

Republic of Iraq
Ministry of Higher Education
and Scientific Research
University of Babylon
College of Education for Pure Sciences
Department of Mathematics



Some Kinds of Mobius Function Graphs

A Thesis

Submitted to the Council of the College of Education for Pure Sciences' University of Babylon as a Partial Fulfillment of the Requirements for the Degree of Master in Education /Mathematics

By

Sanaa Kadum Kamel Yaseen

Supervised

By

Prof. Faez A.AL-maamori (Ph. D.)

Prof. Ahmed Abed Ali Omran (Ph. D.)

2022 A.D

1444 H.D

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿قُلِ اللَّهُمَّ مَالِكِ الْمُلْكِ تُؤْتِي الْمُلْكَ مَنْ تَشَاءُ وَتَنْزِعُ الْمُلْكَ مِمَّنْ
تَشَاءُ وَتُعْزِزُ مَنْ تَشَاءُ وَتُنْزِلُ مَنْ تَشَاءُ بِيَدِكَ الْخَيْرُ إِنَّكَ عَلَىٰ كُلِّ
شَيْءٍ قَدِيرٌ﴾

صدق الله العظيم

سورة آل عمران

الآية (26)

Acknowledgments

*First and foremost, I would like to thank **Allah** Almighty for his graces that countless, without his mercy, I would not have reached what I am ...*

*I extend my deepest thanks to my beloved **Iraq** and to my second home, **University of Babylon**.*

*I would like to express my sincere thanks and gratitude to my supervisor **Prof. Faez A.AL-maamori (Ph.D.)** and supervisor **Prof. Dr. Ahmed Abd Ali Omran** for his invaluable guidance and constant encouragement at every stage of my thesis and for being my spiritual father. I thank him for giving me the opportunity to work with him.*

*I would also like to thank all the staff of the **College of Education for Pure Sciences**, especially the teaching staff.*

*I would like to thank **my family** who encouraged thank my dear husband who stood and supported me throughout my work and were the source of my real inspiration, especially my dear father and mother.*

*Finally, all thanks and appreciation to **my friends** and everyone who encouraged me or helped me morally*

Thank you for all...

Sanaa 2022

DEDICATIONS

To the Prophet of Mercy “Mohammad”

To the moon, my father

To the sun, my mother

To the stars, my brother and sisters

To my family who have endured so much for me ..

Researcher

2022

Supervisor's Certification

I certify that the thesis entitled “Concepts of Number Theory with Graph Theory” by “Sanaa Kadum Kamel Yaseen ” has been prepared under my supervision in Babylon University/ College of Education for Pure Sciences as a partial requirement for the degree of Master in Education / Mathematics.

Signature:

Name: Dr. Faez Ali AL-Maamori

Signature:

Name: Dr. Ahmed Abid Ali Omran

Title: Professor

Date: / / 2022

In view of the available recommendation, I forward this thesis for debate by the examining committee.

Signature:

Name: Dr. Azal Jaafar Musa

Head of Mathematics

Department Title: Assistant Professor

Date: / / 2022

Certification of linguistic Expert

I certify that I have read this thesis entitled “*Concepts of Number Theory with Graph Theory*” and corrected its grammatical mistakes; therefore, it has qualified for debate.

Signature:

Name:

Title: Teacher

Date: / / 2022

Certification of Scientific Expert

I certify that I have read the scientific content of this thesis “*Concepts of Number Theory with Graph Theory*” and I have approved this dissertation is qualified for debate.

Signature:

Name: Dr. Manal Naji Al-Harere

Title: Professor

Date: / /2022

Certification of Scientific Expert

I certify that I have read the scientific content of this thesis “*Concepts of Number Theory with Graph Theory*” and I have approved this dissertation is qualified for debate.

Signature:

Name: Dr. Fatima Faisal Abd Al-Karim

Title: Assistant Professor

Date: / /2022

Examining Committee Certification

We certify that we have read the thesis entitled " **Some Kinds of Mobius Function Graphs**" by" **Sanaa Kadum Kamel Yaseen** " and as a committee examined the student in its contents and, according to our opinion, it is accepted as a thesis for the degree of Master in Education / Mathematics.

Signature:

**Name: Asaad Mohammed Ali
Allah**

Title:

Date: / / 2022

Chairman

Signature:

Name: Khaled Shia Khair

Title:

Date: / / 2022

Member

Signature:

**Name: Faez Ali AL-maamori
Omran**

Title:

Date: / / 2022

Member /

Signature:

Name: Ahmed Abed Ali

Title:

Date: / / 2022

Member / Advisor

Signature:

Name: Ali Younes Shaker

Title:

Date: / / 2022

Member /

Approved by the Dean of College

Signature:

Name: Dr. Bahaa Hussein Salih Rabee

Title: Professor

Address: Dean of the College of Education for Pure Sciences

Date: / / 2022

Contents

Contents	I
List of Symbols	III
List of Figures	IV
Abstract	VI
Introduction	1
List of publications	6
1. Chapter 1: Basic Definitions and Concepts	
1.1 Introduction	7
1.2 Basic Definitions	7
1.3 Domination in Graph Theory	17
2. Chapter 2: Some Properties of Mobius function Graph	
2.1 Analytic the Mobius Function Graph $\mathcal{M}^{(0)}$ in Graph Theory	21
2.2 Some Properties of Mobius Function Graph $\mathcal{M}^{(1)}$	24
2.3 Some Properties of Mobius Function Graph $\mathcal{M}^{(-1)}$	32
3. Chapter 3: Complement the Mobius Function Graph in Graph Theory	

3.1 Complement the Mobius Function Graph $(\mathcal{M}^{(0)})^c$ in Graph Theory	41
3.2 Main results	41
3.3 Complement the Mobius Function Graph $(\mathcal{M}^{(1)})^c$ in Graph Theory	50
3.4 Complement the Mobius Function Graph $(\mathcal{M}^{(-1)})^c$ in Graph Theory	56
3.5 Comparing the six types of Mobius function Graph in Graph theory	62

4. Chapter 4: Conclusion and Future Work

4.1 Conclusions	66
4.2 Future Works	67
Reference	68

List of Symbols

Symbol	Description
$G (V_G, E_G)$	graph of order n and size m
$V(G)$	vertex set in a graph G
$E(G)$	edge set in a graph G
$ V $	number of vertices in a graph G
$ E $	number of edges in a graph G
$ A $	cardinality of a set a
$\lceil A \rceil$	integer $\geq A$ (ceiling)
$\lfloor A \rfloor$	integer $\leq A$ (floor)
$deg(v)$	degree of vertex v in a graph G
$\delta(G)$	minimum degree of vertices in G
$\Delta(G)$	maximum degree of vertices in G
$d(v,u)$	shortest path joining v and u
$N(v)$	open neighborhood of v in a graph G
$N[v]$	closed neighborhood of v in a graph G
$\langle H \rangle$ or $G[H]$	induced subgraph of a graph G
G^c	complement of graph G
P_n	path graph
N_n	Null graph
$\gamma(G)$	Domination number
$\beta(G)$	Independence number
$\chi(G)$	Chromatic number
$\omega(G)$	clique number
$\mathcal{M}(n)$	Mobius function
$\pi(n)$	Gauss's function
C_n	Cycle graph
K_n	Complete graph
D	Dominating set
\bar{D}	Minimal dominating set
N_e^n	The set of even number
S	Isolated vertices
$\text{g.c.d}(u,v)$	greatest common divisor of u,v

$\text{Rad} (\mathcal{M}^{(0) c})$	The radius of complement for mobius function graph
$\text{Diam} (\mathcal{M}^{(0) c})$	The diameter of complement for mobius function graph
$\text{Cent} (\mathcal{M}^{(0) c})$	The center of complement for mobius function graph
$\text{Per} (\mathcal{M}^{(0) c})$	The peripheral of complement for mobius function graph
$e(v)$	eccentricity
S_1	Prime number $\{ \pi_{i=1}^k P_i ,k \text{ is odd} \}$
\square	End of the proof

List of Figures

Abbreviation	Description	Page
Figure 1.1	Seven bridges in Kaliningrad (Konigsberg), Russia	8
Figure 1.2	(6,7) –graph	9
Figure 1.3	A graph has loops and multiple edge	10
Figure 1.4	Cycle graphs C_3, C_4, C_5 and C_6 .	11
Figure 1.5	The union and join two graphs G_1 and G_2 .	12
Figure 1.6	Complete graphs K_1, K_2, K_3, K_4 and K_5 .	12
Figure 1.7	Subgraph, induced subgraph and spanning subgraph.	13
Figure 1.8	A cycle graph and its complement.	14
Figure 1.9	Null graph of K_6 .	14
Figure 1.10	Some tree graphs.	14
Figure 1.11	Three isomorphic graphs.	15
Figure 1.12	Five Queen problem.	17
Figure 2.1.1	The Mobius function graph $\mathcal{M}^{(0)}$ of order 12.	21
Figure 2.2.1.	The Mobius function graph $\mathcal{M}^{(1)}$ of order 12.	24
Figure 2.2.2.	The clique number to Mobius function graph $\mathcal{M}^{(1)}$.	25
Figure 2.2.3.	The chromatic number to Mobius function graph $\mathcal{M}^{(1)}$	26
Figure 2.2.4	The domination number of $\mathcal{M}^{(1)}$ of order n .	29

Figure 2.3.1.	The Mobius function graph $\mathcal{M}^{(-1)}$ of order 12.	32
Figure 2.3.2.	The domination number of $\mathcal{M}^{(-1)}$ of order n	35
Figure 3.2.1.	$MFG\mathcal{M}^{(0)}(A), \mathcal{M}^{(1)}(B), \mathcal{M}^{(-1)}(C)$	43
Figure 3.2.2.	The $MFG(\mathcal{M}^0)^c$ of order 16.	44
Figure 3.2.3	The $MFG(\mathcal{M}^0)^c$ of order 16.	46
Figure 3.3.1.	The $MFG(\mathcal{M}^1)^c$ of order 16.	50
Figure 3.3.2	The $MFG(\mathcal{M}^1)^c$ of order 16.	52
Figure 3.4.1.	The $MFG(\mathcal{M}^{-1})^c$ of order 16.	55

Abstract

Throughout this thesis, many properties of the Mobius function graph \mathcal{M}^0 are discussed. Also, new graphs called the Mobius Function Graph \mathcal{M}^1 and \mathcal{M}^{-1} are introduced. More than three ways of determining the prime-counting function by using these graphs are presented.

Also, some properties of these graphs are proved. Moreover, the domination number, chromatic number, independence number, and clique number of these graphs are determined. Additionally, a comparison was made between the two numbers the domination and independence of these graphs also. After this, we show that the three graphs \mathcal{M}^0 , \mathcal{M}^1 , and \mathcal{M}^{-1} constitute a decomposition of the complete graph of order n , where n is the order of each one of these graphs. So, new graphs are defined depending on the above decomposition which are called the complement Mobius function graphs. The first one is the complete Mobius function graphs $(\mathcal{M}^0)^c$ which is the union of two graphs Mobius Function Graph \mathcal{M}^1 and Mobius Function Graph \mathcal{M}^{-1} , the second is the complete Mobius function graphs $(\mathcal{M}^1)^c$ which is the union of two graphs Mobius Function Graph \mathcal{M}^0 and Mobius Function Graph \mathcal{M}^{-1} , and the last one is the complete Mobius function graphs $(\mathcal{M}^{-1})^c$ which is union of two graphs Mobius Function Graph \mathcal{M}^0 and Mobius Function Graph \mathcal{M}^1 . For each graph of the three graphs mentioned above the relationship between this graph with two graphs that constitute this graph is discussed. Moreover, The clique number, independence number, and domination number are determined. Also, compare this numbers with corresponding numbers in two graphs that constitute this graph. Also, the chromatic number, the Cent (\mathcal{M}^{i^c}) , and Per (\mathcal{M}^{i^c}) ; $i=0,1,-1$ graphs are founded with some properties.

Introduction

In all the constantly evolving sides of life, we need number theory to solve the problems that are related to evolution. The mathematicians create two major branches of number theory, multiplicative number theory which deals with the distribution of the primes and Additive Number Theory which is concerned with the additive structure of the integers.

And as time passes on, it became necessary to introduce analytic number theory by using mathematical analysis to deal with integers. Number theory considers one of the cores of pure mathematics. As a part of it, the study of integers in general "primes in particular" and the integer-valued functions using the mathematical analysis falls under the brunch "Analytic Number Theory". Number Theory was initially one of the ancient Greeks' concerns. Pythagoras's school's mathematicians (500 BC to 300 BC) were interested to study numbers for their mystical and numerological properties. They understood the idea of primality and were interested to study the perfect numbers as well which are defined later. Then, in 1737, the Swiss Euler presented a relation between the prime numbers and zeta function and With this relationship, he put forward a theory "Analytic Number Theory" by using the techniques from mathematical analysis. After one hundred years, the German mathematician Dirichlet, 1837 published his prime number theorem which states that: [for any two prime numbers p and q , there are infinite prime numbers that congruent with p modulo q]. He also used tools from mathematical analysis in the proof. Chebyshev and Riemann later played an important role in improving the concepts and theories about prime numbers. Riemann especially made a breakthrough in analytic math by conjecting that all non-trivial zeros of the zeta function are on the critical line $u = \frac{1}{2}$ and that is what's known now by Riemann hypothesis.

It is considered one of the most important open problems because of its role in different fields of sciences. The Russian Chebyshev, played an important role in Prime Number Theorem proving and proved Bertrand's postulate which states that: [for every integer $n > 1$, there is at least one prime between n and $2n$]. In number theory, we will deal with a number of functions, including: the mobius function, and link this function from the graph theory [1].

Graph theory is a branch of discrete mathematics that is of great importance in solving many complex problems in various fields of science. In 1736, the cornerstone of the graph theory was laid when Swiss scientist Leonard Euler published his paper on the problem "Seven Bridges of Königsberg" where he drew the first graph. In this sense, graphs are very convenient tools to convert any subject represented by vertices while the connections between them are edges. This has opened the way for introducing and publishing many graphs to solve multiple problems including, computer science and networking. The graph theory has become rich and interesting because of the problems that have grown exponentially in recent years which require much effort and time to solve, but the graph theory was present in solving such problems. Therefore, the graph theory proved to be an important mathematical tool. The graph has branched out in applications in almost all fields of science and engineering that can be used for example, electrical networks, encryption, wireless communications, image processing, computer science, especially used for study algorithms and traffic systems. Its beauty has attracted many researchers to adopt a graph theory to develop the model with other areas of applied mathematics and a variety of scientific fields including chemistry, physics, and life sciences. In addition to social sciences such as business administration, sociology, economics, marketing, etc... For more information see books [7,23]. There are many topics in the graph theory such as graphic colors, mutant theory, domination theory, and numbering diagram. After more than a century of Euler's paper, a new

concept was introduced called “Domination”. Where the inspiration was the nucleus of research activity in recent times due to a variety of criteria that can descend from the basic definition of domination. Domains of domination were to answer the question: “what are the minimum pieces (queens specifically) to control the chessboard”. It is noteworthy that the queen can move vertically, horizontally and diagonally, De Jaenisch, in 1862, considered the problem of finding the minimum number of queens that can be placed on a chessboard so that each square is either 2 occupied by a queen or can be occupied by the queen. One step, it turns out that five is the minimum number of queens. Currently, graph theory is one of the most important tools used in proofs for most sciences, including medical, engineering, computer, chemistry, physics, and others. Moreover, most fields in mathematics deal with graph theory to find a new solution or alternative solution such as fuzzy graphs [4] and [16], topological graph [19], labeled graph [23], general graphs [18], [22-29], and [33-34], topological indices graphs [1-2], and others. In this work, the relationship between graph theory and number theory is introduced. There are many numbers in graph theory used such as independence, domination, chromatic, and clique. A subset of the vertex set of a graph is called dominating if each vertex out of this subset is adjacent to at least one vertex in this subset. The domination number is a minimum cardinality of a dominating set and is denoted by $\gamma(G)$ [36]. The independence number denoted by $\beta(G)$ is a maximum cardinality of independent sets. The chromatic number is the smallest number of colors can assignments to all vertices such that each pair of adjacent vertices have different color and denoted by $\chi(G)$ [26].

A clique number is the cardinality of the largest complete subgraph of a graph G and is denoted by $\omega(G)$ [36].

The number theory is an important branch of mathematics where it studied the properties of most numbers, their relationships and their importance, and

suggested a lot of functions that depend on numbers [6-9]. Among these functions is a Mobius function Which is definition as the following, If

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}, \text{ then } \mathcal{M}(n) = \begin{cases} (-1)^r, & \text{if } \alpha_1 = \alpha_2 = \dots = \alpha_r \\ 0, & \text{otherwise} \end{cases}$$

Srimitra and et al. [22] defined the Mobius function $\mathcal{M}^{(0)}$ graph, the vertex of this graph is the natural number and two vertices u and v are adjacent if $\mathcal{M}(uv) = 0$ and get many properties of this graph. Omran and et al. [26] presented the Mobius function $\mathcal{M}^{(1)}$ graph, the vertex of this graph is the natural number and two vertices u and v are adjacent if $\mathcal{M}(uv) = 1$ and discussed various numbers in a graph such as domination, independence, clique, number and many properties of this graph. Moreover, Omran and et al. [28] introduced the Mobius function $\mathcal{M}^{(-1)}$ graph, the vertex of this graph is the natural number and two vertices u and v are adjacent if $\mathcal{M}(uv) = -1$ and again discussed the numbers and properties mentioned above. In this work, two ways are presented to determine the prime-counting function $(\pi(n))$ [23]. Also, the graph Mobius function $(\mathcal{M}^0)^c$ is initiated, the vertex of this graph is the natural number and two vertices u and v are adjacent if $\mathcal{M}(uv) = -1$ or $\mathcal{M}(uv) = 1$. The important numbers in graph theory are determined such as domination, independence, and clique numbers and compare these numbers with numbers obtained in the Mobius function $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(-1)}$ and get interesting results. Additionally, the graphs $\mathcal{M}^0, \mathcal{M}^1, \text{ and } \mathcal{M}^{-1}$ are decomposition of the complete graph of order n is proved. Also, chromatic number, $Rad((\mathcal{M}^0)^c), Diam((\mathcal{M}^0)^c), Cent((\mathcal{M}^0)^c),$ and $Per((\mathcal{M}^0)^c)$ are calculated. The reader can be found all concepts not here in [15] and [14]. For all the following we use MFG as short to Mobius function graph and the labeled function of vertices is defined as follows $f(v_i) = i, \forall i$ and S is the set of the isolated vertices in G .

Chapter one

History the basic subjects used in this study is introduced in chapter one. Furthermore, some basic definitions of graph theory that we will need later are discussed, as well as some definitions of domination in graphs and forms of domination parameters used in this work, in addition to a previous study of some domination types.

- ***Chapter two***

In this chapter, the new concepts entitled “Mobius function graph \mathcal{M}^1 and \mathcal{M}^{-1} and \mathcal{M}^0 ” Also, some properties for Mobius function graph are presented.

Chapter three

In this chapter “Complement the Mobius Function Graph $(\mathcal{M}^0)^c$, $(\mathcal{M}^1)^c$ and $(\mathcal{M}^{-1})^c$ in Graph Theory” are presented. In addition, some properties of them are provided and “Comparing the six types of mobius function graph” are introduced.

List of Publications Arising from This Thesis

1)"**SOME PROPERTIES OF MOBIUS FUNCTION GRAPH \mathcal{M}^1** "
Advanced Mathematical Models & Applications Vol.7, No.1, 2022,
pp.48-54 .

2)" **Some properties of Mobius Function Graph \mathcal{M}^{-1} in Graphs"**
Natural Sciences 01-Applied Mathematics & Information
Sciences Vol.16, No.4 , pp:655-658 (2022)
.

CHAPTER ONE

Chapter One

Basic Definitions and Concepts

1.1 Introduction

In this chapter, the relevant definitions, basic concepts, required background for this thesis are introduced. Two sections in this chapter. The first section contains some basic definitions and notations of graph theory that we need in the thesis. the second section gives definition of domination number with some types and properties of domination number.

1.2 Basic Definitions

The city's people were wondering whether it was possible to pass on the seven bridges only once and return to the starting point without repeating any bridge in 1736, Leonard Euler (1707-1783) came up with an answer to this question, proving that it was not possible to pass on the seven bridges at exactly the same time Use Euler to answer the question a simple way by eliminating all the unnecessary features of the city. He drew a picture of the city where he represented the land b vertices (or points) and bridges by edges (or lines) as shown in Fig.1.1(c). This mathematical structure is called the graph. From these simple assets, the theory of graph has grown into a strong mathematical theory in mathematics and solved many complex life problems.

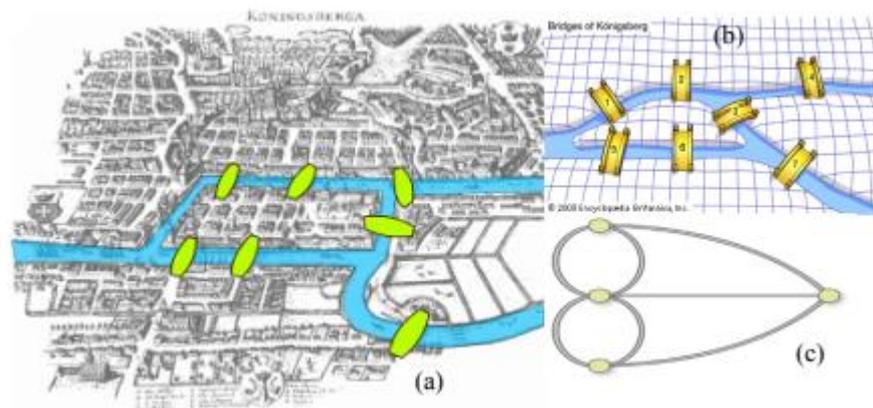


Figure 1.1: Seven bridges in Kaliningrad (Konigsberg), Russia

In view of the above, there is a need to know the basic concepts of the graph on which we depend on in thesis.

Definition 1.2.1. [43] A **graph** $G = (V(G), E(G))$ or $G = (V, E)$ consists of two sets $V(G)$ or V , the vertex set of the graph, which is a non-empty set of elements called vertices (or points) and $E(G)$ or E the edge set of the graph, which is a possibly empty set of elements called edges (or lines), such that each edge e in E is assigned as an unordered pair of vertices called the end vertices of e . A graph G with n vertices and m edges is called a (n, m) -graph.

Definition 1.2.2. [43] A graph G is called **undirect graph** when the pair of vertices representing any edge is unordered.

Example 1.2.3. A graph G of 6 vertices and 7 edges such that

$V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ a set of vertices and

$E(G) = \{v_1v_2, v_1v_5, v_1v_6, v_2v_3, v_2v_4, v_3v_4, v_5v_6\}$ a set of edge of a graph G or $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$.

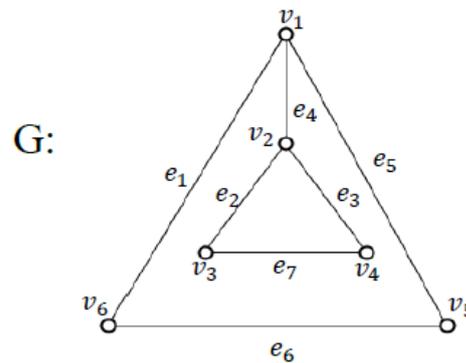


Figure 1.2: $(6,7)$ –graph.

Definition 1.2.4 [6] The **order** of G is $n = |V(G)|$ and the **size** of G is $m = |E(G)|$. In Fig 1.2, order and **size** of G is 6 and 7 respectively.

Definition 1.2.5 [6] A graph G is **trivial** if a vertex set of G is a singleton and its contains no edges.

Definition 1.2.6 [43] A **finite graph** is a graph that has finite number of vertices and finite number of edges. Otherwise, a graph is called infinite graph.

Definition 1.2.7 [20] If two vertices of a graph are joined by an edge then these vertices are called **adjacent vertices**.

Definition 1.2.8 [20] If two or more edges of a graph have a common vertex then these edges are called **adjacent edges**. Note that if $e = vu$ is an edge of G , then e is incident to u and we also say that u and v are the endpoints of e .

Definition 1.2.9 [24] The **open neighborhood** $N(v)$ of the vertex v consists of the set vertices adjacent to v , that is, $N(v) = \{u \in V : vu \in E\}$, and the **closed neighborhood** of v is $N[v] = N(v) \cup \{v\}$.

In Fig.1.2, $N(v_1) = \{v_2, v_5, v_6\}$ and $N[v_1] = \{v_1, v_2, v_5, v_6\}$.

Definition 1.2.10 [20] The **degree** of a vertex v in a graph G , denoted by $\deg(v)$ is the number of edges incident with v . For example, in Fig.1.2, $\deg(v_1) = 3$ i.e. $\deg(v) = |N(v)|$.

Definition 1.2.11 [40] The **maximum degree** of a graph G , denoted by $\Delta(G)$ is the maximum value among the degrees of all the vertices of G , i.e., $\Delta(G) = \max_{v \in V(G)} \deg(v)$.

Similarly, we define the **minimum degree** of a graph G and denote it by $\delta(G)$, i.e., $\delta(G) = \min_{v \in V(G)} \deg(v)$. In Fig.1.2, $\Delta(G) = 3$ and $\delta(G) = 2$.

Definition 1.2.12 [6] A vertex of degree 0 is an **isolated** vertex of G , A vertex of degree 1 is called a **pendant** vertex of G (or end-vertex or leaf), and the unique edge of G incident to such a vertex of G is a pendant edge of G .

Definition 1.2.13 [43] If for some positive integer k , $\deg(v) = k$ for every vertex v of the graph G , then G is called **k -regular**.

Definition 1.2.14 [40] A **loop** is an edge whose end-vertices are the same.

Definition 1.2.15 [40] **Multiple edges** are edges with the same pair of end vertices.

Example 1.2.16 In Fig.1.3 show loops and multiple edge where the edges e_1 and e_2 are loops, while the edges e_3 and e_4 are multiple edges.

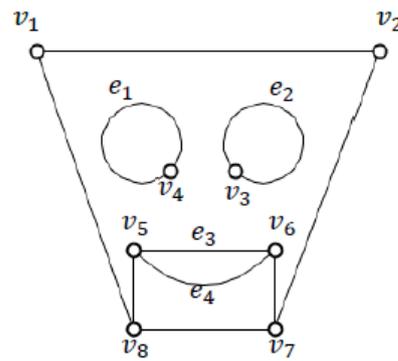


Figure 1.3: A graph has loops and multiple edges.

Definition 1.2.17 [43] Simple graph is **undirected** graph without loops and multiple edges. In other words, simple graph is a pair $G = (V, E)$ where V is an arbitrary set and E is a set of unordered pairs of distinct elements from V .

Definition 1.2.18 [9] A $u - v$ **walk** W in G is a sequence of vertices in G beginning with u and ending at v such that consecutive vertices in the sequence are adjacent.

Definition 1.2.19 [9] A $u - v$ walk in a graph in which no vertices are repeated is a $u - v$ **path** and denoted by P_n .

Definition 1.2.20 [40] A **cycle** graph is one that is obtained by joining the two end-vertices of a path graph. Thus, the degree of each vertex of a cycle graph is two. A cycle graph with n vertices is often denoted by C_n .

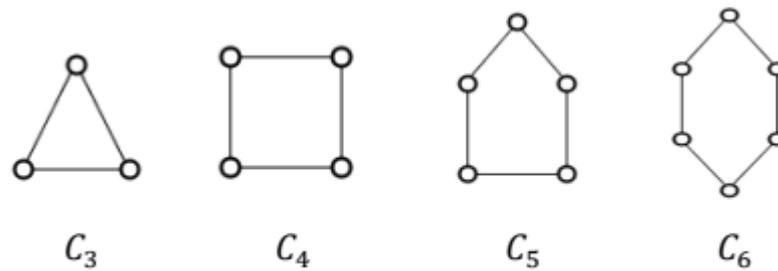


Figure 1.4: Cycle graphs C_3, C_4, C_5 and C_6 .

Definition 1.2.21 [40] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The **union** of G_1 and G_2 , denoted by $G_1 \cup G_2$, is another graph $G = (V, E)$ whose vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. (See Fig.1.5).

Definition 1.2.22 [24] The **join** $G = G_1 + G_2$, where G_1 and G_2 are disjoint graphs is a graph which has a vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{ uv : u \in V(G_1), v \in V(G_2) \}$.

(See Fig. 1.5).

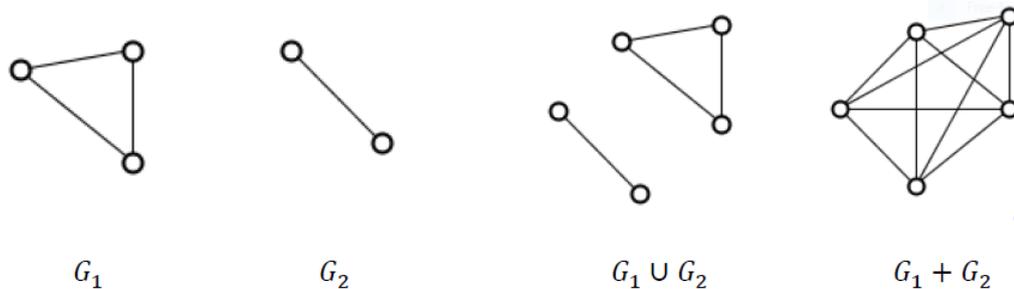


Figure 1.5: The union and join two graphs G_1 and G_2 .

Definition 1.2.23 [9] A graph G is **complete** if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n . (Fig.1.6 (show complete graph where $n = 1, \dots, 5$)).

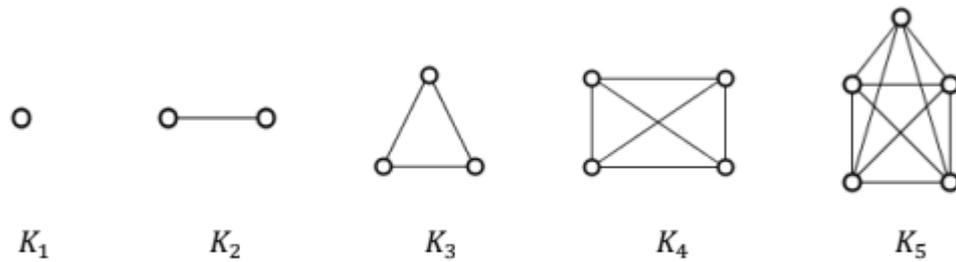


Figure 1.6: Complete graphs K_1, K_2, K_3, K_4 and K_5 .

Definition 1.2.24 [24] A graph G is **connected** if for every pair u, v of vertices there exist a $u - v$ path, otherwise, G is disconnected.

Definition 1.2.25 [43] A graph H is a **subgraph** of G if every vertex of H is a vertex of G , and every edges of H is an edge of G . In other words,

$$V(H) \subseteq V(G) \text{ and } E(H) \subseteq E(G).$$

Definition 1.2.26 [20] A **spanning subgraph** is a subgraph containing all the vertices of G .

Definition 1.2.27 [6] A subgraph H of G is said to be an **induced subgraph** of G if each edge of G having its ends in $V(H)$ is also an edge of H . Such H is denoted by $G[H]$ also denoted by $\langle H \rangle$ G or simply by $\langle H \rangle$.

Example 1.2.28: In Fig.1.8 we show types of **subgraphs** of a graph G such that G_1 and G_2 are induced subgraph, while G_3 and G_4 are spanning subgraph and G_5 is subgraph but neither induced nor spanning subgraph.

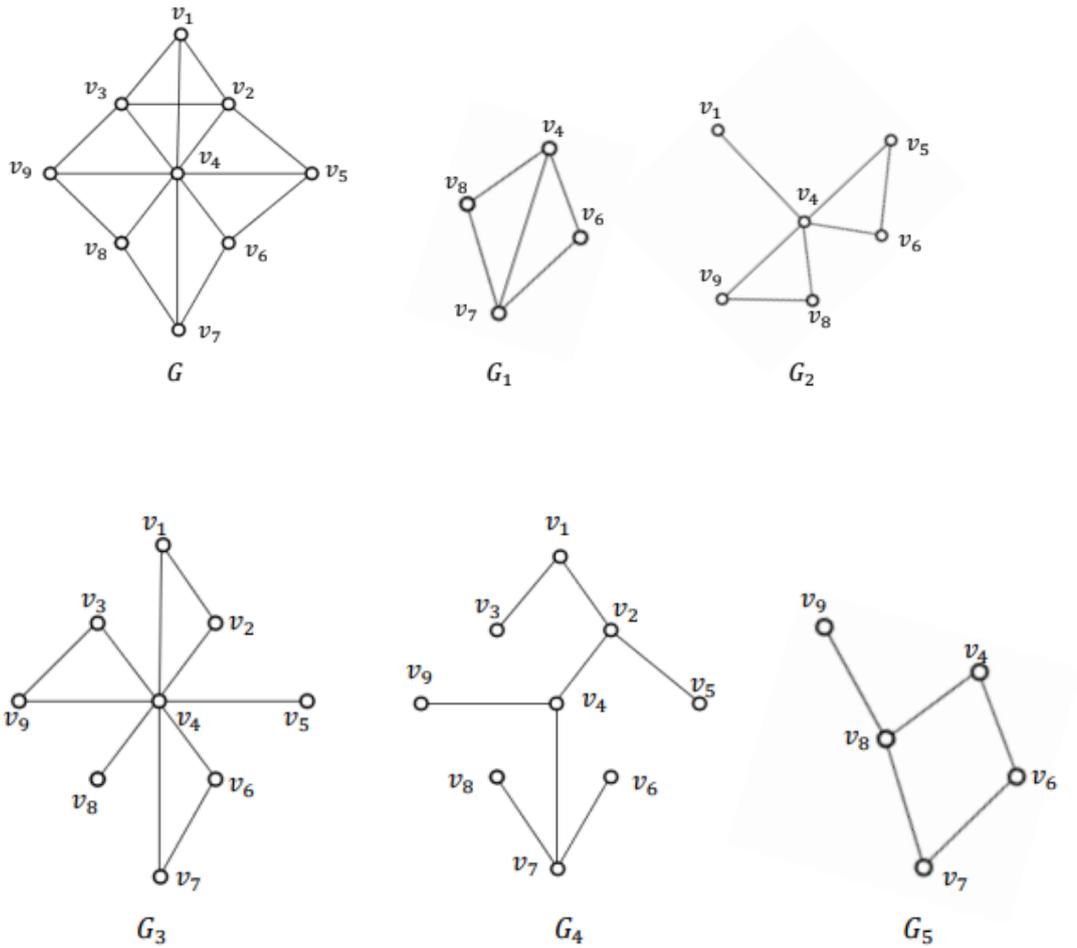


Figure 1.7: G_1 Subgraph, G_2, G_5 induced subgraph and G_3, G_4 spanning subgraph.

Definition 1.2.29 [20] The **complement** G^c of a graph G also has $V(G)$ as its vertex set, but two vertices are adjacent in G^c if and only if they are not adjacent in G . (For example, see Fig.1.8)

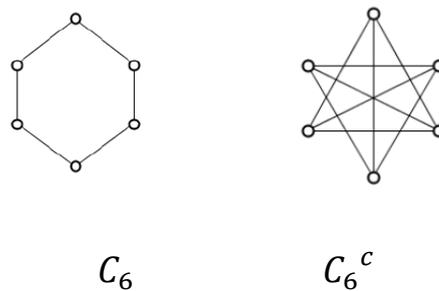


Figure 1.8: A cycle graph and its complement.

Definition 1.2.30 [43] If $E = \emptyset$, in a graph $G (V, E)$ then such a graph without any edges is called a **null graph** and denoted by N_n with n vertices. (An example can be seen in Fig.1.9).

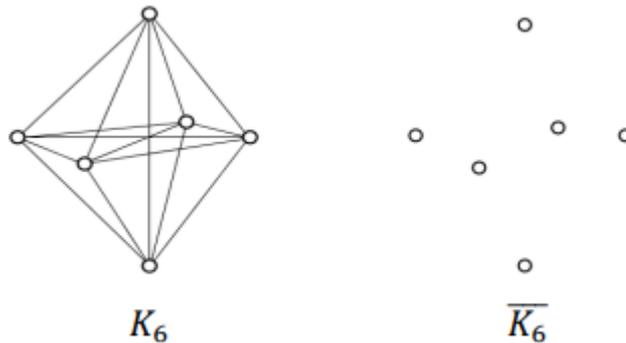


Figure 1.9: Null graph of K_6 .

Definition 1.2.31 [6] A connected graph without cycles is defined as a **tree**.

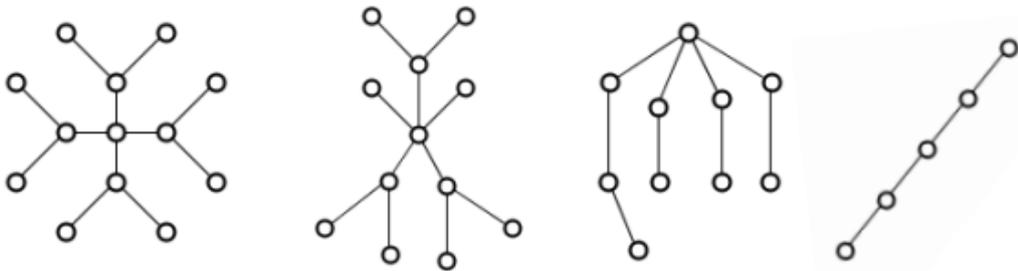


Figure 1.10: Some tree graphs.

Definition 1.2.32 [24] For a graph $G = (V, E)$, a set $S \subseteq V$ is **independent** if no two vertices in S are adjacent.

Definition 1.2.33 [43] A graph $G = (V_{(G)}, E_{(G)})$ is said to be **connected** if there is a path between every pair of its vertices. A graph which is not a connected is called a disconnected graph and every part is called component (as an example. See Figure 1.11).

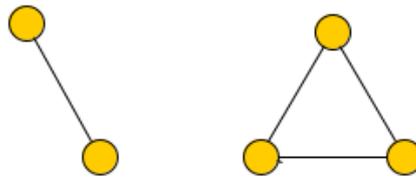


Figure 1.11: Disconnected graph with two components.

Definition 1.2.34 [43] A graph $G_1 = (V_1, E_1)$ is said to be **isomorphic** to the graph $G_2 = (V_2, E_2)$ if there is a one-to-one correspondence between the vertex sets V_1 and V_2 and a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1 is an edge with end vertices u_1 and u_2 in G_1 then the corresponding edge e_2 in G_2 has its end vertices v_1 and v_2 in G_2 which corresponds to u_1 and u_2 , respectively. Such a pair of correspondence is called a graph isomorphism and denoted by $(G_1 \cong G_2)$. (see Fig.1.12) .

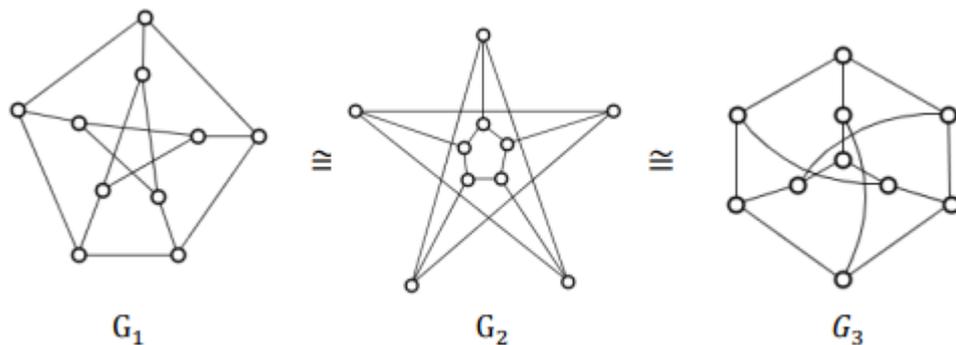


Figure 1.12: Three isomorphic graphs.

Definition 1.2.35 [9]. For a connected graph G , we define the **distance** between two vertices u and v is the smallest length of any $u - v$ path in G and is denoted by $d(u, v)$.

Definition 1.2.36 [9]. The **eccentricity** $e(v)$ of a vertex v of a connected graph G is the number $\max_{u \in V(G)}$. That is, $e(v)$ is the distance between v and a vertex furthest from v .

Definition 1.2.37 [9]. The **diameter** of G denoted $\text{diam}(G)$, is the maximum distance among all pairs of vertices in G .

Definition 1.2.38 [9]. The **radius** of G denoted $\text{rad}(G)$, is the minimum eccentricity among the vertices of G .

Definition 1.2.39 [9]. A vertex v is a **central vertex** if $e(v) = \text{rad } G$ and the center $\text{Cen}(G)$ is the subgraph of G induced by its central vertices.

Definition 1.2.40 [9]. A vertex v is a **peripheral vertex** if $e(v) = \text{diam } G$, while the periphery $\text{Per}(G)$ is the subgraph of G induced by its peripheral vertices.

1.3. Domination in Graph Theory

The origin of the idea of domination in the graph started from the problem of the five queens in the chessboard game. In 1850, several chess players were interested in the minimum number of queen or is attacked by a describes how to determine the minimum number of queen. This problem queens placed on the chessboard so that all chess boxes are controlled by the dominant queens recall that a queen can move any number of squares horizontally, vertically, or diagonally on the chessboard. (as shown Fig.1.13).

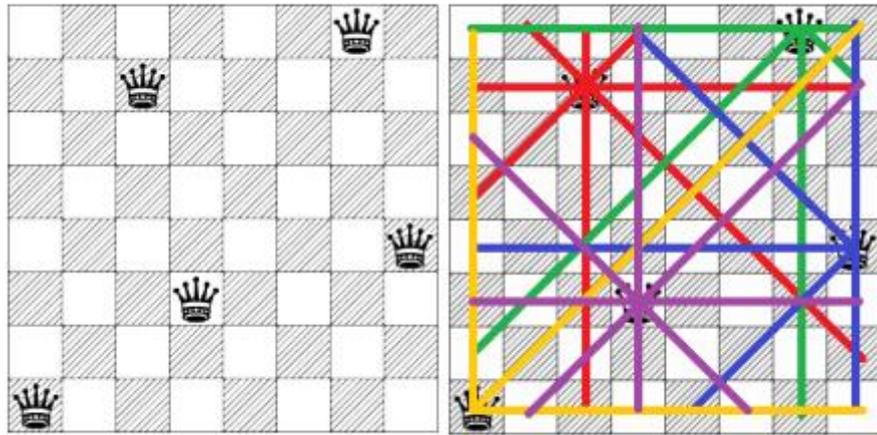


Figure 1.13: Five Queen problem.

Definition 1.3.1 [24] A set $D \subseteq V$ of vertices in a graph $G = (V, E)$ is called a **dominating set** if every vertex $v \in V$ is either an element of D or is adjacent to an element of D .

Definition 1.3.2 [24] A dominating set D is called a **minimal dominating set** if has proper subset $\bar{D} \subset D$ is dominating set.

Definition 1.3.3 [24] The **domination number** $\gamma(G)$ of a graph G equals the minimum cardinality of a set of minimal dominating set of G . Such a set is called the γ -set of G .

Example 1.3.4. In the following Figure

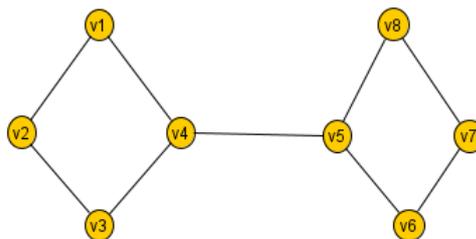


Figure 1.14: Domination number of graph G .

$D_1 = \{v_4, v_5, v_7\}$ is minimal dominating set of G .

$D_2 = \{v_1, v_3, v_8, v_6\}$ is minimal dominating set of G .

$D_3 = \{v_2, v_4, v_5, v_7\}$ is not minimal dominating set of G .

$\gamma(G) = 3$.

Definition 1.3.5 [24]. The maximum cardinality of all independent sets is called the **independence number** of the graph G and denoted by $\beta(G)$ (Haynes et al., 1998). et al., 1998).

Definition 1.3.6 [20]. The **chromatic number** of a vertex-coloring of G is an assignment of colors to all its vertices such that all pairs of adjacent vertices are assigned different colors., denoted by $\chi(G)$ is the smallest number colors necessary for coloring G (Brooks, 1941).

Definition 1.3.7 [24]. The order of largest complete (each vertex in it is adjacent to all other vertices in it) subgraph of a graph G is called the clique number, denoted by $\omega(G)$.

Definition 1.3.8 [7]. Let n be a positive integer. if a and b are integers, then a is said to be congruent to b modulo n , which is written $a \equiv b \pmod{n}$, if n divides $(a-b)$. we call n the modulus of the congruence.

Definition 1.3.9 [18]. Let X be a non-negative real number. The Gauss' s function $\pi(x)$ is defined to be the number of primes not exceeding x . i.e. $\pi(x) = |\{P: P \text{ is prime}, 2 \leq P \leq x\}|$.

Theorem 1.3.10[42]. (The Fundamental Theorem of Mathematics) Every integer number n except $0, \mp 1$ is the product of primes. This prime factorization is unique.

Note that. For all definitions in this thesis we define the labeling function by: $f(v_i) = i$, $i = 1, \dots, n$.

Definition 1.3.11 [31]. The **Mobius function** defined on $n > 1$, where

$$n = P_1^{\alpha_1} P_2^{\alpha_2} \dots P_k^{\alpha_k} = \prod_{i=1}^k P_i^{\alpha_i}$$

$$\text{So, } M(n) = \begin{cases} (-1)^k, & \text{if } \alpha_1 = \alpha_2 = \dots = \alpha_k = 1; \\ 0, & \text{otherwise.} \end{cases}$$

- $M(n) = 1$, if n is a square free positive integer with an even number of prime factors.
- $M(n) = -1$, if n is a square free positive integer with an odd number of prime factors.

$M(n) = 0$, if n has a squared prime factor.

From the table, we notice the mobius function:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
M(n)	1	-1	-1	0	-1	1	-1	0	0	1	-1	0	-1	1	1

CHAPTER TWO

Chapter Two

Analytic the Mobius Function Graph $\mathcal{M}^{(0)}$ in Graph Theory

In this chapter, we improve the work of previous researchers of the Mobius Function Graph $\mathcal{M}^{(0)}$ is introduced. Three ways of determining the prime-counting function by using this graph are presented. Also, some properties of this function are proved. Moreover, the domination number, independence number, chromatic number, and clique number for the graph of Mobius function for '0', $G(\mathcal{M}_n)^{(0)}$, which has the vertex set as the set of first n natural numbers and any two vertices a, b are adjacent if the value of Mobius function $\mathcal{M}(ab) = 0$.

Finally, the relationship between the domination number and the independence number is discussed.

Definition 2.1. Let G be a graph then G is called $\mathcal{M}^{(0)}$ Graph if contain all vertices of G , $v_i, v_j \in V$ and the two vertices v_i , and v_j are adjacent iff $\mathcal{M}(ij) = 0$.

Theorem 2. 1.1. If G is a Mobius function graph $\mathcal{M}^{(0)}$, then

$$\beta(G) = \pi(n) + 1 .$$

Proof. Let $S = \{v_p; p \text{ is prime number}\} \cup \{v_1\}$, it is clear that the vertex v_1 is not adjacent to all other vertices in the set S . Also, let u and v be any two different vertices in the set S such that these vertices not equal to the vertex v_1 . Thus, there are two different prime numbers p and q such that $u = v_p$ and $v = v_q$. It is obvious that $\mathcal{M}(pq) \neq 0$, so the vertex v_p is not

adjacent to the vertex v_q , so $\beta(G) \geq |S|$. Let $v_i \notin S$, then i is not prime and not equal to one, so there is at least two prime p_1 and p_2 such that $p_1 p_2$ is a factor of i . Thus, $\mathcal{M}(ip_1) = \mathcal{M}(ip_2) = 0$, so the vertex v_i is adjacent to at least two vertices v_{p_1} and v_{p_2} . Therefore, the set S is the maximum independent set and $\beta(G) = |S| = \pi(n) + 1$.

(For example, see Figure 2.1.1).

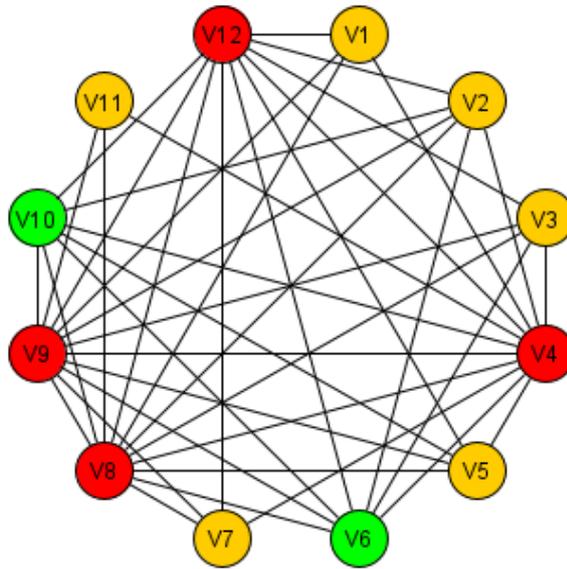


Figure 2.1.1. The Mobius function graph $\mathcal{M}^{(0)}$ of order 12.

Red represents $|S|$, Green represent even number $|N_e^n|$.

Theorem 2.1.2. The clique number to Mobius function graph $\mathcal{M}^{(0)}$ is $\omega(G) = |N_e^n| + |S|$ where $S = \{a; p^\alpha \nmid a, \alpha > 1 \text{ such that } a \notin N_e^n\}$.

Proof. If $a, b \in N_e^n$, then

$$a = 2K_1 \text{ and } b = 2K_2, K_1, K_2 \in \mathbb{Z}^+ - \{0\}$$

So, $ab = 4K_1K_2 = 2^2K_1K_2$ Then $\mathcal{M}(ab) = 0$ So a is adjacent to b

Now if $b \in \{a; p^\alpha \nmid a, \alpha > 1\}$ So $p^\alpha \nmid b$, Thus b is adjacent to all other

vertices Then the induced subgraph $H = \{v_i; i \in S\}$ is a complete graph, so

$\varpi(G) \geq |H|$. Now, let $v_j \notin H$, then j is an odd number and there is no p^α such that $p^\alpha \mid j$, $\alpha > 1$. Thus, the vertex v_j is not adjacent to the vertex v_2 at least. Therefore, the set H is the independent set with maximum cardinality (for example, see Figure 2.1.1), and $\varpi(G) = |H| = |N_e^n| + |S|$.

Proposition 2.1.3. If G is a non-trivial Mobius function graph $\mathcal{M}^{(0)}$, then $\chi(G) = \varpi(G)$.

Proof. According to Theorem 2.1.2, the clique number is equal to $|N_e^n| + |S|$ where $S = \{a; p^\alpha \mid a, \alpha > 1 \text{ such that } a \notin N_e^n\}$, so $\varpi(G)$ is needed to color the largest subgraph isomorphic to complete graph of order $\varpi(G)$.

Proposition 2.1.4.[25]. let G be a Mobius function graph $\mathcal{M}^{(0)}$ of order $n < 4$, then:

- 1) graph G is **connected** otherwise G is disconnected.
- 2) the graph G is isomorphic to **null** graph of order n , if $n \leq 3$.

Theorem 2.1.5.[25]. A graph $\mathcal{M}^{(0)}$ is a tree if $n=4$ or $n=5$.

Theorem 2.1.6.[25]. The domination number of $\mathcal{M}^{(0)}$ is

$$\gamma(\mathcal{M}^{(0)}) = \begin{cases} n, & \text{if } n \leq 3 \\ 1, & \text{if } n > 3 \end{cases}$$

proposition 2.1.7. let G be a Mobius function graph $\mathcal{M}^{(0)}$ and $u, v \in \mathcal{M}^{(0)}$ then $\mathcal{M}(u.v) = \mathcal{M}(u).\mathcal{M}(v)$.

Proof. Let G be a Mobius function graph $\mathcal{M}^{(0)}$ and $u, v \in \mathcal{M}^{(0)}$, $g.c.d(u, v) = 1$, then if $\mathcal{M}(u) = \mathcal{M}(v) = 0$ then $u = \prod_{i=1}^s p^i; i > 1$ for some i and $v = \prod_{j=1}^r p^j; j > 1$ for some j , then it is obvious that $\mathcal{M}(u.v) = 0 = \mathcal{M}(u).\mathcal{M}(v)$.

Theorem 2.1.8. let G be a Mobius function graph $\mathcal{M}^{(0)}$ and non-trivial, then

$$1) \beta(G) + \gamma(G) \leq n, \text{ and } \beta(G) + \gamma(G) = n \text{ if } n = 4$$

$$2) \gamma(G) \leq \beta(G).$$

Proof. 1) According to Theorems 2.1.1 and 2.1.6, the result is obtained when $n \leq 3$. Moreover, if $n = 4$. Then $\beta(G) + \gamma(G) = 3 + 1 = 4 = n$. Now, if $n > 4$, then $\beta(G) + \gamma(G) = \pi(n) + 2 > n$. Thus, the result is obtained.

2) From Theorem 2.1.6 and Proposition 2.1.4(2), the graph G is isomorphic to null graph of order n , if $n \leq 3$, so $\gamma(G) = \beta(G)$. Now, if $n \geq 4$, then $\gamma(G) = 1$, according to Theorem 2.1.6, thus $\gamma(G) < \beta(G)$. Therefore, the result is obtained.

From two cases above, the proof is done.

Theorem 2.1.9.[25]. For a graph $G(\mu_n^{(0)})$, the radius,

$$\text{Rad}(G(\mu_n^{(0)})) = \begin{cases} \infty, & n \leq 3 \\ 1, & n > 3 \end{cases}$$

Theorem 2.1.10.[25]. For a graph $G(\mu_n^{(0)})$, the diameter,

$$\text{Diam}(G(\mu_n^{(0)})) = \begin{cases} \infty, & n \leq 3 \\ 2, & n > 3 \end{cases}$$

2.2 SOME PROPERTIES OF MOBIUS FUNCTION GRAPH $\mathcal{M}^{(1)}$

In this section, a new graph is called the mobius function graph $\mathcal{M}^{(1)}$ is introduced. In Section 2.2 some properties of mobius function graph are presented. Three ways of determining the prime-counting function by using this graph are presented. Also, some properties of this graph are proved. Moreover, the domination number, chromatic number, independence number, and clique number of this graph are determined. Finally, a

comparison was made between the two numbers the domination and independence.

Definition 2.2.1. Let G be a graph then G is called $\mathcal{M}^{(1)}$ **Graph** if contain all vertices of G , $v_i, v_j \in V$ and the two vertices v_i , and v_j are adjacent iff $\mathcal{M}(ij) = 1$.

Theorem 2.2.2. If G is a Mobius function graph $\mathcal{M}^{(1)}$, then

$\pi(n) = \deg(v_2) + 1 - |\{u : u = \prod_{i=1}^k P_i, k \text{ is odd}; P_i \neq 2\forall i\}|$ where P_i are distinct primes numbers and labeled of $f(v_i) = i$.

Proof. If the vertices of a graph take labeled $f(v_i) = i$, then the vertex labeled 2 is adjacent to all vertices that have labeled primes number, since

$\mathcal{M}(2P_i) = 1, \forall i$. Moreover, $\mathcal{M}(2 \prod_{i=1}^k P_i) = 1$, where k is odd and $P_i \neq 2\forall i$, so the vertex labeled 2 is adjacent to all vertices that are labeled with the form $\prod_{i=1}^k P_i$, k is odd and $P_i \neq 2\forall i$. Thus,

$$\pi(n) = \deg(v_2) + 1 - |\{u : u = \prod_{i=1}^k P_i, k \text{ is odd}; P_i \neq 2\forall i\}|$$

(For example, see Figure 2.2.1).

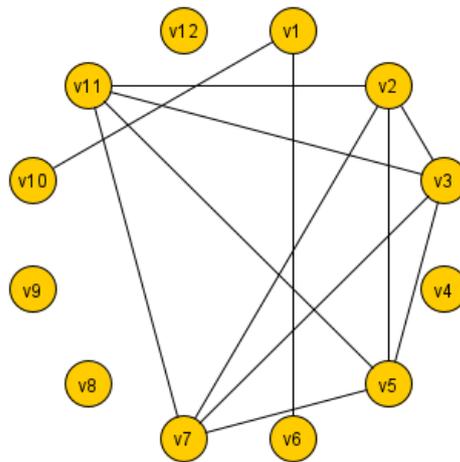


Figure 2.2.1. The Mobius function graph $\mathcal{M}^{(1)}$ of order 12.

Theorem 2.2.3. The clique number to Mobius function graph $M^{(1)}$ is $\pi(n)$.

Proof. Let v_i and v_j be any two vertices such that i and j are prime numbers, then $M(ij) = 1$ (By definition 2.2.1 and ij means i product j), so the vertex v_i is adjacent to the vertex v_j . Thus, the vertices which have prime numbers constitute an induced subgraph isomorphism to a complete. Now, suppose that there is a vertex say v_r such that is adjacent to all vertices mentioned above, then the label of this vertex must be of the form $\prod_{i=1}^{k < n} P_i$ and k is odd greater than one. Thus, the minimum value of k is three, so take $f(v_r) = P_1 P_2 P_3$, where $P_i, i=1,2,3$ are prime numbers. can be concluded that this vertex is not adjacent to the vertices have labeled $P_1, P_2,$ and P_3 and this is a contradiction. Therefore, the clique number to Mobius function graph $M^{(1)}$ is $\pi(n)$. (For example, see figure 2.2.2)

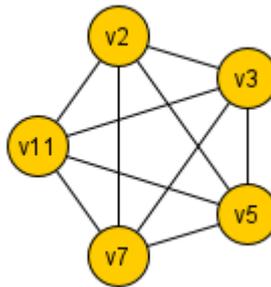


Figure 2.2.2. The clique number to Mobius function graph $\mathcal{M}^{(1)}$, of order 12.

Proposition 2.2.4. If G is a non-trivial Mobius function graph $M^{(1)}$ then $\chi(G) = \pi(n)$.

Proof. According to Theorem 2.2.3. the clique number is equal to $\pi(n)$, so $\pi(n)$ color is needed to color the largest subgraph isomorphic to complete graph of order $\pi(n)$. The other vertices can be colored by the same color which is used previously. Thus, $\chi(G) = \pi(n)$.

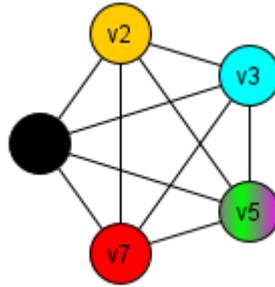


Figure 2.2.3. The chromatic number to Mobius function graph $\mathcal{M}^{(1)}$ of order 12.

Proposition 2.2.5. Let G be a Mobius function graph $M^{(1)}$, then

1. $N(v_1) = \{v_i ; f(v_i) = \prod_{i=1}^k P_i, k \text{ is even}\}$
2. All vertices have labeled that have square prime are isolated vertices.
3. The vertex v is pendant if it is adjacent to the vertex v_1 and there is a common prime factor with labeled of other vertices that adjacent to the vertex v_1 .
4. If G is a Mobius function graph $M^{(1)}$ of order $n > 1$, then graph G is disconnected.

Proof. 1. The labeled of the vertex v_1 is one, so all vertices in the set $\{v_i ; f(v_i) = \prod_{i=1}^k P_i, k \text{ is even}\}$ are adjacent to the vertex v_1 , since $M(1 * \prod_{i=1}^k P_i) = M(\prod_{i=1}^k P_i) = 1$, where k is even. Thus, the required statement is obtained.

2. One can be concluded that each vertex that is labeled has a square prime is not adjacent to all other vertices by definition of Mobius function. Thus, the result is getting.

3. Let the vertex v_r be adjacent to the vertex v_1 , then this vertex has labeled $\prod_{i=1}^k P_i$; k is even. If there is a vertex that adjacent to the vertex v_r say v_s , so the label of this vertex is $\prod_{i=1}^j P_i$; j is even. If there is no common prime factor, then the vertex v_s is adjacent to the vertex v_1 otherwise the two vertices are not adjacent. Therefore, the vertex is a pendant vertex if has common factor prime with all vertices have a characteristic similar to that of the vertex v_s . Thus, the result is obtained.

4. Three cases are discussed below:

1) If $n = 2$, then the graph $M^{(1)}$ is isomorphic to the null graph of order 2, then the graph $M^{(1)}$ is disconnected.

2) If $n = 3$, then $M^{(1)} \equiv K_2 \cup K_1$, so the result is obtained.

3) If $n \geq 4$, then the vertex v_4 is the isolated vertex since it is not a free square.

Proposition 2.2.6. Let G be a Mobius function graph $M^{(1)}$ of order n .

$$\text{Then } \beta(G) = \begin{cases} 1, & \text{if } n = 1; \\ 2, & \text{if } n = 2, 3; \\ 3, & \text{if } n = 4, 5, 6; \\ |N_e^n| + |S - N_e^n|, & \text{if } n \geq 7; \end{cases}$$

S is the set of the isolated vertices in G , and N_e^n is the set of even numbers less than or equal n .

Proof. Two cases appeared as follows:

Case 1. One can conclude the result when $n = 1, 2, \dots, 6$.

Case 2. Let v_r be an isolated vertex, then this vertex belongs to all independent set (I) of G , thus $S \subseteq I$ so, $\beta(G) \geq |S|$. Now, if a vertex v_r is not an isolated vertex, so this vertex is free square prime. Let H be the set of

vertices that labeled even number, let u and v be any two vertices of the set H , so $f(uv) = 2^2 P_z P_w$. Therefore, the vertex u is not adjacent to the vertex v , so the set H is independent. Note that $S \cap H \neq \emptyset$, then $\beta(G) \geq |H| + |S - H|$. Again, let u_1 be a vertex not belonging in the sets S and H , so the labeled of this vertex is $P_j \prod_{i=1}^r P_i$; $P_j \neq 2$. It is obvious that there is at least one P_i , $i = 1, \dots, r$ such that divides one vertex of the set H , since $2 < P_j$. Therefore, $\beta(G) = |H| + |S - H|$.

Theorem 2.2. 7. let G be a Mobius function graph $\mathcal{M}^{(1)}$ of order n , then

$$\gamma(\mathcal{M}^{(1)}) = \begin{cases} 1, & \text{if } n = 1 \\ 2, & \text{if } n = 2, 3 \\ |S| + 2, & \text{otherwise} \end{cases}$$

Proof.

Depending on the number of vertices four cases are discussed as the following.

Case 1. If $n = 1$, then it is obvious that $\gamma(\mathcal{M}^{(1)}) = 1$.

Case 2. If $n = 2$, then the Mobius function graph is isomorphic to null graph of order 2, then $\gamma(\mathcal{M}^{(1)}) = 2$.

Case 3. If $n = 3$, then Mobius function graph is isomorphic to graph $(K_2 \cup K_1)$ and $\gamma(\mathcal{M}^{(1)}) = 2$.

Case 4. If $n \geq 4$, then let S be the set of all isolated vertices, then the set S contains in each dominating set. So, the remained vertices have non-square prime factor and these vertices have two kinds as the following:

Subcases 1. The vertices of the form $\prod_{i=1}^r P_i$ where r is even, all these vertices are adjacent to the vertex v_1 according to proposition 2.2.5(1).

Subcases 2. The vertices of the form $\prod_{i=1}^s P_i$ where s is odd, all these vertices are adjacent to the vertex v_2 according to Theorem 2.2.2. From all cases above, the result is obtained (as an example, see Figure 2.2.4).

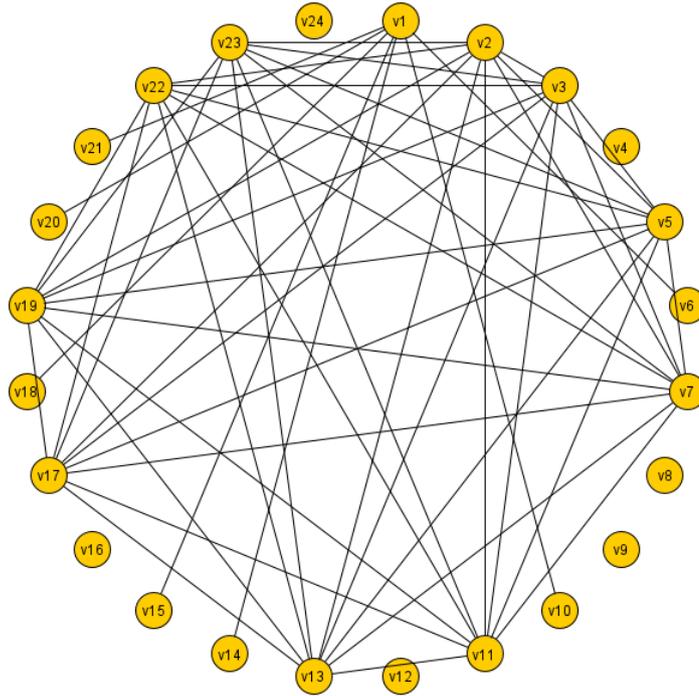


Figure 2.2.4: The domination number of $\mathcal{M}^{(1)}$ of order 24.

Proposition 2.2.8. Let G be a mobius function graph $M^{(1)}$ and $u, v \in M^{(1)}$, $\text{g.c.d}(u, v) = 1$. Then $M(u.v) = M(u)M(v)$.

Proof. Let G be a Mobius function graph $M^{(1)}$ and $u, v \in M^{(1)}$, $\text{g.c.d}(u, v) = 1$. Then four cases are appeared as the following:

Case 1. If $M(u) = M(v) = 1$, then $u = \prod_{i=1}^s P_i$ and $v = \prod_{j=1}^r P_j$, where r and s are even number, so $u.v = \prod_{i=1}^s P_i \prod_{j=1}^r P_j = \prod_{k=1}^{s+r} P_k$ and it is obvious that $s+r$ is even, since $\text{g.c.d}(u, v) = 1$. Thus, $M(u.v) = 1 = M(u)M(v)$.

Case 2. If $M(u) = 1$ and $M(v) = -1$, then $u = \prod_{i=1}^s P_i$; s is even and

$v = \prod_{j=1}^r P_j$; r is odd, then $u.v = \prod_{i=1}^s P_i \prod_{j=1}^r P_j = \prod_{k=1}^{s+r} P_k$ and it is obvious that $s + r$ is odd, since $\text{g.c.d}(u, v) = 1$. Thus,

$$M(u.v) = -1 = M(u).M(v).$$

Case 3. If $M(u) = -1$ and $M(v) = -1$, then $u = \prod_{i=1}^s P_i$; s is odd and

$v = \prod_{j=1}^r P_j$; r is odd, then $u.v = \prod_{i=1}^s P_i \prod_{j=1}^r P_j = \prod_{k=1}^{s+r} P_k$ and it is obvious that $s + r$ is even, since $\text{g.c.d}(u, v) = 1$. Thus, $M(u.v) = 1 = M(u).M(v)$

Case 4. If there is factor prime $P_i^{\alpha_i}$; $\alpha_i > 1$ is divided the labeled of at least one vertex from two vertices u or v say u , then $M(u) = 0$. Now, the factor prime $P_i^{\alpha_i}$ is divided the labeled of the product of two vertices u and v . Thus, $M(u.v) = 0 = M(u).M(v)$. From all cases above, the result is done.

Theorem 2.2.9. Let G be a non-trivial Mobius function graph $M^{(1)}$. Then

1. $\beta(G) + \gamma(G) \geq n$, $\forall n$ except $n = 7$.

2. $\gamma(G) \leq \beta(G)$.

Proof.

(I) The result is obvious if $n = 1, 2, 3$ by proposition 2.2.6, and theorem 2.2.7, so there are cases as follows.

1. If $n = 4$, then the Mobius graph is isomorphic to graph $N_2 \cup K_2$, so this graph contains two isolated vertices which have labeled 1 and 4 and two adjacent vertices labeled 2 and 3. Thus, $\beta(G) + \gamma(G) = 3 + 3 = 6 > n = 4$.

2. If $n = 5$, then the Mobius graph is isomorphic to graph $N_2 \cup K_3$, so this graph contains two isolated vertices which have labeled 1 and 4 and three vertices of the graph K_3 labeled 2, 3, and 5. Thus, $\beta(G) + \gamma(G) = 3 + 3 = 6 > n = 5$.

3. If $n = 6$, then the Mobius graph is isomorphic to graph $N_1 \cup K_2 \cup K_3$, so this graph contains one isolated vertex which have labeled 4 and three vertices of the graph K_3 that labeled 2, 3 and 5 and two vertices of the graph K_2 that labeled 1, 6. Thus, $\beta(G) + \gamma(G) = 3 + 3 = 6 = n$.

4. If $n = 7$, then the Mobius graph is isomorphic to graph $N_1 \cup K_2 \cup K_4$, so this graph contains one isolated vertex which have labeled 4 and four vertices of the graph K_4 that labeled 2, 3, 5 and 7 and two vertices of the graph K_2 that labeled 1, 6. Thus, $\beta(G) + \gamma(G) = 3 + 3 = 6 < n$.

5. If $n \geq 7$, then $\beta(G) + \gamma(G) = |N_e^n| + |S - N_e^n| + |S| + 2 = |S| + |N_e^n| + |N_e^n| + |S| + 2 = 2|N_e^n| + |S| + 2 = 2 \lfloor \frac{n}{2} \rfloor + |S| + 2 \geq n$.

(II) can be conclude easily that the inequality is correct when $n = 1, 2, \dots, 6$. So, $\forall n \geq 7, \gamma(G) = |S| + 2 < |S| + |N_e^n| + |S - N_e^n| = \beta(G)$. Thus, the required statement is obtained.

2.3 Some Properties of Mobius Function Graph

$\mathcal{M}^{(-1)}$

In this section, a new graph is called the Mobius Function Graph $\mathcal{M}^{(-1)}$ is introduced. Three ways of determining the prime-counting function by using this graph are presented. Also, some properties of this function are proved. Moreover, the domination, independence, chromatic, and clique number of this graph are determined. Finally, the relationship between the domination number and the independence number is discussed. Finally, the relationship between the independence number with the domination number is obtained.

Definition 2.3.1. Let G be a graph then G is called $\mathcal{M}^{(-1)}$ Graph if contain all vertices of G , $v_i, v_j \in V$ and the two vertices v_i , and v_j are adjacent iff $\mathcal{M}(ij) = -1$.

Theorem 2.3.2. If G is a Mobius function graph $\mathcal{M}^{(-1)}$, then

$$\pi(n) = \deg(v_1) - |\{u: u = \prod_{i=1}^k P_i, k \text{ is odd}, k > 1\}|,$$

where P_j are distinct primes numbers and labeled of $f(v_i) = i$.

Proof. If the vertices of a graph G take labeled $f(v_i) = i$, then the vertex v_1 is adjacent to all vertices that have labeled primes number, since $f(V_1 \cdot V_i) = f(V_i) = i = P_j$, and P_j is prime number $\forall j$, So $\mathcal{M}(P_j) = -1$, Moreover, $\mathcal{M}(\prod_{j=1}^k P_j) = -1$, where k is odd, so the vertex v_1 is adjacent to all vertices that are labeled with the form $\prod_{j=1}^k P_j$; k is odd. Thus,

$$\pi(n) = \deg(v_1) - |\{u: u = \prod_{j=1}^k P_j, k \text{ is odd}; k > 1\}|$$

(For example, see Figure 2.3.1).

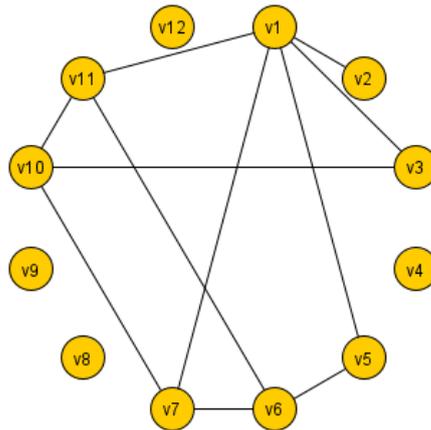


Figure 2.3.1. The Mobius function graph $\mathcal{M}^{(-1)}$ of order 12.

Theorem 2.3.3. The Mobius function graph $\mathcal{M}^{(-1)}$ is a free triangle.

Proof.

Let $v_1, v_2,$ and v_3 are any three vertices, then there are two Cases as follows:

Case 1. If $g.c.d(v_i, v_j) \neq 1$, then there is a prime factor common with different two vertices $v_i,$ and v_j say P_i , so P_i^2 is divided $f(v_i v_j)$. Thus, the vertex v_i is not adjacent to the vertex v_j , so there is no triangle between the three vertices mentioned above.

Case 2. If $g.c.d(v_i, v_j) = 1, \forall i \neq j$, so $f(v_1) = \pi_{i=1}^k P_i$, $f(v_2) = \pi_{j=1}^r P_j$, and $f(v_3) = \pi_{w=1}^s P_w$. Thus, each one of $k, s,$ or w is odd or even, so there are at least two of them are odd or even. In each cases, the product of these numbers is even length and the Mobius of this number is 1. Therefore, these vertices are not adjacent and then there is no triangle.

From the two cases above, the required is obtained.

Corollary 2.3.4. The clique number of the Mobius function graph $\mathcal{M}^{(-1)}$ of order $n; n \geq 2$ is 2.

Proof. It is obvious that the Mobius function graph $\mathcal{M}^{(-1)}$ is not isomorphic to the null graph, so the clique number is greater than or equal to two. According to theorem 2.3.3, this graph is a free triangle. Thus, the clique number is equal to two.

Proposition 2.3.5. If G is a non-trivial Mobius function graph $\mathcal{M}^{(-1)}$, then $\chi(G) = 2$.

Proof. According to Corollary 2.3.4, the clique number is equal to 2 each two adjacent vertices can be two different colors.

Proposition 2.3.6. let G be a Mobius function graph $\mathcal{M}^{(-1)}$, then

1) All vertices have labeled not free square prime are isolated vertices.

2) G is disconnected if $n \geq 4$ otherwise, the graph G is connected.

Proof. Let $f(V_i) = i$, so

1) can be conclude that each vertex that is labeled has a square prime is not adjacent to all other vertices by definition of Mobius function. Thus, the result is getting.

2) if $n = 4$, then $f(V_4) = 4 = 2^2$, thus the vertex V_4 is not adjacent to all other vertices in the Mobius function graph $\mathcal{M}^{(-1)}$ is disconnected. Now, if $n \leq 3$, it is obvious that the Mobius function graph $\mathcal{M}^{(-1)}$ is connected. Therefore, the result is obtained.

Corollary 2.3.7. the Mobius function graph $\mathcal{M}^{(-1)}$ of order n is tree if and only if $n \leq 3$.

Proposition 2.3.8. let G be a Mobius function graph $\mathcal{M}^{(-1)}$ of order n , then

$$\beta(G) = \begin{cases} 1, & \text{if } n = 2 \\ \lfloor \frac{n}{2} \rfloor + 1, & \text{if } n = 1, 3, 4 \\ \lfloor \frac{n}{2} \rfloor + 1, & \text{if } n = 5, 6, \\ |S \cup S_1|, & \forall n \geq 7 \end{cases}$$

where S is the set of the isolated vertices in G and $S_1 = \{ \prod_{i=1}^k p_i, k \text{ is odd} \}$

Proof.

There are two cases as follows:

Case1. One can conclude easily the result when $n=1, 2, \dots, 6$.

Case2. If $n \geq 7$, then let V_r be an isolated vertex, so this vertex belongs to all independent set (I) of G , thus $S \subseteq I$ so, $\beta(G) \geq |S|$. Now if a vertex V_r is not an isolated vertex, so this vertex is free square prime.

Let $S_1 = \{\prod_{i=1}^k p_i, k \text{ is odd}\}$, let u and v be any two vertices of the set S_1 , so there are two subcases as the follows.

Subcase 1. If $\text{g.c.d}(f(v),f(u)) \neq 1$, $\mathcal{M}(f(u)f(v)) = \mathcal{P}_r^2 \pi_{i=1}^k p_i$. Therefore the vertex u is not adjacent to the vertex v .

Subcase 2. If $\text{g.c.d}(f(v),f(u)) = 1$, $\mathcal{M}(f(u)f(v)) = \pi_{i=1}^k p_i$, k is even, so $\mathcal{M}(f(u)f(v)) = 1$.

Therefore the vertex u is not adjacent to the vertex v .

Thus, the set S_1 is an independent set depending on the above two subcases. Note that $S \cap S_1 = \emptyset$, then $\beta(G) \geq S_1 \cup S$. Now, if there is a vertex w ; $w \notin S_1 \cup S$, so

$$\mathcal{M}(f(w)) = \pi_{i=1}^k p_i, k \text{ is even then } \forall z \in S_1 \cup S; \text{g.c.d}(f(w), f(z))=1, \\ \mathcal{M}(f(u)f(v)) = -1,$$

So, the vertex w is adjacent to the vertex z . Thus, the set $S_1 \cup S$ is the maximum independent and the result is obtained.

Theorem 2.3.9. let G be a Mobius function graph $\mathcal{M}^{(-1)}$ of order n , then

$$\gamma(\mathcal{M}^{(-1)}) = \begin{cases} 1, & \text{if } n = 1,2,3 \\ 2, & \text{if } n = 4,5 \\ 2 + |S|, & \text{if } n \geq 6 \end{cases}$$

S is the set of isolated vertices.

Proof. Depending on the number of vertices three cases are discussed as the following.

Case 1. If $n = 1,2,3$ then it is obvious that $\gamma(\mathcal{M}^{(-1)}) = 1$.

Case 2. If $n = 4,5$

Then the vertex V_1 is dominates all other vertices except the vertex V_4 , since this vertex is isolated. thus $\gamma(\mathcal{M}^{(-1)}) = 2$.

Case 3. If $n \geq 6$, then three subcases depend on the factorization of the number $a \leq n$ as :

Subcase 1. If there is $P_i^\alpha, \alpha > 1$ as a factor of a , then the vertex of labeled a is isolated. Let S be the set of all isolated vertices. Thus $\gamma(\mathcal{M}^{(-1)}) \geq |S|$

Subcase 2. If there is no $P_i^\alpha; \alpha > 1$ as a factor of a , then a can be written as the form $\pi_{i=1}^k p_i$ again there are two subcases as follows.

I) If k is odd, then all vertices of this form are adjacent to the vertex v_1 , since $f(v_1) f(v_a) = \pi_{i=1}^k p_i$.

II) If k is even, then let p_s be the largest prime number less than or equal n , so all vertices of this form is adjacent to the vertex of labeled p_s , since $f(v) f(v_a) = p_s \pi_{i=1}^k p_i = \pi_{i=1}^{k+1} p_i$. Thus, all vertices in this case are dominated by only two vertices. Therefore, according to case 1 and case 2, $\gamma(\mathcal{M}^{(-1)}) = 2 + |S|$ (as an example, see Figure 2.3.2).

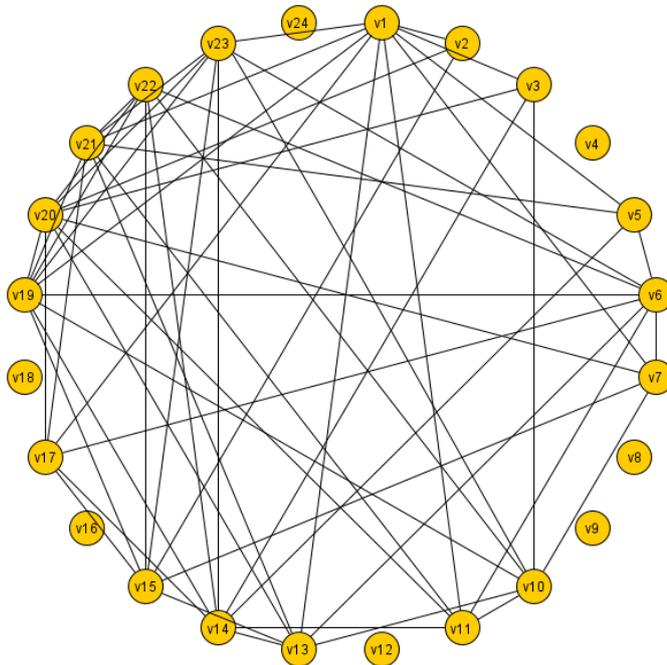


Figure 2.3.2. The domination number of $\mathcal{M}^{(-1)}$ of order 24

Proposition 2.3.10. Assume that G is a Mobius function graph $\mathcal{M}^{(-1)}$ and $u, v \in \mathcal{M}^{(-1)}$, $g. c. d(u, v) = 1$, then $\mathcal{M}(u \cdot v) = \mathcal{M}(u) \cdot \mathcal{M}(v)$.

Proof. Let G be a Mobius function graph $\mathcal{M}^{(-1)}$ and $u, v \in \mathcal{M}^{(-1)}$, $g. c. d(u, v) = 1$, then four cases are appeared as bellow.

Case 1. If $\mathcal{M}(u) = \mathcal{M}(v) = -1$ then $u = \prod_{i=1}^s P_i$ and $v = \prod_{j=1}^r P_j$, where r and s are odd number, so $u \cdot v = \prod_{i=1}^s P_i \prod_{j=1}^r P_j = \prod_{k=1}^{s+r} P_k$ and it is obvious that $s + r$ is even, since $g. c. d(u, v) = 1$. Thus,

$$\mathcal{M}(u \cdot v) = 1 = \mathcal{M}(u) \cdot \mathcal{M}(v).$$

Case 2. If $\mathcal{M}(u) = 1$ and $\mathcal{M}(v) = -1$, then $u = \prod_{i=1}^s P_i$; s is even and $v = \prod_{j=1}^r P_j$; r is odd, then $u \cdot v = \prod_{i=1}^s P_i \prod_{j=1}^r P_j = \prod_{k=1}^{s+r} P_k$ and it is obvious that $s + r$ is odd, since $g. c. d(u, v) = 1$. Thus,

$$\mathcal{M}(u \cdot v) = -1 = \mathcal{M}(u) \cdot \mathcal{M}(v).$$

Case 3. If $\mathcal{M}(u) = 1$ and $\mathcal{M}(v) = 1$, then $u = \prod_{i=1}^s P_i$; s is even and $v = \prod_{j=1}^r P_j$; r is even, then $u \cdot v = \prod_{i=1}^s P_i \prod_{j=1}^r P_j = \prod_{k=1}^{s+r} P_k$ and it is obvious that $s + r$ is even, since $g. c. d(u, v) = 1$. Thus,

$$\mathcal{M}(u \cdot v) = 1 = \mathcal{M}(u) \cdot \mathcal{M}(v).$$

Case 4. If there is factor prime $P_i^{\alpha_i}$; $\alpha_i > 1$ is divided the labeled of at least one vertex from two vertices u and v say u , then $\mathcal{M}(u) = 0$. Now, the factor prime $P_i^{\alpha_i}$ is divided the labeled of the product of two vertices u and v .

Thus, $\mathcal{M}(u \cdot v) = 0 = \mathcal{M}(u) \cdot \mathcal{M}(v)$. According to cases above, the proof is done.

Theorem 2.3.11. Assume that G is a Mobius function graph $\mathcal{M}^{(-1)}$ and non-trivial, then

- 1) $\beta(G) + \gamma(G) \geq n, \forall n$ except $n = 7$.
- 2) $\gamma(G) \leq \beta(G)$.

Proof. 1) Two cases depend on n are classifications as follows.

Case 1. If $1 \leq n \leq 6$, then one can easily conclude the result.

Case 2. If $n \geq 7$, then according to Theorem 2.3.9, $\gamma(\mathcal{M}^{(-1)}) = 2 \cup |S|$ and by Proposition 2.3.8, $\beta(G) = |S \cup S_1|$, where S is the isolated vertices set in G and $S_1 = \{ \pi_{i=1}^k P_i, k \text{ is odd} \}$. Thus,

$$\beta(G) + \gamma(G) = 2 + |S| + |S \cup S_1| = 2 + 2|S| + |S_1|, \text{ so } \beta(G) + \gamma(G) \geq n.$$

2) One can be concluded easily that the inequality is correct when $n = 1, 2, \dots, 6$. So, $\forall n \geq 7, \gamma(G) = |S| + 2 < |S \cup S_1| = \beta(G)$. Thus, the result is obtained.

Proposition 2.3.12. let G be a Mobius function graph $\mathcal{M}^{(-1)}$ and non-trivial, then

$$\text{I) The radius, } Rad(G(\mathcal{M}_n^{(-1)})) = \begin{cases} 1, & n \leq 3 \\ \infty, & n > 3 \end{cases},$$

and the Cent ($G(\mathcal{M}_n^{(-1)})$) is isomorphic to K_1 graph, if $n \leq 3$

II) The diameter, $Diam(G) = \begin{cases} n - 1; & n \leq 3 \\ \infty; & n > 3 \end{cases}$; and the $Per(G)$ is isomorphic to K_2, N_2 graph, if $n = 2, 3$ respectively.

Proof.

I) If $n \leq 3$, it is obvious that $Rad(G) = 1$ and Cent(G) is isomorphic to K_1 . Now, if $n > 3$, then graph G contains an isolated vertex, so graph G is disconnected. Thus, $Rad(G) = \infty$.

II) Three cases are classification as follows.

Case 1. If $n = 2$, then $G \equiv K_2$ and $Diam(G) = 1$, so $Per(G) \equiv K_2$.

Case 2. If $n = 3$, then $G \equiv P_3$ and $Diam(G) = 2$, so $Per(G) \equiv N_2$.

Case 3. if $n > 3$, then the graph G contains an isolated vertex, so the graph G is disconnected. Thus, $Diam(G) = \infty$.

CHAPTER Three

Chapter Three

3.1 Complement the Mobius Function Graph $(\mathcal{M}^0)^c$

In Graph Theory

The purpose of this chapter is introduced new graphs which are called the complete of Mobius Function Graph $(\mathcal{M}^0)^c$, $(\mathcal{M}^1)^c$, and $(\mathcal{M}^{-1})^c$. The relationship between these graphs are discussed. Moreover, the clique number, independence number, domination number are determined for all these graphs. Additionally, compare the results of these graphs (\mathcal{M}^i) , $i = 0, 1, -1$ with its complement. Also, the chromatic number, the $\text{Cent}((\mathcal{M}^i)^c)$, and $\text{Per}((\mathcal{M}^i)^c)$ graphs are founded with some properties.

3.2 Main results.

Proposition 3.2.1 Let G be a graph of order n , then the graphs \mathcal{M}^0 , \mathcal{M}^1 , and \mathcal{M}^{-1} are decomposition of complete graph of order n .

Proof.

Let e_{ij} be an edge that joint the two vertices v_i and v_j , then

1) If $e_{ij} \in E(\mathcal{M}^0) \cap E(\mathcal{M}^1)$ or $e_{ij} \in E(\mathcal{M}^0) \cap E(\mathcal{M}^{-1})$, then $e_{ij} \in E(\mathcal{M}^0)$, this means ij has a square prime factor. Moreover, the edge $e_{ij} \in E(\mathcal{M}^1)$ or $e_{ij} \in E(\mathcal{M}^{-1})$, in each cases ij is a free square prime factor and this is a contradiction. Thus,

$$E(\mathcal{M}^0) \cap E(\mathcal{M}^1) = E(\mathcal{M}^0) \cap E(\mathcal{M}^{-1}) = \emptyset.$$

2) If $e_{ij} \in E(\mathcal{M}^1) \cap E(\mathcal{M}^{-1})$, then $e_{ij} \in \mathcal{M}^1$, this means $\mathcal{M}(v_i v_j) = \prod_{j=1}^k P_j$, k is even and $e_{ij} \in \mathcal{M}^{-1}$, this means $\mathcal{M}(v_i v_j) = \prod_{j=1}^k P_j$, k is odd and this is a contradiction. Thus, $E(\mathcal{M}^1) \cap E(\mathcal{M}^{-1}) = \emptyset$.

3) let e_{ij} be any edge in the graph G , then ij take three options according to the definition of MFG . The first option that ij has a square prime factor so this edge belongs to the $MFG \mathcal{M}^0$. The second option that ij has a free square prime factor so there are two cases as follows.

Case 1. If ij can be written as the form $\prod_{j=1}^k P_j, k$ is even, then the edge e_{ij} belong to the $MFG \mathcal{M}^1$.

Case 2. If ij can be written as the form $\prod_{j=1}^k P_j, k$ is odd, then the edge e_{ij} belong to the $MFG \mathcal{M}^{-1}$. Thus, $E(\mathcal{M}^0) \cup E(\mathcal{M}^1) \cup E(\mathcal{M}^{-1}) = E(K_n)$ (for example, see Figure 3.2.1)

Therefore, depending of all cases above, the result is obtained. \square

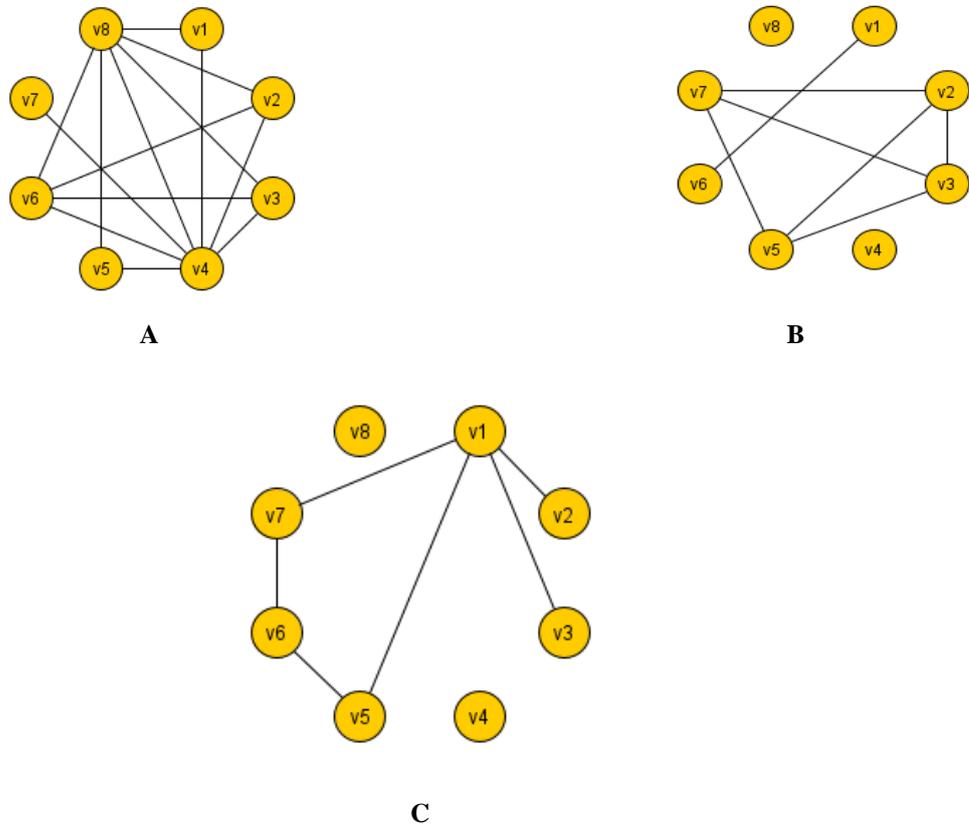


Figure 3.2.1. $MFG \mathcal{M}^{(0)}$ (A), $\mathcal{M}^{(1)}$ (B), and $\mathcal{M}^{(-1)}$ (C)

Corollary 3.2.2.

- 1) $(\mathcal{M}^0)^c = \mathcal{M}^1 \cup \mathcal{M}^{-1}$.
- 2) $(\mathcal{M}^1)^c = \mathcal{M}^0 \cup \mathcal{M}^{-1}$.
- 3) $(\mathcal{M}^{-1})^c = \mathcal{M}^1 \cup \mathcal{M}^0$.

Theorem 3.2.3. Assume that G is the MFG $(\mathcal{M}^0)^c$, then

$$\varpi(G) = \pi(n) + 1.$$

Proof. Take any two vertices v_i and v_j such that i and j are prime numbers, so $\mathcal{M}(ij) = 1$, thus these vertices are adjacent. Therefore, the set of vertices that have prime numbers labeled make an induced subgraph which isomorphic to a complete graph, so $\varpi(G) \geq \pi(n)$. Moreover, the vertex of labeled one is adjacent to all vertices mentioned above, so $\varpi(G) \geq \pi(n) + 1$. let v_r be any other vertex that adjacent to all vertices mentioned above, so there are at least two prime numbers say p and q such that $r = pq$. Thus, this vertex not adjacent to two vertices v_p and v_q and this is a contradiction (as an example, see Figure 3.2.2). Therefore, $\varpi(G) = \pi(n) + 1$. \square

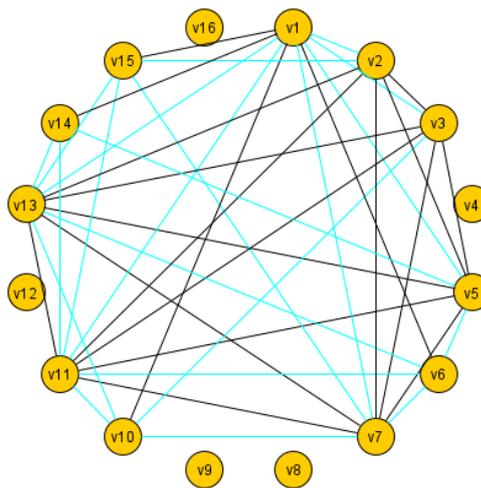


Figure 3.2.2. The MFG $(\mathcal{M}^0)^c$ of order 16, Blue represent \mathcal{M}^{-1} , Black represent \mathcal{M}^1

Proposition 3.2.4. Assume that G is a non-trivial $MFG(\mathcal{M}^0)^c, \mathcal{M}^1$, and \mathcal{M}^{-1} , then

- 1) $\varpi((\mathcal{M}^0)^c) \neq \varpi(\mathcal{M}^1) + \varpi(\mathcal{M}^{-1})$
- 2) $\varpi(\mathcal{M}^1) \leq \varpi((\mathcal{M}^0)^c)$.
- 3) $\varpi(\mathcal{M}^{-1}) = \varpi((\mathcal{M}^0)^c)$ if $n = 1, 2$, otherwise $\varpi(\mathcal{M}^{-1}) < \varpi((\mathcal{M}^0)^c)$.

Proof. According to Theorem 2.2.3, Corollary 2.3.3, and Theorem 3.2.3, the proof is done. \square

Proposition 3.2.5. If G is a $MFG(\mathcal{M}^0)^c$ is non-trivial, then

$$\chi(G) = \pi(n) + 1.$$

Proof. Depending on Theorem 3.2.3, $\varpi((\mathcal{M}^0)^c) = \pi(n) + 1$, then $\pi(n) + 1$ colors which are needed to color the vertices of the largest subgraph that isomorphic to complete graph of order $\pi(n) + 1$. By the same colors that used previously, we can color the remain vertices. Thus, $\chi(G) = \pi(n) + 1$. \square

Proposition 3.2.6. Assume that G is a $MFG(\mathcal{M}^0)^c$ of order n , then

- 1) Each vertex has labeled not free square prime is an isolated vertex.
- 2) Assume that G is a $MFG(\mathcal{M}^0)^c$ of order $n \geq 4$, then $MFG(\mathcal{M}^0)^c$ is disconnected, otherwise, the $MFG(\mathcal{M}^0)^c$ is connected.

Proof. Let $f(v_i) = i$, so

- 1) It is obvious that every vertex has a square prime factor is an isolated depending on the definition of $MFG(\mathcal{M}^0)^c$. Thus, the result is obtained.
- 2) if $n = 4$, then $f(v_4) = 4 = 2^2$, thus the vertex v_4 is not adjacent to all other vertices in the $(\mathcal{M}^0)^c$, so this graph is disconnected. Now, if $n \leq 3$, it is obvious that the $(\mathcal{M}^0)^c$ is connected. Therefore, the result is obtained. \square

Proposition 3.2.7. Assume that G is a MFG $(\mathcal{M}^0)^c$ of order n , then

$$\beta((\mathcal{M}^0)^c) = \begin{cases} 1, & \text{if } n = 1 \\ |N_e^n| + |S - N_e^n|, & \text{otherwise} \end{cases}$$

Proof. Two cases depending on the order of the MFG $(\mathcal{M}^0)^c$ are discussed as follows:

Case 1. If $n = 1$, then $\beta((\mathcal{M}^0)^c) = 1$.

Case 2. Consider $u, v \in N_e^n$, so $f(uv) = 2^2 P_z P_w$. Therefore, the two vertices u and v are not adjacent, then the set N_e^n is an independent, so $\beta(G) \geq |N_e^n|$. Each vertex in the set S belong to every independent set. Note that $S \cap N_e^n \neq \emptyset$, so $\beta(G) \geq |N_e^n| + |S - N_e^n|$. Again, let $u_i \notin S, N_e^n$, so the labeled of this vertex is $f(u_i) = P_j \prod_{i=1}^r P_i, P_j \neq 2$. It is clear that there is at least one $P_i, i = 1, \dots, r$ such that P_i divides one of the labeled of a vertex in the set N_e^n , since $2 < P_j$ (as an example, see Figure 3.2.3). Therefore, $\beta((\mathcal{M}^0)^c) = |N_e^n| + |S - N_e^n|$. \square

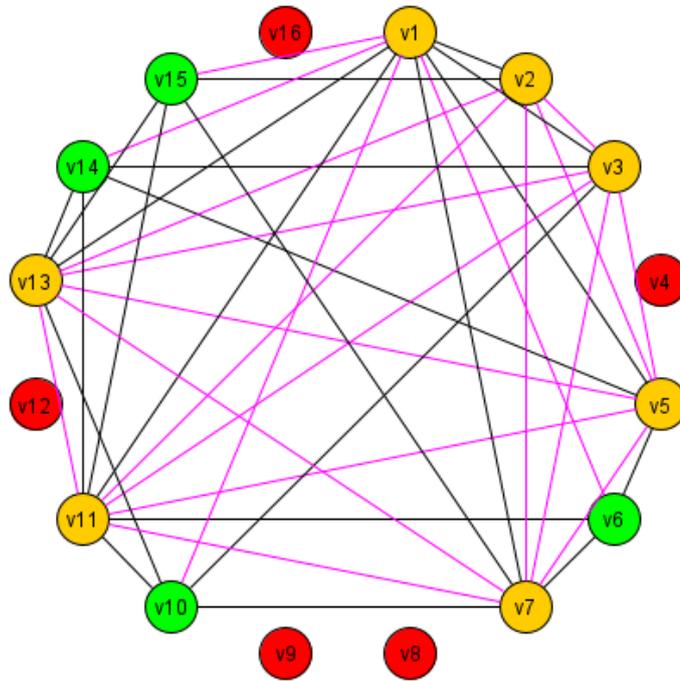


Figure 3.2.3. MFG $(\mathcal{M}^0)^c$, red represent independence number and isolated, Green represent even number

Proposition 3.2.8. Assume that G is a non-trivial MFG $(\mathcal{M}^0)^c$, \mathcal{M}^1 , and \mathcal{M}^{-1} , then

- 1) $\beta((\mathcal{M}^0)^c) \neq \beta(\mathcal{M}^1) + \beta(\mathcal{M}^{-1})$.
- 2) $\beta((\mathcal{M}^0)^c) \neq \beta(\mathcal{M}^1)$ if $n = 2, 3, 4, 5$, otherwise $\beta((\mathcal{M}^0)^c) = \beta(\mathcal{M}^1)$.
- 3) $\beta((\mathcal{M}^0)^c) \leq \beta(\mathcal{M}^1)$
- 4) $\beta((\mathcal{M}^0)^c) = \beta(\mathcal{M}^{-1})$ if $n = 1, 2$, otherwise $\beta((\mathcal{M}^0)^c) \neq \beta(\mathcal{M}^{-1})$.
- 5) $\beta((\mathcal{M}^0)^c) \leq \beta(\mathcal{M}^{-1})$.

Proof.

(1-4) are straightforward from Proposition 2.3.7 and Proposition 3.2.7.

There are two cases as follows.

Case 1. If $n = 1, 2, \dots, 6$, it is obvious by using the Proposition 2.3.7, and Proposition 3.2.7.

Case 2. If $n \geq 7$, then $\beta((\mathcal{M}^0)^c) = |N_e^n| + |S - N_e^n| = |S \cup N_e^n| = |S| + |N_e^n - S|$ and $\beta(\mathcal{M}^{-1}) = |S \cup S_1| = |S| + |S_1|$, since $S \cap S_1 = \emptyset$. Since $|N_e^n - S| \leq |S_1|$, then $\beta((\mathcal{M}^0)^c) \leq \beta(\mathcal{M}^{-1})$. \square

Proposition 3.2.9. Assume that G is a MFG $(\mathcal{M}^0)^c$ of order n , then $\gamma(\mathcal{M}^0)^c = |S| + 1$.

Proof.

Each an isolated vertex belongs to every dominating set, so

$\gamma(\mathcal{M}^0)^c \geq |S|$. The vertex v_1 is adjacent to all vertices which are not isolated. Thus, $\gamma(\mathcal{M}^0)^c = |S| + 1$. \square

Proposition 3.2.10. Assume that G is a MFG $(\mathcal{M}^0)^c$, \mathcal{M}^1 , and \mathcal{M}^{-1} , then

- 1) $\gamma((\mathcal{M}^0)^c) \neq \gamma(\mathcal{M}^1) + \gamma(\mathcal{M}^{-1})$.
- 2) $\gamma((\mathcal{M}^0)^c) \leq \gamma(\mathcal{M}^1)$ and $\gamma((\mathcal{M}^0)^c) \leq \gamma(\mathcal{M}^{-1})$.

Proof. It is straightforward from Theorem 2.2.7, Theorem 2.3.8, and proposition 3.2.9. \square

Proposition 3.2.11. Assume that G is a $MFG (\mathcal{M}^0)^c$ of order $n; n \geq 4$, then $|S| = \lfloor \frac{n}{2} \rfloor - 1$, if $n = 4, 9$ and $|S| \leq \lfloor \frac{n}{2} \rfloor - 2$ otherwise.

Proof.

Two cases are discussed as follows:

Case 1. If $n = 4, 9$, then when $n = 4$, the only isolated vertex is v_4 and when $n = 9$ there are three isolated vertices v_4, v_8 , and v_9 , so for each value of this case, $|S| = \lfloor \frac{n}{2} \rfloor - 1$.

Case 2. If $n \geq 10$, then $|S| \leq \lfloor \frac{n}{2} \rfloor - 2$ by induction .

Thus, according to two above results, the required is obtained. \square

Theorem 3.2.12. Assume that G is a $MFG (\mathcal{M}^0)^c$ of order n and non-trivial, then

- 1) $\beta(G) + \gamma(G) < n$ if $n \neq 2, 4$ and $\beta(G) + \gamma(G) = n$ if $n=2, 4$.
- 2) $\gamma(G) \leq \beta(G)$.

Proof.

1) Two cases deepened on n are classifications as follows:

Case 1: if $1 \leq n \leq 4$, then one can easily conclude the result.

Case 2: if $n \geq 5$, then according to theorem 3.2.9, $\gamma(\mathcal{M}^0)^c = |S| + 1$, and by proposition 3.2.7, $\beta(G) = |N_e^n| + |S - N_e^n| = |S| + |N_e^n - S|$ where S is the isolated vertices set in G and N_e^n is the even number thus, $\beta(G) + \gamma(G) = |S| + |N_e^n - S| + |S| + 1 = 2|S| + |N_e^n - S| + 1 < n$, so $\beta(G) + \gamma(G) < n$. depending on the two cases above, the result is obtained.

2) it is obvious that this is true when $n= 1, \dots, 6$. so, $\forall n \geq 7$,

$\gamma(G) = |S| + 1 < |N_e^n| + |S - N_e^n| = \beta(G)$. thus, the result is obtained. \square

Theorem 3.2.13. Assume that G be a $MFG (\mathcal{M}^0)^c$ of order n and non-trivial, then

1) $deg(v_1) = n - |S|$.

2) $deg(v_p) = n - |S| - 1$, where p is prime number such that $p > \frac{n}{2}$.

Proof.

Let v_m be any not isolated vertex, then m can be written free square of prime factors this means $m = \prod_{j=1}^k P_j$, then

1) The two vertices v_1 and v_m are adjacent to all m . Moreover, the vertex v_m is not adjacent to other vertices since these vertices are isolated. Thus, $deg(v_1) = n - |S|$.

2) By the same manner in (1), the vertex v_p is adjacent to all vertices not isolated except itself, so $deg(v_p) = n - |S| - 1$.

(as an example, see Figure 3.2.4). □

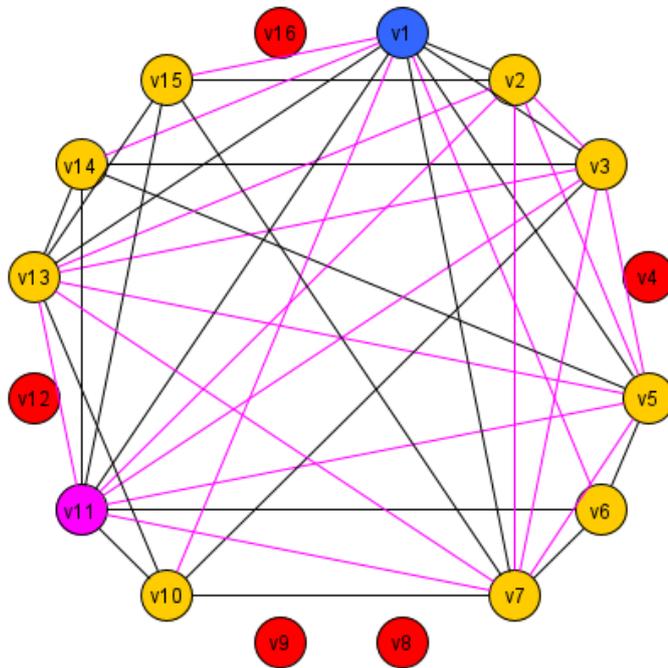


Figure 3.2.4. blue represent deg_{v_1} , Bing represent prime number $> \frac{n}{2}$

Red represents independence number.

Proposition 3.2.14. Assume that G is a MFG $(\mathcal{M}^0)^c$ of order n and $u, v \in (\mathcal{M}^0)^c$, $g. c. d(u, v) = 1$, then $\mathcal{M}(u.v) = \mathcal{M}(u).\mathcal{M}(v)$.

Proof. Three cases are discussed as follows:

Case 1. $u, v \in \mathcal{M}^{(1)}$, then by Proposition 2.2.8, the result is obtained.

Case 2. If $u, v \in \mathcal{M}^{(-1)}$, then by Proposition 2.3.9, the result is obtained.

Case 3. If $u \in \mathcal{M}^{(1)}$ and $v \in \mathcal{M}^{(-1)}$, then $\mathcal{M}(u) = 1$ and $\mathcal{M}(v) = -1$, so $u = \prod_{i=1}^s P_i$; s is even and $v = \prod_{j=1}^r P_j$; r is odd, thus $u.v = \prod_{i=1}^s P_i \prod_{j=1}^r P_j = \prod_{k=1}^{s+r} P_k$ and it is clear that $s+r$ is odd, since $g. c. d(u, v) = 1$. Thus, $\mathcal{M}(u.v) = -1 = \mathcal{M}(u).\mathcal{M}(v)$.

Case 4. If $u \in \mathcal{M}^{(-1)}$ and $v \in \mathcal{M}^{(1)}$, then in the same manner in case 3, the result is obtained.

Thus, according to two above results, the required is obtained. \square

Proposition 3.2.15. Assume that G is a MFG $(\mathcal{M}^0)^c$ of order n and non-trivial, then the radius,

$$Rad((\mathcal{M}^0)^c) = Diam((\mathcal{M}^0)^c) = \begin{cases} 1, & n \leq 3 \\ \infty, & n > 3 \end{cases}$$

and the $Cent((\mathcal{M}^0)^c)$ and $Per((\mathcal{M}^0)^c)$ are isomorphic to K_n graph, if $n \leq 3$.

Proof.

Two cases depend on n are discussed as follows.

Case1. If $n < 4$, then $(\mathcal{M}^0)^c \equiv K_n$, thus $Rad((\mathcal{M}^0)^c) = Diam((\mathcal{M}^0)^c) = 1$ and $Cent((\mathcal{M}^0)^c) \equiv Per((\mathcal{M}^0)^c) \equiv K_n$.

Case 2. If $n \geq 4$, then the graph contains at least one an isolated vertex, so the graph is disconnected. Thus, $Rad((\mathcal{M}^0)^c) = Diam((\mathcal{M}^0)^c) = \infty$.

Thus, according to two above results, the required is obtained. \square

3.3 Complement the Mobius Function Graph $(\mathcal{M}^1)^c$ in Graph Theory

Theorem 3.3.1. Assume that G is the MFG $(\mathcal{M}^1)^c$,

then $\varpi(G) = r; r = |N_e^n \cup \{n_1: n_1 = p^2 m, m \in Z^+ - \{0\}\}|$

Proof.

If $v_i, v_j \in N_e^n$, then $\mathcal{M}(ij) = 0$, so these vertices are adjacent.

Thus, $\varpi(G) \geq |N_e^n|$. Let $S_1 = \{n_1: n_1 = p^2 m, m \in Z\}$ and let

$n_1 \in S_1$, so the vertex v_{n_1} is adjacent to all other vertices in the

graph G . Therefore, the vertices of the set $N_e^n \cup S_1$ is isomorphic

to the complete graph and, $\varpi(G) \geq |N_e^n \cup S_1|$. Let $j \notin N_e^n \cup S_1$,

then index of the vertex v_j is written as $j \in N_o^n - S_1$. Now, for

each vertex in N_e^n say v_t such that $g.c.d(t, j) = 1$, $\mathcal{M}(tj) = -1$, then

the vertex v_j dose not adjacent to all vertices of the form as the vertex v_t .

Thus, the set $N_e^n \cup S_1$ is the maximum set such that the induced subgraph of it isomorphic to complete graph (as an example, see Figure 3.3.1)

and $\varpi(G) = |N_e^n \cup S_1|$. \square

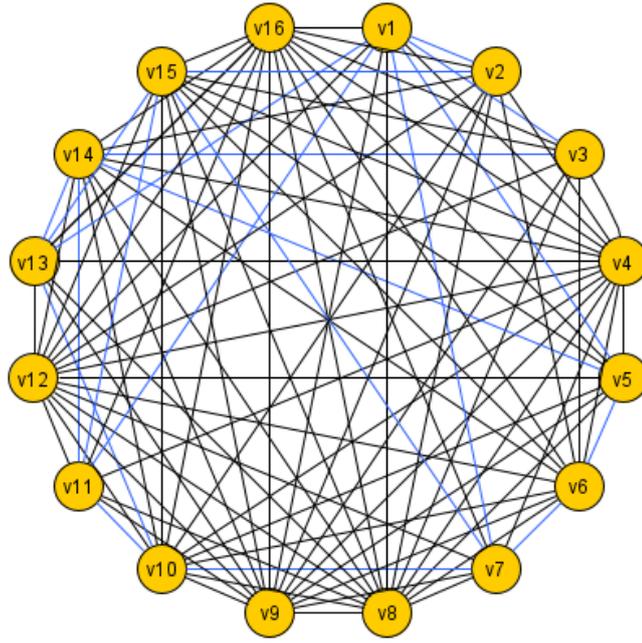


Figure 3.3.1. The $MFG(\mathcal{M}^1)^c$ of order 16.

Proposition 3.3.2. Assume that G is a non-trivial $MFG(\mathcal{M}^1)^c$, \mathcal{M}^0 , and \mathcal{M}^{-1} then

- 1) $\varpi((\mathcal{M}^1)^c) \neq \varpi(\mathcal{M}^0) + \varpi(\mathcal{M}^{-1})$
- 2) $\varpi(\mathcal{M}^0) = \varpi((\mathcal{M}^1)^c)$
- 3) $\varpi(\mathcal{M}^{-1}) \leq \varpi((\mathcal{M}^1)^c)$.

Proof. According to Theorem 2.1.2, Corollary 2.3.3, and Theorem 3.3.1, the proof is done. \square

Proposition 3.3.3. If G is a $MFG(\mathcal{M}^1)^c$ is non-trivial, then

$$\chi(G) = \left| N_e^n \cup \{n_1 : n_1 = p^2 m, m \in Z^+ - \{0\}\} \right|.$$

Proof:

Depending on Theorem 3.3.1, $\varpi((\mathcal{M}^1)^c) = \left| N_e^n \cup \{n_1 : n_1 = p^2 m, m \in Z^+ - \{0\}\} \right|$, then $\left| N_e^n \cup \{n_1 : n_1 = p^2 m, m \in Z^+ - \{0\}\} \right|$. colors which

are needed to color the vertices of the largest subgraph that isomorphic to complete graph of order $|N_e^n \cup \{n_1 : n_1 = p^2 m, m \in Z^+ - \{0\}\}|$. by the same colors that used previously, we can color the remain vertices. Thus,

$$\chi(G) = |N_e^n \cup \{n_1 : n_1 = p^2 m, m \in Z^+ - \{0\}\}|. \quad \square$$

Proposition 3.3.4. Assume that G is a $MFG (\mathcal{M}^1)^c$ of order n , then $MFG (\mathcal{M}^1)^c$ is connected.

Proof. There are two cases as follows.

Case 1. If $n = 1, 2, 3$, then the result is clear.

Case 2. If $n \geq 4$, then the vertex v_4 is adjacent to all other vertices.

Thus, the graph G is connected. \square

Proposition 3.3.5. Assume that G is a $MFG (\mathcal{M}^1)^c$ of order n , then

$$\beta((\mathcal{M}^1)^c) = \begin{cases} 1, & \text{if } n = 1 \\ \pi(n), & \text{if } n \geq 2 \end{cases}$$

Proof.

Case 1. If $n = 1$, then it is obvious that $\beta((\mathcal{M}^1)^c) = 1$.

Case 2. If $n \geq 2$, then let S be the set of all prime number, Therefore, any two vertices u and v of the set S are not adjacent, then the set S is an independent, so $\beta(G) \geq \pi(n)$ (as an example, see Figure 3.3.2). Now, suppose that w is any labeled of the vertex such that w is not prime, so there are at least two primes number say p and q such that $w = pq$. The vertex w is adjacent to two vertices that labeled p and q . Therefore,

$$\beta((\mathcal{M}^0)^c) = \pi(n). \quad \square$$

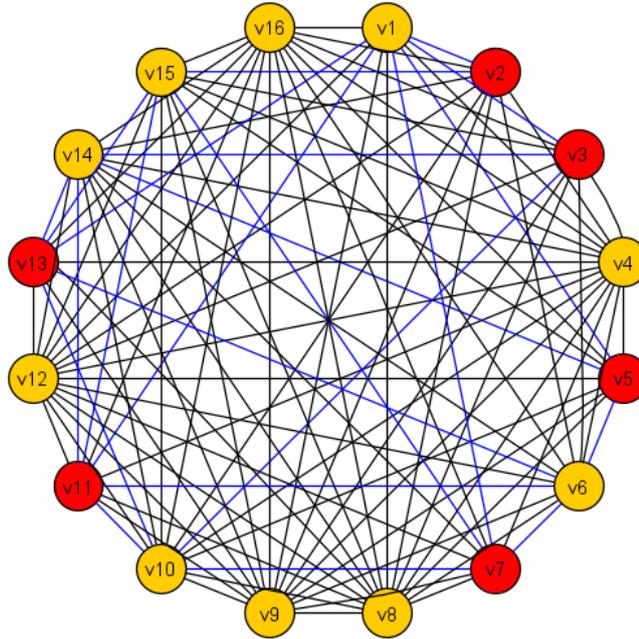


Figure 3.3.2. The $MFG(\mathcal{M}^1)^c$ of order 16, red represent independence number.

Proposition 3.3.6. Assume that G is a non-trivial $MFG(\mathcal{M}^1)^c$, \mathcal{M}^0 , and \mathcal{M}^{-1} then

- 1) $\beta((\mathcal{M}^1)^c) \neq \beta(\mathcal{M}^0) + \beta(\mathcal{M}^{-1})$.
- 2) $\beta((\mathcal{M}^1)^c) \leq \beta(\mathcal{M}^0)$.
- 3) $\beta((\mathcal{M}^1)^c) \leq \beta(\mathcal{M}^{-1})$.

Proof.

(1-3) are straightforward from Proposition 2.3.7, and Proposition 3.3.5. \square

Proposition 3.3.7. Assume that G be a $MFG(\mathcal{M}^1)^c$ of order n , then $\gamma(\mathcal{M}^1)^c = 1$.

Proof. If $n = 1, 2, 3$, one can be concluded that the result is obtained. Otherwise, the vertex v_4 is adjacent to all other vertices. Thus, $\gamma(\mathcal{M}^1)^c = 1$.

Proposition 3.3.8. Assume that G is a MFG $(\mathcal{M}^1)^c$, \mathcal{M}^0 , and \mathcal{M}^{-1} , then

- 1) $\gamma((\mathcal{M}^1)^c) \neq \gamma(\mathcal{M}^0) + \gamma(\mathcal{M}^{-1})$.
- 2) $\gamma((\mathcal{M}^1)^c) \leq \gamma(\mathcal{M}^0)$ and $\gamma((\mathcal{M}^1)^c) \leq \gamma(\mathcal{M}^{-1})$.

Proof. It is straightforward from Theorem 2.2.7, Theorem 2.3.8, and proposition 3.4.7. \square

Theorem 3.3.9. Assume that G is a MFG $(\mathcal{M}^1)^c$ of order n and non-trivial, then

- 1) $\beta(G) + \gamma(G) > n$ if $n = 1$, $\beta(G) + \gamma(G) = n$ if $n = 2, 3$, and $\beta(G) + \gamma(G) \leq n$ if $n \geq 4$
- 2) $\gamma(G) \leq \beta(G)$.

Proof. 1)

There are three cases as follows.

Case 1. If $n=1$, then $\beta(G) = \gamma(G) = 1$, so $\beta(G) + \gamma(G) > n$.

Case 2.

I) If $n=2$, then $\beta(G) = 1, \gamma(G) = 1$, so $\beta(G) + \gamma(G) = n$.

II) If $n = 3$, then $\beta(G) = 2, \gamma(G) = 1$, so $\beta(G) + \gamma(G) = n$.

Case 3. If $n \geq 4$, then $\beta(G) + \gamma(G) = \pi_p + 1 < n$.

From all cases above the result is obtained.

2) For each graph .

Thus, the result is obtained. $\gamma(G) = 1 \leq \begin{cases} \mathbf{1} & , \text{if } n = 1 \\ \pi(n) & , \text{if } n \geq 2 \end{cases} = \beta(G)$

From all cases above, the proof is done. \square

Theorem 3.3.10. Assume that G is a MFG $(\mathcal{M}^1)^c$ of order n and non-trivial, then $\deg(v_1) = n - \left| \prod_{i=1}^k P_i, k \text{ is even} \right|$.

Proof.

Let v_m be any vertex, then m can be written even number this means

$m = \prod_{i=1}^k P_i$, k is even, then the two vertices v_1 and v_m are adjacent to all m . Moreover, the vertex v_m is even number not adjacent to other vertices since these vertices. Thus, $deg(v_1) = n - |\prod_{i=1}^k P_i, k \text{ is even}|$. \square

Proposition 3.3.11. Assume that G be a MFG $(\mathcal{M}^1)^c$ of order n and $u, v \in (\mathcal{M}^1)^c$, $g.c.d(u, v) = 1$, then $\mathcal{M}(u.v) \neq \mathcal{M}(u).\mathcal{M}(v)$.

Proof. Three cases are discussed as follows:

Case 1. $u, v \in \mathcal{M}^{(0)}$, then by Proposition 2.1.7, the result is obtained.

Case 2. If $u, v \in \mathcal{M}^{(-1)}$, then by Proposition 2.3.10, the result is obtained.

Case 3. If $u \in \mathcal{M}^{(0)}$ and $v \in \mathcal{M}^{(-1)}$, then $\mathcal{M}(u) = 0$ and $\mathcal{M}(v) = -1$, so $v = \prod_{j=1}^r P_j$; r is odd, thus, and it is clear that $\mathcal{M}(u.v) \neq \mathcal{M}(u).\mathcal{M}(v)$. \square

Proposition 3.3.12. Assume that G is a MFG $(\mathcal{M}^1)^c$ of order n and non-trivial, then the radius, $Rad((\mathcal{M}^1)^c) = 1, \forall n$

$$Diam((\mathcal{M}^1)^c) = \begin{cases} 1, & \text{if } n = 2 \\ 2, & n \geq 3 \end{cases}$$

and the $Cent((\mathcal{M}^1)^c)$ and $Per((\mathcal{M}^1)^c)$ are isomorphic to K_n graph, if $n \geq 3$.

Proof.

Two cases depend on n are discussed as follows.

Case1. If $n = 2$, then $(\mathcal{M}^1)^c \equiv K_n$, thus $Rad((\mathcal{M}^1)^c) = 1, Diam((\mathcal{M}^1)^c) = 1$ and $Cent((\mathcal{M}^1)^c) \equiv Per((\mathcal{M}^1)^c) \equiv K_n$.

Case 2. If $n \geq 3$, Thus, $Rad((\mathcal{M}^0)^c) = 1, Diam((\mathcal{M}^0)^c) = 2$.

Thus, according to two above results, the required is obtained. \square

3.4 Complement the Mobius Function Graph $(\mathcal{M}^{-1})^c$ in Graph Theory

Theorem 3.4.1. Assume that G is the MFG $(\mathcal{M}^{-1})^c$, then $\varpi(G) = \pi(p) + |\{v_m; p^2 \text{ is a factor of } m \text{ and } p \text{ is a prime}\}|$.

Proof.

Take any two vertices v_i and v_j such that i and j are prime numbers, so $\mathcal{M}(ij) = 1$, thus these vertices are adjacent, thus $\varpi(G) \geq \pi(p)$. Let v_i be a vertex such that $i \in \{v_m; p^2 \text{ is a factor of } m \text{ and } p \text{ is a prime}\}$. It is clear that $\mathcal{M}(ij) = 0, \forall j$, so the vertex v_i is adjacent to all other vertices. Thus, $\varpi(G) \geq \pi(p) + |\{v_m; p^2 \text{ is a factor of } m \text{ and } p \text{ is a prime}\}|$. Now, let v_t be any vertex not belonging to the sets that mentioned above, so, $t \in \{\prod_{i=1}^k P_i, k > 1\}$. Thus, for each prime number say different from primes in the set $\{P_i; i = 1, \dots, k\}$ such that $k + 1$ is an odd the vertex v_t dose not adjacent to the vertex v_{P_r} . (As an example, see Figure 3.4.1.). Therefore, the result is obtained. \square

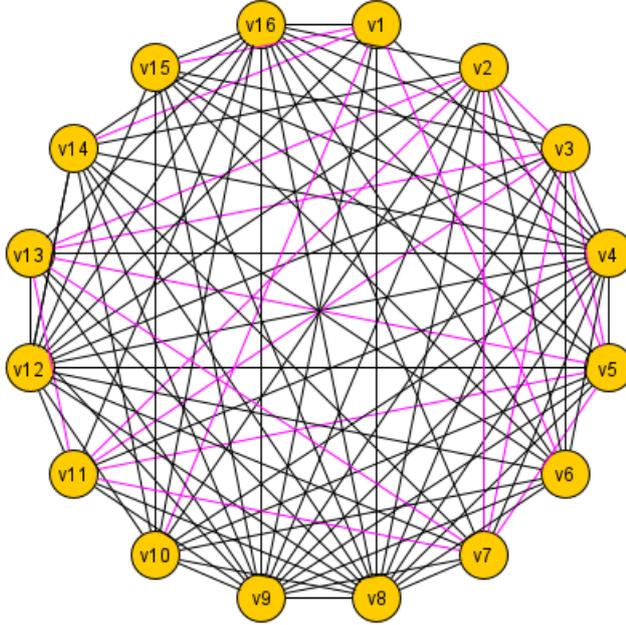


Figure 3.4.1. The $MFG(\mathcal{M}^{-1})^c$ of order 16.

Proposition 3.4.2. Assume that G is a non-trivial $MFG(\mathcal{M}^{-1})^c$, \mathcal{M}^0 , and \mathcal{M}^1 then

- 1) $\varpi((\mathcal{M}^{-1})^c) \neq \varpi(\mathcal{M}^0) + \varpi(\mathcal{M}^1)$
- 2) $\varpi(\mathcal{M}^0) = \varpi((\mathcal{M}^{-1})^c)$
- 3) $\varpi(\mathcal{M}^1) \leq \varpi((\mathcal{M}^{-1})^c)$.

Proof. According to Theorem 2.1.2, Theorem 2.2.3, and Theorem 3.4.1, the proof is done. \square

Proposition 3.4.3. If G is a $MFG(\mathcal{M}^{-1})^c$ is non-trivial, then

$$\chi(G) = |\{v_m; p^2 \text{ is a factor of } m \text{ and } p \text{ is a prime}\}|.$$

Proposition 3.4.4. Assume that G is a $MFG(\mathcal{M}^{-1})^c$ of order n , then

$$\beta((\mathcal{M}^{-1})^c) = \begin{cases} 1, & \text{if } n = 1 \\ 2, & \text{otherwise} \end{cases}.$$

Proof.

Case 1. If $n = 1$, then it is obvious that $\beta((\mathcal{M}^{-1})^c) = 1$.

Case 2. If $n = 2, \dots, 5$ then let $S = \{v_1, v_p\}$, where p is prime number less than or equal n . The set S is independent and it is maximal, since each vertex of prime number is adjacent with the vertex v_p and the vertex v_4 is adjacent to all other vertices. Thus, $\beta((\mathcal{M}^{-1})^c) = 2$.

Case 3. If $n \geq 6$ then let $S = \{v_3, v_6\}$, it is obvious that the set S is independent. The vertex v_1 is adjacent to v_6 so we cannot add this vertex to the set S . The labeled of other vertices (m) were of three forms:

I) If m has a factor as square prime, then the vertex v_m is adjacent to all other vertices, thus $v_m \notin S$.

II) If m has a free factor as square prime, then there are two types as follows:

a) If $m = \pi_{i=1}^k P_i$, k is odd, then the vertex v_m is adjacent to the vertex v_3 , thus $v_m \notin S$.

b) If $m = \pi_{i=1}^k P_i$, k is even, then the vertex v_m is adjacent to the vertex v_6 , thus $v_m \notin S$.

Thus, the set S is maximal independent and $\beta((\mathcal{M}^{-1})^c) = 2$.

From all cases above, the result is obtained. \square

Proposition 3.4.5. Assume that G is a non-trivial MFG $(\mathcal{M}^{-1})^c$, \mathcal{M}^0 , and \mathcal{M}^1 then

$$1) \beta((\mathcal{M}^{-1})^c) \neq \beta(\mathcal{M}^0) + \beta(\mathcal{M}^1).$$

$$2) \beta((\mathcal{M}^{-1})^c) \leq \beta(\mathcal{M}^0).$$

$$3) \beta((\mathcal{M}^{-1})^c) = \beta(\mathcal{M}^1).$$

Proof.

(1-3) are straightforward from Theorem 2.1.1 , Proposition 2.2.6 . \square

Proposition 3.4.6. Assume that G is a MFG $(\mathcal{M}^{-1})^c$ of order n , then

$$\gamma(\mathcal{M}^{-1})^c = \begin{cases} 2, & \text{if } n = 2,3 \\ 1, & \text{if otherwise} \end{cases}$$

Proof: There are two cases as follows.

Case 1. If $n = 2,3$, then it is obvious that $\gamma(\mathcal{M}^{-1})^c = 2$.

Case 2. There are two subcases

Subcase 1. If $n = 1$, then it is clear that $\gamma(\mathcal{M}^{-1}) = 1$.

Subcase 2. If $n \geq 4$, then the vertex v_4 is adjacent to all other vertices, so $\gamma(\mathcal{M}^{-1}) = 1$.

From all cases above, the result is obtained. \square

Proposition 3.4.7. Assume that G is a MFG $(\mathcal{M}^1)^c, \mathcal{M}^0$, and \mathcal{M}^{-1} , then

1) $\gamma((\mathcal{M}^{-1})^c) \neq \gamma(\mathcal{M}^0) + \gamma(\mathcal{M}^1)$.

2) $\gamma((\mathcal{M}^{-1})^c) \leq \gamma(\mathcal{M}^0)$ and $\gamma((\mathcal{M}^{-1})^c) = \gamma(\mathcal{M}^1)$.

Proof. It is straightforward from Theorem 2.1.6, Theorem 2.2.7, and proposition 3.4.6. \square

Theorem 3.4.8. Assume that G is a MFG $(\mathcal{M}^{-1})^c$ of order n and non-trivial, then

1) $\beta(G) + \gamma(G) > n$ if $n \leq 3$ and $\beta(G) + \gamma(G) < n$ if $n > 3$

2) $\gamma(G) \neq \beta(G)$.

Proof.

1) There are two cases as follows.

Case 1. If $n \leq 3$, according to Proposition 3.4.6 and Proposition 3.4.4, the result is clear.

Case 2. If $n > 3$, then $\beta(G) + \gamma(G) = 1 + 2 < n$

2) There is no

2) From proof of (1), the result is obtained.

From two cases above, the result is obtained. \square

Theorem 3.4.9. Assume that G is a $MFG (\mathcal{M}^{-1})^c$ of order n and non-trivial, then $deg(v_1) = n - |\pi_{i=1}^k P_i, k \text{ is odd}|$.

Proof.

Let v_m be any not prime number then m can be written square of prime factors this means $m = \prod_{j=1}^k P_j$, then

The two vertices v_1 and v_m are adjacent to all m . Moreover, the vertex v_1 is not adjacent to prime number since these vertices are isolated. Thus, $deg(v_1) = n - |\pi_{i=1}^k P_i, k \text{ is odd}|$. \square

Proposition 3.4.10. Assume that G is a $MFG (\mathcal{M}^{-1})^c$ of order n and $u, v \in (\mathcal{M}^{-1})^c$, $g. c. d(u, v) = 1$, then $\mathcal{M}(u.v) = \mathcal{M}(u).\mathcal{M}(v)$.

Proof. Three cases are discussed as follows:

Case 1. $u, v \in \mathcal{M}^{(0)}$, then by Proposition 2.1.7, the result is obtained.

Case 2. If $u, v \in \mathcal{M}^{(-1)}$, then by Proposition 2.3.9, the result is obtained.

Case 3. If $u \in \mathcal{M}^{(0)}$ and $v \in \mathcal{M}^{(-1)}$, then $\mathcal{M}(u) = 0$ and $\mathcal{M}(v) = -1$, so $v = \prod_{j=1}^r P_j; r \text{ is odd}$, thus, and it is clear that $\mathcal{M}(u.v) \neq \mathcal{M}(u).\mathcal{M}(v)$. \square

Proposition 3.4.11. Assume that G is a $MFG (\mathcal{M}^{-1})^c$ of order n and non-trivial, then the radius,

$$Rad((\mathcal{M}^{-1})^c) = Diam((\mathcal{M}^{-1})^c) = \begin{cases} \infty, & \text{if } n = 2,3 \\ 1, & n > 2 \end{cases}$$

and the $Cent((\mathcal{M}^{-1})^c)$ and $Per((\mathcal{M}^{-1})^c)$ are isomorphic to K_n graph, if $n > 2$.

Proof.

Two cases depend on n are discussed as follows.

Case 1. If $n \leq 2$, then the graph is disconnected, thus $Rad((\mathcal{M}^{-1})^c) = Diam((\mathcal{M}^{-1})^c) = \infty$.

Case 2. If $n > 2$, Thus, $Rad((\mathcal{M}^{-1})^c) = Diam((\mathcal{M}^{-1})^c) = 1$.

and $Cent((\mathcal{M}^{-1})^c) \equiv Per((\mathcal{M}^{-1})^c) \equiv K_n$

Thus, according to two above results, the required is obtained. \square

3.5 Comparing the six types of mobius function graph in Graph Theory

Through the previous result that we obtained from the study of six types mobius function graph $\mathcal{M}^{(0)}$, mobius function graph $\mathcal{M}^{(1)}$, and mobius function graph $\mathcal{M}^{(-1)}$, $MFG(\mathcal{M}^{(0)})^c$, $MFG(\mathcal{M}^{(1)})^c$, and $MFG(\mathcal{M}^{(-1)})^c$ through which we can compare the six types for each of their properties, as follows:

The following table shows the calculation of each of the properties that previously studied in each graphs

$$\mathcal{M}^{(0)}, \mathcal{M}^{(1)}, \mathcal{M}^{(-1)}, (\mathcal{M}^{(0)})^c, (\mathcal{M}^{(1)})^c \text{ and } (\mathcal{M}^{(-1)})^c .$$

Property	$\mathcal{M}^{(0)}$	$\mathcal{M}^{(1)}$	$\mathcal{M}^{(-1)}$
$\pi(n)$	$\beta(G) - 1$	$\deg(v_2) + 1 - \{u: u = \prod_{i=1}^k P_i, k \text{ is odd}; P_i \neq 2 \forall i\}$	$\pi(n) = \deg(v_1) - \{u: u = \prod_{i=1}^k P_i, k \text{ is odd}, k > 1\}$
$\gamma(G)$	$\begin{cases} n, & \text{if } n \leq 3 \\ 1, & \text{if } n > 3 \end{cases}$	$\begin{cases} 1, & \text{if } n = 1 \\ 2, & \text{if } n = 2, 3 \\ S + 2, & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } n = 1, 2, 3 \\ 2 & \text{if } n = 4, 5 \\ 2 + S , & \text{if } n \geq 6 \end{cases}$
$\beta(G)$	$\pi_p + 1$	$\begin{cases} 1, & \text{if } n = 1 \\ 2, & \text{if } n = 2, 3 \\ 3, & \text{if } n = 4, 5, 6 \\ N_e^n + S - N_e^n , & n \geq 7 \end{cases}$	$\begin{cases} 1, & \text{if } n = 2 \\ \lfloor \frac{n}{2} \rfloor + 1, & \text{if } n = 1, 3, 4 \\ \lfloor \frac{n}{2} \rfloor + 1, & \text{if } n = 5, 6, \forall n \\ S \cup S_1 , & \forall n \geq 7 \end{cases}$
$\varpi(G)$	$ N_e^n + S $ where $S = \{a: p^\alpha \setminus a, \alpha > 1 \text{ s.t. } a \notin N_e^n\}$	$\pi(n)$	$n \geq 2 \text{ is } 2$

$\chi(G)$	$ E \cup \{p^{\alpha>1} - E\}$	$\pi(n)$	2
Rad(G)	$\begin{cases} \infty, & n \leq 3 \\ 1, & n > 3 \end{cases}$	$\begin{cases} 1, & n \leq 3 \\ \infty, & n > 3 \end{cases}$	$\begin{cases} 1, & n \leq 3 \\ \infty, & n > 3 \end{cases}$
Diam(G)	$\begin{cases} \infty, & n \leq 3 \\ 2, & n > 3 \end{cases}$	$\begin{cases} n-1, & n \leq 3 \\ \infty, & n \geq 4 \end{cases}$	$\begin{cases} n-1; & n \leq 3 \\ \infty & n \geq 4 \end{cases}$

Property	$(\mathcal{M}^0)^c$	$(\mathcal{M}^1)^c$	$(\mathcal{M}^{-1})^c$
$\overline{\omega}(G)$	$\pi(n) + 1$	$r; r = N_e^n \cup \{n_1: n_1 = p^2 m, m \in \mathbb{Z}^+ - \{0\}\} $	$\pi(p) + \{v_m: p^2 \text{ is a factor of } m \text{ and } p \text{ is a prime}\} $
$\chi(G)$	$\pi(n) + 1$	$ N_e^n \cup \{n_1: n_1 = p^2 m, m \in \mathbb{Z}^+ - \{0\}\} $	$\pi(p) + \{v_m: p^2 \text{ is a factor of } m \text{ and } p \text{ is a prime}\} $
$\beta(G)$	$\begin{cases} 1, & \text{if } n = 1 \\ N_e^n + S - N_e^n , & \text{o.w.} \end{cases}$	$\begin{cases} 1, & \text{if } n = 1 \\ \pi(n), & \text{if } n \geq 2 \end{cases}$	$\begin{cases} 1, & \text{if } n = 1 \\ 2, & \text{otherwise} \end{cases}$
$\gamma(G)$	$ S + 1$	1	$\begin{cases} 2, & \text{if } n = 2, 3 \\ 1, & \text{otherwise} \end{cases}$
$\text{Deg}(v_1)$	$n \cdot S $	$n \cdot \pi_{i-1}^k p i, k \text{ is even} $	$n \cdot \pi_{i-1}^k p i, k \text{ is odd} $
Rad	$\begin{cases} 1, & \text{if } n \leq 3 \\ \infty, & \text{if } n > 3 \end{cases}$	1	$\begin{cases} \infty, & \text{if } n = 2 \\ 1, & \text{if } n > 2 \end{cases}$
Diam	$\begin{cases} 1, & \text{if } n \leq 3 \\ \infty, & \text{if } n > 3 \end{cases}$	$\begin{cases} 1, & \text{if } n = 2 \\ 2, & \text{if } n \geq 3 \end{cases}$	$\begin{cases} \infty, & \text{if } n = 2 \\ 1, & \text{if } n > 2 \end{cases}$

!) From the above table, we notice that there are six ways to calculate the prime-counting function from the graph \mathcal{M}^1 while there is one way in other graphs. All these ways determined the number of prime numbers less than or equal fixed natural number n . This is one of the important results that we obtained, as there is no method or function to calculate this number in an exact manner, and everyone knows how important the prime numbers are in most of the calculations. It is possible for the reader to follow the calculations of all the other properties, through which we know the comparison between each characteristic in each graph.

The work of the previous researchers was developed in the Mobius function \mathcal{M}^0 , where the independence number, chromatic number, clique number and domination number were calculated prime number was found through the graph exactly. We started our new work in the Mobius function $\mathcal{M}^{(1)}, \mathcal{M}^{(-1)}, (\mathcal{M}^0)^c, (\mathcal{M}^1)^c$ and $(\mathcal{M}^{-1})^c$.

where the exact number of the prime number was found and the relationship between the independence number and the domination number was calculated.

A comparison table has been made for six types of the Mobius function. We note from the table that each type of the Mobius function has its independence in the solution, so this does not mean that it the largest, the smallest, or the equal is the best solution.

Chapter Four

Conclusions and Future works

4.1 Conclusions

The graph theory is a broad research field, and each time new and distinct ideas can be added from this work, the following conclusions can be drawn:

- In this thesis three new ways to determine the prime-counting function by using a new graph is called the Mobius Function Graph $\mathcal{M}^{(1)}$ are given. Also, some properties of this graph are proved. In addition, the domination, independence, and clique numbers are calculated, a comparison between the two numbers the domination and independence is made.
- A new graph is constructed in this work, it is called the Mobius function graph $\mathcal{M}^{(-1)}$. According to the obtained result in this work, the domination, independence, and clique number are determined. Also, some properties of this graph are calculated. Moreover, the relation between the independence number and domination number is discussed.
- New graph is introduced which is called a *MFG* $(\mathcal{M}^0)^c$. In this graph the domination, independence, clique, and chromatic numbers are determined and compare with the same numbers in two graphs the *MFG* $\mathcal{M}^{(1)}$ and *MFGM* $^{(-1)}$ and find the relationship between them. Also, proved that the graphs \mathcal{M}^0 , \mathcal{M}^1 , and \mathcal{M}^{-1} are decomposition of complete graph of order n . Moreover, the $\text{Cent}((\mathcal{M}^0)^c)$ and $\text{Per}((\mathcal{M}^0)^c)$ are founded.
- New graph is introduced which is called a *MFG* $(\mathcal{M}^1)^c$. In this graph the domination, independence, clique, and chromatic numbers are determined and compare with the same numbers in two graphs the *MFG* $\mathcal{M}^{(0)}$ and *MFGM* $^{(-1)}$ and find the relationship between them. Also, proved that the graphs \mathcal{M}^0 , \mathcal{M}^1 , and \mathcal{M}^{-1} are decomposition of complete graph of order n . Moreover, the $\text{Cent}((\mathcal{M}^1)^c)$ and $\text{Per}((\mathcal{M}^1)^c)$ are founded.
- New graph is introduced which is called a *MFG* $(\mathcal{M}^{-1})^c$. In this graph the domination, independence, clique, and chromatic numbers are determined and compare with the same numbers in two graphs the *MFG* $\mathcal{M}^{(1)}$ and *MFGM* $^{(0)}$ and find the relationship between them. Also, proved that the graphs \mathcal{M}^0 , \mathcal{M}^1 , and \mathcal{M}^{-1} are decomposition of complete graph of order n . Moreover, the $\text{Cent}((\mathcal{M}^{-1})^c)$ and $\text{Per}((\mathcal{M}^{-1})^c)$ are founded.

4.2 Future Works

Future work necessary to:

1. Find more properties for mobius function graph and its inverse properties as well as for the domination of the edges and its inverse.
2. . Create a new definition of the mobius function graph by introducing more conditions and finding all the properties that are related to it and its inverse.
3. The search for the possibility of modified mobius function graph by inserting the terms of known domination parameters such as the definition of connected domination, total domination and independent domination, etc.
4. Study mobius function in digraph.
5. Apply the definition of inverse mobius function to special graphs and find their properties.

References

- [1] Alsinai, A., Alwardi, A., Ahmed, H., & Soner, N.D. (2021d). " Leap Zagreb indices for the Central graph of graph". *Journal of Prime Research in Mathematics*, 2(17), 73-78.
- [2] Alsinai, A., Alwardi, A., & Soner, N.D.(2021c)."Topological Properties"
- [3] A. A. Omran and T. Swadi, "Observer Domination Number in Graphs", *Journal of Advanced Research in Dynamical and Control Systems* 11(1 Special Issue), pp. 486-495, 2019.
- [4] A. A. Omran and T. A. Ibrahim, "Fuzzy co-even domination of strong fuzzy graphs", *Int. J. Nonlinear Anal. Appl.* 12(2021) No. 1, 727-734.
- [5] A. A. Omran, M. N. Al-Harere, and Sahib Sh. Kahat, "Equality co-neighborhood domination in graphs, *Discrete Mathematics, Algorithms and Applications*, vol. 14, No. 01, (2022).

- [6] Berge C. , "**Theory of Graphs and its Applications**", Methuen, London, 1962.
- [7] B.Gayathri and S.Kaspar , "**Connected Co-Independence domination of a graph**", *Int.j.Contemp.Math.Sciences*,6(9)(2011),423-429.

- [8] Chartrand G., Zhang P., "**A First Course in Graph Theory**", Dover Publication, USA, 2012.

- [9] Chellson N. L., "**Inverse Domination Number of Graph**", M.Sc thesis, HOD, Department of Mathematics, Government Science College, 2010.

- [10] Cockayne E. J., Dawes R. M., Hedetniemi S. T., "**Total Domination in Graphs**", *Networks*, 10, 211–219, 1980.
- [11] Du D. Z., Wan P. J., "Connected Dominating Set: Theory and Applications", Springer, New York, 77, 2013.
- [12] F. Al-Maamori, T. Hilberdink, "An Example in Beurling's Theory of Generalised Primes", *Acta Arithmetica*, 168(4), pp. 383-395, 2015.
- [13] F.A. Al-Maamori, "Examples of beurling prime systems", *Mathematica Slovaca*, 67(2), pp. 321-344, 2017.
- [14] F. AL-Maamori, S.A. AL-Ameedee, and W.G. Atshan, "on sandwich results of univalent functions defined by a linear operator", *Journal of Interdisciplinary Mathematics*, 23(4), pp. 803-809, 2020.
- [15] F.A. Al-Maamori, S.A. Al-Ameedee, and W.G. Atshan, "**Second Hankel Determinant for Certain Subclasses of Bi-univalent functions**", *Journal of Physics: Conference Series*, 1664(1), 2020. Graphene Using Y_k polynomial: In Proceedings of the Jangjeon Mathematical Society, 3(24), 375-388.
- [16] F. Harary, "**Graph theory**", Addison-Wesley, Reading, Mass., 1969.
- [17] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman & Hall/CRC, 2005.
- [18] G.H. Hardy, E.M. Wright, "**An Introduction to the theory of numbers**", 5 th ed., Clarendon press., Oxford, (2002).
- [19] H. J. Yousif and A. A. Omran, "**2-anti fuzzy domination in anti-fuzzy graphs**", 2020 IOP Conf.
- [20] Haynes T. W., Hedetniemi S. T., Slater P. J., "**Fundamentals of Domination in Graphs**", Marcel Dekker, Inc., New York, 1998.
- [21] I. A. Alwan and A. A. Omran, *Domination Polynomial of the Composition of Complete*.

- [22] K. S. Al'Dzhabri, A. A. Omran, and M. N. Al-Harere, "DG-domination topology in Digraph", Journal of Prime Research in Mathematics, 2021, 17(2), pp. 93-100.
- [23] Kulli V. R., Sigarkanti S.C., "Inverse Domination in Graphs", National Academy Science Letters, India, 14, 473-475, 1991.
- [24] Kulli V. R., Soner N.D., "Complementary Edge Domination in Graphs", Indian J Pure Appl.Math. 28, 917-920, 1997.
- [25] k.k.Srimitra , Shaik Sajana , D.Bharathi., " Some properties of Graph of Mobius Function for 0 " ,international journal of Innovative Research in Science,Engineering and Technology.vol.6,Issue 8, August 2017.
- [26] M. M. Shalaan and A. A. Omran, "Co-Even Domination Number in Some Graphs", IOP Conf. Ser.: Mater. Sci. Eng. 928 042015, 2020.
- [27] M.N. Al-Harere, A. A. Omran, "Binary operation graphs", AIP conference proceeding 2086, 2019.
- [28] Omran A. A., Ibrahim T. A., "Whole Domination in Graph", University of Babylon, Iraq, 2017.
- [29] Ore O., "Theory of Graphs", American Mathematical Society, Providence, R.I., USA, 1962.
- [30] R.L. Brooks, On Coloring the Nodes of a Network, Proc. Camb. Philos. Soc. 37, 194–197, 1941.
- [31] R.H.Aravinth,R.Vignesh, "Mobius Function Graph $\mathcal{M}_n(G)$ " , International journal of Innovative Technology and Exploring Engineering ISSN:2278-3075,vol.8,August 2019.
- [32] S. K. Al-Asadi, A. A. Omran, and F.A. Al-Maamori, "SOME PROPERTIES OF MOBIUS FUNCTION GRAPH $\mathcal{M}^{(1)}$ ", Advanced Mathematical Models and Applications, Vol.7, No.1, pp.48-54, 2022.

- [33] S. K. Al-Asadi, A. A. Omran, and F.A. Al-Maamori, "The Mobius function graph $\mathcal{M}^{(-1)}$ in a graph, Accepted in Applied Mathematics and Information Sciences, 2022.
- [34] S. A. Imran, A. Alsinai, A. A. Omran, A. Khan, and H. A. Othman, "The Stability or Instability of Co-Even Domination in Graphs", Applied Mathematics and Information Sciences, 16, No. 3, 473-478 (2022).
- [35] S. A. Imran and A. A. Omran, "Total co-even domination in graphs in some of engineering project theoretically", AIP Conference Proceedings, 2386, 060012, 2022.
- [36] Sampathkumar E., Walikar H. B., "The Connected Domination Number of a Graph", Journal of Mathematical and Physical Sciences, 13(6), 607-613, 1979.
- [37] Srimitra K. K., Shaik Sajana, Bharathi D., "Adjacency and Degree of Graph of Mobius function", i – Manager's Journal on Mathematics, Vol. 5, No. 1, January – March, 2016, pp 31 – 38.
- [38] S. H. Talib, A. A. Omran, and Y. Rajihy, Additional Properties of Frame Domination in Graphs, 2020 J. Phys.: Conf. Ser. **1664** 012026. doi:10.1088/1742- Sorochan S. V., "Fundamentals of Graph Theory", N. I. Lbachevsky state University of Nizhny Novgorod, 2013.
- [39] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, Fundamentals of Domination in Graphs, Marcel. 6596/1664/1/012026.
- [40] S. H. Talib, A. A. Omran, and Y. Rajihy , "Inverse Frame Domination in Graphs" , 2020 IOPConf.
- [41] Thomas Hungerford , Abstract algebra in introduction , Celeveland State University .
- [42] Yamuna M., Karthika .Dekker, New York, NY, USA, 1998 K., "Medicine Names as a DNA Sequence using Graph Domination", Scholars Research Library Journals, USA, 5(6), 2747- 2756, 2014.

[44] Zmazek B., Zerovnik J., “On Domination numbers of Graph Bundles”, *Journal of Applied Mathematics & Computing*, 22(1-2), 39-48, 2006. 2006.

الخلاصة

خلال هذه الرسالة ، تمت مناقشة العديد من خصائص الرسم البياني لدالة الموبيس الصفر. تم أيضاً تقديم رسوم بيانية جديدة تسمى دالة الموبيس 1 و (-1) يتم عرض أكثر من ثلاث طرق لتحديد وظيفة العد الأولي باستخدام هذه الرسوم البيانية.

أيضاً ، تم إثبات بعض خصائص هذه الرسوم البيانية. علاوة على ذلك ، يتم تحديد رقم الهيمنة والعدد اللوني ورقم الاستقلال وعدد الزمرة لهذه الرسوم البيانية. بالإضافة إلى ذلك ، تم إجراء مقارنة بين الرقمين وهيمنة واستقلال هذه الرسوم البيانية أيضاً. بعد ذلك ، نوضح أن الرسوم البيانية الثلاثة 0 ، 1 ، $\{-1\}$ تشكل تحللاً للرسم البياني الكامل للرتبة n ، حيث n هي ترتيب كل واحد من هذه الرسوم البيانية. لذلك ، يتم تحديد الرسوم البيانية الجديدة اعتماداً على التحليل أعلاه والذي يسمى الرسوم البيانية لدالة الموبيس المكمل. الأول هو الرسوم البيانية الكاملة لدالة الصفر وهو اتحاد رسمين بيانيين 1 و $\{-1\}$ ، والثاني هو الرسوم البيانية الكاملة لدالة الواحد وهو اتحاد اثنين من الرسوم البيانية 0 و $\{-1\}$ ، وآخرها هو الرسوم البيانية الكاملة لدالة $[-1]$ وهو اتحاد بين رسمين بيانيين 0 و 1 . لكل رسم بياني من الرسوم البيانية الثلاثة المذكورة أعلاه ، تمت مناقشة العلاقة بين هذا الرسم البياني مع رسمين بيانيين يشكلان هذا الرسم البياني. علاوة على ذلك ، يتم تحديد رقم الزمرة ورقم الاستقلال ورقم الهيمنة. قارن أيضاً هذه الأرقام بالأرقام المقابلة في رسمين بيانيين يشكلان هذا الرسم البياني. أيضاً ، الرقم اللوني ، $Cent$ و Per تم إنشاء الرسوم البيانية $0, 1, -1$ مع بعض الخصائص.



جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بابل
كلية التربية للعلوم الصرفة قسم الرياضيات

بعض أنواع بيان دالة الموبيس

رسالة

مقدمة الى مجلس كلية التربية للعلوم الصرفة / جامعة بابل
كجزء من متطلبات نيل درجة الماجستير في التربية / الرياضيات

من قبل

سناء كاظم كامل ياسين

بإشراف

أ.د. فائز علي المعموري

أ.د. احمد عبد علي عمران

2022م

1444 هـ

