)$, then the moment about origin of x_T

$$\text{is: } \sum_{i=1}^n \frac{a_i^r F(a_i) - b_{i-1}^r F(b_{i-1}) - r \int_{b_{i-1}}^{a_i} x_T^{r-1} F(x_T) dx_T}{n(F(a_i) - F(b_{i-1}))}$$

$$\text{Proof: } E(X_T^r) = \int_{-\infty}^{\infty} x_T^r \mathfrak{g}_n(x_T) dx_T$$

=

$$\frac{1}{n} \left\{ \frac{a_1^r F(a_1) - b_0^r F(b_0) - r \int_{b_0}^{a_1} x_T^{r-1} F(x_T) dx_T}{F(a_1) - F(b_0)} + \right.$$

$$\frac{a_2^r F(a_2) - b_1^r F(b_1) - r \int_{b_1}^{a_2} x_T^{r-1} F(x_T) dx_T}{F(a_2) - F(b_1)} +$$

$$\dots + \frac{a_n^r F(a_n) - b_{n-1}^r F(b_{n-1}) - r \int_{b_{n-1}}^{a_n} x_T^{r-1} F(x_T) dx_T}{F(a_n) - F(b_{n-1})}$$

For some r , we take:

$$I_1 = a_1^r F(a_1) - b_0^r F(b_0) - r \int_{b_0}^{a_1} x_T^{r-1} F(x_T) dx_T$$

$$= a_1^r F(a_1) - b_0^r F(b_0) - r \int_{b_0}^{a_1} x_T^{r-1} \left(1 - e^{-\frac{x_T^2}{2\sigma^2}}\right) dx_T$$

$$= a_1^r F(a_1) - b_0^r F(b_0) - r \int_{b_0}^{a_1} x_T^{r-1} dx_T + r \int_{b_0}^{a_1} x_T^{r-1} e^{-\frac{x_T^2}{2\sigma^2}} dx_T$$

$$I_1 = -a_1^r e^{-\frac{x_T^2}{2\sigma^2}} + r \int_{b_0}^{a_1} x_T^{r-1} e^{-\frac{x_T^2}{2\sigma^2}} dx_T$$

When $r=1$,

$$I_1 = -a_1 e^{-\frac{x_T^2}{2\sigma^2}} + \int_{b_0}^{a_1} e^{-\frac{x_T^2}{2\sigma^2}} dx_T$$

Integration I_1 by part integration, we get:

$$I_1 = \left(-a_1 e^{-\frac{a_1^2}{2\sigma^2}}\right) + \frac{1}{2} \sqrt{2\sigma^2} \sqrt{\pi} \operatorname{erf}\left(\frac{a_1}{\sqrt{2\sigma^2}}\right), \operatorname{erf}(0) = 0, \quad b_0 < x_T < a_1.$$

$$I_1 = \left(b_{i-1} e^{-\frac{b_{i-1}^2}{2\sigma^2}}\right) - \left(a_i e^{-\frac{a_i^2}{2\sigma^2}}\right)$$

$$+ \frac{1}{2} \sqrt{2\sigma^2} \sqrt{\pi} \left(\operatorname{erf}\left(\frac{a_i}{\sqrt{2\sigma^2}}\right) - \operatorname{erf}\left(\frac{b_{i-1}}{\sqrt{2\sigma^2}}\right)\right), b_{i-1} < x_T < a_i$$

$$\text{Then, } E(X_T) = \left(\frac{1}{n}\right) \sum_{i=0}^{n-1} \left(\frac{\frac{\sqrt{\sigma^2} \sqrt{\pi}}{2} \left(\operatorname{erf}\left(\frac{a_i}{\sqrt{2\sigma^2}}\right) - \operatorname{erf}\left(\frac{b_{i-1}}{\sqrt{2\sigma^2}}\right)\right) - a_i e^{-\frac{a_i^2}{2\sigma^2}} + b_{i-1} e^{-\frac{b_{i-1}^2}{2\sigma^2}}}{F(a_i) - F(b_{i-1})}\right)$$

When $r=2$,

$$I_2 = a_1^2 F(a_1) - b_0^2 F(b_0) + 2 \int_{b_0}^{a_1} x_T e^{-\frac{x_T^2}{2\sigma^2}} dx_T$$

$$= -a_1^2 e^{-\frac{a_1^2}{2\sigma^2}} + b_0^2 e^{-\frac{b_0^2}{2\sigma^2}} - 2\sigma^2 (e^{-\frac{a_1^2}{2\sigma^2}} - e^{-\frac{b_0^2}{2\sigma^2}}),$$

$$I_2 = -a_1^2 e^{-\frac{a_1^2}{2\sigma^2}} + b_0^2 e^{-\frac{b_0^2}{2\sigma^2}} - 2\sigma^2 (e^{-\frac{a_1^2}{2\sigma^2}} - 1), \quad b_0 < x_T < a_1$$

$$I_2 = -a_i^2 e^{-\frac{a_i^2}{2\sigma^2}} + b_{i-1}^2 e^{-\frac{b_{i-1}^2}{2\sigma^2}} - 2\sigma^2 (e^{-\frac{a_i^2}{2\sigma^2}} - e^{-\frac{b_{i-1}^2}{2\sigma^2}}), \quad b_{i-1} < x_T < a_i,$$

$$E(X_T^2) = \frac{1}{n} \sum_{i=1}^n \frac{\left(-e^{-\frac{a_i^2}{2\sigma^2}} (a_i^2 + 2\sigma^2) + e^{-\frac{b_{i-1}^2}{2\sigma^2}} (b_{i-1}^2 + 2\sigma^2) \right)}{e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}}}$$

when $r = 3$

$$I_3 = a_1^3 F(a_1) - b_0^3 F(b_0) - 3 \int_{b_0}^{a_1} x_T^2 dx_T + 3 \int_{b_0}^{a_1} x_T^2 e^{-\frac{x_T^2}{2\sigma^2}} dx_T$$

$$I_3 = -a_1^r e^{-\frac{x_T^2}{2\sigma^2}} + 3 \int_{b_0}^{a_1} x_T^2 e^{-\frac{x_T^2}{2\sigma^2}} dx_T$$

$$I_3 = \left(-a_1^3 e^{-\frac{a_1^2}{2\sigma^2}} \right) + \frac{3(2\sigma^2)}{2} \left(-a_1 e^{-\frac{a_1^2}{2\sigma^2}} \right) + \frac{3(2\sigma^2)}{2} \left(\frac{\sqrt{2\sigma^2}\sqrt{\pi}}{2} \right) \operatorname{erf} \left(\frac{a_1}{\sqrt{2\sigma^2}} \right)$$

$$, \operatorname{erf}(0) = 0, \quad b_0 < x_T < a_1.$$

When $r=4$,

$$I_4 = a_1^4 F(a_1) - b_0^4 F(b_0) + 4 \int_{b_0}^{a_1} x_T^4 e^{-\frac{x_T^2}{2\sigma^2}} dx_T$$

$$= -a_1^4 e^{-\frac{a_1^2}{2\sigma^2}} - 2(2\sigma^2)a_1^2 e^{-\frac{a_1^2}{2\sigma^2}} - 2(2\sigma^2)^2 (e^{-\frac{a_1^2}{2\sigma^2}} - e^{-\frac{b_0^2}{2\sigma^2}}),$$

$$I_4 = -a_1^4 e^{-\frac{a_1^2}{2\sigma^2}} - 2(2\sigma^2)a_1^2 e^{-\frac{a_1^2}{2\sigma^2}} - 2(2\sigma^2)^2 (e^{-\frac{a_1^2}{2\sigma^2}} - 1), \quad b_0 < x_T < a_1$$

when $r = 5$

$$I_5 = a_1^5 F(a_1) - b_0^5 F(b_0) - 5 \int_{b_0}^{a_1} x_T^4 dx_T + 5 \int_{b_0}^{a_1} x_T^4 e^{-\frac{x_T^2}{2\sigma^2}} dx_T$$

$$I_5 = -a_1^5 e^{-\frac{x_T^2}{2\sigma^2}} + 5 \int_{b_0}^{a_1} x_T^4 e^{-\frac{x_T^2}{2\sigma^2}} dx_T$$

$$I_5 = \left(-a_1^5 e^{\frac{-a_1^2}{2\sigma^2}}\right) + \frac{5(2\sigma^2)a_1^3 e^{\frac{-a_1^2}{2\sigma^2}}}{2} - \frac{(3)(5)(2\sigma^2)^2 a_1 e^{\frac{-a_1^2}{2\sigma^2}}}{(2)^2} + \frac{3(5)(2\sigma^2)}{(2)^2} \left(\frac{\sqrt{2\sigma^2}\sqrt{\pi}}{2}\right) \operatorname{erf}\left(\frac{a_1}{\sqrt{2\sigma^2}}\right), \operatorname{erf}(0) = 0, b_0 < x_T < a_1.$$

Corollary 3.1

The value of $E(X_T^r)$ in both theorem (3-1), and (3-2) are equal.

Proof

From theorem (3-1), When $r = 1$,

$$E(X_T) = \frac{1}{n} \sum_{i=1}^n \frac{\left((2\sigma^2)^{\frac{1}{2}} \left[\gamma\left(\frac{1}{2}+1, \frac{a_i^2}{2\sigma^2}\right) - \gamma\left(\frac{1}{2}+1, \frac{b_{i-1}^2}{2\sigma^2}\right) \right]\right)}{e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}}}$$

$$\text{Take: } \gamma\left(\frac{1}{2}+1, \frac{a_i^2}{2\sigma^2}\right) = \frac{1}{2} \gamma\left(\frac{1}{2}, \frac{a_i^2}{2\sigma^2}\right) - e^{\frac{a_i^2}{2\sigma^2}} \left(\frac{a_i^2}{2\sigma^2}\right)^{\frac{1}{2}}$$

$$\gamma\left(\frac{1}{2}, \frac{a_i^2}{2\sigma^2}\right) = \sqrt{\pi} \operatorname{erf}\left(\sqrt{\frac{a_i^2}{2\sigma^2}}\right) = \sqrt{\pi} \operatorname{erf}(a_i/\sqrt{2\sigma^2})$$

$$\gamma\left(\frac{1}{2}+1, \frac{a_i^2}{2\sigma^2}\right) = \frac{1}{2} \sqrt{\pi} \operatorname{erf}(a_i/\sqrt{2\sigma^2}) - e^{\frac{a_i^2}{2\sigma^2}} \left(\frac{a_i^2}{2\sigma^2}\right)^{\frac{1}{2}}$$

$$, \gamma\left(\frac{1}{2}+1, \frac{b_{i-1}^2}{2\sigma^2}\right) = \frac{1}{2} \sqrt{\pi} \operatorname{erf}(b_{i-1}/\sqrt{2\sigma^2}) - e^{\frac{b_{i-1}^2}{2\sigma^2}} \left(\frac{b_{i-1}^2}{2\sigma^2}\right)^{\frac{1}{2}}$$

[since $\gamma\left(\frac{1}{2}, x\right) = \sqrt{\pi} \operatorname{erf}(\sqrt{x})$, and $\gamma(s+1, x) = s\gamma(s, x) - e^{-x} (x)^s$].

$$E(X_T) = \frac{1}{n} \sum_{i=1}^n \frac{\left((2\sigma^2)^{\frac{1}{2}} \left[\gamma\left(\frac{1}{2}+1, \frac{a_i^2}{2\sigma^2}\right) - \gamma\left(\frac{1}{2}+1, \frac{b_{i-1}^2}{2\sigma^2}\right) \right]\right)}{e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}}}$$

$$= \frac{1}{n} \sum_{i=1}^n \frac{(2\sigma^2)^{\frac{1}{2}} \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{a_i}{\sqrt{2\sigma^2}}\right) - e^{\frac{a_i^2}{2\sigma^2}} \left(\frac{a_i^2}{2\sigma^2}\right)^{\frac{1}{2}} - \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{b_{i-1}}{\sqrt{2\sigma^2}}\right) - e^{\frac{b_{i-1}^2}{2\sigma^2}} \left(\frac{b_{i-1}^2}{2\sigma^2}\right)^{\frac{1}{2}} \right]}{e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}}}$$

$$= \frac{1}{n} \sum_{i=1}^n \frac{\left[\frac{(2\sigma^2)^{\frac{1}{2}} \sqrt{\pi}}{2} \operatorname{erf}\left(\frac{a_i}{\sqrt{2\sigma^2}}\right) - \frac{(2\sigma^2)^{\frac{1}{2}} \sqrt{\pi}}{2} \operatorname{erf}\left(\frac{b_{i-1}}{\sqrt{2\sigma^2}}\right) - a_i e^{-\frac{a_i^2}{2\sigma^2}} (2\sigma^2)^{\frac{1}{2}} \left(\frac{1}{2\sigma^2}\right)^{\frac{1}{2}} - b_{i-1} e^{-\frac{b_{i-1}^2}{2\sigma^2}} (2\sigma^2)^{\frac{1}{2}} \left(\frac{1}{2\sigma^2}\right)^{\frac{1}{2}} \right]}{e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}}}$$

$$E(X_T) = \frac{1}{n} \sum_{i=1}^n \frac{\left[\frac{\sqrt{2\sigma^2}\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{a_i}{\sqrt{2\sigma^2}}\right) - \frac{\sqrt{2\sigma^2}\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{b_{i-1}}{\sqrt{2\sigma^2}}\right) - a_i e^{-\frac{a_i^2}{2\sigma^2}} - b_{i-1} e^{-\frac{b_{i-1}^2}{2\sigma^2}} \right]}{1 - e^{-\frac{a_1^2}{2\sigma^2}}}$$

$$E(X_T) = \frac{\left[\frac{(2\sigma^2)^{\frac{1}{2}}\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{a_1}{\sqrt{2\sigma^2}}\right) - a_1 e^{-\frac{a_1^2}{2\sigma^2}} \right]}{1 - e^{-\frac{a_1^2}{2\sigma^2}}}, 0 < x_T < a_1$$

The last equation is equal to formula $E(X_T)$ in theorem (3.1).

When $r = 2$,

$$E(X_T^2) = \frac{1}{n} \sum_{i=1}^n \frac{(2\sigma^2)^{\frac{2}{2}} \left[\gamma\left(2, \frac{a_i^2}{2\sigma^2}\right) - \gamma\left(2, \frac{b_{i-1}^2}{2\sigma^2}\right) \right]}{\mathbb{F}(a_i) - \mathbb{F}(b_{i-1})}$$

$$\text{Take: } \gamma\left(2, \frac{a_1^2}{2\sigma^2}\right) = 1 - e^{-\frac{a_1^2}{2\sigma^2}} - e^{-\frac{a_1^2}{2\sigma^2}} \left(\frac{a_1^2}{2\sigma^2}\right)$$

$$I_2 = -a_i^2 e^{-\frac{a_i^2}{2\sigma^2}} + b_{i-1}^2 e^{-\frac{b_{i-1}^2}{2\sigma^2}} - 2\sigma^2 \left(e^{-\frac{a_i^2}{2\sigma^2}} - e^{-\frac{b_{i-1}^2}{2\sigma^2}} \right),$$

$$E(X_T^2) = \frac{\left[2\sigma^2 - 2\sigma^2 e^{-\frac{a_1^2}{2\sigma^2}} - a_1^2 e^{-\frac{a_1^2}{2\sigma^2}} \right]}{\mathbb{F}(a_1) - \mathbb{F}(b_0)}. \text{ And so, for any } r.$$

Theorem 3.3

let x_T be random variable has probability density function is multi-double truncated Rayleigh distribution, $\mathbb{G}_n(x_T)$ and cumulative distribution function, $\mathbb{G}_n(x_T)$, then the r^{th} moment about a mean of x_T

$$E(x_T - \mu)^r = \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \left(\frac{1}{n} \sum_{i=1}^n \frac{\left((2\sigma^2)^{\frac{1}{2}} \left[\gamma\left(\frac{1}{2}+1, \frac{a_i^2}{2\sigma^2}\right) - \gamma\left(\frac{1}{2}+1, \frac{b_{i-1}^2}{2\sigma^2}\right) \right] \right)}{e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}}} \right)^{r-j} \\ \times \left(\frac{1}{n} \sum_{i=1}^n \frac{\left((2\sigma^2)^{\frac{j}{2}} \left[\gamma\left(\frac{j}{2}+1, \frac{a_i^2}{2\sigma^2}\right) - \gamma\left(\frac{j}{2}+1, \frac{b_{i-1}^2}{2\sigma^2}\right) \right] \right)}{e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}}} \right)$$

Proof:

$$E(x_T - \mu)^r = \int_{-\infty}^{\infty} (x_T - \mu)^r \mathbb{G}_n(x_T) dx_T$$

$$\begin{aligned}
E(x_T - \mu)^r &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} (\mu)^{r-j} \mu_j' \\
E(x_T - \mu)^r &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} (\mu)^{r-j} \int_{-\infty}^{\infty} (x_T)^j \mathbb{G}_n(x_T) dx_T \\
&= \sum_{j=0}^r \frac{(-1)^{r-j}}{n} \binom{r}{j} (\mu)^{r-j} \left\{ \frac{\int_{b_0}^{a_1} x_T^r \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}} dx_T}{F(a_1)-F(b_0)} + \frac{\int_{b_1}^{a_2} x_T^r \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}} dx_T}{F(a_2)-F(b_1)} + \right. \\
&\quad \left. \dots + \frac{\int_{b_{n-1}}^{a_n} x_T^r \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}} dx_T}{F(a_n)-F(b_{n-1})} \right\} \\
E(x_T - \mu)^r &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \left(\frac{1}{n} \sum_{i=1}^n \frac{\left((2\sigma^2)^{\frac{1}{2}} \left[\gamma\left(\frac{1}{2}+1, \frac{a_i^2}{2\sigma^2}\right) - \gamma\left(\frac{1}{2}+1, \frac{b_{i-1}^2}{2\sigma^2}\right) \right] \right)}{e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}}} \right)^{r-j} \\
&\quad \times \left(\frac{1}{n} \sum_{i=1}^n \frac{\left((2\sigma^2)^{\frac{j}{2}} \left[\gamma\left(\frac{j}{2}+1, \frac{a_i^2}{2\sigma^2}\right) - \gamma\left(\frac{j}{2}+1, \frac{b_{i-1}^2}{2\sigma^2}\right) \right] \right)}{e^{-\frac{b_{i-1}^2}{2\sigma^2}} - e^{-\frac{a_i^2}{2\sigma^2}}} \right) \quad (3.9)
\end{aligned}$$

3.2 Moment Generating and Characteristic Functions of MDTRD

The moment generating function of MDTRD random variable is given by:

$$\begin{aligned}
\mathcal{M}_{x_T}(t) &= E(e^{t x_T}) = \frac{1}{n} \left\{ \frac{\int_{b_0}^{a_1} e^{t x_T} f(x_T) dx_T}{F(a_1)-F(b_0)} + \frac{\int_{b_1}^{a_2} e^{t x_T} f(x_T) dx_T}{F(a_2)-F(b_1)} + \right. \\
&\quad \left. \dots + \frac{\int_{b_{n-1}}^{a_n} e^{t x_T} f(x_T) dx_T}{F(a_n)-F(b_{n-1})} \right\} \\
&= \\
&\frac{1}{n} \left\{ \frac{e^{a_1 t} F(a_1) - e^{b_0 t} F(b_0) - t \int_{b_0}^{a_1} e^{t x_T} \left(1 - e^{-\frac{x_T^2}{2\sigma^2}}\right) dx_T}{(F(a_1)-F(b_0))} + \right. \\
&\quad \frac{e^{a_2 t} F(a_2) - e^{b_1 t} F(b_1) - t \int_{b_1}^{a_2} e^{t x_T} \left(1 - e^{-\frac{x_T^2}{2\sigma^2}}\right) dx_T}{(F(a_2)-F(b_1))} + \\
&\quad \left. \dots + \frac{e^{a_n t} F(a_n) - e^{b_{n-1} t} F(b_{n-1}) - t \int_{b_{n-1}}^{a_n} e^{t x_T} \left(1 - e^{-\frac{x_T^2}{2\sigma^2}}\right) dx_T}{(F(a_n)-F(b_{n-1}))} \right\}
\end{aligned}$$

$$\text{Take: } I = \int_{b_0}^{a_1} e^{t x_T} e^{-\frac{x_T^2}{2\sigma^2}} dx_T = \int_{b_0}^{a_1} e^{-\frac{x_T^2 + 2\sigma^2 t x_T}{2\sigma^2}} dx_T$$

$$I = \int_{b_0}^{a_1} e^{\frac{1}{2\sigma^2}(2d x_T - x_T^2)} dx_T = \int_{b_0}^{a_1} e^{\frac{-1}{2\sigma^2}\{(x_T - d)^2 + d^2\}} dx_T, d = \sigma^2 t$$

$$\text{Let } u = x_T - d, du = dx_T, I = e^{\frac{d^2}{2\sigma^2}} \int_{b_0 - d}^{a_1 - d} e^{-\frac{1}{2\sigma^2} u^2} du$$

$$\mathcal{M}_{x_T}(t) = e^{\frac{d^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2}}{2} \sqrt{\pi} \left(\text{erf}\left(a_1 - d\right) \left(\frac{1}{\sqrt{2\sigma^2}}\right) - \text{erf}\left(b_0 - d\right) \left(\frac{1}{\sqrt{2\sigma^2}}\right) \right)$$

$$\mathcal{M}_{x_T}(t) = \frac{1}{n} \sum_{i=1}^n \frac{\left[e^{a_i t} \left(1 - e^{-\frac{a_i^2}{2\sigma^2}}\right) - e^{b_{i-1} t} \left(1 - e^{-\frac{b_{i-1}^2}{2\sigma^2}}\right) - t(a_i - b_{i-1}) + t e^{\frac{(\sigma^2 t)^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2}}{2} \sqrt{\pi} \left(\text{erf}\left(a_i - \sigma^2 t\right) \left(\frac{1}{\sqrt{2\sigma^2}}\right) - \text{erf}\left(b_{i-1} - \sigma^2 t\right) \left(\frac{1}{\sqrt{2\sigma^2}}\right) \right) \right]}{\mathbb{F}(a_i) - \mathbb{F}(b_{i-1})} \quad (3.10)$$

Similarly, obtaining the characteristic function of MDTRD random variable is as follows:

$$\Phi_{x_T}(t) = \frac{1}{n} \sum_{k=1}^n \frac{\left[e^{a_k i t} \left(1 - e^{-\frac{a_k^2}{2\sigma^2}}\right) - e^{b_{k-1} i t} \left(1 - e^{-\frac{b_{k-1}^2}{2\sigma^2}}\right) - i t(a_k - b_{k-1}) + i t e^{\frac{(\sigma^2 i t)^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2}}{2} \sqrt{\pi} \left(\text{erf}\left(a_k - \sigma^2 t\right) \left(\frac{1}{\sqrt{2\sigma^2}}\right) - \text{erf}\left(b_{k-1} - \sigma^2 t\right) \left(\frac{1}{\sqrt{2\sigma^2}}\right) \right) \right]}{\mathbb{F}(a_k) - \mathbb{F}(b_{k-1})} \quad (3.11)$$

3.3 Characteristics of Order Statistics For MDTRD

Let $x_{T_1}, x_{T_2}, x_{T_3}, x_{T_4}, \dots, x_{T_m}$ be the independent and identical distribution of random variables with multi-double truncated Rayleigh distribution. Then $x_{T_{[1]}} \leq x_{T_{[2]}} \leq x_{T_{[3]}} \leq x_{T_{[4]}} \leq \dots \leq x_{T_{[m]}}$, be the order statistic, and the probability density function of $x_{T_{[j]}}$ is:

$$\mathbb{G}_n(x_T)_{[j]} = \frac{m!}{(j-1)!(m-j)!} \left(\frac{1}{n} \sum_{i=1}^n \left(\frac{\mathbb{F}(x_T) I_{(b_{i-1}, a_i)} - \sum_{j=1}^n \mathbb{F}(b_{j-1}) + \sum_{j=1}^n \mathbb{F}(a_j)}{\mathbb{F}(a_i) - \mathbb{F}(b_{i-1})} \right) \right)^{j-1} \left(1 - \frac{1}{n} \sum_{i=1}^n \left(\frac{\mathbb{F}(x_T) I_{(b_{i-1}, a_i)} - \sum_{j=1}^n \mathbb{F}(b_{j-1}) + \sum_{j=1}^n \mathbb{F}(a_j)}{\mathbb{F}(a_i) - \mathbb{F}(b_{i-1})} \right) \right)^{m-j} \left(\sum_{i=1}^n \frac{f(x_T)}{n(\mathbb{F}(a_i) - \mathbb{F}(b_{i-1}))} \right)$$

$$= \frac{m!}{(j-1)!(m-j)!} \left(\sum_{i=1}^n \left(\frac{e^{-\frac{x_T^2}{2\sigma^2}} - \sum_{j=1}^n \left(e^{-\frac{-b_{j-1}^2}{2\sigma^2}} - e^{-\frac{-a_j^2}{2\sigma^2}} \right)}{n \left(e^{-\frac{-b_{i-1}^2}{2\sigma^2}} - e^{-\frac{-a_i^2}{2\sigma^2}} \right)} \right) \right)^{j-1} \left(1 - \frac{1}{n} \sum_{i=1}^n \left(\frac{e^{-\frac{x_T^2}{2\sigma^2}} - \sum_{j=1}^n \left(e^{-\frac{-b_{j-1}^2}{2\sigma^2}} - e^{-\frac{-a_j^2}{2\sigma^2}} \right)}{n \left(e^{-\frac{-b_{i-1}^2}{2\sigma^2}} - e^{-\frac{-a_i^2}{2\sigma^2}} \right)} \right) \right)^{m-j} \left(\sum_{i=1}^n \frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{n \left(e^{-\frac{-b_{i-1}^2}{2\sigma^2}} - e^{-\frac{-a_i^2}{2\sigma^2}} \right)} \right)$$

$$\mathbb{G}_n(x_T)_{[j]} =$$

$$\frac{m!}{(j-1)!(m-j)! n^{j-1}} \left[\sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_{j-1}=1}^n \left(\frac{e^{-\frac{x_T^2}{2\sigma^2}} - \sum_{j=1}^n \left(e^{-\frac{-b_{j-1}^2}{2\sigma^2}} - e^{-\frac{-a_j^2}{2\sigma^2}} \right)}{n \left(e^{-\frac{-b_{i_1-1}^2}{2\sigma^2}} - e^{-\frac{-a_{i_1}^2}{2\sigma^2}} \right)} \right) \right]_{i_1 i_2 \dots i_{j-1}}$$

$$\times \sum_{k=0}^{m-j} \binom{m-j}{k} (-1)^k \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_k=1}^n \left(\frac{e^{-\frac{x_T^2}{2\sigma^2}} - \sum_{j=1}^n \left(e^{-\frac{-b_{j-1}^2}{2\sigma^2}} - e^{-\frac{-a_j^2}{2\sigma^2}} \right)}{n \left(e^{-\frac{-b_{i_1-1}^2}{2\sigma^2}} - e^{-\frac{-a_{i_1}^2}{2\sigma^2}} \right)} \right)_{i_1 \dots i_k}$$

And The smallest order statistics $x_{T[1]}$ is:

$$\mathbb{G}_n(x_T)_{[1]} =$$

$$m \sum_{k=0}^{m-1} \frac{(-1)^k}{n^k} \binom{m-1}{k} \left(\sum_{i=1}^n \frac{e^{-\frac{x_T^2}{2\sigma^2}} + \left(e^{-\frac{-b_{i-1}^2}{2\sigma^2}} - e^{-\frac{-a_i^2}{2\sigma^2}} \right)}{\left(e^{-\frac{-b_{i-1}^2}{2\sigma^2}} - e^{-\frac{-a_i^2}{2\sigma^2}} \right)} \right) \left(\frac{1}{n} \sum_{i=1}^n \frac{x_T e^{-\frac{x_T^2}{2\sigma^2}}}{\left(e^{-\frac{-b_{i-1}^2}{2\sigma^2}} - e^{-\frac{-a_i^2}{2\sigma^2}} \right)} \right)^k$$

=

$$\sum_{k=0}^{m-1} \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n \frac{m \binom{m-1}{k} (-1)^k}{n^{k+1}} \left(\frac{e^{-\frac{x_T^2}{2\sigma^2}} + \left(e^{-\frac{-b_{i_1-1}^2}{2\sigma^2}} - e^{-\frac{-a_{i_1}^2}{2\sigma^2}} \right)}{\left(e^{-\frac{-b_{i_1-1}^2}{2\sigma^2}} - e^{-\frac{-a_{i_1}^2}{2\sigma^2}} \right)} \right) \sum_{i=1}^n \frac{x_T e^{-\frac{x_T^2}{2\sigma^2}}}{e^{-\frac{-b_{i-1}^2}{2\sigma^2}} - e^{-\frac{-a_i^2}{2\sigma^2}}} \quad (3.12)$$

3.4 One Double Truncated of Rayleigh distribution (One-DTRD)

This section introduced an example of the multi-double truncated Rayleigh distribution, which is one –double truncated Rayleigh distribution and some statistical properties.

3.4.1 One Double Truncated Probability Rayleigh Function (One-DTRD)

The probability density function of one double interval truncated of Rayleigh distribution is given by:

$$g_2(x_T) = \begin{cases} \frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{2 \left(\frac{e^{-\frac{b_0^2}{2\sigma^2}}}{2} - \frac{e^{-\frac{a_1^2}{2\sigma^2}}}{2} \right)} & b_0 < x_T < a_1 \\ \frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{2 \left(\frac{e^{-\frac{b_1^2}{2\sigma^2}}}{2} - \frac{e^{-\frac{a_2^2}{2\sigma^2}}}{2} \right)} & b_1 < x_T < a_2 \end{cases} \quad (3.13)$$

that is, the interval (a_1, b_1) , was deleted.

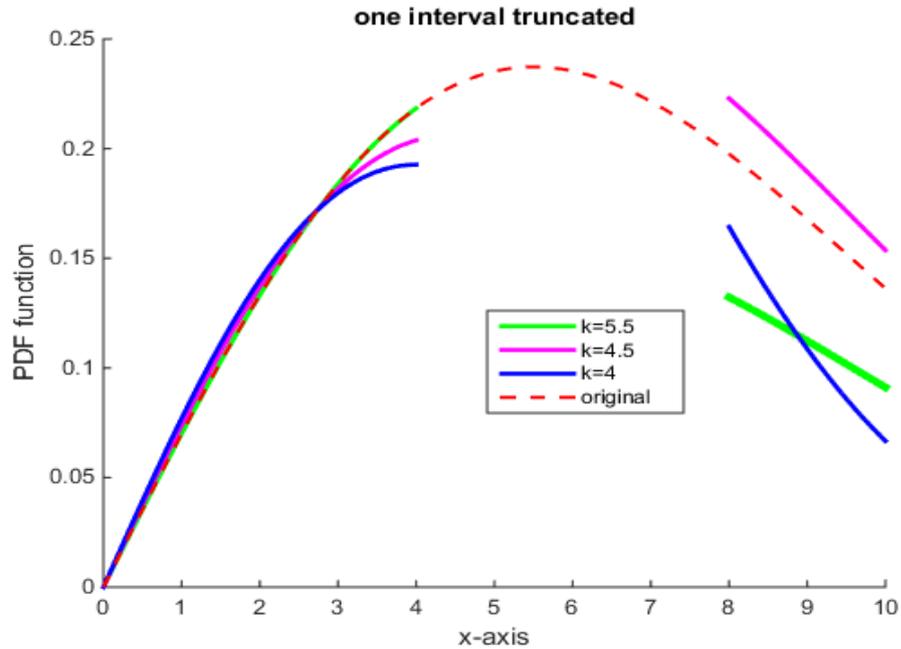


FIGURE 3.1 One interval (4,8) is deleted in DTRD, $\sigma = 5.5,4.5,4$

In figure (3.1), we draw PDF of one double truncated Rayleigh distribution, and the original function when $\sigma = \{5.5,4.5,4\}$ when $(b_0, a_1)=(0,4)$, $(b_1, a_2)=(8,10)$, that is the interval which deleted is (4,8).

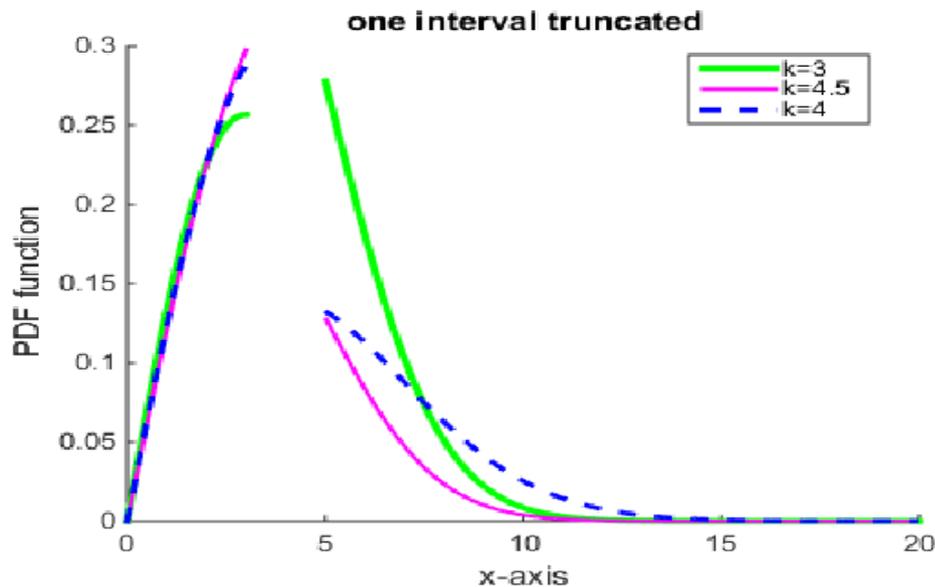


FIGURE 3.2 One interval (3,5) is deleted in DTRD, $\sigma = 3,4.5,4$

shown in figure (3.2), PDF of one-double truncated Rayleigh distribution. When $(b_0, a_1)=(0,3)$, $(b_1, a_2)=(5,10)$, different values of the parameters $\sigma = \{3,4.5,4\}$, such that the interval which deleted is $(a_1, b_1)=(3,5)$.

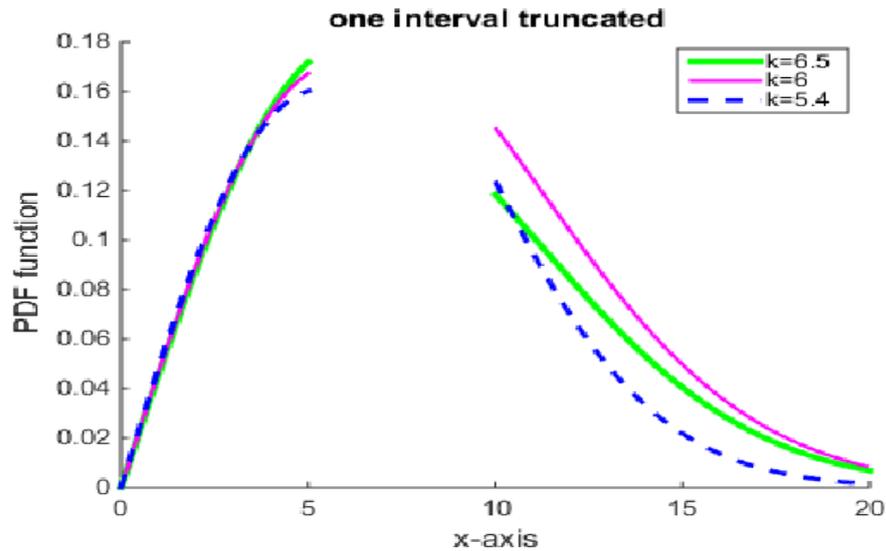


FIGURE 3.3 One interval is deleted in DTRD,

In this figure, we presented PDF of one double truncated Rayleigh such that, the interval that deleted is (5,10) and the parameter $\sigma = \{7,6.5,4\}$.

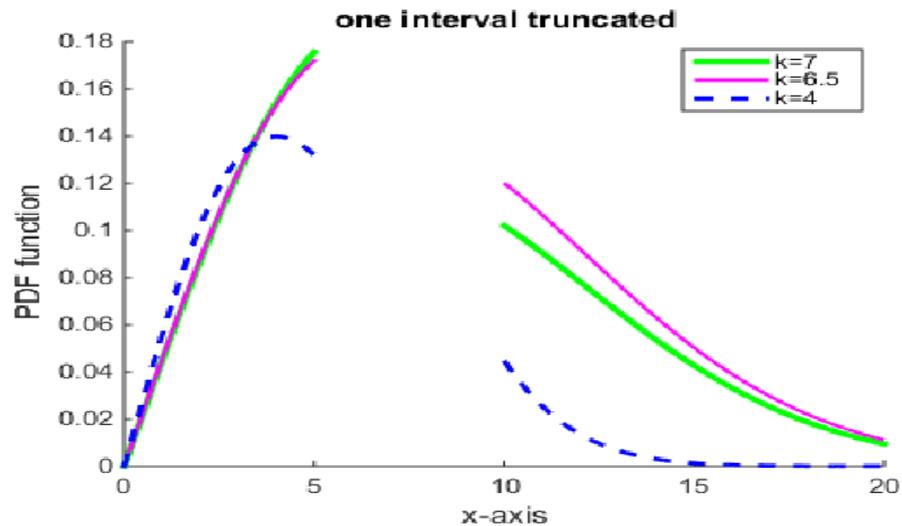


FIGURE 3.4 One interval is deleted in DTRD, $\sigma = 7,6.5,4$

While, in figure(3.4), we drawn PDF of one double truncated Rayleigh the interval that deleted is (5,10) for different values for parameter $\sigma = \{6.5,6,5.4\}$.

3.4.2 One double subinterval truncated for CDF of Rayleigh Distribution

The cumulative distribution function of a one-double truncated Rayleigh distribution is:

$$G_2(x_T) = \begin{cases} 0 & x_T < 0 \\ \left(\frac{e^{-\frac{b_0^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}}}{2e^{-\frac{a_1^2}{2\sigma^2}} - 2e^{-\frac{b_0^2}{2\sigma^2}}} \right) & b_0 \leq x_T \leq a_1 \\ \left(\frac{e^{-\frac{b_0^2}{2\sigma^2}} + e^{-\frac{b_1^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}} - e^{-\frac{a_1^2}{2\sigma^2}}}{2e^{-\frac{a_2^2}{2\sigma^2}} - 2e^{-\frac{b_1^2}{2\sigma^2}}} \right) & b_1 \leq x_T \leq a_2 \\ 1 & x_T > 1 \end{cases} \quad (3.14)$$

In both figures (3.5), and (3.6) the truncated and original distribution has been drawn. For different values of parameter $\sigma = \{3,2.4\}, \{3,3.6\}$ one interval is deleted, where $(b_0, a_1) = (0, 2.5), (0, 3.5)$, $(b_1, a_2) = (4.5, 10), (5, 10)$ that is, the intervals that have been deleted are $(2.5, 5), (3.5, 4.5)$, respectively.

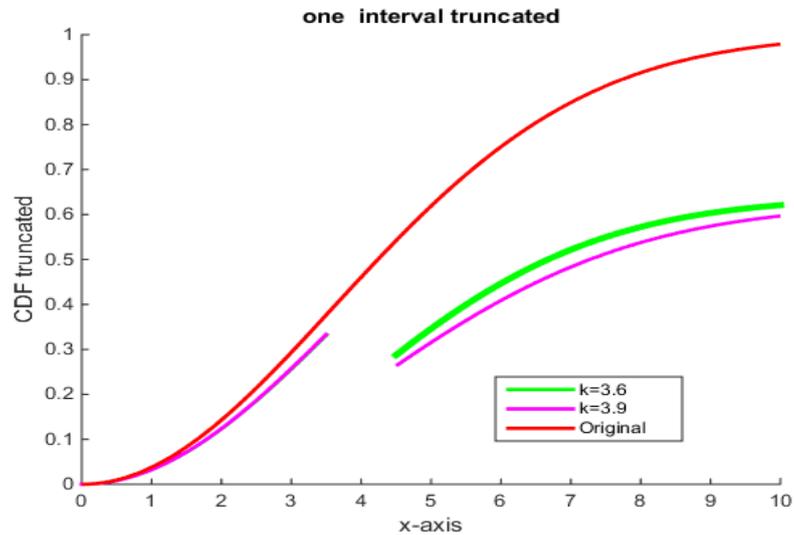


FIGURE 3.5 One interval truncated CDF of DTRD, $\sigma = 3.6, 3.7$

Cumulative distribution function of one double truncated Rayleigh was drawn, where $(b_0, a_1) = (0, 2.5)$, $(b_1, a_2) = (4.5, 10)$. That is, the interval which deleted is $(2.5, 5)$.

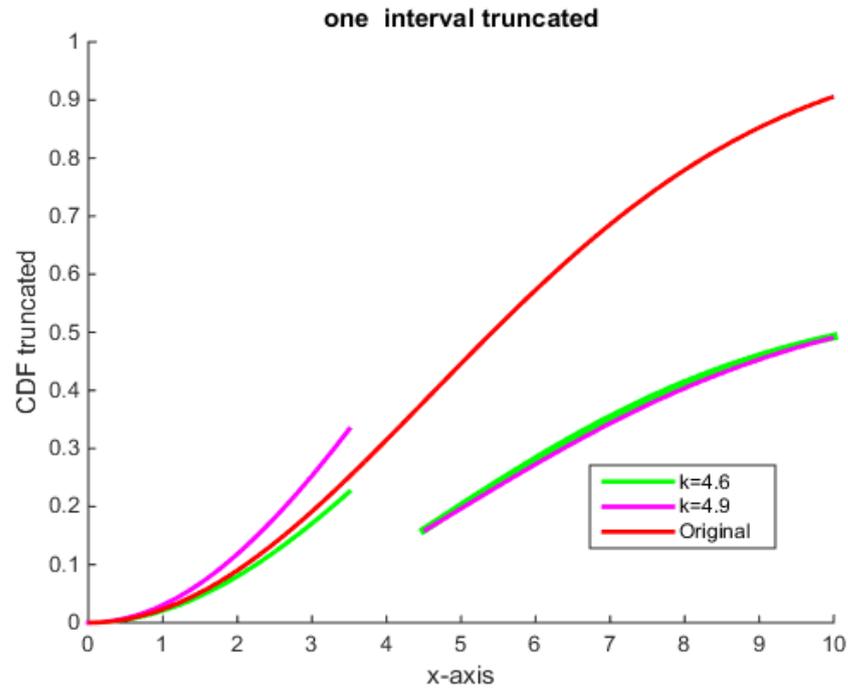


FIGURE 3.6 One interval truncated CDF of DTRD, $\sigma = 3,3.6$

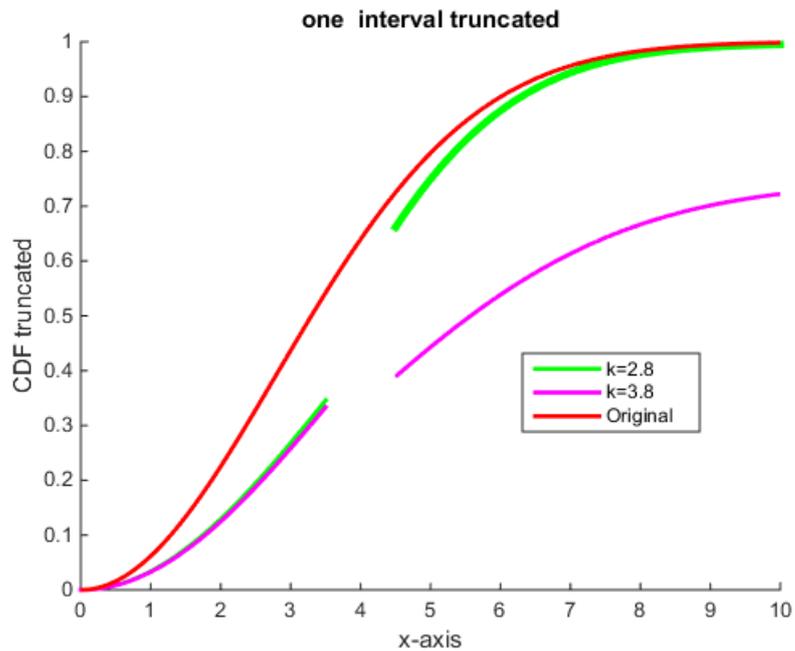


FIGURE 3.7 One interval truncated CDF of DTRD, $\sigma = 2.8,3.8$

In this figure, we drawn CDF of one- DTRD, and with the original distribution, the interval that cut off is (3.5,4.5) for different values of $\sigma = \{2.8,2.9\}$, has been shown in figure (3.7).

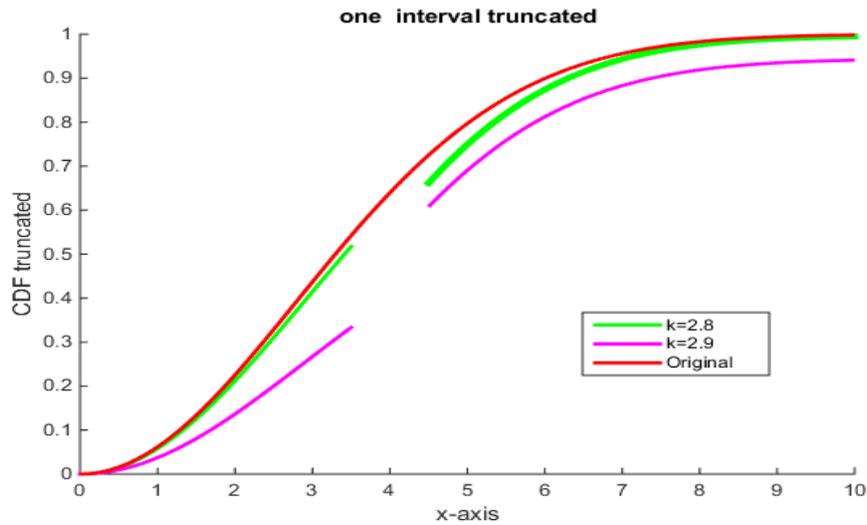


FIGURE 3.8 One interval truncated CDF of DTRD, $\sigma = 2.8, 2.9$

In figure (3.8), we presented CDF of one- DTRD, and the original distribution. The interval that cut off is (3.5,4.5) for different values of $\{2.8, 3.8\}$.

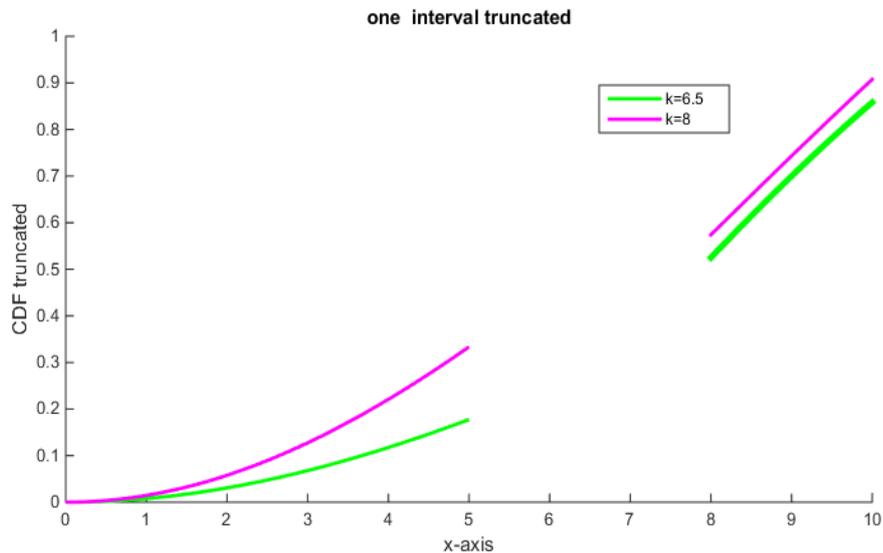


FIGURE 3.9 One interval truncated CDF of DTRD, $\sigma = 6.5, 8$

While in figure (3.9), the interval deleted is (5,8), and a different parameter $\sigma = \{6.5, 8\}$, without the original function.

3.4.3 One double truncated for survival function Rayleigh Distribution

The survival function of one-double truncated Rayleigh distribution is:

$$S_2(x_T) = \begin{cases} 0 & x_T < 0 \\ \left(1 - \frac{e^{-\frac{b_0^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}}}{2e^{-\frac{a_1^2}{2\sigma^2}} - 2e^{-\frac{b_0^2}{2\sigma^2}}} \right) & b_0 \leq x_T \leq a_1 \\ \left(1 - \frac{e^{-\frac{b_0^2}{2\sigma^2}} + e^{-\frac{b_1^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}} - e^{-\frac{a_1^2}{2\sigma^2}}}{2e^{-\frac{a_2^2}{2\sigma^2}} - 2e^{-\frac{b_1^2}{2\sigma^2}}} \right) & b_1 \leq x_T \leq a_2 \\ 1 & x_T > 1 \end{cases} \quad (3.15)$$

Many figures to represent the survival function of one-double truncated Rayleigh has been introduced in figures (3.10), (3.11),and (3.12) for different values for parameters. Such that, the interval which deleted is(4,7).

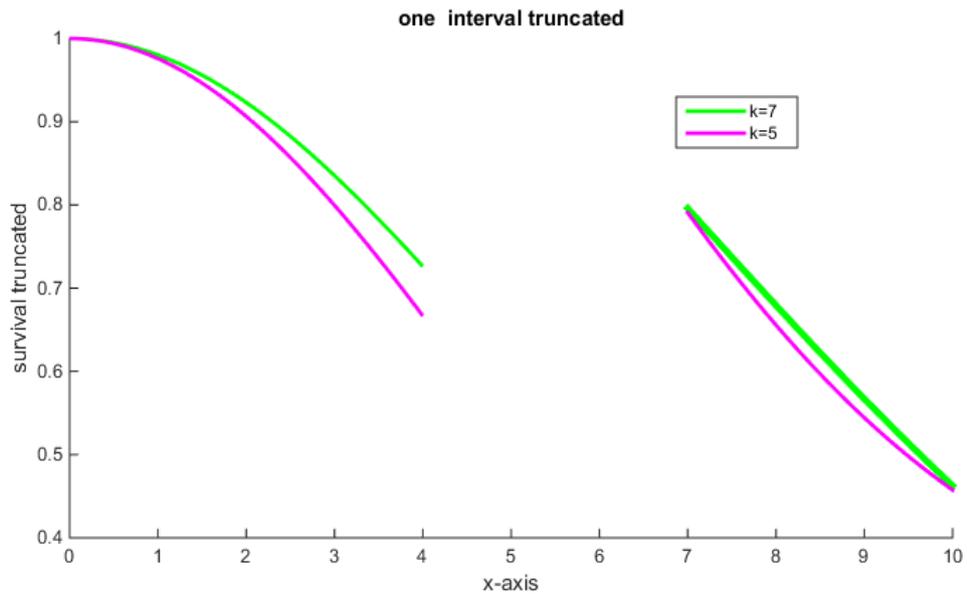


FIGURE 3.10 One interval truncated the survival function of DTRD, $\sigma = 7,5$

The survival function of one-double truncated Rayleigh has been introduced in figure (3.10) for different values for parameters.

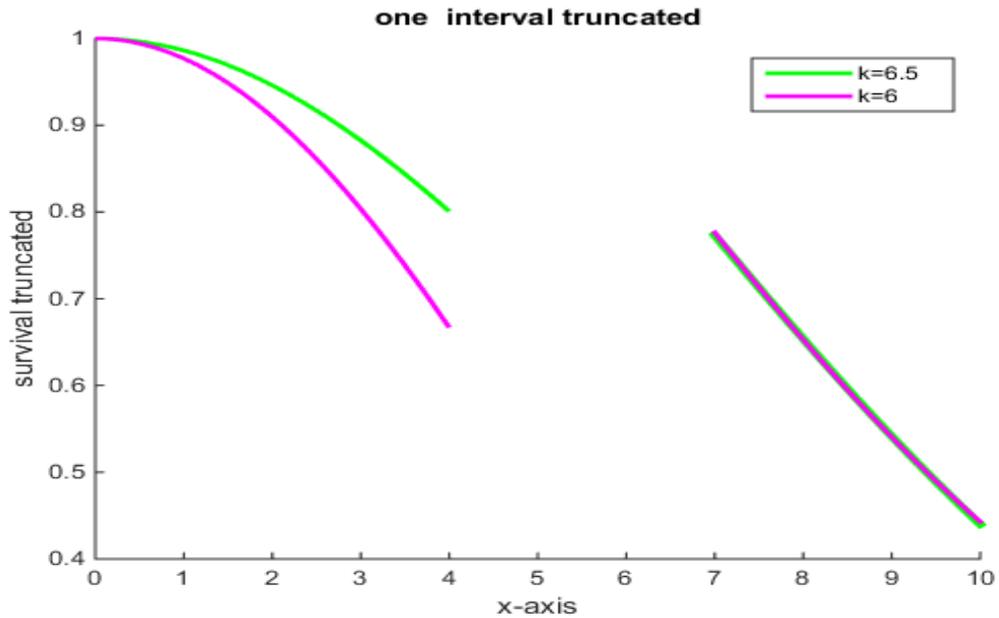


FIGURE 3.11 One interval truncated the survival function of DTRD, $\sigma = 6.5, 6$

In this figure, we presented the survival function of one-double truncated Rayleigh for different values for parameter $\sigma = 6.5, 6$.

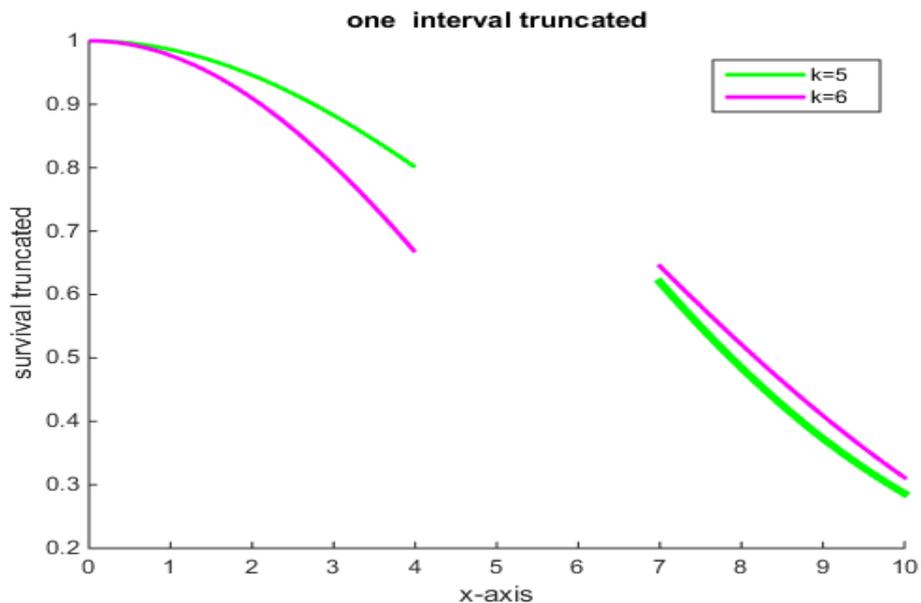


FIGURE 3.12 One interval truncated the survival function of DTRD, $\sigma = 5, 6$

In figure(3.12), we presented the survival function of one-double truncated Rayleigh for different values for parameter $\sigma = 5, 6$.

3.5 Two Double Truncated Rayleigh distribution (Two-DTRD)

This section introduced an example of the multi-double truncated Rayleigh distribution, which is two – double truncated Rayleigh distribution and some statistical properties.

The probability distribution function of two double truncated Rayleigh distribution is:

$$f_3(x_T) = \begin{cases} \frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{3 \left(\frac{-b_0^2}{e^{2\sigma^2}} - \frac{-a_1^2}{e^{2\sigma^2}} \right)} & b_0 < x_T < a_1 \\ \frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{3 \left(\frac{-b_1^2}{e^{2\sigma^2}} - \frac{-a_2^2}{e^{2\sigma^2}} \right)} & b_1 < x_T < a_2 \\ \frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{3 \left(\frac{-b_2^2}{e^{2\sigma^2}} - \frac{-a_3^2}{e^{2\sigma^2}} \right)} & b_2 < x_T < a_3 \end{cases} \quad (3.16)$$

That is, the two intervals $(a_1, b_1), (a_2, b_2)$, are deleted from the original interval for distribution. We can show the blow figures of PDF of (TDTRD)

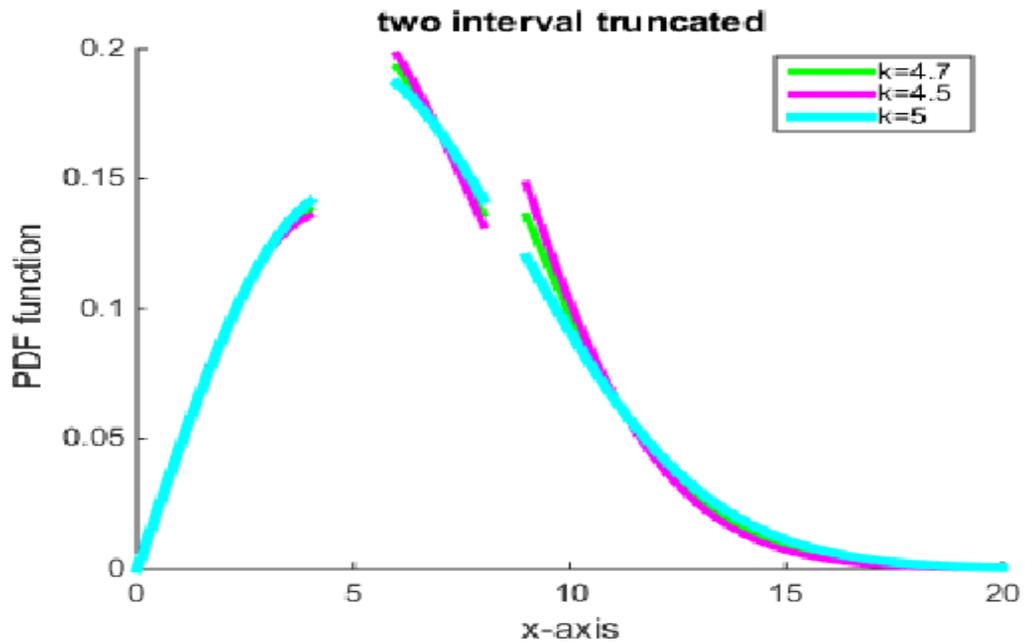


FIGURE 3.13 Two intervals are deleted in DTRD, $\sigma = 4.7, 4.5, 5$

Two intervals (4,6) and (8,9) are truncated for different values of parameters $\sigma = \{5, 6, 5.5\}$ when drawing the curves of PDF, which has been presented in figure (3.13).

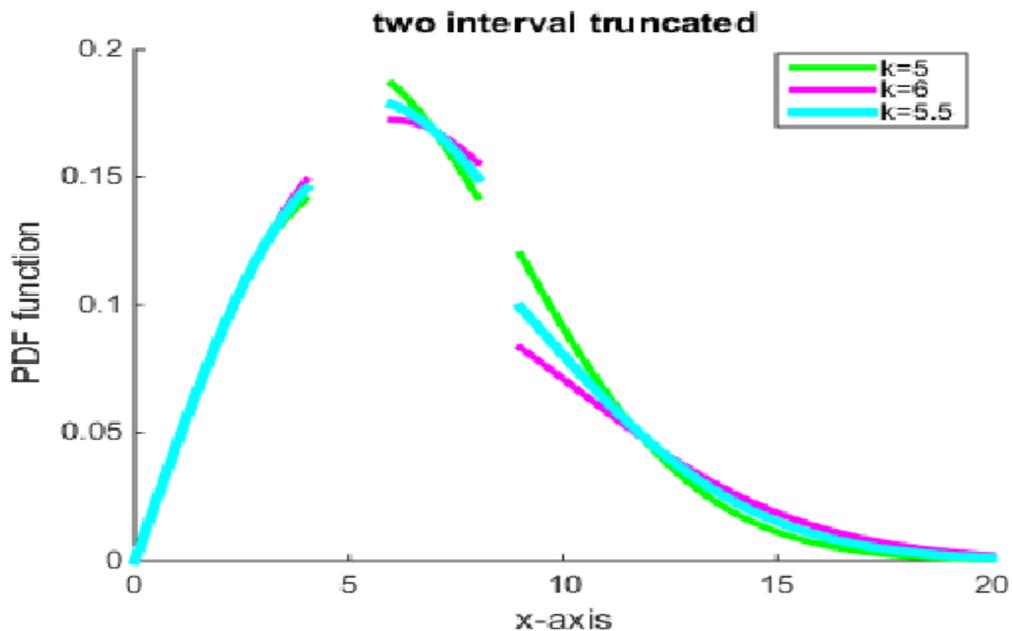


FIGURE 3.14 Two intervals are deleted in DTRD, $\sigma = 5, 6, 5.5$

In figure (3.14), two intervals $(a_1, b_1) = (4, 6)$ and $(a_2, b_2) = (8, 9)$ are deleted from PDF of (TDTRD), such that the values of parameter $\sigma = \{4.7, 4.5, 5\}$.

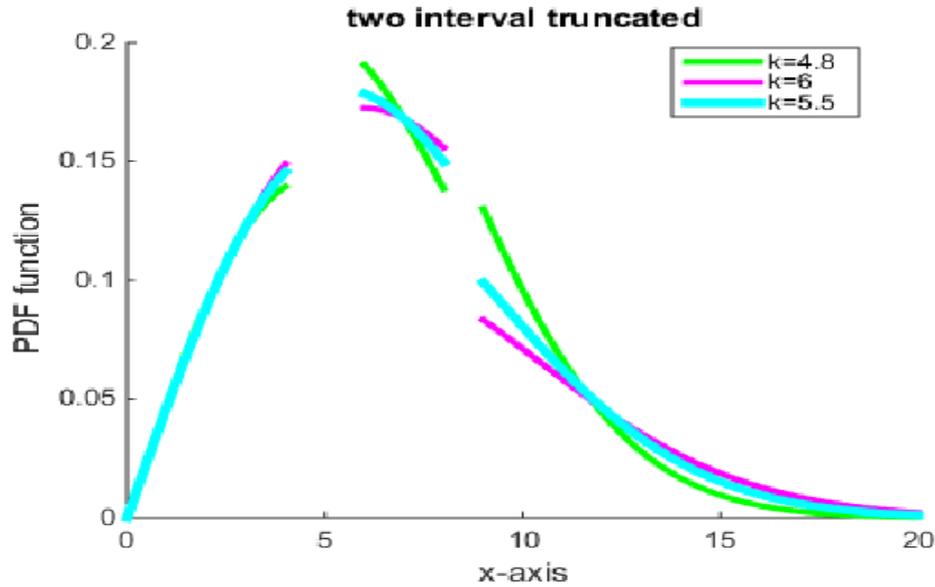


FIGURE 3.15 Two intervals are deleted in DTRD, $\sigma = 4.8, 6, 5.5$

In figure (3.15), we introduced the PDF of (TDTRD) . Where $(a_1, b_1)=(4,6)$ and, $(a_2, b_2)=(8,9)$ are truncated for different values of parameter $\sigma=\{4.8,6,5.5\}$.

3.5.1 Two double intervals truncated CDF of Rayleigh Distribution

The cumulative distribution function of two-double truncated Rayleigh distribution is:

$$\mathbb{G}_3(x_T) = \begin{cases} 0 & x_T < 0 \\ \left(\frac{e^{-\frac{b_0^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}}}{3e^{-\frac{a_1^2}{2\sigma^2}} - 3e^{-\frac{b_0^2}{2\sigma^2}}} \right) & b_0 \leq x_T \leq a_1 \\ \left(\frac{e^{-\frac{b_0^2}{2\sigma^2}} + e^{-\frac{b_1^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}} - e^{-\frac{a_1^2}{2\sigma^2}}}{3e^{-\frac{a_2^2}{2\sigma^2}} - 3e^{-\frac{b_1^2}{2\sigma^2}}} \right) & b_1 \leq x_T \leq a_2 \\ \left(\frac{e^{-\frac{b_0^2}{2\sigma^2}} + e^{-\frac{b_1^2}{2\sigma^2}} + e^{-\frac{b_2^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}} - e^{-\frac{a_1^2}{2\sigma^2}} - e^{-\frac{a_2^2}{2\sigma^2}}}{3e^{-\frac{a_2^2}{2\sigma^2}} - 3e^{-\frac{b_1^2}{2\sigma^2}}} \right) & b_2 \leq x_T \leq a_3 \\ 1 & 1 \leq x_T \end{cases} \quad (3.17)$$

In the following figures, we present some curves of the original cumulative distribution with the truncated cumulative distribution function for different values of parameters σ where $(b_0, a_1)=(0,2)$,

$(b_1, a_2)=(4,5), (4,5)$ and $(b_2, a_3)=(6,10)$. that is, the intervals which have been deleted are $\{(2,4), (4,6)\}$, for different value of $\sigma = \{3.2,3.3\}, \{3.4,3.5\}$ have been shown in figures (3.16), and (3.17), respectively. while in figures (3.18), the intervals are $(b_0, a_1)=(0,2)$, $(b_1, a_2)=(4,4.5)$, and $(b_2, a_3)=(6,10)$, without original distribution function.

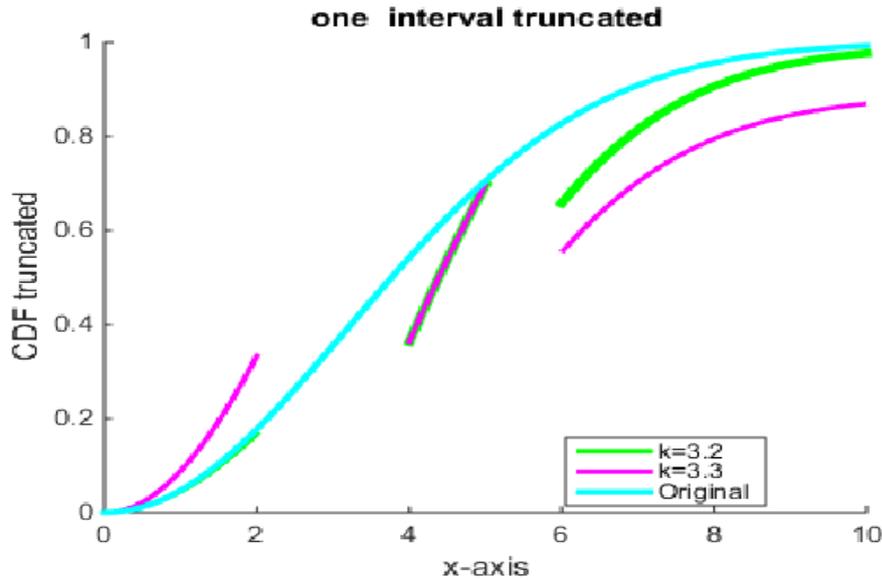


FIGURE 3.16 Two intervals truncated CDF of DTRD, $\sigma = 3.2, 3.3$

In this figure, we introduced the original cumulative distribution with the truncated cumulative distribution function for $\sigma = 3.2, 3.3$ where $(b_0, a_1)=(0,2)$, $(b_1, a_2)=(4,5)$ and $(b_2, a_3)=(6,10)$. that is, the intervals which have been deleted are $\{(2,4), (4,6)\}$.

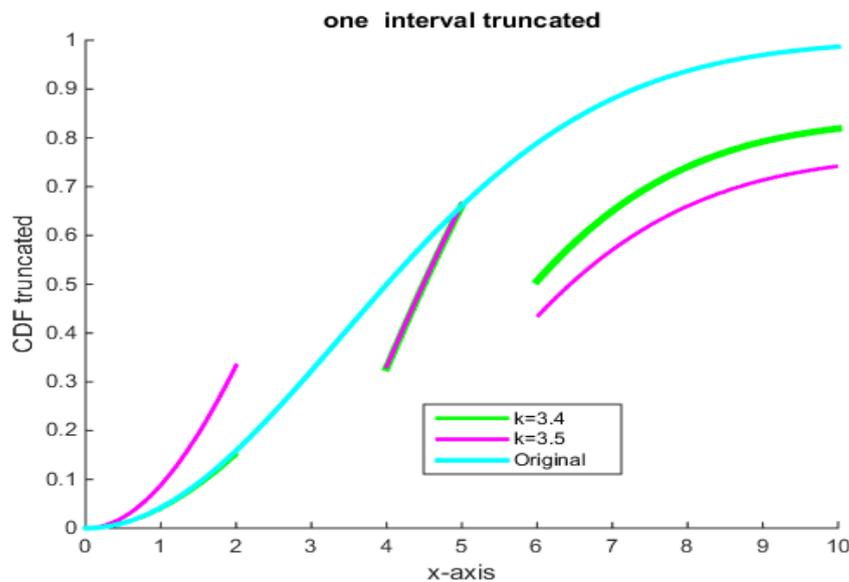


FIGURE 3.17 Two intervals truncated CDF of DTRD, $\sigma = 3.4, 3.5$

In this figure, we introduced the original cumulative distribution with the truncated cumulative distribution function for $\sigma = 3.4, 3.5$.

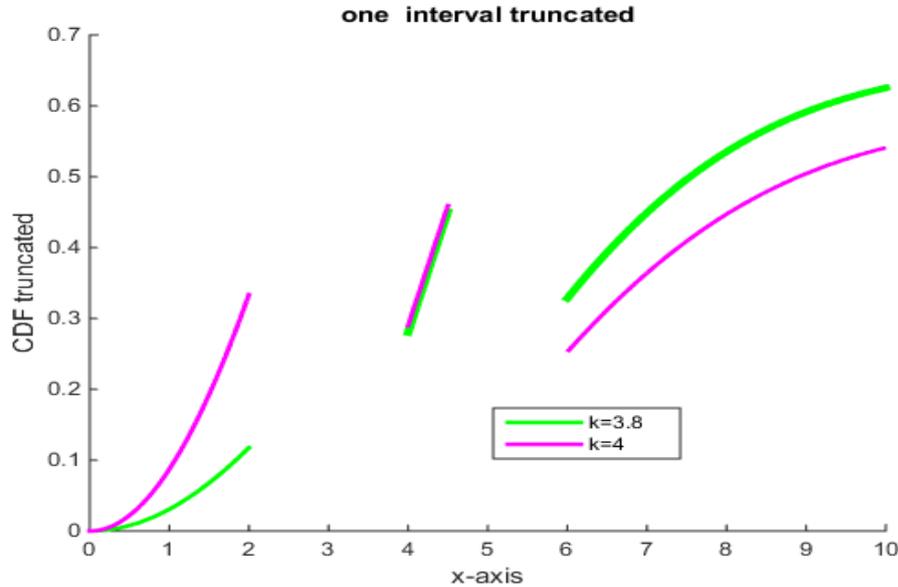


FIGURE 3.18 Two intervals truncated CDF of DTRD, $\sigma = 3.8, 4$

While in this figure, CDF of (TDTRD) with intervals $(0,2)$, $(4,4.5)$, and $(6,10)$.

3.5.2 Two Double Truncated Survival Function Rayleigh Distribution

The survival function of two- double truncated Rayleigh distribution is:

$$\mathbb{S}_n(x_T) = \begin{cases} \left(1 - \frac{e^{-\frac{b_0^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}}}{3e^{-\frac{a_1^2}{2\sigma^2}} - 3e^{-\frac{b_0^2}{2\sigma^2}}} \right) & b_0 \leq x_T \leq a_1 \\ \left(1 - \frac{e^{-\frac{b_0^2}{2\sigma^2}} + e^{-\frac{b_1^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}} - e^{-\frac{a_1^2}{2\sigma^2}}}{3e^{-\frac{a_2^2}{2\sigma^2}} - 3e^{-\frac{b_1^2}{2\sigma^2}}} \right) & b_1 \leq x_T \leq a_2 \\ \left(1 - \frac{e^{-\frac{b_0^2}{2\sigma^2}} + e^{-\frac{b_1^2}{2\sigma^2}} + e^{-\frac{b_2^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}} - e^{-\frac{a_1^2}{2\sigma^2}} - e^{-\frac{a_2^2}{2\sigma^2}}}{3e^{-\frac{a_2^2}{2\sigma^2}} - 3e^{-\frac{b_1^2}{2\sigma^2}}} \right) & b_2 \leq x_T \leq a_3 \end{cases} \quad (3.18)$$

In this subsection, many figures to represent the two double truncated survival Rayleigh have been introduced. Two intervals that deleted are

$\{(2.5,3.5), (5,6)\}, \{(2.5,3), (4.2,6)\}$ for different values for parameter $\{(3.4, 3.6), (3.4,3.5)\}$ and x_T has been shown in figures (3-19), and (3-20), respectively.

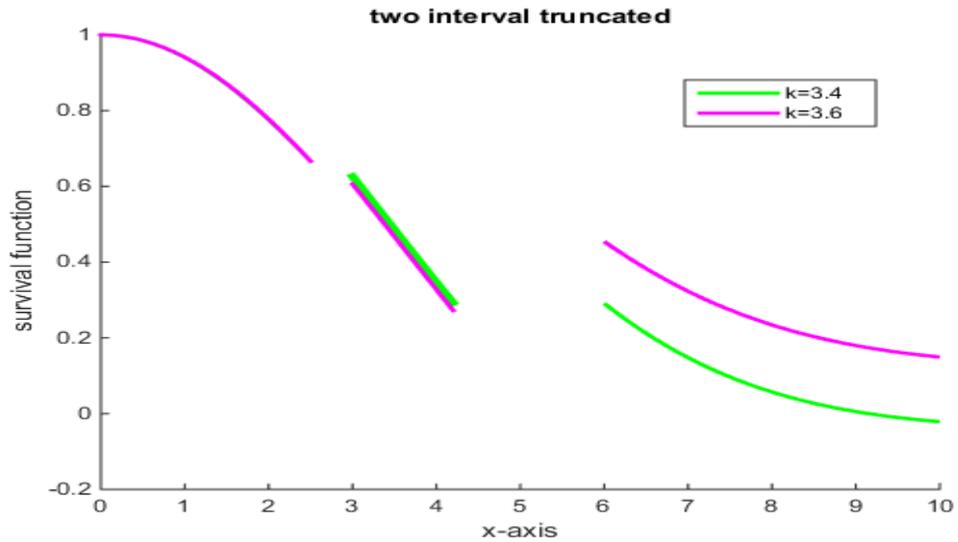


FIGURE 3.19 Two-interval truncated for survival DTRD, $\sigma = 3.4, 3.6$

In this figure, we presented the survival function of two-double truncated Rayleigh with $\sigma = 3.4, 3.6$, such that two intervals that deleted are $\{(2.5,3.5), (5,6)\}$.

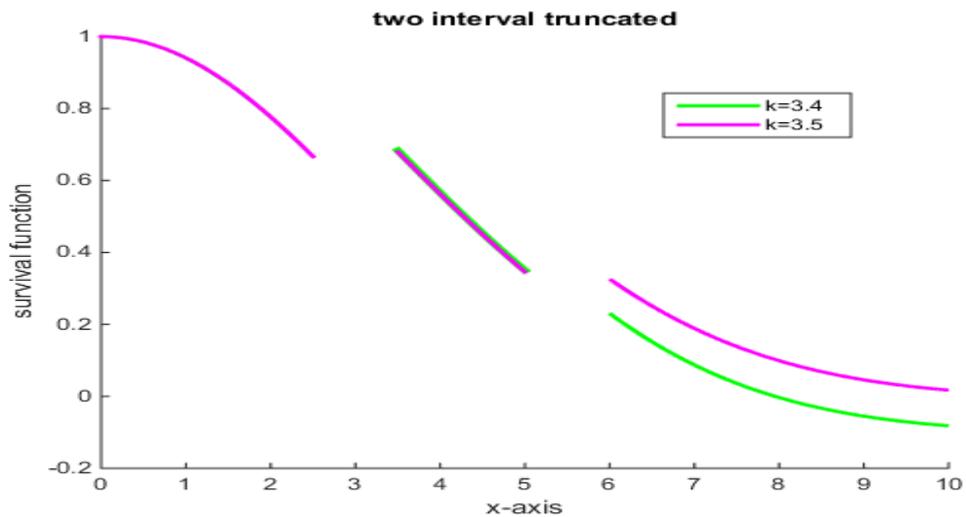


FIGURE 3.20 Two-interval truncated for survival DTRD, $\sigma = 3.4, 3.5$

The survival function of (TDTRD) with $\sigma = 3.4, 3.5$, and the intervals that deleted are $\{(2.5,3), (4.2,6)\}, \sigma = 3.4, 3.5$

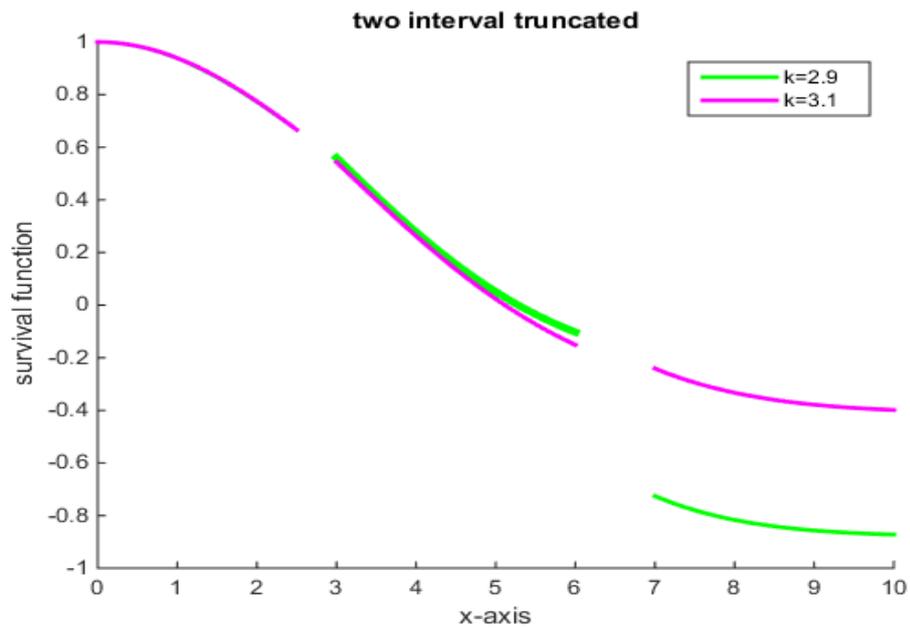


FIGURE 3.21 Two-interval truncated for survival DTRD, $\sigma = 2.9, 3.1$

In figure(3.21), we presented the survival function of (TDTRD) with $\sigma = 2.9, 3.1$, and the intervals that deleted are $\{(2.5, 3), (6, 7)\}$

CHAPTER FOUR

MULTI- LEFT TRUNCATED OF CONTINUOUS DISTRIBUTION

The left truncated distribution is one of the types of truncations, which consider a special case of the survival function. In this chapter, a new study of multi-left truncation on continuous distribution has been derivated. Statistical properties of its such as survival, reliability, hazard function, and r^{th} moment, moment generating function are discussed. Triple left – truncation is developed on Rayleigh distribution, and Exponential distribution with some statistical proprieties and shapes of pdf ,cdf, survival function for both distributions.

4.1 Constriction Multi-Left Truncation Continuous Distribution (MLTCD)

This section aims to create a new truncated distribution with multiple truncations from the left side, (multi-left truncated distribution) (MLTD) within the interval

$$(-\infty, \infty) = (b_n, \infty) \subseteq (b_{n-1}, \infty) \subseteq (b_{n-2}, \infty) \subseteq \dots \subseteq (b_{n-n}, \infty).$$

Let X be a random variable, with a probability density function, $f_X(x)$ and the cumulative distribution function $F_X(x)$ defined on interval $(-\infty, \infty) = (b_i, \infty)$, such that

$$X \in (b_n, \infty) \subseteq (b_{n-1}, \infty) \subseteq (b_{n-2}, \infty) \subseteq \dots \dots \dots (b_1, \infty) \subseteq (b_0, \infty).$$

In this section, we will derive the probability density function $g(x_T)$ of the random variable X_T , after multi subintervals, from the left side have been deleted. If we take any distribution, the truncation occurs from one side (left side) with multi-different intervals. Where the sub-intervals that work us on are: (b_n, ∞) , (b_{n-1}, ∞) , (b_{n-2}, ∞) ,, (b_1, ∞) .

Now, we formulae the probability density function of multi-left truncation,

$$g_n(x_T):$$

After defining the random variable X , we consider the interval

$$\text{is: } (-\infty, \infty) = (b_n, \infty) \subseteq (b_{n-1}, \infty) \subseteq (b_{n-2}, \infty) \subseteq \dots \subseteq (b_{n-n}, \infty).$$

That is, the intervals that have been deleted from the original interval of distribution: the first, second, third,, i^{th} truncations are: (b_0, b_1) , (b_0, b_2) , (b_0, b_3) , ..., (b_0, b_n) , respectively

Then, the probability of density function of multi- left truncated is:

$$g_n(x_T) = \frac{1}{n} \left(\frac{f(x_T)}{F(\infty) - F(b_1)} + \frac{f(x_T)}{F(\infty) - F(b_2)} + \dots + \frac{f(x_T)}{F(\infty) - F(b_n)} \right)$$

$$= \frac{1}{n} \left(\frac{f(x_T)}{1-F(b_1)} + \frac{f(x_T)}{1-F(b_2)} + \dots + \frac{f(x_T)}{1-F(b_n)} \right)$$

When, $b_1 < x_T < \infty, b_2 < x_T < \infty, \dots, b_n < x_T < \infty$.

$$\mathbb{g}_n(x_T) = \begin{cases} \frac{1}{n} \sum_{i=1}^n \frac{f(x_T)}{F(\infty)-F(b_i)} & b_i < x_T < \infty \\ 0 & \text{otherwise} \end{cases} \quad (4.1)$$

The probability density function $\mathbb{g}_n(x_T)$ is a probability density function defined over $(0, \infty), 0 < x_T < \infty, n, F(\infty) - F(b_i) > 0$, then $\mathbb{g}_n(x_T) > 0$ for all $x_T \in (b_i, \infty)$, and $\int_{-\infty}^{\infty} \mathbb{g}_n(x_T) dx_T = 1$.

And the cumulative continuous distribution function of multi-left truncation is:

$$\begin{aligned} \mathbb{G}_n(x_T) &= P(X_T \leq x_T) \\ &= \frac{1}{n} \left(\frac{F(x_T)I_{(b_1, \infty)}}{F(\infty)-F(b_1)} + \frac{F(x_T)I_{(b_2, \infty)}}{F(\infty)-F(b_2)} + \dots + \frac{F(x_T)I_{(b_n, \infty)}}{F(\infty)-F(b_n)} \right) \\ &= \frac{1}{n} \left(\frac{F(x_T)-F(b_1)}{F(\infty)-F(b_1)} + \frac{F(x_T)-F(b_2)}{F(\infty)-F(b_2)} + \dots + \frac{F(x_T)-F(b_n)}{F(\infty)-F(b_n)} \right) = \sum_{i=1}^n \left(\frac{F(x_T)I_{(b_i, \infty)}}{n(F(\infty)-F(b_i))} \right) \\ \mathbb{G}_n(x_T) &= \sum_{i=1}^n \left(\frac{F(x_T)-F(b_i)}{n(F(\infty)-F(b_i))} \right), \end{aligned} \quad (4.2)$$

when $b_1 < x_T < \infty, b_2 < x_T < \infty, \dots, b_i < x_T < \infty$, respectively.

4.2 Other Functions of Multi-Left Truncation of Continuous Distribution

The Survival Function,

$$\begin{aligned} \mathbb{S}_n(x_T) &= 1 - \mathbb{G}_n(x_T) \\ &= 1 - \frac{1}{n} \left(\frac{F(x_T)-F(b_1)}{F(\infty)-F(b_1)} + \frac{F(x_T)-F(b_2)}{F(\infty)-F(b_2)} + \dots + \frac{F(x_T)-F(b_n)}{F(\infty)-F(b_n)} \right) \\ \mathbb{S}_n(x_T) &= 1 - \frac{1}{n} \sum_{i=1}^n \left(\frac{F(x_T)-F(b_i)}{(1-F(b_i))} \right) \end{aligned} \quad (4.3)$$

The Hazard Function,

$$\begin{aligned}
\mathbb{H}_n(X_T) &= \frac{\mathbb{S}_n(X_T)}{\mathbb{S}_n(X_T)} \\
&= \frac{\frac{1}{n} \left(\frac{f(x_T)}{F(\infty)-F(b_1)} + \frac{f(x_T)}{F(\infty)-F(b_2)} + \dots + \frac{f(x_T)}{F(\infty)-F(b_i)} \right)}{1 - \frac{1}{n} \left(\frac{F(x_T)-F(b_1)}{F(\infty)-F(b_1)} + \frac{F(x_T)-F(b_2)}{F(\infty)-F(b_2)} + \dots + \frac{F(x_T)-F(b_n)}{F(\infty)-F(b_n)} \right)} \\
\mathbb{H}_n(X_T) &= \frac{\frac{1}{n} \sum_{i=1}^n \frac{f(x_T)}{1-F(b_i)}}{1 - \frac{1}{n} \sum_{i=1}^n \left(\frac{F(x_T)-F(b_i)}{n(1-F(b_i))} \right)} \tag{4.4}
\end{aligned}$$

Reversed Hazard Function,

$$\begin{aligned}
\widetilde{\mathbb{H}}_n(X_T) &= \frac{\mathbb{S}_n(X_T)}{\mathbb{G}(X_T)} = \frac{\frac{1}{n} \left(\frac{f(x_T)}{F(\infty)-F(b_1)} + \frac{f(x_T)}{F(\infty)-F(b_2)} + \dots + \frac{f(x_T)}{F(\infty)-F(b_i)} \right)}{\frac{1}{n} \left(\frac{F(x_T)I(b_1, \infty)}{F(\infty)-F(b_1)} + \frac{F(x_T)I(b_2, \infty)}{F(\infty)-F(b_2)} + \dots + \frac{F(x_T)I(b_n, \infty)}{F(\infty)-F(b_n)} \right)} \\
\widetilde{\mathbb{H}}_n(X_T) &= \frac{\frac{1}{n} \sum_{i=1}^n \frac{f(x_T)}{1-F(b_i)}}{\frac{1}{n} \sum_{i=1}^n \left(\frac{F(x_T)-F(b_i)}{n(1-F(b_i))} \right)} \tag{4.5}
\end{aligned}$$

4.3 The r^{th} Moment About Origin And About Mean of multi left - Truncated of continuous distribution

The r^{th} moment about origin of multi- left truncated is:

$$\begin{aligned}
E(X_T^r) &= \frac{1}{n} \left(\frac{\int_{b_1}^{\infty} x_T^r f(x_T) dx_T}{F(\infty)-F(b_1)} + \frac{\int_{b_2}^{\infty} x_T^r f(x_T) dx_T}{F(\infty)-F(b_2)} + \dots + \frac{\int_{b_n}^{\infty} x_T^r f(x_T) dx_T}{F(\infty)-F(b_n)} \right) \\
&= \sum_{i=1}^n \left(\frac{\int_{b_i}^{\infty} x_T^r f(x_T) dx_T}{n(1-F(b_i))} \right) \tag{4.6}
\end{aligned}$$

The r^{th} moment about a mean of multi- left truncated is:

$$\begin{aligned}
E(X_T - \mu)^r &= \sum_{j=0}^r \left((-1)^{r-j} \binom{r}{j} (\mu)^{r-j} \mu_j' \right) \\
&= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} (\mu)^{r-j} \int_{-\infty}^{\infty} x_T^j \mathbb{S}_n(x_T) dx_T \\
&= \sum_{j=0}^r \frac{(-1)^{r-j}}{n} \binom{r}{j} (\mu)^{r-j} \sum_{i=1}^n \left(\frac{\int_{b_i}^{\infty} x_T^j f(x_T) dx_T}{n(1-F(b_i))} \right) \\
&= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \left(\sum_{i=1}^n \left(\frac{\int_{b_i}^{\infty} x_T^r f(x_T) dx_T}{n(1-F(b_i))} \right) \right)^{r-j} \sum_{i=1}^n \left(\frac{\int_{b_i}^{\infty} x_T^j f(x_T) dx_T}{n(1-F(b_i))} \right)
\end{aligned}$$

$$E(X_T - \mu)^r = \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \left(\sum_{i=1}^n \left(\frac{\int_{b_i}^{\infty} x_T^r f(x_T) dx_T}{n(1-F(b_i))} \right) \right)^{r-j} \left(\sum_{i=1}^n \left(\frac{\int_{b_i}^{\infty} x_T^j f(x_T) dx_T}{n(1-F(b_i))} \right) \right) \quad (4.7)$$

4.4 Moment Generating and Characteristic Functions of Multi Left -Truncated Continuous Distribution

The moment generating function of MLTCD random variable is:

$$\mathcal{M}_{X_T}(t) = E(e^{tX_T}) = \frac{1}{n} \left(\frac{\int_{b_1}^{\infty} e^{tX_T} f(x_T) dx_T}{1-F(b_1)} + \frac{\int_{b_2}^{\infty} e^{tX_T} f(x_T) dx_T}{1-F(b_2)} + \dots + \frac{\int_{b_n}^{\infty} e^{tX_T} f(x_T) dx_T}{1-F(b_n)} \right) \quad (4.8)$$

$$\mathcal{M}_{X_T}(t) = \sum_{i=1}^n \frac{\int_{b_i}^{\infty} e^{tX_T} f(x_T) dx_T}{n(1-F(b_i))}$$

So, the characteristic function of MLTCD random variable is as follows:

$$\Phi_{X_T}(t) = E(e^{itX_T}) = \frac{1}{n} \left(\frac{\int_{b_1}^{\infty} e^{itX_T} f(x_T) dx_T}{1-F(b_1)} + \frac{\int_{b_2}^{\infty} e^{itX_T} f(x_T) dx_T}{1-F(b_2)} + \dots + \frac{\int_{b_n}^{\infty} e^{itX_T} f(x_T) dx_T}{1-F(b_n)} \right) \quad (4.9)$$

$$\Phi_{X_T}(t) = \sum_{j=1}^n \frac{\int_{b_j}^{\infty} e^{itX_T} f(x_T) dx_T}{n(1-F(b_j))}$$

4-5 Order Statistics of Multi Left -Truncated Continuous Distribution

Let $X_{T1}, X_{T2}, X_{T3}, X_{T4}, \dots, X_{Tm}$ be an independent and identical distribution of random sample with size m from cumulative distribution function. Then $X_{T[1]} \leq X_{T[2]} \leq X_{T[3]} \leq X_{T[4]} \leq \dots \leq X_{T[m]}$, be the order statistic, if the random variable X_T has MLTCD, then the probability density function of $X_{T[j]}$ is:

$$g_n(x_T)_{[j]} = \frac{m!}{(j-1)!(m-j)!} \left(\sum_{i=1}^n \frac{F(x_T) - F(b_i)}{n(1-F(b_i))} \right)^{j-1} \left(1 - \sum_{i=1}^n \frac{F(x_T) - F(b_i)}{n(1-F(b_i))} \right)^{m-j} \left(\sum_{i=1}^n \frac{f(x_T)}{n(1-F(b_i))} \right)$$

$$\text{But, } \left(\sum_{i=1}^n \frac{F(x_T) - F(b_i)}{n(1-F(b_i))} \right)^{j-1} = \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_{(j-1)}=1}^n \left(\frac{F(x_T) - F(b_i)}{n(1-F(b_i))} \right)_{i_1 i_2 \dots i_{(j-1)}}$$

$$\mathbb{G}_n(x_T)_{[j]} = \frac{m!}{(j-1)!(m-j)!} \left[\sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_{(j-1)}=1}^n \left(\frac{F(x_T) - F(b_i)}{n(1-F(b_i))} \right)_{i_1 i_2 \dots i_{(j-1)}} \right]$$

$$\left[\sum_{i=1}^n \frac{f(x_T)}{n(1-F(b_i))} \sum_{k=0}^{m-j} \binom{m-j}{k} (-1)^k \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n \left(\frac{F(x_T) - F(b_i)}{n(1-F(b_i))} \right)_{i_1 i_2 \dots i_k} \right]$$

The smallest order statistics $x_{T[1]}$ is:

$$\mathbb{G}_n(x_T)_{[1]} = m \sum_{k=0}^{m-1} \frac{(-1)^k}{n^k} \binom{m-1}{k} \left(\sum_{i=1}^n \frac{F(x_T) - F(b_i)}{n(1-F(b_i))} \right)^k \left(\sum_{i=1}^n \frac{f(x_T)}{n(1-F(b_i))} \right)$$

$$= m \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n \left(\frac{F(x_T) - F(b_i)}{n(1-F(b_i))} \right)_{i_1 i_2 \dots i_k} \left(\sum_{i=1}^n \frac{f(x_T)}{n(1-F(b_i))} \right) \quad (4.10)$$

4.6 Triple - Left Truncation of Rayleigh Distribution (Triple -LTRD)

4.6.1 Triple Left Truncation of PDF for Rayleigh Distribution

Let x_T be a random variable has PDF, given in equation (4-1), if taking triple-left truncated of Rayleigh distribution, then the probability distribution function of three intervals left truncated of Rayleigh distribution is:

$$\mathbb{G}_3(x_T) = \begin{cases} \frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{3 \left(\frac{-b_1^2}{e \frac{2\sigma^2}{2\sigma^2}} \right)} + \frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{3 \left(\frac{-b_2^2}{e \frac{2\sigma^2}{2\sigma^2}} \right)} + \frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{3 \left(\frac{-b_3^2}{e \frac{2\sigma^2}{2\sigma^2}} \right)} \end{cases} \quad (4.11)$$

where $b_1 < x_T < \infty$, $b_2 < x_T < \infty$, $b_3 < x_T < \infty$, respectively.

That is, the triple intervals (b_1, ∞) , (b_2, ∞) , (b_3, ∞) , have been drawn to represent the triple left truncated. that is, the intervals were deleted from the original interval for distribution and are the first, second, and third truncations (b_0, b_1) , (b_0, b_2) , (b_0, b_3) , respectively. As shown in figures (4.1), (4-2), and (4-3).

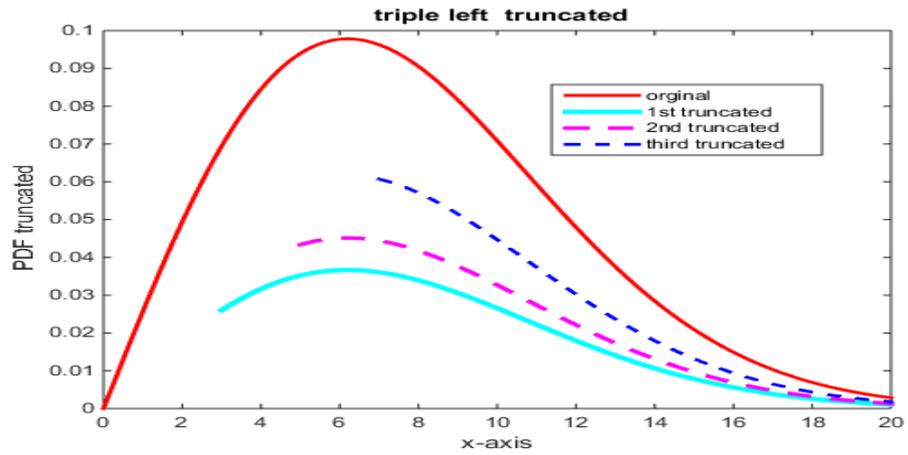


FIGURE 4.1 Three intervals are deleted in LTRD when $\sigma = 5$

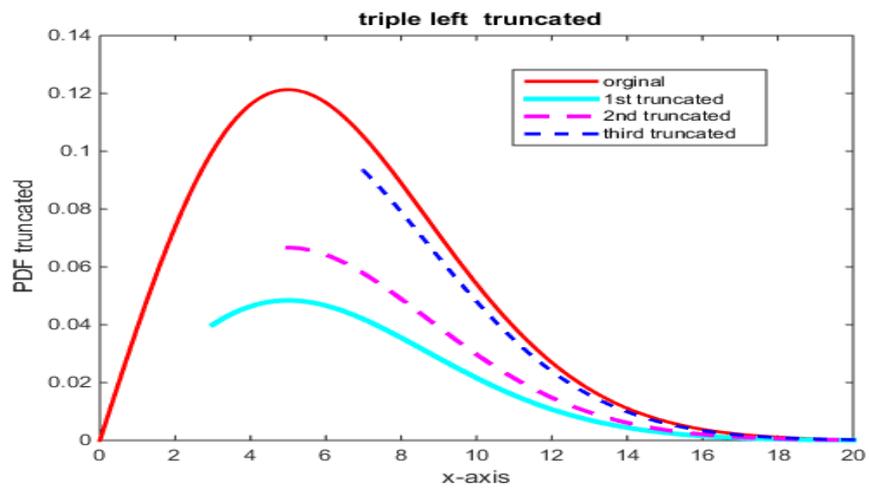


FIGURE 4.2 Three intervals are deleted in LTRD when $\sigma = 6.2$

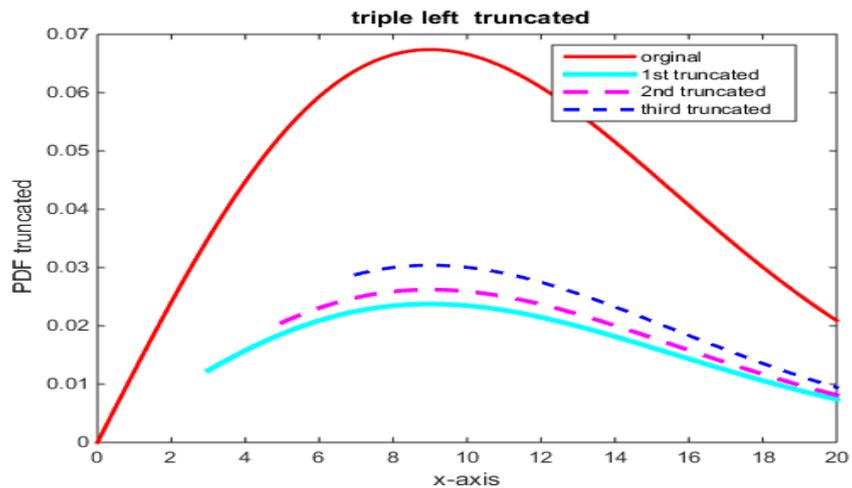


FIGURE 4.3 Three intervals are deleted in LTRD when $\sigma = 8$.

In the above figures, three curves of the triple left truncated and curve of the original function are drawn for three intervals $(3,20), (5,20), (7,20)$, and different values of parameters $\sigma = \{5, 6.2, 8\}$, respectively.

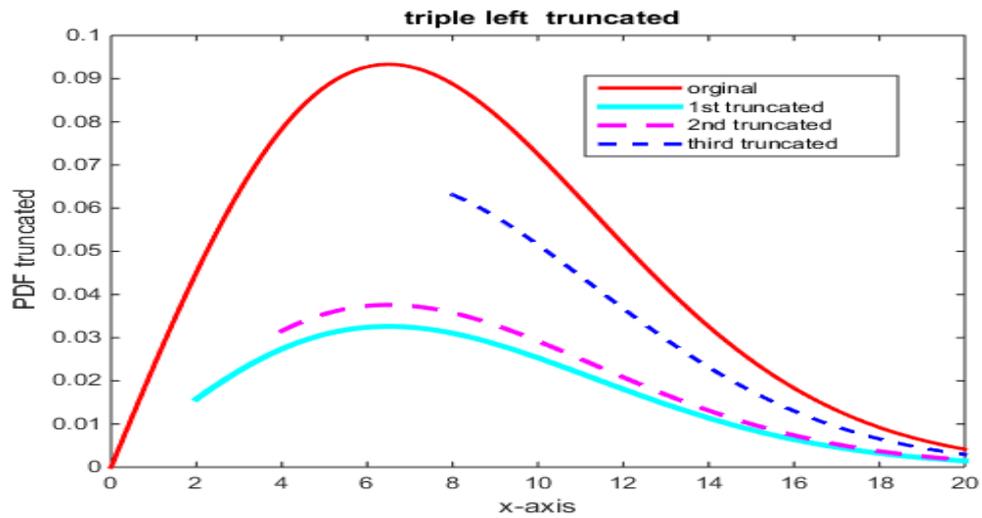


FIGURE 4.4 Three intervals are deleted in LTRD when $\sigma = 6.5$

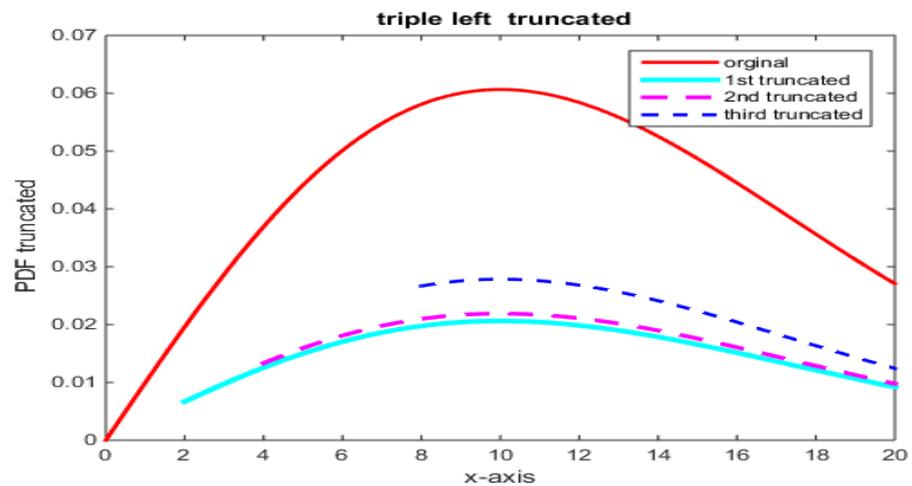


FIGURE 4.5 Three intervals are deleted in LTRD when $\sigma = 10$

Figures (4-4), (4-5), show the original function and the pdf of triple left truncated of Rayleigh distribution, where $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(2,20), (4,20), (8,20)\}, \{(4,20), (8,20), (12,20)\}$ for different values of parameters $\sigma = \{5,10\}$ respectively.

While, figures (4-6),(4-7),(4-8), and (4-9), show the original function and the pdf of triple left truncated of Rayleigh distribution, where $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(5,40), (8,40), (13,40)\}, \{(5,40), (8,40), (10,40)\}, \{(6,30), (9,30), (13,30)\}, \{(4,30), (9,30), (12,30)\}$ for different values of parameters $\sigma = \{9.3, 8.4, 9.2, 10\}$ respectively.

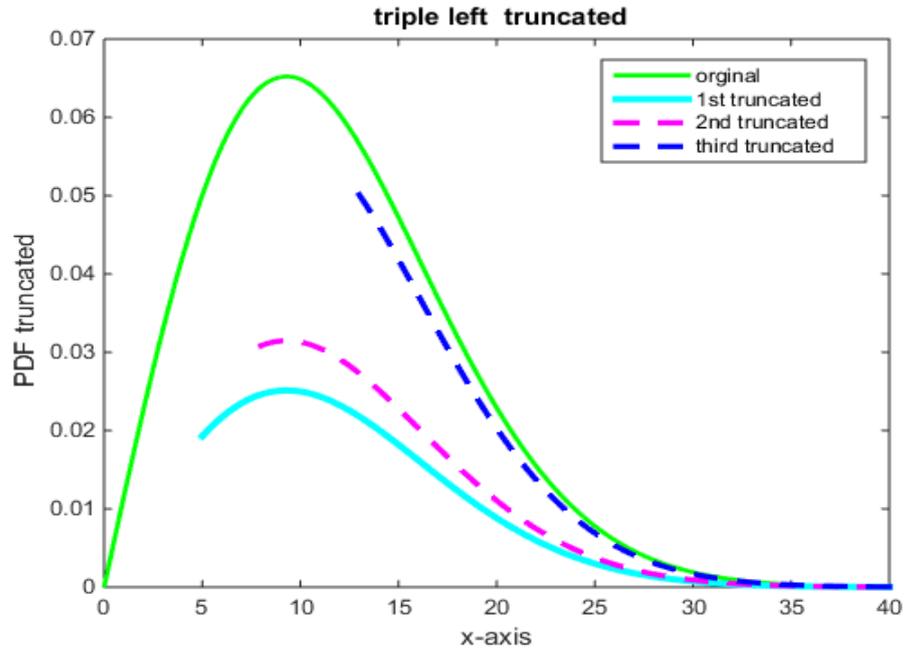


FIGURE 4.6 Three intervals are deleted in LTRD when $\sigma = 9.3$

Show the original function with PDF of triple left truncated of Rayleigh, where $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(5,40), (8,40), (13,40)\}$ $\sigma = 9.3$

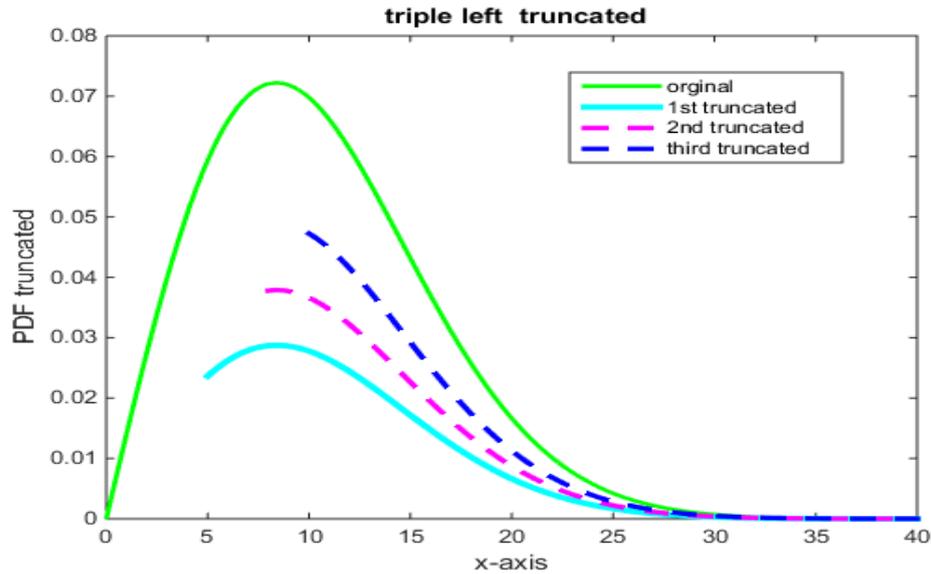


FIGURE 4.7 Three intervals are deleted in LTRD when $\sigma = 8.4$

In this figure, we have drawn the original function with the PDF of triple left truncated of Rayleigh where $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(5,40), (6,30), (9,30), (13,30)\}$ parameter $\sigma = \{8.4\}$

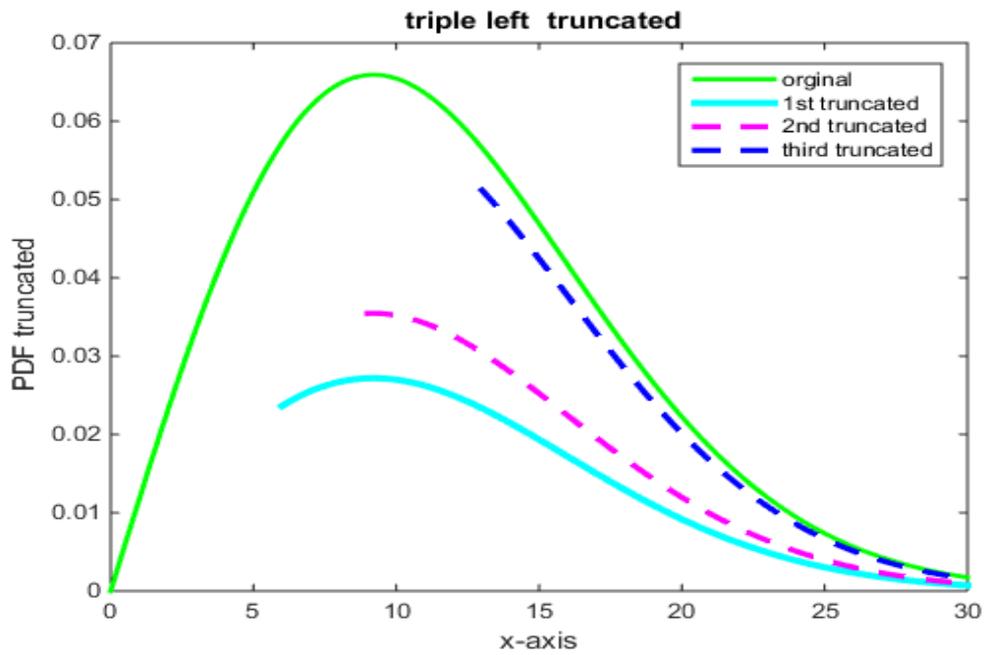


FIGURE 4.8 Three intervals are deleted in LTRD when $\sigma = 9.2$.

show the original function and the PDF of triple left truncated of Rayleigh where $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{6, 30\}, \{9, 30\}, \{13, 30\}$ parameters $\sigma = \{9.2\}$

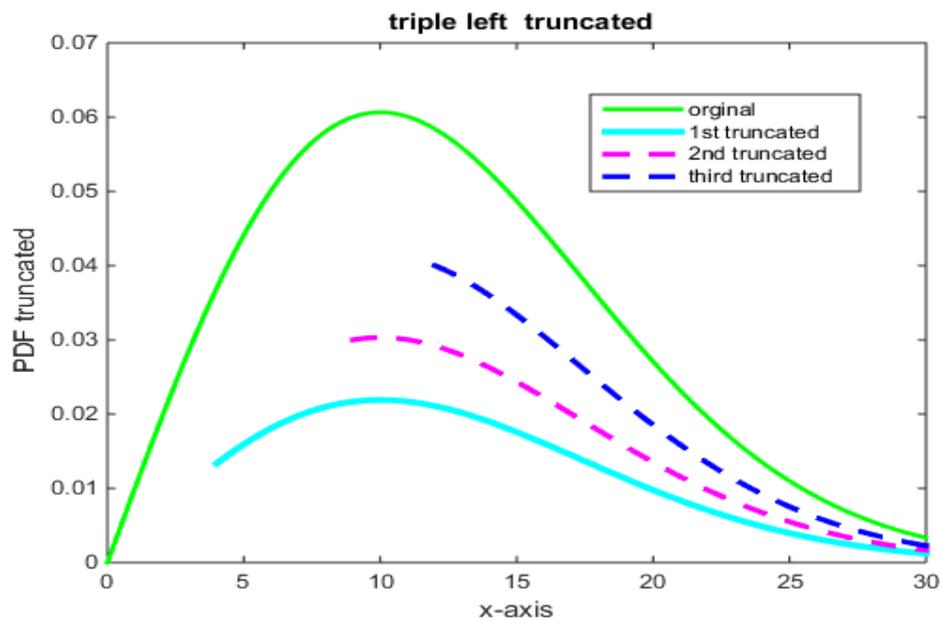


FIGURE 4.9 Three intervals are deleted in LTRD when $\sigma = 10$

show the original function and the pdf of triple left truncated of Rayleigh where $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{4, 30\}, \{9, 30\}, \{12, 30\}$ of parameters $\sigma = \{10\}$

4.6.2 Triple Left Truncation of CDF for Rayleigh Distribution

The cumulative distribution function of Rayleigh distribution after three intervals left truncation is:

$$G_3(x_T) = \begin{cases} \frac{e^{-\frac{b_1^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}}}{3 \left(e^{-\frac{b_1^2}{2\sigma^2}} \right)} + \frac{e^{-\frac{b_2^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}}}{3 \left(e^{-\frac{b_2^2}{2\sigma^2}} \right)} + \frac{e^{-\frac{b_3^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}}}{3 \left(e^{-\frac{b_3^2}{2\sigma^2}} \right)} & (4.12) \\ b_1 < x_T < \infty, \quad b_2 < x_T < \infty, \quad b_3 < x_T < \infty \end{cases}$$

In figures (4.10),(4.11), and (4.12), the curves of the original cumulative distribution function and the cumulative distribution function after three intervals left truncation, for different values of parameters σ have been drawn. Where $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(2,10), (3,10), (6,10)\}$ respectively. That is, the intervals which have been deleted are $\{(0,2), (0,3), (0,6)\}, \{(0,5), (0,8), (0,13)\}, \{(0,2), (0,4), (0,8)\}$, of $\sigma = \{3.5, 9.4, 8\}$, respectively.

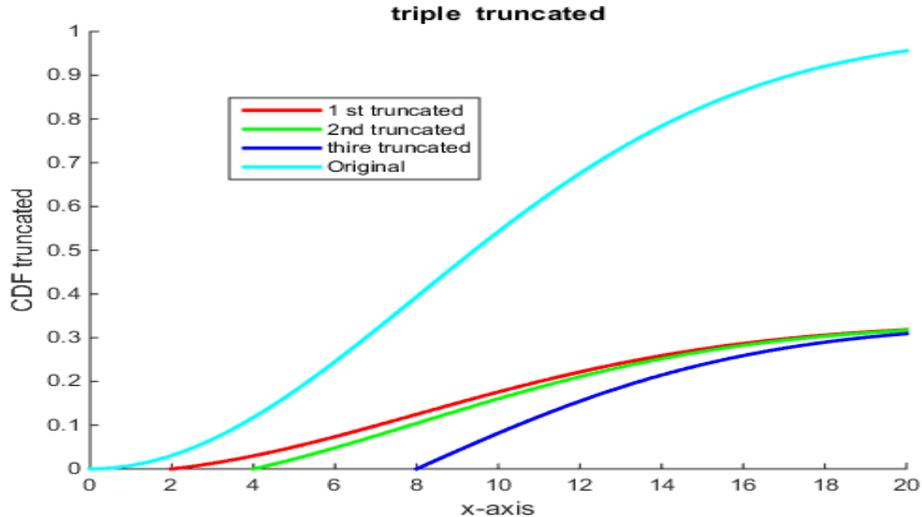


FIGURE 4.10 The CDF of TLTRD when $\sigma = 3.5$

the curves of the original cumulative distribution function and the cumulative distribution function after three intervals left truncation, $\sigma=3.5$. Where $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(2,10), (4,10), (8,10)\}$ respectively. That is, the intervals which have been deleted are $\{(0,2), (0,4), (0,8)\}$

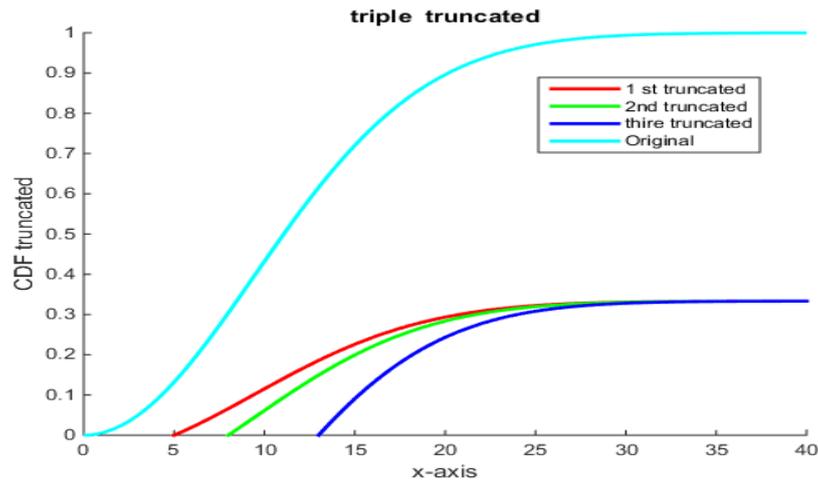


FIGURE 4.11 The CDF of TLTRD when $\sigma = 9.4$

the curves of the original cumulative distribution function and the cumulative distribution function after three intervals lefttruncation, $\sigma=9.4$. Where $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} \{(5,40), (7,40), (12,40)\}$ respectively That is, the intervals that deleted are $\{(0,5), (0,7), (0,12)\}$

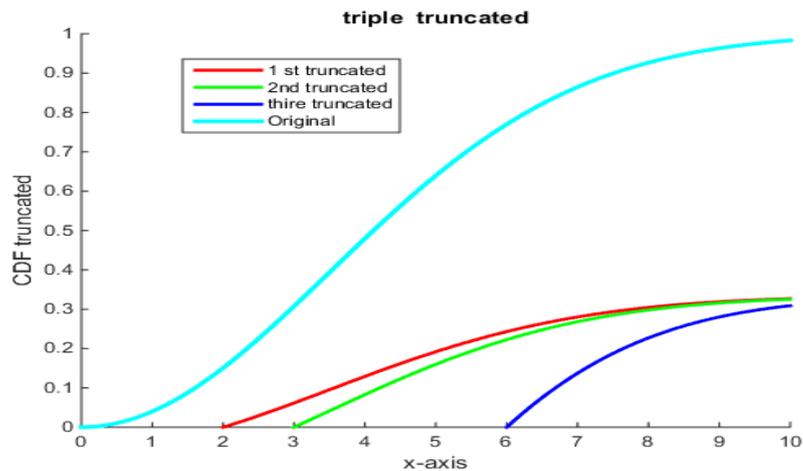


FIGURE 4.12 The CDF of TLTRD when $\sigma = 8$

the curves of the original cumulative distribution function and the cumulative distribution function after three intervals lefttruncation, $\sigma=9.4$. Where $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} \{(2,10), (3,10), (6,10)\}$ respectively That is, the intervals that deleted are $\{(0,2), (0,3), (0,6)\}$

4.6.3 Triple Left Truncation Survival Function of Rayleigh Distribution

The survival function of triple-left truncated for Rayleigh distribution is:

$$S_3(x_T) = 1 - \frac{e^{-\frac{b_1^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}}}{3e^{-\frac{b_1^2}{2\sigma^2}}} - \frac{e^{-\frac{b_2^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}}}{3e^{-\frac{b_2^2}{2\sigma^2}}} - \frac{e^{-\frac{b_3^2}{2\sigma^2}} - e^{-\frac{x_T^2}{2\sigma^2}}}{3e^{-\frac{b_3^2}{2\sigma^2}}} \quad (4.13)$$

$$b_1 < x_T < \infty, \quad b_2 < x_T < \infty, \quad b_3 < x_T < \infty.$$

In this subsection, many figures for representation of the truncation of the triple left truncation of the survival function of Rayleigh distribution have been introduced, for different values for parameter and x_T with the original distribution has been shown in figures(4-13)-(4-17). The value of $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(5, 30), (18, 30), (25, 30)\}$

to figure (4-13),(4-14),(4-15),(4-16) and (4-17) when the values of parameter are $\{3.5, 5, 8, 9, 7.5\}$ respectively.

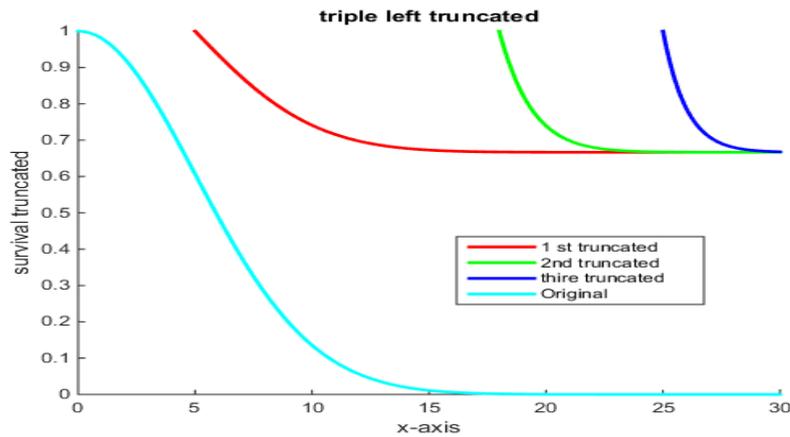


FIGURE 4.13 The survival of TLTRD when $\sigma = 3.5$

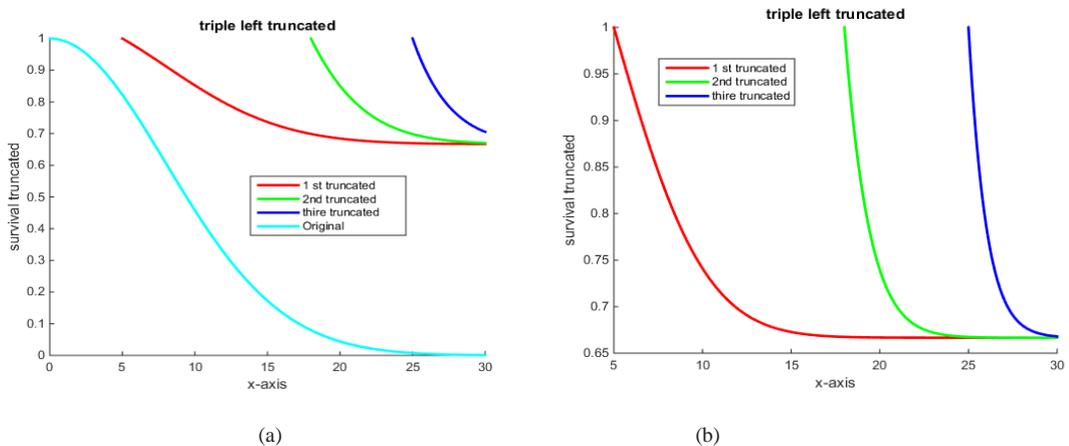


FIGURE 4.14 The survival of TLTRD when $\sigma = 5$, (a) with the original function, (b) without the original function

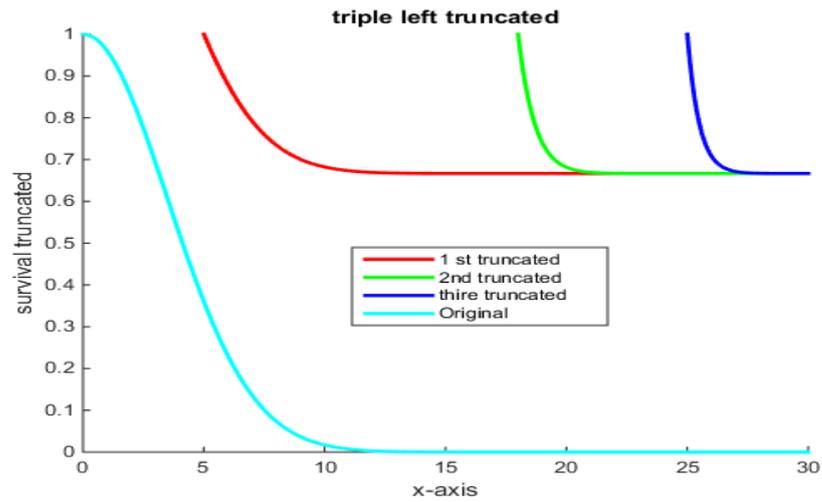


FIGURE 4.15 The survival of TLTRD when $\sigma = 8$

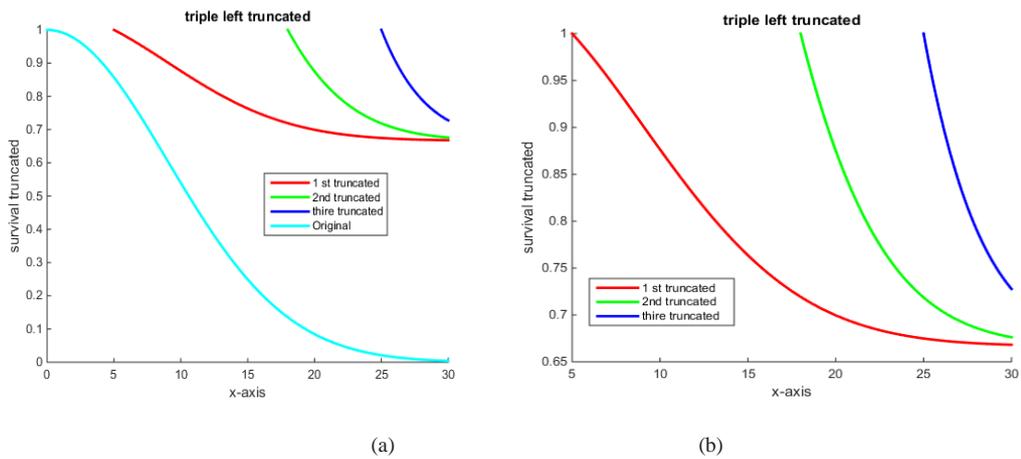


FIGURE 4.16 The survival of TLTRD when $\sigma = 9$, (a) with the original function, (b) without the original function

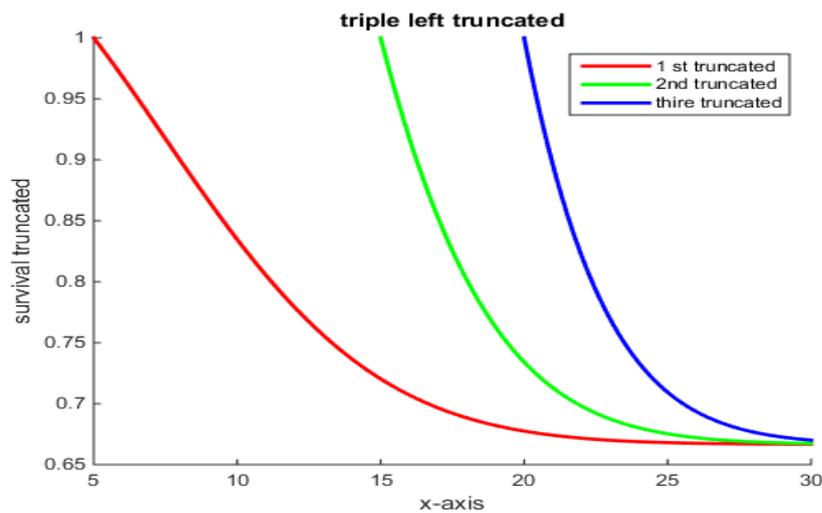


FIGURE 4.17 The survival of TLTRD when $\sigma = 7.4$

In this figure, we presented the survival function of TLTRD when $\sigma = 7$.

4.6.4 The r^{th} Moment about Origin and About Mean of Multi-Left Truncated Rayleigh Distribution

The r^{th} moment about origin is:

$$\begin{aligned} E(X_T^r) &= \frac{1}{n} \left(\frac{\int_{b_1}^{\infty} x_T^r f(x_T) dx_T}{1-F(b_1)} + \frac{\int_{b_2}^{\infty} x_T^r f(x_T) dx_T}{1-F(b_2)} + \dots + \frac{\int_{b_n}^{\infty} x_T^r f(x_T) dx_T}{1-F(b_n)} \right) \\ &= \frac{1}{n} \left(\frac{\int_{b_1}^{\infty} x_T^r \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}} dx_T}{1-F(b_1)} + \frac{\int_{b_2}^{\infty} x_T^r \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}} dx_T}{1-F(b_2)} + \dots + \frac{\int_{b_n}^{\infty} x_T^r \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}} dx_T}{1-F(b_n)} \right) \end{aligned}$$

$$\text{Take } I_1 = \int_{b_1}^{\infty} x_T^r \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}} dx_T$$

$$\text{Let } y = \frac{x_T^2}{2\sigma^2}, x_T = (2\sigma^2 y)^{1/2}, dx_T = (2\sigma^2)^{1/2} \frac{1}{2} (y)^{-1/2} dy$$

$$I_1 = (2\sigma^2)^{\frac{r}{2}} \int_{b_1^2/2\sigma^2}^{\infty} ((y))^{\frac{r}{2}+1-1} e^{-y} dy$$

$$I_1 = (2\sigma^2)^{\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1, b_1^2/2\sigma^2\right),$$

$$I_2 = (2\sigma^2)^{\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1, b_2^2/2\sigma^2\right), \dots, I_n = (2\sigma^2)^{\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1, \frac{b_n^2}{2\sigma^2}\right).$$

$$E(X_T^r) = \sum_{i=1}^n \frac{(2\sigma^2)^{\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1, \frac{b_i^2}{2\sigma^2}\right)}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}}\right)} \quad (4.14)$$

And The r^{th} moment about mean is:

$$\begin{aligned} E(X_T - \mu)^r &= \sum_{j=0}^r \left((-1)^{r-j} \binom{r}{j} (\mu)^{r-j} \mu_j' \right) \\ &= \sum_{j=0}^r \frac{(-1)^{r-j}}{n} \binom{r}{j} (\mu)^{r-j} \sum_{i=1}^n \left(\frac{\int_{b_i}^{\infty} x_T^j f(x_T) dx_T}{n(1-F(b_i))} \right) \\ &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \left(\sum_{i=1}^n \left(\frac{\int_{b_i}^{\infty} x_T^r f(x_T) dx_T}{n(1-F(b_i))} \right) \right)^{r-j} \sum_{i=1}^n \left(\frac{\int_{b_i}^{\infty} x_T^j f(x_T) dx_T}{n(1-F(b_i))} \right) \end{aligned}$$

$$E(X_T - \mu)^r =$$

$$\sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \left(\sum_{i=1}^n \frac{(2\sigma^2)^{\frac{1}{2}} \Gamma\left(\frac{1}{2} + 1, \frac{b_i^2}{2\sigma^2}\right)}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}}\right)} \right)^{r-j} \left(\sum_{i=1}^n \frac{(2\sigma^2)^{\frac{j}{2}} \Gamma\left(\frac{j}{2} + 1, \frac{b_i^2}{2\sigma^2}\right)}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}}\right)} \right)$$

$$E(X_T - \mu)^r =$$

$$\sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_{(r-j)}=1}^n \left(\sum_{i=1}^n \frac{(2\sigma^2)^{\frac{1}{2}} \Gamma(\frac{1}{2}+1, \frac{b_i^2}{2\sigma^2})}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}} \right)} \right)_{i_1 i_2 \dots i_{r-j}} \left(\sum_{i=1}^n \frac{(2\sigma^2)^{\frac{j}{2}} \Gamma(\frac{j}{2}+1, \frac{b_i^2}{2\sigma^2})}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}} \right)} \right), E(X_T) = \sum_{i=1}^n \frac{(2\sigma^2)^{\frac{1}{2}} \Gamma(\frac{1}{2}+1, \frac{b_i^2}{2\sigma^2})}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}} \right)} \quad (4.15)$$

$$\text{Skew}(x_T) = \frac{\sum_{j=0}^3 (-1)^{3-j} \binom{3}{j} \left(\sum_{i=1}^n \frac{(2\sigma^2)^{\frac{1}{2}} \Gamma(\frac{1}{2}+1, \frac{b_i^2}{2\sigma^2})}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}} \right)} \right)^{3-j} \left(\sum_{i=1}^n \frac{(2\sigma^2)^{\frac{j}{2}} \Gamma(\frac{j}{2}+1, \frac{b_i^2}{2\sigma^2})}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}} \right)} \right)}{\left(\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left(\sum_{i=1}^n \frac{(2\sigma^2)^{\frac{1}{2}} \Gamma(\frac{1}{2}+1, \frac{b_i^2}{2\sigma^2})}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}} \right)} \right)^{2-j} \left(\sum_{i=1}^n \frac{(2\sigma^2)^{\frac{j}{2}} \Gamma(\frac{j}{2}+1, \frac{b_i^2}{2\sigma^2})}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}} \right)} \right) \right)^{3/2}} \quad (4.16)$$

$$\text{Kurt}(x_T) = \frac{\sum_{j=0}^4 (-1)^{4-j} \binom{4}{j} \left(\sum_{i=1}^n \frac{(2\sigma^2)^{\frac{1}{2}} \Gamma(\frac{1}{2}+1, \frac{b_i^2}{2\sigma^2})}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}} \right)} \right)^{4-j} \left(\sum_{i=1}^n \frac{(2\sigma^2)^{\frac{j}{2}} \Gamma(\frac{j}{2}+1, \frac{b_i^2}{2\sigma^2})}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}} \right)} \right)}{\left(\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left(\sum_{i=1}^n \frac{(2\sigma^2)^{\frac{1}{2}} \Gamma(\frac{1}{2}+1, \frac{b_i^2}{2\sigma^2})}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}} \right)} \right)^{2-j} \left(\sum_{i=1}^n \frac{(2\sigma^2)^{\frac{j}{2}} \Gamma(\frac{j}{2}+1, \frac{b_i^2}{2\sigma^2})}{n \left(1 - e^{-\frac{b_i^2}{2\sigma^2}} \right)} \right) \right)^2} \quad (4.17)$$

Theorem 4.1 let X_T be a random variable with multi-left truncated Rayleigh distribution, has PDF as shown in equations (4-11), then the moment generating function equal to

$$\frac{1}{n} \sum_{i=1}^n \frac{\frac{1}{\sigma^2} e^{\frac{1}{2\sigma^2} + (\sigma^2 t)^2} \frac{1}{2} \sqrt{\pi} (1 - \text{erf}(b_i - \sigma^2 t))}{e^{-\frac{b_i^2}{2\sigma^2}}}, \text{ and characteristic function is equal}$$

$$\text{to } \frac{1}{n} \sum_{j=1}^n \frac{\frac{1}{\sigma^2} e^{\frac{1}{2\sigma^2} + (\sigma^2 i t)^2} \frac{1}{2} \sqrt{\pi} (1 - \text{erf}(b_j - \sigma^2 i t))}{e^{-\frac{b_j^2}{2\sigma^2}}}.$$

Proof:-

$$\begin{aligned} \mathcal{M}_{X_T}(t) &= E(e^{tx_T}) \\ &= \frac{1}{n} \left(\frac{\int_{b_1}^{\infty} e^{tx_T} f(x_T) dx_T}{1 - F(b_1)} + \frac{\int_{b_2}^{\infty} e^{tx_T} f(x_T) dx_T}{1 - F(b_2)} + \dots + \frac{\int_{b_n}^{\infty} e^{tx_T} f(x_T) dx_T}{1 - F(b_n)} \right) \end{aligned}$$

$$\begin{aligned} \text{Take: } I_1 &= \frac{1}{\sigma^2} \int_{b_1}^{\infty} e^{tx_T} e^{-\frac{x_T^2}{2\sigma^2}} dx_T = \frac{1}{\sigma^2} \int_{b_1}^{\infty} e^{-\frac{x_T^2 + 2\sigma^2 t x_T}{2\sigma^2}} dx_T \\ &= \frac{1}{\sigma^2} \int_{b_1}^{\infty} e^{\frac{-1}{2\sigma^2} \{(x_T - d)^2 + d^2\}} dx_T, \text{ Let } d = \sigma^2 t, u = x_T - d, du = dx_T \end{aligned}$$

$$\begin{aligned} I_1 &= \frac{1}{\sigma^2} e^{\frac{d^2}{2\sigma^2}} \int_{b_1 - d}^{\infty} e^{-\frac{1}{2\sigma^2} u^2} du \\ &= \frac{1}{\sigma^2} e^{\frac{d^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2\pi}}{2} \left(\text{erf}(\infty) \left(\frac{1}{\sqrt{2\sigma^2}} \right) - \text{erf}(b_1 - d) \left(\frac{1}{\sqrt{2\sigma^2}} \right) \right) \end{aligned}$$

$$\begin{aligned} \text{So, if } I_2 &= \frac{1}{\sigma^2} \int_{b_2}^{\infty} e^{tx_T} e^{-\frac{x_T^2}{2\sigma^2}} dx_T = \frac{1}{\sigma^2} e^{\frac{d^2}{2\sigma^2}} \int_{b_2 - d}^{\infty} e^{-\frac{1}{2\sigma^2} u^2} du \\ &= \frac{1}{\sigma^2} e^{\frac{d^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2\pi}}{2} \left(\text{erf}(\infty) \left(\frac{1}{\sqrt{2\sigma^2}} \right) - \text{erf}(b_2 - d) \left(\frac{1}{\sqrt{2\sigma^2}} \right) \right) \end{aligned}$$

$$I_n = \frac{1}{\sigma^2} e^{\frac{d^2}{2\sigma^2}} \int_{b_n - d}^{\infty} e^{-\frac{1}{2\sigma^2} u^2} du = \frac{1}{\sigma^2} e^{\frac{d^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2\pi}}{2} \left(\text{erf}(\infty) \left(\frac{1}{\sqrt{2\sigma^2}} \right) - \text{erf}(b_n - d) \left(\frac{1}{\sqrt{2\sigma^2}} \right) \right)$$

$$\begin{aligned} \mathcal{M}_{X_T}(t) &= \\ &= \frac{1}{n} \left(\frac{\frac{1}{\sigma^2} e^{\frac{d^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2\pi}}{2} \left(\left(\frac{1}{\sqrt{2\sigma^2}} \right) - \text{erf}(b_1 - d) \left(\frac{1}{\sqrt{2\sigma^2}} \right) \right)}{e^{-\frac{b_1^2}{2\sigma^2}}} + \frac{\frac{1}{\sigma^2} e^{\frac{d^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2\pi}}{2} \left(\left(\frac{1}{\sqrt{2\sigma^2}} \right) - \text{erf}(b_2 - d) \left(\frac{1}{\sqrt{2\sigma^2}} \right) \right)}{e^{-\frac{b_2^2}{2\sigma^2}}} + \right. \\ &\quad \left. \frac{\frac{1}{\sigma^2} e^{\frac{d^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2\pi}}{2} \left(\left(\frac{1}{\sqrt{2\sigma^2}} \right) - \text{erf}(b_n - d) \left(\frac{1}{\sqrt{2\sigma^2}} \right) \right)}{e^{-\frac{b_n^2}{2\sigma^2}}} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \frac{\frac{1}{\sigma^2} e^{\frac{d^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2\pi}}{2} \left(\left(\frac{1}{\sqrt{2\sigma^2}} \right) - \text{erf}(b_i - d) \left(\frac{1}{\sqrt{2\sigma^2}} \right) \right)}{e^{-\frac{b_i^2}{2\sigma^2}}} \end{aligned} \tag{4.18}$$

By the same way, we can get the characteristic function of (MLTRD):

$$\begin{aligned}
 \Phi_{X_T}(t) &= E(e^{itx_T}) = \frac{1}{n} \left(\frac{\int_{b_1}^{\infty} e^{itx_T} f(x_T) dx_T}{1-F(b_1)} + \frac{\int_{b_2}^{\infty} e^{itx_T} f(x_T) dx_T}{1-F(b_2)} + \dots + \right. \\
 &\left. \frac{\int_{b_n}^{\infty} e^{itx_T} f(x_T) dx_T}{1-F(b_n)} \right) = \frac{1}{n} \left(\frac{\frac{1}{\sigma^2} e^{\frac{(\sigma^2 it)^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2\pi}}{2} \left(\frac{1}{\sqrt{2\sigma^2}} \right) - \text{erf} \left(b_1 - \sigma^2 it \right) \left(\frac{1}{\sqrt{2\sigma^2}} \right)}{e^{-\frac{b_1^2}{2\sigma^2}}} + \right. \\
 &\left. \frac{\frac{1}{\sigma^2} e^{\frac{(\sigma^2 it)^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2\pi}}{2} \left(\frac{1}{\sqrt{2\sigma^2}} \right) - \text{erf} \left(b_2 - \sigma^2 it \right) \left(\frac{1}{\sqrt{2\sigma^2}} \right)}{e^{-\frac{b_2^2}{2\sigma^2}}} + \dots \dots + \right. \\
 &\left. \frac{\frac{1}{\sigma^2} e^{\frac{(\sigma^2 it)^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2\pi}}{2} \left(\frac{1}{\sqrt{2\sigma^2}} \right) - \text{erf} \left(b_n - \sigma^2 it \right) \left(\frac{1}{\sqrt{2\sigma^2}} \right)}{e^{-\frac{b_n^2}{2\sigma^2}}} \right) \\
 \Phi_{X_T}(t) &= \sum_{j=1}^n \frac{\frac{1}{\sigma^2} e^{\frac{(\sigma^2 it)^2}{2\sigma^2}} \frac{\sqrt{2\sigma^2\pi}}{2} \left(\frac{1}{\sqrt{2\sigma^2}} \right) - \text{erf} \left(b_j - \sigma^2 it \right) \left(\frac{1}{\sqrt{2\sigma^2}} \right)}{n \left(e^{-\frac{b_j^2}{2\sigma^2}} \right)} \tag{4.19}
 \end{aligned}$$

4.7 Triple-Left Truncated for Exponential Distribution (Triple LTED)

In this section, we introduced a formula for the probability density function for exponential distribution after the cut three intervals from the left side. Further, the cumulative distribution, the survival functions, and some statistical properties have been obtained.

4.7.1 Triple Left Truncation for PDF of Exponential Distribution

Let X_T be a random variable has PDF, given in equation (4-1), if taking an exponential distribution, with triple-left truncated. then the probability density function of three intervals left truncated of Exponential distribution is:

$$\mathbb{G}_3(X_T) = \left\{ \frac{\lambda e^{-\lambda x_T}}{3(e^{-\lambda b_1})} + \frac{\lambda e^{-\lambda x_T}}{3(e^{-\lambda b_2})} + \frac{\lambda e^{-\lambda x_T}}{3(e^{-\lambda b_3})} \right. \tag{4.19}$$

where $b_1 < x_T < \infty$, $b_2 < x_T < \infty$, $b_3 < x_T < \infty$, respectively.

That is, the three intervals (b_1, ∞) , (b_2, ∞) , (b_3, ∞) , have been used to represent the triple left truncated probability exponential distribution. the intervals deleted from the original interval are: the first, second, and third (b_0, b_1) , (b_0, b_2) , (b_0, b_3) , respectively, as shown in figures (4.18),(4.19),(4.20), and (4.21).

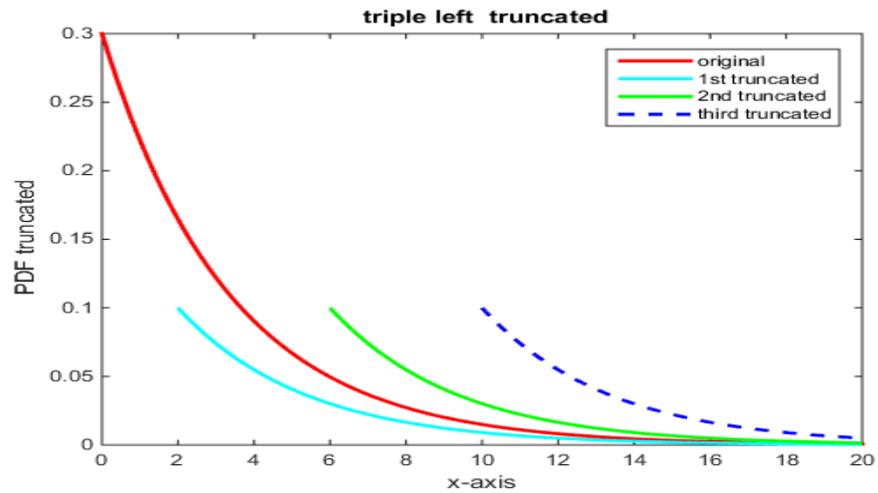


FIGURE 4.18: Three intervals are deleted in LTED when $\lambda = 0.3$

PDF of (TLTED) on intervals $(2,20),(6,20),(10,20)$, Three intervals that deleted in LTED are $(0,2)(0,6)(0,10)$ when $\lambda = 0.3$

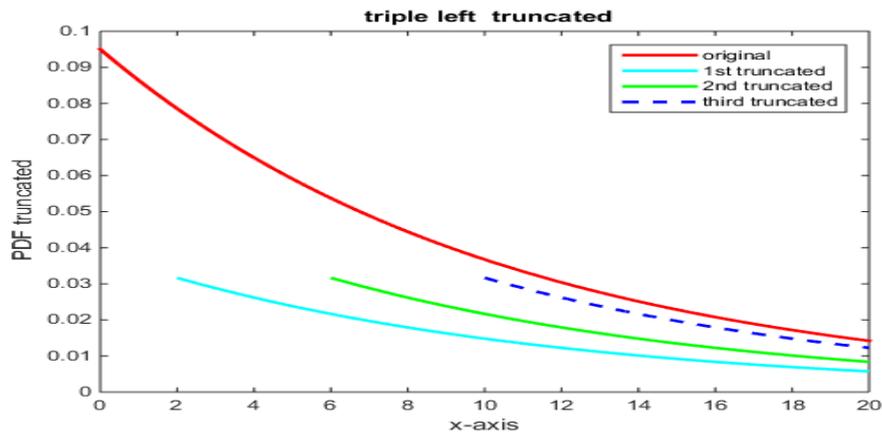


FIGURE 4.19 Three intervals are deleted in LTED when $\lambda = 0.095$

PDF of (TLTED) on intervals $(2,20),(6,20),(10,20)$, Three intervals that deleted in LTED are $(0,2)(0,6)(0,10)$ when $\lambda = 0.095$

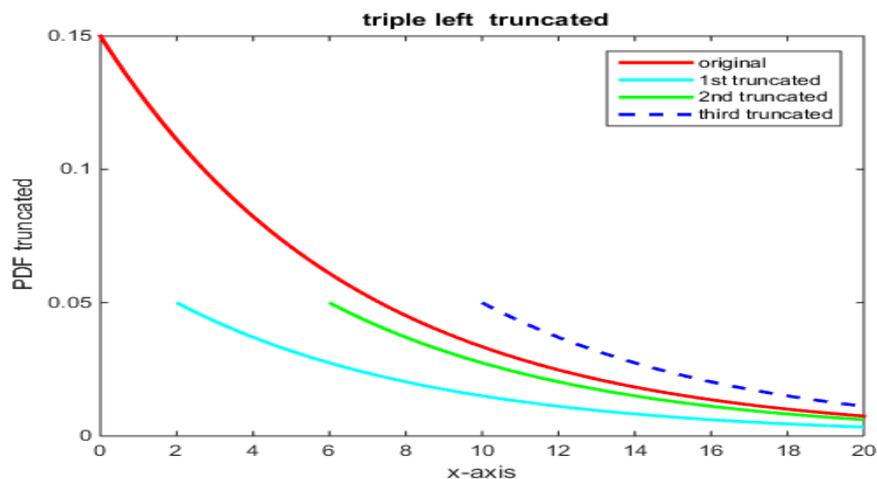


FIGURE 4.20 Three intervals are deleted in LTED when $\lambda = 0.15$

The probability function of (TLTED) on intervals (2,20),(6,20),(10,20), Three intervals that deleted in LTED are(0,2)(0,6)(0,10) when $\lambda = 0.15$

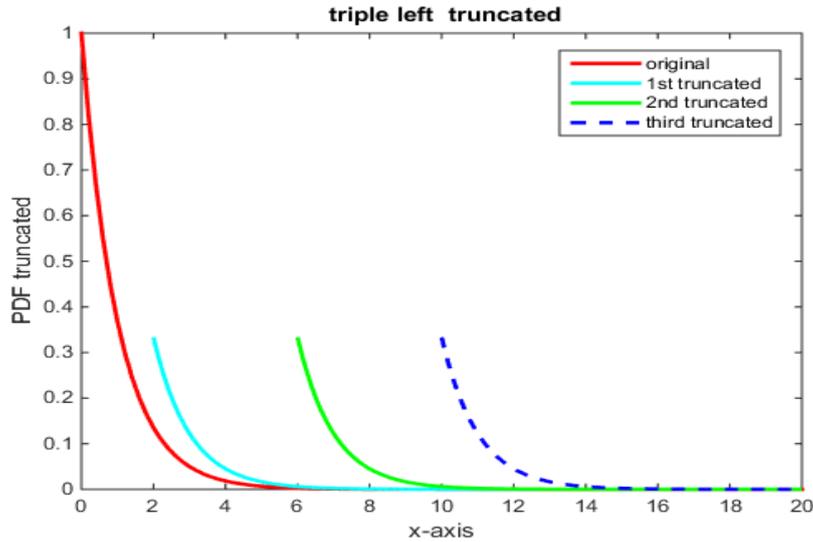


FIGURE 4.21 Three intervals are deleted in LTED when $\lambda = 1$

The probability function of (TLTED) on intervals (2,20),(6,20),(10,20), Three intervals that deleted in LTED are(0,2)(0,6)(0,10) when $\lambda = 1$

4.7.2 Triple Left Truncation of CDF for Exponential Distribution

The cumulative distribution function of three intervals is cut from the left side for exponential distribution is:

$$\mathbb{G}_3(x_T) = \begin{cases} \frac{e^{-\lambda b_1} - e^{-\lambda x_T}}{3(1 - e^{-\lambda b_1})} + \frac{e^{-\lambda b_2} - e^{-\lambda x_T}}{3(1 - e^{-\lambda b_2})} + \frac{e^{-\lambda b_3} - e^{-\lambda x_T}}{3(1 - e^{-\lambda b_3})} & (4.20) \\ b_1 < x_T < \infty, \quad b_2 < x_T < \infty, \quad b_3 < x_T < \infty \end{cases}$$

In figures (4.22),(4.23),(4.24), and (4.25) the curves of the original cumulative distribution and the triple left truncated cumulative distribution function for different values of parameters λ have been drawn. when $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(2,12), (6,12), (10,12)\}$ respectively. That is, the intervals which deleted are $\{(0,2), (0,6), (0,10)\}$, for $\lambda = \{1, 0.6, 0.35, \}$, respectively.

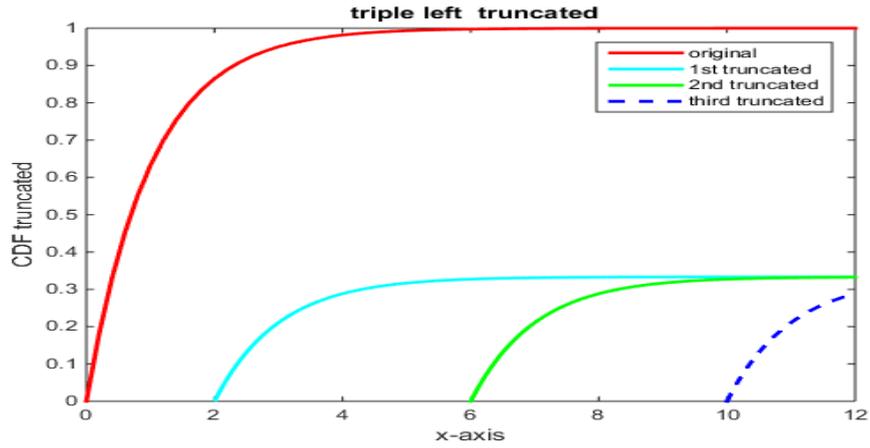


FIGURE 4.22 Three intervals are deleted in LTED when $\lambda = 1$

CDF of (TLTED) has been drawn. when $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(2,12), (6,12), (10,12)\}$ respectively. That is, the intervals which deleted are $\{(0,2), (0,6), (0,10)\}$, for $\lambda = 1$.

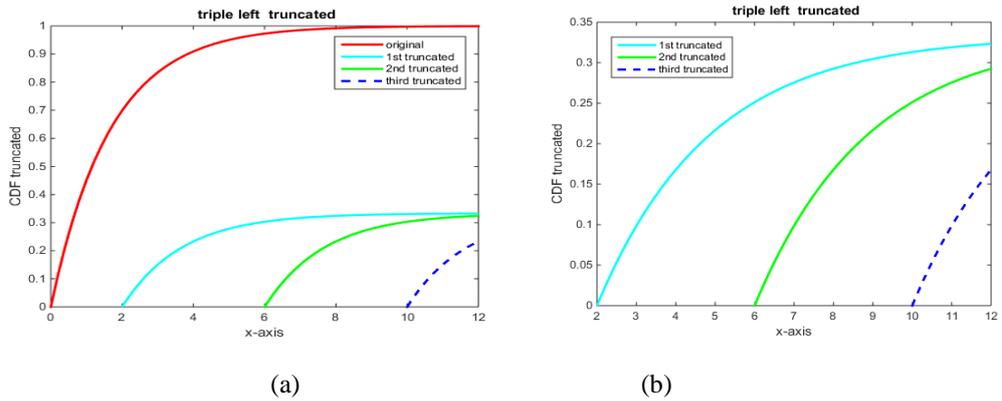


FIGURE 4.23 Three intervals are deleted in LTED when $\lambda = 0.6$. (a) with the original function, (b) without the original function

CDF of (TLTED), when $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(2,12), (6,12), (10,12)\}$. That is, the intervals which deleted are $(0,2), (0,6), (0,10)$.

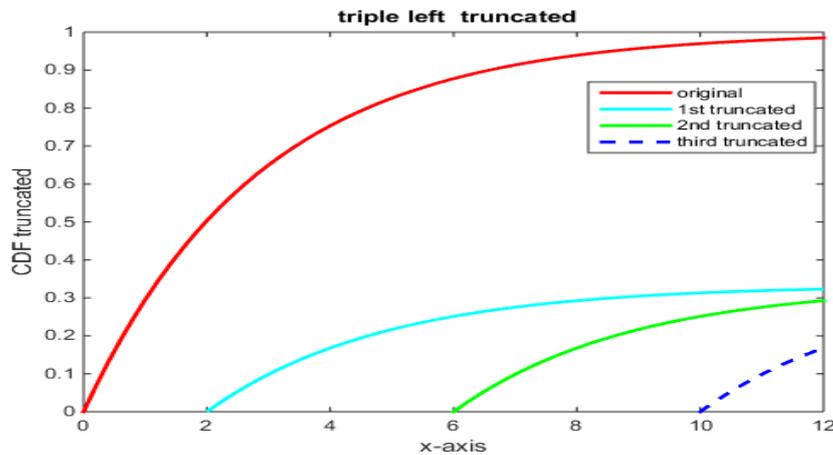


FIGURE 4.24 Three intervals are deleted in LTED when $\lambda = 0.35$

CDF of(TLTED)has been drawn. when $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(2,12),(6,12),(10,12)\}$ respectively. That is, the intervals which deleted are $\{(0,2),(0,6),(0,10)\}$, for $\lambda = 0.35$.

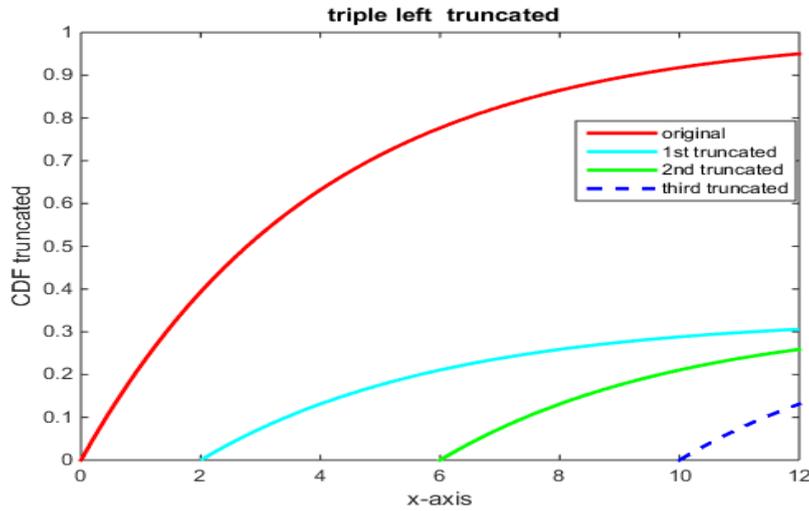


FIGURE 4.25 Three intervals are deleted in LTED when $\lambda = 0.25$

CDF of(TLTED)has been drawn. when $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(2,12),(6,12),(10,12)\}$ respectively. That is, the intervals which deleted are $\{(0,2),(0,6),(0,10)\}$, for $\lambda = 0.25$.

4.7.3 Triple Left Truncation Survival Function Exponential Distribution

The survival function of three intervals left truncated for exponential distribution is:

$$S_3(x_T) = 3 - \frac{e^{-\lambda b_1} - e^{-\lambda x_T}}{3(1 - e^{-\lambda b_1})} - \frac{e^{-\lambda b_2} - e^{-\lambda x_T}}{3(1 - e^{-\lambda b_2})} - \frac{e^{-\lambda b_3} - e^{-\lambda x_T}}{3(1 - e^{-\lambda b_3})} \quad (4.21)$$

$$b_1 < x_T < \infty, \quad b_2 < x_T < \infty, \quad b_3 < x_T < \infty.$$

In this subsection, many shapes of representation of the truncation of survival exponential have been introduced. Triple left truncated for different values for parameter and x_T with the original distribution has been shown in figures(4-26), (4-27). The value of $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(5,30),(18,30),(25,30)\}$ to figure (4-28),(4-29),(4-30), and (4,31) when the value of parameter is $\{0.68,0.25,0.6,1,1.6\}$ respectively.

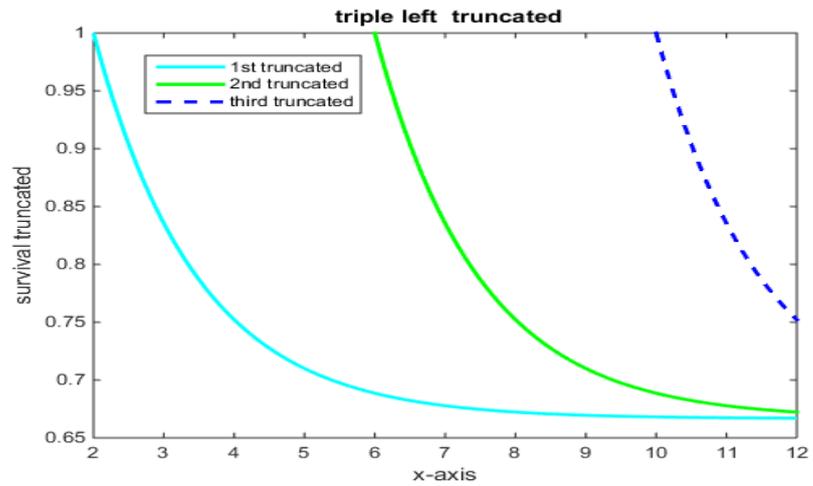


FIGURE 4.26 Three intervals are deleted in LTED when $\lambda = 0.68$

Survival function of(TLTED) has been drawn. $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(5,30), (18,30), (25,30)\}$ for $\lambda = 0.68$.

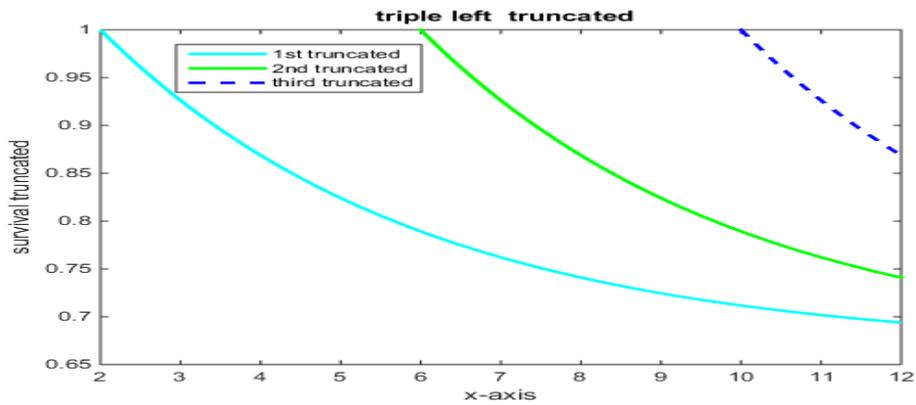


FIGURE 4.27 Three intervals are deleted in LTED when $\lambda = 0.25$

Survival function of(TLTED) has been drawn. $\{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(5,30), (18,30), (25,30)\}$ for $\lambda = 0.25$.

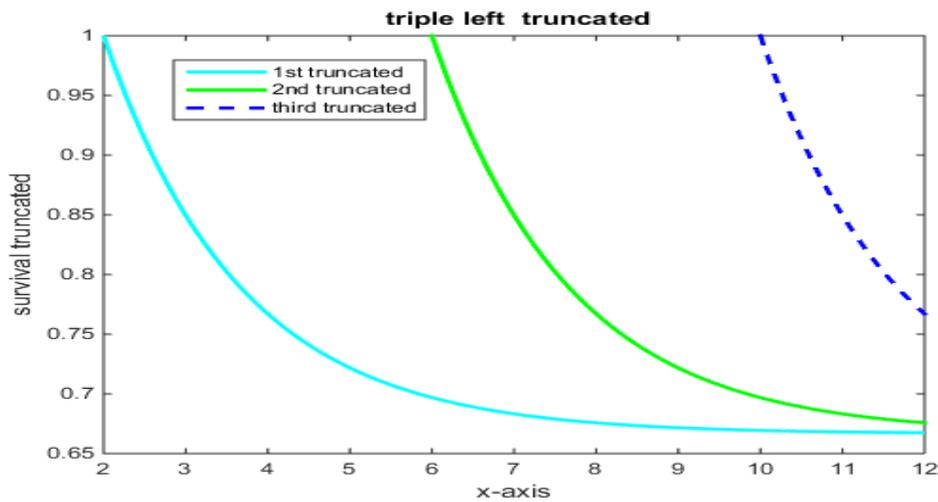


FIGURE 4.28 Three intervals are deleted in LTED when $\lambda = 0.6$

In this figure, we presented the survival function of (LTED) when $\cdot \{(b_1, b_{n+1}), (b_2, b_{n+1}), (b_3, b_{n+1})\} = \{(5,30), (18,30), (25,30)\}$ for $\lambda = 0.6$

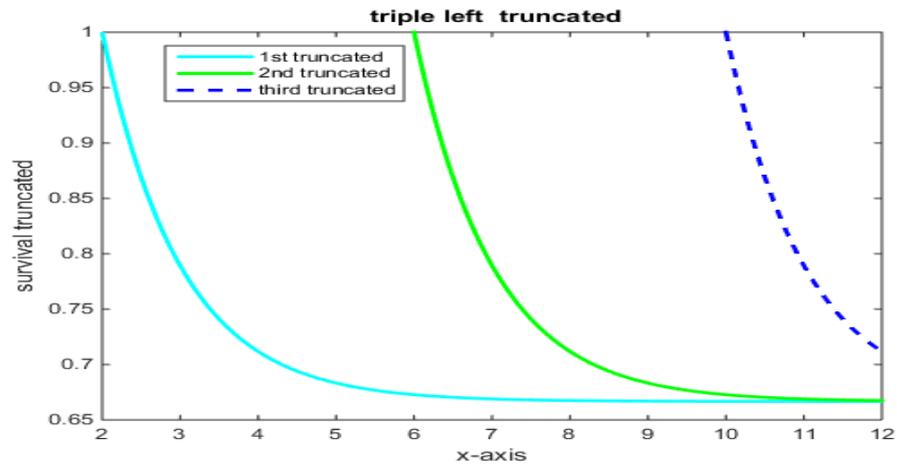


FIGURE 4.29 Three intervals are deleted in LTED when $\lambda = 1$

In this figure, we presented the survival function of (LTED) when $\lambda = 1$

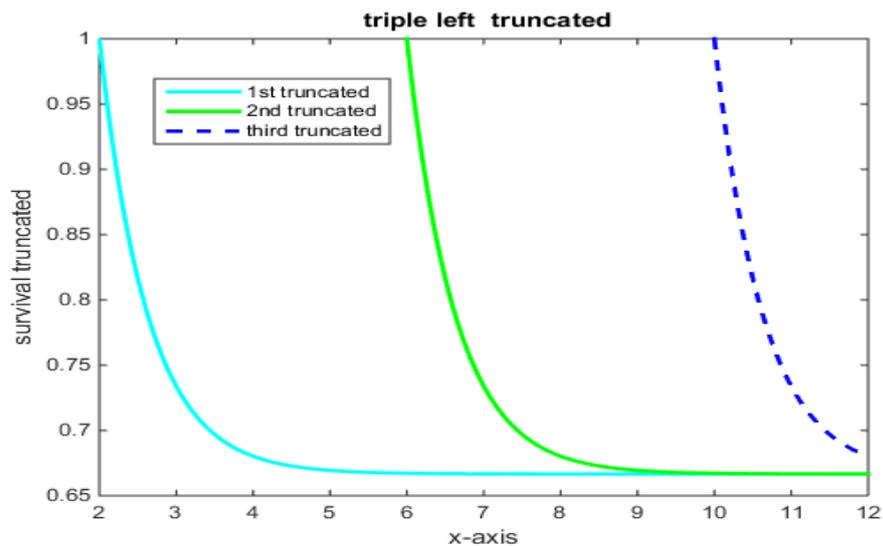


FIGURE 4.30 Three intervals are deleted in LTED when $\lambda = 1.6$

In this figure, we presented the survival function of (LTED) when $\lambda = 1.6$

Theorem 4.2

The r^{th} moment about origin of multi-left truncated exponential distribution is $\sum_{i=1}^n \frac{\frac{1}{\lambda^r}[(r\Gamma(r, \lambda b_i)) + (\lambda b_i)^{r-1} e^{-(\lambda b_i)^2}]}{n(e^{-\lambda b_i})}$, and the r^{th} moment about mean is:

$$\sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(2, \lambda b_i)}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda^j} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right).$$

Proof:-

$$E(X_T^r) = \frac{1}{n} \left(\frac{\lambda \int_{b_1}^{\infty} x_T^r e^{-\lambda x_T} dx_T}{1-F(b_1)} + \frac{\lambda \int_{b_2}^{\infty} x_T^r e^{-\lambda x_T} dx_T}{1-F(b_2)} + \dots + \frac{\lambda \int_{b_n}^{\infty} x_T^r e^{-\lambda x_T} dx_T}{1-F(b_n)} \right)$$

$$\text{Let } I = \lambda \int_{b_1}^{\infty} x_T^r e^{-\lambda x_T} dx_T$$

$$\lambda x_T = w, \quad x_T = w/\lambda, \quad dx_T = dw/\lambda$$

$$\begin{aligned} I &= \frac{1}{\lambda^r} \int_{\lambda b_1}^{\infty} w^r e^{-w} dw = \frac{1}{\lambda^r} \Gamma(r+1, \lambda b_1), \\ &= \frac{1}{n} \left(\frac{\frac{1}{\lambda^r} \Gamma(r+1, \lambda b_1)}{1-F(b_1)} + \frac{\frac{1}{\lambda^r} \Gamma(r+1, \lambda b_2)}{1-F(b_2)} + \dots + \frac{\frac{1}{\lambda^r} \Gamma(r+1, \lambda b_n)}{1-F(b_n)} \right) \\ &= \frac{1}{n} \left(\frac{\frac{1}{\lambda^r} [(r\Gamma(r, \lambda b_1)) + (\lambda b_1)^{r-1} e^{-(\lambda b_1)^2}]}{1-F(b_1)} + \frac{\frac{1}{\lambda^r} [(r\Gamma(r, \lambda b_2)) + (\lambda b_2)^{r-1} e^{-(\lambda b_2)^2}]}{1-F(b_2)} + \right. \\ &\quad \left. \dots + \frac{\frac{1}{\lambda^r} [(r\Gamma(r, \lambda b_n)) + (\lambda b_n)^{r-1} e^{-(\lambda b_n)^2}]}{1-F(b_n)} \right) \end{aligned}$$

$$E(X_T^r) = \sum_{i=1}^n \frac{\frac{1}{\lambda^r} [(r\Gamma(r, \lambda b_i)) + (\lambda b_i)^{r-1} e^{-(\lambda b_i)^2}]}{n(e^{-\lambda b_i})}, \quad (4.22)$$

$$[\text{since, } (\alpha + 1, x) = \alpha \Gamma(\alpha, x) + x^{\alpha-1} e^{-x^2}]$$

And the mean of (MLETD) is:

$$E(X_T) = \sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(2, \lambda b_i)}{n(e^{-\lambda b_i})} = \sum_{i=1}^n \frac{\frac{1}{\lambda} [(\Gamma(1, \lambda b_i)) + (\lambda b_i)^{1-1} e^{-(\lambda b_i)^2}]}{n(e^{-\lambda b_i})} \quad (4.23)$$

And the mean of (triple LETD) is:

$$E(X_T) = \sum_{i=1}^3 \frac{\frac{1}{\lambda} \Gamma(2, \lambda b_i)}{3(e^{-\lambda b_i})} = \sum_{i=1}^3 \frac{\frac{1}{\lambda} [(\Gamma(1, \lambda b_i)) + (\lambda b_i)^{1-1} e^{-(\lambda b_i)^2}]}{3(e^{-\lambda b_i})} \quad (4.24)$$

The r^{th} moment about mean of (MLETD) is:

$$\begin{aligned}
E(X_T - \mu)^r &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} (\mu)^{r-j} \mu'_j \\
&= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(2, \lambda b_i)}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right). \quad (4.25)
\end{aligned}$$

And The r^{th} moment about mean of (TLETD) is:

$$= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} [\Gamma(1, \lambda b_i) + (\lambda b_i)^{r-1} e^{-(\lambda b_i)^2}]}{3(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{3(e^{-\lambda b_i})} \right)$$

According to (4-25), The variance and the standard deviation are respectively:

$$\begin{aligned}
\sigma^2 &= \sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(2, \lambda b_i)}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right) \\
&= \sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} [\Gamma(1, \lambda b_i) + (\lambda b_i)^{r-1} e^{-(\lambda b_i)^2}]}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right) \quad (4.26)
\end{aligned}$$

$$\sigma = \sqrt{\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(2, \lambda b_i)}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right)} \quad (4.27)$$

Therefore the coefficients of variation, kurtosis, and skewness are respectively:

$$\begin{aligned}
CV &= \frac{\left[\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(2, \lambda b_i)}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right) \right]^{1/2}}{\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(2, \lambda b_i)}{n(e^{-\lambda b_i})}} \\
&= \frac{\left[\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} [\Gamma(1, \lambda b_i) + (\lambda b_i)^{r-1} e^{-(\lambda b_i)^2}]}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right) \right]^{1/2}}{\sum_{i=1}^n \frac{\frac{1}{\lambda} [\Gamma(1, \lambda b_i) + (\lambda b_i)^{r-1} e^{-(\lambda b_i)^2}]}{n(e^{-\lambda b_i})}} \quad (4.28)
\end{aligned}$$

$$\begin{aligned}
\text{skew} &= \frac{\sum_{j=0}^4 (-1)^{4-j} \binom{4}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(2, \lambda b_i)}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right)}{\left[\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(2, \lambda b_i)}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right) \right]^2} \\
&= \frac{\sum_{j=0}^4 (-1)^{4-j} \binom{4}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} [\Gamma(1, \lambda b_i) + (\lambda b_i)^{r-1} e^{-(\lambda b_i)^2}]}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right)}{\left[\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} [\Gamma(1, \lambda b_i) + (\lambda b_i)^{r-1} e^{-(\lambda b_i)^2}]}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right) \right]^2} \quad (4.29)
\end{aligned}$$

$$\begin{aligned}
\text{kurt} &= \frac{\sum_{j=0}^3 (-1)^{r-j} \binom{3}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(2, \lambda b_i)}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right)}{\left[\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(2, \lambda b_i)}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right) \right]^{3/2}} \\
&= \frac{\sum_{j=0}^3 (-1)^{r-j} \binom{3}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} [\Gamma(1, \lambda b_i) + (\lambda b_i)^{r-1} e^{-(\lambda b_i)^2}]}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right)}{\left[\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} [\Gamma(1, \lambda b_i) + (\lambda b_i)^{r-1} e^{-(\lambda b_i)^2}]}{n(e^{-\lambda b_i})} \right)^{r-j} \left(\sum_{i=1}^n \frac{\frac{1}{\lambda} \Gamma(j+1, \lambda b_i)}{n(e^{-\lambda b_i})} \right) \right]^{3/2}} \quad (4.30)
\end{aligned}$$

CHAPTER FIVE

***ESTIMATION FOR MULTI-TRUNCATIONS
WITH APPLICATIONS***

In this chapter, we derived the general formula for the maximum likelihood estimation method for multi-double truncated continuous distribution and multi-left truncated continuous distributions. We introduced the estimation of parameters for distributions that have multi-double truncated and multi-left truncated for various distributions by the maximum likelihood method. In this study, we depended on data related to Covid-19 in Babylon city for the period (1-6-2020 to 1-6-2021) through applying one and two intervals for double truncated, and three intervals for left truncated. Also, we implemented an algorithm to solve the maximum likelihood method using Mat Lab (see Appendix B). The study analyzed statistical data for unknown parameters and compared it with some distributions. Furthermore, we tested the fitting of the data to our models. It is worthy to note that data was identical to original distributions before use, see appendix (C).

5.1 The Maximum Likelihood Estimation for Multi-Double Truncated Continuous Distribution

A random variable x_T is said to have the multi-double truncated distribution if its probability density function as in equation (2.1):

$$g_n(x_T) = \frac{1}{n} \sum_{i=1}^n \frac{f(x_T)}{F(b_i) - F(a_i)} \quad \text{when } a_i < x_T < b_i$$

Based on this assumption, we will construct the maximum likelihood function for multi-double truncated continuous distribution in the next theorem:

Theorem 5.1

Consider X_1, X_2, \dots, X_m is a random sample of size m from a population with a probability density function $g_n(x_T; \underline{\theta})$, where $\underline{\theta}$ are unknown parameters, then the likelihood function

$$L(\theta_j, x_T) = \frac{\prod_{j=1}^m f(x_{Tj})}{n^m F_1^{r_1} F_2^{r_2} \dots F_n^{m - \sum_{i=1}^{n-1} r_i}}.$$

Proof: Likelihood function of $\mathfrak{G}_n(x_T; \underline{\theta})$ is

$$\begin{aligned} L(\theta_j, x_T) &= \prod_{j=1}^m \mathfrak{G}_n(x_{Tj}; \underline{\theta}) \\ &= \prod_{j=1}^{r_1} \frac{1}{n} \left(\frac{f(x_T)}{F(b_1) - F(a_1)} \right) \prod_{j=r_1+1}^{r_2} \frac{1}{n} \left(\frac{f(x_T)}{F(b_2) - F(a_2)} \right) \cdots \prod_{j=r_{n-1}+1}^{r_n} \frac{1}{n} \left(\frac{f(x_T)}{F(b_n) - F(a_n)} \right) \\ L(\theta_j, x_T) &= \frac{\prod_{j=1}^m f(x_{Tj})}{n^m F_1^{r_1} F_2^{r_2} \cdots \cdots F_n^{m - \sum_{i=1}^{n-1} r_i}} \end{aligned}$$

Take the log to the above equation to both sides, where $r_n = m$, $F_1 = F(b_1) - F(a_1)$, $F_2 = F(b_2) - F(a_2)$, ..., $F_n = F(b_n) - F(a_n)$

The log-likelihood function is:

$$\begin{aligned} \ell = & \\ & -m \log(m) - r_1 \log(F_1) - (r_2) \log(F_1) - \cdots \cdots - (m - \sum_{i=1}^{n-1} r_i) \log(F_n) + \\ & \sum_{j=1}^m \log(f(x_{Tj})). \end{aligned} \quad (5.1)$$

5.2 Information Criterion and Curve Fitting

To select a convenient model among a set of models that describe the phenomena under study since it is an essential attention of modern sciences. Many statistical models were proposed by researchers dealing with these phenomena such as information criteria and goodness of fit. The goal of calculating such information criteria is to find out how good a model is for explaining the relationship between the variables and identifying the “best model” among a set of specified models.

The curve of fitting is to construct a curve that has the best fit to a series of original data points possibly to constraints with evaluation statistical quantities that measure the goodness of fit.

In this study, we use four different types of information criteria: the Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), Hannan-Quinn information criterion (HQIC) and Bayesian Information Criterion (BIC). Also, we test curve fitting to decide which model exhibited a better curve fit for our models: ODTWD, ODTRD, TDTWD, TLTRD, and TLTED with other models, we examine four quantities:

1. The sum of squares due to error (SSE),
 2. R-squared (RSS),
 3. Adjusted R-squared,
 4. Root mean squared error (RMSE).
- all these are explained in chapter one.

Those quantities are evaluated by curve fitting application offered by the Mat Lab environment. The curve fit application uses a numerical smoothing spline method to perform the fitting and the goodness of fit statistics.

5.3 Real Data Analysis of ODTRD and ODTWD

In this section, we consider real data of the infected with Covid-19 in the city of Babylon-Iraq. The data collection focused on the time of their infection in the virus until their death for the period (1/10/2020-30/12/2020). We illustrated the applicability of ODTRD and ODTWD for this data.

The data set are:

11,2,22,16,17,6,12,7,29,11,7,10,15,8,1,21,8,1,4,17,1,10,21,16,12,1,2,7,4,11,
 ,5,5,8,4,15,1,23,16,8,10,10,11,27,9,18,1,4,1,6,3,26,4,1,3,11,2,7,12,3,32,6,1
 2,8,1,5,10,3,66,51,20,8,15,18,22,4,16,19,8,21,19,12,8,3,9,17,7,12,12,44,11,
 23,12,10,8,11,27,10,4,5,6,11,1,28,22,6,7,31,6,16,2,12,21,19,2,19,8,18,39.

With size is 118.

5.4 Estimation of One-Double Truncated Rayleigh Distribution

5.4.1 Maximum Likelihood Estimates of ODTRD

If X_1, X_2, \dots, X_m denotes a random sample from ODTRD, then the likelihood function is given by:

$$\begin{aligned}
 L &= \prod_{j=1}^m g(x_j; \sigma^2) = \prod_{j=1}^m \left(\frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{2 \left(e^{\frac{-b_0^2}{2\sigma^2}} - e^{\frac{-a_1^2}{2\sigma^2}} \right)} + \frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{2 \left(e^{\frac{-b_1^2}{2\sigma^2}} - e^{\frac{-a_2^2}{2\sigma^2}} \right)} \right) \\
 &= \prod_{j=1}^r \left(\frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{2 \left(e^{\frac{-b_0^2}{2\sigma^2}} - e^{\frac{-a_1^2}{2\sigma^2}} \right)} \right) \prod_{j=r+1}^m \left(\frac{\frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{2 \left(e^{\frac{-b_1^2}{2\sigma^2}} - e^{\frac{-a_2^2}{2\sigma^2}} \right)} \right) \\
 &= \left(\frac{\prod_{j=1}^m \frac{x_T}{\sigma^2} e^{-\frac{x_T^2}{2\sigma^2}}}{2^m \left(e^{\frac{-b_0^2}{2\sigma^2}} - e^{\frac{-a_1^2}{2\sigma^2}} \right)^r \left(e^{\frac{-b_1^2}{2\sigma^2}} - e^{\frac{-a_2^2}{2\sigma^2}} \right)^{m-r}} \right)
 \end{aligned}$$

The log-likelihood function is:

$$\begin{aligned}
 \ell &= \\
 \sum_{j=1}^m \log x_{Tj} - \frac{\sum_{j=1}^m x_{Tj}^2}{2\sigma^2} - m \log(\sigma^2) - m \log(2) - r \left[\log \left(\exp \left(\frac{-b_0^2}{2\sigma^2} \right) - \exp \left(\frac{-a_1^2}{2\sigma^2} \right) \right) \right] - (m-r) \left[\log \left(\exp \left(\frac{-b_1^2}{2\sigma^2} \right) - \exp \left(\frac{-a_2^2}{2\sigma^2} \right) \right) \right] \quad (5.2)
 \end{aligned}$$

By taking the derivatives of ℓ with respect to the parameter σ^2 and the result equals zero,

$$\begin{aligned}
 \frac{d\ell}{d\sigma^2} &= \frac{\sum_{j=1}^m x_{Tj}^2}{2(\sigma^2)^2} - \frac{r \left[\left(\frac{b_0^2}{2(\sigma^2)^2} \right) \exp \left(\frac{-b_0^2}{2\sigma^2} \right) - \left(\frac{a_1^2}{2(\sigma^2)^2} \right) \exp \left(\frac{-a_1^2}{2\sigma^2} \right) \right]}{\exp \left(\frac{-a_1^2}{2\sigma^2} \right) - \exp \left(\frac{-b_0^2}{2\sigma^2} \right)} - \frac{m}{\sigma^2} \\
 &\quad - \frac{(m-r) \left[\left(\frac{b_1^2}{2(\sigma^2)^2} \right) \exp \left(\frac{-b_1^2}{2\sigma^2} \right) - \left(\frac{a_2^2}{2(\sigma^2)^2} \right) \exp \left(\frac{-a_2^2}{2\sigma^2} \right) \right]}{\exp \left(\frac{-a_2^2}{2\sigma^2} \right) - \exp \left(\frac{-b_1^2}{2\sigma^2} \right)} = 0 \quad (5.3)
 \end{aligned}$$

We can obtain the MLE of the parameter σ^2 , by solving the above equation numerically for σ^2 by Newton–Raphson method.

5.4.2 Goodness of ODTRD

In this subsection, we illustrate the applicability of ODTRD distribution by considering real data of the infected with COVID-19 in the city of Babylon-Iraq for the period (1/10/2020-30/12/2020). Fitting by comparison distributions (see appendix (A)) with Maxwell distribution(MAWE) and Chi-Square(CH-SQ) Distribution.

Each parameter of the comparison distributions is estimated by using the maximum likelihood method, and computing the criteria AIC, BIC, CAIC, and HQIC.

The maximal likelihood estimation of all parameters of models for the above data is shown in table (5-1). Table (5-2) shows that the model ODTRD gives the least values for the criteria AIC, BIC, CAIC, and HQIC than the other models (MAWE, CH-SQ), where $r=37$.

Hypotheses:

$H_0 =$ The model ODTRD is fitted the data

$H_1 =$ The model ODTRD is not fitted the data

Table(5.1) Estimations parameters of data for ODTRD

Model	Parameter	Parameter estimates
$ODTR(\sigma^2)$	σ^2^{\wedge}	4.753
$MAWE(\theta)$	θ^{\wedge}	8.996
$CH-SQ(\alpha)$	α^{\wedge}	1.935

The above table shows the values of the estimator parameter of (ODTRD) and all the distributions compared to it, using the maximum likelihood method.

Table (5.2) the values of statistics ℓ^\wedge , AIC, BIC, CAIC, HQIC

Model	ℓ^\wedge	AIC	BIC	CAIC	HQIC
ODTR	-533.079	3.066158	3.138.038	1.100158	1.698858
MAWE	-1.631.689	5.263378	5.335258	3.297378	3.896078
CH-SQ	-794.687	3.589374	3.661254	1.623374	2.222207

In the table above, the numerical values of statistics for all models are given and show the model ODTRD is a better fit of data than other models.

5.5 Estimation of One-Double Truncated Weibull Distribution

5.5.1 Maximum Likelihood Estimates of ODTWD

If X_1, X_2, \dots, X_m denotes a random sample from ODTWD, then the likelihood function is given by:

$$\begin{aligned}
 L &= \prod_{j=1}^m g(x_{Tj}; \delta, \omega), \\
 &= \prod_{j=1}^r \left(\frac{\frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta}}{2 \left(e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta} \right)} \right) \prod_{j=r+1}^n \left(\frac{\frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta}}{2 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta} \right)} \right) \\
 &= \frac{\prod_{j=1}^m \left(\frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta} \right)}{2^m \left(e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta} \right)^r \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta} \right)^{m-r}}
 \end{aligned}$$

The log-likelihood function is:

$$\begin{aligned} \ell = & m \log(\delta) - m\delta \log(\omega) + (\delta - 1) \sum_{j=1}^m \log x_{T_j} - \frac{\sum_{j=1}^m x_{T_j}^\delta}{\omega^\delta} - \\ & m \log(2) - r \left[\log \left(\exp \left(-\left(\frac{a_1}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_1}{\omega}\right)^\delta \right) \right) \right] - (m - \\ & r) \left[\log \left(\exp \left(-\left(\frac{a_2}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_2}{\omega}\right)^\delta \right) \right) \right] \end{aligned} \quad (5.4)$$

Taking the derivatives of ℓ with respect to the parameter δ, ω and the result equal zero, respectively.

$$\begin{aligned} \frac{\partial \ell}{\partial \delta} = & \frac{m}{\delta} + \\ & r \frac{\left[(-1) \left(\frac{a_1}{\omega}\right)^\delta \exp \left(-\left(\frac{a_1}{\omega}\right)^\delta \right) \left(\log \left(\frac{a_1}{\omega} \right) \right) - \left(\frac{b_1}{\omega}\right)^\delta \exp \left(-\left(\frac{b_1}{\omega}\right)^\delta \right) \left(\log \left(\frac{b_1}{\omega} \right) \right) \right]}{\exp \left(-\left(\frac{a_1}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_1}{\omega}\right)^\delta \right)} \\ & - m \log(\omega) - \frac{\sum_{j=1}^m x_{T_j}^\delta \log(x_{T_j}) - \log(\omega) \sum_{j=1}^m x_{T_j}^\delta}{\omega^\delta} + (m - r) \\ & \times \frac{\left[(-1) \left(\frac{a_2}{\omega}\right)^\delta \exp \left(-\left(\frac{a_2}{\omega}\right)^\delta \right) \left(\log \left(\frac{a_2}{\omega} \right) \right) - \left(\frac{b_2}{\omega}\right)^\delta \exp \left(-\left(\frac{b_2}{\omega}\right)^\delta \right) \left(\log \left(\frac{b_2}{\omega} \right) \right) \right]}{\exp \left(-\left(\frac{a_2}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_2}{\omega}\right)^\delta \right)} \\ \frac{\partial \ell}{\partial \omega} = & \frac{-m\delta}{\omega} + \frac{\delta \sum_{j=1}^m x_{T_j}^\delta}{\omega^{\delta+1}} - \frac{r \left[\frac{\delta(a_1)^\delta}{\omega^{\delta+1}} \exp \left(-\left(\frac{a_1}{\omega}\right)^\delta \right) - \frac{\delta(b_1)^\delta}{\omega^{\delta+1}} \exp \left(-\left(\frac{b_1}{\omega}\right)^\delta \right) \right]}{\exp \left(-\left(\frac{a_1}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_1}{\omega}\right)^\delta \right)} - \\ & (m-r) \frac{\left[\left(\frac{a_2}{\omega}\right)^\delta \exp \left(-\left(\frac{a_2}{\omega}\right)^\delta \right) \frac{\delta(a_2)^\delta}{\omega^{\delta+1}} - \left(\frac{b_2}{\omega}\right)^\delta \exp \left(-\left(\frac{b_2}{\omega}\right)^\delta \right) \frac{\delta(b_2)^\delta}{\omega^{\delta+1}} \right]}{\exp \left(-\left(\frac{a_2}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_2}{\omega}\right)^\delta \right)} \\ \frac{\partial \ell}{\partial \omega} = & \frac{-m\delta}{\omega} + \frac{\delta \sum_{j=1}^m x_{T_j}^\delta}{\omega^{\delta+1}} - \frac{r \frac{\delta}{\omega^{\delta+1}} \left[(a_1)^\delta \exp \left(-\left(\frac{a_1}{\omega}\right)^\delta \right) - (b_1)^\delta \exp \left(-\left(\frac{b_1}{\omega}\right)^\delta \right) \right]}{\exp \left(-\left(\frac{a_1}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_1}{\omega}\right)^\delta \right)} \\ & - \frac{(m-r) \frac{\delta}{\omega^{\delta+1}} \left[\left(\frac{a_2}{\omega}\right)^\delta \exp \left(-\left(\frac{a_2}{\omega}\right)^\delta \right) (a_2)^\delta - \left(\frac{b_2}{\omega}\right)^\delta \exp \left(-\left(\frac{b_2}{\omega}\right)^\delta \right) (b_2)^\delta \right]}{\exp \left(-\left(\frac{a_2}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_2}{\omega}\right)^\delta \right)} \end{aligned} \quad (5.5)$$

We can obtain the MLE of the parameters δ, ω , by solving the above equation numerically for δ, ω .

5.5.2 Goodness of ODTWD

In this subsection, we used the real data of the infected with COVID-19 in the city of Babylon-Iraq for the period (1/10/2020-30/12/2020). Compared the results of fitting the importance of the ODTWD with Gamma Rayleigh distribution(GARA), Modified Weibull distribution (MOWE), Rayleigh with two parameters (RATO), and Weibull distribution (WEBU). The maximal likelihood estimation of a model (ODTW) and (WEBU),(GARA),(MOWE), and(RATW) parameters for the above data is shown in table (5.3), where $r = 76$. By using the above criteria, we can note the distribution which has the least criteria is the better fit for the data, as shown in table (5.4).

Hypotheses:

$H_0 =$ The model ODTWD is fitted the data

$H_1 =$ The model ODTWD is not fitted the data

Table(5.3) Estimations parameters of data for ODTWD

Model	Parameters Estimates		
ODTW(δ, ω)	$\hat{\delta} = 9.973$	$\hat{\omega} = 0.85$	-
WEBU(ω, γ)	$\hat{\omega} = 0.94$	$\hat{\gamma} = 8.939$	-
GARA(α, θ, ω)	$\hat{\alpha} = 9.974$	$\hat{\theta} = 1.715$	$\hat{\omega} = 0.791$
MOWE(α, ω, γ)	$\hat{\alpha} = 0.94$	$\hat{\omega} = 1.569$	$\hat{\gamma} = 0.01$
RATW(ω, γ)	$\hat{\omega} = 0.94$	$\hat{\gamma} = 0.988$	-

In this table, we estimated the parameter of the model(ODTWD) , and comparision models

Table (5.4) the values of statistics for (ODTW), (WEBU), (GARA),(MOWE) and(RATW)

Model	$\hat{\ell}$	AIC	BIC	CAIC	HQIC
ODTW	-146.27	296.54	296,6836	296.5747	293.8054
WEBU	-383.786	771.572	771,7156	771.6067	768.8374
GARA	-430.168	864.336	864,4796	864.3707	861.6014
MOWE	-1,688,601	9,377,202	9,592,807	9.587.702	5.275397
RATW	-14,094,60	32,189,212	32,332,812	32.223.912	29.454612

This table shows the statistics $\hat{\ell}$, AIC, BIC, CAIC, and HQIC for (ODTW),(WEBU),(GARA), (MOWE), and (RATW), and the model ODTW is a better fit of data than other models.

Table(5.5) Estimations parameters and $\hat{\ell}$ of data for ODTWD

b_0, a_1, b_1, a_2	statistics	Parameter estimation	
	$\hat{\ell}$	$\hat{\delta}$	$\hat{\omega}$
1,23,26,66	-370.228	0.97	1.997
1,17,20,66	-426.625	0.971	2.00
1,12,15,66	-480.592	0.972	1.994
1,11,14,66	-485.085	0.974	1.991

The estimation parameters of (ODTW), and maximum likelihood estimate for data (1/10/2020-30/12/2020) are shown in table (5.5), for different values of b_0, a_1, b_1, a_2 , where $r=97$.

some of the cutting intervals are found in the original data. We noted that the ODTW has decreased when the values of a_1, b_1 decreased.

5.6 Estimation of Two Double Truncated Weibull distribution

In this section, we derived the maximum likelihood function of two double truncated Weibull distributions and obtained the Goodness fit of data.

5.6.1 Maximum Likelihood Estimates of TDTWD of

The maximum likelihood function of two double truncated Weibull distribution, we can get by using theorem (5-1) as follow:

$$L(\mathbb{g}_3(x_T)) = \prod_{j=1}^{r_1} \left(\frac{\frac{\delta}{\omega^\delta} x_T^{\delta-1} e^{-\left(\frac{x_T}{\omega}\right)^\delta}}{3 \left(e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta} \right)} \right)$$

$$\begin{aligned}
& \prod_{j=r_1+1}^{r_2} \left(\frac{\frac{\delta}{\omega^\delta} x_{T_j}^{\delta-1} e^{-\left(\frac{x_{T_j}}{\omega}\right)^\delta}}{3 \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta} \right)} \right) \prod_{j=r_2+1}^m \left(\frac{\frac{\delta}{\omega^\delta} x_{T_j}^{\delta-1} e^{-\left(\frac{x_{T_j}}{\omega}\right)^\delta}}{3 \left(e^{-\left(\frac{a_3}{\omega}\right)^\delta} - e^{-\left(\frac{b_3}{\omega}\right)^\delta} \right)} \right) \\
&= \frac{\prod_{j=1}^m \left(\frac{\delta}{\omega^\delta} x_{T_j}^{\delta-1} e^{-\left(\frac{x_{T_j}}{\omega}\right)^\delta} \right)}{3^m \left(e^{-\left(\frac{a_1}{\omega}\right)^\delta} - e^{-\left(\frac{b_1}{\omega}\right)^\delta} \right)^{r_1} \left(e^{-\left(\frac{a_2}{\omega}\right)^\delta} - e^{-\left(\frac{b_2}{\omega}\right)^\delta} \right)^{r_2} \left(e^{-\left(\frac{a_3}{\omega}\right)^\delta} - e^{-\left(\frac{b_3}{\omega}\right)^\delta} \right)^{m-r_2-r_1}} \quad (5.6)
\end{aligned}$$

The log-likelihood function of (TDTWD) is:

$$\ell = \log(1) - m \log(3)$$

$$\begin{aligned}
& -r_1 \left[\log \left\{ \exp \left(-\left(\frac{a_1}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_1}{\omega}\right)^\delta \right) \right\} \right] - (r_2) \left[\log \left\{ \exp \left(-\left(\frac{a_2}{\omega}\right)^\delta \right) - \right. \right. \\
& \left. \left. \exp \left(-\left(\frac{b_2}{\omega}\right)^\delta \right) \right\} \right] - (m - r_2 - r_1) \left[\log \left\{ \exp \left(-\left(\frac{a_3}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_3}{\omega}\right)^\delta \right) \right\} \right] \\
& + \left[m \log(\delta) - m\delta \log \omega + (\delta - 1) \sum_{j=1}^m \log x_{T_j} - \frac{\sum_{i=1}^n x_{T_j}^\delta}{\omega^\delta} \right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \omega} &= \frac{-r_1 \left[\frac{-\delta(a_1)^\delta}{(\omega)^{\delta+1}} \exp \left(-\left(\frac{a_1}{\omega}\right)^\delta \right) + \frac{\delta(b_1)^\delta}{(\omega)^{\delta+1}} \exp \left(-\left(\frac{b_1}{\omega}\right)^\delta \right) \right]}{\exp \left(-\left(\frac{a_1}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_1}{\omega}\right)^\delta \right)} \\
& - \frac{(r_2) \left[\frac{-\delta(a_2)^\delta}{(\omega)^{\delta+1}} \exp \left(-\left(\frac{a_2}{\omega}\right)^\delta \right) + \frac{\delta(b_2)^\delta}{(\omega)^{\delta+1}} \exp \left(-\left(\frac{b_2}{\omega}\right)^\delta \right) \right]}{\exp \left(-\left(\frac{a_2}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_2}{\omega}\right)^\delta \right)} \\
& - \frac{(n-r_2-r_1) \left[\frac{-\delta(a_3)^\delta}{(\omega)^{\delta+1}} \exp \left(-\left(\frac{a_3}{\omega}\right)^\delta \right) + \frac{\delta(b_3)^\delta}{(\omega)^{\delta+1}} \exp \left(-\left(\frac{b_3}{\omega}\right)^\delta \right) \right]}{\exp \left(-\left(\frac{a_3}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_3}{\omega}\right)^\delta \right)} - \frac{m\delta}{\omega} - \frac{\delta \sum_{j=1}^m x_j^\delta}{\omega^{\delta+1}} = 0
\end{aligned}$$

$$\frac{\partial \ell}{\partial \delta} = \frac{-r_1 \left[\left(\frac{a_1}{\omega}\right)^\delta \exp \left(-\left(\frac{a_1}{\omega}\right)^\delta \right) \log \left(\frac{a_1}{\omega}\right) + \left(\frac{b_1}{\omega}\right)^\delta \log \left(\frac{b_1}{\omega}\right) \exp \left(-\left(\frac{b_1}{\omega}\right)^\delta \right) \right]}{\exp \left(-\left(\frac{a_1}{\omega}\right)^\delta \right) - \exp \left(-\left(\frac{b_1}{\omega}\right)^\delta \right)}$$

$$\begin{aligned}
& - \frac{(r_2) \left[\left(\frac{a_2}{\omega} \right)^\delta \exp\left(-\left(\frac{a_2}{\omega}\right)^\delta\right) \log\left(\frac{a_2}{\omega}\right) + \left(\frac{b_2}{\omega}\right)^\delta \log\left(\frac{b_2}{\omega}\right) \exp\left(-\left(\frac{b_2}{\omega}\right)^\delta\right) \right]}{\exp\left(-\left(\frac{a_2}{\omega}\right)^\delta\right) - \exp\left(-\left(\frac{b_2}{\omega}\right)^\delta\right)} \\
& - \frac{(m-r_2-r_1) \left[\left(\frac{a_3}{\omega} \right)^\delta \exp\left(-\left(\frac{a_3}{\omega}\right)^\delta\right) \log\left(\frac{a_3}{\omega}\right) + \left(\frac{b_3}{\omega}\right)^\delta \log\left(\frac{b_3}{\omega}\right) \exp\left(-\left(\frac{b_3}{\omega}\right)^\delta\right) \right]}{\exp\left(-\left(\frac{a_3}{\omega}\right)^\delta\right) - \exp\left(-\left(\frac{b_3}{\omega}\right)^\delta\right)} \\
& + \frac{m}{\delta} - m \log \omega + \sum_{j=1}^m \log x_j - \left[\frac{\sum_{j=1}^m x_{Tj}^\delta \log(\omega)}{(\omega)^\delta} + \frac{\sum_{j=1}^m x_{Tj}^\delta \log(x_{Tj})}{(\omega)^\delta} \right] = 0
\end{aligned}$$

5.5.2 Goodness of TDTWD

In this subsection, we used the real data of the infected with COVID-19 in the city of Babylon-Iraq for three sets. Compared the results of fitting the importance of the TDTWD with Weibull (WEBU), Gamma Rayleigh distribution (GARA), Modified Weibull distribution (MOWE), and generalized Rayleigh (GERA).

Hypotheses:

$H_0 =$ The model TDTWD is fitted the data

$H_1 =$ The model TDTWD is not fitted the data

Data set (1)

The following data set are the infected of Covid-19 collected from Babylon city for months (11,12/2020 and 1,2/2021).

$t=[5,10,14,3,66,51,20,8,15,18,22,16,19,8,21,19,12,8,3,9,17,7,12,44,11,23,12,10,8,11,27,10,4,5,6,11,28,22,6,7,31,6,16,2,12,21,19,2,19,8,18,39,22,25,3,11,7,4,14,5,7,20,6,13,14,9,5,13,41,29,3,4,12,17,3,16,6,19,24,4,8,11,13,4,6,9,12,6,31,20]$, with size 85. The intervals cut are: assumption [14,15], [24,27]

Data set (2)

The following data set are the infected of Covid-19 collected from Babylon city for time (1-3 to 30 -5/2021).

$t=[16,36,10,4,24,23,10,15,12,30,6,12,5,7,11,7,12,5,10,4,7,3,2,18,7,1,6,12,23,5,11,1,33,9,4,17,8,5,16,40,6,10,16,8,17,10,9,16,10,4,7,5,2,14,7,15,9]$

The intervals cut are: assumption [10,11], actual [19,22]

Data set (3)

The following data set are the infected of Covid-19 collected from Babylon city for time (1-1 to 28 -2/2021).

$t=[22,25,3,11,7,4,14,5,7,20,6,13,14,9,5,13,41,29,3,4,12,17,3,6,19,4,8,11,13,4,6,9,12, 6,31,20]$ with size 36. The intervals cut are: actual [15,16], [23,24]

Table(5.6) the values of statistics for (TDTW),(WEBU),(MOWE) and(GAWE) with data in set(1)

Model	$\hat{\ell}$	AIC	BIC	CAIC	Estimation parameter
TDTW	-305.2289	614.457767	619.343069	614.652889	$\hat{\delta} = 21.3623$ $\hat{\omega} = 2.0163$
WEBU	8.4335e-163	750.378326	755.263628	750.5734484	$\hat{\omega} = 15.6751$ $\hat{\gamma} = 1.3838$
MOWE	- 5.9525e+09	1190502964 3.546	119050296 50.8748	11905029643. 8432	$\hat{\alpha} = 0.0653$ $\hat{\omega} = 9.2391$ $\hat{\gamma} = 4.6975$
GEWE	-2.8333e+03	5670.546913	5675.43221	5670.74203	$\hat{\alpha} = 126.806$ $\hat{\lambda} = 0.2492$

In this table, we estimated the parameter of the model(TDTWD) , and for comparison models, and find maximum likelihood function for all models with data in set (1). We note, the model (TDTW) has smaller value (AIC,CAIC, BIC) than other models. This inducte the model (TDTWD) is btter than other models.

Table(5.7) the values of statistics for (TDTW),(WEBU),(MOWE) and(GAWE) for data in set (2)

Model	ℓ^{\wedge}	AIC	BIC	CAIC	Estimation parameter
TDTW	-157.2929	318.58578	322.36942 063	318.933606 120	$\delta^{\wedge}=17.7657$ $\omega^{\wedge}=1.7816$
WEBU	1.7729e-83	385.083917	388.86755 855	385.431744 04	$\delta^{\wedge}=12.5236$ $\gamma^{\wedge}=1.3447$
MOW E	- 1.2461e+08	249224999.88	249225005 .55	249225000. 41	$\alpha^{\wedge}=0.0614$ $\omega^{\wedge}=45.328$ $\gamma^{\wedge}=3.6942$
GEWE	-200.8857	405.77144629	409.55508 689	406.119272 3	$\alpha^{\wedge}=0.5540$ $\lambda^{\wedge}=0.0554$

In this table, we estimated the parameter of the model(TDTWD) , and comparison models, and find maximum likelihood function for all models with data in set (2). We note, the model (TDTW) has smaller value (AIC,CAIC, BIC) than other models. This inducte the model (TDTWD) is btter than other models.

Table(5.8) the values of statistics for (TDTW),(WEBU),(MOWE) and

Model	ℓ^{\wedge}	AIC	BIC	CAIC	Estimation parameter
TDTW	-130.14470	264.294033	267.46107	264.77881	$\delta^{\wedge}=20.605$ $\omega^{\wedge}=2.3102$
WEBU	1.4753e-73	339.3997	342.5667	339.8846	$\omega^{\wedge}=13.507$ $\gamma^{\wedge}=1.486$
MOWE	-6.5053e+11	1301054507 668.55	130105450 7673.3	13010545076 69.3	$\omega^{\wedge}=0.084$ $\alpha^{\wedge}=10.087$ $\gamma^{\wedge}=6.623$
GERA	-956.6729	1917.345836	1920.51287 4	1917.830685	$\alpha^{\wedge}=132.72$ $\lambda^{\wedge}=0.2905$

(GERA) for data in set(3)

In this table, we estimated the parameter of the model(TDTWD) , and comparison models, and find maximum likelihood function for all models with data in set (3). We note, the model (TDTW) has smaller value (AIC,CAIC, BIC) than other models. This inducte the model (TDTWD) is btter than other models

Interpretation:-

In all sets, the maximum likelihood estimation of TDTWD and related to the compared models (WEBU),(GARA),(MOWE), and(GEWE), as introduced in tables (5-.6),(5.7), and (5.8). Therefore we presented the estimated values of parameters for all models for data in set (1), set (2), and set (3), respectively. The model TDTWD gives the least values for the criteria AIC, BIC, CAIC, and HQIC than the other models. We found in set (3), the M.L.E of TDTWD is less than set (2) and set (1). Therefore, the values of the criteria of TDTWD are less value as well than the other two.

5.7 Results of Curve Fitting of TDTWD

This section contains results of curve fitting statistics (SSE, R-squared, Adjusted R-squared, and RMSE) to compare the performances of TDTWD to fit data among three sets of given data listed in the next table:

Data	SSE	R-squared	Adj-R squared	RMSE	
TDTWD	Set 1	6.4272e-6	0.999953333	0.999955667	0.13978767
	set 2	0.354333633	0.999966667	0.99992667	0.0982151667
	set 3	1.32406667e-7	0.999966667	0.999866667	8.14433333e-5

Table(5.9) statistics to the goodness of fit (TDTWD)

In this table, we provide the measures of curving goodness of fit (TDTWD) (SSE, R^2 , Adj- R^2 , RMSE)

For data in set (3): TDTW distribution with (SSE= 1.3240667e-7 RMSE= 8.1443333e-5) which is the smallest compared to data in set (1) and data in set (2) fitted the data better than other sets and this result compatible with results of table (5.6),(5.7),(5.8). And SSE with RMSE values closer to 0 indicates a fit that is more useful for fitted the data.

in figures(5.1), (5.2), and (5.3) we presented, the curve fitting for PDF TDTWD after the estimate of its parameter for all sets

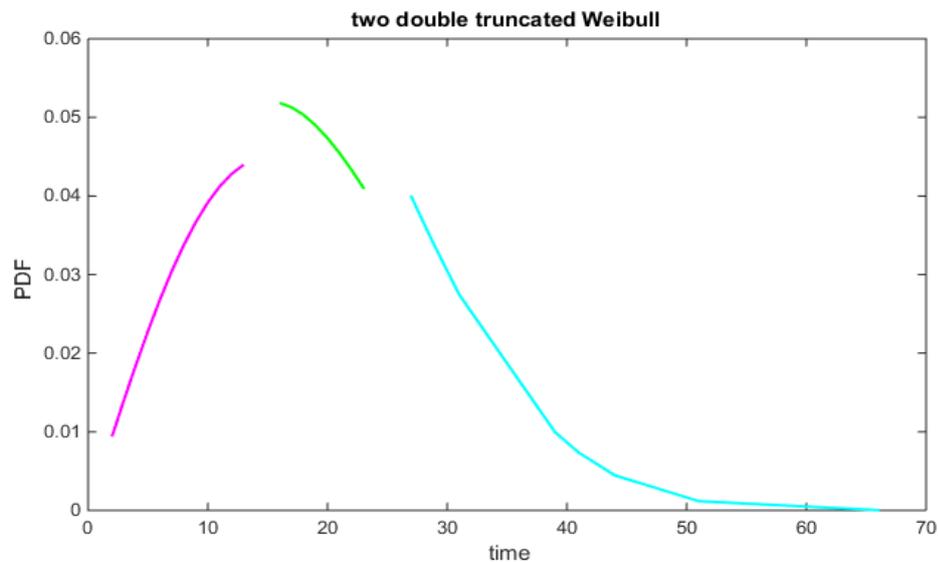


FIGURE 5.1 PDF of TDTWD on data in set (1) with the estimated parameter

In the above figure, we presented PDF of (TDTWD) after estimate the parameter for it in set(1).

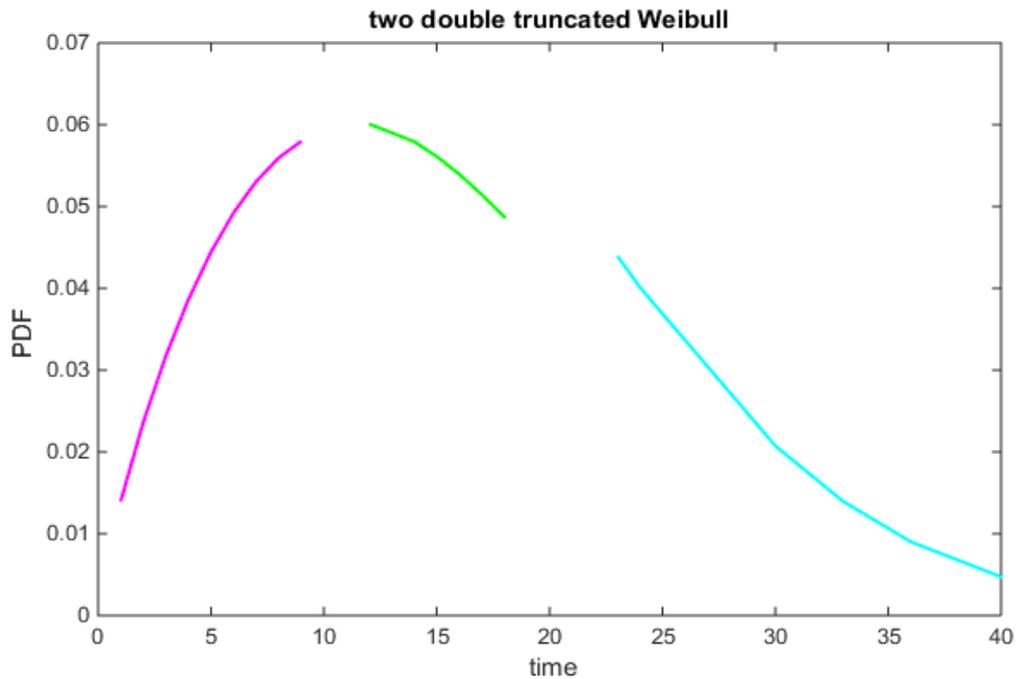


FIGURE 5.2 PDF of TDTWD on data in set (2) with the estimated parameter

In this figure, we presented PDF of (TDTWD) after estimate the parameter for it in set(2).

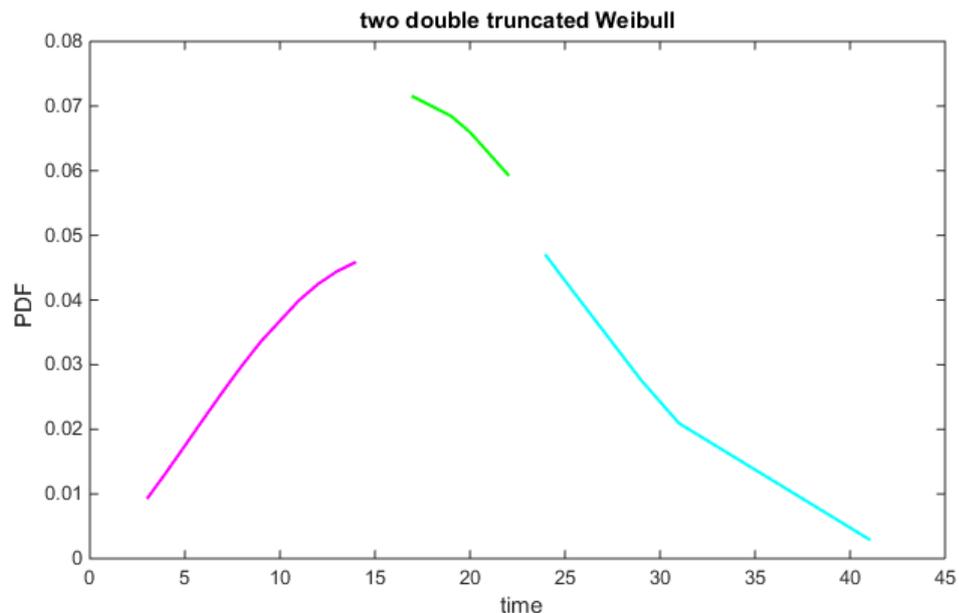


FIGURE 5.3 PDF of TDTWD on data in set (3) with the estimated parameter

In this figure, we presented PDF of (TDTWD) after estimate the parameter for it in set(3).

5.8 The Maximum Likelihood Estimation For The Multi-left Truncation Distribution

In this section, we find an estimation of MLTD by using the maximal likelihood method, for two models (triple left truncated Rayleigh distribution(TLTRD) and triple left truncated Exponential distribution (TLTED)). Goodness fitting for models was presented by two sets of data.

A random variable X_T is said to have the multi-left truncated distribution if its probability density function is:[4-2]

$$g_n(x_T) = \frac{1}{n} \sum_{i=1}^n \frac{f(x_T)}{1-F(b_i)} \quad \text{when } b_i < x_T < \infty$$

Theorem (5-2): Consider X_1, X_2, \dots, X_m is a random sample of size m from a population with a density function $g_n(x_T; \underline{\theta})$, where $\underline{\theta}$ are unknown parameter, then the likelihood function is given by:

$$L(\theta_j, x_T) = \prod_{j=1}^m g_n(x_{T_j}; \underline{\theta}),$$

$$L(\theta_j, x_T) = \frac{\prod_{j=1}^m f(x_{T_j})}{n^m F_1^{r_1} F_2^{r_2} \dots F_n^{m - \sum_{i=1}^{n-1} r_i}}$$

The log-likelihood function is:

$$\begin{aligned} \ell = & -m \log(m) - r_1 \log(F_1) - (r_2) \log(F_2) - \dots - (m - \sum_{i=1}^{n-1} r_i) \log(F_n) + \\ & \sum_{j=1}^m \log(f(x_{T_j})). \end{aligned} \quad (5.7)$$

5.8.1 Maximum Likelihood Estimates of TLTRD

If X_1, X_2, \dots, X_m denotes a random sample from TLTRD, then the likelihood function is given by:

$$\begin{aligned}
 L &= \prod_{j=1}^m \mathfrak{G}_3(x_j; \sigma^2), \\
 &= \prod_{j=1}^{r_1} \left(\frac{\frac{x_j}{\sigma^2} e^{-\frac{x_j^2}{2\sigma^2}}}{3 \left(e^{\frac{-b_1^2}{2\sigma^2}} \right)} \right) \prod_{j=r_1+1}^{r_2} \left(\frac{\frac{x_j}{\sigma^2} e^{-\frac{x_j^2}{2\sigma^2}}}{3 \left(e^{\frac{-b_2^2}{2\sigma^2}} \right)} \right) \prod_{j=r_2+1}^m \left(\frac{\frac{x_j}{\sigma^2} e^{-\frac{x_j^2}{2\sigma^2}}}{3 \left(e^{\frac{-b_3^2}{2\sigma^2}} \right)} \right) \\
 &= \left(\frac{\prod_{j=1}^m \frac{x_j}{\sigma^2} e^{-\frac{x_j^2}{2\sigma^2}}}{3^m \left(e^{\frac{-b_1^2}{2\sigma^2}} \right)^{r_1} \left(e^{\frac{-b_2^2}{2\sigma^2}} \right)^{r_2} \left(e^{\frac{-b_3^2}{2\sigma^2}} \right)^{m-r_2-r_1}} \right)
 \end{aligned}$$

The log-likelihood function is:

$$\begin{aligned}
 \ell &= \\
 &\sum_{j=1}^m \log x_j - \frac{\sum_{j=1}^m x_j^2}{2\sigma^2} - m \log(\sigma^2) - m \log(3) - r_1 \left[\log \left(\exp \left(\frac{-b_1^2}{2\sigma^2} \right) \right) \right] - \\
 &r_2 \left[\log \left(\exp \left(\frac{-b_2^2}{2\sigma^2} \right) \right) \right] - (m - r_2 - r_1) \left[\log \left(\exp \left(\frac{-b_3^2}{2\sigma^2} \right) \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 \ell &= \sum_{j=1}^m \log x_j - \frac{\sum_{j=1}^m x_j^2}{2\sigma^2} - m \log(\sigma^2) - m \log(3) + r_1 \left[\frac{b_1^2}{2\sigma^2} \right] + r_2 \left[\frac{b_2^2}{2\sigma^2} \right] + \\
 &(m - r_2 - r_1) \left[\frac{b_3^2}{2\sigma^2} \right]
 \end{aligned}$$

By taking the derivatives of ℓ with respect to the parameter σ^2 and the result equal to zero,

$$\frac{d\ell}{d\sigma^2} = \frac{\sum_{j=1}^m x_j^2}{2\sigma^4} - \frac{m}{\sigma^2} - r_1 \frac{b_1^2}{2\sigma^4} - r_2 \frac{b_2^2}{2\sigma^4} - (m - r_2 - r_1) \frac{b_3^2}{2\sigma^4} = 0$$

5.8.2 Maximum Likelihood Estimates of TLTED

If X_1, X_2, \dots, X_m denote a random sample from TLTED, then the likelihood function is given by:

$$\begin{aligned} L &= \prod_{j=1}^m \mathbb{G}_3(x_j; \sigma^2), \\ &= \prod_{j=1}^{r_1} \left(\frac{\lambda e^{-\lambda x_j}}{3(e^{-\lambda b_1})} \right) \prod_{j=r_1+1}^{r_2} \left(\frac{\lambda e^{-\lambda x_j}}{3(e^{-\lambda b_2})} \right) \prod_{j=r_2+1}^m \left(\frac{\lambda e^{-\lambda x_j}}{3(e^{-\lambda b_3})} \right) \\ &= \left(\frac{\prod_{j=1}^m \lambda e^{-\lambda x_j}}{3^m (e^{-\lambda b_1})^{r_1} (e^{-\lambda b_2})^{r_2} (e^{-\lambda b_3})^{m-r_2-r_1}} \right) \end{aligned}$$

The log-likelihood function is:

$$\begin{aligned} \ell &= m \log(\lambda) - \sum_{j=1}^m \lambda x_j - m \log(3) + r_1 \lambda b_1 + r_2 \lambda b_2 + \\ &\quad (m - r_2 - r_1) \lambda b_3 \end{aligned} \quad (5.8)$$

By taking the derivatives of ℓ with respect to the parameter λ and the result equal to zero,

$$\frac{d\ell}{d\lambda} = \frac{m}{\lambda} - \sum_{j=1}^m x_j + r_1 b_1 + r_2 b_2 + (m - r_2 - r_1) b_3 = 0$$

5.8.3 Real Data Analysis of Study (TLTRD) and (TLTED)

In this subsection, we illustrate the applicability of TLTRD and TLTED models by considering real data of the infected with Covid-19 in the city of Babylon-Iraq. Data was collected for the period of their infection with the virus until their death for the interval for two sets. We also fit Maxwell distribution, Chi-Square distribution and Rayleigh with two parameters (RATO), see appendix B more detail about compared distribution.

Data in set(1): (1/10/2020-30/12/2020).

The data set are:

$t=[11,2,22,16,17,6,12,7,29,11,7,10,15,8,1,21,8,1,4,17,1,10,21,16,12,1,2,7,4,11,5,5,8,4,15,1,23,16,8,10,10,11,27,9,18,1,4,1,6,3,26,4,1,3,11,2,7,12,3,32,6,12,8,1,5,10,3,66,51,20,8,15,18,22,4,16,19,8,21,19,12,8,3,9,17,7,12,12,44,11,23,12,10,8,11,27,10,4,5,6,11,1,28,22,6,7,31,6,16,2,12,21,19,2,19,8,18,39]$. with size 112

Data in set(2): (1/7/2020-31/8/2020). The data set are:

$w=[5,13,8,5,4,3,4,2,16,7,6,22,24,14,6,4,9,1,1,6,5,2,12,4,2,1,1,7,13,3,1,12,7,14,3,1,2,10,1,17,8,11,10,1,1,1,1,2,6,10,1,13,1,1,1,17,15,6,3,5,19,3,1,1,12,6,3,23,6,14,11,12,8,9,3,1,6,7,1,20,2,3,2,9,8,5,12,6,7,6,1,3,5,1,9,11,1,25,6,5,1,5,1,2,13,19,3,1,3,7,8,5,5,11,3,10,4,13,4,7,4,14,3,9,2,9,9,2,4,17,2,9,6,9,2,3,8,11,1,9,21,12,13,5,6,1,1,6,1,1,13,22,7,4,3,7,16,6,5,4,11,6,24,6,16,4,1,16,6,1,5,12,4,15,3,1,25,1,10,11,2,22,4,9,2,5,1,5,1,2,16,1,6,2,4,21,4,17,12,5,21,7,8,6,7,15,5,7,9,57,6,1,1,3,7,10,2,1,13,3,7,6]$, with size 221.

Data in set(3)

By adding the data for month 6 to data in set (2), which is $t=\{3,3,9,9,25,15,3,3,1,2,5,5,3,9,2,7,1,27,6,1,10,3,10,5,22,26,18,13,18,6,8,2,2,8,10,2,3,1,6,8,10,6,4,1,2,5,17\}$, it becomes total of size data in set (3) is 269.

$H_0 =$ The model TDTEd is fitted the data

$H_1 =$ The model TDTEd is not fitted the data

Table (5-.10) the values of statistics $\hat{\ell}$, AIC, BIC, CAIC, HQIC for data in set (1)

Model	$\hat{\ell}$	AIC	BIC	CAIC	HQIC
TLTED	-165.132	332.264	334.98249	332.30036	334.98249
TLTRD	-678.103	1,358.206	1,360.9249	1,356.24236	1,359.30894
MAWE	-1,592.892	3,187.784	3,190.50249	3,185.82036	3,188.88694
CH-SQ	-783.545	1,567.09	1,571.80849	1,567.12636	1,570.19294
RA-TO	-1.368.956	2,749.912	2,747.34898	2,738.2209	2,750.32375

In this table, found M.L.E, and stastical criteria for (TLTED),(TLTRD) and comparision models with data in set (1). Show the model TLTED is the better fit than TLTRD and other models. And the last is better fit of others models.

Table (5.11) Estimations parameters of TLTRD,TLTED
and others for data in set(1)

Model	parameter	Parameter estimates
TLTED(λ)	λ^{\wedge}	0.622
TLTRD (σ^2)	$\sigma^{2\wedge}$	9.00
CH-SQ	α^{\wedge}	9.00
MAWE (θ)	θ^{\wedge}	1.939
RA-TO (α, δ)	$\alpha^{\wedge}, \delta^{\wedge}$	0.09, 8.937

In this table, we estimated the parameter of the models (TLTED),(TLTRD) , and comparison models, and find maximum likelihood funcnion for all models with data in set (1)

Table (5.12) the values of statistics ℓ^{\wedge} , AIC, BIC, CAIC, HQIC
for TLTRD,TLTED and others for data in set (2)

Model	ℓ^{\wedge}	AIC	BIC	CAIC	HQIC
TLTED	-633.995	1,269.99	1,278.78632	1,268.00826	1,269.67606
TLTRD	-764.487	1,530.974	1,534.37216	1,528.99226	1,530.66006
MAWE	-935.238	1,872.476	1,875.87416	1,870.49926	1,872.81620
CH-SQ	-1,125.799	2,253.598	2,256.99616	2,251.61626	2,253.28406
RA-TO	-1,600,936	3,205,872	3,212,6683	3,208,24412	3,205,24412

In this table, found M.L.E, and stastical criteria for (TLTED),(TLTRD) and comparison models with data in set (2). Show the model TLTED is the better fit than TLTRD and other models. And the last is better fit of others models.

Table (5.13) Estimations parameters for TLTRD and TLTED
of data in set (2)

Model	parameter	Parameter estimates
TLTRD (σ^2)	σ^2^{\wedge}	4.674
TLTED(λ)	λ^{\wedge}	0.2
CH-SQ(α)	α^{\wedge}	1.903
MAWE (θ)	θ^{\wedge}	9.0
<i>RA-TO</i> (α, δ)	$\alpha^{\wedge}, \delta^{\wedge}$	0.2, 7.47

In this table, we estimated the parameter of the models (TLTED),(TLTRD) , and comparison models, and find maximum likelihood function for all models with data in set (1)

By using statistical information criteria, we can say that the (TLTE)distribution which has the least criteria in set (1) is better than a set (2). Furthermore, the (TLTED) is fitting better than (TLTRD), and the lest is better than other models, this show in table(5.10)and (5.12).

The maximal likelihood estimation of all models parameters for the above data for set (1) and set(2) are shown in table (5.11), (5.13),respectively.

Table (5.14) The values of ℓ^{\wedge} and σ^2^{\wedge} for TLTRD with various b_1, b_2, b_3 for data in set (3)

b_1, b_2, b_3	ℓ^{\wedge}	σ^2^{\wedge}
1,5,8	-1.147.424	4.605
5,7,9	-1.089.138	4.362
6,8,10	-1.030.508	4.124
7,10,13	-761.389	3.112
8,11,14	-5.67.842	2.455

This table shows the value of maximum likelihood estimation of (TLTRD) has decreased when the values of b_1, b_2, b_3 increased.

Table (5-15) statistics to the goodness of fit (TLTRD and TLTED)

Data		SSE	R-squared	Adj-R squared	RMSE
TLRD	Set1	3.8342667e-05	0.99983333 3	0.9998	0.000212053
	set 2	0.698692863	0.99983333 3	0.99988	0.0004813333
TLTE	set 1	1.6284933e-5	0.9997	0.999633333	0.00010122
	set 2	6.034e-6	0.9999	0.999833333	0.0003727

For TLTED, the values of (SSE, RMSE) of data in set(1) and set(2) are the smallest compared to TLTRD and it fitted the data better than TLTRD. This result is compatible with the results of Table {(5.10),(5.12)}and SSE, RMSE values are closer to 0 which indicates a fit that is more useful for prediction, this is shown in sets (1), and set(2).

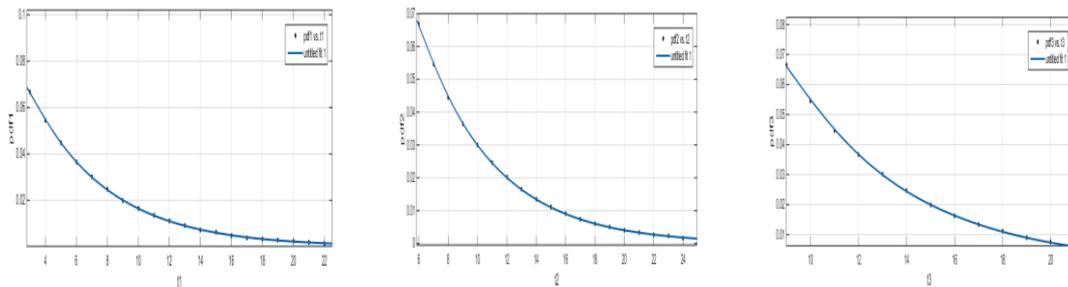


FIGURE 5.4 curving fitting of the model (TLTED) for data in set (1)

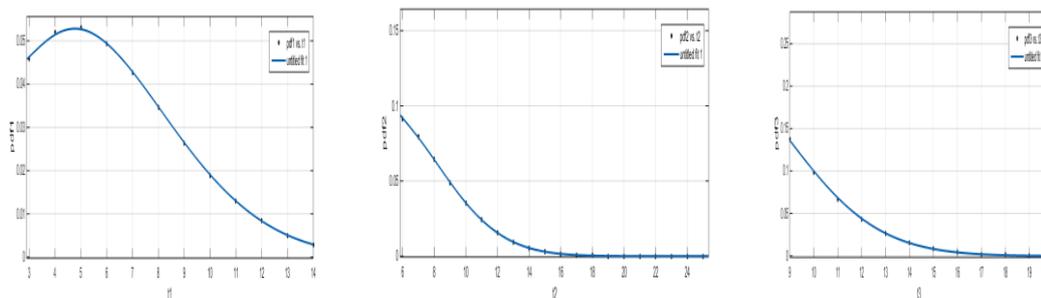


FIGURE 5.5 curving fitting of the model (TLTRD) for data in set (2)

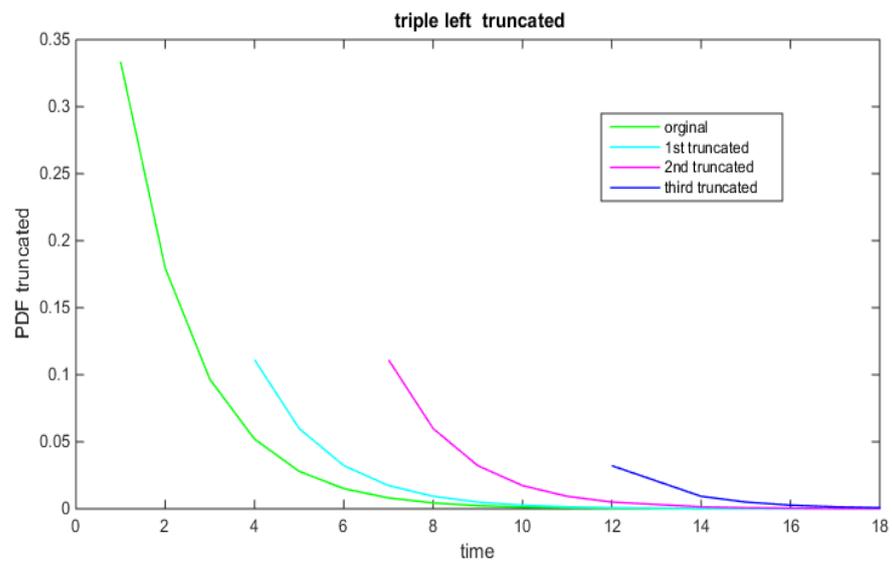


FIGURE 5.6 PDF of TLTED of data set (1) with the estimated parameter

In this figure, we presented PDF of (TLTED) after estimate the parameter for it ,to data in set(1).

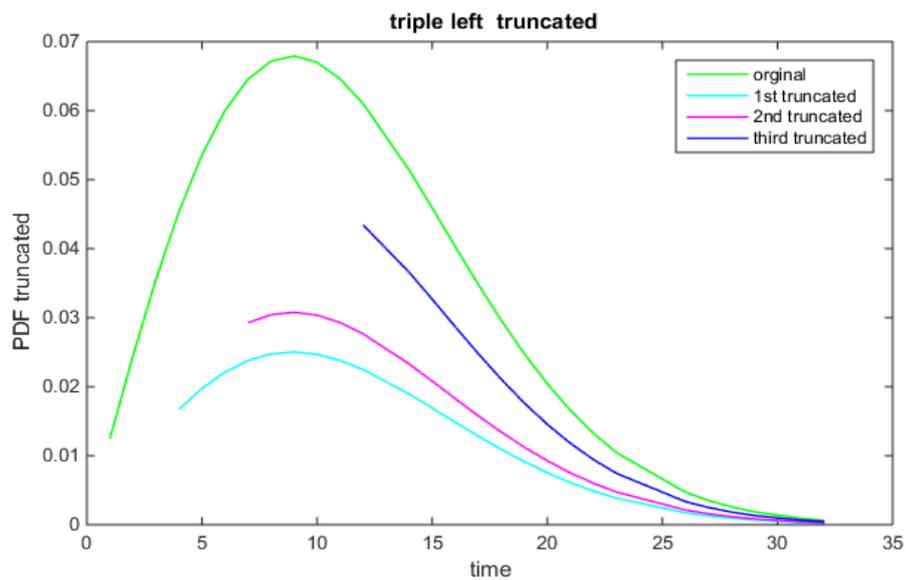


FIGURE 5.7 PDF of TLTRD of data set (1) with the estimated parameter

In this figure, we presented PDF of (TLTRD) after estimate the parameter for it ,to data in set(1).

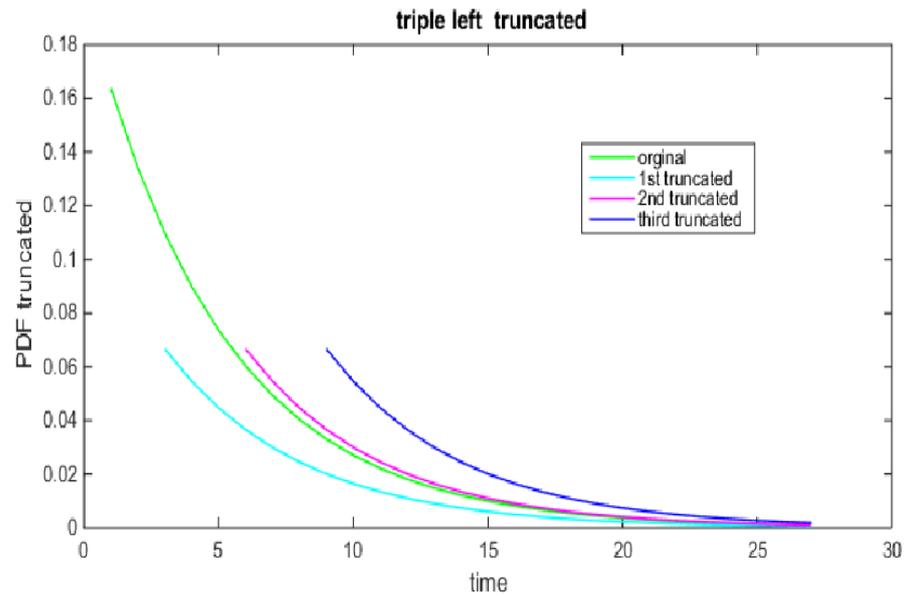


FIGURE 5.8 PDF of TLTED of data set (2) with the estimated parameter

In this figure, we presented PDF of (TLTED) after estimate the parameter for it ,to data in set(2).

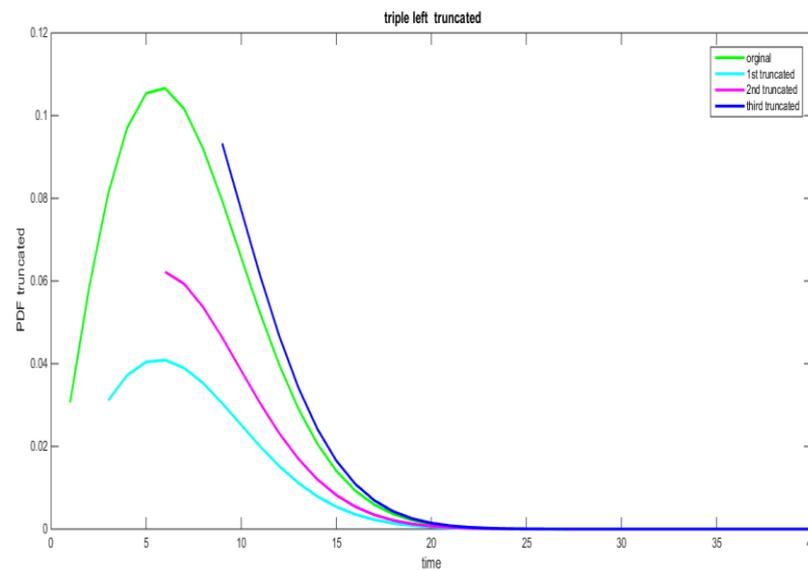


FIGURE 5.9 PDF of TLTRD of data set (2) with estimated parameters

In this figure, we presented PDF of (TLTRD) after estimate the parameter for it ,to data in set(2).

CHAPTER SIX

CONCLUSION AND FUTURE WORKS

Conclusions

1. Construction of new multi- truncation of a continuous distribution, using the double type. Further, found the statistical properties.
2. Multi-double truncated Weibull distribution, and multi-double truncated Rayleigh distribution, are introduced with statistical properties. For example; one double-truncated Weibull distribution, two double-truncated Weibull distribution, and one double-truncated Rayleigh distribution.
3. A new formula to consider the multi-left truncated on continuous distribution, has been employed. Some examples about multi- left truncated with statistical properties are obtained. Such as, triple left truncated Rayleigh distribution , triple left truncated exponential distribution .
4. Derived the r^{th} moment about origin of multi-double Rayleigh, by two way and proved it equal.
5. Graphs for PDF,CDF, and survival function of all previous distributions , have been utilized.
6. Formulating a theory for calculating the maximum likelihood function of a multi-double truncated continuous distribution, and multi-left truncated continuous distribution.
7. The flexibility of one double truncated Weibull distribution , and one double truncated Rayleigh distribution, two double truncated Weibull distribution, triple-left truncated Rayleigh distribution and triple-left truncated exponential distribution are illustrated throughout comparison with other distributions via applications to real data sets. Moreover, the goodness of fit curve of these models is considered, and we clarify the utility of the models proposed by application of real data (data of COVID-19) to Babylon city.

Future works

1. The discrete distributions are very important in solving many problems of lifetime, therefore, we can apply the multi-double truncated, and multi-left truncated on discrete distribution.
2. We can find multi-right truncated for a continuous distribution.
3. Emplly simulation approach on (MDTRD), (MDTWD), (MLTRD), and (MLTED).
4. Using Neutrosophy distribution continuous in multiply truncated distribution.

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Appendix A

Comparisun distributions

1- Maxwell distribution [19]

The probability density function of (MAWE) introduced by (James Maxwell and Boltzmann), is:

$$f(x, \theta) = \frac{4}{\sqrt{\pi}} \frac{x^2 e^{-x^2/\theta}}{\theta^{3/2}} ; \quad x, \theta > 0.$$

2- Chi-Square Distribution[7]

The probability density function of (CH-SQ) was introduced by (James Maxwell and Boltzmann) (CH-SQ) is:

$$f(x, \alpha) = \frac{x^{\frac{\alpha}{2}-1} e^{-x/2}}{2^{\alpha/2} \Gamma(\frac{\alpha}{2})} ; \quad x > 0, \quad \alpha \text{ is degrees of freedom.}$$

3- Gamma Rayleigh distribution [3]

The probability density function of (GARA) introduced by (Aliya Syed Malik and S. P. Ahmad 2019), is:

$$f(x; \alpha, \theta, \omega) = \frac{x}{\theta^2 \Gamma(\alpha) \omega^\alpha} \frac{x^2 e^{-x^2/2\omega\theta^2}}{\theta^{3/2}} \left(\frac{x^2}{2\theta^2} \right)^{\alpha-1} ; \quad x > 0,$$

$\alpha, \theta, \omega > 0.$

4- Modified Weibull distribution[2]

The probability density function of (MOWE) introduced by (Ammar M. and Mazen Z), is:

$$f(x; \alpha, \omega, \gamma) = (\alpha + \omega \gamma x^{\gamma-1}) e^{-\alpha x + \omega x^\gamma} ; \quad x > 0, \quad \alpha, \gamma, \omega > 0.$$

5- Rayleigh with two parameters distribution [39]

The probability density function of (RATO) introduced by (Sanku, Tanujit, and Debasis 2003), is:

$$f(x; \omega, \gamma) = 2\omega(x - \gamma) e^{-\omega(x-\gamma)^2} ; \quad x > 0, \quad \gamma, \omega > 0.$$

ω, γ are shape and scale paramters respectively.

6- Generalized Rayleigh distribution [9][28]

The probability density function of (GERA) introduced by (Sanku, Tanujit, and Debasis 2003), is:

$$f(x; \omega, \gamma) = 2\omega\gamma^2 x e^{-x(\gamma)^2} (1 - e^{-\gamma(x)^2})^{\omega-1} ; \quad x > 0, \quad \gamma, \omega > 0.$$

ω, γ are shape and scale paramters respectively.

Appendix B

1- The algorithm to estimate parameters in maximum likelihood method.

```

%%%%%%%% Babel %%%%%%%%%
clear all
clc

t=[5,10,14,3,66,51,20,8,15,18,22,16,19,8,21,19,12,8,3,9,17,7,12,44,11,23,
12,10,8,11,27,10,4,5,6,11,28,22,6,7,31,6,16,2,12,21,19,2,19,8,18,39, 22,25
,3,11,7,4,14,5,7,20,6,13,14,9,5,13,41,29,3,4,12,17,3,16,6,19,24,4,8,11,13,4,
6,9,12,6,31,20],

b0=1;a1=32;b1=1;a2=66;b2=2;a3=39;

%%%%%%%%%%%%%%

r1=size(t1,2);
r2=size(t2,2);
r3=size(t3,2);
n=r1+r2+r3;
t=[t1 t2 t3];
t=(1./t);

%%%%%%%%%%%%%%

S=["Mean","StandardError","Median","Mode","Standrad
Deviation","Variance","Range","Max","Min"];

s=[mean(t) std(t)/sqrt(n) median(t) mode(t) std(t) std(t)^2 (max(t)-min(t))
max(t) min(t)];

Summary=[S;s]

%%%%%%%%%%%%%%

x0=[1;2];

s1= abs(fsolve(@(x)
mle_babel(x,t',b0,a1,b1,a2,b2,a3,r1,r2,n),x0));%%%%%%%%%%%%%%
% Mle Method

l_mle=s1(1)

```

k_mle=s1(2)

LogL=log(1)-n*log(3)-r1*log(exp(-(b0/l_mle).^k_mle)-exp(-(a1/l_mle).^k_mle))-r2*log(exp(-(b1/l_mle).^k_mle)-exp(-(a2/l_mle).^k_mle))-(n-r2-r1)*log(exp(-(b2/l_mle).^k_mle)-exp(-(a3/l_mle).^k_mle))+n*log(k_mle)-n*k_mle*log(l_mle)+(k_mle-1)*sum(log(t))-sum(t.^k_mle)./(l_mle^k_mle);

AIC_d=-2*LogL+2*2;

AICc_d=(AIC_d+((2*2*4)/(n-2-1)));

BIC_d=-2*LogL+2*log(n);

p_we = wblfit(t);

theta_we=p_we(1);k_we=p_we(2);

L_w=((k_we./theta_we).^n).*prod(t.^(k_we-1)).*exp(-(sum(t.^k_we))./theta_we);

AIC_w=abs(real(-2*log(L_w)+2*2));

AICc_w=abs(AIC_w+((2*2*4)/(n-2-1)));

BIC_w=abs(real(-2*log(L_w)+2*log(n)));

theta_r=raylfit(t);

L_r=(n.*log(2./theta_r)).*sum(log(t)).*(-(sum(t.^2))./theta_r);

AIC_r=abs(real(-2*log(L_r)+2*1));

AICc_r=abs(AIC_r+((2*1*4)/(n-1-1)));

BIC_r=abs(real(-2*log(L_r)+1*log(n)));

x0=[1;2];

s2=abs(fsolve(@(x) mod_babel(x,t',n),x0));%%%%%%%%%

Mle Method

a_mle=s2(1)

b_mle=s2(2)

LogLm=n*log(2)+n*log(a_mle)+2*n*log(b_mle)+sum((b_mle.*t).^2)-2*sum(log(1-(1-a_mle).*exp(-((b_mle.*t).^2)))) ;

AIC_mor=-2*LogLm+2*2;

AICc_mor=(AIC_mor+((2*2*4)/(n-2-1)));

```

BIC_mor=-2*LogLm+2*log(n);
x0=[5;4;9];
s3= abs(fsolve(@(x) gr_babel(x,t',n),x0));%%%%%%%%%%
Mle Method
a_mle=s3(1);
b_mle=s3(2);
c_mle=s3(3);
Loggr=sum(log(t))-2*n*log(c_mle)-n*log(gamma(a_mle))-
n*a_mle*log(b_mle)-sum(((t.^2)./(2*(c_mle^3)*(b_mle))))+(a_mle-
1)*sum(log((t.^2)./(2.*(c_mle^2))));
AIC_gr=-2*Loggr+2*3;
AICc_gr=(AIC_gr+((2*3*4)/(n-3-1)));
BIC_gr=-2*Loggr+3*log(n);
x0=[6;3;12];
s4= abs(fsolve(@(x) mw_babel(x,t'),x0));%%%%%%%%%%
Mle Method
a_mle=s4(1);
b_mle=s4(2);
th_mle=s4(3);
Logmw=-sum(log(a_mle+b_mle.*th_mle.*(t.^(th_mle-1))))-a_mle*sum(t)-
b_mle*sum(t.^th_mle);
AIC_mw=-2*Logmw+2*3;
AICc_mw=(AIC_mw+((2*3*4)/(n-3-1)));
BIC_mw=-2*Logmw+3*log(n);
x0=[2;9];
s5=abs(fsolve(@(x) gray_babel(x,t',n),x0));%%%%%%%%%%
Mle Method
a_mle=s5(1);
l_mle=s5(2);

```

```

Loggray=n*log(a_mle)+2*n*log(l_mle)+sum(log(t))-
(l_mle^2)*sum(t.^2)+(a_mle-1)*sum(log(1-exp(-((l_mle.*t).^2))));
AIC_gray=-2*Loggray+2*2;
AICc_gray=(AIC_gray+((2*2*4)/(n-2-1)));
BIC_gray=-2*Loggray+2*log(n);
x0=[8;4;3];
AIC=[AIC_d AIC_w AIC_r AIC_e AIC_mor AIC_gr AIC_mw AIC_gray
AIC_3w];
AICC=[AICc_d AICc_w AICc_r AICc_e AICc_mor AICc_gr AICc_mw
AICc_gray AICc_3w];
BIC=[BIC_d BIC_w BIC_r BIC_e BIC_mor BIC_gr BIC_mw BIC_gray
BIC_3w];
S1=["truncated","Weibul","Rayleigh","modifiedrayligh","Gamma
Rayleigh","modified weibull","generalized weibull ]
Criteria=[S1;AIC;BIC]

```

2- The program to draw CDF Rayleigh distribution

```

clear all
clc
x1=[0:0.1:2.5];
x2=[3:0.1:4.2];
x3=[6:0.1:10];
x4=[0:0.1:10]
k=3.5
k1=3.7
a1=0, b1=2.5, a2=3, b2=4.2;a3= 6, b3=10
b=2*(k^2)
y1=(1-(exp(-(x1).^2)/b));
y19=(1-(exp(-(b1).^2)/b));
y2=(1-(exp(-(x2).^2)/b));
y29=(1-(exp(-(a2).^2)/b));
y4=(1-(exp(-(b2).^2)/b));
y3=(1-(exp(-(x3).^2)/b));
y39=(1-(exp(-(a3).^2)/b));
yy=(1-(exp(-(x4).^2)/b));
y11=(y1)./(3*(y19));
y22=(y2-y29+ y19)./(3*(y4-y29));

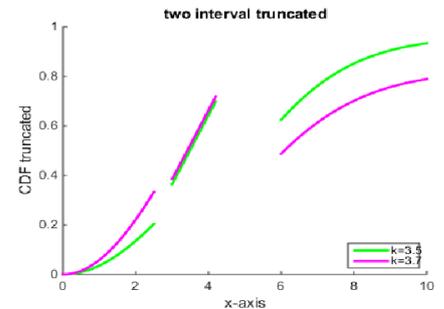
```

Appendix

```

y33=(y3-y39-y29+y4+y19)./(3*(exp(-(a3).^2)/b));
x11=[0:0.1:2.5];
x12=[3:0.1:4.2];
x13=[6:0.1:10];
a1=0, b1=2.5, a2=3, b2=4.2;a3= 6, b3=10
b=2*(k1^2)
y11=(1-(exp(-(x11).^2)/b));
y119=(1-(exp(-(b1).^2)/b));
y12=(1-(exp(-(x12).^2)/b));
y129=(1-(exp(-(a2).^2)/b));
y14=(1-(exp(-(b2).^2)/b));
y13=(1-(exp(-(x13).^2)/b));
y139=(1-(exp(-(a3).^2)/b));
y111=(y11)./(3*(y119));
y122=(y12-y129+y19)./(3*(y14-y129));
y133=(y13-y139-y29+y14+y19)./(3*(exp(-(a3).^2)/b));
hold on;
p1= plot(x1,y11, 'g','linewidth',2);
p2=plot(x2,y22,'g','linewidth',2);
p3=plot(x3,y33,'g','linewidth',2);
p4= plot(x11,y111, 'm','linewidth',2);
p5=plot(x12,y122,'m','linewidth',2);
p6=plot(x13,y133,'m','linewidth',2);
p7=plot(x4,yy,'c','linewidth',2.5);
legend([p1 p4 p7],{'k=3.5','k=3.7','Original'})
title('two interval truncated')
xlabel('x-axis')
ylabel('CDF truncated')

```



Appendix C

Here the approved data is identical to the original distributions

1-Data used to one-double truncated Weibull distribution and one-double truncated Rayleigh distribution (1/10/2020-30/12/2020)

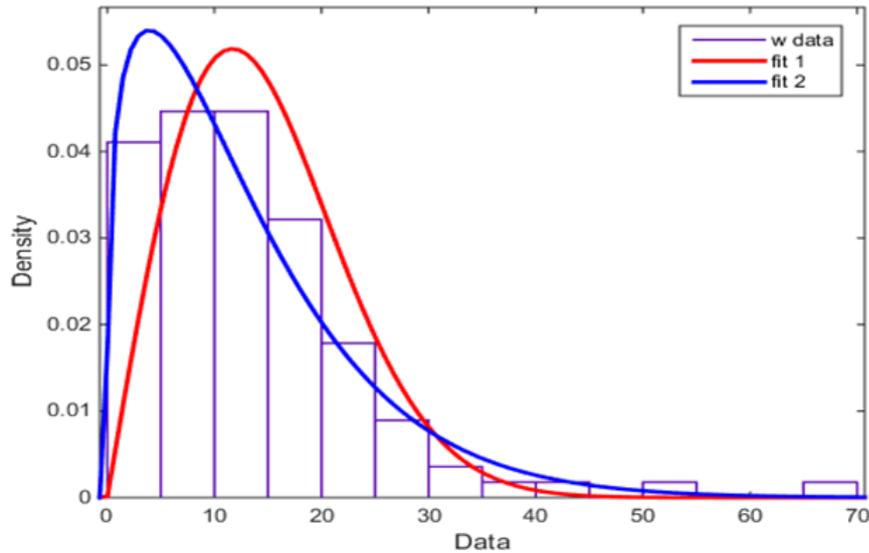


FIGURE A.1 fitting data for Rayleigh distribution, and Weibull distribution

Figure (1) explain the fitting data for patient COVID-19 for period (1/10/2020-30/12/2020) such that, fit1 refers to Rayleigh distribution, while fit2 to Weibull distribution.

2- Data used to two-double truncated Weibull distribution

- a. set 1: for the period (11,12/2020 and 1,2/2021).
- b. Set 2: for period (1-3 to 30 -5/2021).
- c. Set 3: for the period (1-1 to 28 -2/2021).

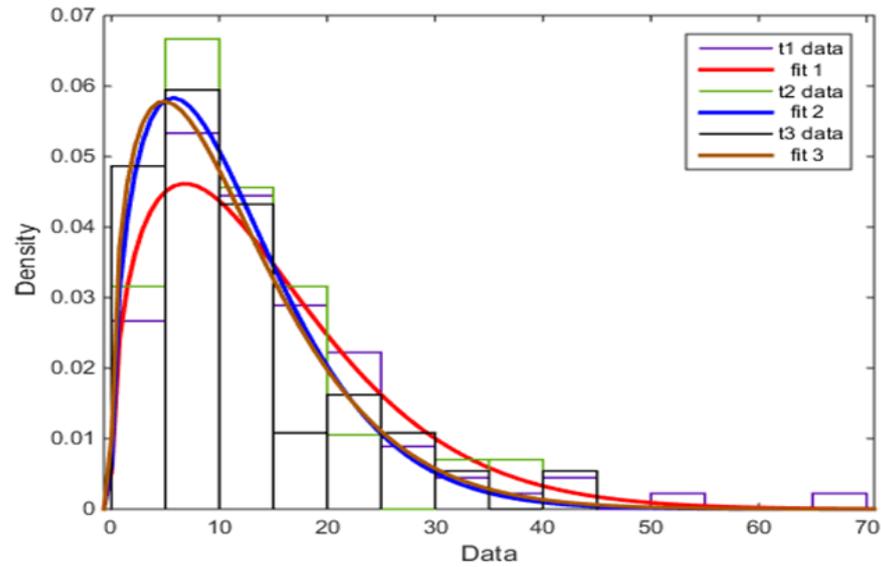


FIGURE A.2 Fitting data of Weibull distribution to set 1, set 2, and set 3 respectively.

In figure (2), the fitting data for patient COVID-19 for period (11,12/2020 and 1,2/2021), (1-3 to 30 -5/2021), and(1-1 to 28 -2/2021). such that, Fit1, fit 2 , and fit 3 refers to PDF of Weibull distribution to (11,12/2020 and 1,2/2021), (1-3 to 30 -5/2021), and(1-1 to 28 -2/2021),respectively.

3-Data used to triple-left truncated exponential distribution and triple-left truncated Rayleigh distribution

- a. set1: (1/10/2020-30/12/2020).
- b. Set 2: (1/7/2020-31/8/2020).
- c. Set 3: (1//2020-31/8/2020).

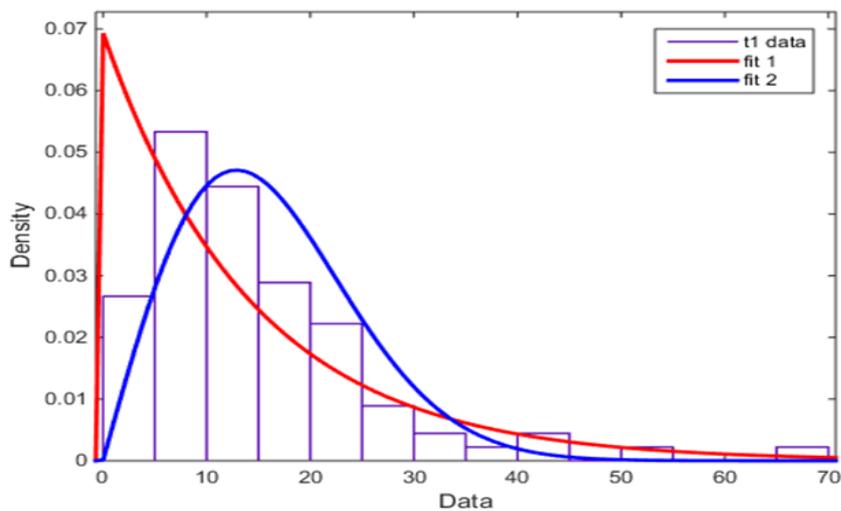


FIGURE A.3 Fitting data of Rayleigh and exponential Distribution, for set 1.

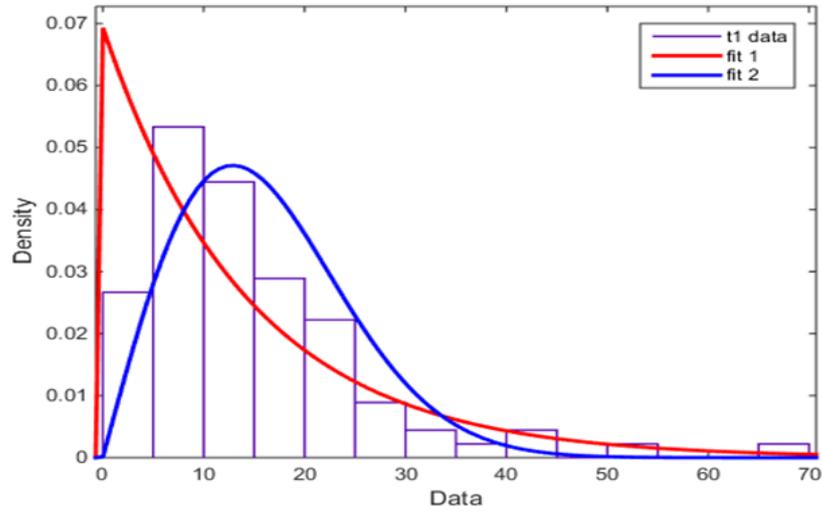


FIGURE A.4 Fitting data of Rayleigh and exponential Distribution, for set 2.

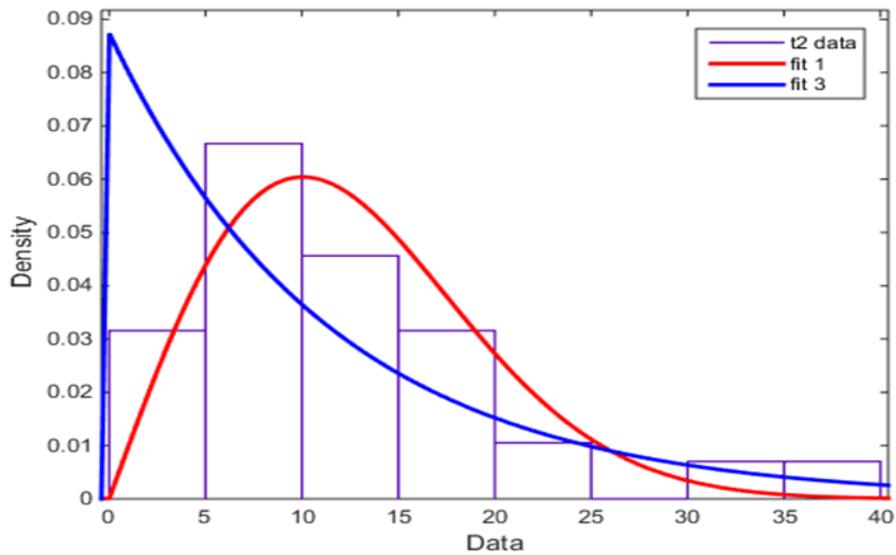


FIGURE A.5 Fitting data of Rayleigh and exponential Distribution, for set 3.

Exponential distribution and Rayleigh distribution are used to apply triple-left truncated. Fit data for these distributions are obtained in Figures (A-3),(A-4), and (A-5), respectively. Such that, fit1 refers to exponential distribution, while fit 2 to Rayleigh distribution in set $i, i=1,2,3$.

المخلص

أنة من الصعب والمعقد نمذجة الظواهر الطبيعية بشكل سهل . ولهذا السبب تعتبر التوزيعات الاحصائية جدا مهم لوصف الظواهر التي يواجهها الباحثون

في هذه الدراسة، ركزنا على التوزيعات المبتورة ذات الاقتطاعات المتعددة لكونها مناسبة في تمثيل الظواهر الطبيعية التي لا يمكن دراستها او تتبعها في جميع مراحل نموها أو تطورها ، على سبيل المثال ، الظواهر المتعلقة بالزراعة والطائرات والصحة والبيئة.

تقدم هذه الدراسة توزيعات متعددة البتر، للبتر المزدوج و للبتر الايسر .

في هذه الدراسة تم بناء صيغة لإيجاد توزيع متعدد البتر ، للبتر المزدوج مع خصائصه الإحصائية. تم استخدام توزيع Weibull و Rayleigh ببتر مزدوج متعدد. تطبيق بعض الأمثلة على البتر المزدوج المتعدد ، بتر مزدوج واحد لتوزيع Weibull ، و بتر مزدوج واحد لتوزيع Rayleigh ، بترين مزدوجين لتوزيع Weibull ، بترين مزدوجين لتوزيع Rayleigh ، بالإضافة إلى الخصائص الإحصائية لكل توزيع.

إلى جانب ذلك ، استخدمنا البتر الأيسر لتقديم صيغة تصف البتر المتعدد له ، مع خصائصه الاحصائية. وضحنا هذا البناء من خلال مثالين ، بتر ايسر ثلاثي.

حيث تم تقديم بتر أيسر ثلاثي لتوزيع Rayleigh ، و بتر أيسر ثلاثي للتوزيع الأسي، مع العزم الرائي حول الصفر وحول المتوسط ، الدالة المولدة للعزوم ، الدالة المميزة ، الاحصاءات المرتبة. بالإضافة الى تمثيل كل التوزيعات اعلاه بيانياً.

أستخدمنا طريقة الامكان الاعظم للتقدير ، كذلك تم اختبار أداء كل توزيع من خلال حساب بعض المعايير الإحصائية (AIC,CAIC,BIC,HQIC) واختبار حسن المطابقة لكل توزيع مع اجراء المقارنات بين التوزيعات واختبارها على بيانات حقيقية لمرضى مصابين ب COVID-19 .



جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بابل
كلية التربية للعلوم الصرفة
قسم الرياضيات

حول التوزيعات المبتورة

أطروحة

مقدمة إلى مجلس كلية التربية للعلوم الصرفة في جامعة بابل
كجزء من متطلبات نيل درجة دكتوراه فلسفة في التربية \ الرياضيات

من قبل

كوثر فوزي حمزة عباس

بإشراف

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