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Controllability for the Solutions of the Dynamical Systems

A Dissertation

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بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

أَلَمْ نَشْرَحْ لَكَ صَدْرَكَ (١) وَوَضَعْنَا عَنْكَ وِزْرَكَ (٢) الَّذِي
أَنْقَضَ ظَهْرَكَ (٣) وَرَفَعْنَا لَكَ ذِكْرَكَ (٤) فَإِنَّ مَعَ الْعُسْرِ يُسْرًا
(٥) إِنَّ مَعَ الْعُسْرِ يُسْرًا (٦) فَإِذَا فَرَغْتَ فَانْصَبْ (٧) وَإِلَىٰ
رَبِّكَ فَارْغَبْ (٨)

صدق الله العلي العظيم

Dedicated To

My daughter MLAK,

my son, MOHAMMED

and all my family.

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Publications

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Abstract

This dissertation aims to study and develop the controllability and observability of fractional control systems with nonsingular kernel in finite and infinite dimensional spaces.

Throughout this work, the controllability of an AB-fractional linear control dynamical system with control delay under sufficient conditions has been proved. Sufficient conditions are set to prove that a nonlinear AB-fractional control dynamical system with control delay is controllable using Schauder Fixed Point Theorem.

The observability of an AB-fractional linear control dynamical system has been investigated. Wherein more than one criterion for it has been introduced. The duality between controllability and observability has been proved.

A mild solution of a nonlinear impulsive control system involving Hattaf-fractional derivative has been introduced in Banach space using fractional calculus and semigroup theory. Under sufficient conditions, we prove the controllability of this system. Our main results are obtained utilizing Nussbaum Fixed Point Theorem. On the other hand, sufficient conditions are introduced to show the existence and uniqueness of the mild solution of the nonlinear system. Also, the approximate controllability of this system is proved in Banach space.

Finally, applications have been shown to illustrate the importance of the main results.

Contents

List of Symbols

Symbol	Meaning
$\dot{y}$	Derivative of a function y for t
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex number
$\mathbb{R}^{\boldsymbol{n}}$	Euclidian space
$\mathbb{R}^{\boldsymbol{n}\times\boldsymbol{m}}$	The set of $n \times m$ real matrices
$C[a,b]$	The set of all continuous functions defined from $[a,b]$ into $\mathbb{R}$
$C_n([a,b])$	The set of all continuous functions defined from $[a,b]$ into $\mathbb{R}^{\boldsymbol{n}}$
$C([a,b];X)$	The set of all continuous functions defined from $[a,b]$ into X
$C^1[a,b]$	The set of all continuously differentiable functions on $[a,b]$
$PC([a,b];X)$	The space of piecewise continuous functions which defined on $[a,b]$
$L_p([a,b];X)$	The space of Lebesgue integrable functions from $[a,b]$ into X
$\mathcal{T}(t)$	Semigroup of bounded linear operators generated by $\mathcal{A}$
C_0-semigroup	Strongly continuous semigroup
$\Gamma(\cdot)$	Gamma function
$\mathcal{L}$	Laplace transform

$D(A)$	The domain of operator A
B^*	The transpose of matrix B
$\mathcal{K}_{\Upsilon}(f)$	The reachable set
$E_\rho(\zeta)$	Mittag-Leffler function of one parameter
$E_{\rho,\omega}(\zeta)$	Mittag-Leffler function of two parameters
${}^{RL}_{a}I^{\rho}$	Riemann-Liouville fractional integral of order ρ
${}^{RL}_{a}D^{\rho}$	Riemann-Liouville fractional derivative of order ρ
${}^{C}_{a}D^{\rho}$	Caputo fractional derivative of order ρ
${}^{ABR}_{a}D^{\rho}$	AB-fractional derivative of Riemann-Liouville sense of order ρ
${}^{ABC}_{a}D^{\rho}$	AB-fractional derivative of Caputo sense of order ρ
${}^{AB}I^{\rho}$	AB-fractional integral of order ρ
${}^{C}_{a}D^{\rho,\omega}$	Hattaf-fractional derivative of Caputo sense of orders ρ,ω
${}_{a}\mathcal{J}^{\rho,\omega}$	Hattaf-fractional integral of orders ρ,ω

Introduction

Control theory is an area of applied mathematics concerned with the behavior of dynamical systems. Mesopotamia (2000 BC) is considered the first to use the theory of control to irrigate agricultural lands. Control mechanisms are found all across nature and are employed by living creatures to keep vital variables like body temperature and blood sugar levels at predetermined ranges. The populations of insects and animals are controlled by a carefully balanced prey-predator relation. There are a variety of basic and complicated man-made control systems in use in our daily lives. [1]

Fractional calculus has received significant interest from researchers because it describes many scientific phenomena with great accuracy. This concept was originally described in 1695 by Leibniz and L'Hospital as a generalization of the integer-order derivative [2]. However, fractional calculus was used in the 1960s. The application of fractional calculus has grown during the past three decades. It is significant to a wide variety of applications in many fields, including physics[3], fluid mechanics[4], biochemical [5], and population growth [6]. Academics increasingly research various forms of fractional differential equations. Numerous definitions of fractional derivatives describe many scientific phenomena, for instance, Riemann-Liouville, Caputo, Hadamard, Grunwald-Letnikov, and Hilfer, for more details; see [7]–[9]

M. Caputo and M. Fabrizio [10] introduced a definition of a derivative with fractional order with the nonsingular exponential kernel. Atangana and Baleanu [11] proposed the AB-fractional derivatives as a concept of fractional derivatives has a nonsingular Mittag-Leffler kernel. In 2020, Hattaf [12] presented a generalization definition of the AB-fractional derivative. The fractional derivatives without singular kernels gave adequately described for models of dissipative phenomena where the classical fractional operators cannot give it, see [13]–[15].

Impulsive differential equations have attracted much research attention due to their significance in modelling processes exposed to short-time changes throughout their development. Many articles deal with impulsive differential equations and their solutions for example; see [16].

Controllability and observability are important properties of dynamical system. They are fundamental elements of modern control theory. If a system is able to transform any initial state to any final state over given time using a control function, then it is said to be controllable. Two forms of controllability are most often considered in practical applications: exactly controllability and approximate controllability. The system is exactly controllable if it reaches a required state at the given time using admissible control. The system is approximate controllable if it reaches a state at the given time lies in a ε-neighborhood of the required state using admissible control. The observability of a system is defined as the ability to determine its initial state from its output behavior.

In 1963, Kalman [17] introduced the concepts of controllability and observability. M. Nawaz, et. al. [18] discussed controllability of nonlinear fractional system with control delay involving the Caputo fractional derivative using fixed point theorem and Mittag-Leffler function. In 2020, Jiale Sheng et al. [19] set sufficient conditions to show that AB-fractional nonlinear dynamical system is controllable using fixed point theorem and Mittag-Leffler function. In 2021, Ghasemi and Nassiri [20] introduced many criterions for the controllability of AB-fractional nonlinear dynamical system provided Caputo derivative of control function exists. In [21], the controllability of AB-fractional systems in a Banach space were discussed based on fixed point theorem and semigroup operator theory.

The controllability problems of impulsive fractional control systems have been studied in many articles, see; [22], [23].

X. Li et al. [24] established sufficient conditions for the approximate controllability of fractional control systems with time delay in Hilbert spaces by using semigroup operator theory and sequence method. In [25] the researcher investigated the approximate controllability for a kind of fractional neutral differential equations with damping in Banach spaces using the approximate sequence method.

In [26], the controllability and observability have been discussed for fractional linear control systems with multiple different orders involving the Caputo fractional derivative using Gramian matrix. K. Balachandran, et al. [27] investigate the observability and controllability of fractional control dynamical system with Grunwald-Letnikov derivative based on Gramian matrix.

This dissertation aims to study the exact and approximate controllability of some types of fractional control systems with Mittag-Leffler kernel in finite and infinite dimensional spaces using fixed point theorems, fractional calculus, and semigroup operator theory. Additionally, the observability of AB-fractional linear control dynamical systems has been discussed and developed in finite dimensional space.

We organized our work into four chapters:

Chapter one deals with the basic concepts and fundamental definitions that helped us achieve our goals, such as functional analysis, semigroup theory, fixed point theorems, fractional calculus, controllability, and observability.

Chapter two contains two sections. The first section investigates the controllability of AB-fractional nonlinear dynamical systems with control delay. We set sufficient conditions to prove that the nonlinear system is controllable using Schauder Fixed Point Theorem. An example is presented to demonstrate our theoretical results. Section two discusses the observability of AB-fractional linear dynamical systems. We present more than one criterion for the observability an AB-fractional linear control dynamical system. Additionally, the duality between controllability and observability has been proved.

Chapter three contains three sections. Section one discusses the controllability of a nonlinear impulsive Hattaf-fractional control system in Banach space using semigroup theory and Nussbaum Fixed Point Theorem. In section two, we prove the existence and uniqueness of the mild solution of the nonlinear system under sufficient conditions using Banach Fixed Point Theorem. Concerning section three, the approximate controllability of the nonlinear system has been proved under sufficient conditions using the approximate sequence method.

Finally, the conclusions and future works have been presented in **chapter four**.

Chapter One

Basic Definitions and Fundamental Theorems

In this chapter, we review some basic concepts, lemmas, and notations that will aid us in establishing our main results later on.

1.1 Functional Analysis

Definition 1.1.1 [28]

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces over the same field F**.** An operator T defined from $\mathbb{X}$ to $\mathbb{Y}$ is called **linear operator** if $T(ax + by) = aT(x) + bT(y)$ for all $x, y \in \mathbb{X}$, and $a, b \in F$.

Definition 1.1.2 [29]

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces and the operator T defined from $\mathbb{X}$ to $\mathbb{Y}$.

1. The operator T is said to be **bounded** if there exists a positive constant $l \in \mathbb{R}$ such that

$$\|Tx\| \leq l\|x\|,$$

for all $x \in \mathbb{X}$.

2. The **norm** of operator T defined as:

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

3. The operator T is called **continuous operator at a point x_0** $\in \mathbb{X}$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|Tx - Tx_0\| < \epsilon, \text{for all } x \in \mathbb{X} \text{ satisfing } \|x - x_0\| < \delta.$$

4. The operator T is called **continuous operator on** $\mathbb{X}$ if it continuous at every point of $\mathbb{X}$ and it called uniformly continuous on $\mathbb{X}$ if δ independ on x.

Remark 1.1.3 [29]

Let T be a bounded linear operator. Then for any $x \in \mathbb{X}$,

$$\|Tx\| \leq \|T\|\|x\|.$$

Lemma 1.1.4 [29]

The linear operator $T: \mathbb{X} \to \mathbb{Y}$ is continuous if and only if it is bounded where $\mathbb{X}$ and $\mathbb{Y}$ are normed spaces.

Definition 1.1.5 [30]

A subset D of a normed space $\mathbb{X}$ is called **relatively compact** if the closure of D is compact.

Definition 1.1.6 [30]

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces. The linear operator $T: \mathbb{X} \to \mathbb{Y}$ is said to be **compact** if T maps each bounded subset $A \subseteq \mathbb{X}$ into a relatively compact set in $\mathbb{Y}$.

Remark 1.1.7 [30]

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces.

i. A linear operator $T: \mathbb{X} \to \mathbb{Y}$ is compact if and only if for any bounded sequence $\{x_n\}$ in $\mathbb{X}$, the sequence $\{Tx_n\}$ contains convergent subsequence in $\mathbb{Y}$.
ii. The compact linear operator is always bounded.

Definition 1.1.8 [28]

A **complete normed space**, also known as **Banach space**, is a normed space where every Cauchy sequence is convergent in it.

Example 1.1.9 [28]

Consider the space of all continuous functions $x: [a, b] \to \mathbb{R}$

$$C[a, b] := \{x : x: [a, b] \to \mathbb{R}, x \text{ is continuous}, a, b \in \mathbb{R}\}.$$

This space is a Banach space with norm given by

$$\|x\| = \max_{t \in [a,b]} |x(t)|.$$

Definition 1.1.10 [30]

Let $(\mathbb{X}, \mu)$ be a measure space and $1 \leq p < \infty$. Then the collection of all measurable function f for which $|f|^p$ is integrable will be denoted by $\boldsymbol{L_p(\mu)}$**.** For each $f \in L_p(\mu)$, set

$$\|f\|_p = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}}$$

the norm $\|f\|_p$is called the $\boldsymbol{L_p}$**-norm** of f.

Lemma 1.1.11 [28]

$(L_p, \|\cdot\|_p)$ is a Banach space.

Definition 1.1.12 [31]

Let $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces. The operator $T: \mathbb{X} \to \mathbb{Y}$ is called **completely continuous** if it is continuous and compact.

Remark 1.1.13 [31]

Let $T: \mathbb{X} \to \mathbb{Y}$ be linear operator, where $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces. Then, if T is compact operator, then it is completely continuous.

Definition 1.1.14 [30]

Assume D is a subset of the space of continuous functions $C[a,b]$.

1- The set D is called **bounded**, if for all $g \in D$ and all $t \in [a,b]$ there exists constant $K > 0$ such that $\|g(t)\| \leq K$.

2- The set D is called **equicontinuous**, if for all $g \in D$ and all $t, s \in [a,b]$ and for each $\epsilon > 0$ there is $\delta > 0$ such that

$$\|g(t) - g(s)\| < \epsilon \quad \text{when } \|t - s\| < \delta.$$

Lemma 1.1.15 [29] "Arzela-Ascoli's Theorem"

Let D be a subset of the space of continuous functions $C[a,b]$. Then D is relatively compact if and only if it is bounded and equicontinuous.

Example 1.1.16 [31]

Consider $\mathbb{X} = C[0,1]$ and let $A: [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Define the operator $T: \mathbb{X} \to \mathbb{X}$ by

$$T(x)(t) = \int_0^1 A(t,s)x(s)\, ds.$$

For any $x \in \mathbb{X}$, we have

$$|T(x)(t)| \leq \int_0^1 |A(t,s)||x(s)|\, ds$$

$$\leq \sup|x(s)| \int_0^1 |A(t,s)|ds$$

$$\leq \|x\| \int_0^1 |A(t,s)|ds.$$

Then T is a bounded operator. Let D be a nonempty and bounded subset of the space $\mathbb{X}$. We show that $T(D)$ is relatively compact. First, let us prove that $T(D)$ is equicontinuous, since A is continuous on compact matric space then it is uniformly continuous, so for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $t_2, t_1 \in [0,1]$ we have

$$|A(t_2,s) - A(t_1,s)| < \epsilon \text{ when } \quad |t_2 - t_1| < \delta$$

For $x \in \mathbb{X}$

$$|T(x)(t_2) - T(x)(t_1)| \leq \int_0^1 |A(t_2,s) - A(t_1,s)||x(s)|\, ds$$

$$\leq \sup|x(t)| \int_0^1 |A(t_2,s) - A(t_1,s)|\, ds$$

$$< \sup|x(t)|\ \epsilon,$$

then $T(D)$ is equicontinuous. According to Arzela-Ascoli's Theorem $T(D)$ is relatively compact, hence the operator T is completely continuous.

Definition 1.1.17 [28]

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces. A linear operator $T: D(T) \subseteq \mathbb{X} \to \mathbb{Y}$ is called **closed operator** when the graph $G(T) = \{(x, T(x)); x \in D(T)\}$ is a closed set in $\mathbb{X} \times \mathbb{Y}$.

Lemma 1.1.18 [28]

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces. An operator $T: \mathbb{X} \to \mathbb{Y}$ is closed if and only if for every sequence $\{x_n\}_{n=0}^{\infty} \subset \mathbb{X}$ such that $x_n \to x$ and $Tx_n \to y$ as $n \to \infty$, $y \in \mathbb{Y}$ then $x \in \mathbb{X}$ and $Tx = y$.

Now, we review some concepts on semigroup operator theory such as uniformly continuous and strongly continuous in a Banach space $\mathbb{X}$.

Definition 1.1.19 [32]

A **semigroup of bounded linear operators** $\mathcal{T}(t), t \geq 0$ on $\mathbb{X}$ is defined as the family of bounded linear operators satisfies the following:

i. $\mathcal{T}(0) = I$,
ii. $\mathcal{T}(t+s) = \mathcal{T}(t) \circ \mathcal{T}(s)$, for every $t, s \geq 0$.

Definition 1.1.20 [32]

The **infinitesimal generator** $\mathcal{A}$ of semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, is the linear operator described by:

$$\mathcal{A}\zeta = \lim_{t\to 0^+} \frac{\mathcal{T}(t)\zeta - \zeta}{t} = \left.\frac{d^+\mathcal{T}(t)\zeta}{dt}\right|_{t=0}, \text{for } \zeta \in D(\mathcal{A})$$

where,

$$D(\mathcal{A}) = \left\{\zeta \in \mathbb{X}; \lim_{t\to 0^+} \frac{\mathcal{T}(t)\zeta - \zeta}{t} \text{ exists}\right\}.$$

Lemma 1.1.21 [32]

There is a unique infinitesimal generator for a semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$,.

Definition 1.1.22 [32]

A semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$,of bounded linear operators on $\mathbb{X}$ is **uniformly continuous** when:

$$\lim_{t\to^+0} \|\mathcal{T}(t) - I\| = 0.$$

Example 1.1.23 [32]

Consider $\mathcal{A}$ is a bounded linear operator on $\mathbb{X}$. Then the exponential function $exp(\mathcal{A}t)$ is a uniformly continuous semigroup generated by $\mathcal{A}$ on $\mathbb{X}$.

Theorem 1.1.24 [32]

Let $\mathcal{A}$ be a linear operator. Then $\mathcal{A}$ is infinitesimal generator of a uniformly continuous semigroup if and only if $\mathcal{A}$ is a bounded linear operator.

Definition 1.1.25 [32]

A **strongly continuous semigroup** (denoted by $\boldsymbol{C_0 - \text{semigroup}}$) is a semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, of bounded linear operators on $\mathbb{X}$ that satisfies:

$$\lim_{t\to 0} \mathcal{T}(t)\zeta = \zeta,$$

for each $\zeta \in \mathbb{X}$.

Examples 1.1.26 [33]

Let $\mathbb{X} = C[0,1]$ such that $\zeta(1) = 0$ for all $\zeta \in \mathbb{X}$. For $t \geq 0$, define

$$(\mathcal{T}(t)\zeta)(s) = \begin{cases} \zeta(s+t) & t+s \leq 1 \\ 0 & t+s > 1 \end{cases}$$

$\mathcal{T}(t)$ is a C_0 −semigroup on $\mathbb{X}$ generated by a linear operator $\mathcal{A}$ which is given by

$$D(\mathcal{A}) = \{\zeta : \zeta \in C^1[0,1] \cap \mathbb{X}, \dot{\zeta} \in \mathbb{X}\}$$

and

$$\mathcal{A}\zeta = \dot{\zeta} \quad \text{for} \quad \zeta \in D(\mathcal{A}).$$

Lemma 1.1.27 [33]

1. Let $\{\mathcal{T}(t)\}_{t\geq 0}$, be C_0-semigroup generated by $\mathcal{A}$. Then for $\zeta \in \mathbb{X}$, the function $t \to \mathcal{T}(t)\zeta$ is continuous from $\mathbb{R}^+$ into $\mathbb{X}$.
2. For $\zeta \in \mathbb{X}, \int_0^t \mathcal{T}(t)\zeta ds \in D(\mathcal{A})$ and $\mathcal{A}\left(\int_0^t \mathcal{T}(s)\zeta ds\right) = \mathcal{T}(t)\zeta - \zeta$.

Definition 1.1.28 [32]

A C_0 −semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, on $\mathbb{X}$ is said to be **differentiable** for $t > 0$ when for any $\zeta \in \mathbb{X}, t \to \mathcal{T}(t)\zeta$ is differentiable for $t > 0$.

Lemma 1.1.29 [32]

Let $\{\mathcal{T}(t)\}_{t\geq 0}$, be C_0-semigroup generated by $\mathcal{A}$. For $\zeta \in D(\mathcal{A}) \subset \mathbb{X}$; $\mathcal{T}(t)\zeta \in D(\mathcal{A})$ and

$$\frac{d}{dt}\mathcal{T}(t)\zeta = \mathcal{A}\mathcal{T}(t)\zeta = \mathcal{T}(t)\mathcal{A}\zeta.$$

Theorem 1.1.30 [32]

Let $\mathcal{A}$ and $\mathcal{B}$ be infinitesimal generators of C_0-semigroups $\mathcal{T}(t)$ and $\mathcal{S}(t)$ respectively. If $\mathcal{A} = \mathcal{B}$, then $\mathcal{T}(t) = \mathcal{S}(t)$ for $t \geq 0$.

Lemma 1.1.31 [33]

Let $\{\mathcal{T}(t)\}_{t\geq 0}$, is C_0-semigroup. Then there is $\sigma \geq 0$ and $K \geq 1$, such that $\|\mathcal{T}(t)\| \leq Ke^{\sigma t}$, for $t \geq 0$.

Definition 1.1.32 [32]

The C_0- semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, is called **compact** if $\mathcal{T}(t)$ is a compact operator for each $t > 0$.

Definition 1.1.33 [32]

The **resolvent set** indicated by $\boldsymbol{p(\mathcal{A})}$ is the set of all complex numbers ξ where $(\xi I - A)^{-1}$ is a bounded linear operator in $\mathbb{X}$.

Definition 1.1.34 [32]

The **resolvent operator** of $\mathcal{A}$ is the family of bounded linear operators $R(\xi;\mathcal{A}) = (\xi I - \mathcal{A})^{-1}, \xi \in p(\mathcal{A})$ and the following equality holds for $x \in \mathbb{X}$

$$R(\xi;\mathcal{A})x = (\xi I - \mathcal{A})^{-1}x = \int_0^\infty e^{-\xi t}\mathcal{T}(t)x dt,$$

where $\{\mathcal{T}(t)\}_{t\geq 0}$, is a C_0 $-$semigroup generated by linear operator $\mathcal{A}, s > 0$.

Theorem 1.1.35 [32] "Hille-Yosida Theorem"

A linear operator $\mathcal{A}$ is the infinitesimal generator of C_0-semigroup $\mathcal{T}(t)$ $(t \geq 0)$, satisfying $\|\mathcal{T}(t)\| \leq K$ $(K \geq 1)$ if and only if

i. $\mathcal{A}$ is closed operator and $\overline{D(\mathcal{A})} = \mathbb{X}$.

ii. The resolvent set $p(\mathcal{A})$ of operator $\mathcal{A}$ contains $\mathbb{R}^+$ and

$$\| R(\xi;\mathcal{A})^n\| \leq \frac{K}{\xi^n}, \text{for } \xi > 0, n \in \mathbb{N}.$$

In the following, we recall some of the fixed point theorems used throughout this work.

If A is an operator of a Banach space $\mathbb{X}$ into itself, then $x \in \mathbb{X}$ is said to be a fixed point of A if $A(x) = x$. Fixed point theorems deal with the existence and attributes of fixed points. Such theorems are the most potent tools for demonstrating the existence and uniqueness of solutions to various mathematical models (differential equations, partial differential equations, fractional order differential equations, etc.).

Definition 1.1.36 [34]

Assume $(\mathbb{X}, \|\cdot\|)$ is a Banach space. An operator Tdefined on $\mathbb{X}$ into itself is called **Lipschitz continuous** if there is $k > 0$, such that

$$\|T(x) - T(y)\| \leq k\|x - y\|$$

for all $x, y \in \mathbb{X}$.

The smallest k is the Lipschitz constant of T. If $k < 1$ then T is called a **contraction**.

Theorem 1.1.37 [31] "Schauder Fixed Point Theorem"

Assume M is a nonempty convex subset of a Banach space $\mathbb{X}$, and $\mathcal{B}$: $M \to M$ is a completely continuous operator. Then $\mathcal{B}$ has at least one fixed point.

Theorem 1.1.38 [35] "Nussbaum Fixed Point Theorem"

Assume G is closed, bounded and convex subset of a Banach space $\mathbb{X}$. If the continuous functions ϕ_1, ϕ_2 from G to $\mathbb{X}$ satisfies the following:

1. $(\phi_1 + \phi_2)G \subset G,$
2. $\|\phi_1 x - \phi_1 y\| \leq \mu\|x - y\|$ for all $x, y \in G$ where $0 < \mu < 1$, i.e. ϕ_1 is contraction,
3. ϕ_2 is completely continuous,

then the operator $\phi_1 + \phi_2$ has a fixed point in G.

Theorem 1.1.39 [36] " Banach Fixed Point Theorem "

Let $\mathbb{X}$ be a Banach space. If $B: \mathbb{X} \to \mathbb{X}$ is a contraction operator, then it has a unique fixed point.

1.2 Fractional Calculus

Fractional calculus is the theory of derivatives and integrals of arbitrary order, which generalizes the concepts of differentiation and integration of integer order. Several fundamental concepts in fractional calculus, such as the definitions, lemmas, and notations, will be reviewed in this section.

In the following, we recall some special functions which are important in fractional calculus.

Definition 1.2.1 [7]

The **Gamma function** is indicated by $\boldsymbol{\Gamma(\alpha)}$, defined as

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} e^{-s}\, ds, \qquad \alpha \in \mathbb{C}, Re(\alpha) > 0$$

which is a generalizes of the factorial function, that is

$$\Gamma(n+1) = n!$$

for $n \in \mathbb{N}$.

Remark 1.2.2 [7]

Some important properties of Gamma function are

i. $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$,

ii. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

The Mittag-Leffler function is essential in fixing problems with fractional differential (integral) equations.

Definition 1.2.3 [7]

The function defined as

$$E_\rho(\zeta) = \sum_{j=0}^{\infty} \frac{\zeta^j}{\Gamma(\rho j + 1)}, \qquad (\zeta \in \mathbb{C}, \rho > 0), \tag{1.1}$$

is called **Mittag-Leffler with one parameter $\boldsymbol{\rho}$**.

The generalization of the **Mittag-Leffler function with two parameters $\boldsymbol{\rho}$ and $\boldsymbol{\omega}$** is given by

$$E_{\rho,\omega}(\zeta) = \sum_{j=0}^{\infty} \frac{\zeta^j}{\Gamma(\rho j + \omega)}, \qquad (\zeta \in \mathbb{C}, \omega, \rho > 0). \tag{1.2}$$

When $\omega = 1,$ then $E_{\rho,\omega}(\zeta)$ coincides with $E_\rho(\zeta)$, i.e. $E_{\rho,1}(\zeta) = E_\rho(\zeta)$.

It is important to note that the Mittag-Leffler function is a generalization of the exponential function. where

$$E_1(\zeta) = \sum_{j=0}^{\infty} \frac{z^j}{j!} = exp(\zeta).$$

Remark 1.2.4 [7]

For $\rho > 0,$

$$\mathrm{E}_{\rho,\rho}(0) = \frac{1}{\Gamma(\rho)}.$$

Now, we recall definition and some properties of Laplace Transform.

Definition 1.2.5 [7]

The **Laplace transform** of a function x is defined by

$$\mathcal{L}\{x(t)\}(s) = \int_0^\infty e^{-st} x(t)\, dt, \qquad s \in \mathbb{C}, t \in \mathbb{R}^+.$$

Lemma 1.2.6 [7]

For $q > -1$ and $s > 0$, the Laplace transform of power function t^q is given by

$$\mathcal{L}\{t^q\}(s) = \frac{\Gamma(q+1)}{s^{q+1}}.$$

Lemma 1.2.7 [7]

Suppose that $x(t)$ and $y(t)$ are two functions, in which the Laplace transforms $\mathcal{L}\{x(t)\}(s)$ and $\mathcal{L}\{y(t)\}(s)$ exists. If the convolution of $x(t)$ and $y(t)$ defined by

$$x(t) * y(t) = \int_0^t x(t-\delta)\ y(\delta)\, d\delta.$$

then the Laplace transform of the convolution of $x(t)$ and $y(t)$ is given by

$$\mathcal{L}\{x(t) * y(t)\}(\lambda) = \mathcal{L}\{x(t)\}(\lambda)\mathcal{L}\{y(t)\}(\lambda).$$

Lemma 1.2.8 [7]

Let $s > 0$, $\theta \in \mathbb{R}, \theta \neq 0$, and $\rho, \omega > 0$. Then the Laplace transform of the Mittag-Leffler functions (1.1) and (1.2) defined as

$$\mathcal{L}\{E_\rho(\theta t^\rho)\}(s) = \frac{s^{\rho-1}}{s^\rho - \theta},$$

$$\mathcal{L}\{t^{\omega-1} E_{\rho,\omega}(\theta t^\rho)\}(s) = \frac{s^{\rho-\omega}}{s^\rho - \theta},$$

respectively.

In the following, we review some definitions and properties of the classic fractional calculus.

Definition 1.2.9 [7]

The fractional integral of order $\rho > 0$ for a function ζ is defined by

$$ {}_{a}^{RL}I^{\rho}\,\zeta(\lambda) = \frac{1}{\Gamma(\rho)}\int_{a}^{\lambda}(\lambda - s)^{\rho-1}\,\zeta(s)ds, \qquad \lambda \in [a, b], $$

is called **Riemann-Liouville fractional integral**, where $\Gamma(\cdot)$ is the Gamma function.

Definition 1.2.10 [7]

Riemann-Liouville fractional derivative of order ρ with the lower limit a for a function ζ is defined by

$$ {}_{a}^{RL}D^{\rho}\,\zeta(\lambda) = \frac{1}{\Gamma(n-\rho)}\frac{d^n}{d\lambda^n}\int_{a}^{\lambda}(\lambda - s)^{n-\rho-1}\,\zeta(s)ds = D^n I_a^{n-\rho}\zeta(t), $$

$$ n - 1 < \rho < n, \quad n \in \mathbb{N}. $$

Example 1.2.11

Assume $0 < \rho < 1$ and $\lambda \in [0, b]$, then

$$ {}_{0}^{RL}I^{\rho}\,\lambda^2 = \frac{1}{\Gamma(\rho)}\int_{0}^{\lambda}(\lambda - s)^{\rho-1}s^2\,ds, $$

by integration of parts, we obtain

$$ \begin{aligned} {}_{0}^{RL}I^{\rho}\,\lambda^2 &= \frac{2}{\Gamma(\rho+1)}\int_{0}^{\lambda}(\lambda - s)^{\rho}s\,ds \\ &= \frac{2}{\Gamma(\rho+2)}\int_{0}^{\lambda}(\lambda - s)^{\rho+1}\,ds \\ &= \frac{2}{\Gamma(\rho+3)}\lambda^{\rho+2}. \end{aligned} $$

Also, we can calculate the Riemann-Liouville derivative of order ρ of the function λ^2 as follows

$$ {}_{0}^{RL}D^{\rho}\,\lambda^2 = D\,{}_{0}^{RL}I^{1-\rho}\lambda^2 = D\,\frac{2}{\Gamma(4-\rho)}\lambda^{(3-\rho)} $$

$$= \frac{2(3-\rho)}{(3-\rho)\Gamma(3-\rho)} \lambda^{(2-\rho)}$$

$$= \frac{2}{\Gamma(3-\rho)} \lambda^{(2-\rho)}.$$

If we choose $\rho = 0.5$, then

$${}_{0}^{RL}I^{0.5}\lambda^{2} = \frac{2}{\Gamma(3.5)} \lambda^{2.5} = 0.6018\, \lambda^{2.5}.$$

$${}_{0}^{RL}D^{0.5}\lambda^{2} = \frac{2}{\Gamma(2.5)} \lambda^{1.5} = 1.5045\, \lambda^{1.5}.$$

Definition 1.2.12 [7]

For a function ζ, the expression

$${}_{a}^{C}D^{\rho}\, \zeta(\lambda) = \frac{1}{\Gamma(n-\rho)} \int_{a}^{\lambda} (\lambda - s)^{n-\rho-1}\, \zeta^{(n)}(s) ds = {}_{a}^{RL}I^{n-\rho} D^{n}\zeta(\lambda),$$

is called the **Caputo fractional derivative** of order ρ, where $\lambda \in [a, b]$, $n - 1 < \rho < n, n \in \mathbb{N}$.

Lemma 1.2.13 [7]

Let $\rho \in \mathbb{R}$ and $n - 1 < \rho < n, n \in \mathbb{N}$. The relationship between the Riemann-Liouville derivative and the Caputo derivative operators is given by

$${}_{a}^{RL}D^{\rho}\, \zeta(\lambda) = {}_{a}^{C}D^{\rho}\, \zeta(\lambda) + \sum_{j=0}^{n-1} \frac{\zeta^{(j)}(a)}{\Gamma(j-\rho+1)} (\lambda - a)^{j-\rho}.$$

In particular, when $0 < \rho < 1$, we have

$${}_{a}^{RL}D^{\rho}\, \zeta(\lambda) = {}_{a}^{C}D^{\rho}\, \zeta(\lambda) + \frac{\zeta(a)}{\Gamma(1-\rho)} (\lambda - a)^{-\rho}.$$

Lemma 1.2.14 [7]

Let $\rho > 0$ and $q > 0$. Then

$${}_{a}^{RL}I^{\rho}\, {}_{a}^{RL}I^{q} \zeta(\lambda) = {}_{a}^{RL}I^{q}\, {}_{a}^{RL}I^{\rho} \zeta(\lambda) = {}_{a}^{RL}I^{\rho+q} \zeta(\lambda).$$

Remark 1.2.15 [7]

For all scalars q, r , we have

$$ {}^{RL}_{a}I^{\rho}\big(q\,x(\lambda)+r\,y(\lambda)\big)=q\,{}^{RL}_{a}I^{\rho}x(\lambda)+r\,{}^{RL}_{a}I^{\rho}y(\lambda). $$

Lemma 1.2.16 [7]

Let $n-1<\rho<n, n\in\mathbb{N}$ and $s>0$ then:

i- the Laplace transform of the Riemann-Liouville fractional integration operator of order ρ is given by

$$ \mathcal{L}\{{}^{RL}_{0}I^{\rho}\zeta(t)\}(s)=s^{-\rho}\mathcal{L}\{\zeta(t)\}(s), $$

ii- the Laplace transform of the Riemann-Liouville fractional differential operator of order ρ is given by

$$ \mathcal{L}\big\{{}^{RL}_{0}D^{\rho}\,\zeta(t)\big\}(s)=s^{\rho}\mathcal{L}\{\zeta(t)\}-\sum_{j=0}^{n-1}s^{n-j-1}D^{(j)}\left({}^{RL}_{0}I^{n-\rho}\zeta\right)(0), $$

iii- the Laplace transform of the Caputo fractional differential operator of order ρ is given by

$$ \mathcal{L}\big\{{}^{C}_{0}D^{\rho}\,\zeta(t)\big\}(s)=s^{\rho}\mathcal{L}\{\zeta(t)\}-\sum_{j=0}^{n-1}s^{\rho-j-1}\big(D^{(j)}\zeta\big)\,(0). $$

Now, we recall some definitions and properties of fractional operators with nonsingular kernel.

Let $\rho\in(0,1), \omega>0, \gamma_{\rho}=\frac{\rho}{1-\rho}$ and $Q(\rho)$ is normalization function satisfies $Q(0)=Q(1)=1$.

Definition 1.2.17 [11]

The **Atangana-Baleanu fractional (AB-fractional) derivative of Riemann-Liouville sense** of order ρ with lower limit a is given by

$$ {}^{ABR}_{a}D^{\rho}\,\zeta(t)=\frac{Q(\rho)}{1-\rho}\frac{d}{dt}\int_{a}^{t}E_{\rho}\big[-\gamma_{\rho}(t-s)^{\rho}\big]\,\zeta(s)\,ds, t\in[a,b]. $$

Definition 1.2.18 [11]

The **Atangana-Baleanu fractional (AB-fractional) derivative of Caputo sense** of order ρ is given by

$$ {}^{ABC}_{\ \ a}D^{\rho}\,\zeta(t) = \frac{Q(\rho)}{1-\rho}\int_a^t E_{\rho}\left[-\gamma_{\rho}(t-s)^{\rho}\right]\left(\frac{d}{ds}\zeta\right)(s)\,ds, t \in [a,b] $$

Definition 1.2.19 [11]

The **fractional integral associated with the Atangana-Baleanu fractional (AB-fractional) derivative** is defined by

$$ {}^{AB}I^{\rho}\,\zeta(t) = \frac{(1-\rho)}{Q(\rho)}\zeta(t) + \rho\ {}^{RL}I^{\rho}\,\zeta(t), \qquad t \in [a,b]. $$

Note: Throughout our work we assume $Q(\rho) = 1$.

Definition 1.2.20 [12]

The **Hattaf fractional derivative of Riemann-Liouville sense** of order ρ with respect to the weight function $\eta \in C^1(a,b)$, $\eta, \acute{\eta} > 0$ with the lower limit a is given by

$$ {}^{RL}_{\ a}D^{\rho,\omega,\lambda}\zeta(t) = \frac{Q(\rho)}{1-\rho}\frac{1}{\eta(t)}\frac{d}{dt}\int_a^t E_{\omega}\left[-\gamma_{\rho}(t-s)^{\lambda}\right](\eta\zeta)(s)\,ds. $$

Definition 1.2.21 [12]

The **Hattaf-fractional derivative of Caputo sense** of order ρ with respect to the weight function $\eta \in C^1(a,b)$, $\eta, \acute{\eta} > 0$ on $[a,b]$ is given by

$$ {}^{C}_{a}D^{\rho,\omega,\lambda}_{\eta}\zeta(t) = \frac{Q(\rho)}{1-\rho}\frac{1}{\eta(t)}\int_a^t E_{\omega}\left[-\gamma_{\rho}(t-s)^{\lambda}\right]\frac{d}{d\zeta}(\eta\zeta)(s)ds, \qquad (1.3) $$

Lemma 1.2.22 [12]

The relationship between the Hattaf-derivative of Riemann-Liouville sense and Hattaf-fractional derivative of Caputo sense operators is given by

$$ {}_{a}^{RL}D^{\rho,\omega,\lambda}\zeta(t) = {}_{a}^{C}D^{\rho,\omega,\lambda}\zeta(t) + \frac{Q(\rho)}{1-\rho}\frac{1}{\eta(t)}E_{\omega}\left[-\gamma_{\rho}(t-a)^{\lambda}\right](\eta\zeta)(a). $$

Remark 1.2.23 [12]

When $\lambda = \omega$ and $Q(\rho) = \eta(t) = 1$, then the fractional derivative (1.3) will be in the form

$$ {}_{a}^{C}D^{\rho,\omega}\zeta(t) = \frac{1}{1-\rho}\int_{a}^{t} E_{\omega}\left[-\gamma_{\rho}(t-s)^{\omega}\right]\frac{d}{d\zeta}\zeta(s)ds. \tag{1.4} $$

Definition 1.2.24 [12]

The **fractional integral corresponding to the Hattaf-fractional derivative (1.4)** is

$$ {}_{a}\mathcal{J}^{\rho,\omega}\zeta(t) = (1-\rho)\zeta(t) + \rho\, {}_{a}^{RL}I^{\omega}\zeta(t) \tag{1.5} $$

Lemma 1.2.25 [11]

The Laplace transform of AB-fractional derivative of Caputo sense is

$$ \mathcal{L}\left\{ {}_{0}^{ABC}\mathrm{D}^{\rho}\, \zeta(t)\right\}(s) = \frac{Q(\rho)}{1-\rho}\frac{s^{\rho}\mathcal{L}\{\zeta(t)\}(s) - s^{\rho-1}\zeta(0)}{s^{\rho}+\gamma_{\rho}}. $$

Lemma 1.2.26 [12]

The Laplace transform of the Hattaf-fractional differential operator (1.4) is

$$ \mathcal{L}\left\{ {}_{0}^{C}D^{\rho,\omega}\zeta(t)\right\}(s) = \frac{1}{1-\rho}\frac{s^{\omega}\mathcal{L}\{\zeta(t)\}(s) - s^{\omega-1}\zeta(0)}{s^{\omega}+\gamma_{\rho}}. $$

Lemma 1.2.27 [12]

The Laplace transform of the Hattaf-fractional integration operator (1.5) is

$$\mathcal{L}\left\{ {}_{0}\mathcal{J}^{\rho,\omega}\zeta(t)\right\}(s) = (1-\rho)\mathcal{L}\{\zeta(t)\}(s) + \frac{\rho}{s^{\omega}}\mathcal{L}\{\zeta(t)\}(s).$$

Lemma 1.2.28 [12]

For Hattaf-fractional differential operator (1.4) and Hattaf-fractional integration operator (1.5),

$${}_{a}^{C}D^{\rho,\omega}\ {}_{a}\mathcal{J}^{\rho,\omega}\zeta(\lambda) = \zeta(\lambda).$$

In the following, some properties for the fractional differential operator (1.4) and fractional integral operator (1.5) are proven.

Lemma 1.2.29

The Hattaf-fractional derivative (1.4) can be written as

$${}_{a}^{C}D^{\rho,\omega}\zeta(t) = \frac{1}{1-\rho}\sum_{k=0}^{\infty}\left(-\gamma_{\rho}\right)^{k}\ {}_{a}^{RL}I^{\omega k+1}\dot{\zeta}(s), \qquad 0 < k < \infty.$$

Proof.

$${}_{a}^{C}D^{\rho,\omega}\zeta(t) = \frac{1}{1-\rho}\int_{a}^{t}\dot{\zeta}(s)E_{\omega}\left(-\gamma_{\rho}(t-s)^{\omega}\right)ds$$

$$= \frac{1}{1-\rho}\int_{a}^{t}\dot{\zeta}(s)\sum_{k=0}^{\infty}\left(-\frac{\rho}{1-\rho}\right)^{k}\frac{(t-\rho)^{\omega k}}{\Gamma(\omega k+1)}ds$$

$$= \frac{1}{1-\rho}\sum_{k=0}^{\infty}\left(\frac{-\rho}{1-\rho}\right)^{k}\frac{1}{\Gamma(\omega k+1)}\int_{a}^{t}\dot{\zeta}(s)(t-s)^{\omega k}\,ds$$

$$= \frac{1}{1-\rho}\sum_{k=0}^{\infty}\left(-\gamma_{\rho}\right)^{k}\ {}_{a}^{RL}I^{\omega k+1}\dot{\zeta}(t)\,. \qquad \blacksquare$$

Lemma 1.2.30

Let $0 < \rho < 1, \omega > 0$. Then

$$ {}_a\mathcal{J}^{\rho,\omega}\, {}^{C}_{a}D^{\rho,\omega}\zeta(t) = \zeta(t) - \zeta(a). $$

Proof.

Since

$$ {}_a\mathcal{J}^{\rho,\omega}\, {}^{C}_{a}D^{\rho,\omega}\zeta(t) = (1-\rho)\, {}^{C}_{a}D^{\rho,\omega}\zeta(t) + \rho\, {}^{RL}_{a}I^{\omega}\, {}^{C}_{a}D^{\rho,\omega}\zeta(t). $$

From Lemma 1.2.29 we have,

$$ {}_a\mathcal{J}^{\rho,\omega}\, {}^{C}_{a}D^{\rho,\omega}\zeta(t) $$
$$ = \sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^{k}\, {}^{RL}_{a}I^{\omega k+1}\dot{\zeta}(t) + {}^{RL}_{a}I^{\omega}\frac{\rho}{1-\rho}\sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^{k}\, {}^{RL}_{a}I^{\omega k+1}\dot{\zeta}(t). $$

By using Lemma 1.2.14, it follows

$$ \mathcal{J}^{\rho,\omega}\ {}^{C}D^{\rho,\omega}\zeta(t) = \sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^{k}\, {}^{RL}_{a}I^{\omega k+1}\dot{\zeta}(t) + \frac{\rho}{1-\rho}\sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^{k}\, {}^{RL}_{a}I^{(1+k)\omega+1}\dot{\zeta}(t) $$
$$ = \sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^{k}\, {}^{RL}_{a}I^{\omega k+1}\dot{\zeta}(t) - \sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^{k+1}\, {}^{RL}_{a}I^{(1+k)\omega+1}\dot{\zeta}(t) $$
$$ = \int_a^t \dot{\zeta}(s)ds = \zeta(t) - \zeta(a). $$ ■

Lemma 1.2.31

Let $0 < \rho < 1, \omega > 0$ and $\zeta \in PC[a,b]$. Then

$$ {}_a\mathcal{J}^{\rho,\omega}\, {}^{C}_{a}D^{\rho,\omega}\zeta(t) = \zeta(t) - \zeta(a) - \sum_{i=1}^{p}\Delta\zeta(t_i) $$

for $i = 1,2,\dots,p$, $\Delta\zeta(t_i) = \zeta(t_i^+) - \zeta(t_i^-)$ and $t \in [a,b]$.

Proof.

Using the same technique as in Lemma (1.3.30), we get

$$ {}_a\mathcal{J}^{\rho,\omega}\, {}^{C}_{a}D^{\rho,\omega}\zeta(t) = \int_a^t \dot{\zeta}(s)ds. $$

$$= \zeta(t) - \zeta(a) - \sum_{i=1}^{p} \Delta\zeta(t_i)$$

for $i = 1,2,\dots,p$ and $t \in [a,b]$. ■

1.3 Cauchy Problem

In this section, we recall the concept of the Cauchy problem because of its importance in solving differential equations.

Let $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{X} \to \mathbb{X}$ be a linear operator, where $\mathbb{X}$ be a Banach space. The abstract Cauchy problem for $\mathcal{A}$ and $y \in \mathbb{X}$ with initial condition y_0 consists of finding a solution $y(t)$ to the initial value problem

$$\begin{cases} \dot{y}(t) = \mathcal{A}y(t) + f(t), \ \ J = [0, \Upsilon] \\ y(0) = y_0. \end{cases} \tag{1.6}$$

where $f: J \to \mathbb{X}$. $\mathcal{A}$ is infinitesimal generator of a C_0 $-$semigroup $\mathcal{T}(t)$.

Definition 1.3.1 [33]

A function $y: J \to \mathbb{X}$ is a **classical solution** (solution) of (1.6) on J if , $y(t) \in C^1(J;\mathbb{X}), y(t) \in D(\mathcal{A})$, for all $t \in J$ and y satisfies (1.6) on J.

Let y be a solution of (1.6). Then the function $q: J \to \mathbb{X}$ defined as $q(s) = \mathcal{T}(t - s)y(s)$ is differentiable for $0 < s < t$ and

$$\frac{dq}{ds} = -\mathcal{A}\mathcal{T}(t-s)y(s) + \mathcal{T}(t-s)\dot{y}(s)$$

$$= -\mathcal{A}\mathcal{T}(t-s)y(s) + \mathcal{T}(t-s)\mathcal{A}y(s) + \mathcal{T}(t-s)f(s)$$

$$= \mathcal{T}(t-s)f(s). \tag{1.7}$$

If $f \in \mathrm{L_p}(J;\mathbb{X})$ then, $\mathcal{T}(t-s)f(s)$ is integrable and integrating (1.7) from 0 to t, yields

$$y(t)=\mathcal{T}(t)y_0+\int_0^t \mathcal{T}(t-s)\,f(s)ds. \tag{1.8}$$

The right-hand side of (1.8) is continuous for any $\mathrm{f} \in \mathrm{L_p}(J;\mathbb{X})$. [32]

It is possible to generalize the solution of (1.6) (mild solution) by removing differentiability condition which defined as follows:

Definition 1.3.2 [33]

Let $\mathcal{T}(t)$ be C_0 $-$semigroup generated by $\mathcal{A}$. Let $\mathrm{f} \in \mathrm{L_p}(J;\mathbb{X})$ and $y_0 \in \mathbb{X}$. The function $y \in C(J;\mathbb{X})$ given by (1.8) is the **mild solution** of System (1.6).

The following example shows that the continuity of f, in general, is not a sufficient condition for the existence of solutions to the Cauchy problem (1.6) for $y_0 \in D(\mathcal{A})$.

Example 1.3.3 [33]

Let $\mathcal{T}(t)$ be C_0 $-$semigroup generated by $\mathcal{A}$, and let $z \in \mathbb{X}$ be such that for any $t \geq 0, \mathcal{T}(t)\, z \notin D(\mathcal{A})$. Suppose that $f(t)=\mathcal{T}(t)z$. Then $f(t)$ is continuous for $t \geq 0$. Consider the initial value problem

$$\begin{cases} \dot{y}(t)=\mathcal{A}y(t)+f(t), \;\; J=[0,\Upsilon] \\ y(0)=0. \end{cases} \tag{1.9}$$

Then the mild solution of (1.9) is

$$y(t)=\int_0^t \mathcal{T}(t-s)\,\mathcal{T}(s)zds$$

$$= \int_0^t \mathcal{T}(t)\, zds$$

$$= t\mathcal{T}(t)z$$

which is not differentiable.

1.4 Controllability and Observability

In this section, we recall some fundamental definitions and lemmas for controllability and observability concepts in finite and infinite dimensional spaces.

Controllability

A linear control dynamical system in finite dimensional space that can be represented by the mathematical model

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t), & t \in J = [0, \Upsilon] \\ y(0) = y_0, \end{cases} \tag{1.10}$$

where $y(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are state and control vectors respectively, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices.

System (1.10) is said to be **controllable** on J, if for any pair of vectors $y_0, y_1 \in \mathbb{R}^n$, there exists a control $u \in C(J; \mathbb{R}^m)$ such that the solution of System (1.10) with given initial condition satisfies $y(\Upsilon) = y_1$ [37]. Since the solution of the System (1.10) is

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}\, Bu(s)ds,$$

then System (1.10) will be controllable on J if for any given $y_1 \in \mathbb{R}^n$, there exists a control u such that

$$y_1 = e^{A\Upsilon}y_0 + \int_0^{\Upsilon} e^{A(\Upsilon-s)}\, Bu(s)ds.$$

Theorem 1.4.1 [33]

The linear System (1.10) is controllable if and only if the controllability Grammian matrix

$$\mathcal{W} = \int_0^{\Upsilon} e^{A(\Upsilon-s)}BB^*e^{A^*(\Upsilon-s)}ds$$

is nonsingular.

Theorem 1.4.2 [37]

The linear System (1.10) is controllable if and only if the rank of the controllability matrix $[B\ AB\ \ A^2B\ \ ...\ A^{n-1}B]_{n\times nm}$ is n.

Example 1.4.3

Consider the linear System

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t)\,, & t \in [0,1] \\ y(0) = y_0 \end{cases}$$

where $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.Then the Grammian matrix

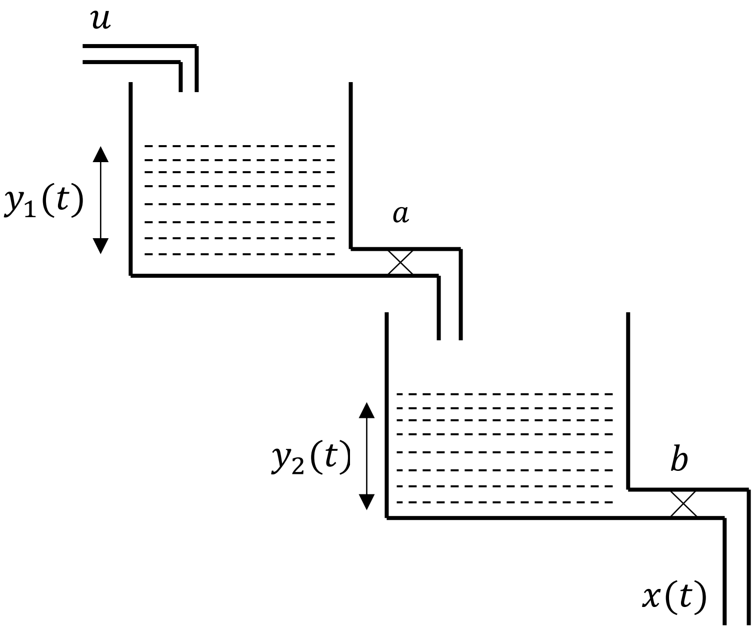
$$\mathcal{W} = \int_0^1 \begin{bmatrix} 2e^{2-2s}+1 \\ e^{1-s}+2 \end{bmatrix} [2e^{2-2s}+1 \quad e^{1-s}+2]ds$$

$$= \begin{bmatrix} 67.376 & 29.22 \\ 29.22 & 14.067 \end{bmatrix}$$

which is nonsingular, then by Theorem (1.4.1), the system is controllable.

Example 1.4.4 [38]

Consider the two tanks problem



Model (1)

Let $y_1(t)$ be denote the level of water in Tank 1, and $y_2(t)$ be denote the level of water in Tank 2. The outflow rates from Tank 1 and Tank 2 are denoted by a and b, respectively. Let u be denote the system's water supply. Then the mathematical model of the system on the time interval $[0,1]$ as follows:

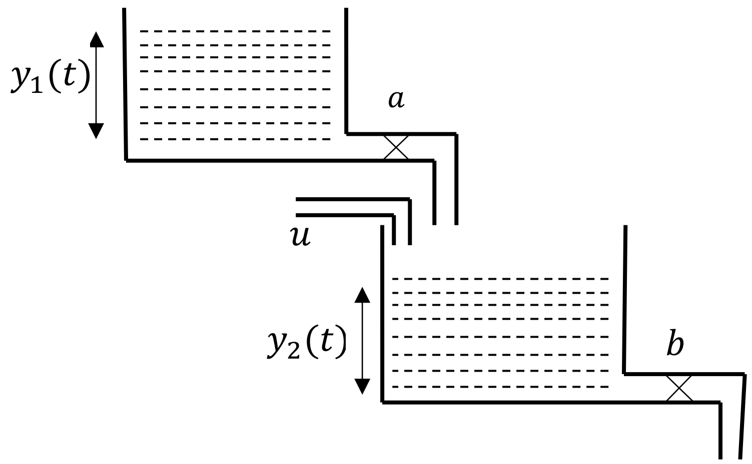
$$\dot{y}(t) = Ay(t) + Bu(t)$$

where $A = \begin{bmatrix} -a & 0 \\ a & -b \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The matrix $[B\ AB] = \begin{bmatrix} 1 & -a \\ 0 & a \end{bmatrix}$ has full rank, then by Theorem (1.4.2) the system is controllable.

Model (2)

Now, if we consider the two Tank problems as following figure:



Then the mathematical model of the system on the time interval $[0,1]$ as follows:

$$\dot{y}(t) = Ay(t) + Bu(t)$$

where $A = \begin{bmatrix} -a & 0 \\ a & -b \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The controllability matrix $[B\ AB] = \begin{bmatrix} 0 & 0 \\ 1 & -b \end{bmatrix}$ hasn't full rank, then by Theorem (1.4.2), the system is not controllable.

On the other hand, the mathematical model of linear control dynamical system in infinite dimensional space can be written as

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t), & t \in J = [0, \Upsilon] \\ y(0) = y_0 \end{cases} \tag{1.11}$$

where y takes values in a Banach space $\mathbb{X}$ and the control function u takes values in a Banach space U. The operator $A: D(A) \subset \mathbb{X} \to \mathbb{X}$ is a closed, linear and densely defined, but not necessarily bounded operator and $B: U \to \mathbb{X}$ is a bounded linear operator.

For any $y_0 \in \mathbb{X}$, the function $y \in C(J; \mathbb{X})$ given by [33]

$$y(t) = \mathcal{T}(t)y_0 + \int_0^t \mathcal{T}(t-s)Bu(s)ds\,, \qquad t \in J = [0, \Upsilon]$$

is called the mild solution of System (1.11), where $\mathcal{T}(t):\mathbb{X}\to\mathbb{X}$ is the C_0 −semigroup generated by the operator $\mathcal{A}$.

In (1977) [39], Triggiani proved that if $\mathcal{A}$ generates a compact C_0-semigroup $\mathcal{T}(t)$, then the linear system could never be exact controllable in an infinite dimensional space.

Definition 1.4.5 [40]

The linear System (1.11) is said to be **approximately controllable** on the interval J , if for given $\epsilon > 0$ and two arbitrary initial and final points y_0 and y_1 in $\mathbb{X}$, there exists an admissible control $u(t)$ on J steering y_0, along a trajectory (solution) $y(t)$ of System (1.11) to ϵ −neighbourad of y_1 such that $\|y(\Upsilon)-y_1\|_X \le \epsilon$.

Definition 1.4.6 [40]

The set $\mathcal{K}_\Upsilon(f)$ defined by

$$\mathcal{K}_\Upsilon(f)=\{\,y(\Upsilon;u),u(t)\in U\},t\in[0,\Upsilon]$$

is called the **reachable set**, which consists of all possible final states.

Definition 1.4.7 [33]

The linear System (1.11) is said to be:

1. **Exact controllable** on $J=[0,\Upsilon]$ if
$$\mathcal{K}_\Upsilon(f)=\mathbb{X}.$$
2. **Approximate controllable** on $J=[0,\Upsilon]$ if
$$\overline{\mathcal{K}_\Upsilon(f)}=\mathbb{X}.$$

Observability

A linear control dynamical system with output in finite dimensional space can be represented by the mathematical model

$$\begin{cases} \dot{y}(t) = Ay(t), & t \in J = [0, \Upsilon] \\ y(0) = y_0, \\ x(t) = Cy(t), \end{cases} \tag{1.12}$$

where the vector $y(t) \in \mathbb{R}^n$ is state vector, $x(\mathrm{t}) \in \mathbb{R}^n$ is the output vector, $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$ are constant matrices. The System is observable over the time interval J if it possible to determine uniquely the initial state $y(0) = y_0$ from knowledge of the output $x(t)$ over J.

Theorem 1.4.8 [41]

The linear dynamical System (1.12) is observable over the interval J if and only if one of the following statements is hold:

(I) The $n \times n$ observability Gramian matrix

$$\mathcal{W}_{ob} = \int_0^{\Upsilon} e^{A^* s} C^* C e^{As} ds$$

is nonsingular.

(II) The observability matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{nm\times n}$$

has full column rank. i.e.,

$$Rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{nm\times n} = n.$$

(III) For each eigenvalue λ of A, we have

$$Rank \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}_{(n+m)\times n} = n.$$

Example. 1.4.9

Consider the system

$$\begin{cases} \dot{y}(t) = Ay(t), & t \in J = [0,1], \\ y(0) = y_0, \\ x(t) = Cy(t), \end{cases}$$

where $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $C = [1 \quad 1]$.

The observability Grammian matrix

$$\mathcal{W}_{ob} = \int_0^1 \begin{bmatrix} e^{2s} + 1 \\ e^{2s} + 1 \end{bmatrix} [\, e^{2s} + 1 \quad e^{2s} + 1] ds$$

$$= \begin{bmatrix} 20.7886 & 20.7886 \\ 20.7886 & 20.7886 \end{bmatrix}$$

which is singular. Then by Theorem (1.4.8) (I), the system is not observable.

Example 1.4.10

Consider the model (1) of the two-tanks system in Example (1.4.4), with the output $x(t) = Cy(t)$

where $C = [0 \quad b]$.

The observability matrix

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & b \\ ab & -b^2 \end{bmatrix}$$

has full rank. Then by Theorem (1.4.8) (II), the system is observable.

Theorem 1.4.11 [41] "Theorem of Duality"

The linear System (1.10) is controllable if and only if the system

$$\begin{cases} \dot{y}(t) = -A^* y(t), \quad y(0) = y_0, \quad t \in J = [0, \Upsilon] \\ \\ \quad x(t) = B^* y(t) \end{cases}$$

is observable.

Chapter Two

Controllability and Observability of AB-Fractional Control Dynamical Systems in Finite Dimensional Space

This chapter discusses the controllability and the observability of AB-fractional control dynamical systems in $\mathbb{R}^{\boldsymbol{n}}$.

The First Section investigates the controllability of non-linear dynamical systems with control delay using the Gramian matrix and Schauder fixed point theorem.

Section Two presents more than one criterion for the observability of linear dynamical systems. Additionally, the duality between controllability and observability has been proven.

2.1 Controllability Results for AB-Fractional Nonlinear Dynamical Systems with Control Delay

In 2020 J. Sheng et al. [19] used the fixed point technique to prove the following nonlinear AB-fractional dynamical system is controllable

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = Ay(t) + B\mathcal{u}(t) + f(t, y(t), \mathcal{u}(t)), t \in I, \\ y(0) = y_0 \end{cases}$$

where ${}^{ABC}D^{\rho}$ is AB-fractional derivative of order ρ, $0 < \rho < 1$, $y(\mathrm{t}) \in \mathbb{R}^n$, $\mathcal{u}(\mathrm{t}) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $b > 0$, $I = [0, b]$ and the function

$f : I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{\boldsymbol{n}}$ is continuous.

Our work aims to study the controllability of the AB-fractional linear dynamical system with control delay

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = Ay(t) + B\mathcal{u}(\hbar(t)), t \in J = [0, \Upsilon] \\ y(0) = y_0, \end{cases} \tag{2.1}$$

and AB-fractional nonlinear dynamical system with control delay

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = Ay(t) + B\mathcal{u}\big(\hbar(t)\big) + f(t, y(t), \mathcal{u}(\mathrm{t})), \qquad t \in J, \\ \qquad\qquad y(0) = y_0, \end{cases} \tag{2.2}$$

where $\Upsilon > 0,\ 0 < \rho < 1$. Here $y(t) \in \mathbb{R}^n$ and $\mathcal{u}(t) \in \mathbb{R}^m$, $y, \mathcal{u}$ are continuous vector valued functions, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are constant matrices, the function $f: J \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous.
The matrix $I - (1-\rho)A$ is nonsingular, and $A_\rho = [I - (1-\rho)A\,]^{-1}$.
. $\hbar: J \to \mathbb{R}$ is strictly increasing in J and twice continuously differentiable. In addition, $\hbar(t) \le t, h(\Upsilon) = \Upsilon$ and $|\mathcal{u}(\hbar(t))| \le |\mathcal{u}(t)|$. Define the time lead function $\sigma: [\hbar(0), \hbar(\Upsilon)] \to J$ such that $\sigma\big(\hbar(t)\big) = t$ for $t \in J$. Assume that $l > 0$ is given and $u: [-l, \Upsilon] \to \mathbb{R}^m$, u_t denote the function defined on $[-l, 0]$ as $\mathcal{u}_t(\wp) = \mathcal{u}(t + \wp)$ for $t \in J$ and $\wp \in [-l, 0]$.

Lemma 2.1.1 [42]

Suppose that continuous function $h: D \times \mathbb{R}^n \to \mathbb{R}^m$ satisfies $\lim_{\|w\|\to\infty} \frac{\|h(v,w)\|}{\|w\|} = 0$ uniformly in $v \in D$ where D is a bounded subset of $\mathbb{R}$, then for each pair of constants a and b, there exists a positive constant e such that if $\|w\| \le e$, then $a\|h(v,w)\| + b \le e$, for all $v \in D$.

In the following, the controllability of The Linear System (2.1) will be discussed.

Firstly, we introduce the solution of System (2.1) in the next theorem.

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Theorem 2.1.2

The solution of System (2.1) given by

$$y(t) = \begin{cases} A_\rho E_\rho\left(\rho A A_\rho t^\rho\right) y_0 + (1-\rho) A_\rho B u(\hbar(t)) \\ +\rho A_\rho^2 \int_0^{\hbar(t)} (t-\sigma(\xi))^{\rho-1} E_{\rho,\rho}\left(\rho A A_\rho (t-\sigma(\xi))^\rho\right) \\ \qquad \times B u(\xi) \dot{\sigma}(\xi) d\xi + G(t) \end{cases} \quad (2.3)$$

where

$$G(t) = \rho A_\rho^2 \int_{\hbar(0)}^0 (t-\sigma(\xi))^{\rho-1} E_{\rho,\rho}\left(\rho A A_\rho (t-\sigma(\xi))^\rho\right) B u_0(\xi) \dot{\sigma}(\xi) d\xi.$$

Poof. By taking Laplace transformation of the both sides of System (2.1)

$$\frac{1}{1-\rho} \cdot \frac{s^\rho Y(s) - s^{\rho-1} y_0}{s^\rho + \frac{\rho}{1-\rho}} = AY(s) + \mathcal{L}\{Bu(\hbar(t))\}(s)$$

where $Y(s) = \mathcal{L}\{y(t)\}(s)$,

then

$$s^\rho Y(s) - s^{\rho-1} y_0 \\ = [(1-\rho)s^\rho + \rho] A Y(s) + [(1-\rho)s^\rho + \rho] \mathcal{L}\{Bu(\hbar(t))\}(s)$$

it follows

$$[(I-(1-\rho)A)s^\rho - \rho A] Y(s) = s^{\rho-1} y_0 + [(1-\rho)s^\rho + \rho] \mathcal{L}\{Bu(\hbar(t))\}(s).$$

By substitute A_ρ, we have

$$\left[A_\rho^{-1} s^\rho - \rho A\right] Y(s) = s^{\rho-1} y_0 + [(1-\rho)s^\rho + \rho] \mathcal{L}\{Bu(\hbar(t))\}(s)$$

$$\left(s^\rho I - \rho A A_\rho\right) A_\rho^{-1} Y(s) = s^{\rho-1} y_0 + [(1-\rho)s^\rho + \rho] \mathcal{L}\{Bu(\hbar(t))\}(s),$$

therefore

$$Y(s) = s^{\rho-1}A_\rho\left(s^\rho I - \rho A A_\rho\right)^{-1} y_0 + \left(s^\rho I - \rho A A_\rho\right)^{-1} A_\rho$$
$$\times\left[(1-\rho)s^\rho + \rho\right]\mathcal{L}\{Bu(\hbar(t))\}(s)$$

$$= A_\rho s^{\rho-1}(s^\rho I - \rho A A_\rho)^{-1} y_0 + (1-\rho)s^\rho(s^\rho I - \rho A A_\rho)^{-1}A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s)$$
$$+\ \rho(s^\rho I - \rho A A_\rho)^{-1}A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s).$$

Note that

$$s^\rho\left(s^\rho I - \rho A A_\rho\right)^{-1} = \left(s^\rho I - \rho A A_\rho + \rho A A_\rho\right)\left(s^\rho I - \rho A A_\rho\right)^{-1}$$
$$= \left(s^\rho I - \rho A A_\rho\right)\left(s^\rho I - \rho A A_\rho\right)^{-1} + \rho A A_\rho\left(s^\rho I - \rho A A_\rho\right)^{-1}$$
$$= I + \rho A A_\rho\left(s^\rho I - \rho A A_\rho\right)^{-1}.$$

Therefore

$$Y(s) = A_\rho s^{\rho-1}(s^\rho I - \rho A A_\rho)^{-1} y_0$$
$$+ (1-\rho)\left(I + \rho A A_\rho\left(s^\rho I - \rho A A_\rho\right)^{-1}\right)A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s)$$
$$+\ \rho(s^\rho I - \rho A A_\rho)^{-1}A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s)$$
$$= A_\rho s^{\rho-1}(s^\rho I - \rho A A_\rho)^{-1} y_0 + (1-\rho)A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s)$$
$$+ (1-\rho)\rho A {A_\rho}^2\left(s^\rho I - \rho A A_\rho\right)^{-1}\mathcal{L}\{Bu(\hbar(t))\}(s)$$
$$+\ \rho(s^\rho I - \rho A A_\rho)^{-1}A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s)$$
$$= A_\rho s^{\rho-1}(s^\rho I - \rho A A_\rho)^{-1} y_0 + (1-\rho)A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s)$$
$$+\left[(1-\rho)A + {A_\rho}^{-1}\right]\rho {A_\rho}^2\ \rho(s^\rho I - \rho A A_\rho)^{-1}\mathcal{L}\{Bu(\hbar(t))\}(s),$$

therefore,

$$Y(s) = A_\rho s^{\rho-1}\left(s^\rho I - \rho A A_\rho\right)^{-1} y_0 + (1-\rho)A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s)$$
$$+\rho {A_\rho}^2\left(s^\rho I - \rho A A_\rho\right)^{-1}\mathcal{L}\{Bu(\hbar(t))\}(s). \qquad (2.4)$$

Now, by taking inverse of Laplace transform for (2.4) we get,

$$\mathcal{L}^{-1}\{Y(s);t\}$$
$$= A_\rho\mathcal{L}^{-1}\{s^{\rho-1}(s^\rho I - \rho A A_\rho)^{-1};t\}y_0 + (1-\rho)A_\rho Bu(\hbar(t))$$
$$+\rho {A_\rho}^2\left[\mathcal{L}^{-1}\{(s^\rho I - \rho A A_\rho)^{-1};t\} * Bu(\hbar(t))\right].$$

From Lemma (1.2.8) we have

$$\mathcal{L}^{-1}\{s^{\rho-1}(s^{\rho}I-\rho AA_{\rho})^{-1};t\}=E_{\rho}(\rho AA_{\rho}t^{\rho})$$

$$\mathcal{L}^{-1}\{(s^{\rho}I-\rho AA_{\rho})^{-1};t\}=t^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}t^{\rho}),$$

therefore,

$$\begin{aligned}y(t)&=A_{\rho}E_{\rho}(\rho AA_{\rho}t^{\rho})y_0+(1-\rho)A_{\rho}Bu(\hbar(t))\\&\quad+\rho {A_{\rho}}^{2}\left[\left(t^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}t^{\rho})\right)*Bu(\hbar(t))\right]\\&=A_{\rho}E_{\rho}(\rho AA_{\rho}t^{\rho})y_0+(1-\rho)A_{\rho}Bu(\hbar(t))\\&\quad+\rho {A_{\rho}}^{2}\int_0^t(t-\xi)^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}(t-\xi)^{\rho})Bu(\hbar(\xi))d\xi.\end{aligned}$$

We use the time lead function $\sigma(t)$ to write the above solution in the following form

$$\begin{aligned}y(t)&=A_{\rho}E_{\rho}(\rho AA_{\rho}t^{\rho})y_0+(1-\rho)A_{\rho}Bu(\hbar(t))\\&\quad+\rho A_{\rho}^{2}\int_{\hbar(0)}^{\hbar(t)}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}(t-\sigma(\xi))^{\rho})Bu(\xi)\dot{\sigma}(\xi)d\xi\end{aligned}$$

$$\begin{aligned}&=A_{\rho}E_{\rho}(\rho AA_{\rho}t^{\rho})y_0+(1-\rho)A_{\rho}Bu(\hbar(t))\\&\quad+\rho A_{\rho}^{2}\int_{\hbar(0)}^{0}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}(t-\sigma(\xi))^{\rho})Bu(\xi)\dot{\sigma}(\xi)d\xi\\&\quad+\rho A_{\rho}^{2}\int_{0}^{\hbar(t)}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}(t-\sigma(\xi))^{\rho})Bu(\xi)\dot{\sigma}(\xi)d\xi.\end{aligned}$$

Therefore,

$$y(t)=\begin{cases}A_{\rho}E_{\rho}(\rho AA_{\rho}t^{\rho})y_0+(1-\rho)A_{\rho}Bu(\hbar(t))\\+\rho A_{\rho}^{2}\int_{0}^{\hbar(t)}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}(t-\sigma(\xi))^{\rho})\\\qquad\times Bu(\xi)\dot{\sigma}(\xi)d\xi+G(t).\end{cases}$$

Note: The completely state of the Linear System (2.1) at $t \in J$ is the set $g(t) = \{y(t), u_t\}$.

Definition 2.1.3

The System (2.1) is said to be controllable on J if for every complete state $g(0)$ and every $y_1 \in \mathbb{R}^n$ there exists a control u(t) defined on J such that the solution $y(t)$ of the system satisfies $y(\Upsilon) = y_1$.

Theorem 2.1.4

The AB-fractional linear dynamical system with control delay (2.1) is controllable on the interval J if the matrix $\mathcal{W}_1$ is nonsingular, such that

$$\mathcal{W}_1 = \frac{1-\rho}{\Gamma(\rho)} BB^* \dot{\sigma}(\Upsilon) + \rho A_\rho \mathcal{W}$$

where

$$\mathcal{W} = \int_0^{\Upsilon} \left(\Upsilon - \sigma(\xi)\right)^{\rho-1} \left[E_{\rho,\rho}(\rho A A_\rho \left(\Upsilon - \sigma(\xi)\right)^{\rho} B\dot{\sigma}(\xi)\right]$$

$$\times \left[E_{\rho,\rho}(\rho A A_\rho \left(\Upsilon - \sigma(\xi)\right)^{\rho} B\dot{\sigma}(\xi)\right]^* d\xi$$

is the controllability-Grammian matrix.

Proof. By hypothesis that $\mathcal{W}_1$ is nonsingular and since the matrix A_ρ is also nonsingular, then the control function may be defined as follows

$$u(t) = B^* E^*_{\rho,\rho}(\rho A A_\rho \left(\Upsilon - \sigma(t)\right)^{\rho} \dot{\sigma}(t) \mathcal{M} {A_\rho}^{-1} {\mathcal{W}_1}^{-1}$$

where

$$\mathcal{M} = y_1 - A_\rho E_\rho\left(\rho A A_\rho t^\rho\right) y_0 - G(\Upsilon).$$

Therefore

$$y(\Upsilon) = A_\rho E_\rho\big(\rho A A_\rho \Upsilon^\rho\big) y_0 + (1-\rho) A_\rho B u(\Upsilon) + \rho A_\rho^2$$

$$\times \int_0^\Upsilon (\Upsilon - \sigma(\xi))^{\rho-1} E_{\rho,\rho}\big(\rho A A_\rho \big(\Upsilon - \sigma(\xi)\big)^\rho\big) B u(\xi) \dot{\sigma}(\xi) d\xi + G(\Upsilon)$$

$$= A_\rho E_\rho\big(\rho A A_\rho \Upsilon^\rho\big) y_0 + (1-\rho) A_\rho B B^* E_{\rho,\rho}^*(\rho A A_\rho (\Upsilon - \Upsilon)^\rho \dot{\sigma}(\Upsilon)$$

$$\times \mathcal{M} {A_\rho}^{-1} {\mathcal{W}_1}^{-1} + \rho A_\rho^2 \int_0^\Upsilon (\Upsilon - \sigma(\xi))^{\rho-1} E_{\rho,\rho}\big(\rho A A_\rho \big(\Upsilon - \sigma(\xi)\big)^\rho\big) B B^*$$

$$\times {E^*}_{\rho,\rho}(\rho A A_\rho \big(\Upsilon - \sigma(\xi)\big)^\rho \dot{\sigma}(\xi) \mathcal{M} {A_\rho}^{-1} {\mathcal{W}_1}^{-1} \dot{\sigma}(\xi) d\xi + G(\Upsilon)$$

$$= y_1.$$

Therefore, The System (2.1) is controllable. ■

In the following, the controllability of The Nonlinear System (2.2) will be discussed.

Assume $\mathcal{X}$ is Banach space of all continuous functions $(y, u): J \times J \to \mathbb{R}^n \times \mathbb{R}^m$ with the norm $\|(y, u)\| = \|y\| + \|u\|$, when $\|y\| = \sup\{|y(t)|; t \in J\}$ and $\|u\| = \sup\{|u(t)|; t \in J\}$.

For each $(x, v) \in \mathcal{X}$, consider the linear system,

$$\begin{cases} {}^{ABC}D^\rho y(t) = A y(t) + B u\big(\hbar(t)\big) + f\big(t, x(t), v(t)\big), \ t \in J, \\ y(0) = y_0. \end{cases} \tag{2.5}$$

Then, the solution of System (2.5) given by

$$y(t)=\begin{cases} A_\rho E_\rho\big(\rho A A_\rho t^\rho\big)y_0+(1-\rho)A_\rho B u\big(\hbar(t)\big) \\ +\rho A_\rho^2\int_0^{\hbar(t)}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}\big(\rho A A_\rho\big(t-\sigma(\xi)\big)^\rho\big) \\ \quad\times Bu(\xi)\dot{\sigma}(\xi)d\xi\ +G(t) \\ +(1-\rho)A_\rho f\big(t,x(t),v(t)\big)+\rho A_\rho^2\int_0^t(t-\xi)^{\rho-1} \\ \quad\times E_{\rho,\rho}\big(\rho A A_\rho(t-\xi)^\rho\big)f\big(\xi,x(\xi),v(\xi)\big)d\xi. \end{cases} \tag{2.6}$$

The controllability Gramian matrix and the control function are described by

$$\mathcal{W}=\int_0^{\Upsilon}\big(\Upsilon-\sigma(\xi)\big)^{\rho-1}\big[E_{\rho,\rho}(\rho A A_\rho\big(\Upsilon-\sigma(\xi)\big)^\rho B\dot{\sigma}(\xi)\big]$$
$$\times\big[E_{\rho,\rho}(\rho A A_\rho\big(\Upsilon-\sigma(\xi)\big)^\rho B\dot{\sigma}(\xi)\big]^* d\xi,$$

and

$$u(t)=B^*E_{\rho,\rho}^*(\rho A A_\rho\big(\Upsilon-\sigma(t)\big)^\rho\dot{\sigma}(t)\psi {A_\rho}^{-1}{\mathcal{W}_1}^{-1},$$

where

$$\psi=\mathcal{M}-(1-\rho)A_\rho f\big(t,x(t),v(t)\big)$$
$$-\rho A_\rho^2\int_0^{\Upsilon}(\Upsilon-\xi)^{\rho-1}E_{\rho,\rho}\big(\rho A A_\rho(\Upsilon-\xi)^\rho\big)$$
$$\times f\big(\xi,x(\xi),v(\xi)\big)d\xi.$$

It is straightforward to demonstrate that the control $u(t)$ steers the Linear System (2.5) from the initial state y_0 to the final state y_1.

Theorem 2.1.5

Assume that the continuous function f satisfies the condition

$$\lim_{\|(y,u)\|\to\infty}\frac{\|f(t,y,u)\|}{\|(y,u)\|}=0, \tag{2.7}$$

uniformly in J, and the matrix $\mathcal{W}_1$ is nonsingular. Then the AB-fractional nonlinear dynamical system with control delay (2.2) is controllable on J.

Proof. Define the operator $\Omega: \mathcal{X} \to \mathcal{X}$ by

$$\Omega(x, v) = (y, u),$$

where

$$u(t) = B^* E^*_{\rho,\rho}(\rho A A_\rho \big(\Upsilon - \sigma(t)\big)^\rho \dot{\sigma}(t) \psi A_\rho{}^{-1} \mathcal{W}_1{}^{-1}$$

$$= \left\{ \begin{array}{c} B^* E^*_{\rho,\rho}(\rho A A_\rho \big(\Upsilon - \sigma(t)\big)^\rho \dot{\sigma}(t) A_\rho{}^{-1} \mathcal{W}_1{}^{-1} \\ \times \big[y_1 - A_\rho E_\rho\big(\rho A A_\rho t^\rho\big) y_0 - G(\Upsilon) \\ -(1-\rho) A_\rho f\big(t, x(t), v(t)\big) - \rho A_\rho^2 \\ \times \int_0^\Upsilon (\Upsilon - \xi)^{\rho-1} E_{\rho,\rho}\big(\rho A A_\rho (\Upsilon - \xi)^\rho\big) f\big(\xi, x(\xi), v(\xi)\big) d\xi \Big] \end{array} \right.$$

and

$$y(t) = \left\{ \begin{array}{c} A_\rho E_\rho\big(\rho A A_\rho t^\rho\big) y_0 + (1-\rho) A_\rho B u(\hbar(t)) \\ + \rho A_\rho^2 \int_0^{\hbar(t)} (t - \sigma(\xi))^{\rho-1} E_{\rho,\rho}\big(\rho A A_\rho \big(t - \sigma(\xi)\big)^\rho\big) \\ \times B u(\xi) \dot{\sigma}(\xi) d\xi + G(t) + (1-\rho) A_\rho f\big(t, x(t), v(t)\big) \\ + \rho A_\rho^2 \int_0^t (t - \xi)^{\rho-1} E_{\rho,\rho}\big(\rho A A_\rho (t - \xi)^\rho\big) \\ \times f\big(\xi, x(\xi), v(\xi)\big) d\xi. \end{array} \right.$$

Let

$$K = sup\|\dot{\sigma}(\xi)\|,$$

$$\eta = sup\|u_0(\xi)\|$$

$$a_1 = sup\big\|E_\rho\big(\rho A A_\rho t^\rho\big) y_0\big\|,$$

$$a_2 = sup\big\|E_{\rho,\rho}\big(\rho A A_\rho \big(\Upsilon - \sigma(\xi)\big)^\rho\big)\big\| K$$

$$\beta = sup\big\|E_{\rho,\rho}\big(\rho A A_\rho (\Upsilon - \xi)^\rho\big)\big\|,$$

$$a_3 = \max\{a_2, \beta\},$$

$$b_1 = (1-\rho)\|A_\rho\| + \Upsilon^\rho \|{A_\rho}^2\| a_3,$$

$$a = \max\{b_1\|B\|, 1\},$$

$$N = \int_{\hbar(0)}^{0} \left(\Upsilon - \sigma(\xi)\right)^{\rho-1} d\xi,$$

$$\theta = \rho \|{A_\rho}^2\| a_3 \|B\| N,$$

$$b_2 = a_3\|B^*\| \|{A_\rho}^{-1}\| \|{\mathcal{W}_1}^{-1}\| b_1,$$

$$c_1 = 4ab_2, \qquad c_2 = 4b_1,$$

$$d_1 = a_3\|B^*\| \|{A_\rho}^{-1}\| \|{\mathcal{W}_1}^{-1}\| \left(|y_1| + \|A_\rho\| a_1 + \eta\theta\right),$$

$$d_2 = 4\left(\|A_\rho\| a_1 + \eta\theta\right)$$

$$d_3 = 4ad_1, \quad d = \max\{d_2, d_3\}$$

$$\|f\| = sup\{\left|f\left(t, x(t), v(t)\right)\right|, t \in J\},$$

$$c = \max\{c_1, c_2\}.$$

Then

$$\begin{aligned} |u(t)| &\le a_2\|B^*\| \|{A_\rho}^{-1}\| \|{\mathcal{W}_1}^{-1}\| \Big[|y_1| + \|A_\rho\| a_1 + \rho \|{A_\rho}^2\| \eta a_2 \|B\| N \\ &\quad + \Big((1-\rho)\|A_\rho\| + \Upsilon^\rho \|{A_\rho}^2\| \beta\Big) \|f\|\Big] \\ &\le a_3\|B^*\| \|{A_\rho}^{-1}\| \|{\mathcal{W}_1}^{-1}\| \left(|y_1| + \|A_\rho\| a_1 + \eta\theta\right) \\ &\quad + a_3\|B^*\| \|{A_\rho}^{-1}\| \|{\mathcal{W}_1}^{-1}\| b_1 \|f\| \\ &= d_1 + b_2\|f\| \\ &= \frac{d_3}{4a} + \frac{c_1}{4a}\|f\|. \end{aligned}$$

It follows,

$$|u(t)| \le \frac{1}{4a}(d + c\,\|f\|). \tag{2.8}$$

and

$$|y(t)| \leq \|A_\rho\| a_1 + (1-\rho)\|A_\rho\| \|B\| |u(t)| + \Upsilon^\rho \|{A_\rho}^2\| a_2 \|B\| |u(t)|$$
$$+ \rho k \|{A_\rho}^2\| \eta a_2 \|B\| N + [(1-\rho)\|A_\rho\| + \Upsilon^\rho \|{A_\rho}^2\| \beta] \|f\|$$
$$\leq \frac{d_2}{4} + \frac{b_1}{4a} \|B\| (d + c\,\|f\|) + \frac{c_2}{4} \|f\|$$
$$\leq \frac{d}{2} + \frac{c}{2} \|f\|.$$

According to Lemma (2.1.1), there exists a positive constant r such that if $\|(y,u)\| \leq r$ then $c\|f\| + d < r$.

Thus if $\|x\| \leq \frac{r}{2}$ and $\|v\| \leq \frac{r}{2}$, it is follows that $c\|f\| + d \leq r$. From (2.8) we have $\|u\| \leq \frac{r}{2}$. Since $|y(t)| \leq \frac{1}{2}(d + c\,\|f\|)$, then $\|y\| \leq \frac{r}{2}$.

Define $\mathcal{X}_r = \left\{(x,v) \in \mathcal{X}: \|x\| \leq \frac{r}{2} \text{ and } \|v\| \leq \frac{r}{2}\right\}$. Then Ω maps $\mathcal{X}_r$ into itself.

Since f is continuous and uniformly bounded for all $t \in J$, then the operator Ω is continuous and uniformly bounded on J. Therefore, by continuity of the Mittag-Leffler function we have $\|u(t_1) - u(t_2)\|$, $\|y(t_1) - y(t_2)\|$ tends to zero when $t_1 \to t_2$ for all $t_1, t_2 \in J$ and for all $(y,u) \in \mathcal{X}$. Thus $\Omega(\mathcal{X}_r)$ is equicontinuous and therefore, the operator Ω is compact by Arzela-Ascoli theorem. Thus Ω is completely continuous. Because $\mathcal{X}_r$ is closed, bounded and convex, then by Schauder Fixed Point Theorem, we get Ω has a fixed point $(x,v) \in \mathcal{X}_r$ that is,

$$\Omega(x,v) = (x,v) = (y,u).$$

Therefore,

$$y(t)=\begin{cases} A_\rho E_\rho\big(\rho A A_\rho t^\rho\big)y_0+(1-\rho)A_\rho B u\big(\hbar(t)\big)+ \\ \rho A_\rho^2\displaystyle\int_0^{\hbar(t)}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}\big(\rho A A_\rho\big(t-\sigma(\xi)\big)^\rho\big) \\ \times Bu(\xi)\dot{\sigma}(\xi)d\xi+G(t)+\big(\rho A A_\rho(t-\xi)^\rho\big)(1-\rho) \\ \times A_\rho f\big(t,y(t),u(t)\big)+\rho A_\rho^2\displaystyle\int_0^{t}(t-\xi)^{\rho-1} \\ \times E_{\rho,\rho}\big(\rho A A_\rho\big(t-\sigma(\xi)\big)^\rho\big)f\big(s,y(\xi),u(\xi)\big)d\xi \end{cases}$$

consequently, $y(t)$ is the solution of System (2.2). Furthermore,

$$\mathrm{y}(\Upsilon)=\begin{cases} A_\rho E_\rho\big(\rho A A_\rho \Upsilon^\rho\big)y_0+\dfrac{(1-\rho)}{\Gamma(\rho)}BB^*\dot{\sigma}(\Upsilon)\mathcal{W}_1{}^{-1}\psi \\ +\rho A_\rho^2\displaystyle\int_0^{\Upsilon}(\Upsilon-\sigma(\xi))^{\rho-1}E_{\rho,\rho}\big(\rho A A_\rho\big(\Upsilon-\sigma(\xi)\big)^\rho\big) \\ \times BB^*E^*{}_{\rho,\rho}\big(\rho A A_\rho\big(\Upsilon-\sigma(\xi)\big)^\rho\big)\dot{\sigma}(\xi)A_\rho^{-1}\mathcal{W}_1{}^{-1}\psi d\xi \\ +(1-\rho)A_\rho f\big(t,y(\Upsilon),u(\Upsilon)\big)+\rho A_\rho^2 \\ \times\displaystyle\int_0^{\Upsilon}(\Upsilon-\xi)^{\rho-1}E_{\rho,\rho}\big(\rho A A_\rho(\Upsilon-\xi)^\rho\big) \\ f\big(\xi,y(\xi),u(\xi)\big)d\xi+G(\Upsilon) \end{cases}$$

$$=\begin{cases} A_\rho E_\rho\big(\rho A A_\rho \mathcal{T}^\rho\big)y_0+\dfrac{(1-\rho)}{\Gamma(\rho)}BB^*\dot{\sigma}(\Upsilon)\mathcal{W}_1{}^{-1}\psi \\ +\rho A_\rho\mathcal{W}\mathcal{W}_1{}^{-1}\psi+(1-\rho)A_\rho f\big(\Upsilon,y(\Upsilon),u(\Upsilon)\big) \\ +\rho A_\rho^2\displaystyle\int_0^{\Upsilon}(\Upsilon-\xi)^{\rho-1}E_{\rho,\rho}\big(\rho A A_\rho(\Upsilon-\xi)^\rho\big) \\ \times f\big(\xi,y(\xi),u(\xi)\big)d\xi+G(\Upsilon) \end{cases}$$

$$=y_1.$$

Therefore, The Nonlinear System (2.2) is controllable on J. ■

In particular, when f is integral operator, the controllability of Nonlinear System (2.2) discussed as follows.

Consider the nonlinear integrodifferential system

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = Ay(t) + B\mathcal{u}\big(\hbar(t)\big) \\ +f\left(t, y(t), \int_0^t g(t,s,y(s))ds\right), t \in J, \\ y(0) = y_0, \end{cases} \tag{2.9}$$

where $f: J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g: J \times J \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous function. Let $C_n(J)$ be Banach space of continuous $\mathbb{R}^n$ valued functions defined on J. For each $x \in C_n(J)$ consider the linear system

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = Ay(t) + B\mathcal{u}\big(\hbar(t)\big) \\ +f\left(t, y(t), \int_0^t g\big(t,s,x(s)\big)ds\right), t, s \in J, \\ y(0) = y_0. \end{cases} \tag{2.10}$$

Then, the solution of system (2.10) is

$$\mathrm{y}(t) = \begin{cases} A_{\rho}E_{\rho}\big(\rho A A_{\rho} t^{\rho}\big)y_0 + (1-\rho)A_{\rho}B\mathcal{u}\big(\hbar(t)\big) \\ +\rho A_{\rho}^2 \int_0^{\hbar(t)} (t-\sigma(\xi))^{\rho-1} \\ \times E_{\rho,\rho}\big(\rho A A_{\rho}\big(t-\sigma(\xi)\big)^{\rho}\big)B\mathcal{u}(\xi)\dot{\sigma}(\xi)d\xi \\ +G(t) + (1-\rho)A_{\rho}f\left(t, x(t), \int_0^t g\big(t,s,x(s)\big)ds\right) \\ +\rho A_{\rho}^2 \int_0^t (t-\xi)^{\rho-1}E_{\rho,\rho}\big(\rho A A_{\rho}(t-\xi)^{\rho}\big) \\ \times f\left(\xi, x(\xi), \int_0^{\xi} g(\xi,s,x(s))ds\right)d\xi. \end{cases} \tag{2.11}$$

for each $t, s \in J, x \in C_n(J)$.

Theorem 2.1.6

Let the matrix $\mathcal{W}_1$ be nonsingular and there exists a constant $L > 0$ such that the continuous function f satisfies the condition

$$\left| f\left(t, x(t), \int_0^t g\big(t, s, x(s)\big) ds \right) \right| \le L. \quad t, s \in J, x \in C_n(J) \qquad (2.12)$$

Then the AB-fractional nonlinear integrodifferential dynamical system with control delay (2.9) is controllable on J.

Proof. Define the operator $\overline{\Omega}: C_n(J) \to C_n(J)$ by

$$\overline{\Omega}\big(x(\mathrm{t})\big) = y(t) = \begin{cases} A_\rho E_\rho\big(\rho A A_\rho t^\rho\big) y_0 + (1-\rho) A_\rho B \mathcal{u}\big(\hbar(t)\big) \\ \qquad +\rho A_\rho^2 \int_0^{\hbar(t)} (t - \sigma(\xi))^{\rho-1} \\ \times E_{\rho,\rho}\big(\rho A A_\rho \big(t - \sigma(\xi)\big)^\rho\big) B \mathcal{u}(\xi) \dot{\sigma}(\xi) d\xi \\ +(1-\rho) A_\rho f\left(t, x(t), \int_0^t g\big(t, s, x(s)\big) ds \right) \\ +\rho A_\rho^2 \int_0^t (t-\xi)^{\rho-1} E_{\rho,\rho}\big(\rho A A_\rho (t-\xi)^\rho\big) \\ \quad \times f\left(\xi, x(\xi), \int_0^\xi g(\xi, s, x(s)) ds \right) d\xi, \end{cases}$$

where the control function $\mathcal{u}(t)$ defined as follows,

$$\mathcal{u}(t) = B^* E^*{}_{\rho,\rho}\big(\rho A A_\rho \big(\Upsilon - \sigma(t)\big)^\rho \dot{\sigma}(t) \psi' A_\rho{}^{-1} \mathcal{W}_1{}^{-1},$$

where

$$\psi' = \mathcal{M} - (1-\rho) A_\rho f\left(t, x(t), \int_0^t g(t, s, x(s)) ds \right)$$

$$-\rho A_\rho^2 \int_0^\Upsilon (\Upsilon - \xi)^{\rho-1} \times E_{\rho,\rho}\big(\rho A A_\rho (\Upsilon - \xi)^\rho\big)$$

$$\times f\left(\xi, x(\xi), \int_0^\xi g(\xi, s, x(s)) ds \right) d\xi.$$

Now, we define a closed convex subset

$$\varphi_r = \{x \in C_n(J) : \|x\| \leq r\}$$

where

$$r = \frac{d}{2} + \frac{c}{2}L$$

Therefore, $\overline{\Omega}$ maps φ_r into itself.

By the same technique in Theorem (2.1.5) we get $\overline{\Omega}$ has a fixed point $x \in \varphi_r$ that is,

$$\Omega(x) = x = y.$$

Therefore,

$$y(t) = \begin{cases} A_\rho E_\rho\left(\rho A A_\rho t^\rho\right) y_0 + (1-\rho) A_\rho B u(\hbar(t)) \\ +\rho A_\rho^2 \int_0^{\hbar(t)} (t-\sigma(\xi))^{\rho-1} E_{\rho,\rho}\left(\rho A A_\rho \left(t-\sigma(\xi)\right)^\rho\right) \\ B u(\xi)\dot{\sigma}(\xi) d\xi + G(t) \\ +(1-\rho) A_\rho f\left(t, x(t), \int_0^t g\left(t, s, y(s)\right) ds\right) \\ +\rho A_\rho^2 \int_0^t (t-\xi)^{\rho-1} E_{\rho,\rho}\left(\rho A A_\rho (t-\xi)^\rho\right) \\ \times f\left(\xi, x(\xi), \int_0^\xi g(\xi, s, y(s)) ds\right) d\xi \end{cases}$$

as the solution of system (2.9), and

$$y(\Upsilon)=\begin{cases} A_\rho E_\rho\left(\rho A A_\rho \Upsilon^\rho\right)y_0+\dfrac{(1-\rho)}{\Gamma(\rho)}BB^*\dot{\sigma}(\Upsilon)\mathcal{W}_1{}^{-1}\psi' \\ +\rho A_\rho^2\displaystyle\int_0^\Upsilon (\Upsilon-\sigma(\xi))^{\rho-1}\,E_{\rho,\rho}\left(\rho A A_\rho\left(\Upsilon-\sigma(\xi)\right)^\rho\right) \\ \times BB^*E^*{}_{\rho,\rho}\left(\rho A A_\rho\left(\Upsilon-\sigma(\xi)\right)^\rho\right)\dot{\sigma}(\xi)A_\rho^{-1}\mathcal{W}_1{}^{-1}\psi' d\xi \\ +(1-\rho)A_\rho f\left(\Upsilon, y(\Upsilon), \displaystyle\int_0^\Upsilon g\left(\Upsilon,s,y(s)\right)ds\right) \\ +\rho A_\rho^2\displaystyle\int_0^\Upsilon (\Upsilon-\xi)^{\rho-1}E_{\rho,\rho}\left(\rho A A_\rho(\Upsilon-\xi)^\rho\right) \\ \times f\left(\xi, y(\xi), \displaystyle\int_0^\xi g(\xi,s,y(s))ds\right)d\xi+G(\Upsilon) \end{cases}$$

$$=\begin{cases} A_\rho E_\rho\left(\rho A A_\rho \Upsilon^\rho\right)y_0+\dfrac{(1-\rho)}{\Gamma(\rho)}BB^*\dot{\sigma}(\Upsilon)\mathcal{W}_1{}^{-1}\psi' \\ +\rho A_\rho \mathcal{W}\mathcal{W}_1{}^{-1}\psi' \ +(1-\rho)A_\rho \\ \times f\left(\Upsilon, y(\Upsilon), \displaystyle\int_0^\Upsilon g\left(\Upsilon,s,y(s)\right)ds\right) \\ +\rho A_\rho^2\displaystyle\int_0^\Upsilon (\Upsilon-\xi)^{\rho-1}E_{\rho,\rho}\left(\rho A A_\rho(\Upsilon-\xi)^\rho\right) \\ \times f\left(\xi, y(\xi), \displaystyle\int_0^\xi g\left(\xi,s,y(s)\right)ds\right)d\xi+G(\Upsilon) \end{cases}$$

$$=y_1$$

Therefore, The Nonlinear System (2.9) is controllable on J. ■

Next, we give examples to illustrate our results.

Example 2.1.7

Consider the following AB-fractional nonlinear system with control delay:

$$\begin{cases} {}^{ABC}D^\rho y(t)=Ay(t)+B\mathcal{u}\left(\hbar(t)\right), \\ \quad +f\left(t,y(t),\mathcal{u}(\mathrm{t})\right), \qquad t\in[0,1] \\ y(0)=y_0, \end{cases} \tag{2.13}$$

where $\hbar: [0,1] \to \mathbb{R}$ defined as $\hbar(t) = 2t - 1$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\rho = 0.5$, and

$$f\big(t, y(t), u(t)\big) = \begin{pmatrix} 0 \\ e^t \cos y_1(t) + \sin u_2(t) \end{pmatrix}$$

Define

$$\sigma: [\hbar(0), \hbar(1)] \to [0,1], \sigma(t) = \frac{t+1}{2},$$

then

$$\dot{\sigma}(t) = \frac{1}{2}.$$

$$A_\rho = [I - (1-\rho)A\,]^{-1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix}\right]^{-1} = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix},$$

and

$$C = \rho A A_\rho = \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix}.$$

The Mittag-Leffler function matrix of the system is:

$$E_{\rho,\rho}\big(C\big(t - \sigma(\xi)\big)^\rho\big) = \sum_{k=0}^{\infty} \frac{C^k((t-\sigma(\xi))^\rho)^k}{\Gamma(k+1)\rho}$$

then,

$$E_{0.5,0.5}\left(C\big(t - \sigma(\xi)\big)^{0.5}\right)$$

$$= \sum_{k=0}^{\infty} \frac{C^k\left(\big(t-\sigma(\xi)\big)^{0.5}\right)^k}{\Gamma(k+1)0.5} = \frac{1}{\Gamma(0.5)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \big(t - \sigma(\xi)\big)^{0.5}$$

$$\times \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{\pi}} & \frac{(t-\sigma(\xi))^{0.5}}{2} \\ 0 & \frac{1}{\sqrt{\pi}} \end{pmatrix}.$$

The controllability-Gramian matrix is

$$\mathcal{W} = \begin{pmatrix} 0.254 & 0.0705 \\ 0.0705 & 0.225 \end{pmatrix}.$$

Therefore

$$\mathcal{W}_1 = \begin{pmatrix} 0.285 & 0.091 \\ 0.035 & 0.253 \end{pmatrix}$$

is nonsingular. Since the nonlinear function $f(t, y(t), \mathcal{u}(t))$ satisfies the hypothesis (2.7), then The System (2.13) is controllable on [0,1] according to Theorem (2.1.5).

Example 2.1.8

Consider the following AB-fractional nonlinear fractional integrodifferential system with control delay

$$\begin{cases} {}^{ABC}D^{\rho} y(t) = Ay(t) + B\mathcal{u}(\hbar(t)) \\ +f\left(t, y(t), \int_0^t g(t, s, y(s))ds\right), t \in [0,1], \\ y(0) = y_0, \end{cases} \tag{2.14}$$

where, $\hbar$, A, B, and ρ are defined as in Example (2.1.6). The function f defined as

$$f\left(t, y(t), \int_0^t g(t, s, y(s))ds\right) = \begin{pmatrix} 0 \\ \frac{\int_0^t e^{-y_1(s)} ds}{1 + {y_1}^2(t) + {y_2}^2(t)} \end{pmatrix}.$$

Due to the fact that the function f meets the condition (2.12) and the matrix $\mathcal{W}_1$ (which was obtained in Example (2.1.7)) is nonsingular, then by Theorem (2.1.6), The Nonlinear System (2.14) is controllable.

2.2 Observability of the AB-fractional Linear Dynamical Systems

In this section, we investigate the observability of the AB-fractional linear dynamical system

$$\begin{cases} {}^{ABC}D^{\rho} y(t) = Ay(t), \ \ t \in J = [0, \Upsilon] \\ y(0) = y_0 \\ x(t) = Cy(t), \end{cases} \tag{2.15}$$

where ${}^{ABC}D^{\rho}$ is AB-fractional derivative of order ρ, $0 < \rho < 1$, the continuous vector valued functions $y(t), x(t) \in \mathbb{R}^n$ are state and output of the system respectively. $A \in \mathbb{R}^{n\times n}$and $C \in \mathbb{R}^{m\times n}$.

Immediately from Theorem (2.1.2), the solution of system (2.15) is given by

$$y(t) = A_{\rho} E_{\rho}\left(\rho A_{\rho} A t^{\rho}\right) y_0. \tag{2.16}$$

Lemma 2.2.1 "Cayley-Hamilton Theorem" [43]

For every $n \times n$ matrix A,

$$a_n A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \cdots + a_1 A + a_0 I_{n\times n}$$

where

$$\Delta(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \cdots + a_1 \lambda + a_0$$

is the characteristic polynomial of A.

Theorem 2.2.2 "Rank-Nullity Theorem" [43]

Let $A \in \mathbb{R}^{m\times n}$, then

$$\dim(\ker(A)) + rank(A) = n.$$

Definition 2.2.3 [44]

Suppose that M is an $n \times n$ matrix. A linear subspace Z of $\mathbb{R}^n$ is said to be $\boldsymbol{M}$ **−invariant** if for every vector $z \in Z$ we have $Mz \in Z$.

Lemma 2.2.4 [44]

Let M be $n \times n$ matrix and Z a nonzero M −invariant subspace of $\mathbb{R}^n$. Then Z contain at least one eigenvector of M.

Now, from (2.16), the output $x(t)$ of System (2.15) is given by

$$x(t) = CA_\rho E_\rho\big(\rho A_\rho A t^\rho\big) y_0. \tag{2.17}$$

Define the observable operator $L: \mathbb{R}^n \to L_2(\mathcal{J}, \mathbb{R}^m)$ such that for any $z \in \mathbb{R}^n$,

$$Lz = CA_\rho E_\rho\big(\rho A_\rho A t^\rho\big) z.$$

The adjoint operator of L is

$$L^*: L_2(\mathcal{J}, \mathbb{R}^m) \to \mathbb{R}^n$$

defined by

$$L^*\varphi = \int_0^\Upsilon \Big(A_\rho E_\rho\big(\rho A_\rho A t^\rho\big)\Big)^* C^* \varphi \, dt,$$

for $\varphi(\cdot) \in L_2(\mathcal{J}, \mathbb{R}^m)$.

Indeed,

$$\langle Lz, \varphi(\cdot)\rangle_{L_2(\mathcal{J},\mathbb{R}^m)} = \int_0^\Upsilon \langle CA_\rho E_\rho\big(\rho A_\rho A t^\rho\big) z, \varphi(t)\rangle_{L_2(\mathcal{J},\mathbb{R}^m)} dt$$

$$= \int_0^\Upsilon \langle z, \Big(A_\rho E_\rho\big(\rho A_\rho A t^\rho\big)\Big)^* C^* \varphi(t)\rangle_{\mathbb{R}^n} \, dt$$

$$= \langle z, \int_0^\Upsilon \Big(A_\rho E_\rho\big(\rho A_\rho A t^\rho\big)\Big)^* C^* \varphi(t) \, dt\rangle_{\mathbb{R}^n}$$

$$= \langle z, L^*\varphi(\cdot)\rangle_{\mathbb{R}^n}.$$

Therefore

$$L^*\varphi = \int_0^{\Upsilon} \left(A_\rho E_\rho(\rho A_\rho A t^\rho)\right)^* C^*\varphi\, dt.$$

Now, we can define the observability Gramian matrix as follows:

$$W_{ob} = L^*L = \int_0^{\Upsilon} \left(A_\rho E_\rho(\rho A_\rho A t^\rho)\right)^* C^* C A_\rho E_\rho(\rho A_\rho A t^\rho)\, dt.$$

Definition 2.2.5

The linear System (2.15) is said to be **observable** over the time interval $\mathcal{J} = [0, \Upsilon]$ if it is possible to determine uniquely the initial state $y(0) = y_0$ from knowledge of the output $x(t)$ over $\mathcal{J}$.

In the following theorem, dependence on observability Gramian matrix W_{ob} we present a criterion for observability of System (2.15).

Theorem 2.2.6

The Linear System (2.15) is observable if and only if the observability Gramian matrix W_{ob} is nonsingular.

Proof. Assume that The Linear System (2.15) is observable**,** and suppose the Gramian matrix W_{ob} is singular then there is a vector $r \in \mathbb{R}^n$ such that $r^* W_{ob} r = 0, r \neq 0$, then

$$\int_0^{\Upsilon} r^* \left(A_\rho E_\rho(\rho A_\rho A t^\rho)\right)^* C^* C A_\rho E_\rho(\rho A_\rho A t^\rho) r\, dt$$

$$= \int_0^{\Upsilon} \left\| C A_\rho E_\rho(\rho A_\rho A t^\rho) r \right\|^2 dt = 0,$$

then

$$C A_\rho E_\rho(\rho A_\rho A t^\rho) r = 0.$$

then by choosing $\hat{y}_0 = y_0 - r$, and from (2.17) we have

$$CA_\rho E_\rho(\rho A_\rho At^\rho)\hat{y}_0 = CA_\rho E_\rho(\rho A_\rho At^\rho)(y_0 - r)$$
$$CA_\rho E_\rho(\rho A_\rho At^\rho)y_0 - CA_\rho E_\rho(\rho A_\rho At^\rho)r$$
$$= x(t).$$

Thus, the same $x(t)$ yields from two different initial states, i.e. we cannot uniquely determine the initial condition y_0, and hence the Linear System (2.15) is not observable which is contradiction.

Conversely, assume that the observability Gramian matrix W_{ob} is invertible. By multiplying the both sides of (2.17) by $\left(A_\rho E_\rho(\rho A_\rho At^\rho)\right)^* C^*$ and integrate over $[0, \Upsilon]$, we have

$$\int_0^\Upsilon \left(A_\rho E_\rho(\rho A_\rho At^\rho)\right)^* C^* x(t)\, dt$$
$$= \int_0^\Upsilon \left(A_\rho E_\rho(\rho A_\rho At^\rho)\right)^* C^* C\, A_\rho E_\rho(\rho A_\rho At^\rho) y_0 dt$$
$$= W_{ob} y_0.$$

Since W_{ob} is invertible, then

$$y_0 = W_{ob}^{-1} \int_0^\Upsilon \left(A_\rho E_\rho(\rho A_\rho At^\rho)\right)^* C^* x(t)\, dt$$

and that is mean y_0 is uniquely determined. Therefore, the Linear System (2.15) is observable.

In the next theorem, we present another criterion condition which is observability matrix in sense of Kalman.

Theorem 2.2.7

The Linear System (2.15) is observable if and only if the matrix

$$G=\begin{bmatrix} CA_\rho \\ \dfrac{CA_\rho\left(\rho A_\rho A\right)}{\Gamma(\rho+1)} \\ \vdots \\ \dfrac{(n-1)!\,CA_\rho\left(\rho A_\rho A\right)^{n-1}}{\Gamma(\rho(n-1)+1)} \end{bmatrix}$$

has full rank.

Proof. For simplicity, we put

$$\gamma_k=\frac{\rho^k k!}{\Gamma(\rho k+1)},k=0,1,2,\dots,n-1,$$

then the matrix G can be written as follows:

$$G=\begin{bmatrix} CA_\rho \\ \gamma_1 CA_\rho\left(A_\rho A\right) \\ \vdots \\ \gamma_{n-1} CA_\rho\left(A_\rho A\right)^{n-1} \end{bmatrix}.$$

Let $t^\rho=s$, then $0\le s=t^\rho\le \Upsilon^\rho\le \Upsilon$. We have

$$x^{(k)}(0)=\frac{k!\,CA_\rho\left(\rho A_\rho A\right)^k}{\Gamma(\rho k+1)}y_0,\qquad k=0,1,2,\dots,n-1,$$

indeed, from (2.17),

$$x(s)=CA_\rho E_\rho\left(\rho A_\rho As\right)y_0=CA_\rho\sum_{k=0}^{\infty}\frac{\left(\rho A_\rho As\right)^k}{\Gamma(\rho k+1)}y_0$$

$$=CA_\rho y_0+CA_\rho\sum_{k=1}^{\infty}\frac{\left(\rho A_\rho As\right)^k}{\Gamma(\rho k+1)}y_0,$$

then

$$x(0)=CA_\rho y_0.$$

$$x^{(1)}(s)=CA_\rho\sum_{k=1}^{\infty}\frac{\left(\rho A_\rho A\right)^k}{\Gamma(\rho k+1)}ks^{k-1}\,y_0$$

$$= CA_\rho(\rho A_\rho A)\sum_{k=1}^{\infty}\frac{(\rho A_\rho A)^{k-1}}{\Gamma(\rho k+1)}ks^{k-1}\,y_0$$

$$= \frac{CA_\rho(\rho A_\rho A)}{\Gamma(\rho+1)}y_0 + CA_\rho(\rho A_\rho A)\sum_{k=2}^{\infty}\frac{(\rho A_\rho A)^{k-1}}{\Gamma(\rho k+1)}ks^{k-1}\,y_0,$$

then

$$x^{(1)}(0) = \frac{CA_\rho(\rho A_\rho A)}{\Gamma(\rho+1)}y_0.$$

$$x^{(2)}(s) = CA_\rho(\rho A_\rho A)\sum_{k=2}^{\infty}\frac{(\rho A_\rho A)^{k-1}}{\Gamma(\rho k+1)}k(k-1)s^{k-2}\,y_0$$

$$= CA_\rho(\rho A_\rho A)^2\sum_{k=2}^{\infty}\frac{(\rho A_\rho A)^{k-2}}{\Gamma(\rho k+1)}k(k-1)s^{k-2}y_0$$

$$= 2\frac{CA_\rho(\rho A_\rho A)^2}{\Gamma(2\rho+1)}y_0 + CA_\rho(\rho A_\rho A)^2\sum_{k=3}^{\infty}\frac{(\rho A_\rho A)^{k-2}}{\Gamma(\rho k+1)}k(k-1)s^{k-2}y_0,$$

then,

$$x^{(2)}(0) = 2\frac{CA_\rho(\rho A_\rho A)^2}{\Gamma(2\rho+1)}y_0.$$

By repeating the processes, we have

$$x^{(k)}(0) = \frac{k!\,CA_\rho(\rho A_\rho A)^k}{\Gamma(\rho k+1)}y_0, \qquad k = 0, 1, 2, \dots, n-1.$$

Now, by Cayley-Hamilton Theorem, the matrix $(\rho A_\rho A)^n$ can be written as a linear combinations of lowers powers of $A_\rho A$.

Therefore

$$\begin{bmatrix} x(0) \\ x^{(1)}(0) \\ \vdots \\ x^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} CA_\rho \\ \gamma_1 CA_\rho(A_\rho A) \\ \vdots \\ \gamma_{n-1} CA_\rho(A_\rho A)^{n-1} \end{bmatrix} y_0,$$

since $rank(G) = n$, then G has left inverse, therefore y_0 is uniquely determined which means the Linear System (2.15) is observable.

Conversely, suppose that $rank(G) < n$, then there exists a nonzero vector z such that $Gz = 0$, then

$$CA_\rho z = 0, CA_\rho(A_\rho A)z = 0, \dots, CA_\rho(A_\rho A)^{n-1} z = 0.$$

By Cayley-Hamilton Theorem, the matrix $E_\rho(\rho A_\rho As)$ can be written as a linear combinations of lowers powers of $A_\rho As$. Put $0 \neq z = y_0$ and since

$$x(t) = CA_\rho E_\rho(\rho A_\rho As) y_0$$

$$= CA_\rho \left[\zeta_0 I + \zeta_1 A_\rho As + \cdots + \zeta_{n-1}(A_\rho A)^{n-1} s^{n-1}\right] y_0$$

where $\zeta_k \neq 0, k = 0,1,2, \dots, n-1$.

Then

$$x(t) = \zeta_0 CA_\rho z + \zeta_1 CA_\rho(A_\rho As)z + \cdots + \zeta_{n-1} CA_\rho(A_\rho As)^{n-1} z = 0.$$

Also, when $y_0 = 0$, then

$$x(t) = 0,$$

therefore, y_0 is not uniquely determined which means the Linear System (2.15) is not observable.

Based on eigenvectors of the matrix $A_\rho A$, we present observability test for the Linear System (2.15) in the following theorem.

Theorem 2.2.8

The Linear System (2.15) is observable if and only if every eigenvector of $A_\rho A$ is not belong to $\ker(CA_\rho)$.

Proof. Suppose the Linear System (2.15) is not observable, then by Theorem (2.2.7), $rank(G) < n$ where

$$G = \begin{bmatrix} CA_\rho \\ \gamma_1 CA_\rho(A_\rho A) \\ \vdots \\ \gamma_{n-1} CA_\rho(A_\rho A)^{n-1} \end{bmatrix},$$

and by Rank-Nullity Theorem, we have $\dim(\ker(G)) \geq 1$, which means that there exists a nonzero vector $z \in R^n$ such that $z \in \ker(G)$, $i.e.\, Gz = 0$.

Now, we show that $\ker(G)$ is $(A_\rho A) -$ invariant.

Assume that $z \in \ker(G)$, then

$$Gz = \begin{bmatrix} CA_\rho \\ \gamma_1 CA_\rho(A_\rho A) \\ \vdots \\ \gamma_{n-1} CA_\rho(A_\rho A)^{n-1} \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.18}$$

and

$$G(A_\rho A)z = \begin{bmatrix} CA_\rho(A_\rho A) \\ \gamma_1 CA_\rho(A_\rho A)^2 \\ \vdots \\ \gamma_{n-1} CA_\rho(A_\rho A)^n \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \gamma_{n-1} CA_\rho(A_\rho A)^n z \end{bmatrix}.$$

By using Cayley-Hamilton Theorem, $CA_\rho(A_\rho A)^n z$ can be written as a linear combination of the terms

$$CA_\rho(A_\rho A)^n z = CA_\rho z, CA_\rho(A_\rho A)z, \dots, CA_\rho(A_\rho A)^{n-1} z$$

and they are all zeros from (2.18). We conclude that $A_\rho A z$ belong to $\ker(G)$ which means that $\ker(G)$ is $(A_\rho A) -$ invariant. Then by Lemma (2.2.4), $\ker(G)$ must contain at least one eigenvector z of $A_\rho A$. But $Gz = 0$ implies $CA_\rho z = 0$ which means $z \in \ker(CA_\rho)$.

Conversely, assume that the Linear System (2.15) is observable. If there exists an eigenvalue λ such that $A_\rho A z = \lambda z$, with $z \neq 0$ for which $z \in \ker\left(CA_\rho\right)$, then

$$Gz = \begin{bmatrix} CA_\rho \\ \gamma_1 CA_\rho\left(A_\rho A\right) \\ \vdots \\ \gamma_{n-1} CA_\rho\left(A_\rho A\right)^{n-1} \end{bmatrix} z = \begin{bmatrix} CA_\rho z \\ \gamma_1 CA_\rho \lambda z \\ \vdots \\ \gamma_{n-1} CA_\rho \lambda^{n-1} z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

which means $z \in \ker(G)$ then by Rank-Nullity Theorem, $rank(G) < n$. Then from Theorem (2.2.7) we have contradiction with the observability of the Linear System (2.15).

Another test of controllability for Linear System (2.15) is presented in the following theorem.

Theorem 2.2.9

The Linear System (2.15) is observable if and only if for each $\lambda \in \mathbb{C}$, we have

$$rank \begin{bmatrix} A_\rho A - \lambda I \\ CA_\rho \end{bmatrix} = n.$$

Proof. Assume the Linear System (2.15) is observable. If there exists $\lambda \in \mathbb{C}$, such that

$$rank \begin{bmatrix} A_\rho A - \lambda I \\ CA_\rho \end{bmatrix} < n,$$

then from Rank-Nullity Theorem, there exists a nonzero vector $z \in R^n$ such that $z \in \ker \begin{bmatrix} A_\rho A - \lambda I \\ CA_\rho \end{bmatrix}$. Then $\left(A_\rho A - \lambda I\right)z = 0$ and $\gamma CA_\rho z = 0$ implies that z is an eigenvector of $A_\rho A$ and $z \in \ker(CA_\rho)$. Then by Theorem (2.2.8) that is contradiction with the observability of the Linear System (2.15).

Conversely, Suppose the system is not observable, then from Theorem (2.2.8), there exists a nonzero eigenvector z of $A_\rho A$ which belong to $\ker(CA_\rho)$ for some $\lambda \in \mathbb{C}$. This means

$$\begin{bmatrix} A_\rho A - \lambda I \\ C A_\rho \end{bmatrix} z = 0,$$

which means $z \in \ker \begin{bmatrix} A_\rho A - \lambda I \\ C A_\rho \end{bmatrix}$.

Therefore

$$rank \begin{bmatrix} A_\rho A - \lambda I \\ C A_\rho \end{bmatrix} < n.$$

In 2021, Ghasemi and Nassiri [20] studied the controllability of the AB-fractional linear control system

$$\begin{cases} {}^{ABC}D^\rho y(t) = Ay(t) + Bu(t), \;\; t \in J = [0, \Upsilon] \\ \qquad y(0) = y_0. \end{cases} \tag{2.19}$$

They set a condition on the system that the Caputo fractional derivative (of order ρ) for the control function u exists.

Lemma 2.2.10 [20]

The Linear System (2.19) is controllable on J if and only if the rank of the $n \times nm$ controllability matrix $\mathcal{R}$ is n, where

$$\mathcal{R} = [A_\alpha B \;\; (A_\alpha A) A_\alpha B \; \dots \; (A_\alpha A)^{n-1} \;\; A_\alpha B].$$

The following theorem shows the relationship between the controllability and the observability of an AB-fractional linear dynamical system.

Theorem 2.2.11 "Theorem of Duality"

The linear AB-fractional control dynamical system

$$\begin{cases} {}^{ABC}D^\rho y(t) = Ay(t) + Bu(\mathrm{t}), \;\; t \in J = [0, \Upsilon] \\ y(0) = y_0, \end{cases} \tag{2.20}$$

is controllable if and only if the system

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = A^{*}y(t), & t \in J = [0, \Upsilon] \\ y(0) = y_0 \\ x(t) = B^{*}y(t) \end{cases} \tag{2.21}$$

is observable.

Proof. According to Lemma (2.2.10), the Linear System (2.20) is controllable if and only if the matrix $\mathcal{R} = \left[(A_\rho B) \quad (A_\rho A)(A_\rho B) \ \dots \ (A_\rho A)^{n-1}(A_\rho B) \right]$ has full rank. The transpose

$$\mathcal{R}^{*} = \begin{bmatrix} (A_\rho B)^{*} \\ (A_\rho B)^{*}(A_\rho A)^{*} \\ \vdots \\ (A_\rho B)^{*}[(A_\rho A)^{*}]^{n-1} \end{bmatrix}$$

has full rank if and only if $\mathcal{R}$ has full rank. By Theorem (2.2.7), The System (2.21) is observable if and only if $\mathcal{R}^{*}$ has full rank.

Next, we give examples to illustrate our results.

Example 2.2.13 Consider the following AB-fractional linear system

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = Ay(t), & t \in [0,1] \\ y(0) = y_0 \\ x(t) = Cy(t), \end{cases} \tag{2.22}$$

where $A = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$, $C = (1 \quad 1)$, and $\rho = 0.5$.

Then,

$$A_\rho = [I - (1-\rho)A]^{-1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 0.5 \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \right]^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$M = \rho A_\rho A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

The Mittag-Leffler function matrix of the system is:

$$E_\rho(Mt^\rho) = \sum_{k=0}^{\infty} \frac{M^k (t^\rho)^k}{\Gamma(k\rho+1)},$$

$$= I + \frac{t^{0.5}}{\Gamma(1.5)} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + 1.1284t^{0.5} & -1.1284t^{0.5} \\ 1.1284t^{0.5} & 1 - 1.1284t^{0.5} \end{pmatrix}.$$

Then

$$W_{ob} = \int_0^1 \left(A_\rho E_\rho(Mt^\rho)\right)^* C^* C A_\rho E_\rho(Mt^\rho)\, dt$$

$$= \int_0^1 \begin{pmatrix} 2.2568t^{0.5} + 3 \\ -2.2568t^{0.5} - 1 \end{pmatrix} (2.2568t^{0.5} + 3 \quad -2.2568t^{0.5} - 1)\, dt$$

$$= \begin{pmatrix} 20.5737 & -11.5647 \\ -11.5647 & 6.5556 \end{pmatrix}$$

which is nonsingular. Therefore, by Theorem (2.2.6), the system is observable.

Example 2.2.14 Consider the following AB-fractional linear system

$$\begin{cases} {}^{ABC}D^\rho y(t) = Ay(t) + Bu(t), \quad t \in [0,1] \\ y(0) = y_0 \end{cases} \tag{2.23}$$

where $A = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\rho = 0.3$.

Then

$$A_\rho = [I - (1-\rho)A]^{-1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1.4 & 0.7 \\ -2.8 & -1.4 \end{pmatrix}\right]^{-1}$$

$$= \begin{pmatrix} 2.4 & 0.7 \\ -2.8 & -0.4 \end{pmatrix}.$$

To prove the system (2.23) is controllable by Theorem (2.2.11), we have to show that the following AB-fractional linear system is observable

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = A^{*}y(t), \qquad t \in [0,1] \\ \quad y(0) = y_0 \\ \quad x(t) = B^{*}y(t). \end{cases} \tag{2.24}$$

$$A_\rho A = \begin{pmatrix} 2.4 & 0.7 \\ -2.8 & -0.4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$$

$$A_\rho B = \begin{pmatrix} 2.4 & 0.7 \\ -2.8 & -0.4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.4 \\ -2.8 \end{pmatrix}.$$

Therefore

$$G = \begin{bmatrix} \left(A_\rho B\right)^{*} \\ \dfrac{\rho\left(A_\rho B\right)^{*}\left(AA_\rho\right)^{*}}{\Gamma(\rho+1)} \end{bmatrix} = \begin{bmatrix} 2.4 & -2.8 \\ 0.6685 & -1.3371 \end{bmatrix}$$

which is has full rank, then by Theorem (2.2.7), the System (2.24) is observable. Therefore, the System (2.23) is controllable.

Chapter Three

Exact and Approximate Controllability of Hattaf-Fractional Control Systems in a Banach Space

This chapter aims to investigate the exact and approximate controllability of Hattaf-fractional nonlinear control systems in a Banach space. In the first section, we discuss the controllability of the nonlinear system using semigroup theory and Nussbaum Fixed Point Theorem. Under sufficient conditions, the existence and uniqueness of the mild solution of the nonlinear system are proved in section two. The third section deals with the approximate controllability of the nonlinear system under sufficient conditions.

3.1 Exact Controllability of Impulsive Hattaf-Fractional Nonlinear Control System in Banach Space

This section investigates the exact controllability of the Hattaf-fractional impulsive nonlinear control system

$$\begin{cases} {}^{C}D^{\rho,\omega}\left[y(t)-h\big(t,y(t)\big)\right]=\mathcal{A}y(t)+\mathrm{B}\mathcal{u}(t) \\ \quad +f\big(t,y(t)\big), \quad t\in\mathcal{J}=[0,\Upsilon], t\neq t_{\gamma}, \gamma=1,2,\dots.p, \\ \Delta y\big(t_{\gamma}\big)=q_{\gamma}\left(y\big(t_{\gamma}^{-}\big)\right), \\ y(0)=y_{0,} \end{cases} \tag{3.1}$$

where $\rho,\omega\in(0,1)$, ${}^{C}D^{\rho,\omega}$ is Hattaf-fractional derivative in the sense of Caputo of order ρ. $y(t)$ takes values in Banach space X, $\mathcal{A}: D(\mathcal{A})\subset \mathrm{X}\to\mathrm{X}$ is closed linear operator, the control $\mathcal{u}\in L_P(\mathcal{J},U)$ with U as a Banach space, $\mathrm{B}: L_P(\mathcal{J},U)\to L_P(\mathcal{J},X)$ is bounded linear operator, the functions $f:\mathcal{J}\times\mathrm{X}\to\mathrm{X}$ and $h:\mathcal{J}\times\mathrm{X}\to\mathrm{D}(\mathcal{A})$ are continuous, $q_{\gamma}: X\to X$ is the jump operator, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = \Upsilon$, $y(t_\gamma^+)$ and $y(t_\gamma^-)$ indicate to the right and left limits of $y(t)$ at $t = t_\gamma$ respectively and $\Delta y(t_\gamma) = y(t_\gamma^+) - y(t_\gamma^-)$.

Assume the linear operator $\mathcal{A}: D(\mathcal{A}) \subset X \to X$ is the generator of C_0-semigroup $\{\mathcal{G}(t), t \geq 0\}$ on a Banach space X, where $sup_{t\geq 0}\|\mathcal{G}(t)\| = \mathcal{S}, \mathcal{S} \geq 1$. We consider the linear operator $E \coloneqq \rho\mathcal{A}_\rho\mathcal{A}$ where $\mathcal{A}_\rho = \left[(1-\rho)\left(\frac{1}{1-\rho}I - \mathcal{A}\right)\right]^{-1}$. Then E is the generator of C_0 −semigroup $\{\mathcal{T}(t), t \geq 0\}$ and $sup_{t\geq 0}\|\mathcal{T}(t)\| = \mathcal{S}$ [32].

We introduce the family of functions:

$$PC(\mathcal{J}, \mathrm{X}) = \{y: \mathcal{J} \to \mathrm{X}: \text{y is continuous at } t \in \mathcal{J}\backslash\{t_1, t_2, \dots, t_r\}$$
$$\text{, and there exist } y(t_\gamma^+) \text{ and } y(t_\gamma^-) \text{ with } y(t_\gamma^-) = y(t) \text{ for } \gamma = 1,2, \dots, r\}.$$

It is clear that $(PC(\mathcal{J}, \mathrm{X}), \|\cdot\|_{PC})$ is a Banach space with the norm $\|y\|_{PC} = sup_{t\in\mathcal{J}}\|y(t)\|$.

The mild solution of the Nonlinear System (3.1) will introduced in the following.

Assume the one-sided stable probability density function [45]

$$\psi_\omega(\delta) = \frac{1}{\pi}\sum_{i=1}^{\infty}(-1)^{i-1}\delta^{-\omega i-1}\frac{\Gamma(i\omega+1)}{i!}\sin(i\pi\omega)$$

then the Laplace transform of $\psi_\omega(\delta)$ is

$$\mathcal{L}\{\psi_\omega(\delta)\}(\lambda) = e^{-\lambda^\omega} \tag{3.2}$$

where $0 < \delta < \infty, 0 < \omega < 1$ and $\lambda > 0$.

Additionally, assume the probability density function

$$\varphi_\omega(\delta) = \frac{1}{\omega}\delta^{-1-\frac{1}{\omega}}\psi_\omega\left(\delta^{-\frac{1}{\omega}}\right), 0 < \delta < \infty, 0 < \omega < 1, \tag{3.3}$$

then, for $0 \le \xi \le 1$

$$\int_0^\infty \delta^\xi \varphi_\omega(\delta)d\delta = \int_0^\infty \frac{1}{\delta^{\omega\xi}}\psi_\omega(\delta)d\delta = \frac{\Gamma(1+\xi)}{\Gamma(1+\omega\xi)}, 0 < \delta < \infty, 0 < \omega < 1.$$

We begin by proving the following lemma before defining a mild solution of System (3.1).

Lemma 3.1.1

If $y \in PC(\mathcal{J}, \mathrm{X})$ is a solution of System (3.1), then y satisfies the following

$$y(t) = \begin{cases} \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} E L_\omega(t-\zeta) h\big(\zeta, y(\zeta)\big) d\zeta \\ + \mathcal{A}_\rho \mathcal{T}_\omega(t)\big(y_0 - h(0, y_0)\big) + (1-\rho)\mathcal{A}_\rho\big[Bu(t) + f\big(t, y(t)\big)\big] \\ + \rho {\mathcal{A}_\rho}^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \quad t \in [0, t_1] \\ \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} E L_\omega(t-\zeta) h\big(\zeta, y(\zeta)\big) d\zeta \\ + \mathcal{A}_\rho \mathcal{T}_\omega(t)\big(y_0 - h(0, y_0)\big) + (1-\rho)\mathcal{A}_\rho\big[Bu(t) + f\big(t, y(t)\big)\big] \\ + \rho {\mathcal{A}_\rho}^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \\ + \mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega\big(t - t_\gamma\big)\Delta y\big(t_\gamma\big) \qquad t \in \big(t_\gamma, t_{\gamma+1}\big] \end{cases}$$

where

$$L_\omega(t) = \omega \int_0^\infty \theta \varphi_\omega(\theta) \mathcal{T}(\theta t^\omega) d\theta,$$

$$\mathcal{T}_\omega(t) = \int_0^\infty \varphi_\omega(\theta) \mathcal{T}(\theta t^\omega) d\,\theta,$$

, $\varphi_\omega(\theta)$ given in identically (3.3) and $0 < \rho < 1$.

Proof.

We see that $y(\cdot)$ is decomposable into $m(\cdot)+n(\cdot)$, where m is a continuous mild solution of the system

$$\begin{cases} {}^{C}D^{\rho,\omega}m(t) = {}^{c}D^{\rho,\omega}h\big(t,y(t)\big) + \mathcal{A}m(t) + \mathrm{B}u(t) \\ \qquad +f\big(t,y(t)\big) \quad t \in \mathcal{J} \\ m(0) = y_0 \end{cases} \tag{3.4}$$

and n is picewise continuous solution of the system

$$\begin{cases} {}^{C}D^{\rho,\omega}n(t) = \mathcal{A}n(t) \quad t \in \mathcal{J}, t \neq t_\gamma \\ \Delta y(t_\gamma) = q_\gamma\left(n(t_\gamma^-)\right) \ \ \gamma = 1,2,\dots,p, \\ n(0) = 0. \end{cases} \tag{3.5}$$

To find the mild solution $m(t)$, we applying fractional integral (1.5) for both sides of System (3.4),

$$\mathcal{J}^{\rho,\omega}\, {}^{c}D^{\rho,\omega}m(t) = \mathcal{J}^{\rho,\omega}\, {}^{c}D^{\rho,\omega}h(t,y(t)) + \mathcal{J}^{\rho,\omega}\big[\mathcal{A}m(t) + \mathrm{B}u(t) + f\big(t,y(t)\big)\big].$$

using Lemma (1.2.30), we have

$$m(t) = h\big(t,y(t)\big) - h(0,y_0) + y_0 + \mathcal{J}^{\rho,\omega}[\mathcal{A}m(t) + \mathrm{B}u(t) + f\big(t,y(t)\big)] \tag{3.6}$$

By taking Laplace transform for both sides of (3.6), we have

$$\mathrm{M}(\lambda) = H(\lambda) - \frac{h(0,y_0)}{\lambda} + \frac{y_0}{\lambda} + (1-\rho)\mathcal{L}\big\{\mathcal{A}m(t) + \mathrm{B}u(t) + f\big(t,y(t)\big)\big\}(\lambda) + \frac{\rho}{\lambda^\omega}\mathcal{L}\big\{\mathcal{A}m(t) + \mathrm{B}u(t) + f\big(t,y(t)\big)\big\}(\lambda)$$

where $\mathrm{M}(\lambda) = \mathcal{L}\{m(t)\}(\lambda)$ and $H(\lambda) = \mathcal{L}\{h(t,y(t))\}(\lambda)$.

It follows

$$\mathrm{M}(\lambda) - (1-\rho)\mathcal{A}\mathrm{M}(\lambda) - \frac{\rho}{\lambda^{\omega}}\mathcal{A}\mathrm{M}(\lambda)$$
$$= H(\lambda) - \frac{h(0,y_0)}{\lambda} + \frac{y_0}{\lambda} + \left[(1-\rho) + \frac{\rho}{\lambda^{\omega}}\right]F(\lambda)$$

where $F(\lambda) = \mathcal{L}\{\mathrm{B}u(t) + f(t,y(t))\}(\lambda)$, then

$$\left[(I-(1-\rho)\mathcal{A}) - \frac{\rho}{\lambda^{\omega}}\mathcal{A}\right]\mathrm{M}(\lambda)$$
$$= H(\lambda) + \frac{y_0 - h(0,y_0)}{\lambda} + \left[(1-\rho) + \frac{\rho}{\lambda^{\omega}}\right]F(\lambda)$$

$$\left(\mathcal{A}_{\rho}^{-1} - \frac{\rho}{\lambda^{\omega}}\mathcal{A}\right)\mathrm{M}(\lambda) = H(\lambda) + \frac{y_0 - h(0,y_0)}{\lambda} + \left[(1-\rho) + \frac{\rho}{\lambda^{\omega}}\right]F(\lambda)$$

$$\left(\lambda^{\omega}\mathcal{A}_{\rho}^{-1} - \rho\mathcal{A}\right)\mathrm{M}(\lambda)$$
$$= \lambda^{\omega}H(\lambda) + \lambda^{\omega-1}\big(y_0 - h(0,y_0)\big) + [\lambda^{\omega}(1-\rho)+\rho]F(\lambda)$$

$$\left[\left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho}\right)\mathcal{A}_{\rho}^{-1}\right]\mathrm{M}(\lambda)$$
$$= \lambda^{\omega}H(\lambda) + \lambda^{\omega-1}[y_0 - h(0,y_0)] + [\lambda^{\omega}(1-\rho)+\rho]F(\lambda)$$

$$\begin{aligned}\mathrm{M}(\lambda) = {} & \mathcal{A}_{\rho}\left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}\lambda^{\omega}H(\lambda) \\ & + \mathcal{A}_{\rho}\left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}\lambda^{\omega-1}[y_0 - h(0,y_0)] \\ & + \mathcal{A}_{\rho}\left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}\lambda^{\omega}(1-\rho)F(\lambda) \\ & + \mathcal{A}_{\rho}\left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}\rho F(\lambda).\end{aligned}$$

$$\lambda^{\omega}\left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1} = \left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho} + \rho\mathcal{A}\mathcal{A}_{\rho}\right)\left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}$$
$$= \left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho}\right)\left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1} + \rho\mathcal{A}\mathcal{A}_{\rho}\left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}$$
$$= I + \rho\mathcal{A}\mathcal{A}_{\rho}\left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}$$

then

$$\begin{aligned}\mathrm{M}(\lambda) = {} & \mathcal{A}_{\rho}\left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}\lambda^{\omega}H(\lambda) \\ & + \mathcal{A}_{\rho}\left(\lambda^{\omega}I - \rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}\lambda^{\omega-1}\big(y_0 - h(0,y_0)\big)\end{aligned}$$

$$+(1-\rho)\mathcal{A}_\rho\left[I+\rho\mathcal{A}\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}\right]F(\lambda)$$
$$+\rho\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}F(\lambda)$$

$$=\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}\lambda^\omega H(\lambda)+\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}\lambda^{\omega-1}\left(y_0-h(0,y_0)\right)$$
$$+(1-\rho)\mathcal{A}_\rho F(\lambda)+(1-\rho)\rho\mathcal{A}\mathcal{A}_\rho^2\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}F(\lambda)$$
$$+\rho\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}F(\lambda)$$

$$=\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}\lambda^\omega H(\lambda)+\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}\lambda^{\omega-1}\left(y_0-h(0,y_0)\right)$$
$$+(1-\rho)\mathcal{A}_\rho F(\lambda)$$
$$+\left[(1-\rho)\mathcal{A}+\mathcal{A}_\rho^{-1}\right]\rho\mathcal{A}_\rho^2\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}F(\lambda).$$

Since

$$\mathcal{A}_\rho=(I-(1-\rho)\mathcal{A})^{-1}$$

then

$$I=(1-\rho)\mathcal{A}+\mathcal{A}_\rho^{-1}.$$

Therefore

$M(\lambda)$

$$=\begin{cases}\mathcal{A}_\rho(\lambda^\omega I-E)^{-1}\lambda^\omega H(\lambda)+\mathcal{A}_\rho(\lambda^\omega I-E)^{-1}\lambda^{\omega-1}\left(y_0-h(0,y_0)\right)\\ \quad +(1-\rho)\mathcal{A}_\rho F(\lambda)+\rho\mathcal{A}_\rho^2(\lambda^\omega I-E)^{-1}F(\lambda).\end{cases} \tag{3.7}$$

Now,

$$\mathcal{A}_\rho(\lambda^\omega I-\mathrm{E})^{-1}\lambda^\omega H(\lambda)=\mathcal{A}_\rho\lambda^\omega\int_0^\infty e^{-\lambda^\omega s}\mathcal{T}(s)H(\lambda)ds$$
$$=\mathcal{A}_\rho\lambda^\omega\int_0^\infty e^{-(\lambda u)^\omega}\mathcal{T}(u^\omega)H(\lambda)\omega u^{\omega-1}du$$
$$=\mathcal{A}_\rho\int_0^\infty\lambda\omega(\lambda u)^{\omega-1}e^{-(\lambda u)^\omega}\mathcal{T}(u^\omega)H(\lambda)du$$
$$=\mathcal{A}_\rho\int_0^\infty-\frac{de^{-(\lambda u)^\omega}}{du}\mathcal{T}(u^\omega)H(\lambda)du,$$

using integration by part, we obtain

$$\mathcal{A}_\rho(\lambda^\omega I-\mathrm{E})^{-1}\lambda^\omega H(\lambda)$$
$$=\mathcal{A}_\rho\left[\left[-\mathcal{T}(u^\omega)H(\lambda)e^{-(\lambda u)^\omega}\right]_0^\infty - \int_0^\infty -e^{-(\lambda u)^\omega}\mathrm{E}\mathcal{T}(u^\omega)\omega u^{\omega-1}\,H(\lambda)du\right]$$

$$=\mathcal{A}_\rho H(\lambda)+\mathcal{A}_\rho\int_0^\infty e^{-(\lambda u)^\omega}\mathrm{E}\mathcal{T}(u^\omega)\omega u^{\omega-1}\,H(\lambda)du,$$

using (3.2), we have

$$\mathcal{A}_\rho(\lambda^\omega I-\mathrm{E})^{-1}\lambda^\omega H(\lambda)$$
$$=\mathcal{A}_\rho H(\lambda)+\mathcal{A}_\rho\int_0^\infty\int_0^\infty e^{-\lambda u\delta}\psi_\omega(\delta)\mathrm{E}\mathcal{T}(u^\omega)H(\lambda)\omega u^{\omega-1}d\delta du$$
$$=\mathcal{A}_\rho H(\lambda)+\mathcal{A}_\rho\int_0^\infty\int_0^\infty \omega e^{-\lambda v}\psi_\omega(\delta)\mathrm{E}\mathcal{T}(\frac{v}{\delta})^\omega\,\frac{v^{\omega-1}}{\delta^\omega}\,H(\lambda)d\delta dv$$
$$=\mathcal{A}_\rho H(\lambda)$$
$$+\mathcal{A}_\rho\int_0^\infty\int_0^\infty\int_0^\infty \omega e^{-\lambda(v+\zeta)`}\psi_\omega(\delta)\mathrm{E}\mathcal{T}\left(\frac{v}{\delta}\right)^\omega\frac{v^{\omega-1}}{\delta^\omega}h\big(\zeta,y(\zeta)\big)d\zeta d\delta dv$$
$$=\mathcal{A}_\rho H(\lambda)$$
$$+\mathcal{A}_\rho\int_0^\infty\int_0^\infty\int_\zeta^\infty \omega e^{-\lambda t`}\psi_\omega(\delta)\mathrm{E}\mathcal{T}\left(\frac{t-\zeta}{\delta}\right)^\omega\frac{(t-\zeta)^{\omega-1}}{\delta^\omega}h\big(\zeta,y(\zeta)\big)dtd\zeta d\delta$$
$$=\mathcal{A}_\rho H(\lambda)$$
$$+\mathcal{A}_\rho\int_0^\infty e^{-\lambda t}\left[\omega\int_0^t\int_0^\infty \psi_\omega(\delta)\mathrm{E}\mathcal{T}\left(\frac{t-\zeta}{\delta}\right)^\omega\frac{(t-\zeta)^{\omega-1}}{\delta^\omega}h\big(\zeta,y(\zeta)\big)d\delta d\zeta\right]dt$$

$$= \mathcal{A}_\rho H(\lambda)$$
$$+ \mathcal{A}_\rho \mathcal{L}\left\{\omega \int_0^t \int_0^\infty \psi_\omega(\delta) \mathrm{E}\mathcal{T}\left(\frac{t-\zeta}{\delta}\right)^\omega \frac{(t-\zeta)^{\omega-1}}{\delta^\omega} h\big(\zeta, y(\zeta)\big) d\delta d\zeta\right\}(\lambda)$$

using (3.3), we have

$$\mathcal{A}_\rho(\lambda^\omega I - \mathrm{E})^{-1}\lambda^\omega H(\lambda) =$$
$$\mathcal{A}_\rho H(\lambda) + \mathcal{A}_\rho \mathcal{L}\left\{\omega \int_0^t \int_0^\infty \varphi_\omega(\theta) \mathrm{E}T(\theta(t-\zeta)^\omega)\theta(t - \zeta)^{\omega-1} h\big(\zeta, y(\zeta)\big) d\theta d\zeta\right\}(\lambda).$$

Therefore

$$\mathcal{A}_\rho(\lambda^\omega I - \mathrm{E})^{-1}\lambda^\omega H(\lambda) = \mathcal{A}_\rho H(\lambda) + \mathcal{A}_\rho \mathcal{L}\left\{\int_0^t (t-\zeta)^{\omega-1} \mathrm{E}\, L_\omega(t-\zeta) h\big(\zeta, y(\zeta)\big) d\zeta\right\}(\lambda). \tag{3.8}$$

$$\mathcal{A}_\rho \lambda^{\omega-1}(\lambda^\omega I - \mathrm{E})^{-1}[y_0 - h(0, y_0)]$$
$$= \mathcal{A}_\rho \lambda^{\omega-1} \int_0^\infty e^{-\lambda^\omega s} \mathcal{T}(s)\big(y_0 - h(0, y_0)\big) ds$$
$$= \mathcal{A}_\rho \lambda^{\omega-1} \int_0^\infty e^{-(\lambda u)^\omega} \mathcal{T}(u^\omega)[y_0 - h(0, y_0)] \omega u^{\omega-1} du$$
$$= \mathcal{A}_\rho \int_0^\infty \omega(\lambda u)^{\omega-1} e^{-(\lambda u)^\omega} \mathcal{T}(u^\omega)[y_0 - h(0, y_0)] du$$
$$= \mathcal{A}_\rho \int_0^\infty -\frac{1}{\lambda} \frac{de^{-(\lambda u)^\omega}}{du} \mathcal{T}(u^\omega)[y_0 - h(0, y_0)] du,$$

from (3.2), we have

$$e^{-(\lambda u)^\omega} = \int_0^\infty e^{-\lambda u \delta} \psi_\omega(\delta) d\delta$$

then

$$\frac{de^{-(\lambda u)^{\omega}}}{du} = \int_0^{\infty} -\lambda\delta e^{-\lambda u\delta}\psi_{\omega}(\delta)d\delta.$$

Therefore

$$\mathcal{A}_{\rho}\lambda^{\omega-1}(\lambda^{\omega}I - \mathrm{E})^{-1}[y_0 - h(0, y_0)]$$
$$= \mathcal{A}_{\rho}\int_0^{\infty}\int_0^{\infty} \delta e^{-\lambda u\delta}\, \psi_{\omega}(\delta)\mathcal{T}(u^{\omega})[y_0 - h(0, y_0)]d\delta du$$
$$= \mathcal{A}_{\rho}\int_0^{\infty}\int_0^{\infty} e^{-\lambda t}\, \psi_{\omega}(\delta)\mathcal{T}\left(\left(\frac{t}{\delta}\right)^{\omega}\right)[y_0 - h(0, y_0)]d\delta dt$$
$$= \mathcal{A}_{\rho}\int_0^{\infty}\int_0^{\infty} -\frac{1}{\omega}\theta^{-\frac{1}{\omega}-1}e^{-\lambda t}\, \psi_{\omega}\left(\theta^{-\frac{1}{\omega}}\right)\mathcal{T}(\theta t^{\omega})[y_0 - h(0, y_0)]d\theta dt$$

using (3.3), we have

$$\mathcal{A}_{\rho}\lambda^{\omega-1}(\lambda^{\omega}I - \mathrm{E})^{-1}[y_0 - h(0, y_0)]$$
$$= \mathcal{A}_{\rho}\int_0^{\infty}\int_0^{\infty} e^{-\lambda t}\, \varphi_{\omega}(\theta)\mathcal{T}(\theta t^{\omega})[y_0 - h(0, y_0)]d\theta dt$$
$$= \mathcal{A}_{\rho}\mathcal{L}\left\{\int_0^{\infty} \varphi_{\omega}(\theta)\mathcal{T}(\theta t^{\omega})\big(y_0 - h(0, y_0)\big)d\theta\right\}(\lambda).$$

Therefore,

$$\mathcal{A}_{\rho}\lambda^{\omega-1}(\lambda^{\omega}I - \mathrm{E})^{-1}[y_0 - h(0, y_0)]$$
$$= \mathcal{A}_{\rho}\mathcal{L}\{\mathcal{T}_{\omega}(t)[y_0 - h(0, y_0)]\}(\lambda). \tag{3.9}$$

$$\rho\mathcal{A}_{\rho}^2(\lambda^{\omega}I - \mathrm{E})^{-1}\, F(\lambda) = \rho\mathcal{A}_{\rho}^2\int_0^{\infty} e^{-\lambda^{\omega}s}\mathcal{T}(s)F(\lambda)ds$$
$$= \rho\mathcal{A}_{\rho}^2\int_0^{\infty} e^{-(\lambda u)^{\omega}}\mathcal{T}(u^{\omega})F(\lambda)\omega u^{\omega-1}du$$

using (3.2), we have

$$\rho\mathcal{A}_{\rho}^2(\lambda^{\omega}I - \mathrm{E})^{-1}\, F(\lambda) = \rho\mathcal{A}_{\rho}^2\int_0^{\infty}\int_0^{\infty} \omega e^{-\lambda u\delta}\psi_{\omega}(\delta)\mathcal{T}(u^{\omega})F(\lambda)u^{\omega-1}d\delta du$$
$$= \rho\mathcal{A}_{\rho}^2\int_0^{\infty}\int_0^{\infty} \omega e^{-\lambda v}\psi_{\omega}(\delta)\mathcal{T}\left(\left(\frac{v}{\delta}\right)^{\omega}\right)\frac{v^{\omega-1}}{\delta^{\omega}}F(\lambda)d\delta dv$$

$$= \rho\mathcal{A}_\rho^2 \int_0^\infty \int_0^\infty \int_0^\infty \omega e^{-\lambda(v+\zeta)} \psi_\omega(\delta)\mathcal{T}\left(\left(\frac{v}{\delta}\right)^\omega\right)\frac{v^{\omega-1}}{\delta^\omega}[Bu(\zeta) + f(\zeta, y(\zeta)]\, d\zeta d\delta dv$$

$$= \rho\mathcal{A}_\rho^2 \int_0^\infty \int_0^\infty \int_\zeta^\infty \omega e^{-\lambda t} \psi_\omega(\delta)\mathcal{T}\left(\left(\frac{(t-\zeta)}{\delta}\right)^\omega\right)\frac{(t-\zeta)^{\omega-1}}{\delta^\omega}[Bu(\zeta) + f(\zeta, y(\zeta)]\, dt d\zeta d\,\delta$$

$$= \rho\mathcal{A}_\rho^2 \int_0^\infty e^{-\lambda t}\left[\omega \int_0^t \int_0^\infty \psi_\omega(\delta)\mathcal{T}\left(\left(\frac{(t-\zeta)}{\delta}\right)^\omega\right)\frac{(t-\zeta)^{\omega-1}}{\delta^\omega}[Bu(\zeta) + f(\zeta, y(\zeta)] d\delta d\zeta\right] dt$$

$$= \rho\mathcal{A}_\rho^2 \mathcal{L}\left\{\omega \int_0^t \int_0^\infty \psi_\omega(\delta)\mathcal{T}\left(\left(\frac{(t-\zeta)}{\delta}\right)^\omega\right)\frac{(t-\zeta)^{\omega-1}}{\delta^\omega}[Bu(\zeta) + f(\zeta, y(\zeta)] d\delta d\zeta\right\}(\lambda)$$

using (3.3), we have

$$\rho\mathcal{A}_\rho^2(\lambda^\omega I - \mathrm{E})^{-1} F(\lambda) = \rho\mathcal{A}_\rho^2 \mathcal{L}\left\{\omega \int_0^t \int_0^\infty \theta\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)(t-\zeta)^{\omega-1}[Bu(\zeta) + f(\zeta, y(\zeta)] d\theta d\zeta\right\}(\lambda),$$

$$\rho\mathcal{A}_\rho^2(\lambda^\omega I - \mathrm{E})^{-1} F(\lambda) = \rho\mathcal{A}_\rho^2 \mathcal{L}\left\{\int_0^t L_\omega(t-\zeta)(t-\zeta)^{\omega-1}\,[Bu(\zeta) + f(\zeta, y(\zeta)] d\zeta\right\}(\lambda). \quad (3.10)$$

From (3.8), (3.9) and (3.10) we get

$$M(\lambda) = \begin{cases} \mathcal{A}_\rho H(\lambda) + \mathcal{A}_\rho \mathcal{L}\left\{\int_0^t (t-\zeta)^{\omega-1} \mathrm{E}\, L_\omega(t-\zeta) h\big(\zeta, y(\zeta)\big) d\zeta\right\}(\lambda) \\ +\mathcal{A}_\rho \mathcal{L}\{\mathcal{T}_\omega(t)\big(y_0 - h(0,y_0)\big)\}(\lambda) + (1-\rho)\mathcal{A}_\rho F(\lambda) \\ +\rho\mathcal{A}_\rho^2 \mathcal{L}\left\{\int_0^t L_\omega(t-\zeta)(t-\zeta)^{\omega-1}\,[Bu(\zeta) + f(\zeta, y(\zeta)])d\zeta\right\}(\lambda). \end{cases} \tag{3.11}$$

Taking the inverse of Laplace transform for both sides of (3.11), we get

$$m(t) = \begin{cases} \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} \mathrm{E}\, L_\omega(t-\zeta) h\big(\zeta, y(\zeta)\big) d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)[y_0 - h(0,y_0)] + (1-\rho)\mathcal{A}_\rho[Bu(t) + f(t, y(t)] \\ +\rho\mathcal{A}_\rho^2 \int_0^t L_\omega(t-\zeta)(t-\zeta)^{\omega-1}\,\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big]d\zeta. \end{cases} \tag{3.12}$$

Now, to find the mild solution $n(t)$, we applying fractional integral (1.5) for both sides of System (3.5),

$$\mathcal{J}^{\rho,\omega} D^{\rho,\omega} n(t) = \mathcal{J}^{\rho,\omega}[\mathcal{A}n(t)].$$

By using Lemma (1.2.31) we have,

$$n(t) = \sum_{\gamma=1}^{p} \Delta y\big(t_\gamma\big)\sigma_\gamma(t) + \mathcal{J}^{\rho,\omega}[\mathcal{A}n(t)] \tag{3.13}$$

where

$$\sigma_\gamma(t) = \begin{cases} 0 & t \in [0, t_1] \\ 1 & t \in \big(t_\gamma, t_{\gamma+1}\big] \qquad \gamma = 1,2,\dots,p \end{cases} \tag{3.14}$$

and

$$\mathcal{L}\{\sigma_\gamma(t)\}(\lambda) = \frac{e^{-\lambda t_\gamma}}{\lambda}.$$

Taking Laplace transform for both sides of (3.13),

$$N(\lambda)=\sum_{\gamma=1}^{p}\Delta y(t_\gamma)\lambda^{-1}e^{-\lambda t_\gamma}+(1-\rho)\mathcal{A}N(\lambda)+\frac{\rho}{\lambda^\omega}\mathcal{A}N(\lambda)$$

where $N(\lambda)$ denotes the Laplace transform of $n(t)$.

It follows,

$$N(\lambda)-(1-\rho)\mathcal{A}N(\lambda)-\frac{\rho}{\lambda^\omega}\mathcal{A}N(\lambda)=\sum_{\gamma=1}^{p}\Delta y(t_\gamma)\lambda^{-1}e^{-\lambda t_\gamma}$$

$$\left[I-(1-\rho)\mathcal{A}-\frac{\rho}{\lambda^\omega}\mathcal{A}\right]N(\lambda)=\sum_{\gamma=1}^{p}\Delta y(t_\gamma)\lambda^{-1}e^{-\lambda t_\gamma}$$

$$\left[\mathcal{A}_\rho^{-1}-\frac{\rho}{\lambda^\omega}\mathcal{A}\right]N(\lambda)=\sum_{\gamma=1}^{p}\Delta y(t_\gamma)\lambda^{-1}e^{-\lambda t_\gamma}$$

$$\left[\lambda^\omega\mathcal{A}_\rho^{-1}-\rho\mathcal{A}\right]N(\lambda)=\sum_{\gamma=1}^{p}\Delta y(t_\gamma)\lambda^{\omega-1}e^{-\lambda t_\gamma}$$

$$\left[(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho)\mathcal{A}_\rho^{-1}\right]N(\lambda)=\sum_{\gamma=1}^{p}\Delta y(t_\gamma)\lambda^{\omega-1}e^{-\lambda t_\gamma}$$

$$N(\lambda)=\mathcal{A}_\rho(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho)^{-1}\lambda^{\omega-1}\sum_{\gamma=1}^{p}\Delta y(t_\gamma)e^{-\lambda t_\gamma}$$

$$=\mathcal{A}(\lambda^\omega I-E)^{-1}\lambda^{\omega-1}\sum_{\gamma=1}^{p}\Delta y(t_\gamma)e^{-\lambda t_\gamma}$$

$$=\mathcal{A}_\rho\lambda^{\omega-1}\sum_{\gamma=1}^{p}\int_0^\infty\Delta y(t_\gamma)e^{-\lambda^\omega s}\mathcal{T}(s)\,e^{-\lambda t_\gamma}ds.$$

$$=\mathcal{A}_\rho\lambda^{\omega-1}\sum_{\gamma=1}^{p}\int_0^\infty\Delta y(t_\gamma)e^{-(\lambda u)^\omega}\mathcal{T}(u^\omega)\,e^{-\lambda t_\gamma}\omega u^{\omega-1}du$$

$$= \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty \Delta y(t_\gamma)\omega(\lambda u)^{\omega-1} e^{-(\lambda u)^\omega} \mathcal{T}(u^\omega)\, e^{-\lambda t_\gamma} du$$

$$= \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty -\Delta y(t_\gamma) \frac{1}{\lambda} \frac{d e^{-(\lambda u)^\omega}}{du} \mathcal{T}(u^\omega)\, e^{-\lambda t_\gamma} du$$

using (3.2), we have

$$N(\lambda) = \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y(t_\gamma) \delta e^{-\lambda u \delta} \psi_\omega(\delta) \mathcal{T}(u^\omega)\, e^{-\lambda t_\gamma} d\delta du.$$

$$= \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y(t_\gamma) e^{-\lambda v} \psi_\omega(\delta) \mathcal{T}\left(\frac{v^{\ \omega}}{\delta^\omega}\right) e^{-\lambda t_\gamma} d\delta dv$$

$$= \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y(t_\gamma) e^{-\lambda(v+t_\gamma)} \psi_\omega(\delta) \mathcal{T}\left(\frac{v^{\ \omega}}{\delta^\omega}\right) d\delta dv.$$

$$= \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y(t_\gamma) e^{-\lambda t} \psi_\omega(\delta) \mathcal{T}\left(\frac{\left(t - t_\gamma\right)^\omega}{\delta^\omega}\right) d\delta dt.$$

using (3.3), we have

$$N(\lambda) = \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y(t_\gamma) e^{-\lambda t} \varphi_\omega(\theta) \mathcal{T}\left(\theta\left(t - t_\gamma\right)^\omega\right) d\theta dt$$

$$= \mathcal{A}_\rho \mathcal{L}\left\{\sum_{\gamma=1}^{p} \int_0^\infty \Delta y(t_\gamma) \varphi_\omega(\theta) \mathcal{T}\left(\theta\left(t - t_\gamma\right)^\omega\right) d\theta\right\}(\lambda)$$

$$N(\lambda) = \mathcal{A}_\rho \mathcal{L}\left\{\sum_{\gamma=1}^{p} \Delta y(t_\gamma) \mathcal{T}_\omega\left(t - t_\gamma\right)^\omega\right\}(\lambda). \tag{3.15}$$

By taking the inverse of Laplace transform for both sides of (3.15), we get

$$n(t) = \mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega\left(t - t_\gamma\right)^\omega \Delta y(t_\gamma). \tag{3.16}$$

From (3.12) and (3.16), we have

$$y(t)=\begin{cases}\mathcal{A}_\rho h(t,y(t))+\mathcal{A}_\rho\int_0^t(t-\zeta)^{\omega-1}EL_\omega(t-\zeta)h(\zeta,y(\zeta))d\zeta\\ +\mathcal{A}_\rho\mathcal{T}_\omega(t)[y_0-h(0,y_0)]+(1-\rho)\mathcal{A}_\rho[Bu(t)+f(t,y(t))]\\ +\rho\mathcal{A}_\rho^2\int_0^t(t-\zeta)^{\omega-1}L_\omega(t-\zeta)[Bu(\zeta)+f(\zeta,y(\zeta))]d\zeta \quad t\in[0,t_1]\\ \mathcal{A}_\rho h(t,y(t))+\mathcal{A}_\rho\int_0^t(t-\zeta)^{\omega-1}EL_\omega(t-\zeta)h(\zeta,y(\zeta))d\zeta\\ +\mathcal{A}_\rho\mathcal{T}_\omega(t)(y_0-h(0,y_0))+(1-\rho)\mathcal{A}_\rho[Bu(t)+f(t,y(t))]\\ +\rho\mathcal{A}_\rho^2\int_0^t(t-\zeta)^{\omega-1}L_\omega(t-\zeta)[Bu(\zeta)+f(\zeta,y(\zeta))]d\zeta\\ +\mathcal{A}_\rho\sum_{\gamma=1}^{r}\mathcal{T}_\omega(t-t_\gamma)\Delta y(t_\gamma) \qquad t\in(t_\gamma,t_{\gamma+1}].\end{cases}$$

■

Lemma 3.1.2 [46]

The operators L_ω and $\mathcal{T}_\omega$ have the following properties.

i. For any fixed $t\geq 0$, $L_\omega(t)$ and $\mathcal{T}_\omega(t)$ are linear and bounded operators, i.e. for any $y\in X$,

$$\|L_\omega(t)y\|=\left\|\omega\int_0^\infty\theta\varphi_\omega(\theta)\mathcal{T}(\theta t^\omega)yd\theta\right\|\leq\frac{\omega\mathcal{S}}{\Gamma(1+\omega)}\|y\|$$
$$=\frac{\mathcal{S}}{\Gamma(\omega)}\|y\|$$
$$\|\mathcal{T}_\omega(t)y\|=\left\|\int_0^\infty\varphi_\omega(\theta)\mathcal{T}(\theta t^\omega)yd\theta\right\|\leq\mathcal{S}\|y\|$$

where $\mathcal{S}=sup_{t\geq0}\|\mathcal{T}(t)\|$.

ii. The operators $\{L_\omega(t)\}_{t\geq0}$ and $\{\mathcal{T}_\omega(t)\}_{t\geq0}$ are strongly continuous.

iii. If $\mathcal{T}(t)$ is compact, then the operators $\{L_\omega(t)\}_{t\geq0}$ and $\{\mathcal{T}_\omega(t)\}_{t\geq0}$ are compact.

From Lemma (1.1.29), for $y \in D(\mathcal{A})$ we have

$$\frac{d}{dt}\mathcal{T}(t)y = E\mathcal{T}(t)y = \mathcal{T}(t)Ey.$$

Definition 3.1.3

The **mild solution** $y(t)$ of System (3.1) is

$$y(t) = \begin{cases} \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)\big(y_0 - h(0, y_0)\big) + (1-\rho)\mathcal{A}_\rho\big[Bu(t) + f\big(t, y(t)\big)\big] \\ +\rho\mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \quad t \in [0, t_1] \\ \\ \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)\big(y_0 - h(0, y_0)\big) + (1-\rho)\mathcal{A}_\rho\big[Bu(t) + f\big(t, y(t)\big)\big] \\ +\rho\mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \\ +\mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega\big(t - t_\gamma\big)\Delta y\big(t_\gamma\big), \qquad t \in \big(t_\gamma, t_{\gamma+1}\big]. \end{cases}$$

Next, we discuss the controllability of Nonlinear System (3.1).

Assume that the C_0-semigroup $T(t)$ is continuous in the uniform operator topology and also assume the following conditions:

H1The linear operator $W_\rho: L_p(\mathcal{J}, U) \to X$ defined as

$$W_\rho u = (1-\rho)\mathcal{A}_\rho B u + \rho\mathcal{A}_\rho^2 W u$$

has an inverse operator ${W_\rho}^{-1}$ on $L_p(\mathcal{J}, U)/kerW_\rho$ and $\|{W_\rho}^{-1}\| \le K,\ K > 0$, $\|\mathcal{A}_\rho\| \le \eta, \eta > 0$.

where W is linear operator from $L_p(\mathcal{J}, U)$ into X such that

$$Wu = \int_0^{\Upsilon} (\Upsilon - s)^{\omega-1} L_\omega(\Upsilon - s)Bu(s)ds.$$

H2: There exist constants $\mathcal{M}_h, \widehat{\mathcal{M}_h} > 0$ such that

$$\left\|Eh\big(t, y_1(t)\big) - Eh\big(t, y_2(t)\big)\right\| \leq \mathcal{M}_h \|y_1 - y_2\|$$

and

$$\widehat{\mathcal{M}_h} = sup_{t\in\mathcal{J}} \|Eh(t,0)\|.$$

H3: The continuous function $f: \mathcal{J} \times X \to X$ satisfies Lipchitz condition i.e. there exist a constant $\mathcal{M}_f > 0$ such that

$$\|f(t, y_1(t)) - f(t, y_2(t))\| \leq \mathcal{M}_f \|y_1 - y_2\|$$

and

$$\widehat{\mathcal{M}_f} = sup_{t\in\mathcal{J}} \|f(t,0)\|.$$

where $\widehat{\mathcal{M}_f} > 0$.

H4: The function $q_\gamma: X \to X, \gamma = 1,2,\dots,p$ is continuous and satisfies Lipchitz condition, i.e. there exists a constant $\mathcal{M}_\gamma > 0$ such that

$$\left\|q_\gamma(y_1) - q_\gamma(y_2)\right\| \leq \mathcal{M}_\gamma \|y_1 - y_2\|$$

and

$$\sum_{\gamma=1}^{p} \mathcal{M}_\gamma = \mathcal{M}$$

where $\mathcal{M} > 0$.

H5: For all bounded subsets Θ, the set

$$(\mathcal{K}_r^{\upsilon})(t) = \left\{\omega\rho\mathcal{A}_\rho^2 \int_0^{t-\upsilon} \int_r^{\infty} \theta(t-\zeta)^{\omega-1} \varphi_\omega(\theta) \mathcal{T}(\theta(t-\zeta)^\omega) f\big(\zeta, y(\zeta)\big) d\theta d\zeta \,, y \in \Theta\right\}$$

is relatively compact in X for arbitrary $\upsilon \in (0,t)$ and a real number $r > 0$.

Definition 3.1.4

System (3.1) is **exact controllable** on $\mathcal{J} = [0, \Upsilon]$ if for each initial state $y_0 \in X$ and each $y_1 \in X$, there is suitable control $u \in L_p(\mathcal{J}, U)$ such that the mild solution $y(t)$ of System (3.1) satisfies $y(\Upsilon) = y_1$.

Theorem 3.1.5

System (3.1) is exact controllable on $\mathcal{J} = [0, \Upsilon]$ if it satisfies the conditions **H1-H5**, and

$$0 < D + \eta K \|B\| D \left[(1-\rho) + \frac{\rho \eta \mathcal{S} \Upsilon^{\omega}}{\Gamma(\omega+1)} \right] < 1 \tag{3.17}$$

where

$$D = \eta \left[\|E^{-1}\| \mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)} \mathcal{M}_h + (1-\rho)\mathcal{M}_f + \eta\rho \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)} \mathcal{M}_f + \mathcal{S}\mathcal{M} \right].$$

Proof. Using condition **H1**, define the control $u_y(t)$ for an arbitrary function $y(\cdot) \in PC(\mathcal{J}, X)$ as

$$u_y(t) = \begin{cases} W_\rho^{-1} \Big[y_1 - \mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) - \mathcal{A}_\rho \int_0^{\Upsilon} (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \\ \quad - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)[y_0 - h(0, y_0)] - (1-\rho)\mathcal{A}_\rho f(\Upsilon, y(\Upsilon)) \\ \quad - \rho \mathcal{A}_\rho^2 \int_0^{\Upsilon} (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \Big], & t \in [0, t_1] \\ W_\rho^{-1} \Big[y_1 - \mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) - \mathcal{A}_\rho \int_0^{\Upsilon} (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \\ \quad - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) - (1-\rho)\mathcal{A}_\rho f(\Upsilon, y(\Upsilon)) \\ \quad - \rho \mathcal{A}_\rho^2 \int_0^{\Upsilon} (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \\ \quad - \mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega(\Upsilon - t_\gamma) \Delta y(t_\gamma) \Big], & t \in \big(t_\gamma, t_{\gamma+1}] \end{cases}$$

By using (3.14), we can write $u_y(t)$ in the form

$$u_y(t) = \begin{cases} W_\rho^{-1}\Big[y_1 - \mathcal{A}_\rho h(\Upsilon, y(\Upsilon)) - \mathcal{A}_\rho \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) E h(\zeta, y(\zeta)) d\zeta \\ \quad -\mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) - (1-\rho)\mathcal{A}_\rho f(\Upsilon, y(\Upsilon)) \\ \quad -\rho \mathcal{A}_\rho^2 \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) f(\zeta, y(\zeta)) d\zeta \\ \quad -\mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega(\Upsilon - t_\gamma)\Delta y(t_\gamma)\sigma_\gamma(t)\Big], \quad t \in [0, \Upsilon] \end{cases}$$

We have to show that the operator $\Phi: PC(\mathcal{J}, X) \to PC(\mathcal{J}, X)$ has a fixed point when applying this control, where

$$(\Phi y)(t) = \begin{cases} \mathcal{A}_\rho h(t, y(t)) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h(\zeta, y(\zeta)) d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)[y_0 - h(0, y_0)] + (1-\rho)\mathcal{A}_\rho[B u_y(t) + f(t, y(t))] \\ +\rho \mathcal{A}_\rho{}^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)[B u_y(\zeta) + f(\zeta, y(\zeta))] d\zeta \quad t \in [0, t_1] \\ \mathcal{A}_\rho h(t, y(t)) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h(\zeta, y(\zeta)) d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)[y_0 - h(0, y_0)] + (1-\rho)\mathcal{A}_\rho[B u_y(t) + f(t, y(t))] \\ +\rho \mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)[B u_y(\zeta) + f(\zeta, y(\zeta))] d\zeta \\ +\mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega(t - t_\gamma)\Delta y(t_\gamma) \qquad t \in (t_\gamma, t_{\gamma+1}]. \end{cases}$$

By using (3.14), we can write $(\Phi y)(t)$ in the form

$$(\Phi y)(t) = \mathcal{A}_\rho h(t, y(t)) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h(\zeta, y(\zeta)) d\zeta$$

$$+\mathcal{A}_\rho \mathcal{T}_\omega(t)[y_0 - h(0, y_0)] + (1-\rho)\mathcal{A}_\rho[B u_y(t) + f(t, y(t))]$$

$$+\rho\mathcal{A}_\rho^2\int_0^t (t-\zeta)^{\omega-1}L_\omega(t-\zeta)\big[B u_y(\zeta)+f\big(\zeta,y(\zeta)\big)\big]d\zeta$$

$$+\mathcal{A}_\rho\sum_{\gamma=1}^{p}\mathcal{T}_\omega\big(t-t_\gamma\big)\Delta y\big(t_\gamma\big)\sigma_\gamma(t),\qquad t\in\mathcal{J}.$$

By using the control $u_y(t)$, we have $(\Phi y)(\Upsilon)=y_1$, indeed,

$$(\Phi y)(\Upsilon)=\mathcal{A}_\rho h\big(\Upsilon,y(\Upsilon)\big)+\mathcal{A}_\rho\int_0^\Upsilon(\Upsilon-\zeta)^{\omega-1}\,L_\omega(\Upsilon-\zeta)Eh\big(\zeta,y(\zeta)\big)d\zeta$$

$$+\mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)[y_0-h(0,y_0)]+(1-\rho)\mathcal{A}_\rho BW_\rho^{-1}\Lambda(\Upsilon)$$

$$+(1-\rho)\mathcal{A}_\rho f\big(\Upsilon,y(\Upsilon)\big)$$

$$+\rho\mathcal{A}_\rho^2\int_0^\Upsilon(\Upsilon-\zeta)^{\omega-1}L_\omega(\Upsilon-\zeta)BW_\rho^{-1}\Lambda(\Upsilon)d\zeta$$

$$+\rho\mathcal{A}_\rho^2\int_0^\Upsilon(\Upsilon-\zeta)^{\omega-1}L_\omega(\Upsilon-\zeta)f\big(\zeta,y(\zeta)\big)d\zeta$$

$$+\mathcal{A}_\rho\sum_{\gamma=1}^{p}\mathcal{T}_\omega\big(\Upsilon-t_\gamma\big)\Delta y\big(t_\gamma\big)\sigma_\gamma(t),$$

where

$$\Lambda(t)=y_1-\mathcal{A}_\rho h\big(\Upsilon,y(\Upsilon)\big)-\mathcal{A}_\rho\int_0^\Upsilon(\Upsilon-\zeta)^{\omega-1}L_\omega(\Upsilon-\zeta)Eh\big(\zeta,y(\zeta)\big)d\zeta$$

$$-\mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)[y_0-h(0,y_0)]-(1-\rho)\mathcal{A}_\rho f\big(\Upsilon,\mathrm{y}(\Upsilon)\big)$$

$$-\rho\mathcal{A}_\rho^2\int_0^\Upsilon(\Upsilon-\zeta)^{\omega-1}L_\omega(\Upsilon-\zeta)f\big(\zeta,y(\zeta)\big)d\zeta$$

$$-\mathcal{A}_\rho\sum_{\gamma=1}^{p}\mathcal{T}_\omega\big(\Upsilon-t_\gamma\big)\Delta y\big(t_\gamma\big)\sigma_\gamma(t).$$

It follows,

$$(\Phi y)(\Upsilon) = \mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) + \mathcal{A}_\rho \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta$$
$$+ \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)[y_0 - h(0, y_0)] + \big[(1-\rho)\mathcal{A}_\rho B + \rho \mathcal{A}_\rho^2 W\big] W_\rho^{-1} \Lambda(\Upsilon)$$
$$+ (1-\rho)\mathcal{A}_\rho f\big(\Upsilon, y(\Upsilon)\big)$$
$$+ \rho \mathcal{A}_\rho^2 \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta$$
$$+ \mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega(\Upsilon - t_\gamma) \Delta y(t_\gamma) \sigma_\gamma(t)$$

$$= \mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) + \mathcal{A}_\rho \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta$$
$$+ \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)[y_0 - h(0, y_0)]$$
$$+ W_\rho W_\rho^{-1} \Bigg[y_1 - \mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) - \mathcal{A}_\rho \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta$$
$$- \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)[y_0 - h(0, y_0)] - (1-\rho)\mathcal{A}_\rho f\big(\Upsilon, \mathrm{y}(\Upsilon)\big)$$
$$- \rho \mathcal{A}_\rho^2 \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta - \mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega(\Upsilon$$
$$- t_\gamma) \Delta y(t_\gamma) \sigma_\gamma(t) \Bigg] + (1-\rho)\mathcal{A}_\rho f\big(\Upsilon, y(\Upsilon)\big)$$
$$+ \rho \mathcal{A}_\rho^2 \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta$$
$$+ \mathcal{A}_\rho \sum_{\gamma=1}^{p} T_\omega(\Upsilon - t_\gamma) \Delta y(t_\gamma) \sigma_\gamma(t).$$

$$= y_1.$$

Now, for $t \in \mathcal{J}$,

$$\|u_y(t)\| \le K\Bigg[\|y_1\| + \|\mathcal{A}_\rho\|\|h(\Upsilon, y(\Upsilon))\|$$

$$+ \|\mathcal{A}_\rho\|\frac{\mathcal{S}}{\Gamma(\omega)}\int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1}\|Eh(\zeta, y(\zeta))\|d\zeta$$

$$+ \|\mathcal{A}_\rho\|\mathcal{S}\|y_0 - h(0,y_0)\| + (1-\rho)\|\mathcal{A}_\rho\|\|f(\Upsilon, \mathrm{y}(\Upsilon))\|$$

$$+ \rho\|\mathcal{A}_\rho^2\|\frac{\mathcal{S}}{\Gamma(\omega)}\int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1}\|f(\zeta, \mathrm{y}(\zeta))\|d\zeta$$

$$+ \|\mathcal{A}_\rho\|\left\|\sum_{\gamma=1}^{p}\mathcal{T}_\omega(\Upsilon - t_\gamma)\Delta y(t_\gamma)\sigma_\gamma(t)\right\|\Bigg]$$

$$\le K\Bigg[\|y_1\| + \eta\Bigg[\|E^{-1}\|\mathcal{M}_h\|y\| + \|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)}\left(\mathcal{M}_h\|y\| + \widehat{\mathcal{M}_h}\right)$$

$$+ \mathcal{S}\left(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\right)$$

$$+ (1-\rho)\left(\mathcal{M}_f\|y\| + \widehat{\mathcal{M}_f}\right) + \eta\rho\frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)}\left(\mathcal{M}_f\|y\| + \widehat{\mathcal{M}_f}\right) + \mathcal{S}\mathcal{M}\|y\|$$

$$+ \mathcal{S}\sum_{\gamma=1}^{p}\|q_\gamma(0)\|\Bigg]\Bigg]$$

$$= K\Big[\|y_1\| + \eta\|y\|\Big[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f$$

$$+\eta\rho\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{S}\mathcal{M}\Big] + \eta\Big[\|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\widehat{\mathcal{M}_h}$$

$$+ \mathcal{S}\left(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\right) + (1-\rho)\widehat{\mathcal{M}_f}$$

$$+ \eta\rho\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\widehat{\mathcal{M}_f} + \mathcal{S}\sum_{\gamma=1}^{p}\|q_\gamma(0)\|\Big]\Big]$$

$$= K\left[D\|y\| + \|y_1\| + \widehat{D}\right],$$

where D is given in assumption and

$$\widehat{D} = \eta \left[\|E^{-1}\| \widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)} \widehat{\mathcal{M}_h} + \mathcal{S}\left(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\right) + (1-\rho)\widehat{\mathcal{M}_f} \right.$$

$$\left. + \eta\rho \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)} \widehat{\mathcal{M}_f} + \mathcal{S} \sum_{\gamma=1}^{p} \left\| q_{\gamma}(0) \right\| \right].$$

Also, for $y, \hat{y} \in PC(\mathcal{J}, X)$, we have

$$\left\| u_y - u_{\hat{y}} \right\| \le \left\| W_{\rho}^{-1} \right\| \left\| \mathcal{A}_{\rho} \right\| \left[\int_0^{\Upsilon} (\Upsilon - \zeta)^{\omega-1} \|L_{\omega}(\Upsilon - \zeta)\| \left\| Eh\left(\zeta, y(\zeta)\right) \right.\right.$$

$$\left. - Eh\left(\zeta, \hat{y}(\zeta)\right) \right\| d\zeta$$

$$+ \rho \left\| \mathcal{A}_{\rho} \right\| \int_0^{\Upsilon} (\Upsilon - \zeta)^{\omega-1} \|L_{\omega}(\Upsilon - \zeta)\| \left\| f\left(\zeta, y(\zeta)\right) \right.$$

$$\left.\left. - f\left(\zeta, \hat{y}(\zeta)\right) \right\| d\zeta + \sum_{\gamma=1}^{p} \|\mathcal{T}_{\omega}(\Upsilon - \zeta)\| \left\| q_{\gamma} y(t_{\gamma}) - q_{\gamma} \hat{y}(t_{\gamma}) \right\| \right]$$

$$\le K\eta \left[\frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)} \mathcal{M}_h \|y - \hat{y}\| + \rho\eta \frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)} \mathcal{M}_f \|y - \hat{y}\| + \mathcal{S}\mathcal{M}\|y - \hat{y}\| \right]$$

$$= K\eta\mathcal{S} \left[\frac{\Upsilon^{\omega}}{\Gamma(\omega+1)} \mathcal{M}_h + \rho\eta \frac{\Upsilon^{\omega}}{\Gamma(\omega+1)} \mathcal{M}_f + \mathcal{M} \right] \|y - \hat{y}\|.$$

Define the set

$$C_{\epsilon} = \{y : y \in PC(\mathcal{J}, X), \|y\| \le \epsilon \text{ for each } t \in \mathcal{J}\},$$

then C_{ϵ} is closed, convex and bounded subset of $PC(\mathcal{J}, X)$ for each ϵ.

Define the operators

$$(\Phi_1)y(t) = \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta$$
$$+ \mathcal{A}_\rho \mathcal{T}_\omega(t)\big(y_0 - h(0, y_0)\big) + (1-\rho)\mathcal{A}_\rho\big[B\mathcal{u}_y(t) + f\big(t, y(t)\big)\big]$$
$$+ \rho\mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) B\mathcal{u}_y(\zeta) d\zeta + \mathcal{A}_\rho \sum_{\gamma=1}^{p} \Delta y\big(t_\gamma\big)\mathcal{T}_\omega\big(t$$
$$- t_\gamma\big), \quad t \in \mathcal{J}.$$
$$(\Phi_2)y(t) = \rho\mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta, \quad t \in \mathcal{J}.$$

It is clear that

$$(\Phi_1 + \Phi_2)y = \Phi y.$$

To show operator Φ has a fixed point on C_ϵ, we need to choose $\epsilon_0 > 0$, such that $(\Phi_1 + \Phi_2)y$ has a fixed point on C_{ϵ_0}.

Taking

$$\epsilon_0 = \frac{\eta K\|B\|\left[(1-\rho) + \frac{\rho\eta\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right]\left(\|y_1\| + \widehat{D}\right) + \widehat{D}}{1 - \left[D + \eta K\|B\| D\left[(1-\rho) + \frac{\rho\eta\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right]\right]}.$$

We will show the operator $(\Phi_1 + \Phi_2)y$ has a fixed point on C_{ϵ_0}. Our proof consists of three steps:

Claim I: We will show $\Phi C_{\epsilon_0} \subset C_{\epsilon_0}$.

Let $y \in C_{\epsilon_0}$, then

$$\|(\Phi y)(t)\| \leq \|\mathcal{A}_\rho\| \Bigg[\|h(t,y(t))\| + \frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t (t-\zeta)^{\omega-1}\|Eh(\zeta,y(\zeta)\|d\zeta$$

$$+ \mathcal{S}\big\|\big(y_0 - h(0,y_0)\big)\big\| + (1-\rho)\big[\big\|B\mathcal{u}_y(t)\big\| + \big\|f\big(t,y(t)\big)\big\|\big]$$

$$+ \rho\big\|\mathcal{A}_\rho\big\|\frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t (t-\zeta)^{\omega-1}\big\|B\mathcal{u}_y(\zeta)\big\|d\zeta$$

$$+ \rho\big\|\mathcal{A}_\rho\big\|\frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t (t-\zeta)^{\omega-1}\big\|f\big(\zeta,y(\zeta)\big)\big\|d\zeta + \mathcal{S}\sum_{\gamma=1}^{p}\big\|\Delta y\big(t_\gamma\big)\big\|\Bigg]$$

$$\leq \eta\Bigg[\|E^{-1}\|\mathcal{M}_h\epsilon_0 + \|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)}\big(\mathcal{M}_h\epsilon_0 + \widehat{\mathcal{M}_h}\big)$$

$$+ \mathcal{S}\big(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\big) + (1-\rho)\|B\|\big\|\mathcal{u}_y(t)\big\|$$

$$+ (1-\rho)\big(\mathcal{M}_f\epsilon_0 + \widehat{\mathcal{M}_f}\big) + \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)}\|B\|\big\|\mathcal{u}_y(t)\big\|$$

$$+ \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)}\big(\mathcal{M}_f\epsilon_0 + \widehat{\mathcal{M}_f}\big) + \mathcal{S}\mathcal{M}\epsilon_0 + \mathcal{S}\sum_{\gamma=1}^{p}\big\|q_\gamma(0)\big\|\Bigg]$$

$$= \eta\Bigg[\epsilon_0\bigg[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{S}\mathcal{M}\bigg]$$

$$+ \|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\widehat{\mathcal{M}_h} + \mathcal{S}\big(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\big)$$

$$+ (1-\rho)\widehat{\mathcal{M}_f} + \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\widehat{\mathcal{M}_f} + \mathcal{S}\sum_{\gamma=1}^{p}\big\|q_\gamma(0)\big\|$$

$$+ \bigg[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\bigg]\|B\|\big\|\mathcal{u}_y(t)\big\|\Bigg]$$

$$= D\epsilon_0 + \widehat{D} + \bigg[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\bigg]\eta\|B\|\big\|\mathcal{u}_y(t)\big\|$$

$$\leq D\epsilon_0 + \widehat{D} + \bigg[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\bigg]\eta K\|B\|\big(D\epsilon_0 + \|y_1\| + \widehat{D}\big)$$

$$= \left[D + \eta K \|B\| D \left[(1-\rho) + \rho\eta \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)} \right] \right] \epsilon_0$$

$$+ \eta K \|B\| \left[(1-\rho) + \rho\eta \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)} \right] \left(\|y_1\| + \widehat{D} \right) + \widehat{D}$$

$$= \epsilon_0.$$

Therefore $(\Phi_1 + \Phi_2) C_{\epsilon_0} = \Phi C_{\epsilon_0} \subset C_{\epsilon_0}$.

Claim II We prove the operator Φ_1 is contraction on C_{ϵ_0}.

Let $y, \hat{y} \in C_{\epsilon_0}$, then

$$\|\Phi_1 y - \Phi_1 \hat{y}\|$$

$$\leq \left\|\mathcal{A}_\rho\right\| \Bigg[\left\| h\big(t, y(t)\big) - h(t, \hat{y}(t)) \right\|$$

$$+ \int_0^t (t-\zeta)^{\omega-1} \|L_\omega(t-\zeta)\| \|Eh(\zeta, y(\zeta) - Eh(\zeta, \hat{y}(\zeta)\| d\zeta$$

$$+ (1-\rho)\|B\| \left\| u_y(t) - u_{\hat{y}}(t) \right\|$$

$$+ (1-\rho) \left\| f\big(t, y(t)\big) - f\big(t, \hat{y}(t)\big) \right\|$$

$$+ \rho \left\|\mathcal{A}_\rho\right\| \int_0^t (t-\zeta)^{\omega-1} \|L_\omega(t-\zeta)\| \, \|B\| \left\| u_y(\zeta) \right.$$

$$\left. - u_{\hat{y}}(\zeta) \right\| d\zeta + \sum_{\gamma=1}^{p} \|\mathcal{T}_\omega\| \left\| q_\gamma y(t_\gamma) - q_\gamma \hat{y}(t_\gamma) \right\| \Bigg]$$

$$\leq \eta \Bigg[\left\| E^{-1} \right\| \mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)} \mathcal{M}_h$$

$$+ (1-\rho)\|B\| K\eta\mathcal{S} \left(\frac{\Upsilon^{\omega}}{\omega\Gamma(\omega)} \mathcal{M}_h + \rho\eta \frac{\Upsilon^{\omega}}{\omega\Gamma(\omega)} \mathcal{M}_f + \mathcal{M} \right)$$

$$+ (1-\rho)\mathcal{M}_f$$

$$+ \rho\eta \frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)} \|B\| K\eta\mathcal{S} \left(\frac{\Upsilon^{\omega}}{\omega\Gamma(\omega)} \mathcal{M}_h + \rho\eta \frac{\Upsilon^{\omega}}{\omega\Gamma(\omega)} \mathcal{M}_f + \mathcal{M} \right)$$

$$+ \mathcal{S}\mathcal{M} \Bigg] \|y - \hat{y}\|$$

$$= \eta \left[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \mathcal{S}\mathcal{M} + \|B\|K\eta\mathcal{S}\left((1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right)\left(\frac{\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + \rho\eta\frac{\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{M}\right)\right]\|y-\hat{y}\|.$$

Let

$$\mathcal{N} = \eta \left[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \mathcal{S}\mathcal{M} + \|B\|K\eta\mathcal{S}\left((1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right)\left(\frac{\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + \rho\eta\frac{\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{M}\right)\right].$$

From (3.17), we can observe that $0 < \mathcal{N} < 1$ which mean Φ_1 is a contraction.

Claim III We prove the operator Φ_2 is completely continuous.

Firstly, we prove Φ_2 is continuous.

Let y_n be a sequence in $PC(\mathcal{J}, X)$ which converge to y. Since f is continuous function, then

$$\|\Phi_2 y_n - \Phi_2 y\| \le \rho\|\mathcal{A}_\rho^2\| \int_0^t (t-\zeta)^{\omega-1}\|L_\omega(t-\zeta)\|\|f(\zeta, y_n(\zeta)) - f(\zeta, y(\zeta))\|d\zeta$$

$$\le \rho\eta^2 \frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t (t-\zeta)^{\omega-1}\|f(\zeta, y_n(\zeta)) - f(\zeta, y(\zeta))\|d\zeta$$

which converge to zero as $n \to \infty$. Therefore $\Phi_2 y_n \to \Phi_2 y$ in $PC(\mathcal{J}, X)$.

Next, we prove the family $\{\Phi_2 y: y \in C_{\epsilon_0}\}$ is relatively compact. According to Arzela – Ascoli Theorem it is suffices to prove:

- $\{\Phi_2 y: y \in C_{\epsilon_0}\}$ is uniformly bounded.
- $\{\Phi_2 y: y \in C_{\epsilon_0}\}$ is equicontinuous.
- For each $t \in \mathcal{J}$ then $\{(\Phi_2 y)(t): y \in C_{\epsilon_0}\}$ is relatively compact in X.

By definition of C_{ϵ_0} we have $\|\Phi_2 y\| \le \epsilon_0$ for any $y \in C_{\epsilon_0}$, therefore $\{\Phi_2 y: y \in C_{\epsilon_0}\}$ is uniformly bounded.

To prove $\{\Phi_2 y: y \in C_{\epsilon_0}\}$ is equicontinuous, let $t_1, t_2 \in \mathcal{J}, t_1 < t_2$ then

$$\begin{aligned}
&\|\Phi_2 y(t_2) - \Phi_2 y(t_1)\| \\
&= \left\| \int_0^{t_2} (t_2-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right. \\
&\qquad \left. - \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega(t_1-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right\| \\
&= \left\| \int_{t_1}^{t_2} (t_2-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right. \\
&\quad + \int_0^{t_1} (t_2-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \\
&\quad + \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \\
&\quad - \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \\
&\quad \left. - \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega(t_1-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right\|
\end{aligned}$$

$$\leq \left\| \int_{t_1}^{t_2} (t_2-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right\|$$

$$+ \left\| \int_0^{t_1} (t_2-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta - \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right\|$$

$$+ \left\| \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta - \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega(t_1-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right\|$$

$$= \left\| \int_{t_1}^{t_2} (t_2-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right\|$$

$$+ \left\| \int_0^{t_1} [(t_2-\zeta)^{\omega-1} - (t_1-\zeta)^{\omega-1}] L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right\|$$

$$+ \left\| \int_0^{t_1} (t_1-\zeta)^{\omega-1} [L_\omega(t_2-\zeta) - L_\omega(t_1-\zeta)] f\big(\zeta, y(\zeta)\big) d\zeta \right\|.$$

Let

$$O_1 = \left\| \int_{t_1}^{t_2} (t_2-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right\|,$$

$$O_2 = \left\| \int_0^{t_1} [(t_2-\zeta)^{\omega-1} - (t_1-\zeta)^{\omega-1}] L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right\|,$$

$$O_3 = \left\| \int_0^{t_1} (t_1-\zeta)^{\omega-1} [L_\omega(t_2-\zeta) - L_\omega(t_1-\zeta)] f\big(\zeta, y(\zeta)\big) d\zeta \right\|.$$

We have to prove O_1, O_2 and O_3 tend to zero when $t_2 \to t_1$.

By using Lemma (3.1.2) and condition **H3**, we have

$$O_1 \leq \frac{\mathcal{S}}{\Gamma(\omega)} \int_{t_1}^{t_2} (t_2-\zeta)^{\omega-1} \left\| f\big(\zeta, y(\zeta)\big) \right\| d\zeta$$

$$\leq \frac{\mathcal{S}}{\Gamma(\omega+1)} (t_2-t_1)^{\omega} \big(\mathcal{M}_f \epsilon_0 + \widehat{\mathcal{M}_f}\big)$$

which tend to zero as $t_2 \to t_1$.

Also,

$$O_2 \leq \frac{\mathcal{S}}{\Gamma(\omega)} \frac{(t_2 - t_1)^{\omega}}{\omega} \left(\mathcal{M}_f \epsilon_0 + \widehat{\mathcal{M}_f}\right)$$

which is tend to zero as $t_2 \to t_1$.

Now, for O_3, if $t_1 = 0$ then $O_3 = 0$.

If $t_1 > 0$ and s is a small enough, then

$$O_3 \leq \int_0^{t_1 - s} (t_1 - \zeta)^{\omega - 1} \|L_\omega(t_2 - \zeta) - L_\omega(t_1 - \zeta)\| \left\|f\left(\zeta, y(\zeta)\right)\right\| d\zeta$$

$$+ \int_{t_1 - s}^{t_1} (t_1 - \zeta)^{\omega - 1} \|L_\omega(t_2 - \zeta) - L_\omega(t_1 - \zeta)\| \left\|f\left(\zeta, y(\zeta)\right)\right\| d\zeta$$

$$\leq sup_{\zeta \in [0, t_1 - s]} \|L_\omega(t_2 - \zeta) - L_\omega(t_1 - \zeta)\| \frac{t_1^{\omega} - s^{\omega}}{\omega} \left(\mathcal{M}_f \epsilon_0 + \widehat{\mathcal{M}_f}\right)$$

$$+ sup_{\zeta \in [t_1 - s, t_1]} \|L_\omega(t_2 - \zeta) - L_\omega(t_1 - \zeta)\| \frac{s^{\omega}}{\omega} \left(\mathcal{M}_f \epsilon_0 + \widehat{\mathcal{M}_f}\right).$$

Since $\mathcal{T}(t), t > 0$ is continuous in the uniform operator topology, we have L_ω is continuous in the uniform operator topology, then O_3 tend to zero as $t_2 \to t_1$, $s \to 0$.

Therefore $\|\Phi_2 y(t_2) - \Phi_2 y(t_1)\|$ tend to zero independently of $y \in C_{\epsilon_0}$ as $t_2 \to t_1$ which mean the family $\left\{\Phi_2 y : y \in C_{\sigma_0}\right\}$ is equicontinuous.

Finally, we prove the set $R(t) = \left\{(\Phi_2 y)(t), y \in C_{\epsilon_0}\right\}$ for any $t \in \mathcal{J}$ is relatively compact in X.

If $t = 0$, then $R(0) = \{(\Phi_2 y)(0)\} = \{0\}$ which is compact set.

If $t \in (0, \Upsilon]$, we choose $\upsilon \in (0, t)$ and a real number $r > 0$ to define the operator

$$(\Phi_r^v y)(t) = \omega\rho\mathcal{A}_\rho^2 \int_0^{t-v}\int_r^{\infty} \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f\big(\zeta, y(\zeta)\big)d\theta d\zeta,$$

$y \in C_{\epsilon_0}$. By condition **H5** for any $t \in \mathcal{J}$, the set $(\mathcal{K}_r^v)(t) = \{(\Phi_r^v y)(t), y \in C_{\epsilon_0}\}$ is relatively compact in X. Moreover, for any $y \in C_{\epsilon_0}$ we have

$$\begin{aligned}
&\|(\Phi_2 y)(t) - (\Phi_r^v y)(t)\| \\
&= \Big\| \omega\rho\mathcal{A}_\rho^2 \int_0^{t}\int_0^{\infty} \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f\big(\zeta, y(\zeta)\big)d\theta d\zeta \\
&- \omega\rho\mathcal{A}_\rho^2 \int_0^{t-v}\int_r^{\infty} \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f\big(\zeta, y(\zeta)\big)d\theta d\zeta \Big\| \\
&= \Big\| \omega\rho\mathcal{A}_\rho^2 \int_0^{t}\int_0^{r} \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f\big(\zeta, y(\zeta)\big)d\theta d\zeta \\
&+ \omega\rho\mathcal{A}_\rho^2 \int_0^{t}\int_r^{\infty} \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f\big(\zeta, y(\zeta)\big)d\theta d\zeta \\
&- \omega\rho\mathcal{A}_\rho^2 \int_0^{t-v}\int_r^{\infty} \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f\big(\zeta, y(\zeta)\big)d\theta d\zeta \Big\| \\
&\le \Big\| \omega\rho\mathcal{A}_\rho^2 \int_0^{t}\int_0^{r} \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f\big(\zeta, y(\zeta)\big)d\theta d\zeta \Big\| \\
&+ \Big\| \omega\rho\mathcal{A}_\rho^2 \int_{t-v}^{t}\int_r^{\infty} \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f\big(\zeta, y(\zeta)\big)d\theta d\zeta \Big\| \\
&\le \omega\rho\eta^2\mathcal{S}\int_0^t (t-\zeta)^{\omega-1}\|f(\zeta, y(\zeta)\|d\zeta \int_0^r \theta\varphi_\omega(\theta)d\theta \\
&\qquad + \omega\rho\eta^2\mathcal{S}\int_{t-v}^t (t-\zeta)^{\omega-1}\|f(\zeta, y(\zeta)\|d\zeta \int_r^\infty \theta\varphi_\omega(\theta)d\theta \\
&\le \rho\eta^2\mathcal{S}\Upsilon^\omega\big(\mathcal{M}_f\epsilon_0 + \widehat{\mathcal{M}_f}\big)\int_0^r \theta\varphi_\omega(\theta)d\theta \\
&\qquad + \rho\eta^2\mathcal{S}v^\omega\big(\mathcal{M}_f\epsilon_0 + \widehat{\mathcal{M}_f}\big)\int_r^\infty \theta\varphi_\omega(\theta)d\theta.
\end{aligned}$$

Therefore, $\|(\Phi_2 y)(t) - (\Phi_r^v y)(t)\|$ tends to zero as $v, r \to 0$. Consequently, there are relatively compact sets arbitrary close to the set $R(t)$. Thus, $R(t)$ is relatively compact in X. Based on the above, the operator Φ_2 is completely continuous. According to Nussbaum Fixed Point Theorem, operator Φ has a fixed point in C_{ϵ_0}. Therefore, The System (3.1) is exact controllable on $\mathcal{J}$. ■

The next example illustrates our result.

Example 3.1.6

Consider

$$\begin{cases} {}^{C}D^{\rho,\omega}[y(t,\gamma) - h(t, y(t,\gamma))] = \mathcal{A}y(t,\gamma) + Bu(t) + f\big(t, y(t,\gamma)\big), \\ \qquad \rho, \omega \in (0,1), \gamma \in [0,\pi], t \in [0,t_1) \cup (t_1, 1], \\ y(t,0) = y(t,\pi) = 0, t \in [0,1], \\ \qquad \Delta y\big(t_1\big) = q_1\big(y(t_1^-)\big), \qquad t_1 = \dfrac{1}{2}, \end{cases}$$

and $X = L^2([0,\pi], R)$. Define

$$\mathcal{A}y(t,\gamma) = \frac{\partial^2 y}{\partial \gamma^2}(t,\gamma).$$

where

$$D(\mathcal{A}) = \left\{ y \in X\colon \frac{\partial y}{\partial \gamma}, \frac{\partial^2 y}{\partial \gamma^2} \in X \text{ and } \gamma(0) = \gamma(\pi) = 0 \right\}.$$

For $y \in D(\mathcal{A})$ then $\mathcal{A}$ can be written as the following

$$\mathcal{A}y = \sum_{s=1}^{\infty} -s^2 \langle y, y_s \rangle y_s,$$

where $y_s(\gamma) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(s\gamma)$, s=1,2,3.... Therefore $\mathcal{A}$ is the generator of C_0 −semigroup $\{\mathcal{T}(t), t \geq 0\}$ in $L^2[0,\pi]$ such that $\mathcal{T}(t)y = \sum_{s=1}^{\infty} e^{-s^2 t} \langle y, y_s \rangle y_s$, $y \in D(\mathcal{A})$ [33]. The functions h, f and q_1 defined as follows:

- $h: [0,1] \times X \to D(\mathcal{A})$ such that

$$h\big(t, y(t,\gamma)\big) = siny(t,\gamma), t \in [0,1], \gamma \in [0,\pi], y \in X.$$

- $f: [0,1] \times X \to X$ such that

$$f(t, y(t,\gamma)) = \frac{t^2 e^{-t}|y(t,\gamma)|}{b}, \qquad t \in [0,1], \gamma \in [0,\pi], y \in X, b > 0.$$

- $q_1: X \to X$ such that

$$q_1(y(t_1,\gamma)) = \frac{\left|y\left(\frac{1}{2}^-, \gamma\right)\right|}{2\left(1 + \left|y(\frac{1}{2}^-, \gamma)\right|\right)}, t \in [0,1], \gamma \in [0,\pi], y \in X.$$

For $y_1, y_2 \in X$,

$$\big\|f\big(t, y_1(t,\gamma)\big) - f\big(t, y_2(t,\gamma)\big)\big\| = \left\|\frac{t^2 e^{-t}|y_1(t,\gamma)|}{b} - \frac{t^2 e^{-t}|y_2(t,\gamma)|}{b}\right\|$$

$$\le \frac{1}{b}\|y_1 - y_2\|$$

and

$$\big\|Eh\big(t, y_1(t,\gamma)\big) - Eh\big(t, y_2(t,\gamma)\big)\big\| = \|E \sin y_1(t,\gamma) - E \sin y_2(t,\gamma)\|$$

$$\le \|E\|\|\sin y_1(t,\gamma) - \sin y_2(t,\gamma)\|$$

$$\le \|E\|\|y_1 - y_2\|.$$

Also,

$$\big\|q_1\big(y_1(t_1,\gamma)\big) - q_1\big(y_2(t_1,\gamma)\big)\big\| \le \frac{1}{2}\|y_1 - y_2\|.$$

If

$$0 < D + \eta K\|B\|D\left[(1-\rho) + \frac{\rho\eta\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right] < 1$$

where

$$D = \eta\left[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \eta\rho\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{S}\mathcal{M}\right],$$

and the condition **H5** is hold, then the hypothesis of Theorem (3.1.5) are fulfilled. Therefore, the system is exact controllable.

3.2 The Existence and Uniqueness of the Mild Solution of Impulsive Hattaf-Fractional Nonlinear Control System in Banach Space

This section investigates the existence and uniqueness of the mild solution of the impulsive Hattaf-fractional Nonlinear Control System (3.1) using Banach Fixed Point Theorem. We assume that The System (3.1) satisfies the conditions **H2**, **H3**, and **H4**, given in Section 3.1, $\|A_\rho\| \leq \eta$ and the C_0 −semigroup $T(t), t > 0$ is compact.

Theorem 3.2.1

Assume that

$$D = \eta\left[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \eta\rho\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{S}\mathcal{M}\right] < 1,$$

then The System (3.1) has a unique mild solution on $PC(\mathcal{J}, \mathrm{X})$ for each $u \in L_P(\mathcal{J}, U)$.

Proof. Define the operator

$$\left(\widehat{\Phi}y\right)(t) =$$

$$\begin{cases} \mathcal{A}_\rho h\left(t,y(t)\right) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h\left(\zeta,y(\zeta)\right) d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)\left(y_0 - h(0,y_0)\right) + (1-\rho)\mathcal{A}_\rho\left[Bu(t) + f\left(t,y(t)\right)\right] \\ +\rho\mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\left[Bu(\zeta) + f\left(\zeta,y(\zeta)\right)\right] d\zeta \quad t \in [0,t_1] \\ \mathcal{A}_\rho h\left(t,y(t)\right) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h\left(\zeta,y(\zeta)\right) d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)\left(y_0 - h(0,y_0)\right) + (1-\rho)\mathcal{A}_\rho\left[Bu(t) + f\left(t,y(t)\right)\right] \\ +\rho\mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\left[Bu(\zeta) + f\left(\zeta,y(\zeta)\right)\right] d\zeta \\ +\mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega\left(t-t_\gamma\right)\Delta y\left(t_\gamma\right), \qquad t \in \left(t_\gamma, t_{\gamma+1}\right]. \end{cases}$$

Step 1. We show the operator $\widehat{\Phi}$ maps $PC(\mathcal{J},\mathrm{X})$ into itself.

For $0 \le s < s_1 \le t_1$,

$$\begin{aligned} &\left\|\left(\widehat{\Phi}y\right)(s) - \left(\widehat{\Phi}y\right)(s_1)\right\| \\ &= \Bigg\| \mathcal{A}_\rho h\left(s,y(s)\right) + \mathcal{A}_\rho \int_0^s (s-\zeta)^{\omega-1} L_\omega(s-\zeta) E h\left(\zeta,y(\zeta)\right) d\zeta \\ &+ \mathcal{A}_\rho \mathcal{T}_\omega(s)\left(y_0 - h(0,y_0)\right) + (1-\rho)\mathcal{A}_\rho\left[Bu(s) + f\left(s,y(s)\right)\right] \\ &+ \rho\mathcal{A}_\rho^2 \int_0^s (s-\zeta)^{\omega-1} L_\omega(s-\zeta)\left[Bu(\zeta) + f\left(\zeta,y(\zeta)\right)\right] d\zeta \\ &- \mathcal{A}_\rho h\left(s_1,y(s_1)\right) - \mathcal{A}_\rho \int_0^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta) \end{aligned}$$

$$Eh\big(\zeta, y(\zeta)\big)d\zeta - \mathcal{A}_\rho \mathcal{T}_\omega(s_1)\big(y_0 - h(0, y_0)\big)$$

$$-(1-\rho)\mathcal{A}_\rho\big[Bu(s_1) + f\big(s_1, y(s_1)\big)\big]$$

$$-\rho\mathcal{A}_\rho^2 \int_0^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big]d\zeta\Bigg\|$$

$$\leq \Big\|\mathcal{A}_\rho \Big(h\big(s, y(s)\big) - h\big(s_1, y(s_1)\big)\Big)\Big\|$$

$$+ \left\|\mathcal{A}_\rho \left[\int_0^{s} (s-\zeta)^{\omega-1} L_\omega(s-\zeta) Eh\big(\zeta, y(\zeta)\big)d\zeta \right.\right.$$

$$\left.\left. - \int_0^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta) Eh\big(\zeta, y(\zeta)\big)d\zeta\right]\right\|$$

$$+ \big\|\mathcal{A}_\rho\big(y_0 - h(0, y_0)\big)\big(\mathcal{T}_\omega(s) - \mathcal{T}_\omega(s_1)\big)\big\|$$

$$+ \big\|(1-\rho)\mathcal{A}_\rho\big[B\big(u(s) - u(s_1)\big) + f\big(s, y(s)\big) - f\big(s_1, y(s_1)\big)\big]\big\|$$

$$+ \left\|\rho\mathcal{A}_\rho^2 \left[\int_0^{s} (s-\zeta)^{\omega-1} L_\omega(s-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big]d\zeta \right.\right.$$

$$\left.\left. - \int_0^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big]d\zeta\right]\right\|$$

$$\leq \|\mathcal{A}_\rho\| \left[h\big(s, y(s)\big) - h\big(s_1, y(s_1)\big) + \left\|\int_0^{s} (s-\zeta)^{\omega-1} L_\omega(s-\zeta) Eh\big(\zeta, y(\zeta)\big)d\zeta \right.\right.$$

$$- \int_0^{s} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta) Eh\big(\zeta, y(\zeta)\big)d\zeta - \int_s^{s_1} (s_1-\zeta)^{\omega-1}$$

$$L_\omega(s_1-\zeta) Eh\big(\zeta, y(\zeta)\big)d\zeta + \int_0^{s} (s_1-\zeta)^{\omega-1} L_\omega(s-\zeta) Eh\big(\zeta, y(\zeta)\big)d\zeta$$

$$- \int_0^{s} (s_1-\zeta)^{\omega-1} L_\omega(s-\zeta) Eh\big(\zeta, y(\zeta)\big)d\zeta\Bigg\|$$

$$+ \|y_0 - h(0, y_0)\|\|\mathcal{T}_\omega(s) - \mathcal{T}_\omega(s_1)\|$$

$$+(1-\rho)\big[\|B\|\|u(s)-u(s_1)\|+\|f(s,y(s))-f(s_1,y(s_1))\|\big]$$

$$+\rho\|\mathcal{A}_\rho\|\Bigg\|\int_0^s (s-\zeta)^{\omega-1}L_\omega(s-\zeta)\big[Bu(\zeta)+f(\zeta,y(\zeta))\big]d\zeta$$

$$-\int_0^s (s_1-\zeta)^{\omega-1}L_\omega(s_1-\zeta)\big[Bu(\zeta)+f(\zeta,y(\zeta))\big]d\zeta$$

$$-\int_s^{s_1} (s_1-\zeta)^{\omega-1}L_\omega(s_1-\zeta)\big[Bu(\zeta)+f(\zeta,y(\zeta))\big]d\zeta$$

$$+\int_0^s (s_1-\zeta)^{\omega-1}L_\omega(s-\zeta)\big[Bu(\zeta)+f(\zeta,y(\zeta))\big]d\zeta$$

$$-\int_0^s (s_1-\zeta)^{\omega-1}L_\omega(s-\zeta)\big[Bu(\zeta)+f(\zeta,y(\zeta))\big]d\zeta\Bigg\|$$

$$\leq \|\mathcal{A}_\rho\|\Bigg[\|h(s,y(s))-h(s_1,y(s_1))\|$$

$$+\Bigg\|\int_s^{s_1} (s_1-\zeta)^{\omega-1}L_\omega(s_1-\zeta)Eh(\zeta,y(\zeta))d\zeta\Bigg\|$$

$$+\int_0^s \|(s_1-\zeta)^{\omega-1}\|L_\omega(s-\zeta)-L_\omega(s_1-\zeta)\|Eh(\zeta,y(\zeta))\|d\zeta$$

$$+\int_0^s \|(s-\zeta)^{\omega-1}-(s_1-\zeta)^{\omega-1}\|\|L_\omega(s-\zeta)\|\|Eh(\zeta,y(\zeta))\|d\zeta$$

$$+\|y_0-h(0,y_0)\|\|\mathcal{T}_\omega(s)-\mathcal{T}_\omega(s_1)\|$$

$$+(1-\rho)\|B\|\|u(s)-u(s_1)\|$$

$$+(1-\rho)\|f(s,y(s))-f(s_1,y(s_1))\|$$

$$+\rho\|\mathcal{A}_\rho\|\Bigg\|\int_s^{s_1} (s_1-\zeta)^{\omega-1}\|L_\omega(s_1-\zeta)\|\|Bu(\zeta)$$

$$+f(\zeta,y(\zeta))\|d\zeta\Bigg\|$$

$$+\rho\|\mathcal{A}_\rho\|\int_0^s (s_1-\zeta)^{\omega-1}\|L_\omega(s-\zeta)-L_\omega(s_1-\zeta)\|\|Bu(\zeta)$$
$$+f(\zeta,y(\zeta))\|d\zeta+\rho\|\mathcal{A}_\rho\|\int_0^s \|(s-\zeta)^{\omega-1}$$
$$-(s_1-\zeta)^{\omega-1}\|\|L_\omega(s-\zeta)\|\|Bu(\zeta)+f(\zeta,y(\zeta))\|d\zeta\Big]$$

$$\leq \eta\Big[\|h(s,y(s))-h(s_1,y(s_1))\|+\frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)}sup_{\xi\in[s,s_1]}\|Eh(\xi,y(\xi))\|$$
$$+sup_{\xi\in[0,s]}\|L_\omega(s-\zeta)-L_\omega(s_1-\zeta)\|sup_{\xi\in[0,s]}\|Eh(\zeta,y(\zeta))\|$$
$$\times\left[\frac{s_1^\omega}{\omega}-\frac{(s_1-s)^\omega}{\omega}\right]+\frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)}sup_{\xi\in[0,s]}\|Eh(\zeta,y(\zeta))\|+\|y_0-h(0,y_0)\|$$
$$\|\mathcal{T}_\omega(s)-\mathcal{T}_\omega(s_1)\|+(1-\rho)\|B\|\|u(s)-u(s_1)\|$$
$$+(1-\rho)\|f(s,y(s))-f(s_1,y(s_1))\|$$
$$+\rho\eta\frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)}sup_{\xi\in[s,s_1]}\|Bu(\zeta)+f(\zeta,y(\zeta))\|$$
$$+\rho\eta sup_{\xi\in[0,s]}\|L_\omega(s-\zeta)-L_\omega(s_1-\zeta)\|sup_{\xi\in[0,s]}\|Bu(\zeta)$$
$$+f(\zeta,y(\zeta))\|\left[\frac{s_1^\omega}{\omega}-\frac{(s_1-s)^\omega}{\omega}\right]$$
$$+\rho\eta\frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)}sup_{\xi\in[0,s]}\|Bu(\zeta)+f(\zeta,y(\zeta))\|\Big].$$

Let

$$O_1=\eta\|h(s,y(s))-h(s_1,y(s_1))\|$$

$$O_2=\eta\frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)}sup_{\xi\in[s,s_1]}\|Eh(\xi,y(\xi))\|$$

$$O_3 = \eta sup_{\xi\in[0,s]}\|L_\omega(s-\zeta) - L_\omega(s_1-\zeta)\| sup_{\xi\in[0,s]}\left\|Eh\big(\zeta, y(\zeta)\big)\right\| \left[\frac{s_1^\omega}{\omega} - \frac{(s_1-s)^\omega}{\omega}\right]$$

$$O_4 = \eta \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[0,s]}\left\|Eh\big(\zeta, y(\zeta)\big)\right\|$$

$$O_5 = \eta\|y_0 - h(0,y_0)\|\|\mathcal{T}_\omega(s) - \mathcal{T}_\omega(s_1)\|$$

$$O_6 = (1-\rho)\eta\|B\|\|u(s) - u(s_1)\|$$

$$O_7 = (1-\rho)\eta\left\|f\big(s,y(s)\big) - f\big(s_1,y(s_1)\big)\right\|$$

$$O_8 = \rho\eta^2 \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[s,s_1]}\left\|Bu(\zeta) + f\big(\zeta,y(\zeta)\big)\right\|$$

$$O_9 = \rho\eta^2 sup_{\xi\in[0,s]}\|L_\omega(s-\zeta) - L_\omega(s_1-\zeta)\| sup_{\xi\in[0,s]}\left\|Bu(\zeta) + f\big(\zeta,y(\zeta)\big)\right\| \left[\frac{s_1^\omega}{\omega} - \frac{(s_1-s)^\omega}{\omega}\right]$$

$$O_{10} = \rho\eta^2 \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[0,s]}\left\|Bu(\zeta) + f\big(\zeta,y(\zeta)\big)\right\|.$$

Since f, h are continuous functions on $\mathcal{J}$, then O_1, O_7 tend to zero as $s \to s_1$.

Since $L_\omega, \mathcal{T}_\omega$ are continuous in the uniform operator topology, then

O_3, O_5 and O_9 tend to zero as $s \to s_1$.

Since u is measurable, then $u(s) \to u(s_1)$ almost every where $s \to s_1$, then O_6 tend to zero.

Clearly O_2, O_4, O_8 and O_{10} tend to zero as $s \to s_1$. Therefore

$\left\|(\widehat{\Phi}y)(s)-(\widehat{\Phi}y)(s_1)\right\| \to 0$ as $s \to s_1$. Thus $(\widehat{\Phi}y)(t) \in C[0,t_1]$.

Now, for $t_\gamma < s < s_1 \le t_{\gamma+1}$, we have

$$
\begin{aligned}
&\left\|(\widehat{\Phi}y)(s)-(\widehat{\Phi}y)(s_1)\right\| \\
&\qquad \le \|\mathcal{A}_\rho\| \Bigg[\left\|h(s,y(s))-h(s_1,y(s_1))\right\| \\
&\qquad + \left\| \int_s^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta) E h(\zeta,y(\zeta)) d\zeta \right\| \\
&\qquad + \int_0^s \|(s_1-\zeta)^{\omega-1}\| L_\omega(s-\zeta) - L_\omega(s_1-\zeta) \left\| E h(\zeta,y(\zeta)) \right\| d\zeta \\
&\qquad + \int_0^s \|(s-\zeta)^{\omega-1}-(s_1-\zeta)^{\omega-1}\| \|L_\omega(s-\zeta)\| \left\| E h(\zeta,y(\zeta)) \right\| d\zeta \\
&\qquad + \|y_0 - h(0,y_0)\| \|\mathcal{T}_\omega(s)-\mathcal{T}_\omega(s_1)\| \\
&\qquad + (1-\rho)\|B\| \|u(s)-u(s_1)\| \\
&\qquad + (1-\rho)\left\| f(s,y(s)) - f(s_1,y(s_1)) \right\| \\
&\qquad + \rho\|\mathcal{A}_\rho\| \left\| \int_s^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta) \left[Bu(\zeta)+f(\zeta,y(\zeta))\right] d\zeta \right\| \\
&\qquad + \rho\|\mathcal{A}_\rho\| \int_0^s (s_1-\zeta)^{\omega-1} \|L_\omega(s-\zeta)-L_\omega(s_1-\zeta)\| \big[Bu(\zeta) \\
&\qquad + f(\zeta,y(\zeta))\big] d\zeta \\
&\qquad + \rho\|\mathcal{A}_\rho\| \int_0^s \|(s-\zeta)^{\omega-1}-(s_1-\zeta)^{\omega-1}\| L_\omega(s-\zeta) \big[Bu(\zeta) \\
&\qquad + f(\zeta,y(\zeta))\big] d\zeta + \Sigma \Bigg],
\end{aligned}
$$

where

$$
\Sigma = \sum_{\gamma=1}^{p} \left\| \mathcal{T}_\omega(s_1-t_\gamma) - \mathcal{T}_\omega(s-t_\gamma) \right\| \left\| \Delta y(t_\gamma) \right\|.
$$

Since $\mathcal{T}_\omega$ is continuous in the uniform operator topology, then Σ tend to zero as $s \to s_1$, and from above we have $\|(\widehat{\Phi}y)(s) - (\widehat{\Phi}y)(s_1)\|$ tend to zero as $s \to s_1$. Therefore $\widehat{\Phi}y \in PC[0, \Upsilon]$.

Step 2. We show the operator $\widehat{\Phi}$ is contraction on $PC(\mathcal{J}, X)$.

For $y_1, y_2 \in PC(\mathcal{J}, X)$, and for each $t \in [0, t_1]$,

$$\begin{aligned}
&\|(\widehat{\Phi}y_1)(t) - (\widehat{\Phi}y_2)(t)\| \\
&\qquad \le \|\mathcal{A}_\rho[h(t, y_1(t)) - h(t, y_2(t))]\| \\
&\qquad + \left\|\mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E[h(\zeta, y_1(\zeta)) - h(\zeta, y_2(\zeta))] d\zeta\right\| \\
&\qquad + \|(1-\rho)\mathcal{A}_\rho[f(t, y_1(t)) - f(t, y_2(t))]\| \\
&\qquad + \left\|\rho \mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)[f(\zeta, y_1(\zeta)) - f(\zeta, y_2(\zeta))] d\zeta\right\|
\end{aligned}$$

$$\begin{aligned}
&\le \eta \|h(t, y_1(t)) - h(t, y_2(t))\| + \eta \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_h \|y_1 - y_2\| \\
&\qquad + (1-\rho)\eta \|f(t, y_1(t)) - f(t, y_2(t))\| \\
&\qquad + \rho\eta^2 \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_f \|y_1 - y_2\|
\end{aligned}$$

$$\begin{aligned}
&\le \eta \mathcal{M}_h E^{-1} \|y_1 - y_2\| + \eta \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_h \|y_1 - y_2\| + (1-\rho)\eta \mathcal{M}_f \|y_1 - y_2\| \\
&\qquad + \rho\eta^2 \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_f \|y_1 - y_2\|
\end{aligned}$$

$$\begin{aligned}
&= \eta \left[\mathcal{M}_h E^{-1} + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_h + (1-\rho)\mathcal{M}_f + \rho \|\mathcal{A}_\rho\| \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_f\right] \|y_1 \\
&\qquad - y_2\|.
\end{aligned}$$

Now, for $t \in (t_\gamma, t_{\gamma+1}]$, using our assumption we have,

$$\left\|\left(\widehat{\Phi}y_1\right)(t)-\left(\widehat{\Phi}y_2\right)(t)\right\| \leq$$

$$= \eta\left[\mathcal{M}_h E^{-1}+\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h+(1-\rho)\mathcal{M}_f\right.$$

$$\left.+\rho\left\|\mathcal{A}_\rho\right\|\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f\right]\|y_1-y_2\|+\eta\mathcal{M}\mathcal{S}\|y_1-y_2\|$$

$$= D\|y_1-y_2\|$$

and by our assumption, then $\widehat{\Phi}$ is contraction. According to Banach Fixed Point Theorem, the operator $\widehat{\Phi}$ has unique fixed point y such that $\widehat{\Phi}y = y$. Therefore the system (3.1) has a unique mild solution for any control $u \in L_P(\mathcal{J}, U)$. ■

3.3. Approximate Controllability of Impulsive Hattaf-Fractional Nonlinear Control System in Banach Space

The approximate controllability of the impulsive Hattaf-Fractional Nonlinear Control System (3.1) investigated throughout this section. Assume System (3.1) meets the conditions **H2**, **H3**, and **H4** outlined in Section 3.1. Define the bounded linear operator $\Lambda: L_P(\mathcal{J}, X) \to X$ as

$$\Lambda(\mathcal{y}) = (1-\rho)\mathcal{A}_\rho \mathcal{y}(\Upsilon)+\rho\mathcal{A}_\rho^2\int_0^{\Upsilon}(\Upsilon-s)^{\omega-1}L_\omega(\Upsilon-s)\mathcal{y}(s)ds.$$

The following condition is important to prove the approximate controllability of System (3.1),

H6 $\forall\, \epsilon > 0, \forall\, \mathcal{y} \in L_P(\mathcal{J}, X), \exists\, \mathcal{u} \in L_P(\mathcal{J}, U)$ such that

$$\|\Lambda(\mathcal{y})-\Lambda(B\mathcal{u})\| < \epsilon$$

and

$$\|B\mathcal{u}(\cdot)\| < \lambda\|\mathcal{y}(\cdot)\|$$

where $\lambda > 0$.

Definition 3.3.1

The System (3.1) is **approximately controllable** on $\mathcal{J}$ if $\overline{\mathcal{K}_{\Upsilon}(y)} = X$, where $\mathcal{K}_{\Upsilon}(y) = \{y(\Upsilon; u): u(\mathrm{t}) \in U\}$ is a reachable set of System (3.1).

Lemma 3.3.2

Assume the condition **H6** is hold, then

i. $\|\mathrm{y(t)}\| \leq D\|y(t)\| + \widehat{D} + \left[(1-\rho) + \rho\eta \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right]\eta\|B\|\|u(t)\|.$

where

$$\widehat{D} = \eta\left[\|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_h} + \mathcal{S}\big(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\big)\right.$$
$$\left. +(1-\rho)\widehat{\mathcal{M}_f} + \eta\rho\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_f} + \mathcal{S}\sum_{\gamma=1}^{p}\|q_{\gamma}(0)\|\right].$$

ii. For $y_1, y_2 \in X$, then

$$\|y_2(t) - y_1(t)\| \leq \frac{\eta}{1-D}\left[(1-\rho)\|B\| + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\|B\|\right]\|u_2(t) - u_1(t)\|.$$

Proof.

i. $\|\mathrm{y(t)}\| \leq \|\mathcal{A}_{\rho}\|\Big[\|h(t, y(t))\| + \frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t (t-\zeta)^{\omega-1}\|Eh(\zeta, y(\zeta)\|d\zeta + \mathcal{S}\|\big(y_0 - h(0, y_0)\big)\| + (1-\rho)\big[\|Bu(t)\| + \|f\big(t, y(t)\big)\|\big] + \rho\|\mathcal{A}_{\rho}\|\frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t (t-\zeta)^{\omega-1}\|Bu(\zeta)\|d\zeta + \rho\|\mathcal{A}_{\rho}\|\frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t (t-\zeta)^{\omega-1}\|f\big(\zeta, y(\zeta)\big)\|d\zeta + \mathcal{S}\sum_{\gamma=1}^{p}\|\Delta y\big(t_{\gamma}\big)\|\Big]$

$$\leq \eta \Bigg[\|E^{-1}\|\mathcal{M}_h\|y\| + \|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\left(\mathcal{M}_h\|y\| + \widehat{\mathcal{M}_h}\right)$$

$$+ \mathcal{S}\left(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\right) + (1-\rho)\|B\|\|u(t)\|$$

$$+ (1-\rho)\left(\mathcal{M}_f\|y\| + \widehat{\mathcal{M}_f}\right) + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\|B\|\|u(t)\|$$

$$+ \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\left(\mathcal{M}_f\|y\| + \widehat{\mathcal{M}_f}\right) + \mathcal{S}\mathcal{M}\|y\| + \mathcal{S}\sum_{\gamma=1}^{p}\left\|q_\gamma(0)\right\| \Bigg]$$

$$= \eta \Bigg[\|y\| \left[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f \right.$$

$$\left. + \mathcal{S}\mathcal{M} \right] + \|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_h}$$

$$+ \mathcal{S}\left(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\right) + (1-\rho)\widehat{\mathcal{M}_f} + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_f}$$

$$+ \mathcal{S}\sum_{\gamma=1}^{p}\left\|q_\gamma(0)\right\| + \left[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right]\|B\|\|u(t)\| \Bigg]$$

$$= D\|y\| + \widehat{D} + \left[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right]\eta\|B\|\|u(t)\|$$

ii. $\|y_2(t) - y_1(t)\| \leq \left\|\mathcal{A}_\rho\right\| \left[\left\|h\left(t, y_2(t)\right) - h\left(t, y_1(t)\right)\right\| + \int_0^t (t - \zeta)^{\omega-1} \|L_\omega(t-\zeta)\|\|Eh(\zeta, y_2(\zeta) - Eh(\zeta, y_1(\zeta)\|d\zeta + (1 - \rho)\|B\|\|u_2(t) - u_1(t)\| + (1-\rho)\left\|f\left(t, y_2(t)\right) - f\left(t, y_1(t)\right)\right\| + \rho\left\|\mathcal{A}_\rho\right\| \int_0^t (t-\zeta)^{\omega-1}\|L_\omega(t-\zeta)\|\,\|B\|\|u_2(\xi) - u_1(\xi)\|d\zeta + \sum_{\gamma=1}^{p}\|\mathcal{T}_\omega\|\left\|q_\gamma y_2\left(t_\gamma\right) - q_\gamma y_1\left(t_\gamma\right)\right\| \right]$

$$\leq \eta\left[\|E^{-1}\|\mathcal{M}_h\|y_2-y_1\|+\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h\|y_2-y_1\|\right.$$
$$+(1-\rho)\|B\|\|u_2(t)-u_1(t)\|+(1-\rho)\mathcal{M}_f\|y_2-y_1\|$$
$$+\rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\left[\|B\|\|u_2(t)-u_1(t)\|+\mathcal{M}_f\|y_2-y_1\|\right]$$
$$\left.+\mathcal{S}\mathcal{M}\|y_2-y_1\|\right]$$
$$=\eta\left[\|E^{-1}\|\mathcal{M}_h+\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h+(1-\rho)\mathcal{M}_f+\rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f+\mathcal{S}\mathcal{M}\right]\|y_2$$
$$-y_1\|+\eta\left[(1-\rho)\|B\|+\rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\|B\|\right]\|u_2(t)-u_1(t)\|$$
$$=D\|y_2-y_1\|+\eta\left[(1-\rho)\|B\|+\rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\|B\|\right]\|u_2(t)-u_1(t)\|.$$

It follows,

$$\|y_2-y_1\|-D\|y_2-y_1\|$$
$$\leq \eta\left[(1-\rho)\|B\|+\rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\|B\|\right]\|u_2(t)-u_1(t)\|$$

Thus,

$$\|y_2-y_1\|\leq\frac{\eta}{1-D}\left[(1-\rho)\|B\|+\rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\|B\|\right]\|u_2(t)-u_1(t)\|.$$

Theorem 3.3.3

Suppose the condition **H6** is holds, then The System (3.1) is approximate controllability provided

$$\left(\mathcal{M}_f+\frac{\|\mathcal{A}_\rho^{-1}\|}{\rho}\mathcal{M}_h\right)\lambda\frac{\eta}{1-D}\left[(1-\rho)+\rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right]\|B\|<1, \qquad (3.18)$$

where $\lambda>0$.

Proof. Since the domain $D(\mathcal{A})$ of operator $\mathcal{A}$ is dense in X [33], i.e. $\overline{D(\mathcal{A})} = X$. It sufficient to prove $D(\mathcal{A}) \subset \mathcal{K}_\Upsilon(y)$, that is mean we must show for any $\epsilon > 0$ and $x \in D(\mathcal{A})$, there exists $u \in L_P(\mathcal{J}, U)$ such that

$$\left\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) - \mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) + \frac{1-\rho}{\rho} Eh\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathcal{A}_\rho^{-1}}{\rho} \Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big) \right\|$$

$$= \left\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) + \left(\frac{1-\rho}{\rho} \rho \mathcal{A}\mathcal{A}_\rho - \mathcal{A}_\rho\right) h\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathcal{A}_\rho^{-1}}{\rho} \Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big) \right\|$$

$$= \left\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) + \big((1-\rho)\mathcal{A} - I\big)\mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathcal{A}_\rho^{-1}}{\rho} \Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big) \right\|$$

$$= \left\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) - \mathcal{A}_\rho^{-1}\mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathcal{A}_\rho^{-1}}{\rho} \Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big) \right\|$$

$$= \left\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) - h\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathcal{A}_\rho^{-1}}{\rho} \Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big) \right\| < \epsilon$$

where $\Sigma_\Upsilon = \sum_{\gamma=1}^{p} \mathcal{T}_\omega(\Upsilon - t_\gamma) \Delta y(t_\gamma) \sigma_\gamma(\Upsilon)$.

For any initial $y_0 \in X$, since $\mathcal{T}(t)$ is differentiability semigroup for each

$t > 0$ then

$$\big[\mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) + h\big(\Upsilon, y(\Upsilon)\big) + \Sigma_\Upsilon\big] \in D(\mathcal{A})$$

and we can see there exists a function $\mathcal{Q} \in L_p(\mathcal{J}, X)$ such that

$$\Lambda(\mathcal{Q}(\cdot)) = x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) - h(\Upsilon, y(\Upsilon)) - \Sigma_\Upsilon.$$

For example,

$\mathcal{Q}(t) =$

$$\begin{cases} 0 & t = \Upsilon \\ \dfrac{(\Upsilon - t)^{1-\omega}}{\Upsilon\rho} (\Gamma(\omega))^2 \mathcal{A}_\rho^{-2} \left(L_\omega(\Upsilon - t) + 2t \dfrac{d}{dt} L_\omega(\Upsilon - t) \right) & t \in [0, \Upsilon) \\ (x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) - h(\Upsilon, y(\Upsilon)) - \Sigma_\Upsilon) & \end{cases}$$

then,

$$\begin{aligned} \Lambda(\mathcal{Q}(t)) &= \rho \mathcal{A}_\rho^2 \int_0^\Upsilon (\Upsilon - s)^{\omega-1} L_\omega(\Upsilon - s) \frac{(\Upsilon - s)^{1-\omega}}{\Upsilon\rho} (\Gamma(\omega))^2 \mathcal{A}_\rho^{-2} \Bigg(L_\omega(\Upsilon \\ &\quad - s) + 2s \frac{d}{ds} L_\omega(\Upsilon - s) \Bigg) \big(x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) \\ &\quad - h(\Upsilon, y(\Upsilon)) - \Sigma_\Upsilon\big) ds \\ &= \frac{(\Gamma(\omega))^2}{\Upsilon} \int_0^\Upsilon \left((L_\omega(\Upsilon - s))^2 + 2s L_\omega(\Upsilon - s) \frac{d}{ds} L_\omega(\Upsilon - s) \right) \big(x \\ &\quad - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) - h(\Upsilon, y(\Upsilon)) - \Sigma_\Upsilon\big) ds \\ &= \frac{(\Gamma(\omega))^2}{\Upsilon} \Bigg[\int_0^\Upsilon (L_\omega(\Upsilon - s))^2 \big(x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) - h(\Upsilon, y(\Upsilon)) \\ &\quad - \Sigma_\Upsilon\big) ds \\ &\quad + \int_0^\Upsilon s \frac{d}{ds} (L_\omega(\Upsilon - s))^2 \big(x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) \\ &\quad - h(\Upsilon, y(\Upsilon)) - \Sigma_\Upsilon\big) ds \Bigg]. \end{aligned}$$

Using integral by parts, we have

$$\Lambda\big(\mathcal{Q}(t)\big) = \frac{\big(\Gamma(\omega)\big)^2}{\Upsilon}\Bigg[\int_0^{\Upsilon}\big(L_\omega(\Upsilon - s)\big)^2\big(x - \mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon\big)ds + \frac{\Upsilon}{\big(\Gamma(\omega)\big)^2}\big(x - \mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon\big) - \int_0^{\Upsilon}\big(L_\omega(\Upsilon - s)\big)^2\big(x - \mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon\big)ds\Bigg]$$

$$= \big(x - \mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon\big).$$

Now, for any given $\epsilon > 0$ and by **H6** there exists a control u such that

$$\big\|x - \mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\big(Bu(t)\big)\big\| < \frac{\epsilon}{2^3}. \tag{3.19}$$

Let $u_1 \in L_p(\mathcal{J}, U)$, then by **H6**, there exists $u_2 \in L_p(\mathcal{J}, U)$, such that

$$\left\|\Lambda\left[Bu(t) - f\big(t, y_1(t)\big) - \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t, y_1(t)\big)\right] - \Lambda\big(Bu_2(t)\big)\right\| < \frac{\epsilon}{2^3} \tag{3.20}$$

where $y_1(t) = y(\mathrm{t}; u_1), t \in \mathcal{J}$.

From (3.19) and (3.20) we have

$$\left\|x - \mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\left[f\big(t, y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t, y_1(t)\big)\right] - \Lambda\big(Bu_2(t)\big)\right\|$$

$$= \Big\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\big(Bu(t)\big)$$

$$+ \Lambda\big(Bu(t)\big) - \Lambda\left[f\big(t, y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_1(t)\big)\right]$$

$$- \Lambda\big(Bu_2(t)\big)\Big\|$$

$$\leq \left\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\big(Bu(t)\big)\right\|$$

$$+ \Big\| \Lambda\big(Bu(t)\big) - \Lambda\left[f\big(t, y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_1(t)\big)\right]$$

$$- \Lambda\big(Bu_2(t)\big)\Big\|$$

$$\leq \frac{\epsilon}{2^2}.$$

Denote $y_2 = y(\mathrm{t}; u_2), t \in \mathcal{J}$, then by **H6**, there exists $w_2 \in L_p(\mathcal{J}, U)$ such that

$$\Big\| \Lambda\left[f\big(t, y_2(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_2(t)\big)\right] - \Lambda\left[f\big(t, y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_1(t)\big)\right]$$

$$- \Lambda\big(Bw_2(t)\big)\Big\| < \frac{\epsilon}{2^3},$$

and

$$\|Bw_2(t)\| < \lambda \Big\| f\big(t, y_2(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_2(t)\big) - f\big(t, y_1(t)\big)$$

$$- \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_1(t)\big)\Big\|$$

$$\leq \lambda \left\| f\big(t, y_2(t)\big) - f\big(t, y_1(t)\big)\right\| + \lambda \frac{\left\|\mathcal{A}_\rho^{-1}\right\|}{\rho} \left\| Eh\big(t, y_2(t)\big) - Eh\big(t, y_1(t)\big)\right\|$$

$$\leq \left(\mathcal{M}_f + \frac{\|\mathcal{A}_\rho^{-1}\|}{\rho}\mathcal{M}_h\right)\lambda\|y_2 - y_1\|$$

$$\leq \left(\mathcal{M}_f + \frac{\|\mathcal{A}_\rho^{-1}\|}{\rho}\mathcal{M}_h\right)\lambda\frac{\|\mathcal{A}_\rho\|}{1-D}\left[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right]\|B\|\|u_2(t) - u_1(t)\|.$$

Let $u_3(t) = u_2(t) - w_2(t)$, $u_3(\cdot) \in L_p(\mathcal{J}, U)$. It follows

$$\Bigg\|x - \mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\left[f\big(t,y_2(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_2(t)\big)\right] - \Lambda\big(Bu_3(t)\big)\Bigg\|$$

$$= \Bigg\|x - \mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon + \Lambda\left[f\big(t,y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_1(t)\big)\right] - \Lambda\left[f\big(t,y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_1(t)\big)\right] - \Lambda\left[f\big(t,y_2(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_2(t)\big)\right] - \Lambda\big(Bu_3(t)\big)\Bigg\|$$

$$\leq \Bigg\|x - \mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\left[f\big(t,y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_1(t)\big)\right] - \Lambda\big(Bu_2(t)\big)\Bigg\| + \Bigg\|\Lambda\left[f\big(t,y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_1(t)\big)\right] - \Lambda\left[f\big(t,y_2(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_2(t)\big)\right] + \Lambda\big(Bw_2(t)\big)\Bigg\|$$

$$< \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3}.$$

By Mathematical Induction, we can see that the sequence $\{u_n, n = 0,1,2,\dots\} \subset L_p(\mathcal{J}, U)$, consequently,

$$\Bigg\|x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\left[f\big(t, y_n(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_n(t)\big)\right] - \Lambda\big(B\mathcal{U}_{n+1}(t)\big)\Bigg\| < \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3} + \cdots + \frac{\epsilon}{2^{n+1}}$$

where $y_n = y(\mathrm{t}; \mathcal{U}_n)$ and

$$\|Bu_{n+1} - Bu_n\| \leq \left(\mathcal{M}_f + \frac{\|\mathcal{A}_\rho^{-1}\|}{\rho} \mathcal{M}_h\right) \lambda \frac{\|\mathcal{A}_\rho\|}{1 - D} \left((1 - \rho) + \rho\eta \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega + 1)}\right) \|Bu_n(t) - Bu_{n-1}(t)\|$$

and from our assumption we get the sequence $\{B\mathcal{U}_n(t), n = 1,2,3, \ldots\}$ is a Cauchy sequence on X. Since X is a Banach space, then there exists a point $\delta(t) \in X$ such that $B\mathcal{U}_n \to \delta(t)$ as $n \to \infty$. Then for any $\epsilon > 0$, there exists a positive integer k such that

$$\begin{aligned}
&\Bigg\|x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\left[f\big(t, y_k(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_k(t)\big)\right] - \Lambda\big(B\mathcal{U}_k(t)\big)\Bigg\| \\
&\leq \Bigg\|x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\big[f\big(t, y_k(t)\big)\big] - \Lambda\left[\frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_k(t)\big)\right] - \Lambda\big(B\mathcal{U}_{k+1}(t)\big)\Bigg\| \\
&\quad + \big\|\Lambda\big(B\mathcal{U}_{k+1}(t)\big) - \Lambda\big(B\mathcal{U}_k(t)\big)\big\| \\
&< \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3} + \cdots + \frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Therefore, we get a sequence $\{y_k, k = 1,2, \ldots\} \subset \mathcal{K}_\Upsilon(f)$ converge to $x \in D(\mathcal{A})$, thus $x \in \overline{\mathcal{K}_\Upsilon(f)}$, which mean $\overline{\mathcal{K}_\Upsilon(f)} = X$. ■

The next example illustrates our result.

Example 3.3.4

Consider the following nonlinear Hattaf-fractional control system

$$\begin{cases} {}^{C}D^{\frac{1}{2},\frac{1}{3}}\left[y(t,\gamma)-h\big(t,y(t,\gamma)\big)\right] = \mathcal{A}y(t,y)+Bu(t)+f\big(t,y(t,\gamma)\big), \\ \qquad \gamma\in[0,\pi], t\in[0,t_1)\cup(t_1,1], \\ y(t,0)=y(t,\pi)=0, t\in[0,1], \\ \qquad \Delta y(t_1^+)=q_1\big(y(t_1^-)\big), \qquad t_1=\dfrac{1}{2}, \end{cases} \tag{3.21}$$

Setting $X=L_2([0.\pi],R)=U$, and define the operator $\mathcal{A}: D(\mathcal{A})\subset X\to X$ by

$$\mathcal{A}y(t,\gamma)=\frac{\partial^2 y}{\partial\gamma^2}(t,\gamma).$$

where

$$D(\mathcal{A})=\left\{y\in X: \frac{\partial y}{\partial\gamma},\frac{\partial^2 y}{\partial\gamma^2}\in X \text{ and } \gamma(0)=\gamma(\pi)=0\right\}.$$

For $y\in D(\mathcal{A})$ then $\mathcal{A}$ can be written as the following

$$\mathcal{A}y=\sum_{s=1}^{\infty}-s^2\langle y,y_s\rangle y_s,$$

where $y_s(\gamma)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin(s\gamma)$, $s=1,2,3$ .... Then $\{y_s(\gamma)\}$ is an orthonormal basis for X and y_s is an eigenfunction corresponding to the eigenvalue $\lambda_s=-s^2$ of the operator $\mathcal{A}$, $s=1,2,3$ ....

Therefore, $\mathcal{A}$ is the generator of C_0 −semigroup $\{\mathcal{T}(t), t\geq 0\}$ in $L^2[0,\pi]$ such that $\mathcal{T}(t)y=\sum_{s=1}^{\infty}e^{-s^2t}\langle y,y_s\rangle y_s$, $y\in D(\mathcal{A})$, and $\|\mathcal{T}(t)\|<1=\mathcal{S}$. The functions h, f and q_1 defined as follows:

- $h:[0,1]\times X\to D(\mathcal{A})$ such that

$$h\big(t,y(t,\gamma)\big)=\int_0^{\gamma}\sin y(t,\zeta)\,d\zeta\,, t\in[0,1], \gamma\in[0,\pi], y\in X.$$

- $f:[0,1]\times X\to X$ such that

$$f(t, y(t, \gamma)) = \frac{t^2 e^{-t}|y(t, \gamma)|}{b}, \qquad t \in [0,1], \gamma \in [0, \pi], y \in X, b > 0.$$

- $q_1: X \to X$ such that

$$q_1(y(t_1, \gamma)) = \frac{\left|y\left(\frac{1}{2}^{-}, \gamma\right)\right|}{2\left(1 + \left|y(\frac{1}{2}^{-}, \gamma)\right|\right)}, t \in [0,1], \gamma \in [0, \pi], y \in X.$$

According to Hille-Yosida Theorem $\|\mathcal{G}(t)\| = \|\mathcal{T}(t)\| = \mathcal{S}$.

In Example (3.1.6) we showed that The System (3.21) satisfies the conditions **H2, H3** and **H4**.

For every $u(\cdot) \in L_P(J, U)$ of the form $u(t) = \sum_{s=1}^{\infty} u_s(t) y_s$, define

$$Bu(t) = \sum_{s=1}^{\infty} \hat{u}_s(t) y_s$$

where

$$\hat{u}_s(t) = \begin{cases} 0 & 0 \leq t < 1 - \dfrac{1}{s^2} \\ u_s(t) & 1 - \dfrac{1}{s^2} \leq t \leq 1 \end{cases}.$$

So $\|Bu\| \leq \|u(\cdot)\|$. Therefore B is a bounded linear operator from $L_P(J, U)$ into X.

Now we shall prove The Condition **H6**. Consider the corresponding linear system of The System (3.21) as follows:

$$\begin{cases} {}^{C}D^{\frac{1}{2}, \frac{1}{3}} y_s(t) + s^2 y_s(t) = \hat{u}_s(t), \qquad 1 - \dfrac{1}{s^2} \leq t \leq 1 \\ \Delta y(t_1^+) = q_1\big(y(t_1^-)\big), \end{cases}$$

Let $\wp(\cdot)$ be arbitrary element in $L_p(J, X)$ and $k \in X$ defined as

$$k = (1 - \rho)\mathcal{A}_\rho \wp(1) + \rho \mathcal{A}_\rho^2 \int_0^1 (1 - \xi)^{\omega - 1} L_\omega (1 - \xi) \wp(\xi) d\xi.$$

Assume that $\wp(t) = \sum_{s=1}^{\infty} \wp_s(t) y_s$ and $K = \sum_{s=1}^{\infty} k_s y_s$, where $k_s = \int_0^1 e^{-s^2(1-\zeta)} \wp_s(\zeta) d\zeta$

we can choose the control function

$$u_s(t) = \frac{2s^2}{1-e^{-2}} k_s e^{-s^2(1-t)}, 1 - \frac{1}{s^2} \le t \le 1,$$

then

$$(1-\rho)\mathcal{A}_\rho Bu(1) + \rho\mathcal{A}_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega(1-\xi) Bu(\xi) d\xi$$

$$= (1-\rho)\mathcal{A}_\rho Bu(1) + \rho\mathcal{A}_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega(1-\xi) \sum_{s=1}^{\infty} \tilde{u}_s(\xi) y_s \, d\xi$$

$$= (1-\rho)\mathcal{A}_\rho Bu(1)$$

$$+ \rho\mathcal{A}_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega(1-\xi) \sum_{s=1}^{\infty} \frac{2s^2}{1-e^{-2}} k_s e^{-s^2(1-\xi)} y_s \, d\xi$$

$$= k = (1-\rho)\mathcal{A}_\rho \wp(1) + \rho\mathcal{A}_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega(1-\xi) \wp(\xi) d\xi.$$

Therefore, the first part of Condition **H6** is hold.

Now,

$$\|Bu(t)\|^2 \le \sum_{s=1}^{\infty} \int_{1-\frac{1}{s^2}}^{1} |\tilde{u}_s(t)|^2 dt$$

$$= \sum_{s=1}^{\infty} \int_{1-\frac{1}{s^2}}^{1} \left| \frac{2s^2}{1-e^{-2}} k_s e^{-s^2(1-t)} \right|^2 dt$$

$$= \sum_{s=1}^{\infty} \int_{1-\frac{1}{s^2}}^{1} \frac{4s^4}{(1-e^{-2})^2} k_s^2 e^{-2s^2(1-t)} dt$$

$$= \sum_{s=1}^{\infty} \frac{2s^2}{1-e^{-2}} k_s^2$$

$$= \frac{1}{1-e^{-2}} \sum_{s=1}^{\infty} \left(1 - e^{-2s^2}\right) \int_0^1 |\hat{\wp}_s(t)|^2 dt$$

$$\le \frac{1}{1-e^{-2}} \|\wp(\cdot)\|^2.$$

Therefore, the Condition **H6** holds.

If

$$D = \eta\left[\left(\|E^{-1}\| + \frac{1}{\Gamma\left(\frac{4}{3}\right)}\right)\mathcal{M}_h + \frac{1}{2}\left(1 + \eta\frac{1}{\Gamma\left(\frac{4}{3}\right)}\right)\mathcal{M}_f + \mathcal{M}\right] < 1,$$

and

$$\left(\mathcal{M}_f + 2\left\|\mathcal{A}_\rho^{-1}\right\|\mathcal{M}_h\right)\frac{1}{2(1-e^{-2})}\frac{\eta}{1-D}\|B\| < 1$$

then The System (3.21) is approximately controllable. ■

Chapter Four

Conclusions and Future Works

4.1 Conclusions

- This work has investigated the controllability for linear System (2.1) and nonlinear System (2.2) in finite dimensional space. The solutions were obtained using fractional calculus, Laplace transform, and the Mittag-Leffler function. The use of the controllable for the linear system together with sufficient condition on $(f(t, y(t), u(t))$ has helped us to prove the controllability of System (2.2) via Schauder Fixed Point Theorem.

- Observability of System (2.15) in finite dimension space $\mathbb{R}^n$ has been investigated in this work. First, we showed that the nonsingularity of Gramian observability matrix W_{ob} is a necessary and sufficient condition for the linear System (2.15) to be observable. For the linear System (2.15) to be observable, another condition has been established based on the rank of the matrix

$$G = \begin{bmatrix} C \\ \frac{C(\rho A_\rho A)}{\Gamma(\rho+1)} \\ \vdots \\ \frac{(n-1)!\, C(\rho A_\rho A)^{n-1}}{\Gamma(\rho(n-1)+1)} \end{bmatrix}.$$

One can show that the observability of System (2.15) could be determined without depends on time t. This is done by constant matrices A and B only. Two tests for the observability of System (2.15) are introduced which them: eigenvector of the matrix $A_\rho A$ (eigenvector test), and rank the matrix $\begin{bmatrix} A_\rho A - \lambda A \\ \gamma C \end{bmatrix}$. An important relationship between the observability and controllability of System (2.16) (Duality Theorem) is given. Examples are given for our main results.

- The exact controllability System (3.1) in a Banach space has been investigated. The mild solutions were obtained using semigroup theory, fractional calculus, and Laplace transform.

- Sufficient conditions have been established to prove the exact controllability of the nonlinear system (3.1). It has been proven by employing the Nussbaum Fixed Point Theorem. To illustrate the main results, an example was given

- The mild solution of System (3.1) has been proved exist and unique in a Banach space by employing Banach Fixed Point Theorem.

- For System (3.1), The approximate controllability was discussed using the approximate sequence method. The efficacy of our result has been shown using an example.

4.2 Future works

We will study:

1. Stability of Hattaf-fractional nonlinear dynamical system in finite dimensional space.

2. Observability of the AB-fractional linear dynamical systems in banach space.

3. Optimal control of Hattaf-fractional nonlinear control system in Banach space.

References

[1] N. Sukavanam and Divya, “Exact and approximate controllability of abstract semilinear control systems,” *Indian Journal of Pure and Applied Mathematics*, 33 (12) (2002) pp. 1827–1835.

[2] B. Ross, “The development of fractional calculus 1695-1900,” *Historia Mathematica*, 4 (1) (1977) pp. 75–89.

[3] R. Hilfer, "Applications of fractional calculus in physics", *World scientific*, Singapore, 2000.

[4] V. V. Kulish and J. L. Lage, “Application of fractional calculus to fluid mechanics,” *J Fluids Eng*, 124 (3) (2002) pp. 803–806.

[5] E. Ahmed, A. Hashish, and F. A. Rihan, “On fractional order cancer model,” *Journal of Fractional Calculus and Applied Analysis*, 3 (2) 2012 pp. 1–6.

[6] K. Krishnaveni, K. Kannan, and S. R. Balachandar, “Approximate analytical solution for fractional population growth model,” *International Journal of Engineering and Technology*, 5 (3) (2013) pp. 2832–2836.

[7] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, "Theory and applications of fractional differential equations", *Elsevier* Netherlands, 204 (2006).

[8] I. Podlubny, "An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications", *Academic Press*, Inc., San Diego CA, 198 (1999).

[9] K. Diethelm, “Analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type,” *springer-Verlag Berlin*, 2004 (2010).

[10] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," *Progr. Fract. Differ. Appl*, 1 (2) (2015) pp. 1–13.

[11] A. Atangana and D. Baleanu, "New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model," *Thermal Science*, 20 (2) (2016) pp.763-769.

[12] K. Hattaf, "A new generalized definition of fractional derivative with non-singular kernel," *Computation*, 8 (2) (2020), pp. 1–9.

[13] J. Hristov, "On the Atangana – Baleanu derivative and Its relation to the fading memory concept: The diffusion equation," *Springer International Publishing*, 194 (2019) pp. 175–193.

[14] J. Hristov, "Derivatives with non-Singular kernels from the Caputo-Fabrizio definition and beyond: Appraising analysis with emphasis on diffusion models," *Frontiers*, 1 (2018) pp. 269–341.

[15] M. I. Syam and M. Al-Refai, "Fractional differential equations with Atangana–Baleanu fractional derivative: Analysis and applications," *Chaos, Solitons and Fractals: X*, 2 (2019), pp. 3–7.

[16] M. Fečkan, J.-R. Wang, and Y. Zhou, "On the new concept of solutions and existence results for impulsive fractional evolution equations," *Dynamics of Partial Differential Equations*, 8 (4) (2011), pp. 345–361.

[17] R. E. Kalman, "Mathematical description of linear dynamical systems," *Journal of the Society for Industrial and Applied Mathematics, Series A: Control*, 1 (2) (1963) pp. 152–192.

[18] M. Nawaz, W. Jiang, J. Sheng, A. U. K. Niazi, and Y. Lichang, "On the controllability of nonlinear fractional system with control delay," *Hacettepe Journal of Mathematics and Statistics*, 49, (1) (2020) pp. 294–302.

[19] J. Sheng, W. Jiang, D. Pang, and S. Wang, “Controllability of nonlinear fractional dynamical systems with a Mittag–Leffler kernel,” *Mathematics*, 8, (12) (2020), pp. 2139.

[20] M. Ghasemi and K. Nassiri, “On controllability of fractional continuous-time systems,” *Mathematical Problems in Engineering*, 2021 (2021).

[21] D. Aimene, D. Baleanu, and D. Seba, “Controllability of semilinear impulsive Atangana-Baleanu fractional differential equations with delay,” *Chaos, Solitons & Fractals*, 128 (2019) pp. 51–57.

[22] H. Qin, X. Zuo, and J. Liu, “Existence and controllability results for fractional impulsive integrodifferential systems in Banach spaces,” *Abstract and Applied Analysis*, 2013 (2013).

[23] V. S. Muni and R. K. George, “Controllability of semilinear impulsive control systems with multiple time delays in control,” *IMA Journal of Mathematical Control and Information*, 36 (3) (2019) pp. 869–899.

[24] X. Li, Z. Liu, and C. C. Tisdell, “Approximate controllability of fractional control systems with time delay using the sequence method,” *Electron. J. Differential Equations*, 272 (2017) pp. 1–11.

[25] J. Du, D. Cui, Y. Sun, and J. Xu, “Approximate controllability for a kind of fractional neutral differential equations with damping,” *Mathematical Problems in Engineering*, 2020 (2020).

[26] T. Kaczorek and Ł. Sajewski, “Relationship between controllability and observability of standard and fractional different orders discrete-time linear system,” *Fractional Calculus and Applied Analysis*, 22 (1) (2019), pp. 158–169.

[27] K. Balachandran, V. Govindaraj, M. D. Ortigueira, M. Rivero, and J. J. Trujillo, “Observability and controllability of fractional linear dynamical systems,” *IFAC Proceedings Volumes*, 46 (1) (2013) pp. 893–898.

[28] E. Kreyszig, "Introductory functional analysis with applications", *John Wiley & Sons*, USA, 17 (1991).

[29] C. Heil, "Metrics, Norms, Inner Products, and Operator Theory," *Springer*, Switzerland, 6 (2018).

[30] R. Kress, V. Maz'ya, and V. Kozlov, "Linear integral equations," *Springer*, Berlin, 82 (1989).

[31] R. Precup, "Methods in nonlinear integral equations". S*pringer Science & Business Media*, Netherlands, 2002.

[32] A. Pazy, "Semigroups of linear operators and applications to partial differential equations*"*, *Springer Science & Business Media*, USA, 44 (2012).

[33] R. F. Curtain and H. Zwart, "An introduction to infinite-dimensional linear systems theory," *Springer Science & Business Media*, New York, 21 (2012).

[34] V. I. Istrăţescu, "Fixed point theory,*" of Mathematics and its Applications, D*. Reidel Publishing Co., Dordrecht, 7 (1981).

[35] R. D. Nussbaum, "The fixed point index and asymptotic fixed point theorems for k-set-contractions," *Bulletin of the American Mathematical Society*, 75 (3) (1969) pp. 490–495.

[36] D. Zezislaw and S. Migórski, "An Introduction to Nonlinear Analysis: Applications". *Kluwer Academic*, Dordrecht, 2003.

[37] J. Macki and A. Strauss, "Introduction to optimal control theory". S*pringer Science & Business Media*, New York, 2012.

[38] R. K. George and T. IIST, "Controllability, observability, stability and stabilizability of linear systems," 2015.

[39] R. Triggiani, “A note on the lack of exact controllability for mild solutions in Banach spaces,” *SIAM Journal on Control and Optimization*, 15 (3) (1977) pp. 407–411.

[40] K. Naito, “Controllability of semilinear control systems dominated by the linear part,” *SIAM Journal on control and Optimization*, 25 (3) (1987) pp. 715–722.

[41] C.-T. Chen, “Linear system theory and design,” *Oxford University Press Inc*, USA 1999.

[42] J. P. Dauer, “Nonlinear perturbations of quasi-linear control systems,” *Journal of Mathematical Analysis and Applications*, 54, (3) (1976) pp. 717–725.

[43] G. Strang, "Linear algebra and its applications". *Thomson, Brooks/Cole*, Belmont, CA, 2006.

[44] A. J. Laub, "Matrix analysis for scientists and engineers,". *Siam*, 91 (2005).

[45] F. Mainardi, P. Paradisi, and R. Gorenflo, “Probability distributions generated by fractional diffusion equations,” *Econophysics: An Emerging Science (J. Kertesz & I. Kondor eds). Dordrecht: Kluwer*, 2003.

[46] Zhou, Yong, and Feng Jiao. "Existence of mild solutions for fractional neutral evolution equations." *Computers & Mathematics with Applications* 59 (3) (2010) pp.1063-1077.

المستخلص

تهدف هذه الأطروحة إلى دراسة وتطوير قابلية التحكم وقابلية المراقبة لأنظمة تحكم كسرية ذات نواة غير شاذة في فضاءات ذات أبعاد منتهية وغير منتهية.

خلال هذا العمل، تم إثبات قابلية التحكم في نظام تحكم ديناميكي خطي كسري من النوع AB مع تأخير التحكم في ظل شرط كاف. تم وضع شروط كافية لإثبات أن نظام تحكم ديناميكي غير خطي كسري من النوع AB مع تأخير التحكم يمكن التحكم فيه باستخدام نظرية النقطة الصامدة لشودر.

تم التحقق من قابلية مراقبة نظام تحكم ديناميكي خطي كسري من نوع AB. حيث تم تقديم أكثر من معيار واحد لها. تم إثبات الازدواجية بين قابلية التحكم وقابلية المراقبة.

تم تقديم حل معتدل لنظام تحكم تسارعي غير خطي يتضمن مشتق هتاف الكسري في فضاء باناخ باستخدام حساب التفاضل والتكامل الكسري ونظريات شبه الزمرة. في ظل شروط كافية، أثبتنا قابلية التحكم في هذا النظام. تم الحصول على نتائجنا الرئيسية باستخدام نظرية النقطة الصامدة لنوسباوم. من ناحية أخرى، تم تقديم شروط كافية لإثبات وجود ووحدانية الحل المعتدل للنظام غير الخطي. أيضًا، تم إثبات قابلية التحكم التقريبية على هذا النظام في فضاء باناخ.

أخيرًا، تم عرض تطبيقات لتوضيح أهمية النتائج الرئيسية.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
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قســـــــم الرياضـــــــيات

قابلية التحكم في حلول النظم الدينميه

أطروحة

مقدمة الى مجلس كلية التربية للعلوم الصرفة في جامعة بابل كجزء
من متطلبات نيل درجة الدكتوراه فلسفة في التربية / الرياضيات

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Controllability for the Solutions of the Dynamical Systems

A Dissertation

Submitted to College of Education for Pure Sciences -University of Babylon in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Education / Mathematics

By

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2022 A.D 1444 A.H

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

أَلَمْ نَشْرَحْ لَكَ صَدْرَكَ (١) وَوَضَعْنَا عَنْكَ وِزْرَكَ (٢) الَّذِي
أَنْقَضَ ظَهْرَكَ (٣) وَرَفَعْنَا لَكَ ذِكْرَكَ (٤) فَإِنَّ مَعَ الْعُسْرِ يُسْرًا
(٥) إِنَّ مَعَ الْعُسْرِ يُسْرًا (٦) فَإِذَا فَرَغْتَ فَانْصَبْ (٧) وَإِلَىٰ
رَبِّكَ فَارْغَبْ (٨)

صدق الله العلي العظيم

Dedicated To

My daughter MLAK,

my son, MOHAMMED

and all my family.

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Publications

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2. F. A. Naji and I. Al-Sharaa, “Controllability of impulsive fractional nonlinear control system with Mittag-Leffler kernel in Banach space”, *International Journal of Nonlinear Analysis and Applications*, vol. 13, no. 1, pp. 3257–3280, 2022.

3. F. A. Naji and I. Al-Sharaa, "Existence, uniqueness and approximate controllability of impulsive fractional nonlinear control system with nonsingular kernel", *Iraqi Journal of Science*, Vol. 63, no 10, 2022. (acceptable for publication).

Abstract

This dissertation aims to study and develop the controllability and observability of fractional control systems with nonsingular kernel in finite and infinite dimensional spaces.

Throughout this work, the controllability of an AB-fractional linear control dynamical system with control delay under sufficient conditions has been proved. Sufficient conditions are set to prove that a nonlinear AB-fractional control dynamical system with control delay is controllable using Schauder Fixed Point Theorem.

The observability of an AB-fractional linear control dynamical system has been investigated. Wherein more than one criterion for it has been introduced. The duality between controllability and observability has been proved.

A mild solution of a nonlinear impulsive control system involving Hattaf-fractional derivative has been introduced in Banach space using fractional calculus and semigroup theory. Under sufficient conditions, we prove the controllability of this system. Our main results are obtained utilizing Nussbaum Fixed Point Theorem. On the other hand, sufficient conditions are introduced to show the existence and uniqueness of the mild solution of the nonlinear system. Also, the approximate controllability of this system is proved in Banach space.

Finally, applications have been shown to illustrate the importance of the main results.

Contents

List of Symbols

Symbol	Meaning
$\dot{y}$	Derivative of a function y for t
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex number
$\mathbb{R}^{\boldsymbol{n}}$	Euclidian space
$\mathbb{R}^{\boldsymbol{n}\times\boldsymbol{m}}$	The set of $n \times m$ real matrices
$C[a,b]$	The set of all continuous functions defined from $[a,b]$ into $\mathbb{R}$
$C_n([a,b])$	The set of all continuous functions defined from $[a,b]$ into $\mathbb{R}^{\boldsymbol{n}}$
$C([a,b];X)$	The set of all continuous functions defined from $[a,b]$ into X
$C^1[a,b]$	The set of all continuously differentiable functions on $[a,b]$
$PC([a,b];X)$	The space of piecewise continuous functions which defined on $[a,b]$
$L_p([a,b];X)$	The space of Lebesgue integrable functions from $[a,b]$ into X
$\mathcal{T}(t)$	Semigroup of bounded linear operators generated by $\mathcal{A}$
C_0-semigroup	Strongly continuous semigroup
$\Gamma(\cdot)$	Gamma function
$\mathcal{L}$	Laplace transform

$D(A)$	The domain of operator A
B^*	The transpose of matrix B
$\mathcal{K}_\Upsilon(f)$	The reachable set
$E_\rho(\zeta)$	Mittag-Leffler function of one parameter
$E_{\rho,\omega}(\zeta)$	Mittag-Leffler function of two parameters
${}^{RL}_{a}I^{\rho}$	Riemann-Liouville fractional integral of order ρ
${}^{RL}_{a}D^{\rho}$	Riemann-Liouville fractional derivative of order ρ
${}^{C}_{a}D^{\rho}$	Caputo fractional derivative of order ρ
${}^{ABR}_{a}D^{\rho}$	AB-fractional derivative of Riemann-Liouville sense of order ρ
${}^{ABC}_{a}D^{\rho}$	AB-fractional derivative of Caputo sense of order ρ
${}^{AB}I^{\rho}$	AB-fractional integral of order ρ
${}^{C}_{a}D^{\rho,\omega}$	Hattaf-fractional derivative of Caputo sense of orders ρ,ω
${}_{a}\mathcal{J}^{\rho,\omega}$	Hattaf-fractional integral of orders ρ,ω

Introduction

Control theory is an area of applied mathematics concerned with the behavior of dynamical systems. Mesopotamia (2000 BC) is considered the first to use the theory of control to irrigate agricultural lands. Control mechanisms are found all across nature and are employed by living creatures to keep vital variables like body temperature and blood sugar levels at predetermined ranges. The populations of insects and animals are controlled by a carefully balanced prey-predator relation. There are a variety of basic and complicated man-made control systems in use in our daily lives. [1]

Fractional calculus has received significant interest from researchers because it describes many scientific phenomena with great accuracy. This concept was originally described in 1695 by Leibniz and L'Hospital as a generalization of the integer-order derivative [2]. However, fractional calculus was used in the 1960s. The application of fractional calculus has grown during the past three decades. It is significant to a wide variety of applications in many fields, including physics[3], fluid mechanics[4], biochemical [5], and population growth [6]. Academics increasingly research various forms of fractional differential equations. Numerous definitions of fractional derivatives describe many scientific phenomena, for instance, Riemann-Liouville, Caputo, Hadamard, Grunwald-Letnikov, and Hilfer, for more details; see [7]–[9]

M. Caputo and M. Fabrizio [10] introduced a definition of a derivative with fractional order with the nonsingular exponential kernel. Atangana and Baleanu [11] proposed the AB-fractional derivatives as a concept of fractional derivatives has a nonsingular Mittag-Leffler kernel. In 2020, Hattaf [12] presented a generalization definition of the AB-fractional derivative. The fractional derivatives without singular kernels gave adequately described for models of dissipative phenomena where the classical fractional operators cannot give it, see [13]–[15].

Impulsive differential equations have attracted much research attention due to their significance in modelling processes exposed to short-time changes throughout their development. Many articles deal with impulsive differential equations and their solutions for example; see [16].

Controllability and observability are important properties of dynamical system. They are fundamental elements of modern control theory. If a system is able to transform any initial state to any final state over given time using a control function, then it is said to be controllable. Two forms of controllability are most often considered in practical applications: exactly controllability and approximate controllability. The system is exactly controllable if it reaches a required state at the given time using admissible control. The system is approximate controllable if it reaches a state at the given time lies in a ε-neighborhood of the required state using admissible control. The observability of a system is defined as the ability to determine its initial state from its output behavior.

In 1963, Kalman [17] introduced the concepts of controllability and observability. M. Nawaz, et. al. [18] discussed controllability of nonlinear fractional system with control delay involving the Caputo fractional derivative using fixed point theorem and Mittag-Leffler function. In 2020, Jiale Sheng et al. [19] set sufficient conditions to show that AB-fractional nonlinear dynamical system is controllable using fixed point theorem and Mittag-Leffler function. In 2021, Ghasemi and Nassiri [20] introduced many criterions for the controllability of AB-fractional nonlinear dynamical system provided Caputo derivative of control function exists. In [21], the controllability of AB-fractional systems in a Banach space were discussed based on fixed point theorem and semigroup operator theory.

The controllability problems of impulsive fractional control systems have been studied in many articles, see; [22], [23].

X. Li et al. [24] established sufficient conditions for the approximate controllability of fractional control systems with time delay in Hilbert spaces by using semigroup operator theory and sequence method. In [25] the researcher investigated the approximate controllability for a kind of fractional neutral differential equations with damping in Banach spaces using the approximate sequence method.

In [26], the controllability and observability have been discussed for fractional linear control systems with multiple different orders involving the Caputo fractional derivative using Gramian matrix. K. Balachandran, et al. [27] investigate the observability and controllability of fractional control dynamical system with Grunwald-Letnikov derivative based on Gramian matrix.

This dissertation aims to study the exact and approximate controllability of some types of fractional control systems with Mittag-Leffler kernel in finite and infinite dimensional spaces using fixed point theorems, fractional calculus, and semigroup operator theory. Additionally, the observability of AB-fractional linear control dynamical systems has been discussed and developed in finite dimensional space.

We organized our work into four chapters:

Chapter one deals with the basic concepts and fundamental definitions that helped us achieve our goals, such as functional analysis, semigroup theory, fixed point theorems, fractional calculus, controllability, and observability.

Chapter two contains two sections. The first section investigates the controllability of AB-fractional nonlinear dynamical systems with control delay. We set sufficient conditions to prove that the nonlinear system is controllable using Schauder Fixed Point Theorem. An example is presented to demonstrate our theoretical results. Section two discusses the observability of AB-fractional linear dynamical systems. We present more than one criterion for the observability an AB-fractional linear control dynamical system. Additionally, the duality between controllability and observability has been proved.

Chapter three contains three sections. Section one discusses the controllability of a nonlinear impulsive Hattaf-fractional control system in Banach space using semigroup theory and Nussbaum Fixed Point Theorem. In section two, we prove the existence and uniqueness of the mild solution of the nonlinear system under sufficient conditions using Banach Fixed Point Theorem. Concerning section three, the approximate controllability of the nonlinear system has been proved under sufficient conditions using the approximate sequence method.

Finally, the conclusions and future works have been presented in **chapter four**.

Chapter One

Basic Definitions and Fundamental Theorems

In this chapter, we review some basic concepts, lemmas, and notations that will aid us in establishing our main results later on.

1.1 Functional Analysis

Definition 1.1.1 [28]

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces over the same field F**.** An operator T defined from $\mathbb{X}$ to $\mathbb{Y}$ is called **linear operator** if $T(ax + by) = aT(x) + bT(y)$ for all $x, y \in \mathbb{X}$, and $a, b \in F$.

Definition 1.1.2 [29]

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces and the operator T defined from $\mathbb{X}$ to $\mathbb{Y}$.

1. The operator T is said to be **bounded** if there exists a positive constant $l \in \mathbb{R}$ such that

$$\|Tx\| \leq l\|x\|,$$

for all $x \in \mathbb{X}$.

2. The **norm** of operator T defined as:

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

3. The operator T is called **continuous operator at a point x_0** $\in \mathbb{X}$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|Tx - Tx_0\| < \epsilon, \text{for all } x \in \mathbb{X} \text{ satisfing } \|x - x_0\| < \delta.$$

4. The operator T is called **continuous operator on** $\mathbb{X}$ if it continuous at every point of $\mathbb{X}$ and it called uniformly continuous on $\mathbb{X}$ if δ independ on x.

Remark 1.1.3 [29]

Let T be a bounded linear operator. Then for any $x \in \mathbb{X}$,

$$\|Tx\| \leq \|T\|\|x\|.$$

Lemma 1.1.4 [29]

The linear operator $T: \mathbb{X} \to \mathbb{Y}$ is continuous if and only if it is bounded where $\mathbb{X}$ and $\mathbb{Y}$ are normed spaces.

Definition 1.1.5 [30]

A subset D of a normed space $\mathbb{X}$ is called **relatively compact** if the closure of D is compact.

Definition 1.1.6 [30]

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces. The linear operator $T: \mathbb{X} \to \mathbb{Y}$ is said to be **compact** if T maps each bounded subset $A \subseteq \mathbb{X}$ into a relatively compact set in $\mathbb{Y}$.

Remark 1.1.7 [30]

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces.

i. A linear operator $T: \mathbb{X} \to \mathbb{Y}$ is compact if and only if for any bounded sequence $\{x_n\}$ in $\mathbb{X}$, the sequence $\{Tx_n\}$ contains convergent subsequence in $\mathbb{Y}$.
ii. The compact linear operator is always bounded.

Definition 1.1.8 [28]

A **complete normed space**, also known as **Banach space**, is a normed space where every Cauchy sequence is convergent in it.

Example 1.1.9 [28]

Consider the space of all continuous functions $x: [a, b] \to \mathbb{R}$

$$C[a, b] := \{x : x: [a, b] \to \mathbb{R}, x \text{ is continuous}, a, b \in \mathbb{R}\}.$$

This space is a Banach space with norm given by

$$\|x\| = \max_{t \in [a,b]} |x(t)|.$$

Definition 1.1.10 [30]

Let $(\mathbb{X}, \mu)$ be a measure space and $1 \leq p < \infty$. Then the collection of all measurable function f for which $|f|^p$ is integrable will be denoted by $\boldsymbol{L_p(\mu)}$**.** For each $f \in L_p(\mu)$, set

$$\|f\|_p = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}}$$

the norm $\|f\|_p$is called the $\boldsymbol{L_p}$**-norm** of f.

Lemma 1.1.11 [28]

$(L_p, \|\cdot\|_p)$ is a Banach space.

Definition 1.1.12 [31]

Let $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces. The operator $T: \mathbb{X} \to \mathbb{Y}$ is called **completely continuous** if it is continuous and compact.

Remark 1.1.13 [31]

Let $T: \mathbb{X} \to \mathbb{Y}$ be linear operator, where $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces. Then, if T is compact operator, then it is completely continuous.

Definition 1.1.14 [30]

Assume D is a subset of the space of continuous functions $C[a, b]$.

1- The set D is called **bounded**, if for all $g \in D$ and all $t \in [a, b]$ there exists constant $K > 0$ such that $\|g(t)\| \leq K$.

2- The set D is called **equicontinuous**, if for all $g \in D$ and all $t, s \in [a, b]$ and for each $\epsilon > 0$ there is $\delta > 0$ such that

$$\|g(t) - g(s)\| < \epsilon \quad \text{when } \|t - s\| < \delta.$$

Lemma 1.1.15 [29] "Arzela-Ascoli's Theorem"

Let D be a subset of the space of continuous functions $C[a, b]$. Then D is relatively compact if and only if it is bounded and equicontinuous.

Example 1.1.16 [31]

Consider $\mathbb{X} = C[0,1]$ and let $A: [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Define the operator $T: \mathbb{X} \to \mathbb{X}$ by

$$T(x)(t) = \int_0^1 A(t, s)x(s)\, ds.$$

For any $x \in \mathbb{X}$, we have

$$|T(x)(t)| \leq \int_0^1 |A(t, s)||x(s)|\, ds$$

$$\leq \sup|x(s)| \int_0^1 |A(t,s)|ds$$

$$\leq \|x\| \int_0^1 |A(t,s)|ds.$$

Then T is a bounded operator. Let D be a nonempty and bounded subset of the space $\mathbb{X}$. We show that $T(D)$ is relatively compact. First, let us prove that $T(D)$ is equicontinuous, since A is continuous on compact matric space then it is uniformly continuous, so for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $t_2, t_1 \in [0,1]$ we have

$$|A(t_2,s) - A(t_1,s)| < \epsilon \text{ when } \quad |t_2 - t_1| < \delta$$

For $x \in \mathbb{X}$

$$|T(x)(t_2) - T(x)(t_1)| \leq \int_0^1 |A(t_2,s) - A(t_1,s)||x(s)|\, ds$$

$$\leq \sup|x(t)| \int_0^1 |A(t_2,s) - A(t_1,s)|\, ds$$

$$< \sup|x(t)|\ \epsilon,$$

then $T(D)$ is equicontinuous. According to Arzela-Ascoli's Theorem $T(D)$ is relatively compact, hence the operator T is completely continuous.

Definition 1.1.17 [28]

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces. A linear operator $T: D(T) \subseteq \mathbb{X} \to \mathbb{Y}$ is called **closed operator** when the graph $G(T) = \{(x, T(x)); x \in D(T)\}$ is a closed set in $\mathbb{X} \times \mathbb{Y}$.

Lemma 1.1.18 [28]

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces. An operator $T: \mathbb{X} \to \mathbb{Y}$ is closed if and only if for every sequence $\{x_n\}_{n=0}^{\infty} \subset \mathbb{X}$ such that $x_n \to x$ and $Tx_n \to y$ as $n \to \infty$, $y \in \mathbb{Y}$ then $x \in \mathbb{X}$ and $Tx = y$.

Now, we review some concepts on semigroup operator theory such as uniformly continuous and strongly continuous in a Banach space $\mathbb{X}$.

Definition 1.1.19 [32]

A **semigroup of bounded linear operators** $\mathcal{T}(t), t \geq 0$ on $\mathbb{X}$ is defined as the family of bounded linear operators satisfies the following:

i. $\mathcal{T}(0) = I$,

ii. $\mathcal{T}(t+s) = \mathcal{T}(t) \circ \mathcal{T}(s)$, for every $t, s \geq 0$.

Definition 1.1.20 [32]

The **infinitesimal generator** $\mathcal{A}$ of semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, is the linear operator described by:

$$\mathcal{A}\zeta = \lim_{t \to 0^+} \frac{\mathcal{T}(t)\zeta - \zeta}{t} = \left. \frac{d^+\mathcal{T}(t)\zeta}{dt} \right|_{t=0}, \text{for } \zeta \in D(\mathcal{A})$$

where,

$$D(\mathcal{A}) = \left\{ \zeta \in \mathbb{X}; \lim_{t \to 0^+} \frac{\mathcal{T}(t)\zeta - \zeta}{t} \text{ exists} \right\}.$$

Lemma 1.1.21 [32]

There is a unique infinitesimal generator for a semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$.

Definition 1.1.22 [32]

A semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$,of bounded linear operators on $\mathbb{X}$ is **uniformly continuous** when:

$$\lim_{t\to^+0} \|\mathcal{T}(t) - I\| = 0.$$

Example 1.1.23 [32]

Consider $\mathcal{A}$ is a bounded linear operator on $\mathbb{X}$. Then the exponential function $exp(\mathcal{A}t)$ is a uniformly continuous semigroup generated by $\mathcal{A}$ on $\mathbb{X}$.

Theorem 1.1.24 [32]

Let $\mathcal{A}$ be a linear operator. Then $\mathcal{A}$ is infinitesimal generator of a uniformly continuous semigroup if and only if $\mathcal{A}$ is a bounded linear operator.

Definition 1.1.25 [32]

A **strongly continuous semigroup** (denoted by $\boldsymbol{C_0}$ **− semigroup**) is a semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, of bounded linear operators on $\mathbb{X}$ that satisfies:

$$\lim_{t\to 0} \mathcal{T}(t)\zeta = \zeta,$$

for each $\zeta \in \mathbb{X}$.

Examples 1.1.26 [33]

Let $\mathbb{X} = C[0,1]$ such that $\zeta(1) = 0$ for all $\zeta \in \mathbb{X}$. For $t \geq 0$, define

$$(\mathcal{T}(t)\zeta)(s) = \begin{cases} \zeta(s+t) & t+s \leq 1 \\ 0 & t+s > 1 \end{cases}$$

$\mathcal{T}(t)$ is a C_0 −semigroup on $\mathbb{X}$ generated by a linear operator $\mathcal{A}$ which is given by

$$D(\mathcal{A}) = \{\zeta : \zeta \in C^1[0,1] \cap \mathbb{X}, \dot{\zeta} \in \mathbb{X}\}$$

and

$$\mathcal{A}\zeta = \dot{\zeta} \qquad \text{for} \quad \zeta \in D(\mathcal{A}).$$

Lemma 1.1.27 [33]

1. Let $\{\mathcal{T}(t)\}_{t\geq 0}$, be C_0-semigroup generated by $\mathcal{A}$. Then for $\zeta \in \mathbb{X}$, the function $t \to \mathcal{T}(t)\zeta$ is continuous from $\mathbb{R}^+$ into $\mathbb{X}$.
2. For $\zeta \in \mathbb{X}, \int_0^t \mathcal{T}(t)\zeta ds \in D(\mathcal{A})$ and $\mathcal{A}\left(\int_0^t \mathcal{T}(s)\zeta ds\right) = \mathcal{T}(t)\zeta - \zeta$.

Definition 1.1.28 [32]

A C_0 −semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, on $\mathbb{X}$ is said to be **differentiable** for $t > 0$ when for any $\zeta \in \mathbb{X}, t \to \mathcal{T}(t)\zeta$ is differentiable for $t > 0$.

Lemma 1.1.29 [32]

Let $\{\mathcal{T}(t)\}_{t\geq 0}$, be C_0-semigroup generated by $\mathcal{A}$. For $\zeta \in D(\mathcal{A}) \subset \mathbb{X}$; $\mathcal{T}(t)\zeta \in D(\mathcal{A})$ and

$$\frac{d}{dt}\mathcal{T}(t)\zeta = \mathcal{A}\mathcal{T}(t)\zeta = \mathcal{T}(t)\mathcal{A}\zeta.$$

Theorem 1.1.30 [32]

Let $\mathcal{A}$ and $\mathcal{B}$ be infinitesimal generators of C_0-semigroups $\mathcal{T}(t)$ and $\mathcal{S}(t)$ respectively. If $\mathcal{A} = \mathcal{B}$, then $\mathcal{T}(t) = \mathcal{S}(t)$ for $t \geq 0$.

Lemma 1.1.31 [33]

Let $\{\mathcal{T}(t)\}_{t\geq 0}$, is C_0-semigroup. Then there is $\sigma \geq 0$ and $K \geq 1$, such that $\|\mathcal{T}(t)\| \leq Ke^{\sigma t}$, for $t \geq 0$.

Definition 1.1.32 [32]

The $C_0 -$ semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, is called **compact** if $\mathcal{T}(t)$ is a compact operator for each $t > 0$.

Definition 1.1.33 [32]

The **resolvent set** indicated by $\boldsymbol{p(\mathcal{A})}$ is the set of all complex numbers ξ where $(\xi I - A)^{-1}$ is a bounded linear operator in $\mathbb{X}$.

Definition 1.1.34 [32]

The **resolvent operator** of $\mathcal{A}$ is the family of bounded linear operators $R(\xi;\mathcal{A}) = (\xi I - \mathcal{A})^{-1}, \xi \in p(\mathcal{A})$ and the following equality holds for $x \in \mathbb{X}$

$$R(\xi;\mathcal{A})x = (\xi I - \mathcal{A})^{-1}x = \int_0^\infty e^{-\xi t}\mathcal{T}(t)x dt,$$

where $\{\mathcal{T}(t)\}_{t\geq 0}$, is a C_0 −semigroup generated by linear operator $\mathcal{A}, s > 0$.

Theorem 1.1.35 [32] "Hille-Yosida Theorem"

A linear operator $\mathcal{A}$ is the infinitesimal generator of C_0-semigroup $\mathcal{T}(t)$ $(t \geq 0)$, satisfying $\|\mathcal{T}(t)\| \leq K$ $(K \geq 1)$ if and only if

i. $\mathcal{A}$ is closed operator and $\overline{D(\mathcal{A})} = \mathbb{X}$.

ii. The resolvent set $p(\mathcal{A})$ of operator $\mathcal{A}$ contains $\mathbb{R}^+$ and

$$\| R(\xi;\mathcal{A})^n\| \leq \frac{K}{\xi^n}, \text{for } \xi > 0, n \in \mathbb{N}.$$

In the following, we recall some of the fixed point theorems used throughout this work.

If A is an operator of a Banach space $\mathbb{X}$ into itself, then $x \in \mathbb{X}$ is said to be a fixed point of A if $A(x) = x$. Fixed point theorems deal with the existence and attributes of fixed points. Such theorems are the most potent tools for demonstrating the existence and uniqueness of solutions to various mathematical models (differential equations, partial differential equations, fractional order differential equations, etc.).

Definition 1.1.36 [34]

Assume $(\mathbb{X}, \|\cdot\|)$ is a Banach space. An operator Tdefined on $\mathbb{X}$ into itself is called **Lipschitz continuous** if there is $k > 0$, such that

$$\|T(x) - T(y)\| \leq k\|x - y\|$$

for all $x, y \in \mathbb{X}$.

The smallest k is the Lipschitz constant of T. If $k < 1$ then T is called a **contraction**.

Theorem 1.1.37 [31] "Schauder Fixed Point Theorem"

Assume M is a nonempty convex subset of a Banach space $\mathbb{X}$, and $\mathcal{B}$: $M \to M$ is a completely continuous operator. Then $\mathcal{B}$ has at least one fixed point.

Theorem 1.1.38 [35] "Nussbaum Fixed Point Theorem"

Assume G is closed, bounded and convex subset of a Banach space $\mathbb{X}$. If the continuous functions ϕ_1, ϕ_2 from G to $\mathbb{X}$ satisfies the following:

1. $(\phi_1 + \phi_2)G \subset G$,
2. $\|\phi_1 x - \phi_1 y\| \leq \mu\|x - y\|$ for all $x, y \in G$ where $0 < \mu < 1$, i.e. ϕ_1 is contraction,
3. ϕ_2 is completely continuous,

then the operator $\phi_1 + \phi_2$ has a fixed point in G.

Theorem 1.1.39 [36] " Banach Fixed Point Theorem "

Let $\mathbb{X}$ be a Banach space. If $B: \mathbb{X} \to \mathbb{X}$ is a contraction operator, then it has a unique fixed point.

1.2 Fractional Calculus

Fractional calculus is the theory of derivatives and integrals of arbitrary order, which generalizes the concepts of differentiation and integration of integer order. Several fundamental concepts in fractional calculus, such as the definitions, lemmas, and notations, will be reviewed in this section.

In the following, we recall some special functions which are important in fractional calculus.

Definition 1.2.1 [7]

The **Gamma function** is indicated by $\boldsymbol{\Gamma(\alpha)}$, defined as

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} e^{-s}\, ds, \qquad \alpha \in \mathbb{C}, Re(\alpha) > 0$$

which is a generalizes of the factorial function, that is

$$\Gamma(n+1) = n!$$

for $n \in \mathbb{N}$.

Remark 1.2.2 [7]

Some important properties of Gamma function are

i. $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$,

ii. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

The Mittag-Leffler function is essential in fixing problems with fractional differential (integral) equations.

Definition 1.2.3 [7]

The function defined as

$$E_{\rho}(\zeta) = \sum_{j=0}^{\infty} \frac{\zeta^j}{\Gamma(\rho j + 1)}, \qquad (\zeta \in \mathbb{C}, \rho > 0), \tag{1.1}$$

is called **Mittag-Leffler with one parameter $\boldsymbol{\rho}$**.

The generalization of the **Mittag-Leffler function with two parameters $\boldsymbol{\rho}$ and $\boldsymbol{\omega}$** is given by

$$E_{\rho,\omega}(\zeta) = \sum_{j=0}^{\infty} \frac{\zeta^j}{\Gamma(\rho j + \omega)}, \qquad (\zeta \in \mathbb{C}, \omega, \rho > 0). \tag{1.2}$$

When $\omega = 1$, then $E_{\rho,\omega}(\zeta)$ coincides with $E_{\rho}(\zeta)$, i.e. $E_{\rho,1}(\zeta) = E_{\rho}(\zeta)$.

It is important to note that the Mittag-Leffler function is a generalization of the exponential function. where

$$E_1(\zeta) = \sum_{j=0}^{\infty} \frac{z^j}{j!} = exp(\zeta).$$

Remark 1.2.4 [7]

For $\rho > 0$,

$$\mathrm{E}_{\rho,\rho}(0) = \frac{1}{\Gamma(\rho)}.$$

Now, we recall definition and some properties of Laplace Transform.

Definition 1.2.5 [7]

The **Laplace transform** of a function x is defined by

$$\mathcal{L}\{x(t)\}(s) = \int_0^\infty e^{-st} x(t)\, dt, \qquad s \in \mathbb{C}, t \in \mathbb{R}^+.$$

Lemma 1.2.6 [7]

For $q > -1$ and $s > 0$, the Laplace transform of power function t^q is given by

$$\mathcal{L}\{t^q\}(s) = \frac{\Gamma(q+1)}{s^{q+1}}.$$

Lemma 1.2.7 [7]

Suppose that $x(t)$ and $y(t)$ are two functions, in which the Laplace transforms $\mathcal{L}\{x(t)\}(s)$ and $\mathcal{L}\{y(t)\}(s)$ exists. If the convolution of $x(t)$ and $y(t)$ defined by

$$x(t) * y(t) = \int_0^t x(t-\delta)\ y(\delta)\, d\delta.$$

then the Laplace transform of the convolution of $x(t)$ and $y(t)$ is given by

$$\mathcal{L}\{x(t) * y(t)\}(\lambda) = \mathcal{L}\{x(t)\}(\lambda)\mathcal{L}\{y(t)\}(\lambda).$$

Lemma 1.2.8 [7]

Let $s > 0$, $\theta \in \mathbb{R}, \theta \neq 0$, and $\rho, \omega > 0$. Then the Laplace transform of the Mittag-Leffler functions (1.1) and (1.2) defined as

$$\mathcal{L}\{E_\rho(\theta t^\rho)\}(s) = \frac{s^{\rho-1}}{s^\rho - \theta},$$

$$\mathcal{L}\{t^{\omega-1} E_{\rho,\omega}(\theta t^\rho)\}(s) = \frac{s^{\rho-\omega}}{s^\rho - \theta},$$

respectively.

In the following, we review some definitions and properties of the classic fractional calculus.

Definition 1.2.9 [7]

The fractional integral of order $\rho > 0$ for a function ζ is defined by

$$ {}_{a}^{RL}I^{\rho}\,\zeta(\lambda) = \frac{1}{\Gamma(\rho)}\int_{a}^{\lambda}(\lambda - s)^{\rho-1}\,\zeta(s)ds, \qquad \lambda \in [a,b], $$

is called **Riemann-Liouville fractional integral**, where $\Gamma(\cdot)$ is the Gamma function.

Definition 1.2.10 [7]

Riemann-Liouville fractional derivative of order ρ with the lower limit a for a function ζ is defined by

$$ {}_{a}^{RL}D^{\rho}\,\zeta(\lambda) = \frac{1}{\Gamma(n-\rho)}\frac{d^{n}}{d\lambda^{n}}\int_{a}^{\lambda}(\lambda - s)^{n-\rho-1}\,\zeta(s)ds = D^{n}I_{a}^{n-\rho}\zeta(t), $$

$$ n-1 < \rho < n, \quad n \in \mathbb{N}. $$

Example 1.2.11

Assume $0 < \rho < 1$ and $\lambda \in [0,b]$, then

$$ {}_{0}^{RL}I^{\rho}\,\lambda^{2} = \frac{1}{\Gamma(\rho)}\int_{0}^{\lambda}(\lambda - s)^{\rho-1}s^{2}\,ds, $$

by integration of parts, we obtain

$$ {}_{0}^{RL}I^{\rho}\,\lambda^{2} = \frac{2}{\Gamma(\rho+1)}\int_{0}^{\lambda}(\lambda - s)^{\rho}s\,ds $$

$$ = \frac{2}{\Gamma(\rho+2)}\int_{0}^{\lambda}(\lambda - s)^{\rho+1}\,ds $$

$$ = \frac{2}{\Gamma(\rho+3)}\lambda^{\rho+2}. $$

Also, we can calculate the Riemann-Liouville derivative of order ρ of the function λ^{2} as follows

$$ {}_{0}^{RL}D^{\rho}\,\lambda^{2} = D\,{}_{0}^{RL}I^{1-\rho}\lambda^{2} = D\frac{2}{\Gamma(4-\rho)}\lambda^{(3-\rho)} $$

$$= \frac{2(3-\rho)}{(3-\rho)\Gamma(3-\rho)} \lambda^{(2-\rho)}$$

$$= \frac{2}{\Gamma(3-\rho)} \lambda^{(2-\rho)}.$$

If we choose $\rho = 0.5$, then

$${}_{0}^{RL}I^{0.5}\lambda^2 = \frac{2}{\Gamma(3.5)} \lambda^{2.5} = 0.6018\, \lambda^{2.5}.$$

$${}_{0}^{RL}D^{0.5}\lambda^2 = \frac{2}{\Gamma(2.5)} \lambda^{1.5} = 1.5045\, \lambda^{1.5}.$$

Definition 1.2.12 [7]

For a function ζ, the expression

$${}_{a}^{C}D^{\rho}\, \zeta(\lambda) = \frac{1}{\Gamma(n-\rho)} \int_a^{\lambda} (\lambda - s)^{n-\rho-1}\, \zeta^{(n)}(s) ds = {}_{a}^{RL}I^{n-\rho} D^n \zeta(\lambda),$$

is called the **Caputo fractional derivative** of order ρ, where $\lambda \in [a, b]$,

$n - 1 < \rho < n, n \in \mathbb{N}$.

Lemma 1.2.13 [7]

Let $\rho \in \mathbb{R}$ and $n - 1 < \rho < n, n \in \mathbb{N}$. The relationship between the Riemann-Liouville derivative and the Caputo derivative operators is given by

$${}_{a}^{RL}D^{\rho}\, \zeta(\lambda) = {}_{a}^{C}D^{\rho}\, \zeta(\lambda) + \sum_{j=0}^{n-1} \frac{\zeta^{(j)}(a)}{\Gamma(j-\rho+1)} (\lambda - a)^{j-\rho}.$$

In particular, when $0 < \rho < 1$, we have

$${}_{a}^{RL}D^{\rho}\, \zeta(\lambda) = {}_{a}^{C}D^{\rho}\, \zeta(\lambda) + \frac{\zeta(a)}{\Gamma(1-\rho)} (\lambda - a)^{-\rho}.$$

Lemma 1.2.14 [7]

Let $\rho > 0$ and $q > 0$. Then

$${}_{a}^{RL}I^{\rho}\, {}_{a}^{RL}I^{q} \zeta(\lambda) = {}_{a}^{RL}I^{q}\, {}_{a}^{RL}I^{\rho} \zeta(\lambda) = {}_{a}^{RL}I^{\rho+q} \zeta(\lambda).$$

Remark 1.2.15 [7]

For all scalars q, r , we have

$${}_{a}^{RL}I^{\rho}\big(q\,x(\lambda)+r\,y(\lambda)\big)=q\;{}_{a}^{RL}I^{\rho}x(\lambda)+r\;{}_{a}^{RL}I^{\rho}y(\lambda).$$

Lemma 1.2.16 [7]

Let $n-1<\rho<n, n\in\mathbb{N}$ and $s>0$ then:

i- the Laplace transform of the Riemann-Liouville fractional integration operator of order ρ is given by

$$\mathcal{L}\{{}_{0}^{RL}I^{\rho}\zeta(t)\}(s)=s^{-\rho}\mathcal{L}\{\zeta(t)\}(s),$$

ii- the Laplace transform of the Riemann-Liouville fractional differential operator of order ρ is given by

$$\mathcal{L}\left\{{}_{0}^{RL}D^{\rho}\,\zeta(t)\right\}(s)=s^{\rho}\mathcal{L}\{\zeta(t)\}-\sum_{j=0}^{n-1}s^{n-j-1}D^{(j)}\left({}_{0}^{RL}I^{n-\rho}\zeta\right)(0),$$

iii- the Laplace transform of the Caputo fractional differential operator of order ρ is given by

$$\mathcal{L}\left\{{}_{0}^{C}D^{\rho}\,\zeta(t)\right\}(s)=s^{\rho}\mathcal{L}\{\zeta(t)\}-\sum_{j=0}^{n-1}s^{\rho-j-1}\left(D^{(j)}\zeta\right)(0).$$

Now, we recall some definitions and properties of fractional operators with nonsingular kernel.

Let $\rho\in(0,1), \omega>0, \gamma_{\rho}=\frac{\rho}{1-\rho}$ and $Q(\rho)$ is normalization function satisfies $Q(0)=Q(1)=1$.

Definition 1.2.17 [11]

The **Atangana-Baleanu fractional (AB-fractional) derivative of Riemann-Liouville sense** of order ρ with lower limit a is given by

$${}_{a}^{ABR}D^{\rho}\,\zeta(t)=\frac{Q(\rho)}{1-\rho}\frac{d}{dt}\int_{a}^{t}E_{\rho}\left[-\gamma_{\rho}(t-s)^{\rho}\right]\zeta(s)\,ds, t\in[a,b].$$

Definition 1.2.18 [11]

The **Atangana-Baleanu fractional (AB-fractional) derivative of Caputo sense** of order ρ is given by

$$ {}^{ABC}_{a}D^{\rho}\,\zeta(t) = \frac{Q(\rho)}{1-\rho}\int_{a}^{t} E_{\rho}\left[-\gamma_{\rho}(t-s)^{\rho}\right]\left(\frac{d}{ds}\zeta\right)(s)\,ds, t \in [a,b] $$

Definition 1.2.19 [11]

The **fractional integral associated with the Atangana-Baleanu fractional (AB-fractional) derivative** is defined by

$$ {}^{AB}I^{\rho}\,\zeta(t) = \frac{(1-\rho)}{Q(\rho)}\zeta(t) + \rho\,{}^{RL}I^{\rho}\,\zeta(t), \qquad t \in [a,b]. $$

Note: Throughout our work we assume $Q(\rho) = 1$.

Definition 1.2.20 [12]

The **Hattaf fractional derivative of Riemann-Liouville sense** of order ρ with respect to the weight function $\eta \in C^{1}(a,b)$, $\eta, \acute{\eta} > 0$ with the lower limit a is given by

$$ {}^{RL}_{a}D^{\rho,\omega,\lambda}\zeta(t) = \frac{Q(\rho)}{1-\rho}\frac{1}{\eta(t)}\frac{d}{dt}\int_{a}^{t} E_{\omega}\left[-\gamma_{\rho}(t-s)^{\lambda}\right](\eta\zeta)(s)\,ds. $$

Definition 1.2.21 [12]

The **Hattaf-fractional derivative of Caputo sense** of order ρ with respect to the weight function $\eta \in C^{1}(a,b), \eta, \acute{\eta} > 0$ on $[a,b]$ is given by

$$ {}^{C}_{a}D^{\rho,\omega,\lambda}_{\eta}\zeta(t) = \frac{Q(\rho)}{1-\rho}\frac{1}{\eta(t)}\int_{a}^{t} E_{\omega}\left[-\gamma_{\rho}(t-s)^{\lambda}\right]\frac{d}{d\zeta}(\eta\zeta)(s)ds, \qquad (1.3) $$

Lemma 1.2.22 [12]

The relationship between the Hattaf-derivative of Riemann-Liouville sense and Hattaf-fractional derivative of Caputo sense operators is given by

$$ {}_{a}^{RL}D^{\rho,\omega,\lambda}\zeta(t) = {}_{a}^{C}D^{\rho,\omega,\lambda}\zeta(t) + \frac{Q(\rho)}{1-\rho}\frac{1}{\eta(t)}E_{\omega}\left[-\gamma_{\rho}(t-a)^{\lambda}\right](\eta\zeta)(a). $$

Remark 1.2.23 [12]

When $\lambda = \omega$ and $Q(\rho) = \eta(t) = 1$, then the fractional derivative (1.3) will be in the form

$$ {}_{a}^{C}D^{\rho,\omega}\zeta(t) = \frac{1}{1-\rho}\int_{a}^{t} E_{\omega}\left[-\gamma_{\rho}(t-s)^{\omega}\right]\frac{d}{d\zeta}\zeta(s)ds. \tag{1.4} $$

Definition 1.2.24 [12]

The **fractional integral corresponding to the Hattaf-fractional derivative (1.4)** is

$$ {}_{a}\mathcal{J}^{\rho,\omega}\zeta(t) = (1-\rho)\zeta(t) + \rho\, {}_{a}^{RL}I^{\omega}\zeta(t) \tag{1.5} $$

Lemma 1.2.25 [11]

The Laplace transform of AB-fractional derivative of Caputo sense is

$$ \mathcal{L}\left\{ {}_{0}^{ABC}\mathrm{D}^{\rho}\, \zeta(t)\right\}(s) = \frac{Q(\rho)}{1-\rho}\frac{s^{\rho}\mathcal{L}\{\zeta(t)\}(s) - s^{\rho-1}\zeta(0)}{s^{\rho}+\gamma_{\rho}}. $$

Lemma 1.2.26 [12]

The Laplace transform of the Hattaf-fractional differential operator (1.4) is

$$ \mathcal{L}\left\{ {}_{0}^{C}D^{\rho,\omega}\zeta(t)\right\}(s) = \frac{1}{1-\rho}\frac{s^{\omega}\mathcal{L}\{\zeta(t)\}(s) - s^{\omega-1}\zeta(0)}{s^{\omega}+\gamma_{\rho}}. $$

Lemma 1.2.27 [12]

The Laplace transform of the Hattaf-fractional integration operator (1.5) is

$$\mathcal{L}\{\,{}_0\mathcal{J}^{\rho,\omega}\zeta(t)\}(s) = (1-\rho)\mathcal{L}\{\zeta(t)\}(s) + \frac{\rho}{s^{\omega}}\mathcal{L}\{\zeta(t)\}(s).$$

Lemma 1.2.28 [12]

For Hattaf-fractional differential operator (1.4) and Hattaf-fractional integration operator (1.5),

$${}_a^C D^{\rho,\omega}\ {}_a\mathcal{J}^{\rho,\omega}\zeta(\lambda) = \zeta(\lambda).$$

In the following, some properties for the fractional differential operator (1.4) and fractional integral operator (1.5) are proven.

Lemma 1.2.29

The Hattaf-fractional derivative (1.4) can be written as

$${}_a^C D^{\rho,\omega}\zeta(t) = \frac{1}{1-\rho}\sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^k\,{}_a^{RL}I^{\omega k+1}\dot{\zeta}(s), \qquad 0 < k < \infty.$$

Proof.

$${}_a^C D^{\rho,\omega}\zeta(t) = \frac{1}{1-\rho}\int_a^t \dot{\zeta}(s)E_\omega\left(-\gamma_\rho(t-s)^\omega\right)ds$$

$$= \frac{1}{1-\rho}\int_a^t \dot{\zeta}(s)\sum_{k=0}^{\infty}\left(-\frac{\rho}{1-\rho}\right)^k\frac{(t-\rho)^{\omega k}}{\Gamma(\omega k+1)}ds$$

$$= \frac{1}{1-\rho}\sum_{k=0}^{\infty}\left(\frac{-\rho}{1-\rho}\right)^k\frac{1}{\Gamma(\omega k+1)}\int_a^t \dot{\zeta}(s)(t-s)^{\omega k}\,ds$$

$$= \frac{1}{1-\rho}\sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^k\,{}_a^{RL}I^{\omega k+1}\dot{\zeta}(t).$$

■

Lemma 1.2.30

Let $0 < \rho < 1, \omega > 0$. Then

$$ {}_a\mathcal{J}^{\rho,\omega}\,{}_a^C D^{\rho,\omega}\zeta(t) = \zeta(t) - \zeta(a). $$

Proof.

Since

$$ {}_a\mathcal{J}^{\rho,\omega}\,{}_a^C D^{\rho,\omega}\zeta(t) = (1-\rho)\,{}_a^C D^{\rho,\omega}\zeta(t) + \rho\,{}_a^{RL}I^{\omega}\,{}_a^C D^{\rho,\omega}\zeta(t). $$

From Lemma 1.2.29 we have,

$$ {}_a\mathcal{J}^{\rho,\omega}\,{}_a^C D^{\rho,\omega}\zeta(t) $$
$$ = \sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^k {}_a^{RL}I^{\omega k+1}\dot{\zeta}(t) + {}_a^{RL}I^{\omega}\frac{\rho}{1-\rho}\sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^k {}_a^{RL}I^{\omega k+1}\dot{\zeta}(t). $$

By using Lemma 1.2.14, it follows

$$ \mathcal{J}^{\rho,\omega}\ {}^C D^{\rho,\omega}\zeta(t) = \sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^k {}_a^{RL}I^{\omega k+1}\dot{\zeta}(t) + \frac{\rho}{1-\rho}\sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^k {}_a^{RL}I^{(1+k)\omega+1}\dot{\zeta}(t) $$
$$ = \sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^k {}_a^{RL}I^{\omega k+1}\dot{\zeta}(t) - \sum_{k=0}^{\infty}\left(-\gamma_\rho\right)^{k+1} {}_a^{RL}I^{(1+k)\omega+1}\dot{\zeta}(t) $$
$$ = \int_a^t \dot{\zeta}(s)ds = \zeta(t) - \zeta(a). $$ ■

Lemma 1.2.31

Let $0 < \rho < 1, \omega > 0$ and $\zeta \in PC[a,b]$. Then

$$ {}_a\mathcal{J}^{\rho,\omega}\,{}_a^C D^{\rho,\omega}\zeta(t) = \zeta(t) - \zeta(a) - \sum_{i=1}^{p}\Delta\zeta(t_i) $$

for $i = 1,2,\dots,p$, $\Delta\zeta(t_i) = \zeta(t_i^+) - \zeta(t_i^-)$ and $t \in [a,b]$.

Proof.

Using the same technique as in Lemma (1.3.30), we get

$$ {}_a\mathcal{J}^{\rho,\omega}\,{}_a^C D^{\rho,\omega}\zeta(t) = \int_a^t \dot{\zeta}(s)ds. $$

$$= \zeta(t) - \zeta(a) - \sum_{i=1}^{p} \Delta\zeta(t_i)$$

for $i = 1,2, \dots, p$ and $t \in [a, b]$. ■

1.3 Cauchy Problem

In this section, we recall the concept of the Cauchy problem because of its importance in solving differential equations.

Let $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{X} \to \mathbb{X}$ be a linear operator, where $\mathbb{X}$ be a Banach space. The abstract Cauchy problem for $\mathcal{A}$ and $y \in \mathbb{X}$ with initial condition y_0 consists of finding a solution $y(t)$ to the initial value problem

$$\begin{cases} \dot{y}(t) = \mathcal{A}y(t) + f(t), \ J = [0, \Upsilon] \\ y(0) = y_0. \end{cases} \tag{1.6}$$

where $f: J \to \mathbb{X}$. $\mathcal{A}$ is infinitesimal generator of a C_0 −semigroup $\mathcal{T}(t)$.

Definition 1.3.1 [33]

A function $y: J \to \mathbb{X}$ is a **classical solution** (solution) of (1.6) on J if , $y(t) \in C^1(J; \mathbb{X}), y(t) \in D(\mathcal{A})$, for all $t \in J$ and y satisfies (1.6) on J.

Let y be a solution of (1.6). Then the function $q: J \to \mathbb{X}$ defined as $q(s) = \mathcal{T}(t - s)y(s)$ is differentiable for $0 < s < t$ and

$$\frac{dq}{ds} = -\mathcal{A}\mathcal{T}(t-s)y(s) + \mathcal{T}(t-s)\dot{y}(s)$$

$$= -\mathcal{A}\mathcal{T}(t-s)y(s) + \mathcal{T}(t-s)\mathcal{A}y(s) + \mathcal{T}(t-s)f(s)$$

$$= \mathcal{T}(t-s)f(s). \tag{1.7}$$

If $f \in L_p(J;\mathbb{X})$ then, $\mathcal{T}(t-s)f(s)$ is integrable and integrating (1.7) from 0 to t, yields

$$y(t) = \mathcal{T}(t)y_0 + \int_0^t \mathcal{T}(t-s)\, f(s)ds. \quad (1.8)$$

The right-hand side of (1.8) is continuous for any $f \in L_p(J;\mathbb{X})$. [32]

It is possible to generalize the solution of (1.6) (mild solution) by removing differentiability condition which defined as follows:

Definition 1.3.2 [33]

Let $\mathcal{T}(t)$ be C_0 −semigroup generated by $\mathcal{A}$. Let $f \in L_p(J;\mathbb{X})$ and $y_0 \in \mathbb{X}$. The function $y \in C(J;\mathbb{X})$ given by (1.8) is the **mild solution** of System (1.6).

The following example shows that the continuity of f, in general, is not a sufficient condition for the existence of solutions to the Cauchy problem (1.6) for $y_0 \in D(\mathcal{A})$.

Example 1.3.3 [33]

Let $\mathcal{T}(t)$ be C_0 −semigroup generated by $\mathcal{A}$, and let $z \in \mathbb{X}$ be such that for any $t \geq 0, \mathcal{T}(t)\, z \notin D(\mathcal{A})$. Suppose that $f(t) = \mathcal{T}(t)z$. Then $f(t)$ is continuous for $t \geq 0$. Consider the initial value problem

$$\begin{cases} \dot{y}(t) = \mathcal{A}y(t) + f(t), \;\; J = [0, \Upsilon] \\ y(0) = 0. \end{cases} \quad (1.9)$$

Then the mild solution of (1.9) is

$$y(t) = \int_0^t \mathcal{T}(t-s)\, \mathcal{T}(s)zds$$

$$= \int_0^t \mathcal{T}(t)\, z ds$$

$$= t\mathcal{T}(t)z$$

which is not differentiable.

1.4 Controllability and Observability

In this section, we recall some fundamental definitions and lemmas for controllability and observability concepts in finite and infinite dimensional spaces.

Controllability

A linear control dynamical system in finite dimensional space that can be represented by the mathematical model

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t), & t \in J = [0, \Upsilon] \\ y(0) = y_0, \end{cases} \tag{1.10}$$

where $y(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are state and control vectors respectively, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices.

System (1.10) is said to be **controllable** on J, if for any pair of vectors $y_0, y_1 \in \mathbb{R}^n$, there exists a control $u \in C(J; \mathbb{R}^m)$ such that the solution of System (1.10) with given initial condition satisfies $y(\Upsilon) = y_1$ [37]. Since the solution of the System (1.10) is

$$y(t) = e^{At} y_0 + \int_0^t e^{A(t-s)} Bu(s) ds,$$

then System (1.10) will be controllable on J if for any given $y_1 \in \mathbb{R}^n$, there exists a control u such that

$$y_1 = e^{A\Upsilon}y_0 + \int_0^{\Upsilon} e^{A(\Upsilon-s)} Bu(s)ds.$$

Theorem 1.4.1 [33]

The linear System (1.10) is controllable if and only if the controllability Grammian matrix

$$\mathcal{W} = \int_0^{\Upsilon} e^{A(\Upsilon-s)}BB^* e^{A^*(\Upsilon-s)}ds$$

is nonsingular.

Theorem 1.4.2 [37]

The linear System (1.10) is controllable if and only if the rank of the controllability matrix $[B\ AB\ \ A^2B\ \ \dots\ A^{n-1}B]_{n\times nm}$ is n.

Example 1.4.3

Consider the linear System

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t)\,, & t \in [0,1] \\ y(0) = y_0 \end{cases}$$

where $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.Then the Grammian matrix

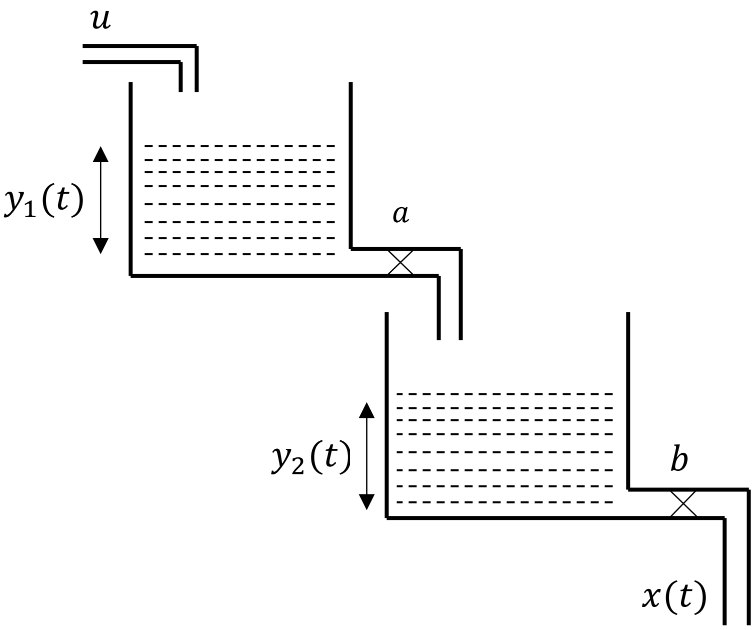
$$\mathcal{W} = \int_0^1 \begin{bmatrix} 2e^{2-2s}+1 \\ e^{1-s}+2 \end{bmatrix} [2e^{2-2s}+1 \quad e^{1-s}+2]ds$$

$$= \begin{bmatrix} 67.376 & 29.22 \\ 29.22 & 14.067 \end{bmatrix}$$

which is nonsingular, then by Theorem (1.4.1), the system is controllable.

Example 1.4.4 [38]

Consider the two tanks problem



Model (1)

Let $y_1(t)$ be denote the level of water in Tank 1, and $y_2(t)$ be denote the level of water in Tank 2. The outflow rates from Tank 1 and Tank 2 are denoted by a and b, respectively. Let u be denote the system's water supply. Then the mathematical model of the system on the time interval $[0,1]$ as follows:

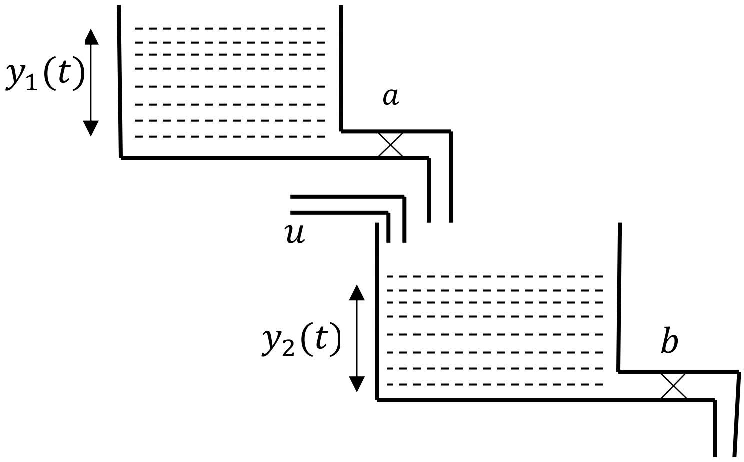
$$\dot{y}(t) = Ay(t) + Bu(t)$$

where $A = \begin{bmatrix} -a & 0 \\ a & -b \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The matrix $[B\ AB] = \begin{bmatrix} 1 & -a \\ 0 & a \end{bmatrix}$ has full rank, then by Theorem (1.4.2) the system is controllable.

Model (2)

Now, if we consider the two Tank problems as following figure:



Then the mathematical model of the system on the time interval $[0,1]$ as follows:

$$\dot{y}(t) = Ay(t) + Bu(t)$$

where $A = \begin{bmatrix} -a & 0 \\ a & -b \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The controllability matrix $[B\ AB] = \begin{bmatrix} 0 & 0 \\ 1 & -b \end{bmatrix}$ hasn't full rank, then by Theorem (1.4.2), the system is not controllable.

On the other hand, the mathematical model of linear control dynamical system in infinite dimensional space can be written as

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t), & t \in J = [0, \Upsilon] \\ y(0) = y_0 \end{cases} \tag{1.11}$$

where y takes values in a Banach space $\mathbb{X}$ and the control function u takes values in a Banach space U. The operator $A: D(A) \subset \mathbb{X} \to \mathbb{X}$ is a closed, linear and densely defined, but not necessarily bounded operator and $B: U \to \mathbb{X}$ is a bounded linear operator.

For any $y_0 \in \mathbb{X}$, the function $y \in C(J; \mathbb{X})$ given by [33]

$$y(t) = \mathcal{T}(t)y_0 + \int_0^t \mathcal{T}(t-s)Bu(s)ds, \qquad t \in J = [0, \Upsilon]$$

is called the mild solution of System (1.11), where $\mathcal{T}(t):\mathbb{X}\to\mathbb{X}$ is the C_0 −semigroup generated by the operator $\mathcal{A}$.

In (1977) [39], Triggiani proved that if $\mathcal{A}$ generates a compact C_0-semigroup $\mathcal{T}(t)$, then the linear system could never be exact controllable in an infinite dimensional space.

Definition 1.4.5 [40]

The linear System (1.11) is said to be **approximately controllable** on the interval J , if for given $\epsilon > 0$ and two arbitrary initial and final points y_0 and y_1 in $\mathbb{X}$, there exists an admissible control $u(t)$ on J steering y_0, along a trajectory (solution) $y(t)$ of System (1.11) to ϵ −neighbourad of y_1 such that $\|y(\Upsilon)-y_1\|_X \le \epsilon$.

Definition 1.4.6 [40]

The set $\mathcal{K}_\Upsilon(f)$ defined by

$$\mathcal{K}_\Upsilon(f) = \{\, y(\Upsilon;u), u(t)\in U\}, t\in[0,\Upsilon]$$

is called the **reachable set**, which consists of all possible final states.

Definition 1.4.7 [33]

The linear System (1.11) is said to be:

1. **Exact controllable** on $J=[0,\Upsilon]$ if
$$\mathcal{K}_\Upsilon(f)=\mathbb{X}.$$
2. **Approximate controllable** on $J=[0,\Upsilon]$ if
$$\overline{\mathcal{K}_\Upsilon(f)} = \mathbb{X}.$$

Observability

A linear control dynamical system with output in finite dimensional space can be represented by the mathematical model

$$\begin{cases} \dot{y}(t) = Ay(t), & t \in J = [0, \Upsilon] \\ y(0) = y_0, \\ x(t) = Cy(t), \end{cases} \tag{1.12}$$

where the vector $y(t) \in \mathbb{R}^n$ is state vector, $x(\mathrm{t}) \in \mathbb{R}^n$ is the output vector, $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times n}$ are constant matrices. The System is observable over the time interval J if it possible to determine uniquely the initial state $y(0) = y_0$ from knowledge of the output $x(t)$ over J.

Theorem 1.4.8 [41]

The linear dynamical System (1.12) is observable over the interval J if and only if one of the following statements is hold:

(I) The $n \times n$ observability Gramian matrix

$$\mathcal{W}_{ob} = \int_0^{\Upsilon} e^{A^* s} C^* C e^{As} ds$$

is nonsingular.

(II) The observability matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{nm \times n}$$

has full column rank. i.e.,

$$Rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{nm \times n} = n.$$

(III) For each eigenvalue λ of A, we have

$$Rank \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}_{(n+m) \times n} = n.$$

Example. 1.4.9

Consider the system

$$\begin{cases} \dot{y}(t) = Ay(t), & t \in J = [0,1], \\ y(0) = y_0, \\ x(t) = Cy(t), \end{cases}$$

where $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $C = [1 \quad 1]$.

The observability Grammian matrix

$$\mathcal{W}_{ob} = \int_0^1 \begin{bmatrix} e^{2s}+1 \\ e^{2s}+1 \end{bmatrix} [\, e^{2s}+1 \quad e^{2s}+1] ds$$
$$= \begin{bmatrix} 20.7886 & 20.7886 \\ 20.7886 & 20.7886 \end{bmatrix}$$

which is singular. Then by Theorem (1.4.8) (I), the system is not observable.

Example 1.4.10

Consider the model (1) of the two-tanks system in Example (1.4.4), with the output $x(t) = Cy(t)$

where $C = [0 \quad b]$.

The observability matrix

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & b \\ ab & -b^2 \end{bmatrix}$$

has full rank. Then by Theorem (1.4.8) (II), the system is observable.

Theorem 1.4.11 [41] "Theorem of Duality"

The linear System (1.10) is controllable if and only if the system

$$\begin{cases} \dot{y}(t) = -A^* y(t), \quad y(0) = y_0, \quad t \in J = [0, \Upsilon] \\ \\ \quad x(t) = B^* y(t) \end{cases}$$

is observable.

Chapter Two

Controllability and Observability of AB-Fractional Control Dynamical Systems in Finite Dimensional Space

This chapter discusses the controllability and the observability of AB-fractional control dynamical systems in $\mathbb{R}^{\boldsymbol{n}}$.

The First Section investigates the controllability of non-linear dynamical systems with control delay using the Gramian matrix and Schauder fixed point theorem.

Section Two presents more than one criterion for the observability of linear dynamical systems. Additionally, the duality between controllability and observability has been proven.

2.1 Controllability Results for AB-Fractional Nonlinear Dynamical Systems with Control Delay

In 2020 J. Sheng et al. [19] used the fixed point technique to prove the following nonlinear AB-fractional dynamical system is controllable

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = Ay(t) + B\mathcal{u}(t) + f(t, y(t), \mathcal{u}(t)), t \in I, \\ y(0) = y_0 \end{cases}$$

where ${}^{ABC}D^{\rho}$ is AB-fractional derivative of order $\rho,\ 0 < \rho < 1,\ y(\mathrm{t}) \in \mathbb{R}^n$, $\mathcal{u}(\mathrm{t}) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $b > 0$, $I = [0, b]$ and the function

$f : I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{\boldsymbol{n}}$ is continuous.

Our work aims to study the controllability of the AB-fractional linear dynamical system with control delay

$$\begin{cases} {}^{ABC}D^{\rho} y(t) = Ay(t) + Bu(\hbar(t)), t \in J = [0, \Upsilon] \\ y(0) = y_0, \end{cases} \tag{2.1}$$

and AB-fractional nonlinear dynamical system with control delay

$$\begin{cases} {}^{ABC}D^{\rho} y(t) = Ay(t) + Bu\big(\hbar(t)\big) + f(t, y(t), u(\mathrm{t})), \qquad t \in J, \\ \qquad\qquad y(0) = y_0, \end{cases} \tag{2.2}$$

where $\Upsilon > 0,\ 0 < \rho < 1$. Here $y(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, y, u are continuous vector valued functions, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are constant matrices, the function $f: J \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous.
The matrix $I - (1 - \rho)A$ is nonsingular, and $A_\rho = [I - (1 - \rho)A\ ]^{-1}$.
. $\hbar: J \to \mathbb{R}$ is strictly increasing in J and twice continuously differentiable. In addition, $\hbar(t) \le t, h(\Upsilon) = \Upsilon$ and $|u(\hbar(t))| \le |u(t)|$. Define the time lead function $\sigma: [\hbar(0), \hbar(\Upsilon)] \to J$ such that $\sigma\big(\hbar(t)\big) = t$ for $t \in J$. Assume that $l > 0$ is given and $u: [-l, \Upsilon] \to \mathbb{R}^m$, u_t denote the function defined on $[-l, 0]$ as $u_t(\wp) = u(t + \wp)$ for $t \in J$ and $\wp \in [-l, 0]$.

Lemma 2.1.1 [42]

Suppose that continuous function $h: D \times \mathbb{R}^n \to \mathbb{R}^m$ satisfies $\lim\limits_{\|w\|\to\infty} \frac{\|h(v,w)\|}{\|w\|} = 0$ uniformly in $v \in D$ where D is a bounded subset of $\mathbb{R}$, then for each pair of constants a and b, there exists a positive constant e such that if $\|w\| \le e$, then $a\|h(v, w)\| + b \le e,$ for all $v \in D$.

In the following, the controllability of The Linear System (2.1) will be discussed.

Firstly, we introduce the solution of System (2.1) in the next theorem.

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Theorem 2.1.2

The solution of System (2.1) given by

$$y(t)=\begin{cases} A_\rho E_\rho\left(\rho A A_\rho t^\rho\right)y_0+(1-\rho)A_\rho B u(\hbar(t)) \\ +\rho A_\rho^2\int_0^{\hbar(t)}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}\left(\rho A A_\rho\left(t-\sigma(\xi)\right)^\rho\right) \\ \qquad\qquad \times Bu(\xi)\dot{\sigma}(\xi)d\xi+G(t)\end{cases} \tag{2.3}$$

where

$$G(t)=\rho A_\rho^2\int_{\hbar(0)}^{0}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}\left(\rho A A_\rho\left(t-\sigma(\xi)\right)^\rho\right)Bu_0(\xi)\dot{\sigma}(\xi)d\xi.$$

Poof. By taking Laplace transformation of the both sides of System (2.1)

$$\frac{1}{1-\rho}\cdot\frac{s^\rho Y(s)-s^{\rho-1}y_0}{s^\rho+\frac{\rho}{1-\rho}}=AY(s)+\mathcal{L}\{Bu(\hbar(t))\}(s)$$

where $Y(s)=\mathcal{L}\{y(t)\}(s)$,

then

$$s^\rho Y(s)-s^{\rho-1}y_0$$
$$=[(1-\rho)s^\rho+\rho]AY(s)+[(1-\rho)s^\rho+\rho]\mathcal{L}\{Bu(\hbar(t))\}(s)$$

it follows

$$[(I-(1-\rho)A)s^\rho-\rho A]Y(s)=s^{\rho-1}y_0+[(1-\rho)s^\rho+\rho]\mathcal{L}\{Bu(\hbar(t))\}(s).$$

By substitute A_ρ, we have

$$\left[{A_\rho}^{-1}s^\rho-\rho A\right]Y(s)=s^{\rho-1}y_0+[(1-\rho)s^\rho+\rho]\mathcal{L}\{Bu(\hbar(t))\}(s)$$

$$\left(s^\rho I-\rho A A_\rho\right){A_\rho}^{-1}Y(s)=s^{\rho-1}y_0+[(1-\rho)s^\rho+\rho]\mathcal{L}\{Bu(\hbar(t))\}(s),$$

therefore

$$Y(s) = s^{\rho-1}A_\rho\left(s^\rho I - \rho AA_\rho\right)^{-1}y_0 + \left(s^\rho I - \rho AA_\rho\right)^{-1}A_\rho \times [(1-\rho)s^\rho + \rho]\mathcal{L}\{Bu(\hbar(t))\}(s)$$

$$= A_\rho s^{\rho-1}(s^\rho I - \rho AA_\rho)^{-1}y_0 + (1-\rho)s^\rho(s^\rho I - \rho AA_\rho)^{-1}A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s) + \rho(s^\rho I - \rho AA_\rho)^{-1}A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s).$$

Note that

$$s^\rho\left(s^\rho I - \rho AA_\rho\right)^{-1} = \left(s^\rho I - \rho AA_\rho + \rho AA_\rho\right)\left(s^\rho I - \rho AA_\rho\right)^{-1}$$
$$= \left(s^\rho I - \rho AA_\rho\right)\left(s^\rho I - \rho AA_\rho\right)^{-1} + \rho AA_\rho\left(s^\rho I - \rho AA_\rho\right)^{-1}$$
$$= I + \rho AA_\rho\left(s^\rho I - \rho AA_\rho\right)^{-1}.$$

Therefore

$$Y(s) = A_\rho s^{\rho-1}(s^\rho I - \rho AA_\rho)^{-1}y_0 + (1-\rho)\left(I + \rho AA_\rho\left(s^\rho I - \rho AA_\rho\right)^{-1}\right)A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s) + \rho(s^\rho I - \rho AA_\rho)^{-1}A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s)$$

$$= A_\rho s^{\rho-1}(s^\rho I - \rho AA_\rho)^{-1}y_0 + (1-\rho)A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s) + (1-\rho)\rho A{A_\rho}^2\left(s^\rho I - \rho AA_\rho\right)^{-1}\mathcal{L}\{Bu(\hbar(t))\}(s) + \rho(s^\rho I - \rho AA_\rho)^{-1}A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s)$$

$$= A_\rho s^{\rho-1}(s^\rho I - \rho AA_\rho)^{-1}y_0 + (1-\rho)A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s) + \left[(1-\rho)A + {A_\rho}^{-1}\right]\rho{A_\rho}^2\,\rho(s^\rho I - \rho AA_\rho)^{-1}\mathcal{L}\{Bu(\hbar(t))\}(s),$$

therefore,

$$Y(s) = A_\rho s^{\rho-1}\left(s^\rho I - \rho AA_\rho\right)^{-1}y_0 + (1-\rho)A_\rho\mathcal{L}\{Bu(\hbar(t))\}(s) + \rho{A_\rho}^2\left(s^\rho I - \rho AA_\rho\right)^{-1}\mathcal{L}\{Bu(\hbar(t))\}(s). \tag{2.4}$$

Now, by taking inverse of Laplace transform for (2.4) we get,

$$\mathcal{L}^{-1}\{Y(s);t\} = A_\rho\mathcal{L}^{-1}\{s^{\rho-1}(s^\rho I - \rho AA_\rho)^{-1};t\}y_0 + (1-\rho)A_\rho Bu(\hbar(t)) + \rho{A_\rho}^2\left[\mathcal{L}^{-1}\{(s^\rho I - \rho AA_\rho)^{-1};t\} * Bu(\hbar(t))\right].$$

From Lemma (1.2.8) we have

$$\mathcal{L}^{-1}\{s^{\rho-1}(s^{\rho}I-\rho AA_{\rho})^{-1};t\}=E_{\rho}(\rho AA_{\rho}t^{\rho})$$

$$\mathcal{L}^{-1}\{(s^{\rho}I-\rho AA_{\rho})^{-1};t\}=t^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}t^{\rho}),$$

therefore,

$$\begin{aligned}y(t)&=A_{\rho}E_{\rho}(\rho AA_{\rho}t^{\rho})y_0+(1-\rho)A_{\rho}Bu(\hbar(t))\\&\quad+\rho {A_{\rho}}^{2}\left[\left(t^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}t^{\rho})\right)*Bu(\hbar(t))\right]\\&=A_{\rho}E_{\rho}(\rho AA_{\rho}t^{\rho})y_0+(1-\rho)A_{\rho}Bu(\hbar(t))\\&\quad+\rho {A_{\rho}}^{2}\int_0^t(t-\xi)^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}(t-\xi)^{\rho})Bu(\hbar(\xi))d\xi.\end{aligned}$$

We use the time lead function $\sigma(t)$ to write the above solution in the following form

$$\begin{aligned}y(t)&=A_{\rho}E_{\rho}(\rho AA_{\rho}t^{\rho})y_0+(1-\rho)A_{\rho}Bu(\hbar(t))\\&\quad+\rho A_{\rho}^{2}\int_{\hbar(0)}^{\hbar(t)}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}(t-\sigma(\xi))^{\rho})Bu(\xi)\dot{\sigma}(\xi)d\xi\end{aligned}$$

$$\begin{aligned}&=A_{\rho}E_{\rho}(\rho AA_{\rho}t^{\rho})y_0+(1-\rho)A_{\rho}Bu(\hbar(t))\\&\quad+\rho A_{\rho}^{2}\int_{\hbar(0)}^{0}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}(t-\sigma(\xi))^{\rho})Bu(\xi)\dot{\sigma}(\xi)d\xi\\&\quad+\rho A_{\rho}^{2}\int_{0}^{\hbar(t)}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}(t-\sigma(\xi))^{\rho})Bu(\xi)\dot{\sigma}(\xi)d\xi.\end{aligned}$$

Therefore,

$$y(t)=\begin{cases}A_{\rho}E_{\rho}(\rho AA_{\rho}t^{\rho})y_0+(1-\rho)A_{\rho}Bu(\hbar(t))\\+\rho A_{\rho}^{2}\int_{0}^{\hbar(t)}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}(\rho AA_{\rho}(t-\sigma(\xi))^{\rho})\\\qquad\times Bu(\xi)\dot{\sigma}(\xi)d\xi+G(t).\end{cases}$$

Note: The completely state of the Linear System (2.1) at $t \in J$ is the set $g(t) = \{y(t), \mathcal{u}_t\}$.

Definition 2.1.3

The System (2.1) is said to be controllable on J if for every complete state $g(0)$ and every $y_1 \in \mathbb{R}^n$ there exists a control $\mathcal{u}$(t) defined on J such that the solution $y(t)$ of the system satisfies $y(\Upsilon) = y_1$.

Theorem 2.1.4

The AB-fractional linear dynamical system with control delay (2.1) is controllable on the interval J if the matrix $\mathcal{W}_1$ is nonsingular, such that

$$\mathcal{W}_1 = \frac{1-\rho}{\Gamma(\rho)} BB^* \dot{\sigma}(\Upsilon) + \rho A_\rho \mathcal{W}$$

where

$$\mathcal{W} = \int_0^{\Upsilon} \left(\Upsilon - \sigma(\xi)\right)^{\rho-1} \left[E_{\rho,\rho}(\rho A A_\rho \left(\Upsilon - \sigma(\xi)\right)^{\rho} B \dot{\sigma}(\xi)\right]$$

$$\times \left[E_{\rho,\rho}(\rho A A_\rho \left(\Upsilon - \sigma(\xi)\right)^{\rho} B \dot{\sigma}(\xi)\right]^* d\xi$$

is the controllability-Grammian matrix.

Proof. By hypothesis that $\mathcal{W}_1$ is nonsingular and since the matrix A_ρ is also nonsingular, then the control function may be defined as follows

$$\mathcal{u}(t) = B^* E^*_{\rho,\rho}(\rho A A_\rho \left(\Upsilon - \sigma(t)\right)^{\rho} \dot{\sigma}(t) \mathcal{M} {A_\rho}^{-1} {\mathcal{W}_1}^{-1}$$

where

$$\mathcal{M} = y_1 - A_\rho E_\rho\left(\rho A A_\rho t^\rho\right) y_0 - G(\Upsilon).$$

Therefore

$$y(\Upsilon) = A_\rho E_\rho\big(\rho A A_\rho \Upsilon^\rho\big) y_0 + (1-\rho) A_\rho B \mathcal{u}(\Upsilon) + \rho A_\rho^2$$

$$\times \int_0^\Upsilon (\Upsilon - \sigma(\xi))^{\rho-1} E_{\rho,\rho}\big(\rho A A_\rho \big(\Upsilon - \sigma(\xi)\big)^\rho\big) B\mathcal{u}(\xi)\dot{\sigma}(\xi) d\xi + G(\Upsilon)$$

$$= A_\rho E_\rho\big(\rho A A_\rho \Upsilon^\rho\big) y_0 + (1-\rho) A_\rho B B^* E^*_{\rho,\rho}(\rho A A_\rho (\Upsilon - \Upsilon)^\rho \dot{\sigma}(\Upsilon)$$

$$\times \mathcal{M} {A_\rho}^{-1} {\mathcal{W}_1}^{-1} + \rho A_\rho^2 \int_0^\Upsilon (\Upsilon - \sigma(\xi))^{\rho-1} E_{\rho,\rho}\big(\rho A A_\rho \big(\Upsilon - \sigma(\xi)\big)^\rho\big) B B^*$$

$$\times {E^*}_{\rho,\rho}(\rho A A_\rho \big(\Upsilon - \sigma(\xi)\big)^\rho \dot{\sigma}(\xi) \mathcal{M} {A_\rho}^{-1} {\mathcal{W}_1}^{-1} \dot{\sigma}(\xi) d\xi + G(\Upsilon)$$

$$= y_1.$$

Therefore, The System (2.1) is controllable. ■

In the following, the controllability of The Nonlinear System (2.2) will be discussed.

Assume $\mathcal{X}$ is Banach space of all continuous functions $(y, u): J \times J \to \mathbb{R}^n \times \mathbb{R}^m$ with the norm $\|(y, \mathcal{u})\| = \|y\| + \|\mathcal{u}\|$, when $\|y\| = \sup\{|y(t)|; t \in J\}$ and $\|\mathcal{u}\| = \sup\{|\mathcal{u}(t)|; t \in J\}$.

For each $(x, v) \in \mathcal{X}$, consider the linear system,

$$\begin{cases} {}^{ABC}D^\rho y(t) = Ay(t) + B\mathcal{u}\big(\hbar(t)\big) + f\big(t, x(t), v(\mathrm{t})\big), \ t \in J, \\ y(0) = y_0. \end{cases} \tag{2.5}$$

Then, the solution of System (2.5) given by

$$y(t)=\begin{cases} A_\rho E_\rho\left(\rho A A_\rho t^\rho\right)y_0+(1-\rho)A_\rho B u\left(\hbar(t)\right) \\ +\rho A_\rho^2\int_0^{\hbar(t)}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}\left(\rho A A_\rho\left(t-\sigma(\xi)\right)^\rho\right) \\ \times B u(\xi)\dot{\sigma}(\xi)d\xi+G(t) \\ +(1-\rho)A_\rho f\left(t,x(t),v(t)\right)+\rho A_\rho^2\int_0^t(t-\xi)^{\rho-1} \\ \times E_{\rho,\rho}\left(\rho A A_\rho(t-\xi)^\rho\right)f\left(\xi,x(\xi),v(\xi)\right)d\xi. \end{cases} \tag{2.6}$$

The controllability Gramian matrix and the control function are described by

$$\mathcal{W}=\int_0^\Upsilon\left(\Upsilon-\sigma(\xi)\right)^{\rho-1}\left[E_{\rho,\rho}(\rho A A_\rho\left(\Upsilon-\sigma(\xi)\right)^\rho B\dot{\sigma}(\xi)\right]$$
$$\times\left[E_{\rho,\rho}(\rho A A_\rho\left(\Upsilon-\sigma(\xi)\right)^\rho B\dot{\sigma}(\xi)\right]^*d\xi,$$

and

$$u(t)=B^*E_{\rho,\rho}^*(\rho A A_\rho\left(\Upsilon-\sigma(t)\right)^\rho\dot{\sigma}(t)\psi {A_\rho}^{-1}{\mathcal{W}_1}^{-1},$$

where

$$\psi=\mathcal{M}-(1-\rho)A_\rho f\left(t,x(t),v(t)\right)$$
$$-\rho A_\rho^2\int_0^\Upsilon(\Upsilon-\xi)^{\rho-1}E_{\rho,\rho}\left(\rho A A_\rho(\Upsilon-\xi)^\rho\right)$$
$$\times f\left(\xi,x(\xi),v(\xi)\right)d\xi.$$

It is straightforward to demonstrate that the control $u(t)$ steers the Linear System (2.5) from the initial state y_0 to the final state y_1.

Theorem 2.1.5

Assume that the continuous function f satisfies the condition

$$\lim_{\|(y,u)\|\to\infty}\frac{\|f(t,y,u)\|}{\|(y,u)\|}=0, \tag{2.7}$$

uniformly in J, and the matrix $\mathcal{W}_1$ is nonsingular. Then the AB-fractional nonlinear dynamical system with control delay (2.2) is controllable on J.

Proof. Define the operator $\Omega: \mathcal{X} \to \mathcal{X}$ by

$$\Omega(x, v) = (y, u),$$

where

$$u(t) = B^* E^*_{\rho,\rho}(\rho A A_\rho \left(\Upsilon - \sigma(t)\right)^\rho \dot{\sigma}(t) \psi {A_\rho}^{-1} {\mathcal{W}_1}^{-1}$$

$$= \begin{cases} B^* E^*_{\rho,\rho}(\rho A A_\rho \left(\Upsilon - \sigma(t)\right)^\rho \dot{\sigma}(t) {A_\rho}^{-1} {\mathcal{W}_1}^{-1} \\ \times \left[y_1 - A_\rho E_\rho\left(\rho A A_\rho t^\rho\right) y_0 - G(\Upsilon) \right. \\ -(1-\rho) A_\rho f\left(t, x(t), v(t)\right) - \rho A_\rho^2 \\ \left. \times \int_0^\Upsilon (\Upsilon - \xi)^{\rho-1} E_{\rho,\rho}\left(\rho A A_\rho (\Upsilon - \xi)^\rho\right) f\left(\xi, x(\xi), v(\xi)\right) d\xi \right] \end{cases}$$

and

$$y(t) = \begin{cases} A_\rho E_\rho\left(\rho A A_\rho t^\rho\right) y_0 + (1-\rho) A_\rho B u(\hbar(t)) \\ +\rho A_\rho^2 \int_0^{\hbar(t)} (t - \sigma(\xi))^{\rho-1} E_{\rho,\rho}\left(\rho A A_\rho \left(t - \sigma(\xi)\right)^\rho\right) \\ \times B u(\xi) \dot{\sigma}(\xi) d\xi + G(t) + (1-\rho) A_\rho f\left(t, x(t), v(t)\right) \\ +\rho A_\rho^2 \int_0^t (t - \xi)^{\rho-1} E_{\rho,\rho}\left(\rho A A_\rho (t - \xi)^\rho\right) \\ \times f\left(\xi, x(\xi), v(\xi)\right) d\xi. \end{cases}$$

Let

$$K = sup\|\dot{\sigma}(\xi)\|,$$

$$\eta = sup\|u_0(\xi)\|$$

$$a_1 = sup\left\|E_\rho\left(\rho A A_\rho t^\rho\right) y_0\right\|,$$

$$a_2 = sup\left\|E_{\rho,\rho}\left(\rho A A_\rho \left(\Upsilon - \sigma(\xi)\right)^\rho\right)\right\| K$$

$$\beta = sup\left\|E_{\rho,\rho}\left(\rho A A_\rho (\Upsilon - \xi)^\rho\right)\right\|,$$

$$a_3 = \max\{a_2, \beta\},$$

$$b_1 = (1-\rho)\|A_\rho\| + \Upsilon^\rho \|A_\rho{}^2\| a_3,$$

$$a = \max\{b_1\|B\|, 1\},$$

$$N = \int_{\hbar(0)}^{0} \left(\Upsilon - \sigma(\xi)\right)^{\rho-1} d\xi,$$

$$\theta = \rho \|A_\rho{}^2\| a_3 \|B\| N,$$

$$b_2 = a_3\|B^*\| \|A_\rho{}^{-1}\| \|\mathcal{W}_1{}^{-1}\| b_1,$$

$$c_1 = 4ab_2, \qquad c_2 = 4b_1,$$

$$d_1 = a_3\|B^*\| \|A_\rho{}^{-1}\| \|\mathcal{W}_1{}^{-1}\| \left(|y_1| + \|A_\rho\| a_1 + \eta\theta\right),$$

$$d_2 = 4\left(\|A_\rho\| a_1 + \eta\theta\right)$$

$$d_3 = 4ad_1, \quad d = \max\{d_2, d_3\}$$

$$\|f\| = sup\left\{\left|f\left(t, x(t), v(t)\right)\right|, t \in J\right\},$$

$$c = \max\{c_1, c_2\}.$$

Then

$$|u(t)| \le a_2\|B^*\| \|A_\rho{}^{-1}\| \|\mathcal{W}_1{}^{-1}\| \Big[|y_1| + \|A_\rho\| a_1 + \rho \|A_\rho{}^2\| \, \eta a_2 \|B\| N$$
$$+ \Big((1-\rho)\|A_\rho\| + \Upsilon^\rho \|A_\rho{}^2\| \beta\Big) \|f\|\Big]$$
$$\le a_3\|B^*\| \|A_\rho{}^{-1}\| \|\mathcal{W}_1{}^{-1}\| \left(|y_1| + \|A_\rho\| a_1 + \eta\theta\right)$$
$$+ a_3\|B^*\| \|A_\rho{}^{-1}\| \|\mathcal{W}_1{}^{-1}\| b_1 \|f\|$$
$$= d_1 + b_2\|f\|$$
$$= \frac{d_3}{4a} + \frac{c_1}{4a}\|f\|.$$

It follows,

$$|u(t)| \le \frac{1}{4a}\left(d + c\,\|f\|\right). \tag{2.8}$$

and

$$|y(t)| \leq \|A_\rho\|a_1 + (1-\rho)\|A_\rho\|\|B\||u(t)| + \Upsilon^\rho\|{A_\rho}^2\|a_2\|B\||u(t)|$$
$$+ \rho k\|{A_\rho}^2\|\,\eta a_2\|B\|N + [(1-\rho)\|A_\rho\| + \Upsilon^\rho\|{A_\rho}^2\|\beta]\|f\|$$
$$\leq \frac{d_2}{4} + \frac{b_1}{4a}\|B\|(d + c\,\|f\|) + \frac{c_2}{4}\|f\|$$
$$\leq \frac{d}{2} + \frac{c}{2}\|f\|.$$

According to Lemma (2.1.1), there exists a positive constant r such that if $\|(y,u)\| \leq r$ then $c\|f\| + d < r$.

Thus if $\|x\| \leq \frac{r}{2}$ and $\|v\| \leq \frac{r}{2}$, it is follows that $c\|f\| + d \leq r$. From (2.8) we have $\|u\| \leq \frac{r}{2}$. Since $|y(t)| \leq \frac{1}{2}(d + c\,\|f\|)$, then $\|y\| \leq \frac{r}{2}$.

Define $\mathcal{X}_r = \left\{(x,v) \in \mathcal{X} \colon \|x\| \leq \frac{r}{2} \text{ and } \|v\| \leq \frac{r}{2}\right\}$. Then Ω maps $\mathcal{X}_r$ into itself.

Since f is continuous and uniformly bounded for all $t \in J$, then the operator Ω is continuous and uniformly bounded on J. Therefore, by continuity of the Mittag-Leffler function we have $\|u(t_1) - u(t_2)\|$, $\|y(t_1) - y(t_2)\|$ tends to zero when $t_1 \to t_2$ for all $t_1, t_2 \in J$ and for all $(y,u) \in \mathcal{X}$. Thus $\Omega(\mathcal{X}_r)$ is equicontinuous and therefore, the operator Ω is compact by Arzela-Ascoli theorem. Thus Ω is completely continuous. Because $\mathcal{X}_r$ is closed, bounded and convex, then by Schauder Fixed Point Theorem, we get Ω has a fixed point $(x,v) \in \mathcal{X}_r$ that is,

$$\Omega(x,v) = (x,v) = (y,u).$$

Therefore,

$$y(t)=\begin{cases} A_\rho E_\rho\big(\rho A A_\rho t^\rho\big)y_0+(1-\rho)A_\rho B u\big(\hbar(t)\big)+ \\ \rho A_\rho^2\int_0^{\hbar(t)}(t-\sigma(\xi))^{\rho-1}E_{\rho,\rho}\big(\rho A A_\rho\big(t-\sigma(\xi)\big)^\rho\big) \\ \times Bu(\xi)\dot{\sigma}(\xi)d\xi+G(t)+\big(\rho A A_\rho(t-\xi)^\rho\big)(1-\rho) \\ \times A_\rho f\big(t,y(t),u(t)\big)+\rho A_\rho^2\int_0^t(t-\xi)^{\rho-1} \\ \times E_{\rho,\rho}\big(\rho A A_\rho\big(t-\sigma(\xi)\big)^\rho\big)f\big(s,y(\xi),u(\xi)\big)d\xi \end{cases}$$

consequently, $y(t)$ is the solution of System (2.2). Furthermore,

$$\mathrm{y}(\Upsilon)=\begin{cases} A_\rho E_\rho\big(\rho A A_\rho \Upsilon^\rho\big)y_0+\frac{(1-\rho)}{\Gamma(\rho)}BB^*\dot{\sigma}(\Upsilon)\mathcal{W}_1{}^{-1}\psi \\ +\rho A_\rho^2\int_0^{\Upsilon}(\Upsilon-\sigma(\xi))^{\rho-1}E_{\rho,\rho}\big(\rho A A_\rho\big(\Upsilon-\sigma(\xi)\big)^\rho\big) \\ \times BB^*E^*{}_{\rho,\rho}\big(\rho A A_\rho\big(\Upsilon-\sigma(\xi)\big)^\rho\big)\dot{\sigma}(\xi)A_\rho^{-1}\mathcal{W}_1{}^{-1}\psi d\xi \\ +(1-\rho)A_\rho f\big(t,y(\Upsilon),u(\Upsilon)\big)+\rho A_\rho^2 \\ \times\int_0^{\Upsilon}(\Upsilon-\xi)^{\rho-1}E_{\rho,\rho}\big(\rho A A_\rho(\Upsilon-\xi)^\rho\big) \\ f\big(\xi,y(\xi),u(\xi)\big)d\xi+G(\Upsilon) \end{cases}$$

$$=\begin{cases} A_\rho E_\rho\big(\rho A A_\rho \mathcal{T}^\rho\big)y_0+\frac{(1-\rho)}{\Gamma(\rho)}BB^*\dot{\sigma}(\Upsilon)\mathcal{W}_1{}^{-1}\psi \\ +\rho A_\rho\mathcal{W}\mathcal{W}_1{}^{-1}\psi+(1-\rho)A_\rho f\big(\Upsilon,y(\Upsilon),u(\Upsilon)\big) \\ +\rho A_\rho^2\int_0^{\Upsilon}(\Upsilon-\xi)^{\rho-1}E_{\rho,\rho}\big(\rho A A_\rho(\Upsilon-\xi)^\rho\big) \\ \times f\big(\xi,y(\xi),u(\xi)\big)d\xi+G(\Upsilon) \end{cases}$$

$$=y_1.$$

Therefore, The Nonlinear System (2.2) is controllable on J. ■

In particular, when f is integral operator, the controllability of Nonlinear System (2.2) discussed as follows.

Consider the nonlinear integrodifferential system

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = Ay(t) + B\mathcal{u}\big(\hbar(t)\big) \\ +f\left(t, y(t), \int_0^t g(t,s,y(s))ds\right), t \in J, \\ y(0) = y_0, \end{cases} \tag{2.9}$$

where $f: J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g: J \times J \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous function. Let $C_n(J)$ be Banach space of continuous $\mathbb{R}^n$ valued functions defined on J. For each $x \in C_n(J)$ consider the linear system

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = Ay(t) + B\mathcal{u}\big(\hbar(t)\big) \\ +f\left(t, y(t), \int_0^t g\big(t,s,x(s)\big)ds\right), t, s \in J, \\ y(0) = y_0. \end{cases} \tag{2.10}$$

Then, the solution of system (2.10) is

$$\mathrm{y}(t) = \begin{cases} A_{\rho}E_{\rho}\big(\rho A A_{\rho} t^{\rho}\big)y_0 + (1-\rho)A_{\rho}B\mathcal{u}\big(\hbar(t)\big) \\ +\rho A_{\rho}^2 \int_0^{\hbar(t)} (t-\sigma(\xi))^{\rho-1} \\ \times E_{\rho,\rho}\big(\rho A A_{\rho}\big(t-\sigma(\xi)\big)^{\rho}\big)B\mathcal{u}(\xi)\dot{\sigma}(\xi)d\xi \\ +G(t) + (1-\rho)A_{\rho}f\left(t, x(t), \int_0^t g\big(t,s,x(s)\big)ds\right) \\ +\rho A_{\rho}^2 \int_0^t (t-\xi)^{\rho-1}E_{\rho,\rho}\big(\rho A A_{\rho}(t-\xi)^{\rho}\big) \\ \times f\left(\xi, x(\xi), \int_0^{\xi} g(\xi,s,x(s))ds\right)d\xi. \end{cases} \tag{2.11}$$

for each $t, s \in J, x \in C_n(J)$.

Theorem 2.1.6

Let the matrix $\mathcal{W}_1$ be nonsingular and there exists a constant $L > 0$ such that the continuous function f satisfies the condition

$$\left| f\left(t, x(t), \int_0^t g\big(t, s, x(s)\big) ds\right) \right| \leq L. \quad t, s \in J, x \in C_n(J) \qquad (2.12)$$

Then the AB-fractional nonlinear integrodifferential dynamical system with control delay (2.9) is controllable on J.

Proof. Define the operator $\overline{\Omega}: C_n(J) \to C_n(J)$ by

$$\overline{\Omega}\big(x(\mathrm{t})\big) = y(t) = \begin{cases} A_\rho E_\rho\big(\rho A A_\rho t^\rho\big) y_0 + (1-\rho) A_\rho B u\big(\hbar(t)\big) \\ \qquad +\rho A_\rho^2 \int_0^{\hbar(t)} (t - \sigma(\xi))^{\rho-1} \\ \quad \times E_{\rho,\rho}\big(\rho A A_\rho \big(t - \sigma(\xi)\big)^\rho\big) B u(\xi) \dot{\sigma}(\xi) d\xi \\ +(1-\rho) A_\rho f\left(t, x(t), \int_0^t g\big(t, s, x(s)\big) ds\right) \\ +\rho A_\rho^2 \int_0^t (t-\xi)^{\rho-1} E_{\rho,\rho}\big(\rho A A_\rho (t-\xi)^\rho\big) \\ \quad \times f\left(\xi, x(\xi), \int_0^\xi g(\xi, s, x(s)) ds\right) d\xi, \end{cases}$$

where the control function $u(t)$ defined as follows,

$$u(t) = B^* {E^*}_{\rho,\rho}\big(\rho A A_\rho \big(\Upsilon - \sigma(t)\big)^\rho \dot{\sigma}(t) \psi' {A_\rho}^{-1} {\mathcal{W}_1}^{-1},$$

where

$$\psi' = \mathcal{M} - (1-\rho) A_\rho f\left(t, x(t), \int_0^t g(t, s, x(s)) ds\right)$$

$$-\rho A_\rho^2 \int_0^\Upsilon (\Upsilon - \xi)^{\rho-1} \times E_{\rho,\rho}\big(\rho A A_\rho (\Upsilon - \xi)^\rho\big)$$

$$\times f\left(\xi, x(\xi), \int_0^\xi g(\xi, s, x(s)) ds\right) d\xi.$$

Now, we define a closed convex subset

$$\varphi_r = \{x \in C_n(J) : \|x\| \leq r\}$$

where

$$r = \frac{d}{2} + \frac{c}{2}L$$

Therefore, $\overline{\Omega}$ maps φ_r into itself.

By the same technique in Theorem (2.1.5) we get $\overline{\Omega}$ has a fixed point $x \in \varphi_r$ that is,

$$\Omega(x) = x = y.$$

Therefore,

$$y(t) = \begin{cases} A_\rho E_\rho\left(\rho A A_\rho t^\rho\right) y_0 + (1-\rho) A_\rho B u(\hbar(t)) \\ +\rho A_\rho^2 \int_0^{\hbar(t)} (t-\sigma(\xi))^{\rho-1} E_{\rho,\rho}\left(\rho A A_\rho \left(t-\sigma(\xi)\right)^\rho\right) \\ B u(\xi)\dot{\sigma}(\xi) d\xi + G(t) \\ +(1-\rho) A_\rho f\left(t, x(t), \int_0^t g\left(t,s,y(s)\right) ds\right) \\ +\rho A_\rho^2 \int_0^t (t-\xi)^{\rho-1} E_{\rho,\rho}\left(\rho A A_\rho (t-\xi)^\rho\right) \\ \times f\left(\xi, x(\xi), \int_0^\xi g(\xi,s,y(s)) ds\right) d\xi \end{cases}$$

as the solution of system (2.9), and

$$y(\Upsilon)=\begin{cases} A_\rho E_\rho\big(\rho A A_\rho \Upsilon^\rho\big)y_0+\dfrac{(1-\rho)}{\Gamma(\rho)}BB^*\dot{\sigma}(\Upsilon)\mathcal{W}_1{}^{-1}\psi' \\ +\rho A_\rho^2\displaystyle\int_0^\Upsilon (\Upsilon-\sigma(\xi))^{\rho-1}\,E_{\rho,\rho}\big(\rho A A_\rho\big(\Upsilon-\sigma(\xi)\big)^\rho\big) \\ \times BB^*E^*{}_{\rho,\rho}\big(\rho A A_\rho\big(\Upsilon-\sigma(\xi)\big)^\rho\big)\dot{\sigma}(\xi)A_\rho^{-1}\mathcal{W}_1{}^{-1}\psi' d\xi \\ +(1-\rho)A_\rho f\left(\Upsilon, y(\Upsilon), \displaystyle\int_0^\Upsilon g\big(\Upsilon,s,y(s)\big)ds\right) \\ +\rho A_\rho^2\displaystyle\int_0^\Upsilon (\Upsilon-\xi)^{\rho-1}E_{\rho,\rho}\big(\rho A A_\rho(\Upsilon-\xi)^\rho\big) \\ \times f\left(\xi, y(\xi), \displaystyle\int_0^\xi g(\xi,s,y(s))ds\right)d\xi+G(\Upsilon) \end{cases}$$

$$=\begin{cases} A_\rho E_\rho\big(\rho A A_\rho \Upsilon^\rho\big)y_0+\dfrac{(1-\rho)}{\Gamma(\rho)}BB^*\dot{\sigma}(\Upsilon)\mathcal{W}_1{}^{-1}\psi' \\ +\rho A_\rho \mathcal{W}\mathcal{W}_1{}^{-1}\psi' \;+(1-\rho)A_\rho \\ \times f\left(\Upsilon, y(\Upsilon), \displaystyle\int_0^\Upsilon g\big(\Upsilon,s,y(s)\big)ds\right) \\ +\rho A_\rho^2\displaystyle\int_0^\Upsilon (\Upsilon-\xi)^{\rho-1}E_{\rho,\rho}\big(\rho A A_\rho(\Upsilon-\xi)^\rho\big) \\ \times f\left(\xi, y(\xi), \displaystyle\int_0^\xi g\big(\xi,s,y(s)\big)ds\right)d\xi+G(\Upsilon) \end{cases}$$

$$=y_1$$

Therefore, The Nonlinear System (2.9) is controllable on J. ■

Next, we give examples to illustrate our results.

Example 2.1.7

Consider the following AB-fractional nonlinear system with control delay:

$$\begin{cases} {}^{ABC}D^\rho y(t)=Ay(t)+B\mathcal{u}\big(\hbar(t)\big), \\ \quad +f\big(t,y(t),\mathcal{u}(\mathrm{t})\big), \qquad t\in[0,1] \\ y(0)=y_0, \end{cases} \tag{2.13}$$

where $\hbar: [0,1] \to \mathbb{R}$ defined as $\hbar(t) = 2t - 1$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\rho = 0.5$, and

$$f\big(t, y(t), u(t)\big) = \begin{pmatrix} 0 \\ e^t \cos y_1(t) + \sin u_2(t) \end{pmatrix}$$

Define

$$\sigma: [\hbar(0), \hbar(1)] \to [0,1], \sigma(t) = \frac{t+1}{2},$$

then

$$\dot{\sigma}(t) = \frac{1}{2}.$$

$$A_\rho = [I - (1-\rho)A\,]^{-1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix}\right]^{-1} = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix},$$

and

$$C = \rho A A_\rho = \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix}.$$

The Mittag-Leffler function matrix of the system is:

$$E_{\rho,\rho}\big(C\big(t - \sigma(\xi)\big)^\rho\big) = \sum_{k=0}^{\infty} \frac{C^k((t-\sigma(\xi))^\rho)^k}{\Gamma(k+1)\rho}$$

then,

$$E_{0.5,0.5}\left(C\big(t - \sigma(\xi)\big)^{0.5}\right)$$

$$= \sum_{k=0}^{\infty} \frac{C^k\left(\big(t-\sigma(\xi)\big)^{0.5}\right)^k}{\Gamma(k+1)0.5} = \frac{1}{\Gamma(0.5)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \big(t - \sigma(\xi)\big)^{0.5}$$

$$\times \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{\pi}} & \frac{(t-\sigma(\xi))^{0.5}}{2} \\ 0 & \frac{1}{\sqrt{\pi}} \end{pmatrix}.$$

The controllability-Gramian matrix is

$$\mathcal{W} = \begin{pmatrix} 0.254 & 0.0705 \\ 0.0705 & 0.225 \end{pmatrix}.$$

Therefore

$$\mathcal{W}_1 = \begin{pmatrix} 0.285 & 0.091 \\ 0.035 & 0.253 \end{pmatrix}$$

is nonsingular. Since the nonlinear function $f(t, y(t), u(t))$ satisfies the hypothesis (2.7), then The System (2.13) is controllable on [0,1] according to Theorem (2.1.5).

Example 2.1.8

Consider the following AB-fractional nonlinear fractional integrodifferential system with control delay

$$\begin{cases} {}^{ABC}D^{\rho} y(t) = Ay(t) + Bu(\hbar(t)) \\ +f\left(t, y(t), \int_0^t g(t, s, y(s))ds\right), t \in [0,1], \\ y(0) = y_0, \end{cases} \tag{2.14}$$

where, $\hbar$, A, B, and ρ are defined as in Example (2.1.6). The function f defined as

$$f\left(t, y(t), \int_0^t g(t, s, y(s))ds\right) = \begin{pmatrix} 0 \\ \frac{\int_0^t e^{-y_1(s)} ds}{1 + {y_1}^2(t) + {y_2}^2(t)} \end{pmatrix}.$$

Due to the fact that the function f meets the condition (2.12) and the matrix $\mathcal{W}_1$ (which was obtained in Example (2.1.7)) is nonsingular, then by Theorem (2.1.6), The Nonlinear System (2.14) is controllable.

2.2 Observability of the AB-fractional Linear Dynamical Systems

In this section, we investigate the observability of the AB-fractional linear dynamical system

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = Ay(t), \quad t \in J = [0, \Upsilon] \\ y(0) = y_0 \\ x(t) = Cy(t), \end{cases} \tag{2.15}$$

where ${}^{ABC}D^{\rho}$ is AB-fractional derivative of order ρ, $0 < \rho < 1$, the continuous vector valued functions $y(t), x(t) \in \mathbb{R}^n$ are state and output of the system respectively. $A \in \mathbb{R}^{n\times n}$and $C \in \mathbb{R}^{m\times n}$.

Immediately from Theorem (2.1.2), the solution of system (2.15) is given by

$$y(t) = A_{\rho}E_{\rho}\left(\rho A_{\rho}At^{\rho}\right)y_0. \tag{2.16}$$

Lemma 2.2.1 "Cayley-Hamilton Theorem" [43]

For every $n \times n$ matrix A,

$$a_nA^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1A + a_0I_{n\times n}$$

where

$$\Delta(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0$$

is the characteristic polynomial of A.

Theorem 2.2.2 "Rank-Nullity Theorem" [43]

Let $A \in \mathbb{R}^{m\times n}$, then

$$\dim(\ker(A)) + rank(A) = n.$$

Definition 2.2.3 [44]

Suppose that M is an $n \times n$ matrix. A linear subspace Z of $\mathbb{R}^n$ is said to be $\boldsymbol{M}$ **−invariant** if for every vector $z \in Z$ we have $Mz \in Z$.

Lemma 2.2.4 [44]

Let M be $n \times n$ matrix and Z a nonzero M −invariant subspace of $\mathbb{R}^n$. Then Z contain at least one eigenvector of M.

Now, from (2.16), the output $x(t)$ of System (2.15) is given by

$$x(t) = CA_\rho E_\rho\big(\rho A_\rho A t^\rho\big) y_0. \tag{2.17}$$

Define the observable operator $L: \mathbb{R}^n \to L_2(\mathcal{J}, \mathbb{R}^m)$ such that for any $z \in \mathbb{R}^n$,

$$Lz = CA_\rho E_\rho\big(\rho A_\rho A t^\rho\big) z.$$

The adjoint operator of L is

$$L^*: L_2(\mathcal{J}, \mathbb{R}^m) \to \mathbb{R}^n$$

defined by

$$L^*\varphi = \int_0^\Upsilon \Big(A_\rho E_\rho\big(\rho A_\rho A t^\rho\big)\Big)^* C^* \varphi\, dt,$$

for $\varphi(\cdot) \in L_2(\mathcal{J}, \mathbb{R}^m)$.

Indeed,

$$\begin{aligned}\langle Lz, \varphi(\cdot)\rangle_{L_2(\mathcal{J},\mathbb{R}^m)} &= \int_0^\Upsilon \langle CA_\rho E_\rho\big(\rho A_\rho A t^\rho\big) z, \varphi(t)\rangle_{L_2(\mathcal{J},\mathbb{R}^m)} dt \\ &= \int_0^\Upsilon \langle z, \Big(A_\rho E_\rho\big(\rho A_\rho A t^\rho\big)\Big)^* C^* \varphi(t)\rangle_{\mathbb{R}^n}\, dt \\ &= \langle z, \int_0^\Upsilon \Big(A_\rho E_\rho\big(\rho A_\rho A t^\rho\big)\Big)^* C^* \varphi(t)\, dt\rangle_{\mathbb{R}^n} \\ &= \langle z, L^*\varphi(\cdot)\rangle_{\mathbb{R}^n}.\end{aligned}$$

Therefore

$$L^*\varphi = \int_0^{\Upsilon} \left(A_\rho E_\rho(\rho A_\rho A t^\rho)\right)^* C^*\varphi\, dt.$$

Now, we can define the observability Gramian matrix as follows:

$$W_{ob} = L^*L = \int_0^{\Upsilon} \left(A_\rho E_\rho(\rho A_\rho A t^\rho)\right)^* C^* C A_\rho E_\rho(\rho A_\rho A t^\rho)\, dt.$$

Definition 2.2.5

The linear System (2.15) is said to be **observable** over the time interval $\mathcal{J} = [0, \Upsilon]$ if it is possible to determine uniquely the initial state $y(0) = y_0$ from knowledge of the output $x(t)$ over $\mathcal{J}$.

In the following theorem, dependence on observability Gramian matrix W_{ob} we present a criterion for observability of System (2.15).

Theorem 2.2.6

The Linear System (2.15) is observable if and only if the observability Gramian matrix W_{ob} is nonsingular.

Proof. Assume that The Linear System (2.15) is observable**,** and suppose the Gramian matrix W_{ob} is singular then there is a vector $r \in \mathbb{R}^n$ such that $r^* W_{ob} r = 0, r \neq 0$, then

$$\int_0^{\Upsilon} r^* \left(A_\rho E_\rho(\rho A_\rho A t^\rho)\right)^* C^* C A_\rho E_\rho(\rho A_\rho A t^\rho) r\, dt$$

$$= \int_0^{\Upsilon} \left\| C A_\rho E_\rho(\rho A_\rho A t^\rho) r \right\|^2 dt = 0,$$

then

$$C A_\rho E_\rho(\rho A_\rho A t^\rho) r = 0.$$

then by choosing $\hat{y}_0 = y_0 - r$, and from (2.17) we have

$$CA_\rho E_\rho(\rho A_\rho At^\rho)\hat{y}_0 = CA_\rho E_\rho(\rho A_\rho At^\rho)(y_0 - r)$$
$$CA_\rho E_\rho(\rho A_\rho At^\rho)y_0 - CA_\rho E_\rho(\rho A_\rho At^\rho)r$$
$$= x(t).$$

Thus, the same $x(t)$ yields from two different initial states, i.e. we cannot uniquely determine the initial condition y_0, and hence the Linear System (2.15) is not observable which is contradiction.

Conversely, assume that the observability Gramian matrix W_{ob} is invertible. By multiplying the both sides of (2.17) by $\left(A_\rho E_\rho(\rho A_\rho At^\rho)\right)^* C^*$ and integrate over $[0, \Upsilon]$, we have

$$\int_0^\Upsilon \left(A_\rho E_\rho(\rho A_\rho At^\rho)\right)^* C^* x(t)\, dt$$
$$= \int_0^\Upsilon \left(A_\rho E_\rho(\rho A_\rho At^\rho)\right)^* C^* C\, A_\rho E_\rho(\rho A_\rho At^\rho) y_0 dt$$
$$= W_{ob} y_0.$$

Since W_{ob} is invertible, then

$$y_0 = W_{ob}^{-1} \int_0^\Upsilon \left(A_\rho E_\rho(\rho A_\rho At^\rho)\right)^* C^* x(t)\, dt$$

and that is mean y_0 is uniquely determined. Therefore, the Linear System (2.15) is observable.

In the next theorem, we present another criterion condition which is observability matrix in sense of Kalman.

Theorem 2.2.7

The Linear System (2.15) is observable if and only if the matrix

$$G=\begin{bmatrix} CA_\rho \\ \frac{CA_\rho(\rho A_\rho A)}{\Gamma(\rho+1)} \\ \vdots \\ \frac{(n-1)!\,CA_\rho(\rho A_\rho A)^{n-1}}{\Gamma(\rho(n-1)+1)} \end{bmatrix}$$

has full rank.

Proof. For simplicity, we put

$$\gamma_k=\frac{\rho^k k!}{\Gamma(\rho k+1)}, k=0,1,2,\dots,n-1,$$

then the matrix G can be written as follows:

$$G=\begin{bmatrix} CA_\rho \\ \gamma_1 CA_\rho(A_\rho A) \\ \vdots \\ \gamma_{n-1}CA_\rho(A_\rho A)^{n-1} \end{bmatrix}.$$

Let $t^\rho=s$, then $0\le s=t^\rho\le \Upsilon^\rho\le\Upsilon$. We have

$$x^{(k)}(0)=\frac{k!\,CA_\rho(\rho A_\rho A)^k}{\Gamma(\rho k+1)}y_0, \qquad k=0,1,2,\dots,n-1,$$

indeed, from (2.17),

$$x(s)=CA_\rho E_\rho(\rho A_\rho As)y_0=CA_\rho\sum_{k=0}^{\infty}\frac{(\rho A_\rho As)^k}{\Gamma(\rho k+1)}y_0$$

$$=CA_\rho y_0+CA_\rho\sum_{k=1}^{\infty}\frac{(\rho A_\rho As)^k}{\Gamma(\rho k+1)}y_0,$$

then

$$x(0)=CA_\rho y_0.$$

$$x^{(1)}(s)=CA_\rho\sum_{k=1}^{\infty}\frac{(\rho A_\rho A)^k}{\Gamma(\rho k+1)}ks^{k-1}\,y_0$$

$$= CA_\rho(\rho A_\rho A)\sum_{k=1}^{\infty}\frac{(\rho A_\rho A)^{k-1}}{\Gamma(\rho k+1)}ks^{k-1}\,y_0$$

$$= \frac{CA_\rho(\rho A_\rho A)}{\Gamma(\rho+1)}y_0 + CA_\rho(\rho A_\rho A)\sum_{k=2}^{\infty}\frac{(\rho A_\rho A)^{k-1}}{\Gamma(\rho k+1)}ks^{k-1}\,y_0,$$

then

$$x^{(1)}(0) = \frac{CA_\rho(\rho A_\rho A)}{\Gamma(\rho+1)}y_0.$$

$$x^{(2)}(s) = CA_\rho(\rho A_\rho A)\sum_{k=2}^{\infty}\frac{(\rho A_\rho A)^{k-1}}{\Gamma(\rho k+1)}k(k-1)s^{k-2}\,y_0$$

$$= CA_\rho(\rho A_\rho A)^2\sum_{k=2}^{\infty}\frac{(\rho A_\rho A)^{k-2}}{\Gamma(\rho k+1)}k(k-1)s^{k-2}y_0$$

$$= 2\frac{CA_\rho(\rho A_\rho A)^2}{\Gamma(2\rho+1)}y_0 + CA_\rho(\rho A_\rho A)^2\sum_{k=3}^{\infty}\frac{(\rho A_\rho A)^{k-2}}{\Gamma(\rho k+1)}k(k-1)s^{k-2}y_0,$$

then,

$$x^{(2)}(0) = 2\frac{CA_\rho(\rho A_\rho A)^2}{\Gamma(2\rho+1)}y_0.$$

By repeating the processes, we have

$$x^{(k)}(0) = \frac{k!\,CA_\rho(\rho A_\rho A)^k}{\Gamma(\rho k+1)}y_0, \qquad k = 0, 1, 2, \dots, n-1.$$

Now, by Cayley-Hamilton Theorem, the matrix $(\rho A_\rho A)^n$ can be written as a linear combinations of lowers powers of $A_\rho A$.

Therefore

$$\begin{bmatrix} x(0) \\ x^{(1)}(0) \\ \vdots \\ x^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} CA_\rho \\ \gamma_1 CA_\rho(A_\rho A) \\ \vdots \\ \gamma_{n-1} CA_\rho(A_\rho A)^{n-1} \end{bmatrix} y_0,$$

since $rank(G) = n$, then G has left inverse, therefore y_0 is uniquely determined which means the Linear System (2.15) is observable.

Conversely, suppose that $rank(G) < n$, then there exists a nonzero vector z such that $Gz = 0$, then

$$CA_\rho z = 0, CA_\rho(A_\rho A)z = 0, \dots, CA_\rho(A_\rho A)^{n-1} z = 0.$$

By Cayley-Hamilton Theorem, the matrix $E_\rho(\rho A_\rho As)$ can be written as a linear combinations of lowers powers of $A_\rho As$. Put $0 \neq z = y_0$ and since

$$x(t) = CA_\rho E_\rho(\rho A_\rho As) y_0$$

$$= CA_\rho \left[\zeta_0 I + \zeta_1 A_\rho As + \dots + \zeta_{n-1}(A_\rho A)^{n-1} s^{n-1}\right] y_0$$

where $\zeta_k \neq 0, k = 0,1,2,\dots,n-1$.

Then

$$x(t) = \zeta_0 CA_\rho z + \zeta_1 CA_\rho(A_\rho As)z + \dots + \zeta_{n-1} CA_\rho(A_\rho As)^{n-1} z = 0.$$

Also, when $y_0 = 0$, then

$$x(t) = 0,$$

therefore, y_0 is not uniquely determined which means the Linear System (2.15) is not observable.

Based on eigenvectors of the matrix $A_\rho A$, we present observability test for the Linear System (2.15) in the following theorem.

Theorem 2.2.8

The Linear System (2.15) is observable if and only if every eigenvector of $A_\rho A$ is not belong to $\ker(CA_\rho)$.

Proof. Suppose the Linear System (2.15) is not observable, then by Theorem (2.2.7), $rank(G) < n$ where

$$G = \begin{bmatrix} CA_\rho \\ \gamma_1 CA_\rho(A_\rho A) \\ \vdots \\ \gamma_{n-1} CA_\rho(A_\rho A)^{n-1} \end{bmatrix},$$

and by Rank-Nullity Theorem, we have $\dim(\ker(G)) \geq 1$, which means that there exists a nonzero vector $z \in R^n$ such that $z \in \ker(G)\, , i.e. Gz = 0$.

Now, we show that $\ker(G)$ is $(A_\rho A) -$ invariant.

Assume that $z \in \ker(G)$, then

$$Gz = \begin{bmatrix} CA_\rho \\ \gamma_1 CA_\rho(A_\rho A) \\ \vdots \\ \gamma_{n-1} CA_\rho(A_\rho A)^{n-1} \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (2.18)$$

and

$$G(A_\rho A)z = \begin{bmatrix} CA_\rho(A_\rho A) \\ \gamma_1 CA_\rho(A_\rho A)^2 \\ \vdots \\ \gamma_{n-1} CA_\rho(A_\rho A)^{n} \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \gamma_{n-1} CA_\rho(A_\rho A)^{n} z \end{bmatrix}.$$

By using Cayley-Hamilton Theorem, $CA_\rho(A_\rho A)^n z$ can be written as a linear combination of the terms

$$CA_\rho(A_\rho A)^n z = CA_\rho z, CA_\rho(A_\rho A)z, \dots, CA_\rho(A_\rho A)^{n-1} z$$

and they are all zeros from (2.18). We conclude that $A_\rho A z$ belong to $\ker(G)$ which means that $\ker(G)$ is $(A_\rho A) -$ invariant. Then by Lemma (2.2.4), $\ker(G)$ must contain at least one eigenvector z of $A_\rho A$. But $Gz = 0$ implies $CA_\rho z = 0$ which means $z \in \ker(CA_\rho)$.

Conversely, assume that the Linear System (2.15) is observable. If there exists an eigenvalue λ such that $A_\rho A z = \lambda z$, with $z \neq 0$ for which $z \in \ker(CA_\rho)$, then

$$Gz = \begin{bmatrix} CA_\rho \\ \gamma_1 CA_\rho (A_\rho A) \\ \vdots \\ \gamma_{n-1} CA_\rho (A_\rho A)^{n-1} \end{bmatrix} z = \begin{bmatrix} CA_\rho z \\ \gamma_1 CA_\rho \lambda z \\ \vdots \\ \gamma_{n-1} CA_\rho \lambda^{n-1} z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

which means $z \in \ker(G)$ then by Rank-Nullity Theorem, $rank(G) < n$. Then from Theorem (2.2.7) we have contradiction with the observability of the Linear System (2.15).

Another test of controllability for Linear System (2.15) is presented in the following theorem.

Theorem 2.2.9

The Linear System (2.15) is observable if and only if for each $\lambda \in \mathbb{C}$, we have

$$rank \begin{bmatrix} A_\rho A - \lambda I \\ CA_\rho \end{bmatrix} = n.$$

Proof. Assume the Linear System (2.15) is observable. If there exists $\lambda \in \mathbb{C}$, such that

$$rank \begin{bmatrix} A_\rho A - \lambda I \\ CA_\rho \end{bmatrix} < n,$$

then from Rank-Nullity Theorem, there exists a nonzero vector $z \in R^n$ such that $z \in \ker \begin{bmatrix} A_\rho A - \lambda I \\ CA_\rho \end{bmatrix}$. Then $(A_\rho A - \lambda I)z = 0$ and $\gamma CA_\rho z = 0$ implies that z is an eigenvector of $A_\rho A$ and $z \in \ker(CA_\rho)$. Then by Theorem (2.2.8) that is contradiction with the observability of the Linear System (2.15).

Conversely, Suppose the system is not observable, then from Theorem (2.2.8), there exists a nonzero eigenvector z of $A_\rho A$ which belong to $\ker(CA_\rho)$ for some $\lambda \in \mathbb{C}$. This means

$$\begin{bmatrix} A_\rho A - \lambda I \\ CA_\rho \end{bmatrix} z = 0,$$

which means $z \in \ker \begin{bmatrix} A_\rho A - \lambda I \\ CA_\rho \end{bmatrix}$.

Therefore

$$rank \begin{bmatrix} A_\rho A - \lambda I \\ CA_\rho \end{bmatrix} < n.$$

In 2021, Ghasemi and Nassiri [20] studied the controllability of the AB-fractional linear control system

$$\begin{cases} {}^{ABC}D^\rho y(t) = Ay(t) + Bu(t), \;\; t \in J = [0, \Upsilon] \\ \qquad y(0) = y_0. \end{cases} \tag{2.19}$$

They set a condition on the system that the Caputo fractional derivative (of order ρ) for the control function u exists.

Lemma 2.2.10 [20]

The Linear System (2.19) is controllable on J if and only if the rank of the $n \times nm$ controllability matrix $\mathcal{R}$ is n, where

$$\mathcal{R} = [A_\alpha B \;\; (A_\alpha A)A_\alpha B \; \dots \; (A_\alpha A)^{n-1} \; A_\alpha B].$$

The following theorem shows the relationship between the controllability and the observability of an AB-fractional linear dynamical system.

Theorem 2.2.11 "Theorem of Duality"

The linear AB-fractional control dynamical system

$$\begin{cases} {}^{ABC}D^\rho y(t) = Ay(t) + Bu(\mathrm{t}), \;\; t \in J = [0, \Upsilon] \\ y(0) = y_0, \end{cases} \tag{2.20}$$

is controllable if and only if the system

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = A^{*}y(t), & t \in J = [0,\Upsilon] \\ y(0) = y_0 \\ x(t) = B^{*}y(t) \end{cases} \tag{2.21}$$

is observable.

Proof. According to Lemma (2.2.10), the Linear System (2.20) is controllable if and only if the matrix $\mathcal{R} = \left[(A_\rho B) \quad (A_\rho A)(A_\rho B) \ \dots \ (A_\rho A)^{n-1}(A_\rho B)\right]$ has full rank. The transpose

$$\mathcal{R}^{*} = \begin{bmatrix} (A_\rho B)^{*} \\ (A_\rho B)^{*}(A_\rho A)^{*} \\ \vdots \\ (A_\rho B)^{*}[(A_\rho A)^{*}]^{n-1} \end{bmatrix}$$

has full rank if and only if $\mathcal{R}$ has full rank. By Theorem (2.2.7), The System (2.21) is observable if and only if $\mathcal{R}^{*}$ has full rank.

Next, we give examples to illustrate our results.

Example 2.2.13 Consider the following AB-fractional linear system

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = Ay(t), & t \in [0,1] \\ y(0) = y_0 \\ x(t) = Cy(t), \end{cases} \tag{2.22}$$

where $A = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$, $C = (1 \quad 1)$, and $\rho = 0.5$.

Then,

$$A_\rho = [I - (1-\rho)A]^{-1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 0.5\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}\right]^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$M = \rho A_\rho A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

The Mittag-Leffler function matrix of the system is:

$$E_\rho(Mt^\rho) = \sum_{k=0}^{\infty} \frac{M^k (t^\rho)^k}{\Gamma(k\rho + 1)},$$

$$= I + \frac{t^{0.5}}{\Gamma(1.5)} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + 1.1284t^{0.5} & -1.1284t^{0.5} \\ 1.1284t^{0.5} & 1 - 1.1284t^{0.5} \end{pmatrix}.$$

Then

$$W_{ob} = \int_0^1 \left(A_\rho E_\rho(Mt^\rho)\right)^* C^* C A_\rho E_\rho(Mt^\rho)\, dt$$

$$= \int_0^1 \begin{pmatrix} 2.2568t^{0.5} + 3 \\ -2.2568t^{0.5} - 1 \end{pmatrix} (2.2568t^{0.5} + 3 \quad - 2.2568t^{0.5} - 1)\, dt$$

$$= \begin{pmatrix} 20.5737 & -11.5647 \\ -11.5647 & 6.5556 \end{pmatrix}$$

which is nonsingular. Therefore, by Theorem (2.2.6), the system is observable.

Example 2.2.14 Consider the following AB-fractional linear system

$$\begin{cases} {}^{ABC}D^\rho y(t) = Ay(t) + Bu(t), \quad t \in [0,1] \\ \\ \quad y(0) = y_0 \end{cases} \tag{2.23}$$

where $A = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\rho = 0.3$.

Then

$$A_\rho = [I - (1-\rho)A]^{-1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1.4 & 0.7 \\ -2.8 & -1.4 \end{pmatrix}\right]^{-1}$$

$$= \begin{pmatrix} 2.4 & 0.7 \\ -2.8 & -0.4 \end{pmatrix}.$$

To prove the system (2.23) is controllable by Theorem (2.2.11), we have to show that the following AB-fractional linear system is observable

$$\begin{cases} {}^{ABC}D^{\rho}y(t) = A^{*}y(t), \qquad t \in [0,1] \\ \quad y(0) = y_0 \\ \quad x(t) = B^{*}y(t). \end{cases} \tag{2.24}$$

$$A_{\rho}A = \begin{pmatrix} 2.4 & 0.7 \\ -2.8 & -0.4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$$

$$A_{\rho}B = \begin{pmatrix} 2.4 & 0.7 \\ -2.8 & -0.4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.4 \\ -2.8 \end{pmatrix}.$$

Therefore

$$G = \begin{bmatrix} (A_{\rho}B)^{*} \\ \dfrac{\rho(A_{\rho}B)^{*}(AA_{\rho})^{*}}{\Gamma(\rho+1)} \end{bmatrix} = \begin{bmatrix} 2.4 & -2.8 \\ 0.6685 & -1.3371 \end{bmatrix}$$

which is has full rank, then by Theorem (2.2.7), the System (2.24) is observable. Therefore, the System (2.23) is controllable.

Chapter Three

Exact and Approximate Controllability of Hattaf-Fractional Control Systems in a Banach Space

This chapter aims to investigate the exact and approximate controllability of Hattaf-fractional nonlinear control systems in a Banach space. In the first section, we discuss the controllability of the nonlinear system using semigroup theory and Nussbaum Fixed Point Theorem. Under sufficient conditions, the existence and uniqueness of the mild solution of the nonlinear system are proved in section two. The third section deals with the approximate controllability of the nonlinear system under sufficient conditions.

3.1 Exact Controllability of Impulsive Hattaf-Fractional Nonlinear Control System in Banach Space

This section investigates the exact controllability of the Hattaf-fractional impulsive nonlinear control system

$$\begin{cases} {}^{C}D^{\rho,\omega}\left[y(t)-h\big(t,y(t)\big)\right]=\mathcal{A}y(t)+\mathrm{B}\mathcal{u}(t) \\ \quad +f\big(t,y(t)\big), \quad t\in\mathcal{J}=[0,\Upsilon], t\neq t_{\gamma}, \gamma=1,2,\ldots.p, \\ \Delta y(t_{\gamma})=q_{\gamma}\left(y(t_{\gamma}^{-})\right), \\ y(0)=y_{0,} \end{cases} \tag{3.1}$$

where $\rho,\omega\in(0,1)$, ${}^{C}D^{\rho,\omega}$ is Hattaf-fractional derivative in the sense of Caputo of order ρ. $y(t)$ takes values in Banach space X, $\mathcal{A}: D(\mathcal{A})\subset \mathrm{X}\to\mathrm{X}$ is closed linear operator, the control $\mathcal{u}\in L_P(\mathcal{J},U)$ with U as a Banach space, $\mathrm{B}: L_P(\mathcal{J},U)\to L_P(\mathcal{J},X)$ is bounded linear operator, the functions $f:\mathcal{J}\times\mathrm{X}\to\mathrm{X}$ and $h:\mathcal{J}\times\mathrm{X}\to\mathrm{D}(\mathcal{A})$ are continuous, $q_{\gamma}: X\to X$ is the jump operator, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = \Upsilon$, $y(t_\gamma^+)$ and $y(t_\gamma^-)$ indicate to the right and left limits of $y(t)$ at $t = t_\gamma$ respectively and $\Delta y(t_\gamma) = y(t_\gamma^+) - y(t_\gamma^-)$.

Assume the linear operator $\mathcal{A}: D(\mathcal{A}) \subset X \to X$ is the generator of C_0-semigroup $\{\mathcal{G}(t), t \geq 0\}$ on a Banach space X, where $sup_{t\geq 0}\|\mathcal{G}(t)\| = \mathcal{S}, \mathcal{S} \geq 1$. We consider the linear operator $E := \rho\mathcal{A}_\rho\mathcal{A}$ where $\mathcal{A}_\rho = \left[(1-\rho)\left(\frac{1}{1-\rho}I - \mathcal{A}\right)\right]^{-1}$. Then E is the generator of C_0 −semigroup $\{\mathcal{T}(t), t \geq 0\}$ and $sup_{t\geq 0}\|\mathcal{T}(t)\| = \mathcal{S}$ [32].

We introduce the family of functions:

$$PC(\mathcal{J}, \mathrm{X}) = \{y: \mathcal{J} \to \mathrm{X}: \text{y is continuous at } t \in \mathcal{J}\backslash\{t_1, t_2, \dots, t_r\}$$
$$\text{, and there exist } y(t_\gamma^+) \text{ and } y(t_\gamma^-) \text{ with } y(t_\gamma^-) = y(t) \text{ for } \gamma = 1,2,\dots,r\}.$$

It is clear that $(PC(\mathcal{J}, \mathrm{X}), \|\cdot\|_{PC})$ is a Banach space with the norm $\|y\|_{PC} = sup_{t\in\mathcal{J}}\|y(t)\|$.

The mild solution of the Nonlinear System (3.1) will introduced in the following.

Assume the one-sided stable probability density function [45]

$$\psi_\omega(\delta) = \frac{1}{\pi}\sum_{i=1}^{\infty}(-1)^{i-1}\delta^{-\omega i-1}\frac{\Gamma(i\omega+1)}{i!}\sin(i\pi\omega)$$

then the Laplace transform of $\psi_\omega(\delta)$ is

$$\mathcal{L}\{\psi_\omega(\delta)\}(\lambda) = e^{-\lambda^\omega} \tag{3.2}$$

where $0 < \delta < \infty, 0 < \omega < 1$ and $\lambda > 0$.

Additionally, assume the probability density function

$$\varphi_\omega(\delta) = \frac{1}{\omega}\delta^{-1-\frac{1}{\omega}}\psi_\omega\left(\delta^{-\frac{1}{\omega}}\right), 0 < \delta < \infty, 0 < \omega < 1, \tag{3.3}$$

then, for $0 \leq \xi \leq 1$

$$\int_0^\infty \delta^\xi \varphi_\omega(\delta)d\delta = \int_0^\infty \frac{1}{\delta^{\omega\xi}}\psi_\omega(\delta)d\delta = \frac{\Gamma(1+\xi)}{\Gamma(1+\omega\xi)}, 0 < \delta < \infty, 0 < \omega < 1.$$

We begin by proving the following lemma before defining a mild solution of System (3.1).

Lemma 3.1.1

If $y \in PC(\mathcal{J}, \mathrm{X})$ is a solution of System (3.1), then y satisfies the following

$$y(t) = \begin{cases} \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} E L_\omega(t-\zeta) h\big(\zeta, y(\zeta)\big)d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)\big(y_0 - h(0, y_0)\big) + (1-\rho)\mathcal{A}_\rho\big[Bu(t) + f\big(t, y(t)\big)\big] \\ +\rho {\mathcal{A}_\rho}^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big]d\zeta \quad t \in [0, t_1] \\ \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} E L_\omega(t-\zeta) h\big(\zeta, y(\zeta)\big)d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)\big(y_0 - h(0, y_0)\big) + (1-\rho)\mathcal{A}_\rho\big[Bu(t) + f\big(t, y(t)\big)\big] \\ +\rho {\mathcal{A}_\rho}^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big]d\zeta \\ +\mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega\big(t - t_\gamma\big)\Delta y\big(t_\gamma\big) \qquad t \in \big(t_\gamma, t_{\gamma+1}\big] \end{cases}$$

where

$$L_\omega(t) = \omega \int_0^\infty \theta \varphi_\omega(\theta)\mathcal{T}(\theta t^\omega)d\theta,$$

$$\mathcal{T}_\omega(t) = \int_0^\infty \varphi_\omega(\theta)\mathcal{T}(\theta t^\omega)d\,\theta,$$

, $\varphi_\omega(\theta)$ given in identically (3.3) and $0 < \rho < 1$.

Proof.

We see that $y(\cdot)$ is decomposable into $m(\cdot)+n(\cdot)$, where m is a continuous mild solution of the system

$$\begin{cases} {}^{C}D^{\rho,\omega}m(t) = {}^{c}D^{\rho,\omega}h\big(t,y(t)\big) + \mathcal{A}m(t) + \mathrm{B}u(t) \\ \qquad\qquad +f\big(t,y(t)\big) \quad t\in\mathcal{J} \\ \quad m(0) = y_0 \end{cases} \tag{3.4}$$

and n is picewise continuous solution of the system

$$\begin{cases} {}^{C}D^{\rho,\omega}n(t) = \mathcal{A}n(t) \quad t\in\mathcal{J}, t\neq t_\gamma \\ \quad \Delta y\big(t_\gamma\big) = q_\gamma\left(n\big(t_\gamma^-\big)\right) \;\; \gamma = 1,2,\ldots,p, \\ \quad n(0) = 0. \end{cases} \tag{3.5}$$

To find the mild solution $m(t)$, we applying fractional integral (1.5) for both sides of System (3.4),

$$\mathcal{J}^{\rho,\omega}\, {}^{c}D^{\rho,\omega}m(t) = \mathcal{J}^{\rho,\omega}\, {}^{c}D^{\rho,\omega}h(t,y(t)) + \mathcal{J}^{\rho,\omega}\big[\mathcal{A}m(t) + \mathrm{B}u(t) + f\big(t,y(t)\big)\big].$$

using Lemma (1.2.30), we have

$$m(t) = h\big(t,y(t)\big) - h(0,y_0) + y_0 + \mathcal{J}^{\rho,\omega}[\mathcal{A}m(t) + \mathrm{B}u(t) + f\big(t,y(t)\big)] \tag{3.6}$$

By taking Laplace transform for both sides of (3.6), we have

$$\mathrm{M}(\lambda) = H(\lambda) - \frac{h(0,y_0)}{\lambda} + \frac{y_0}{\lambda} + (1-\rho)\mathcal{L}\big\{\mathcal{A}m(t) + \mathrm{B}u(t) + f\big(t,y(t)\big)\big\}(\lambda) + \frac{\rho}{\lambda^\omega}\mathcal{L}\big\{\mathcal{A}m(t) + \mathrm{B}u(t) + f\big(t,y(t)\big)\big\}(\lambda)$$

where $\mathrm{M}(\lambda) = \mathcal{L}\{m(t)\}(\lambda)$ and $H(\lambda) = \mathcal{L}\{h(t,y(t))\}(\lambda)$.

It follows

$$\mathrm{M}(\lambda)-(1-\rho)\mathcal{A}\mathrm{M}(\lambda)-\frac{\rho}{\lambda^{\omega}}\mathcal{A}\mathrm{M}(\lambda)$$
$$=H(\lambda)-\frac{h(0,y_0)}{\lambda}+\frac{y_0}{\lambda}+\left[(1-\rho)+\frac{\rho}{\lambda^{\omega}}\right]F(\lambda)$$

where $F(\lambda)=\mathcal{L}\{\mathrm{B}u(t)+f(t,y(t))\}(\lambda)$, then

$$\left[(I-(1-\rho)\mathcal{A})-\frac{\rho}{\lambda^{\omega}}\mathcal{A}\right]\mathrm{M}(\lambda)$$
$$=H(\lambda)+\frac{y_0-h(0,y_0)}{\lambda}+\left[(1-\rho)+\frac{\rho}{\lambda^{\omega}}\right]F(\lambda)$$

$$\left(\mathcal{A}_{\rho}^{-1}-\frac{\rho}{\lambda^{\omega}}\mathcal{A}\right)\mathrm{M}(\lambda)=H(\lambda)+\frac{y_0-h(0,y_0)}{\lambda}+\left[(1-\rho)+\frac{\rho}{\lambda^{\omega}}\right]F(\lambda)$$

$$\left(\lambda^{\omega}\mathcal{A}_{\rho}^{-1}-\rho\mathcal{A}\right)\mathrm{M}(\lambda)$$
$$=\lambda^{\omega}H(\lambda)+\lambda^{\omega-1}\big(y_0-h(0,y_0)\big)+[\lambda^{\omega}(1-\rho)+\rho]F(\lambda)$$

$$\left[\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}\right)\mathcal{A}_{\rho}^{-1}\right]\mathrm{M}(\lambda)$$
$$=\lambda^{\omega}H(\lambda)+\lambda^{\omega-1}[y_0-h(0,y_0)]+[\lambda^{\omega}(1-\rho)+\rho]F(\lambda)$$

$$\begin{aligned}\mathrm{M}(\lambda)=&\ \mathcal{A}_{\rho}\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}\lambda^{\omega}H(\lambda)\\&+\mathcal{A}_{\rho}\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}\lambda^{\omega-1}[y_0-h(0,y_0)]\\&+\mathcal{A}_{\rho}\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}\lambda^{\omega}(1-\rho)F(\lambda)\\&+\mathcal{A}_{\rho}\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}\rho F(\lambda).\end{aligned}$$

$$\lambda^{\omega}\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}=\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}+\rho\mathcal{A}\mathcal{A}_{\rho}\right)\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}$$
$$=\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}\right)\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}+\rho\mathcal{A}\mathcal{A}_{\rho}\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}$$
$$=I+\rho\mathcal{A}\mathcal{A}_{\rho}\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}$$

then

$$\begin{aligned}\mathrm{M}(\lambda)=&\ \mathcal{A}_{\rho}\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}\lambda^{\omega}H(\lambda)\\&+\mathcal{A}_{\rho}\left(\lambda^{\omega}I-\rho\mathcal{A}\mathcal{A}_{\rho}\right)^{-1}\lambda^{\omega-1}\big(y_0-h(0,y_0)\big)\end{aligned}$$

$$+(1-\rho)\mathcal{A}_\rho\left[I+\rho\mathcal{A}\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}\right]F(\lambda)$$
$$+\rho\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}F(\lambda)$$

$$=\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}\lambda^\omega H(\lambda)+\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}\lambda^{\omega-1}\left(y_0-h(0,y_0)\right)$$
$$+(1-\rho)\mathcal{A}_\rho F(\lambda)+(1-\rho)\rho\mathcal{A}\mathcal{A}_\rho^2\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}F(\lambda)$$
$$+\rho\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}F(\lambda)$$

$$=\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}\lambda^\omega H(\lambda)+\mathcal{A}_\rho\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}\lambda^{\omega-1}\left(y_0-h(0,y_0)\right)$$
$$+(1-\rho)\mathcal{A}_\rho F(\lambda)$$
$$+\left[(1-\rho)\mathcal{A}+\mathcal{A}_\rho^{-1}\right]\rho\mathcal{A}_\rho^2\left(\lambda^\omega I-\rho\mathcal{A}\mathcal{A}_\rho\right)^{-1}F(\lambda).$$

Since

$$\mathcal{A}_\rho=(I-(1-\rho)\mathcal{A})^{-1}$$

then

$$I=(1-\rho)\mathcal{A}+\mathcal{A}_\rho^{-1}.$$

Therefore

$M(\lambda)$

$$=\begin{cases}\mathcal{A}_\rho(\lambda^\omega I-E)^{-1}\lambda^\omega H(\lambda)+\mathcal{A}_\rho(\lambda^\omega I-E)^{-1}\lambda^{\omega-1}\left(y_0-h(0,y_0)\right)\\ \qquad +(1-\rho)\mathcal{A}_\rho F(\lambda)+\rho\mathcal{A}_\rho^2(\lambda^\omega I-E)^{-1}F(\lambda).\end{cases}\qquad(3.7)$$

Now,

$$\mathcal{A}_\rho(\lambda^\omega I-\mathrm{E})^{-1}\lambda^\omega H(\lambda)=\mathcal{A}_\rho\lambda^\omega\int_0^\infty e^{-\lambda^\omega s}\mathcal{T}(s)H(\lambda)ds$$
$$=\mathcal{A}_\rho\lambda^\omega\int_0^\infty e^{-(\lambda u)^\omega}\mathcal{T}(u^\omega)H(\lambda)\omega u^{\omega-1}du$$
$$=\mathcal{A}_\rho\int_0^\infty\lambda\omega(\lambda u)^{\omega-1}e^{-(\lambda u)^\omega}\mathcal{T}(u^\omega)H(\lambda)du$$
$$=\mathcal{A}_\rho\int_0^\infty-\frac{de^{-(\lambda u)^\omega}}{du}\mathcal{T}(u^\omega)H(\lambda)du,$$

using integration by part, we obtain

$$\mathcal{A}_\rho(\lambda^\omega I-\mathrm{E})^{-1}\lambda^\omega H(\lambda)$$
$$=\mathcal{A}_\rho\left[\left[-\mathcal{T}(u^\omega)H(\lambda)e^{-(\lambda u)^\omega}\right]_0^\infty\right.$$
$$\left.-\int_0^\infty -e^{-(\lambda u)^\omega}\mathrm{E}\mathcal{T}(u^\omega)\omega u^{\omega-1}\,H(\lambda)du\right]$$

$$=\mathcal{A}_\rho H(\lambda)+\mathcal{A}_\rho\int_0^\infty e^{-(\lambda u)^\omega}\mathrm{E}\mathcal{T}(u^\omega)\omega u^{\omega-1}\,H(\lambda)du,$$

using (3.2), we have

$$\mathcal{A}_\rho(\lambda^\omega I-\mathrm{E})^{-1}\lambda^\omega H(\lambda)$$
$$=\mathcal{A}_\rho H(\lambda)+\mathcal{A}_\rho\int_0^\infty\int_0^\infty e^{-\lambda u\delta}\psi_\omega(\delta)\mathrm{E}\mathcal{T}(u^\omega)H(\lambda)\omega u^{\omega-1}d\delta du$$
$$=\mathcal{A}_\rho H(\lambda)+\mathcal{A}_\rho\int_0^\infty\int_0^\infty \omega e^{-\lambda v}\psi_\omega(\delta)\mathrm{E}\mathcal{T}(\frac{v}{\delta})^\omega\,\frac{v^{\omega-1}}{\delta^\omega}\,H(\lambda)d\delta dv$$
$$=\mathcal{A}_\rho H(\lambda)$$
$$+\mathcal{A}_\rho\int_0^\infty\int_0^\infty\int_0^\infty \omega e^{-\lambda(v+\zeta)`}\psi_\omega(\delta)\mathrm{E}\mathcal{T}\left(\frac{v}{\delta}\right)^\omega\frac{v^{\omega-1}}{\delta^\omega}\,h\big(\zeta,y(\zeta)\big)d\zeta d\delta dv$$
$$=\mathcal{A}_\rho H(\lambda)$$
$$+\mathcal{A}_\rho\int_0^\infty\int_0^\infty\int_\zeta^\infty \omega e^{-\lambda t`}\psi_\omega(\delta)\mathrm{E}\mathcal{T}\left(\frac{t-\zeta}{\delta}\right)^\omega\frac{(t-\zeta)^{\omega-1}}{\delta^\omega}\,h\big(\zeta,y(\zeta)\big)dtd\zeta d\delta$$
$$=\mathcal{A}_\rho H(\lambda)$$
$$+\mathcal{A}_\rho\int_0^\infty e^{-\lambda t}\left[\omega\int_0^t\int_0^\infty \psi_\omega(\delta)\mathrm{E}\mathcal{T}\left(\frac{t-\zeta}{\delta}\right)^\omega\frac{(t-\zeta)^{\omega-1}}{\delta^\omega}\,h\big(\zeta,y(\zeta)\big)d\delta d\zeta\right]dt$$

$$= \mathcal{A}_\rho H(\lambda)$$
$$+ \mathcal{A}_\rho \mathcal{L}\left\{\omega \int_0^t \int_0^\infty \psi_\omega(\delta) \mathrm{E}\mathcal{T}\left(\frac{t-\zeta}{\delta}\right)^\omega \frac{(t-\zeta)^{\omega-1}}{\delta^\omega} h\big(\zeta, y(\zeta)\big) d\delta d\zeta\right\}(\lambda)$$

using (3.3), we have

$$\mathcal{A}_\rho (\lambda^\omega I - \mathrm{E})^{-1} \lambda^\omega H(\lambda) =$$
$$\mathcal{A}_\rho H(\lambda) + \mathcal{A}_\rho \mathcal{L}\left\{\omega \int_0^t \int_0^\infty \varphi_\omega(\theta) \mathrm{E}T(\theta(t-\zeta)^\omega)\theta(t-\zeta)^{\omega-1} h\big(\zeta, y(\zeta)\big) d\theta d\zeta\right\}(\lambda).$$

Therefore

$$\mathcal{A}_\rho (\lambda^\omega I - \mathrm{E})^{-1} \lambda^\omega H(\lambda) = \mathcal{A}_\rho H(\lambda)$$
$$+ \mathcal{A}_\rho \mathcal{L}\left\{\int_0^t (t-\zeta)^{\omega-1} \mathrm{E}\, L_\omega(t-\zeta) h\big(\zeta, y(\zeta)\big) d\zeta\right\}(\lambda). \tag{3.8}$$

$$\mathcal{A}_\rho \lambda^{\omega-1} (\lambda^\omega I - \mathrm{E})^{-1} [y_0 - h(0, y_0)]$$
$$= \mathcal{A}_\rho \lambda^{\omega-1} \int_0^\infty e^{-\lambda^\omega s} \mathcal{T}(s) \big(y_0 - h(0, y_0)\big) ds$$
$$= \mathcal{A}_\rho \lambda^{\omega-1} \int_0^\infty e^{-(\lambda u)^\omega} \mathcal{T}(u^\omega)[y_0 - h(0, y_0)] \omega u^{\omega-1} du$$
$$= \mathcal{A}_\rho \int_0^\infty \omega (\lambda u)^{\omega-1} e^{-(\lambda u)^\omega} \mathcal{T}(u^\omega)[y_0 - h(0, y_0)] du$$
$$= \mathcal{A}_\rho \int_0^\infty -\frac{1}{\lambda} \frac{d e^{-(\lambda u)^\omega}}{du} \mathcal{T}(u^\omega)[y_0 - h(0, y_0)] du,$$

from (3.2), we have

$$e^{-(\lambda u)^\omega} = \int_0^\infty e^{-\lambda u \delta} \psi_\omega(\delta) d\delta$$

then

$$\frac{de^{-(\lambda u)^{\omega}}}{du} = \int_0^{\infty} -\lambda\delta e^{-\lambda u\delta}\psi_{\omega}(\delta)d\delta.$$

Therefore

$$\begin{aligned}\mathcal{A}_{\rho}\lambda^{\omega-1}(\lambda^{\omega}I - \mathrm{E})^{-1}[y_0 - h(0,y_0)] \\ = \mathcal{A}_{\rho}\int_0^{\infty}\int_0^{\infty}\delta e^{-\lambda u\delta}\,\psi_{\omega}(\delta)\mathcal{T}(u^{\omega})[y_0 - h(0,y_0)]d\delta du \\ = \mathcal{A}_{\rho}\int_0^{\infty}\int_0^{\infty}e^{-\lambda t}\,\psi_{\omega}(\delta)\mathcal{T}\left(\left(\frac{t}{\delta}\right)^{\omega}\right)[y_0 - h(0,y_0)]d\delta dt \\ = \mathcal{A}_{\rho}\int_0^{\infty}\int_0^{\infty}-\frac{1}{\omega}\theta^{-\frac{1}{\omega}-1}e^{-\lambda t}\,\psi_{\omega}\left(\theta^{-\frac{1}{\omega}}\right)\mathcal{T}(\theta t^{\omega})[y_0 - h(0,y_0)]d\theta dt\end{aligned}$$

using (3.3), we have

$$\begin{aligned}\mathcal{A}_{\rho}\lambda^{\omega-1}(\lambda^{\omega}I - \mathrm{E})^{-1}[y_0 - h(0,y_0)] \\ = \mathcal{A}_{\rho}\int_0^{\infty}\int_0^{\infty}e^{-\lambda t}\,\varphi_{\omega}(\theta)\mathcal{T}(\theta t^{\omega})[y_0 - h(0,y_0)]d\theta dt \\ = \mathcal{A}_{\rho}\mathcal{L}\left\{\int_0^{\infty}\varphi_{\omega}(\theta)\mathcal{T}(\theta t^{\omega})\big(y_0 - h(0,y_0)\big)d\theta\right\}(\lambda).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{A}_{\rho}\lambda^{\omega-1}(\lambda^{\omega}I - \mathrm{E})^{-1}[y_0 - h(0,y_0)] \\ = \mathcal{A}_{\rho}\mathcal{L}\{\mathcal{T}_{\omega}(t)[y_0 - h(0,y_0)]\}(\lambda).\end{aligned} \tag{3.9}$$

$$\begin{aligned}\rho\mathcal{A}_{\rho}^2(\lambda^{\omega}I - \mathrm{E})^{-1}\,F(\lambda) = \rho\mathcal{A}_{\rho}^2\int_0^{\infty}e^{-\lambda^{\omega}s}\mathcal{T}(s)F(\lambda)ds \\ = \rho\mathcal{A}_{\rho}^2\int_0^{\infty}e^{-(\lambda u)^{\omega}}\mathcal{T}(u^{\omega})F(\lambda)\omega u^{\omega-1}du\end{aligned}$$

using (3.2), we have

$$\begin{aligned}\rho\mathcal{A}_{\rho}^2(\lambda^{\omega}I - \mathrm{E})^{-1}\,F(\lambda) = \rho\mathcal{A}_{\rho}^2\int_0^{\infty}\int_0^{\infty}\omega e^{-\lambda u\delta}\psi_{\omega}(\delta)\mathcal{T}(u^{\omega})F(\lambda)u^{\omega-1}d\delta du \\ = \rho\mathcal{A}_{\rho}^2\int_0^{\infty}\int_0^{\infty}\omega e^{-\lambda v}\psi_{\omega}(\delta)\mathcal{T}\left(\left(\frac{v}{\delta}\right)^{\omega}\right)\frac{v^{\omega-1}}{\delta^{\omega}}F(\lambda)d\delta dv\end{aligned}$$

$$= \rho\mathcal{A}_\rho^2 \int_0^\infty \int_0^\infty \int_0^\infty \omega e^{-\lambda(v+\zeta)} \psi_\omega(\delta) \mathcal{T}\left(\left(\frac{v}{\delta}\right)^\omega\right) \frac{v^{\omega-1}}{\delta^\omega} [Bu(\zeta) + f(\zeta, y(\zeta)]\, d\zeta d\delta dv$$

$$= \rho\mathcal{A}_\rho^2 \int_0^\infty \int_0^\infty \int_\zeta^\infty \omega e^{-\lambda t} \psi_\omega(\delta) \mathcal{T}\left(\left(\frac{(t-\zeta)}{\delta}\right)^\omega\right) \frac{(t-\zeta)^{\omega-1}}{\delta^\omega} [Bu(\zeta) + f(\zeta, y(\zeta)]\, dt d\zeta d\,\delta$$

$$= \rho\mathcal{A}_\rho^2 \int_0^\infty e^{-\lambda t} \left[\omega \int_0^t \int_0^\infty \psi_\omega(\delta) \mathcal{T}\left(\left(\frac{(t-\zeta)}{\delta}\right)^\omega\right) \frac{(t-\zeta)^{\omega-1}}{\delta^\omega} [Bu(\zeta) + f(\zeta, y(\zeta)] d\delta d\zeta\right] dt$$

$$= \rho\mathcal{A}_\rho^2 \mathcal{L}\left\{\omega \int_0^t \int_0^\infty \psi_\omega(\delta) \mathcal{T}\left(\left(\frac{(t-\zeta)}{\delta}\right)^\omega\right) \frac{(t-\zeta)^{\omega-1}}{\delta^\omega} [Bu(\zeta) + f(\zeta, y(\zeta)] d\delta d\zeta\right\}(\lambda)$$

using (3.3), we have

$$\rho\mathcal{A}_\rho^2 (\lambda^\omega I - \mathrm{E})^{-1} F(\lambda) = \rho\mathcal{A}_\rho^2 \mathcal{L}\left\{\omega \int_0^t \int_0^\infty \theta\varphi_\omega(\theta) \mathcal{T}(\theta(t-\zeta)^\omega)(t-\zeta)^{\omega-1}[Bu(\zeta) + f(\zeta, y(\zeta)] d\theta d\zeta\right\}(\lambda),$$

$$\rho\mathcal{A}_\rho^2 (\lambda^\omega I - \mathrm{E})^{-1} F(\lambda) = \rho\mathcal{A}_\rho^2 \mathcal{L}\left\{\int_0^t L_\omega(t-\zeta)(t-\zeta)^{\omega-1} [Bu(\zeta) + f(\zeta, y(\zeta)] d\zeta\right\}(\lambda). \quad (3.10)$$

From (3.8), (3.9) and (3.10) we get

$$M(\lambda) = \begin{cases} \mathcal{A}_\rho H(\lambda) + \mathcal{A}_\rho \mathcal{L}\left\{\int_0^t (t-\zeta)^{\omega-1} \mathrm{E}\, L_\omega(t-\zeta) h\big(\zeta, y(\zeta)\big) d\zeta\right\}(\lambda) \\ +\mathcal{A}_\rho \mathcal{L}\big\{\mathcal{T}_\omega(t)\big(y_0 - h(0, y_0)\big)\big\}(\lambda) + (1-\rho)\mathcal{A}_\rho F(\lambda) \\ +\rho\mathcal{A}_\rho^2 \mathcal{L}\left\{\int_0^t L_\omega(t-\zeta)(t-\zeta)^{\omega-1}\,[Bu(\zeta) + f(\zeta, y(\zeta)])d\zeta\right\}(\lambda). \end{cases} \tag{3.11}$$

Taking the inverse of Laplace transform for both sides of (3.11), we get

$$m(t) = \begin{cases} \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} \mathrm{E}\, L_\omega(t-\zeta) h\big(\zeta, y(\zeta)\big) d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)[y_0 - h(0, y_0)] + (1-\rho)\mathcal{A}_\rho[Bu(t) + f(t, y(t)] \\ +\rho\mathcal{A}_\rho^2 \int_0^t L_\omega(t-\zeta)(t-\zeta)^{\omega-1}\,\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big]d\zeta. \end{cases} \tag{3.12}$$

Now, to find the mild solution $n(t)$, we applying fractional integral (1.5) for both sides of System (3.5),

$$\mathcal{J}^{\rho,\omega} D^{\rho,\omega} n(t) = \mathcal{J}^{\rho,\omega}[\mathcal{A}n(t)].$$

By using Lemma (1.2.31) we have,

$$n(t) = \sum_{\gamma=1}^{p} \Delta y\big(t_\gamma\big)\sigma_\gamma(t) + \mathcal{J}^{\rho,\omega}[\mathcal{A}n(t)] \tag{3.13}$$

where

$$\sigma_\gamma(t) = \begin{cases} 0 & t \in [0, t_1] \\ 1 & t \in \big(t_\gamma, t_{\gamma+1}\big] \qquad \gamma = 1,2,\dots,p \end{cases} \tag{3.14}$$

and

$$\mathcal{L}\big\{\sigma_\gamma(t)\big\}(\lambda) = \frac{e^{-\lambda t_\gamma}}{\lambda}.$$

Taking Laplace transform for both sides of (3.13),

$$N(\lambda) = \sum_{\gamma=1}^{p} \Delta y(t_\gamma)\lambda^{-1}e^{-\lambda t_\gamma} + (1-\rho)\mathcal{A}N(\lambda) + \frac{\rho}{\lambda^\omega}\mathcal{A}N(\lambda)$$

where $N(\lambda)$ denotes the Laplace transform of $n(t)$.

It follows,

$$N(\lambda) - (1-\rho)\mathcal{A}N(\lambda) - \frac{\rho}{\lambda^\omega}\mathcal{A}N(\lambda) = \sum_{\gamma=1}^{p} \Delta y(t_\gamma)\lambda^{-1}e^{-\lambda t_\gamma}$$

$$\left[I - (1-\rho)\mathcal{A} - \frac{\rho}{\lambda^\omega}\mathcal{A}\right]N(\lambda) = \sum_{\gamma=1}^{p} \Delta y(t_\gamma)\lambda^{-1}e^{-\lambda t_\gamma}$$

$$\left[\mathcal{A}_\rho^{-1} - \frac{\rho}{\lambda^\omega}\mathcal{A}\right]N(\lambda) = \sum_{\gamma=1}^{p} \Delta y(t_\gamma)\lambda^{-1}e^{-\lambda t_\gamma}$$

$$\left[\lambda^\omega\mathcal{A}_\rho^{-1} - \rho\mathcal{A}\right]N(\lambda) = \sum_{\gamma=1}^{p} \Delta y(t_\gamma)\lambda^{\omega-1}e^{-\lambda t_\gamma}$$

$$\left[(\lambda^\omega I - \rho\mathcal{A}\mathcal{A}_\rho)\mathcal{A}_\rho^{-1}\right]N(\lambda) = \sum_{\gamma=1}^{p} \Delta y(t_\gamma)\lambda^{\omega-1}e^{-\lambda t_\gamma}$$

$$N(\lambda) = \mathcal{A}_\rho(\lambda^\omega I - \rho\mathcal{A}\mathcal{A}_\rho)^{-1}\lambda^{\omega-1}\sum_{\gamma=1}^{p} \Delta y(t_\gamma)e^{-\lambda t_\gamma}$$

$$= \mathcal{A}(\lambda^\omega I - E)^{-1}\lambda^{\omega-1}\sum_{\gamma=1}^{p} \Delta y(t_\gamma)e^{-\lambda t_\gamma}$$

$$= \mathcal{A}_\rho\lambda^{\omega-1}\sum_{\gamma=1}^{p}\int_0^\infty \Delta y(t_\gamma)e^{-\lambda^\omega s}\mathcal{T}(s)\,e^{-\lambda t_\gamma}ds.$$

$$= \mathcal{A}_\rho\lambda^{\omega-1}\sum_{\gamma=1}^{p}\int_0^\infty \Delta y(t_\gamma)e^{-(\lambda u)^\omega}\mathcal{T}(u^\omega)\,e^{-\lambda t_\gamma}\omega u^{\omega-1}du$$

$$= \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty \Delta y(t_\gamma)\omega(\lambda u)^{\omega-1} e^{-(\lambda u)^\omega} \mathcal{T}(u^\omega)\, e^{-\lambda t_\gamma} du$$

$$= \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty -\Delta y(t_\gamma) \frac{1}{\lambda} \frac{de^{-(\lambda u)^\omega}}{du} \mathcal{T}(u^\omega)\, e^{-\lambda t_\gamma} du$$

using (3.2), we have

$$N(\lambda) = \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y(t_\gamma)\delta e^{-\lambda u \delta} \psi_\omega(\delta) \mathcal{T}(u^\omega)\, e^{-\lambda t_\gamma} d\delta du.$$

$$= \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y(t_\gamma) e^{-\lambda v} \psi_\omega(\delta) \mathcal{T}\left(\frac{v^{\ \omega}}{\delta^\omega}\right) e^{-\lambda t_\gamma} d\delta dv$$

$$= \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y(t_\gamma) e^{-\lambda(v+t_\gamma)} \psi_\omega(\delta) \mathcal{T}\left(\frac{v^{\ \omega}}{\delta^\omega}\right) d\delta dv.$$

$$= \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y(t_\gamma) e^{-\lambda t} \psi_\omega(\delta) \mathcal{T}\left(\frac{\left(t-t_\gamma\right)^\omega}{\delta^\omega}\right) d\delta dt.$$

using (3.3), we have

$$N(\lambda) = \mathcal{A}_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y(t_\gamma) e^{-\lambda t} \varphi_\omega(\theta) \mathcal{T}\left(\theta\left(t-t_\gamma\right)^\omega\right) d\theta dt$$

$$= \mathcal{A}_\rho \mathcal{L}\left\{\sum_{\gamma=1}^{p} \int_0^\infty \Delta y(t_\gamma) \varphi_\omega(\theta) \mathcal{T}\left(\theta\left(t-t_\gamma\right)^\omega\right) d\theta\right\}(\lambda)$$

$$N(\lambda) = \mathcal{A}_\rho \mathcal{L}\left\{\sum_{\gamma=1}^{p} \Delta y(t_\gamma) \mathcal{T}_\omega\left(t-t_\gamma\right)^\omega\right\}(\lambda). \tag{3.15}$$

By taking the inverse of Laplace transform for both sides of (3.15), we get

$$n(t) = \mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega\left(t-t_\gamma\right)^\omega \Delta y(t_\gamma). \tag{3.16}$$

From (3.12) and (3.16), we have

$$y(t)=\begin{cases}\mathcal{A}_\rho h\big(t,y(t)\big)+\mathcal{A}_\rho\displaystyle\int_0^t(t-\zeta)^{\omega-1}EL_\omega(t-\zeta)h\big(\zeta,y(\zeta)\big)d\zeta\\ +\mathcal{A}_\rho\mathcal{T}_\omega(t)[y_0-h(0,y_0)]+(1-\rho)\mathcal{A}_\rho\big[Bu(t)+f\big(t,y(t)\big)\big]\\ +\rho\mathcal{A}_\rho^2\displaystyle\int_0^t(t-\zeta)^{\omega-1}L_\omega(t-\zeta)\big[Bu(\zeta)+f\big(\zeta,y(\zeta)\big)\big]d\zeta \quad t\in[0,t_1]\\ \mathcal{A}_\rho h\big(t,y(t)\big)+\mathcal{A}_\rho\displaystyle\int_0^t(t-\zeta)^{\omega-1}EL_\omega(t-\zeta)h\big(\zeta,y(\zeta)\big)d\zeta\\ +\mathcal{A}_\rho\mathcal{T}_\omega(t)\big(y_0-h(0,y_0)\big)+(1-\rho)\mathcal{A}_\rho\big[Bu(t)+f\big(t,y(t)\big)\big]\\ +\rho\mathcal{A}_\rho^2\displaystyle\int_0^t(t-\zeta)^{\omega-1}L_\omega(t-\zeta)\big[Bu(\zeta)+f\big(\zeta,y(\zeta)\big)\big]d\zeta\\ +\mathcal{A}_\rho\displaystyle\sum_{\gamma=1}^{r}\mathcal{T}_\omega\big(t-t_\gamma\big)\Delta y\big(t_\gamma\big) \qquad t\in\big(t_\gamma,t_{\gamma+1}\big].\end{cases}$$

■

Lemma 3.1.2 [46]

The operators L_ω and $\mathcal{T}_\omega$ have the following properties.

i. For any fixed $t\geq 0$, $L_\omega(t)$ and $\mathcal{T}_\omega(t)$ are linear and bounded operators, i.e. for any $y\in$ X,

$$\|L_\omega(t)\mathrm{y}\|=\left\|\omega\int_0^\infty\theta\varphi_\omega(\theta)\mathcal{T}(\theta t^\omega)yd\theta\right\|\leq\frac{\omega\mathcal{S}}{\Gamma(1+\omega)}\|y\|$$
$$=\frac{\mathcal{S}}{\Gamma(\omega)}\|y\|$$
$$\|\mathcal{T}_\omega(t)\mathrm{y}\|=\left\|\int_0^\infty\varphi_\omega(\theta)\mathcal{T}(\theta t^\omega)yd\,\theta\right\|\leq\mathcal{S}\|y\|$$

where $\mathcal{S}=sup_{t\geq0}\|\mathcal{T}(t)\|$.

ii. The operators $\{L_\omega(t)\}_{t\geq0}$ and $\{\mathcal{T}_\omega(t)\}_{t\geq0}$ are strongly continuous.

iii. If $\mathcal{T}(t)$ is compact, then the operators $\{L_\omega(t)\}_{t\geq0}$ and $\{\mathcal{T}_\omega(t)\}_{t\geq0}$ are compact.

From Lemma (1.1.29), for $y \in D(\mathcal{A})$ we have

$$\frac{d}{dt}\mathcal{T}(t)y = E\mathcal{T}(t)y = \mathcal{T}(t)Ey.$$

Definition 3.1.3

The **mild solution** $y(t)$ of System (3.1) is

$$y(t) = \begin{cases} \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)\big(y_0 - h(0, y_0)\big) + (1-\rho)\mathcal{A}_\rho\big[Bu(t) + f\big(t, y(t)\big)\big] \\ +\rho\mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \quad t \in [0, t_1] \\ \\ \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)\big(y_0 - h(0, y_0)\big) + (1-\rho)\mathcal{A}_\rho\big[Bu(t) + f\big(t, y(t)\big)\big] \\ +\rho\mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \\ +\mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega\big(t - t_\gamma\big)\Delta y\big(t_\gamma\big), \qquad t \in \big(t_\gamma, t_{\gamma+1}\big]. \end{cases}$$

Next, we discuss the controllability of Nonlinear System (3.1).

Assume that the C_0-semigroup $T(t)$ is continuous in the uniform operator topology and also assume the following conditions:

H1The linear operator $W_\rho: L_p(\mathcal{J}, U) \to X$ defined as

$$W_\rho u = (1-\rho)\mathcal{A}_\rho B u + \rho\mathcal{A}_\rho^2 W u$$

has an inverse operator ${W_\rho}^{-1}$ on $L_p(\mathcal{J}, U)/kerW_\rho$ and $\|{W_\rho}^{-1}\| \leq K,\ K > 0$, $\|\mathcal{A}_\rho\| \leq \eta, \eta > 0$.

where W is linear operator from $L_p(\mathcal{J}, U)$ into X such that

$$Wu = \int_0^{\Upsilon} (\Upsilon - s)^{\omega-1} L_\omega(\Upsilon - s) Bu(s) ds.$$

H2: There exist constants $\mathcal{M}_h, \widehat{\mathcal{M}_h} > 0$ such that

$$\|Eh(t, y_1(t)) - Eh(t, y_2(t))\| \leq \mathcal{M}_h \|y_1 - y_2\|$$

and

$$\widehat{\mathcal{M}_h} = sup_{t\in\mathcal{J}} \|Eh(t, 0)\|.$$

H3: The continuous function $f: \mathcal{J} \times X \to X$ satisfies Lipchitz condition i.e. there exist a constant $\mathcal{M}_f > 0$ such that

$$\|f(t, y_1(t)) - f(t, y_2(t))\| \leq \mathcal{M}_f \|y_1 - y_2\|$$

and

$$\widehat{\mathcal{M}_f} = sup_{t\in\mathcal{J}} \|f(t, 0)\|.$$

where $\widehat{\mathcal{M}_f} > 0$.

H4: The function $q_\gamma: X \to X, \gamma = 1,2, \dots, p$ is continuous and satisfies Lipchitz condition, i.e. there exists a constant $\mathcal{M}_\gamma > 0$ such that

$$\|q_\gamma(y_1) - q_\gamma(y_2)\| \leq \mathcal{M}_\gamma \|y_1 - y_2\|$$

and

$$\sum_{\gamma=1}^{p} \mathcal{M}_\gamma = \mathcal{M}$$

where $\mathcal{M} > 0$.

H5: For all bounded subsets Θ, the set

$$(\mathcal{K}_r^v)(t) = \left\{ \omega\rho\mathcal{A}_\rho^2 \int_0^{t-v} \int_r^{\infty} \theta(t-\zeta)^{\omega-1} \varphi_\omega(\theta) \mathcal{T}(\theta(t-\zeta)^\omega) f(\zeta, y(\zeta)) d\theta d\zeta \,, y \in \Theta \right\}$$

is relatively compact in X for arbitrary $v \in (0, t)$ and a real number $r > 0$.

Definition 3.1.4

System (3.1) is **exact controllable** on $\mathcal{J} = [0, \Upsilon]$ if for each initial state $y_0 \in X$ and each $y_1 \in X$, there is suitable control $u \in L_p(\mathcal{J}, U)$ such that the mild solution $y(t)$ of System (3.1) satisfies $y(\Upsilon) = y_1$.

Theorem 3.1.5

System (3.1) is exact controllable on $\mathcal{J} = [0, \Upsilon]$ if it satisfies the conditions **H1-H5**, and

$$0 < D + \eta K\|B\|D\left[(1-\rho) + \frac{\rho\eta\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right] < 1 \qquad (3.17)$$

where

$$D = \eta\left[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \eta\rho\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{S}\mathcal{M}\right].$$

Proof. Using condition **H1**, define the control $u_y(t)$ for an arbitrary function $y(\cdot) \in PC(\mathcal{J}, X)$ as

$$u_y(t) = \begin{cases} W_\rho^{-1}\Big[y_1 - \mathcal{A}_\rho h(\Upsilon, y(\Upsilon)) - \mathcal{A}_\rho \int_0^{\Upsilon}(\Upsilon-\zeta)^{\omega-1}L_\omega(\Upsilon-\zeta)Eh(\zeta, y(\zeta))d\zeta \\ \quad -\mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)[y_0 - h(0, y_0)] - (1-\rho)\mathcal{A}_\rho f(\Upsilon, y(\Upsilon)) \\ \quad -\rho\mathcal{A}_\rho^2\int_0^{\Upsilon}(\Upsilon-\zeta)^{\omega-1}L_\omega(\Upsilon-\zeta)f(\zeta, y(\zeta))d\zeta\Big], \quad t \in [0, t_1] \\ W_\rho^{-1}\Big[y_1 - \mathcal{A}_\rho h(\Upsilon, y(\Upsilon)) - \mathcal{A}_\rho \int_0^{\Upsilon}(\Upsilon-\zeta)^{\omega-1}L_\omega(\Upsilon-\zeta)Eh(\zeta, y(\zeta))d\zeta \\ \quad -\mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) - (1-\rho)\mathcal{A}_\rho f(\Upsilon, y(\Upsilon)) \\ \quad -\rho\mathcal{A}_\rho^2\int_0^{\Upsilon}(\Upsilon-\zeta)^{\omega-1}L_\omega(\Upsilon-\zeta)f(\zeta, y(\zeta))d\zeta \\ \quad -\mathcal{A}_\rho\sum_{\gamma=1}^{p}\mathcal{T}_\omega(\Upsilon-t_\gamma)\Delta y(t_\gamma)\Big], \quad t \in (t_\gamma, t_{\gamma+1}] \end{cases}$$

By using (3.14), we can write $u_y(t)$ in the form

$$u_y(t) = \begin{cases} W_\rho^{-1}\Big[y_1 - \mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) - \mathcal{A}_\rho \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \\ \quad -\mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - (1-\rho)\mathcal{A}_\rho f\big(\Upsilon, y(\Upsilon)\big) \\ \quad -\rho \mathcal{A}_\rho^2 \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \\ \quad -\mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega(\Upsilon - t_\gamma)\Delta y(t_\gamma)\sigma_\gamma(t)\Big], \quad t \in [0,\Upsilon] \end{cases}$$

We have to show that the operator $\Phi: PC(\mathcal{J},X) \to PC(\mathcal{J},X)$ has a fixed point when applying this control, where

$$(\Phi y)(t) = \begin{cases} \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \\ \quad +\mathcal{A}_\rho \mathcal{T}_\omega(t)[y_0 - h(0,y_0)] + (1-\rho)\mathcal{A}_\rho\big[B u_y(t) + f\big(t, y(t)\big)\big] \\ \quad +\rho \mathcal{A}_\rho{}^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\big[B u_y(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \quad t \in [0,t_1] \\ \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \\ \quad +\mathcal{A}_\rho \mathcal{T}_\omega(t)[y_0 - h(0,y_0)] + (1-\rho)\mathcal{A}_\rho\big[B u_y(t) + f\big(t, y(t)\big)\big] \\ \quad +\rho \mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\big[B u_y(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \\ \quad +\mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega\big(t - t_\gamma\big)\Delta y\big(t_\gamma\big) \qquad t \in \big(t_\gamma, t_{\gamma+1}\big]. \end{cases}$$

By using (3.14), we can write $(\Phi y)(t)$ in the form

$$(\Phi y)(t) = \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta$$

$$+\mathcal{A}_\rho \mathcal{T}_\omega(t)[y_0 - h(0,y_0)] + (1-\rho)\mathcal{A}_\rho\big[B u_y(t) + f\big(t, y(t)\big)\big]$$

$$+\rho\mathcal{A}_\rho^2\int_0^t (t-\zeta)^{\omega-1}L_\omega(t-\zeta)\big[B\mathcal{u}_y(\zeta)+f\big(\zeta,y(\zeta)\big)\big]d\zeta$$

$$+\mathcal{A}_\rho\sum_{\gamma=1}^{p}\mathcal{T}_\omega\big(t-t_\gamma\big)\Delta y\big(t_\gamma\big)\sigma_\gamma(t), \qquad t\in\mathcal{J}.$$

By using the control $\mathcal{u}_y(t)$, we have $(\Phi y)(\Upsilon)=y_1$, indeed,

$$(\Phi y)(\Upsilon)=\mathcal{A}_\rho h\big(\Upsilon,y(\Upsilon)\big)+\mathcal{A}_\rho\int_0^\Upsilon(\Upsilon-\zeta)^{\omega-1}\,L_\omega(\Upsilon-\zeta)Eh\big(\zeta,y(\zeta)\big)d\zeta$$

$$+\,\mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)[y_0-h(0,y_0)]+(1-\rho)\mathcal{A}_\rho BW_\rho^{-1}\Lambda(\Upsilon)$$

$$+\,(1-\rho)\mathcal{A}_\rho f\big(\Upsilon,y(\Upsilon)\big)$$

$$+\,\rho\mathcal{A}_\rho^2\int_0^\Upsilon(\Upsilon-\zeta)^{\omega-1}L_\omega(\Upsilon-\zeta)BW_\rho^{-1}\Lambda(\Upsilon)d\zeta$$

$$+\,\rho\mathcal{A}_\rho^2\int_0^\Upsilon(\Upsilon-\zeta)^{\omega-1}L_\omega(\Upsilon-\zeta)f\big(\zeta,y(\zeta)\big)d\zeta$$

$$+\,\mathcal{A}_\rho\sum_{\gamma=1}^{p}\mathcal{T}_\omega\big(\Upsilon-t_\gamma\big)\Delta y\big(t_\gamma\big)\sigma_\gamma(t),$$

where

$$\Lambda(t)=y_1-\mathcal{A}_\rho h\big(\Upsilon,y(\Upsilon)\big)-\mathcal{A}_\rho\int_0^\Upsilon(\Upsilon-\zeta)^{\omega-1}L_\omega(\Upsilon-\zeta)Eh\big(\zeta,y(\zeta)\big)d\zeta$$

$$-\mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)[y_0-h(0,y_0)]-(1-\rho)\mathcal{A}_\rho f\big(\Upsilon,\mathrm{y}(\Upsilon)\big)$$

$$-\rho\mathcal{A}_\rho^2\int_0^\Upsilon(\Upsilon-\zeta)^{\omega-1}L_\omega(\Upsilon-\zeta)f\big(\zeta,y(\zeta)\big)d\zeta$$

$$-\,\mathcal{A}_\rho\sum_{\gamma=1}^{p}\mathcal{T}_\omega\big(\Upsilon-t_\gamma\big)\Delta y\big(t_\gamma\big)\sigma_\gamma(t).$$

It follows,

$$(\Phi y)(\Upsilon) = \mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) + \mathcal{A}_\rho \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta$$
$$+ \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)[y_0 - h(0,y_0)] + \big[(1-\rho)\mathcal{A}_\rho B + \rho \mathcal{A}_\rho^2 W\big] W_\rho^{-1} \Lambda(\Upsilon)$$
$$+ (1-\rho)\mathcal{A}_\rho f\big(\Upsilon, y(\Upsilon)\big)$$
$$+ \rho \mathcal{A}_\rho^2 \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta$$
$$+ \mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega(\Upsilon - t_\gamma)\Delta y(t_\gamma)\sigma_\gamma(t)$$

$$= \mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) + \mathcal{A}_\rho \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta$$
$$+ \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)[y_0 - h(0,y_0)]$$
$$+W_\rho W_\rho^{-1} \Bigg[y_1 - \mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) - \mathcal{A}_\rho \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta$$
$$-\mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)[y_0 - h(0,y_0)] - (1-\rho)\mathcal{A}_\rho f\big(\Upsilon, \mathrm{y}(\Upsilon)\big)$$
$$- \rho \mathcal{A}_\rho^2 \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta - \mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega(\Upsilon$$
$$- t_\gamma)\Delta y(t_\gamma)\sigma_\gamma(t) \Bigg] + (1-\rho)\mathcal{A}_\rho f\big(\Upsilon, y(\Upsilon)\big)$$
$$+\rho \mathcal{A}_\rho^2 \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega(\Upsilon-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta$$
$$+ \mathcal{A}_\rho \sum_{\gamma=1}^{p} T_\omega(\Upsilon - t_\gamma)\Delta y(t_\gamma)\sigma_\gamma(t).$$

$$= y_1.$$

Now, for $t \in \mathcal{J}$,

$$\|u_y(t)\| \leq K\Bigg[\|y_1\| + \|\mathcal{A}_\rho\|\|h(\Upsilon, y(\Upsilon))\|$$

$$+ \|\mathcal{A}_\rho\|\frac{\mathcal{S}}{\Gamma(\omega)}\int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1}\|Eh(\zeta, y(\zeta))\|d\zeta$$

$$+ \|\mathcal{A}_\rho\|\mathcal{S}\|y_0 - h(0, y_0)\| + (1-\rho)\|\mathcal{A}_\rho\|\|f(\Upsilon, \mathrm{y}(\Upsilon))\|$$

$$+ \rho\|\mathcal{A}_\rho^2\|\frac{\mathcal{S}}{\Gamma(\omega)}\int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1}\|f(\zeta, \mathrm{y}(\zeta))\|d\zeta$$

$$+ \|\mathcal{A}_\rho\|\left\|\sum_{\gamma=1}^{p}\mathcal{T}_\omega(\Upsilon - t_\gamma)\Delta y(t_\gamma)\sigma_\gamma(t)\right\|\Bigg]$$

$$\leq K\Bigg[\|y_1\| + \eta\Bigg[\|E^{-1}\|\mathcal{M}_h\|y\| + \|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)}\left(\mathcal{M}_h\|y\| + \widehat{\mathcal{M}_h}\right)$$

$$+ \mathcal{S}\left(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\right)$$

$$+ (1-\rho)\left(\mathcal{M}_f\|y\| + \widehat{\mathcal{M}_f}\right) + \eta\rho\frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)}\left(\mathcal{M}_f\|y\| + \widehat{\mathcal{M}_f}\right) + \mathcal{S}\mathcal{M}\|y\|$$

$$+ \mathcal{S}\sum_{\gamma=1}^{p}\|q_\gamma(0)\|\Bigg]\Bigg]$$

$$= K\Bigg[\|y_1\| + \eta\|y\|\Bigg[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f$$

$$+\eta\rho\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{S}\mathcal{M}\Bigg] + \eta\Bigg[\|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\widehat{\mathcal{M}_h}$$

$$+ \mathcal{S}\left(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\right) + (1-\rho)\widehat{\mathcal{M}_f}$$

$$+ \eta\rho\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\widehat{\mathcal{M}_f} + \mathcal{S}\sum_{\gamma=1}^{p}\|q_\gamma(0)\|\Bigg]\Bigg]$$

$$= K\left[D\|y\| + \|y_1\| + \widehat{D}\right],$$

where D is given in assumption and

$$\widehat{D} = \eta\left[\|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_h} + \mathcal{S}\big(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\big) + (1-\rho)\widehat{\mathcal{M}_f}\right.$$

$$\left. + \eta\rho\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_f} + \mathcal{S}\sum_{\gamma=1}^{p}\|q_\gamma(0)\|\right].$$

Also, for $y, \hat{y} \in PC(\mathcal{J}, X)$, we have

$$\|u_y - u_{\hat{y}}\| \le \|W_\rho^{-1}\|\|\mathcal{A}_\rho\|\left[\int_0^{\Upsilon}(\Upsilon-\zeta)^{\omega-1}\|L_\omega(\Upsilon-\zeta)\|\|Eh(\zeta, y(\zeta))\right.$$

$$- Eh(\zeta, \hat{y}(\zeta))\|d\zeta$$

$$+ \rho\|\mathcal{A}_\rho\|\int_0^{\Upsilon}(\Upsilon-\zeta)^{\omega-1}\|L_\omega(\Upsilon-\zeta)\|\|f(\zeta, y(\zeta))$$

$$\left.- f(\zeta, \hat{y}(\zeta))\|d\zeta + \sum_{\gamma=1}^{p}\|\mathcal{T}_\omega(\Upsilon-\zeta)\|\,\|q_\gamma y(t_\gamma) - q_\gamma \hat{y}(t_\gamma)\|\right]$$

$$\le K\eta\left[\frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\mathcal{M}_h\|y-\hat{y}\| + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\mathcal{M}_f\|y-\hat{y}\| + \mathcal{S}\mathcal{M}\|y-\hat{y}\|\right]$$

$$= K\eta\mathcal{S}\left[\frac{\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + \rho\eta\frac{\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{M}\right]\|y-\hat{y}\|.$$

Define the set

$$C_\epsilon = \{y : y \in PC(\mathcal{J}, X), \|y\| \le \epsilon \text{ for each } t \in \mathcal{J}\},$$

then C_ϵ is closed, convex and bounded subset of $PC(\mathcal{J}, X)$ for each ϵ.

Define the operators

$$(\Phi_1)y(t) = \mathcal{A}_\rho h\big(t, y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta$$
$$+ \mathcal{A}_\rho \mathcal{T}_\omega(t)\big(y_0 - h(0, y_0)\big) + (1-\rho)\mathcal{A}_\rho\big[B u_y(t) + f\big(t, y(t)\big)\big]$$
$$+ \rho \mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) B u_y(\zeta) d\zeta + \mathcal{A}_\rho \sum_{\gamma=1}^{p} \Delta y\big(t_\gamma\big)\mathcal{T}_\omega\big(t$$
$$- t_\gamma\big), \quad t \in \mathcal{J}.$$
$$(\Phi_2)y(t) = \rho \mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta, \quad t \in \mathcal{J}.$$

It is clear that

$$(\Phi_1 + \Phi_2)y = \Phi y.$$

To show operator Φ has a fixed point on C_ϵ, we need to choose $\epsilon_0 > 0$, such that $(\Phi_1 + \Phi_2)y$ has a fixed point on C_{ϵ_0}.

Taking

$$\epsilon_0 = \frac{\eta K\|B\|\left[(1-\rho) + \frac{\rho\eta\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right]\left(\|y_1\| + \widehat{D}\right) + \widehat{D}}{1 - \left[D + \eta K\|B\| D\left[(1-\rho) + \frac{\rho\eta\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right]\right]}.$$

We will show the operator $(\Phi_1 + \Phi_2)y$ has a fixed point on C_{ϵ_0}. Our proof consists of three steps:

Claim I: We will show $\Phi C_{\epsilon_0} \subset C_{\epsilon_0}$.

Let $y \in C_{\epsilon_0}$, then

$$\|(\Phi y)(t)\| \leq \|\mathcal{A}_\rho\| \Bigg[\|h(t,y(t))\| + \frac{\mathcal{S}}{\Gamma(\omega)} \int_0^t (t-\zeta)^{\omega-1} \|Eh(\zeta,y(\zeta)\| d\zeta$$

$$+ \mathcal{S}\|(y_0 - h(0,y_0))\| + (1-\rho)[\|B\mathcal{u}_y(t)\| + \|f(t,y(t))\|]$$

$$+ \rho\|\mathcal{A}_\rho\| \frac{\mathcal{S}}{\Gamma(\omega)} \int_0^t (t-\zeta)^{\omega-1} \|B\mathcal{u}_y(\zeta)\| d\zeta$$

$$+ \rho\|\mathcal{A}_\rho\| \frac{\mathcal{S}}{\Gamma(\omega)} \int_0^t (t-\zeta)^{\omega-1} \|f(\zeta,y(\zeta))\| d\zeta + \mathcal{S} \sum_{\gamma=1}^{p} \|\Delta y(t_\gamma)\| \Bigg]$$

$$\leq \eta \Bigg[\|E^{-1}\|\mathcal{M}_h\epsilon_0 + \|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)}(\mathcal{M}_h\epsilon_0 + \widehat{\mathcal{M}_h})$$

$$+ \mathcal{S}(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}) + (1-\rho)\|B\|\|\mathcal{u}_y(t)\|$$

$$+ (1-\rho)(\mathcal{M}_f\epsilon_0 + \widehat{\mathcal{M}_f}) + \rho\eta \frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)} \|B\|\|\mathcal{u}_y(t)\|$$

$$+ \rho\eta \frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)}(\mathcal{M}_f\epsilon_0 + \widehat{\mathcal{M}_f}) + \mathcal{S}\mathcal{M}\epsilon_0 + \mathcal{S}\sum_{\gamma=1}^{p} \|q_\gamma(0)\| \Bigg]$$

$$= \eta \Bigg[\epsilon_0 \left[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{S}\mathcal{M} \right]$$

$$+ \|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\widehat{\mathcal{M}_h} + \mathcal{S}(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h})$$

$$+ (1-\rho)\widehat{\mathcal{M}_f} + \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\widehat{\mathcal{M}_f} + \mathcal{S}\sum_{\gamma=1}^{p} \|q_\gamma(0)\|$$

$$+ \left[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right] \|B\|\|\mathcal{u}_y(t)\| \Bigg]$$

$$= D\epsilon_0 + \widehat{D} + \left[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right] \eta\|B\|\|\mathcal{u}_y(t)\|$$

$$\leq D\epsilon_0 + \widehat{D} + \left[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right] \eta K\|B\|(D\epsilon_0 + \|y_1\| + \widehat{D})$$

$$= \left[D + \eta K\|B\|D\left[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right]\right]\epsilon_0$$

$$+ \eta K\|B\|\left[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right]\left(\|y_1\| + \widehat{D}\right) + \widehat{D}$$

$$= \epsilon_0.$$

Therefore $(\Phi_1 + \Phi_2)C_{\epsilon_0} = \Phi C_{\epsilon_0} \subset C_{\epsilon_0}$.

Claim II We prove the operator Φ_1 is contraction on C_{ϵ_0}.

Let $y, \hat{y} \in C_{\epsilon_0}$, then

$$\|\Phi_1 y - \Phi_1 \hat{y}\|$$

$$\leq \|\mathcal{A}_\rho\|\Bigg[\|h(t, y(t)) - h(t, \hat{y}(t))\|$$

$$+ \int_0^t (t-\zeta)^{\omega-1}\|L_\omega(t-\zeta)\|\|Eh(\zeta, y(\zeta) - Eh(\zeta, \hat{y}(\zeta))\|d\zeta$$

$$+ (1-\rho)\|B\|\left\|u_y(t) - u_{\hat{y}}(t)\right\|$$

$$+ (1-\rho)\left\|f(t, y(t)) - f(t, \hat{y}(t))\right\|$$

$$+ \rho\|\mathcal{A}_\rho\|\int_0^t (t-\zeta)^{\omega-1}\|L_\omega(t-\zeta)\|\,\|B\|\left\|u_y(\zeta)\right.$$

$$\left. - u_{\hat{y}}(\zeta)\right\|d\zeta + \sum_{\gamma=1}^{p}\|\mathcal{T}_\omega\|\left\|q_\gamma y(t_\gamma) - q_\gamma \hat{y}(t_\gamma)\right\|\Bigg]$$

$$\leq \eta\Bigg[\left\|E^{-1}\right\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\mathcal{M}_h$$

$$+ (1-\rho)\|B\|K\eta\mathcal{S}\left(\frac{\Upsilon^{\omega}}{\omega\Gamma(\omega)}\mathcal{M}_h + \rho\eta\frac{\Upsilon^{\omega}}{\omega\Gamma(\omega)}\mathcal{M}_f + \mathcal{M}\right)$$

$$+ (1-\rho)\mathcal{M}_f$$

$$+ \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\|B\|K\eta\mathcal{S}\left(\frac{\Upsilon^{\omega}}{\omega\Gamma(\omega)}\mathcal{M}_h + \rho\eta\frac{\Upsilon^{\omega}}{\omega\Gamma(\omega)}\mathcal{M}_f + \mathcal{M}\right)$$

$$+ \mathcal{S}\mathcal{M}\Bigg]\|y - \hat{y}\|$$

$$= \eta \left[\|E^{-\mathbf{1}}\| \mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)} \mathcal{M}_h + (1-\rho)\mathcal{M}_f + \mathcal{S}\mathcal{M} + \|B\| K \eta \mathcal{S} \left((1-\rho) + \rho\eta \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)} \right) \left(\frac{\Upsilon^{\omega}}{\Gamma(\omega+1)} \mathcal{M}_h + \rho\eta \frac{\Upsilon^{\omega}}{\Gamma(\omega+1)} \mathcal{M}_f + \mathcal{M} \right) \right] \|y - \hat{y}\|.$$

Let

$$\mathcal{N} = \eta \left[\|E^{-\mathbf{1}}\| \mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)} \mathcal{M}_h + (1-\rho)\mathcal{M}_f + \mathcal{S}\mathcal{M} + \|B\| K \eta \mathcal{S} \left((1-\rho) + \rho\eta \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)} \right) \left(\frac{\Upsilon^{\omega}}{\Gamma(\omega+1)} \mathcal{M}_h + \rho\eta \frac{\Upsilon^{\omega}}{\Gamma(\omega+1)} \mathcal{M}_f + \mathcal{M} \right) \right].$$

From (3.17), we can observe that $0 < \mathcal{N} < 1$ which mean Φ_1 is a contraction.

Claim III We prove the operator Φ_2 is completely continuous.

Firstly, we prove Φ_2 is continuous.

Let y_n be a sequence in $PC(\mathcal{J}, X)$ which converge to y. Since f is continuous function, then

$$\|\Phi_2 y_n - \Phi_2 y\| \le \rho \|\mathcal{A}_\rho^2\| \int_0^t (t-\zeta)^{\omega-1} \|L_\omega(t-\zeta)\| \|f(\zeta, y_n(\zeta)) - f(\zeta, y(\zeta))\| d\zeta$$

$$\le \rho\eta^2 \frac{\mathcal{S}}{\Gamma(\omega)} \int_0^t (t-\zeta)^{\omega-1} \|f(\zeta, y_n(\zeta)) - f(\zeta, y(\zeta))\| d\zeta$$

which converge to zero as $n \to \infty$. Therefore $\Phi_2 y_n \to \Phi_2 y$ in $PC(\mathcal{J}, X)$.

Next, we prove the family $\{\Phi_2 y : y \in C_{\epsilon_0}\}$ is relatively compact. According to Arzela – Ascoli Theorem it is suffices to prove:

- $\{\Phi_2 y : y \in C_{\epsilon_0}\}$ is uniformly bounded.
- $\{\Phi_2 y : y \in C_{\epsilon_0}\}$ is equicontinuous.
- For each $t \in \mathcal{J}$ then $\{(\Phi_2 y)(t) : y \in C_{\epsilon_0}\}$ is relatively compact in X.

By definition of C_{ϵ_0} we have $\|\Phi_2 y\| \leq \epsilon_0$ for any $y \in C_{\epsilon_0}$, therefore $\{\Phi_2 y : y \in C_{\epsilon_0}\}$ is uniformly bounded.

To prove $\{\Phi_2 y : y \in C_{\epsilon_0}\}$ is equicontinuous, let $t_1, t_2 \in \mathcal{J}$, $t_1 < t_2$ then

$$
\begin{aligned}
&\|\Phi_2 y(t_2) - \Phi_2 y(t_1)\| \\
&\quad = \left\| \int_0^{t_2} (t_2-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right. \\
&\qquad\qquad \left. - \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega(t_1-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right\| \\
&\quad = \left\| \int_{t_1}^{t_2} (t_2-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right. \\
&\qquad + \int_0^{t_1} (t_2-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \\
&\qquad + \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \\
&\qquad - \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega(t_2-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \\
&\qquad \left. - \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega(t_1-\zeta) f\big(\zeta, y(\zeta)\big) d\zeta \right\|
\end{aligned}
$$

$$\leq \left\|\int_{t_1}^{t_2}(t_2-\zeta)^{\omega-1}L_\omega(t_2-\zeta)f(\zeta,y(\zeta))d\zeta\right\|$$
$$+\left\|\int_0^{t_1}(t_2-\zeta)^{\omega-1}L_\omega(t_2-\zeta)f(\zeta,y(\zeta))d\zeta - \int_0^{t_1}(t_1-\zeta)^{\omega-1}L_\omega(t_2-\zeta)f(\zeta,y(\zeta))d\zeta\right\|$$
$$+\left\|\int_0^{t_1}(t_1-\zeta)^{\omega-1}L_\omega(t_2-\zeta)f(\zeta,y(\zeta))d\zeta - \int_0^{t_1}(t_1-\zeta)^{\omega-1}L_\omega(t_1-\zeta)f(\zeta,y(\zeta))d\zeta\right\|$$

$$= \left\|\int_{t_1}^{t_2}(t_2-\zeta)^{\omega-1}L_\omega(t_2-\zeta)f(\zeta,y(\zeta))d\zeta\right\|$$
$$+\left\|\int_0^{t_1}[(t_2-\zeta)^{\omega-1}-(t_1-\zeta)^{\omega-1}]L_\omega(t_2-\zeta)f(\zeta,y(\zeta))d\zeta\right\|$$
$$+\left\|\int_0^{t_1}(t_1-\zeta)^{\omega-1}[L_\omega(t_2-\zeta)-L_\omega(t_1-\zeta)]f(\zeta,y(\zeta))d\zeta\right\|.$$

Let

$$O_1 = \left\|\int_{t_1}^{t_2}(t_2-\zeta)^{\omega-1}L_\omega(t_2-\zeta)f(\zeta,y(\zeta))d\zeta\right\|,$$
$$O_2 = \left\|\int_0^{t_1}[(t_2-\zeta)^{\omega-1}-(t_1-\zeta)^{\omega-1}]L_\omega(t_2-\zeta)f(\zeta,y(\zeta))d\zeta\right\|,$$
$$O_3 = \left\|\int_0^{t_1}(t_1-\zeta)^{\omega-1}[L_\omega(t_2-\zeta)-L_\omega(t_1-\zeta)]f(\zeta,y(\zeta))d\zeta\right\|.$$

We have to prove O_1, O_2 and O_3 tend to zero when $t_2 \to t_1$.

By using Lemma (3.1.2) and condition **H3**, we have

$$O_1 \leq \frac{\mathcal{S}}{\Gamma(\omega)}\int_{t_1}^{t_2}(t_2-\zeta)^{\omega-1}\|f(\zeta,y(\zeta))\|d\zeta$$
$$\leq \frac{\mathcal{S}}{\Gamma(\omega+1)}(t_2-t_1)^{\omega}\left(\mathcal{M}_f\epsilon_0+\widehat{\mathcal{M}_f}\right)$$

which tend to zero as $t_2 \to t_1$.

Also,

$$O_2 \leq \frac{\mathcal{S}}{\Gamma(\omega)} \frac{(t_2 - t_1)^{\omega}}{\omega} \left(\mathcal{M}_f \epsilon_0 + \widehat{\mathcal{M}_f}\right)$$

which is tend to zero as $t_2 \to t_1$.

Now, for O_3, if $t_1 = 0$ then $O_3 = 0$.

If $t_1 > 0$ and s is a small enough, then

$$O_3 \leq \int_0^{t_1 - s} (t_1 - \zeta)^{\omega - 1} \|L_\omega(t_2 - \zeta) - L_\omega(t_1 - \zeta)\| \big\|f\big(\zeta, y(\zeta)\big)\big\| d\zeta$$

$$+ \int_{t_1 - s}^{t_1} (t_1 - \zeta)^{\omega - 1} \|L_\omega(t_2 - \zeta) - L_\omega(t_1 - \zeta)\| \big\|f\big(\zeta, y(\zeta)\big)\big\| d\zeta$$

$$\leq sup_{\zeta \in [0, t_1 - s]} \|L_\omega(t_2 - \zeta) - L_\omega(t_1 - \zeta)\| \frac{t_1^{\omega} - s^{\omega}}{\omega} \left(\mathcal{M}_f \epsilon_0 + \widehat{\mathcal{M}_f}\right)$$

$$+ sup_{\zeta \in [t_1 - s, t_1]} \|L_\omega(t_2 - \zeta) - L_\omega(t_1 - \zeta)\| \frac{s^{\omega}}{\omega} \left(\mathcal{M}_f \epsilon_0 + \widehat{\mathcal{M}_f}\right).$$

Since $\mathcal{T}(t), t > 0$ is continuous in the uniform operator topology, we have L_ω is continuous in the uniform operator topology, then O_3 tend to zero as $t_2 \to t_1$, $s \to 0$.

Therefore $\|\Phi_2 y(t_2) - \Phi_2 y(t_1)\|$ tend to zero independently of $y \in C_{\epsilon_0}$ as $t_2 \to t_1$ which mean the family $\left\{\Phi_2 y \colon y \in C_{\sigma_0}\right\}$ is equicontinuous.

Finally, we prove the set $R(t) = \left\{(\Phi_2 y)(t), y \in C_{\epsilon_0}\right\}$ for any $t \in \mathcal{J}$ is relatively compact in X.

If $t = 0$, then $R(0) = \{(\Phi_2 y)(0)\} = \{0\}$ which is compact set.

If $t \in (0, \Upsilon]$, we choose $\upsilon \in (0, t)$ and a real number $r > 0$ to define the operator

$$(\Phi_r^v y)(t) = \omega\rho\mathcal{A}_\rho^2 \int_0^{t-v}\int_r^\infty \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f(\zeta, y(\zeta))d\theta d\zeta,$$

$y \in C_{\epsilon_0}$. By condition **H5** for any $t \in \mathcal{J}$, the set $(\mathcal{K}_r^v)(t) = \{(\Phi_r^v y)(t), y \in C_{\epsilon_0}\}$ is relatively compact in X. Moreover, for any $y \in C_{\epsilon_0}$ we have

$$\begin{aligned}
&\|(\Phi_2 y)(t) - (\Phi_r^v y)(t)\| \\
&= \left\| \omega\rho\mathcal{A}_\rho^2 \int_0^t\int_0^\infty \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f(\zeta, y(\zeta))d\theta d\zeta \right. \\
&\quad \left. - \omega\rho\mathcal{A}_\rho^2 \int_0^{t-v}\int_r^\infty \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f(\zeta, y(\zeta))d\theta d\zeta \right\| \\
&= \left\| \omega\rho\mathcal{A}_\rho^2 \int_0^t\int_0^r \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f(\zeta, y(\zeta))d\theta d\zeta \right. \\
&\quad + \omega\rho\mathcal{A}_\rho^2 \int_0^t\int_r^\infty \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f(\zeta, y(\zeta))d\theta d\zeta \\
&\quad \left. - \omega\rho\mathcal{A}_\rho^2 \int_0^{t-v}\int_r^\infty \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f(\zeta, y(\zeta))d\theta d\zeta \right\| \\
&\le \left\| \omega\rho\mathcal{A}_\rho^2 \int_0^t\int_0^r \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f(\zeta, y(\zeta))d\theta d\zeta \right\| \\
&\quad + \left\| \omega\rho\mathcal{A}_\rho^2 \int_{t-v}^t\int_r^\infty \theta(t-\zeta)^{\omega-1}\varphi_\omega(\theta)\mathcal{T}(\theta(t-\zeta)^\omega)f(\zeta, y(\zeta))d\theta d\zeta \right\| \\
&\le \omega\rho\eta^2\mathcal{S}\int_0^t (t-\zeta)^{\omega-1}\|f(\zeta, y(\zeta)\| d\zeta \int_0^r \theta\varphi_\omega(\theta)d\theta \\
&\qquad + \omega\rho\eta^2\mathcal{S}\int_{t-v}^t (t-\zeta)^{\omega-1}\|f(\zeta, y(\zeta)\| d\zeta \int_r^\infty \theta\varphi_\omega(\theta)d\theta \\
&\le \rho\eta^2\mathcal{S}\Upsilon^\omega(\mathcal{M}_f\epsilon_0 + \widehat{\mathcal{M}_f})\int_0^r \theta\varphi_\omega(\theta)d\theta \\
&\qquad + \rho\eta^2\mathcal{S}v^\omega(\mathcal{M}_f\epsilon_0 + \widehat{\mathcal{M}_f})\int_r^\infty \theta\varphi_\omega(\theta)d\theta.
\end{aligned}$$

Therefore, $\|(\Phi_2 y)(t) - (\Phi_r^v y)(t)\|$ tends to zero as $v, r \to 0$. Consequently, there are relatively compact sets arbitrary close to the set $R(t)$. Thus, $R(t)$ is relatively compact in X. Based on the above, the operator Φ_2 is completely continuous. According to Nussbaum Fixed Point Theorem, operator Φ has a fixed point in C_{ϵ_0}. Therefore, The System (3.1) is exact controllable on $\mathcal{J}$. ■

The next example illustrates our result.

Example 3.1.6

Consider

$$\begin{cases} {}^{C}D^{\rho,\omega}[y(t,\gamma) - h(t, y(t,\gamma))] = \mathcal{A}y(t,\gamma) + Bu(t) + f\big(t, y(t,\gamma)\big), \\ \qquad \rho, \omega \in (0,1), \gamma \in [0,\pi], t \in [0, t_1) \cup (t_1, 1], \\ y(t,0) = y(t,\pi) = 0, t \in [0,1], \\ \qquad \Delta y\big(t_1\big) = q_1\big(y(t_1^-)\big), \qquad t_1 = \dfrac{1}{2}, \end{cases}$$

and $X = L^2([0,\pi], R)$. Define

$$\mathcal{A}y(t,\gamma) = \frac{\partial^2 y}{\partial \gamma^2}(t,\gamma).$$

where

$$D(\mathcal{A}) = \left\{ y \in X : \frac{\partial y}{\partial \gamma}, \frac{\partial^2 y}{\partial \gamma^2} \in X \text{ and } \gamma(0) = \gamma(\pi) = 0 \right\}.$$

For $y \in D(\mathcal{A})$ then $\mathcal{A}$ can be written as the following

$$\mathcal{A}y = \sum_{s=1}^{\infty} -s^2 \langle y, y_s \rangle y_s,$$

where $y_s(\gamma) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(s\gamma)$, s=1,2,3…. Therefore $\mathcal{A}$ is the generator of C_0 –semigroup $\{\mathcal{T}(t), t \geq 0\}$ in $L^2[0,\pi]$ such that $\mathcal{T}(t)y = \sum_{s=1}^{\infty} e^{-s^2 t} \langle y, y_s \rangle y_s$, $y \in D(\mathcal{A})$ [33]. The functions h, f and q_1 defined as follows:

- $h: [0,1] \times X \to D(\mathcal{A})$ such that

$$h\big(t, y(t,\gamma)\big) = siny(t,\gamma), t \in [0,1], \gamma \in [0,\pi], y \in X.$$

- $f: [0,1] \times X \to X$ such that

$$f(t, y(t,\gamma)) = \frac{t^2 e^{-t}|y(t,\gamma)|}{b}, \qquad t \in [0,1], \gamma \in [0,\pi], y \in X, b > 0.$$

- $q_1: X \to X$ such that

$$q_1(y(t_1,\gamma)) = \frac{\left|y\left(\frac{1}{2}^{-}, \gamma\right)\right|}{2\left(1 + \left|y(\frac{1}{2}^{-}, \gamma)\right|\right)}, t \in [0,1], \gamma \in [0,\pi], y \in X.$$

For $y_1, y_2 \in X$,

$$\left\|f\big(t, y_1(t,\gamma)\big) - f\big(t, y_2(t,\gamma)\big)\right\| = \left\|\frac{t^2 e^{-t}|y_1(t,\gamma)|}{b} - \frac{t^2 e^{-t}|y_2(t,\gamma)|}{b}\right\|$$

$$\leq \frac{1}{b}\|y_1 - y_2\|$$

and

$$\left\|Eh\big(t, y_1(t,\gamma)\big) - Eh\big(t, y_2(t,\gamma)\big)\right\| = \|E \sin y_1(t,\gamma) - E \sin y_2(t,\gamma)\|$$

$$\leq \|E\|\|\sin y_1(t,\gamma) - \sin y_2(t,\gamma)\|$$

$$\leq \|E\|\|y_1 - y_2\|.$$

Also,

$$\left\|q_1\big(y_1(t_1,\gamma)\big) - q_1\big(y_2(t_1,\gamma)\big)\right\| \leq \frac{1}{2}\|y_1 - y_2\|.$$

If

$$0 < D + \eta K\|B\|D\left[(1-\rho) + \frac{\rho\eta\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right] < 1$$

where

$$D = \eta\left[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \eta\rho\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{S}\mathcal{M}\right],$$

and the condition **H5** is hold, then the hypothesis of Theorem (3.1.5) are fulfilled. Therefore, the system is exact controllable.

3.2 The Existence and Uniqueness of the Mild Solution of Impulsive Hattaf-Fractional Nonlinear Control System in Banach Space

This section investigates the existence and uniqueness of the mild solution of the impulsive Hattaf-fractional Nonlinear Control System (3.1) using Banach Fixed Point Theorem. We assume that The System (3.1) satisfies the conditions **H2**, **H3**, and **H4**, given in Section 3.1, $\|A_\rho\| \leq \eta$ and the C_0 −semigroup $T(t), t > 0$ is compact.

Theorem 3.2.1

Assume that

$$D = \eta\left[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \eta\rho\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{S}\mathcal{M}\right] < 1,$$

then The System (3.1) has a unique mild solution on $PC(\mathcal{J}, \mathrm{X})$ for each $u \in L_P(\mathcal{J}, U)$.

Proof. Define the operator

$$\left(\widehat{\Phi}y\right)(t) =$$

$$\begin{cases} \mathcal{A}_\rho h\big(t,y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h\big(\zeta,y(\zeta)\big) d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)\big(y_0 - h(0,y_0)\big) + (1-\rho)\mathcal{A}_\rho\big[Bu(t) + f\big(t,y(t)\big)\big] \\ +\rho\mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\big[Bu(\zeta) + f\big(\zeta,y(\zeta)\big)\big] d\zeta \quad t \in [0,t_1] \\ \mathcal{A}_\rho h\big(t,y(t)\big) + \mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h\big(\zeta,y(\zeta)\big) d\zeta \\ +\mathcal{A}_\rho \mathcal{T}_\omega(t)\big(y_0 - h(0,y_0)\big) + (1-\rho)\mathcal{A}_\rho\big[Bu(t) + f\big(t,y(t)\big)\big] \\ +\rho\mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)\big[Bu(\zeta) + f\big(\zeta,y(\zeta)\big)\big] d\zeta \\ +\mathcal{A}_\rho \sum_{\gamma=1}^{p} \mathcal{T}_\omega\big(t-t_\gamma\big)\Delta y\big(t_\gamma\big), \qquad t \in \big(t_\gamma, t_{\gamma+1}\big]. \end{cases}$$

Step 1. We show the operator $\widehat{\Phi}$ maps $PC(\mathcal{J},\mathrm{X})$ into itself.

For $0 \le s < s_1 \le t_1$,

$$\begin{aligned} &\left\|\left(\widehat{\Phi}y\right)(s) - \left(\widehat{\Phi}y\right)(s_1)\right\| \\ &= \Bigg\| \mathcal{A}_\rho h\big(s,y(s)\big) + \mathcal{A}_\rho \int_0^s (s-\zeta)^{\omega-1} L_\omega(s-\zeta) E h\big(\zeta,y(\zeta)\big) d\zeta \\ &+ \mathcal{A}_\rho \mathcal{T}_\omega(s)\big(y_0 - h(0,y_0)\big) + (1-\rho)\mathcal{A}_\rho\big[Bu(s) + f\big(s,y(s)\big)\big] \\ &+ \rho\mathcal{A}_\rho^2 \int_0^s (s-\zeta)^{\omega-1} L_\omega(s-\zeta)\big[Bu(\zeta) + f\big(\zeta,y(\zeta)\big)\big] d\zeta \\ &- \mathcal{A}_\rho h\big(s_1,y(s_1)\big) - \mathcal{A}_\rho \int_0^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta) \end{aligned}$$

$$Eh\big(\zeta, y(\zeta)\big)d\zeta - \mathcal{A}_\rho \mathcal{T}_\omega(s_1)\big(y_0 - h(0, y_0)\big)$$

$$-(1-\rho)\mathcal{A}_\rho\big[Bu(s_1) + f\big(s_1, y(s_1)\big)\big]$$

$$-\rho\mathcal{A}_\rho^2 \int_0^{s_1} (s_1 - \zeta)^{\omega-1} L_\omega(s_1 - \zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big]d\zeta\Bigg\|$$

$$\begin{aligned}
\leq \Big\|\mathcal{A}_\rho\Big(h\big(s, y(s)\big) - h\big(s_1, y(s_1)\big)\Big)\Big\| \\
&+ \Bigg\|\mathcal{A}_\rho\Bigg[\int_0^{s} (s-\zeta)^{\omega-1} L_\omega(s-\zeta)Eh\big(\zeta, y(\zeta)\big)d\zeta \\
&- \int_0^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta)Eh\big(\zeta, y(\zeta)\big)d\zeta\Bigg]\Bigg\| \\
&+ \big\|\mathcal{A}_\rho\big(y_0 - h(0, y_0)\big)\big(\mathcal{T}_\omega(s) - \mathcal{T}_\omega(s_1)\big)\big\| \\
&+ \big\|(1-\rho)\mathcal{A}_\rho\big[B\big(u(s) - u(s_1)\big) + f\big(s, y(s)\big) - f\big(s_1, y(s_1)\big)\big]\big\| \\
&+ \Bigg\|\rho\mathcal{A}_\rho^2\Bigg[\int_0^{s} (s-\zeta)^{\omega-1} L_\omega(s-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big]d\zeta \\
&- \int_0^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big]d\zeta\Bigg]\Bigg\|
\end{aligned}$$

$$\leq \|\mathcal{A}_\rho\|\Bigg[h\big(s, y(s)\big) - h\big(s_1, y(s_1)\big) + \Bigg\|\int_0^{s} (s-\zeta)^{\omega-1} L_\omega(s-\zeta)Eh\big(\zeta, y(\zeta)\big)d\zeta$$

$$- \int_0^{s} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta)Eh\big(\zeta, y(\zeta)\big)d\zeta - \int_s^{s_1} (s_1-\zeta)^{\omega-1}$$

$$L_\omega(s_1-\zeta)Eh\big(\zeta, y(\zeta)\big)d\zeta + \int_0^{s} (s_1-\zeta)^{\omega-1} L_\omega(s-\zeta)Eh\big(\zeta, y(\zeta)\big)d\zeta$$

$$- \int_0^{s} (s_1-\zeta)^{\omega-1} L_\omega(s-\zeta)Eh\big(\zeta, y(\zeta)\big)d\zeta\Bigg\|$$

$$+ \|y_0 - h(0, y_0)\|\|\mathcal{T}_\omega(s) - \mathcal{T}_\omega(s_1)\|$$

$$+(1-\rho)\left[\|B\|\|u(s)-u(s_1)\|+\left\|f\big(s,y(s)\big)-f\big(s_1,y(s_1)\big)\right\|\right]$$

$$+\rho\left\|\mathcal{A}_\rho\right\|\left\|\int_0^s (s-\zeta)^{\omega-1}L_\omega(s-\zeta)\left[Bu(\zeta)+f\big(\zeta,y(\zeta)\big)\right]d\zeta\right.$$

$$-\int_0^s (s_1-\zeta)^{\omega-1}L_\omega(s_1-\zeta)\left[Bu(\zeta)+f\big(\zeta,y(\zeta)\big)\right]d\zeta$$

$$-\int_s^{s_1} (s_1-\zeta)^{\omega-1}L_\omega(s_1-\zeta)\left[Bu(\zeta)+f\big(\zeta,y(\zeta)\big)\right]d\zeta$$

$$+\int_0^s (s_1-\zeta)^{\omega-1}L_\omega(s-\zeta)\left[Bu(\zeta)+f\big(\zeta,y(\zeta)\big)\right]d\zeta$$

$$\left.-\int_0^s (s_1-\zeta)^{\omega-1}L_\omega(s-\zeta)\left[Bu(\zeta)+f\big(\zeta,y(\zeta)\big)\right]d\zeta\right\|$$

$$\leq \left\|\mathcal{A}_\rho\right\|\left[\left\|h\big(s,y(s)\big)-h\big(s_1,y(s_1)\big)\right\|\right.$$

$$+\left\|\int_s^{s_1} (s_1-\zeta)^{\omega-1}L_\omega(s_1-\zeta)Eh\big(\zeta,y(\zeta)\big)d\zeta\right\|$$

$$+\int_0^s \|(s_1-\zeta)^{\omega-1}\|L_\omega(s-\zeta)-L_\omega(s_1-\zeta)\|Eh\big(\zeta,y(\zeta)\big)\|d\zeta$$

$$+\int_0^s \|(s-\zeta)^{\omega-1}-(s_1-\zeta)^{\omega-1}\|\|L_\omega(s-\zeta)\|\left\|Eh\big(\zeta,y(\zeta)\big)\right\|d\zeta$$

$$+\|y_0-h(0,y_0)\|\|\mathcal{T}_\omega(s)-\mathcal{T}_\omega(s_1)\|$$

$$+(1-\rho)\|B\|\|u(s)-u(s_1)\|$$

$$+(1-\rho)\left\|f\big(s,y(s)\big)-f\big(s_1,y(s_1)\big)\right\|$$

$$+\rho\left\|\mathcal{A}_\rho\right\|\left\|\int_s^{s_1} (s_1-\zeta)^{\omega-1}\|L_\omega(s_1-\zeta)\|\right\|Bu(\zeta)$$

$$+f\big(\zeta,y(\zeta)\big)\|d\zeta\Big\|$$

$$+\rho\|\mathcal{A}_\rho\|\int_0^s (s_1-\zeta)^{\omega-1}\|L_\omega(s-\zeta)-L_\omega(s_1-\zeta)\|\|Bu(\zeta)$$

$$+f(\zeta,y(\zeta))\|d\zeta+\rho\|\mathcal{A}_\rho\|\int_0^s \|(s-\zeta)^{\omega-1}$$

$$-(s_1-\zeta)^{\omega-1}\|\|L_\omega(s-\zeta)\|\|Bu(\zeta)+f(\zeta,y(\zeta))\|d\zeta\Bigg]$$

$$\leq \eta\Bigg[\|h(s,y(s))-h(s_1,y(s_1))\|+\frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)}sup_{\xi\in[s,s_1]}\|Eh(\xi,y(\xi))\|$$

$$+sup_{\xi\in[0,s]}\|L_\omega(s-\zeta)-L_\omega(s_1-\zeta)\|sup_{\xi\in[0,s]}\|Eh(\zeta,y(\zeta))\|$$

$$\times\left[\frac{s_1^\omega}{\omega}-\frac{(s_1-s)^\omega}{\omega}\right]+\frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)}sup_{\xi\in[0,s]}\|Eh(\zeta,y(\zeta))\|+\|y_0-h(0,y_0)\|$$

$$\|\mathcal{T}_\omega(s)-\mathcal{T}_\omega(s_1)\|+(1-\rho)\|B\|\|u(s)-u(s_1)\|$$

$$+(1-\rho)\|f(s,y(s))-f(s_1,y(s_1))\|$$

$$+\rho\eta\frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)}sup_{\xi\in[s,s_1]}\|Bu(\zeta)+f(\zeta,y(\zeta))\|$$

$$+\rho\eta sup_{\xi\in[0,s]}\|L_\omega(s-\zeta)-L_\omega(s_1-\zeta)\|sup_{\xi\in[0,s]}\|Bu(\zeta)$$

$$+f(\zeta,y(\zeta))\|\left[\frac{s_1^\omega}{\omega}-\frac{(s_1-s)^\omega}{\omega}\right]$$

$$+\rho\eta\frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)}sup_{\xi\in[0,s]}\|Bu(\zeta)+f(\zeta,y(\zeta))\|\Bigg].$$

Let

$$O_1=\eta\|h(s,y(s))-h(s_1,y(s_1))\|$$

$$O_2=\eta\frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)}sup_{\xi\in[s,s_1]}\|Eh(\xi,y(\xi))\|$$

$$O_3 = \eta sup_{\xi\in[0,s]}\|L_\omega(s-\zeta) - L_\omega(s_1-\zeta)\| sup_{\xi\in[0,s]}\left\|Eh\big(\zeta, y(\zeta)\big)\right\| \left[\frac{s_1^\omega}{\omega} - \frac{(s_1-s)^\omega}{\omega}\right]$$

$$O_4 = \eta \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[0,s]}\left\|Eh\big(\zeta, y(\zeta)\big)\right\|$$

$$O_5 = \eta\|y_0 - h(0,y_0)\|\|\mathcal{T}_\omega(s) - \mathcal{T}_\omega(s_1)\|$$

$$O_6 = (1-\rho)\eta\|B\|\|u(s) - u(s_1)\|$$

$$O_7 = (1-\rho)\eta\left\|f\big(s,y(s)\big) - f\big(s_1,y(s_1)\big)\right\|$$

$$O_8 = \rho\eta^2 \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[s,s_1]}\left\|Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\right\|$$

$$O_9 = \rho\eta^2 sup_{\xi\in[0,s]}\|L_\omega(s-\zeta) - L_\omega(s_1-\zeta)\| sup_{\xi\in[0,s]}\left\|Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\right\| \left[\frac{s_1^\omega}{\omega} - \frac{(s_1-s)^\omega}{\omega}\right]$$

$$O_{10} = \rho\eta^2 \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[0,s]}\left\|Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\right\|.$$

Since f, h are continuous functions on $\mathcal{J}$, then O_1, O_7 tend to zero as $s \to s_1$.

Since $L_\omega, \mathcal{T}_\omega$ are continuous in the uniform operator topology, then

O_3, O_5 and O_9 tend to zero as $s \to s_1$.

Since u is measurable, then $u(s) \to u(s_1)$ almost every where $s \to s_1$, then O_6 tend to zero.

Clearly O_2, O_4, O_8 and O_{10} tend to zero as $s \to s_1$. Therefore

$\left\|\left(\widehat{\Phi}y\right)(s)-\left(\widehat{\Phi}y\right)(s_1)\right\|\to 0$ as $s\to s_1$. Thus $\left(\widehat{\Phi}y\right)(t)\in C[0,t_1]$.

Now, for $t_\gamma < s < s_1 \le t_{\gamma+1}$, we have

$$
\begin{aligned}
&\left\|\left(\widehat{\Phi}y\right)(s)-\left(\widehat{\Phi}y\right)(s_1)\right\| \\
&\le \left\|\mathcal{A}_\rho\right\|\Bigg[\left\|h\big(s,y(s)\big)-h\big(s_1,y(s_1)\big)\right\| \\
&+\left\|\int_s^{s_1}(s_1-\zeta)^{\omega-1}L_\omega(s_1-\zeta)Eh\big(\zeta,y(\zeta)\big)d\zeta\right\| \\
&+\int_0^s\|(s_1-\zeta)^{\omega-1}\|L_\omega(s-\zeta)-L_\omega(s_1-\zeta)\|Eh\big(\zeta,y(\zeta)\big)\|d\zeta \\
&+\int_0^s\|(s-\zeta)^{\omega-1}-(s_1-\zeta)^{\omega-1}\|\|L_\omega(s-\zeta)\|\|Eh\big(\zeta,y(\zeta)\big)\|d\zeta \\
&+\|y_0-h(0,y_0)\|\|\mathcal{T}_\omega(s)-\mathcal{T}_\omega(s_1)\| \\
&+(1-\rho)\|B\|\|u(s)-u(s_1)\| \\
&+(1-\rho)\left\|f\big(s,y(s)\big)-f\big(s_1,y(s_1)\big)\right\| \\
&+\rho\left\|\mathcal{A}_\rho\right\|\left\|\int_s^{s_1}(s_1-\zeta)^{\omega-1}L_\omega(s_1-\zeta)\big[Bu(\zeta)+f\big(\zeta,y(\zeta)\big)\big]d\zeta\right\| \\
&+\rho\left\|\mathcal{A}_\rho\right\|\int_0^s(s_1-\zeta)^{\omega-1}\|L_\omega(s-\zeta)-L_\omega(s_1-\zeta)\|\big[Bu(\zeta) \\
&+f\big(\zeta,y(\zeta)\big)\big]d\zeta \\
&+\rho\left\|\mathcal{A}_\rho\right\|\int_0^s\|(s-\zeta)^{\omega-1}-(s_1-\zeta)^{\omega-1}\|L_\omega(s-\zeta)\big[Bu(\zeta) \\
&+f\big(\zeta,y(\zeta)\big)\big]d\zeta+\Sigma\Bigg],
\end{aligned}
$$

where

$$\Sigma=\sum_{\gamma=1}^{p}\left\|\mathcal{T}_\omega\big(s_1-t_\gamma\big)-\mathcal{T}_\omega\big(s-t_\gamma\big)\right\|\left\|\Delta y\big(t_\gamma\big)\right\|.$$

Since $\mathcal{T}_\omega$ is continuous in the uniform operator topology, then Σ tend to zero as $s \to s_1$, and from above we have $\|(\widehat{\Phi}y)(s) - (\widehat{\Phi}y)(s_1)\|$ tend to zero as $s \to s_1$. Therefore $\widehat{\Phi}y \in PC[0, \Upsilon]$.

Step 2. We show the operator $\widehat{\Phi}$ is contraction on $PC(\mathcal{J}, X)$.

For $y_1, y_2 \in PC(\mathcal{J}, X)$, and for each $t \in [0, t_1]$,

$$\begin{aligned}
&\|(\widehat{\Phi}y_1)(t) - (\widehat{\Phi}y_2)(t)\| \\
&\qquad \le \|\mathcal{A}_\rho[h(t, y_1(t)) - h(t, y_2(t))]\| \\
&\qquad + \left\|\mathcal{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E[h(\zeta, y_1(\zeta)) - h(\zeta, y_2(\zeta))] d\zeta\right\| \\
&\qquad + \|(1-\rho)\mathcal{A}_\rho[f(t, y_1(t)) - f(t, y_2(t))]\| \\
&\qquad + \left\|\rho\mathcal{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)[f(\zeta, y_1(\zeta)) - f(\zeta, y_2(\zeta))] d\zeta\right\|
\end{aligned}$$

$$\begin{aligned}
&\le \eta\|h(t, y_1(t)) - h(t, y_2(t))\| + \eta \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_h \|y_1 - y_2\| \\
&\qquad + (1-\rho)\eta\|f(t, y_1(t)) - f(t, y_2(t))\| \\
&\qquad + \rho\eta^2 \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_f \|y_1 - y_2\|
\end{aligned}$$

$$\begin{aligned}
&\le \eta\mathcal{M}_h E^{-1}\|y_1 - y_2\| + \eta \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_h \|y_1 - y_2\| + (1-\rho)\eta\mathcal{M}_f\|y_1 - y_2\| \\
&\qquad + \rho\eta^2 \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_f \|y_1 - y_2\|
\end{aligned}$$

$$\begin{aligned}
&= \eta\left[\mathcal{M}_h E^{-1} + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_h + (1-\rho)\mathcal{M}_f + \rho\|\mathcal{A}_\rho\| \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_f\right] \|y_1 \\
&\qquad - y_2\|.
\end{aligned}$$

Now, for $t \in (t_\gamma, t_{\gamma+1}]$, using our assumption we have,

$$\left\|\left(\widehat{\Phi}y_1\right)(t)-\left(\widehat{\Phi}y_2\right)(t)\right\| \le$$

$$= \eta\left[\mathcal{M}_h E^{-1} + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \rho\left\|\mathcal{A}_\rho\right\|\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f\right]\|y_1-y_2\| + \eta\mathcal{M}\mathcal{S}\|y_1-y_2\|$$

$$= D\|y_1-y_2\|$$

and by our assumption, then $\widehat{\Phi}$ is contraction. According to Banach Fixed Point Theorem, the operator $\widehat{\Phi}$ has unique fixed point y such that $\widehat{\Phi}y = y$. Therefore the system (3.1) has a unique mild solution for any control $u \in L_P(\mathcal{J}, U)$. ■

3.3. Approximate Controllability of Impulsive Hattaf-Fractional Nonlinear Control System in Banach Space

The approximate controllability of the impulsive Hattaf-Fractional Nonlinear Control System (3.1) investigated throughout this section. Assume System (3.1) meets the conditions **H2**, **H3**, and **H4** outlined in Section 3.1. Define the bounded linear operator $\Lambda: L_P(\mathcal{J}, X) \to X$ as

$$\Lambda(\mathcal{y}) = (1-\rho)\mathcal{A}_\rho\mathcal{y}(\Upsilon) + \rho\mathcal{A}_\rho^2\int_0^{\Upsilon}(\Upsilon-s)^{\omega-1}L_\omega(\Upsilon-s)\mathcal{y}(s)ds.$$

The following condition is important to prove the approximate controllability of System (3.1),

H6 $\forall\, \epsilon > 0, \forall\, \mathcal{y} \in L_P(\mathcal{J}, X), \exists\, \mathcal{u} \in L_P(\mathcal{J}, U)$ such that

$$\|\Lambda(\mathcal{y}) - \Lambda(B\mathcal{u})\| < \epsilon$$

and

$$\|B\mathcal{u}(\cdot)\| < \lambda\|\mathcal{y}(\cdot)\|$$

where $\lambda > 0$.

Definition 3.3.1

The System (3.1) is **approximately controllable** on $\mathcal{J}$ if $\overline{\mathcal{K}_{\Upsilon}(y)} = X$, where $\mathcal{K}_{\Upsilon}(y) = \{y(\Upsilon; u) : u(\mathrm{t}) \in U\}$ is a reachable set of System (3.1).

Lemma 3.3.2

Assume the condition **H6** is hold, then

i. $\|\mathrm{y}(\mathrm{t})\| \leq D\|y(t)\| + \widehat{D} + \left[(1-\rho) + \rho\eta \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right]\eta\|B\|\|u(t)\|.$

where

$$\widehat{D} = \eta\Bigg[\|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_h} + \mathcal{S}\big(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\big)$$

$$+(1-\rho)\widehat{\mathcal{M}_f} + \eta\rho\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_f} + \mathcal{S}\sum_{\gamma=1}^{p}\|q_{\gamma}(0)\|\Bigg].$$

ii. For $y_1, y_2 \in X$, then

$$\|y_2(t) - y_1(t)\| \leq \frac{\eta}{1-D}\left[(1-\rho)\|B\| + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\|B\|\right]\|u_2(t) - u_1(t)\|.$$

Proof.

i. $\|\mathrm{y}(\mathrm{t})\| \leq \left\|\mathcal{A}_{\rho}\right\|\Big[\|h(t,y(t))\| + \frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t (t-\zeta)^{\omega-1}\|Eh(\zeta,y(\zeta)\|d\zeta +$ $\mathcal{S}\left\|\big(y_0 - h(0,y_0)\big)\right\| + (1-\rho)\big[\|Bu(t)\| + \left\|f\big(t,y(t)\big)\right\|\big] +$ $\rho\left\|\mathcal{A}_{\rho}\right\|\frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t (t-\zeta)^{\omega-1}\|Bu(\zeta)\|d\zeta + \rho\left\|\mathcal{A}_{\rho}\right\|\frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t (t-$ $\zeta)^{\omega-1}\left\|f\big(\zeta,y(\zeta)\big)\right\|d\zeta + \mathcal{S}\sum_{\gamma=1}^{p}\left\|\Delta y\big(t_{\gamma}\big)\right\|\Big]$

$$\leq \eta\Bigg[\|E^{-1}\|\mathcal{M}_h\|y\| + \|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\left(\mathcal{M}_h\|y\| + \widehat{\mathcal{M}_h}\right)$$

$$+\mathcal{S}\left(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\right) + (1-\rho)\|B\|\|u(t)\|$$

$$+(1-\rho)\left(\mathcal{M}_f\|y\| + \widehat{\mathcal{M}_f}\right) + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\|B\|\|u(t)\|$$

$$+\rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\left(\mathcal{M}_f\|y\| + \widehat{\mathcal{M}_f}\right) + \mathcal{S}\mathcal{M}\|y\| + \mathcal{S}\sum_{\gamma=1}^{p}\left\|q_\gamma(0)\right\|\Bigg]$$

$$= \eta\Bigg[\|y\|\left[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f\right.$$

$$\left.+\mathcal{S}\mathcal{M}\right] + \|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_h}$$

$$+\mathcal{S}\left(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\right) + (1-\rho)\widehat{\mathcal{M}_f} + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_f}$$

$$+\mathcal{S}\sum_{\gamma=1}^{p}\left\|q_\gamma(0)\right\| + \left[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right]\|B\|\|u(t)\|\Bigg]$$

$$= D\|y\| + \widehat{D} + \left[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right]\eta\|B\|\|u(t)\|$$

ii. $\|y_2(t) - y_1(t)\| \leq \left\|\mathcal{A}_\rho\right\|\Big[\left\|h\big(t, y_2(t)\big) - h\big(t, y_1(t)\big)\right\| + \int_0^t (t - \zeta)^{\omega-1}\|L_\omega(t-\zeta)\|\|Eh(\zeta, y_2(\zeta) - Eh(\zeta, y_1(\zeta)\|d\zeta + (1 - \rho)\|B\|\|u_2(t) - u_1(t)\| + (1-\rho)\left\|f\big(t, y_2(t)\big) - f\big(t, y_1(t)\big)\right\| + \rho\left\|\mathcal{A}_\rho\right\| \int_0^t (t-\zeta)^{\omega-1}\|L_\omega(t-\zeta)\|\,\|B\|\|u_2(\xi) - u_1(\xi)\|d\zeta + \sum_{\gamma=1}^{p}\|\mathcal{T}_\omega\|\left\|q_\gamma y_2\big(t_\gamma\big) - q_\gamma y_1\big(t_\gamma\big)\right\|\Big]$

$$\leq \eta \Bigg[\|E^{-1}\| \mathcal{M}_h \|y_2 - y_1\| + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_h \|y_2 - y_1\|$$

$$+ (1-\rho)\|B\| \|u_2(t) - u_1(t)\| + (1-\rho)\mathcal{M}_f \|y_2 - y_1\|$$

$$+ \rho\eta \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \left[\|B\| \|u_2(t) - u_1(t)\| + \mathcal{M}_f \|y_2 - y_1\| \right]$$

$$+ \mathcal{S}\mathcal{M} \|y_2 - y_1\| \Bigg]$$

$$= \eta \left[\|E^{-1}\| \mathcal{M}_h + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_h + (1-\rho)\mathcal{M}_f + \rho\eta \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_f + \mathcal{S}\mathcal{M} \right] \|y_2$$

$$- y_1\| + \eta \left[(1-\rho)\|B\| + \rho\eta \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \|B\| \right] \|u_2(t) - u_1(t)\|$$

$$= D\|y_2 - y_1\| + \eta \left[(1-\rho)\|B\| + \rho\eta \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \|B\| \right] \|u_2(t) - u_1(t)\|.$$

It follows,

$$\|y_2 - y_1\| - D\|y_2 - y_1\|$$

$$\leq \eta \left[(1-\rho)\|B\| + \rho\eta \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \|B\| \right] \|u_2(t) - u_1(t)\|$$

Thus,

$$\|y_2 - y_1\| \leq \frac{\eta}{1-D} \left[(1-\rho)\|B\| + \rho\eta \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \|B\| \right] \|u_2(t) - u_1(t)\|.$$

Theorem 3.3.3

Suppose the condition **H6** is holds, then The System (3.1) is approximate controllability provided

$$\left(\mathcal{M}_f + \frac{\|\mathcal{A}_\rho^{-1}\|}{\rho} \mathcal{M}_h \right) \lambda \frac{\eta}{1-D} \left[(1-\rho) + \rho\eta \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \right] \|B\| < 1, \qquad (3.18)$$

where $\lambda > 0$.

Proof. Since the domain $D(\mathcal{A})$ of operator $\mathcal{A}$ is dense in X [33], i.e. $\overline{D(\mathcal{A})} = X$. It sufficient to prove $D(\mathcal{A}) \subset \mathcal{K}_{\Upsilon}(y)$, that is mean we must show for any $\epsilon > 0$ and $x \in D(\mathcal{A})$, there exists $u \in L_P(\mathcal{J}, U)$ such that

$$\Bigg\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) - \mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) + \frac{1-\rho}{\rho} Eh\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathcal{A}_\rho^{-1}}{\rho} \Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big) \Bigg\|$$

$$= \Bigg\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) + \left(\frac{1-\rho}{\rho} \rho \mathcal{A}\mathcal{A}_\rho - \mathcal{A}_\rho\right) h\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathcal{A}_\rho^{-1}}{\rho} \Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big) \Bigg\|$$

$$= \Bigg\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) + \big((1-\rho)\mathcal{A} - I\big)\mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathcal{A}_\rho^{-1}}{\rho} \Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big) \Bigg\|$$

$$= \Bigg\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) - \mathcal{A}_\rho^{-1}\mathcal{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathcal{A}_\rho^{-1}}{\rho} \Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big) \Bigg\|$$

$$= \Bigg\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) - h\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathcal{A}_\rho^{-1}}{\rho} \Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big) \Bigg\| < \epsilon$$

where $\Sigma_\Upsilon = \sum_{\gamma=1}^{p} \mathcal{T}_\omega\big(\Upsilon - t_\gamma\big)\Delta y\big(t_\gamma\big)\sigma_\gamma(\Upsilon)$.

For any initial $y_0 \in X$, since $\mathcal{T}(t)$ is differentiability semigroup for each

$t > 0$ then

$$\big[\mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) + h\big(\Upsilon, y(\Upsilon)\big) + \Sigma_\Upsilon\big] \in D(\mathcal{A})$$

and we can see there exists a function $\mathcal{Q} \in L_p(\mathcal{J}, X)$ such that

$$\Lambda(\mathcal{Q}(\cdot)) = x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) - h(\Upsilon, y(\Upsilon)) - \Sigma_\Upsilon.$$

For example,

$\mathcal{Q}(t) =$

$$\begin{cases} 0 & t = \Upsilon \\ \dfrac{(\Upsilon - t)^{1-\omega}}{\Upsilon\rho}\big(\Gamma(\omega)\big)^2 \mathcal{A}_\rho^{-2}\left(L_\omega(\Upsilon - t) + 2t\dfrac{d}{dt}L_\omega(\Upsilon - t)\right) & t \in [0, \Upsilon) \\ (x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) - h(\Upsilon, y(\Upsilon)) - \Sigma_\Upsilon) & \end{cases}$$

then,

$$\Lambda(\mathcal{Q}(t)) = \rho\mathcal{A}_\rho^2 \int_0^\Upsilon (\Upsilon - s)^{\omega-1} L_\omega(\Upsilon - s)\frac{(\Upsilon - s)^{1-\omega}}{\Upsilon\rho}\big(\Gamma(\omega)\big)^2 \mathcal{A}_\rho^{-2}\left(L_\omega(\Upsilon - s) + 2s\frac{d}{ds}L_\omega(\Upsilon - s)\right)\big(x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) - h(\Upsilon, y(\Upsilon)) - \Sigma_\Upsilon\big)ds$$

$$= \frac{\big(\Gamma(\omega)\big)^2}{\Upsilon}\int_0^\Upsilon \left(\big(L_\omega(\Upsilon - s)\big)^2 + 2sL_\omega(\Upsilon - s)\frac{d}{ds}L_\omega(\Upsilon - s)\right)\big(x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) - h(\Upsilon, y(\Upsilon)) - \Sigma_\Upsilon\big)ds$$

$$= \frac{\big(\Gamma(\omega)\big)^2}{\Upsilon}\bigg[\int_0^\Upsilon \big(L_\omega(\Upsilon - s)\big)^2\big(x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) - h(\Upsilon, y(\Upsilon)) - \Sigma_\Upsilon\big)ds + \int_0^\Upsilon s\frac{d}{ds}\big(L_\omega(\Upsilon - s)\big)^2\big(x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)(y_0 - h(0, y_0)) - h(\Upsilon, y(\Upsilon)) - \Sigma_\Upsilon\big)ds\bigg].$$

Using integral by parts, we have

$$\Lambda(\mathcal{Q}(t)) = \frac{(\Gamma(\omega))^2}{\Upsilon}\Bigg[\int_0^{\Upsilon}(L_\omega(\Upsilon-s))^2\big(x-\mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)(y_0-h(0,y_0))$$
$$-h(\Upsilon,y(\Upsilon))-\Sigma_\Upsilon\big)ds$$
$$+\frac{\Upsilon}{(\Gamma(\omega))^2}\big(x-\mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)(y_0-h(0,y_0))-h(\Upsilon,y(\Upsilon))-\Sigma_\Upsilon\big)$$
$$-\int_0^{\Upsilon}(L_\omega(\Upsilon-s))^2\big(x-\mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)(y_0-h(0,y_0))-h(\Upsilon,y(\Upsilon))$$
$$-\Sigma_\Upsilon\big)ds\Bigg]$$
$$=\big(x-\mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)(y_0-h(0,y_0))-h(\Upsilon,y(\Upsilon))-\Sigma_\Upsilon\big).$$

Now, for any given $\epsilon > 0$ and by **H6** there exists a control u such that

$$\|x-\mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)(y_0-h(0,y_0))-h(\Upsilon,y(\Upsilon))-\Sigma_\Upsilon-\Lambda(Bu(t))\| < \frac{\epsilon}{2^3}. \quad (3.19)$$

Let $u_1 \in L_p(\mathcal{J},U)$, then by **H6**, there exists $u_2 \in L_p(\mathcal{J},U)$, such that

$$\left\|\Lambda\left[Bu(t)-f(t,y_1(t))-\frac{\mathcal{A}_\rho^{-1}}{\rho}Eh(t,y_1(t))\right]-\Lambda(Bu_2(t))\right\| < \frac{\epsilon}{2^3} \quad (3.20)$$

where $y_1(t) = y(\text{t};u_1), t \in \mathcal{J}$.

From (3.19) and (3.20) we have

$$\Bigg\|x-\mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)(y_0-h(0,y_0))-h(\Upsilon,y(\Upsilon))-\Sigma_\Upsilon$$
$$-\Lambda\left[f(t,y_1(t))+\frac{\mathcal{A}_\rho^{-1}}{\rho}Eh(t,y_1(t))\right]-\Lambda(Bu_2(t))\Bigg\|$$

$$= \left\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\big(Bu(t)\big) + \Lambda\big(Bu(t)\big) - \Lambda\left[f\big(t, y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_1(t)\big)\right] - \Lambda\big(Bu_2(t)\big)\right\|$$

$$\leq \left\| x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\big(Bu(t)\big)\right\| + \left\|\Lambda\big(Bu(t)\big) - \Lambda\left[f\big(t, y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_1(t)\big)\right] - \Lambda\big(Bu_2(t)\big)\right\|$$

$$\leq \frac{\epsilon}{2^2}.$$

Denote $y_2 = y(\mathrm{t}; u_2), t \in \mathcal{J}$, then by **H6**, there exists $w_2 \in L_p(\mathcal{J}, U)$ such that

$$\left\|\Lambda\left[f\big(t, y_2(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_2(t)\big)\right] - \Lambda\left[f\big(t, y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_1(t)\big)\right] - \Lambda\big(Bw_2(t)\big)\right\| < \frac{\epsilon}{2^3},$$

and

$$\|Bw_2(t)\| < \lambda \left\| f\big(t, y_2(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_2(t)\big) - f\big(t, y_1(t)\big) - \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_1(t)\big)\right\|$$

$$\leq \lambda \left\| f\big(t, y_2(t)\big) - f\big(t, y_1(t)\big)\right\| + \lambda \frac{\left\|\mathcal{A}_\rho^{-1}\right\|}{\rho} \left\| Eh\big(t, y_2(t)\big) - Eh\big(t, y_1(t)\big)\right\|$$

$$\leq \left(\mathcal{M}_f + \frac{\|\mathcal{A}_\rho^{-1}\|}{\rho}\mathcal{M}_h\right)\lambda\|y_2 - y_1\|$$

$$\leq \left(\mathcal{M}_f + \frac{\|\mathcal{A}_\rho^{-1}\|}{\rho}\mathcal{M}_h\right)\lambda\frac{\|\mathcal{A}_\rho\|}{1-D}\left[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right]\|B\|\|\mathcal{u}_2(t) - \mathcal{u}_1(t)\|.$$

Let $\mathcal{u}_3(t) = \mathcal{u}_2(t) - w_2(t)$, $\mathcal{u}_3(\cdot) \in L_p(\mathcal{J}, U)$. It follows

$$\Bigg\|x - \mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\left[f\big(t,y_2(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_2(t)\big)\right] - \Lambda\big(B\mathcal{u}_3(t)\big)\Bigg\|$$

$$= \Bigg\|x - \mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon + \Lambda\left[f\big(t,y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_1(t)\big)\right] - \Lambda\left[f\big(t,y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_1(t)\big)\right] - \Lambda\left[f\big(t,y_2(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_2(t)\big)\right] - \Lambda\big(B\mathcal{u}_3(t)\big)\Bigg\|$$

$$\leq \Bigg\|x - \mathcal{A}_\rho\mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\left[f\big(t,y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_1(t)\big)\right] - \Lambda\big(B\mathcal{u}_2(t)\big)\Bigg\| + \Bigg\|\Lambda\left[f\big(t,y_1(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_1(t)\big)\right] - \Lambda\left[f\big(t,y_2(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho}Eh\big(t,y_2(t)\big)\right] + \Lambda\big(Bw_2(t)\big)\Bigg\|$$

$$< \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3}.$$

By Mathematical Induction, we can see that the sequence $\{\mathcal{u}_n, n = 0,1,2,\ldots\} \subset L_p(\mathcal{J}, U)$, consequently,

$$\Bigg\|x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\left[f\big(t, y_n(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_n(t)\big)\right] - \Lambda\big(B\mathcal{u}_{n+1}(t)\big)\Bigg\| < \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3} + \cdots + \frac{\epsilon}{2^{n+1}}$$

where $y_n = y(\mathrm{t}; \mathcal{u}_n)$ and

$$\|Bu_{n+1} - Bu_n\| \leq \left(\mathcal{M}_f + \frac{\|\mathcal{A}_\rho^{-1}\|}{\rho}\mathcal{M}_h\right)\lambda \frac{\|\mathcal{A}_\rho\|}{1-D}\left((1-\rho) + \rho\eta \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right)\|Bu_n(t) - Bu_{n-1}(t)\|$$

and from our assumption we get the sequence $\{B\mathcal{u}_n(t), n = 1,2,3,\dots\}$ is a Cauchy sequence on X. Since X is a Banach space, then there exists a point $\delta(t) \in X$ such that $B\mathcal{u}_n \to \delta(t)$ as $n \to \infty$. Then for any $\epsilon > 0$, there exists a positive integer k such that

$$\begin{aligned}
&\Bigg\|x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\left[f\big(t, y_k(t)\big) + \frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_k(t)\big)\right] - \Lambda\big(B\mathcal{u}_k(t)\big)\Bigg\| \\
&\leq \Bigg\|x - \mathcal{A}_\rho \mathcal{T}_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\big[f\big(t, y_k(t)\big)\big] - \Lambda\left[\frac{\mathcal{A}_\rho^{-1}}{\rho} Eh\big(t, y_k(t)\big)\right] - \Lambda\big(B\mathcal{u}_{k+1}(t)\big)\Bigg\| \\
&\quad + \big\|\Lambda\big(B\mathcal{u}_{k+1}(t)\big) - \Lambda\big(B\mathcal{u}_k(t)\big)\big\| \\
&< \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3} + \cdots + \frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Therefore, we get a sequence $\{y_k, k = 1,2,\dots\} \subset \mathcal{K}_\Upsilon(f)$ converge to $x \in D(\mathcal{A})$, thus $x \in \overline{\mathcal{K}_\Upsilon(f)}$, which mean $\overline{\mathcal{K}_\Upsilon(f)} = X$. ■

The next example illustrates our result.

Example 3.3.4

Consider the following nonlinear Hattaf-fractional control system

$$\begin{cases} {}^{C}D^{\frac{1}{2},\frac{1}{3}}\left[y(t,\gamma) - h\big(t,y(t,\gamma)\big)\right] = \mathcal{A}y(t,y) + Bu(t) + f\big(t,y(t,\gamma)\big), \\ \qquad \gamma \in [0,\pi], t \in [0,t_1) \cup (t_1,1], \\ y(t,0) = y(t,\pi) = 0, t \in [0,1], \\ \qquad \Delta y(t_1^+) = q_1\big(y(t_1^-)\big), \qquad t_1 = \dfrac{1}{2}, \end{cases} \tag{3.21}$$

Setting $X = L_2([0.\pi], R) = U$, and define the operator $\mathcal{A}: D(\mathcal{A}) \subset X \to X$ by

$$\mathcal{A}y(t,\gamma) = \frac{\partial^2 y}{\partial \gamma^2}(t,\gamma).$$

where

$$D(\mathcal{A}) = \left\{y \in X: \frac{\partial y}{\partial \gamma}, \frac{\partial^2 y}{\partial \gamma^2} \in X \text{ and } \gamma(0) = \gamma(\pi) = 0\right\}.$$

For $y \in D(\mathcal{A})$ then $\mathcal{A}$ can be written as the following

$$\mathcal{A}y = \sum_{s=1}^{\infty} -s^2 \langle y, y_s \rangle y_s,$$

where $y_s(\gamma) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(s\gamma)$, $s = 1,2,3$ .... Then $\{y_s(\gamma)\}$ is an orthonormal basis for X and y_s is an eigenfunction corresponding to the eigenvalue $\lambda_s = -s^2$ of the operator $\mathcal{A}$, $s = 1,2,3$ ....

Therefore, $\mathcal{A}$ is the generator of C_0 −semigroup $\{\mathcal{T}(t), t \geq 0\}$ in $L^2[0,\pi]$ such that $\mathcal{T}(t)y = \sum_{s=1}^{\infty} e^{-s^2 t} \langle y, y_s \rangle y_s$, $y \in D(\mathcal{A})$, and $\|\mathcal{T}(t)\| < 1 = \mathcal{S}$. The functions h, f and q_1 defined as follows:

- $h: [0,1] \times X \to D(\mathcal{A})$ such that

$$h\big(t, y(t,\gamma)\big) = \int_0^{\gamma} \sin y(t,\zeta)\, d\zeta\,, t \in [0,1], \gamma \in [0,\pi], y \in X.$$

- $f: [0,1] \times X \to X$ such that

$$f(t, y(t,\gamma)) = \frac{t^2 e^{-t}|y(t,\gamma)|}{b}, \qquad t \in [0,1], \gamma \in [0,\pi], y \in X, b > 0.$$

- $q_1: X \to X$ such that

$$q_1(y(t_1,\gamma)) = \frac{\left|y\left(\frac{1}{2}^{-},\gamma\right)\right|}{2\left(1+\left|y(\frac{1}{2}^{-},\gamma)\right|\right)}, t \in [0,1], \gamma \in [0,\pi], y \in X.$$

According to Hille-Yosida Theorem $\|\mathcal{G}(t)\| = \|\mathcal{T}(t)\| = \mathcal{S}$.

In Example (3.1.6) we showed that The System (3.21) satisfies the conditions **H2, H3** and **H4**.

For every $u(\cdot) \in L_P(J,U)$ of the form $u(t) = \sum_{s=1}^{\infty} u_s(t) y_s$, define

$$Bu(t) = \sum_{s=1}^{\infty} \hat{u}_s(t) y_s$$

where

$$\hat{u}_s(t) = \begin{cases} 0 & 0 \le t < 1 - \frac{1}{s^2} \\ u_s(t) & 1 - \frac{1}{s^2} \le t \le 1 \end{cases}.$$

So $\|Bu\| \le \|u(\cdot)\|$. Therefore B is a bounded linear operator from $L_P(J,U)$ into X.

Now we shall prove The Condition **H6**. Consider the corresponding linear system of The System (3.21) as follows:

$$\begin{cases} {}^{C}D^{\frac{1}{2},\frac{1}{3}} y_s(t) + s^2 y_s(t) = \hat{u}_s(t), & 1 - \frac{1}{s^2} \le t \le 1 \\ \Delta y(t_1^+) = q_1\left(y(t_1^-)\right), & \end{cases}$$

Let $\wp(\cdot)$ be arbitrary element in $L_p(J,X)$ and $k \in X$ defined as

$$k = (1-\rho)\mathcal{A}_\rho \wp(1) + \rho \mathcal{A}_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega (1-\xi) \wp(\xi) d\xi.$$

Assume that $\wp(t) = \sum_{s=1}^{\infty} \wp_s(t) y_s$ and $K = \sum_{s=1}^{\infty} k_s y_s$, where $k_s = \int_0^1 e^{-s^2(1-\zeta)} \wp_s(\zeta) d\zeta$

we can choose the control function

$$u_s(t) = \frac{2s^2}{1-e^{-2}} k_s e^{-s^2(1-t)}, 1 - \frac{1}{s^2} \le t \le 1,$$

then

$$(1-\rho)\mathcal{A}_\rho Bu(1) + \rho\mathcal{A}_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega(1-\xi) Bu(\xi) d\xi$$

$$= (1-\rho)\mathcal{A}_\rho Bu(1) + \rho\mathcal{A}_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega(1-\xi) \sum_{s=1}^{\infty} \tilde{u}_s(\xi) y_s \, d\xi$$

$$= (1-\rho)\mathcal{A}_\rho Bu(1)$$

$$+ \rho\mathcal{A}_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega(1-\xi) \sum_{s=1}^{\infty} \frac{2s^2}{1-e^{-2}} k_s e^{-s^2(1-\xi)} y_s \, d\xi$$

$$= k = (1-\rho)\mathcal{A}_\rho \wp(1) + \rho\mathcal{A}_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega(1-\xi) \wp(\xi) d\xi.$$

Therefore, the first part of Condition **H6** is hold.

Now,

$$\|Bu(t)\|^2 \le \sum_{s=1}^{\infty} \int_{1-\frac{1}{s^2}}^{1} |\tilde{u}_s(t)|^2 dt$$

$$= \sum_{s=1}^{\infty} \int_{1-\frac{1}{s^2}}^{1} \left| \frac{2s^2}{1-e^{-2}} k_s e^{-s^2(1-t)} \right|^2 dt$$

$$= \sum_{s=1}^{\infty} \int_{1-\frac{1}{s^2}}^{1} \frac{4s^4}{(1-e^{-2})^2} k_s^2 e^{-2s^2(1-t)} dt$$

$$= \sum_{s=1}^{\infty} \frac{2s^2}{1-e^{-2}} k_s^2$$

$$= \frac{1}{1-e^{-2}} \sum_{s=1}^{\infty} \left(1 - e^{-2s^2}\right) \int_0^1 |\hat{\wp}_s(t)|^2 dt$$

$$\le \frac{1}{1-e^{-2}} \|\wp(\cdot)\|^2.$$

Therefore, the Condition **H6** holds.

If

$$D = \eta\left[\left(\|E^{-1}\| + \frac{1}{\Gamma\left(\frac{4}{3}\right)}\right)\mathcal{M}_h + \frac{1}{2}\left(1 + \eta\frac{1}{\Gamma\left(\frac{4}{3}\right)}\right)\mathcal{M}_f + \mathcal{M}\right] < 1,$$

and

$$\left(\mathcal{M}_f + 2\left\|\mathcal{A}_\rho^{-1}\right\|\mathcal{M}_h\right)\frac{1}{2(1-e^{-2})}\frac{\eta}{1-D}\|B\| < 1$$

then The System (3.21) is approximately controllable. ■

Chapter Four

Conclusions and Future Works

4.1 Conclusions

- This work has investigated the controllability for linear System (2.1) and nonlinear System (2.2) in finite dimensional space. The solutions were obtained using fractional calculus, Laplace transform, and the Mittag-Leffler function. The use of the controllable for the linear system together with sufficient condition on $(f(t, y(t), u(t))$ has helped us to prove the controllability of System (2.2) via Schauder Fixed Point Theorem.

- Observability of System (2.15) in finite dimension space $\mathbb{R}^n$ has been investigated in this work. First, we showed that the nonsingularity of Gramian observability matrix W_{ob} is a necessary and sufficient condition for the linear System (2.15) to be observable. For the linear System (2.15) to be observable, another condition has been established based on the rank of the matrix

$$G = \begin{bmatrix} C \\ \frac{C\left(\rho A_\rho A\right)}{\Gamma(\rho+1)} \\ \vdots \\ \frac{(n-1)!\, C\left(\rho A_\rho A\right)^{n-1}}{\Gamma(\rho(n-1)+1)} \end{bmatrix}.$$

One can show that the observability of System (2.15) could be determined without depends on time t. This is done by constant matrices A and B only. Two tests for the observability of System (2.15) are introduced which them: eigenvector of the matrix $A_\rho A$ (eigenvector test), and rank the matrix $\begin{bmatrix} A_\rho A - \lambda A \\ \gamma C \end{bmatrix}$. An important relationship between the observability and controllability of System (2.16) (Duality Theorem) is given. Examples are given for our main results.

- The exact controllability System (3.1) in a Banach space has been investigated. The mild solutions were obtained using semigroup theory, fractional calculus, and Laplace transform.

- Sufficient conditions have been established to prove the exact controllability of the nonlinear system (3.1). It has been proven by employing the Nussbaum Fixed Point Theorem. To illustrate the main results, an example was given

- The mild solution of System (3.1) has been proved exist and unique in a Banach space by employing Banach Fixed Point Theorem.

- For System (3.1), The approximate controllability was discussed using the approximate sequence method. The efficacy of our result has been shown using an example.

4.2 Future works

We will study:

1. Stability of Hattaf-fractional nonlinear dynamical system in finite dimensional space.

2. Observability of the AB-fractional linear dynamical systems in banach space.

3. Optimal control of Hattaf-fractional nonlinear control system in Banach space.

References

[1] N. Sukavanam and Divya, “Exact and approximate controllability of abstract semilinear control systems,” *Indian Journal of Pure and Applied Mathematics*, 33 (12) (2002) pp. 1827–1835.

[2] B. Ross, “The development of fractional calculus 1695-1900,” *Historia Mathematica*, 4 (1) (1977) pp. 75–89.

[3] R. Hilfer, "Applications of fractional calculus in physics", *World scientific*, Singapore, 2000.

[4] V. V. Kulish and J. L. Lage, “Application of fractional calculus to fluid mechanics,” *J Fluids Eng*, 124 (3) (2002) pp. 803–806.

[5] E. Ahmed, A. Hashish, and F. A. Rihan, “On fractional order cancer model,” *Journal of Fractional Calculus and Applied Analysis*, 3 (2) 2012 pp. 1–6.

[6] K. Krishnaveni, K. Kannan, and S. R. Balachandar, “Approximate analytical solution for fractional population growth model,” *International Journal of Engineering and Technology*, 5 (3) (2013) pp. 2832–2836.

[7] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, "Theory and applications of fractional differential equations", *Elsevier* Netherlands, 204 (2006).

[8] I. Podlubny, "An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications", *Academic Press*, Inc., San Diego CA, 198 (1999).

[9] K. Diethelm, “Analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type,” *springer-Verlag Berlin*, 2004 (2010).

[10] M. Caputo and M. Fabrizio, “A new definition of fractional derivative without singular kernel,” *Progr. Fract. Differ. Appl*, 1 (2) (2015) pp. 1–13.

[11] A. Atangana and D. Baleanu, “New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model,” *Thermal Science*, 20 (2) (2016) pp.763-769.

[12] K. Hattaf, “A new generalized definition of fractional derivative with non-singular kernel,” *Computation*, 8 (2) (2020), pp. 1–9.

[13] J. Hristov, “On the Atangana – Baleanu derivative and Its relation to the fading memory concept: The diffusion equation,” *Springer International Publishing*, 194 (2019) pp. 175–193.

[14] J. Hristov, “Derivatives with non-Singular kernels from the Caputo-Fabrizio definition and beyond: Appraising analysis with emphasis on diffusion models,” *Frontiers*, 1 (2018) pp. 269–341.

[15] M. I. Syam and M. Al-Refai, “Fractional differential equations with Atangana–Baleanu fractional derivative: Analysis and applications,” *Chaos, Solitons and Fractals: X*, 2 (2019), pp. 3–7.

[16] M. Fečkan, J.-R. Wang, and Y. Zhou, “On the new concept of solutions and existence results for impulsive fractional evolution equations,” *Dynamics of Partial Differential Equations*, 8 (4) (2011), pp. 345–361.

[17] R. E. Kalman, “Mathematical description of linear dynamical systems,” *Journal of the Society for Industrial and Applied Mathematics, Series A: Control*, 1 (2) (1963) pp. 152–192.

[18] M. Nawaz, W. Jiang, J. Sheng, A. U. K. Niazi, and Y. Lichang, “On the controllability of nonlinear fractional system with control delay,” *Hacettepe Journal of Mathematics and Statistics*, 49, (1) (2020) pp. 294–302.

[19] J. Sheng, W. Jiang, D. Pang, and S. Wang, "Controllability of nonlinear fractional dynamical systems with a Mittag–Leffler kernel," *Mathematics*, 8, (12) (2020), pp. 2139.

[20] M. Ghasemi and K. Nassiri, "On controllability of fractional continuous-time systems," *Mathematical Problems in Engineering*, 2021 (2021).

[21] D. Aimene, D. Baleanu, and D. Seba, "Controllability of semilinear impulsive Atangana-Baleanu fractional differential equations with delay," *Chaos, Solitons & Fractals*, 128 (2019) pp. 51–57.

[22] H. Qin, X. Zuo, and J. Liu, "Existence and controllability results for fractional impulsive integrodifferential systems in Banach spaces," *Abstract and Applied Analysis*, 2013 (2013).

[23] V. S. Muni and R. K. George, "Controllability of semilinear impulsive control systems with multiple time delays in control," *IMA Journal of Mathematical Control and Information*, 36 (3) (2019) pp. 869–899.

[24] X. Li, Z. Liu, and C. C. Tisdell, "Approximate controllability of fractional control systems with time delay using the sequence method," *Electron. J. Differential Equations*, 272 (2017) pp. 1–11.

[25] J. Du, D. Cui, Y. Sun, and J. Xu, "Approximate controllability for a kind of fractional neutral differential equations with damping," *Mathematical Problems in Engineering*, 2020 (2020).

[26] T. Kaczorek and Ł. Sajewski, "Relationship between controllability and observability of standard and fractional different orders discrete-time linear system," *Fractional Calculus and Applied Analysis*, 22 (1) (2019), pp. 158–169.

[27] K. Balachandran, V. Govindaraj, M. D. Ortigueira, M. Rivero, and J. J. Trujillo, "Observability and controllability of fractional linear dynamical systems," *IFAC Proceedings Volumes*, 46 (1) (2013) pp. 893–898.

[28] E. Kreyszig, "Introductory functional analysis with applications", *John Wiley & Sons*, USA, 17 (1991).

[29] C. Heil, "Metrics, Norms, Inner Products, and Operator Theory," *Springer*, Switzerland, 6 (2018).

[30] R. Kress, V. Maz'ya, and V. Kozlov, "Linear integral equations," *Springer*, Berlin, 82 (1989).

[31] R. Precup, "Methods in nonlinear integral equations". S*pringer Science & Business Media*, Netherlands, 2002.

[32] A. Pazy, "Semigroups of linear operators and applications to partial differential equations*", Springer Science & Business Media*, USA, 44 (2012).

[33] R. F. Curtain and H. Zwart, "An introduction to infinite-dimensional linear systems theory," *Springer Science & Business Media*, New York, 21 (2012).

[34] V. I. Istrăţescu, "Fixed point theory,*" of Mathematics and its Applications, D*. Reidel Publishing Co., Dordrecht, 7 (1981).

[35] R. D. Nussbaum, "The fixed point index and asymptotic fixed point theorems for k-set-contractions," *Bulletin of the American Mathematical Society*, 75 (3) (1969) pp. 490–495.

[36] D. Zezislaw and S. Migórski, "An Introduction to Nonlinear Analysis: Applications". *Kluwer Academic*, Dordrecht, 2003.

[37] J. Macki and A. Strauss, "Introduction to optimal control theory". S*pringer Science & Business Media*, New York, 2012.

[38] R. K. George and T. IIST, "Controllability, observability, stability and stabilizability of linear systems," 2015.

[39] R. Triggiani, “A note on the lack of exact controllability for mild solutions in Banach spaces,” *SIAM Journal on Control and Optimization,* 15 (3) (1977) pp. 407–411.

[40] K. Naito, “Controllability of semilinear control systems dominated by the linear part,” *SIAM Journal on control and Optimization,* 25 (3) (1987) pp. 715–722.

[41] C.-T. Chen, “Linear system theory and design,” *Oxford University Press Inc,* USA 1999.

[42] J. P. Dauer, “Nonlinear perturbations of quasi-linear control systems,” *Journal of Mathematical Analysis and Applications*, 54, (3) (1976) pp. 717–725.

[43] G. Strang, "Linear algebra and its applications". *Thomson, Brooks/Cole*, Belmont, CA, 2006.

[44] A. J. Laub, "Matrix analysis for scientists and engineers,". *Siam,* 91 (2005).

[45] F. Mainardi, P. Paradisi, and R. Gorenflo, “Probability distributions generated by fractional diffusion equations,” *Econophysics: An Emerging Science (J. Kertesz & I. Kondor eds). Dordrecht: Kluwer*, 2003.

[46] Zhou, Yong, and Feng Jiao. "Existence of mild solutions for fractional neutral evolution equations." *Computers & Mathematics with Applications* 59 (3) (2010) pp.1063-1077.

المستخلص

تهدف هذه الأطروحة إلى دراسة وتطوير قابلية التحكم وقابلية المراقبة لأنظمة تحكم كسرية ذات نواة غير شاذة في فضاءات ذات أبعاد منتهية وغير منتهية.

خلال هذا العمل، تم إثبات قابلية التحكم في نظام تحكم ديناميكي خطي كسري من النوع AB مع تأخير التحكم في ظل شرط كاف. تم وضع شروط كافية لإثبات أن نظام تحكم ديناميكي غير خطي كسري من النوع AB مع تأخير التحكم يمكن التحكم فيه باستخدام نظرية النقطة الصامدة لشودر.

تم التحقق من قابلية مراقبة نظام تحكم ديناميكي خطي كسري من نوع AB. حيث تم تقديم أكثر من معيار واحد لها. تم إثبات الازدواجية بين قابلية التحكم وقابلية المراقبة.

تم تقديم حل معتدل لنظام تحكم تسارعي غير خطي يتضمن مشتق هتاف الكسري في فضاء باناخ باستخدام حساب التفاضل والتكامل الكسري ونظريات شبه الزمرة. في ظل شروط كافية، أثبتنا قابلية التحكم في هذا النظام. تم الحصول على نتائجنا الرئيسية باستخدام نظرية النقطة الصامدة لنوسباوم. من ناحية أخرى، تم تقديم شروط كافية لإثبات وجود ووحدانية الحل المعتدل للنظام غير الخطي. أيضًا، تم إثبات قابلية التحكم التقريبية على هذا النظام في فضاء باناخ.

أخيرًا، تم عرض تطبيقات لتوضيح أهمية النتائج الرئيسية.

جمهورية العراق
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جامعـــــــة بابـــــــل
كليـــة التربيـــة للعلـــوم الصـــرفة
قســـــــم الرياضـــــــيات

قابلية التحكم في حلول النظم الدينميه

أطروحة

مقدمة الى مجلس كلية التربية للعلوم الصرفة في جامعة بابل كجزء
من متطلبات نيل درجة الدكتوراه فلسفة في التربية / الرياضيات

من قبل
فاضل عباس ناجي عنيزان

بإشراف
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