Republic of Iraq
Ministry of Higher Education
and Scientific Research
University of Babylon
College of Education for Pure Sciences
Department of Mathematics

Solving Constrained and Unconstrained Problems of Numerical Optimization

Research

Submitted to the Council of Department of Mathematics the College of Education, for Pure Science in the University of Babylon in Fulfillment of the Requirement for the Degree of Higher Diploma \ Mathematics

By

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2022 A.D. **1444 A.H.**

بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

وَقُلِ ٱعۡمَلُواْ فَسَيَرَى ٱللَّهُ عَمَلَكُمۡ وَرَسُولُهُۥ وَٱلۡمُؤۡمِنُونَۖ
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Dedication

To whom I prefer, and why not; she sacrificed for me

She spared no effort to always make me happy

(My mother's love).

We walk the paths of life, and it remains who controls our minds in every path we take
The owner of the good face, and good deeds.
He did not spare me for his whole life

(My dear father).
To my professors, respected, and colleagues

To my friends, and everyone who stood next to me and
helped me with all their possessions, in many levels

This research is presented to you, and I hope that it
Will satisfy you.

Ali Hussein Mohammed-Sharba
2022

Acknowledgments

In the name of **God,** the merciful praise be to Allah, and peace and blessing be upon. The prophet Mohammed and His Messenger.

First and foremost, I must acknowledge the unlimited thanks to God, the everlasting terrible; everlasting thanks, for his help. I am absolutely certain that this work will never become a reality without his guidance.

My sincere thanks to my supervisor **Prof. Ahmed Sabah Aljilawi,** for guiding, supporting, motivating and encouraging me throughout the implementation of this work.

My sincere thanks and gratitude to my father, mother and wife for their love and support throughout my life. Thank you all for giving me the power to reach the start and chase my dream. My brothers and friends deserve my sincere thanks.

Last but not last, I sincerely thank all the people involved in making this thesis read.

Ali Hussein Mohammed-Sharba

2022

Abstract

This research deals with solving the constrained and unconstrained problems of numerical optimization. The problem of linear programming for numerical improvement was verified through a basic mathematical tool that aims to obtain the best value for the variables that provide the minimum value or the maximum of the mathematical function (objective function), which includes all improvement problems to find optimal solutions of the linear programming problem problems. The most important concepts have been addressed with the graphical method. The basic ones that meet the special needs of the work and the use of many methods, including restricted and unrestricted, to reach the presentation of the problem through the diagram and to obtain the optimal solution for the results. It was concluded that the graph solution method is the best mathematical equation and results to show how to get the restrictions that limit your goal to get to your desired goal faster and in the least possible time.

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INTRODUCTION

Optimization has been one of the most fundamental and successful tools in our daily lives. Optimization is an essential mathematical tool that aims to find the best value of variables that provide the minimum value or the maximum for a mathematical function (the objective function). Optimization algorithms are a fundamental and successful tool in mathematical programming to reach a solution, generally with the assistance of a computer. Optimization algorithms start with an initial estimate of the value of the variables and by an iterative technique generates a sequence of improved estimates, or iterates, until an optimal solution is reached. A great algorithm should be efficient, fast, accurate, and robust. It should generate a good approximation of an optimal solution. [1]
Although there are examples of unconstrained optimizations in economics, for example finding the optimal profit, maximum revenue, minimum cost, etc., constrained optimization is one of the fundamental tools in economics and in real life. Consumers maximize their utility subject to many constraints, and one significant constraint is their budget constraint. Similarly, while maximizing profit or minimizing costs, the producers face several economic constraints in real life, for example, resource constraints, production constraints, etc.

The commonly used mathematical technique of constrained optimizations involves the use of Lagrange multiplier and Lagrange function to solve these problems followed by checking the second order conditions using the Bordered Hessian. When the objective function is a function of two variables, and there is only one equality constraint, the constrained optimization problem can also be solved using the geometric approach discussed earlier given that the optimum point is an interior optimum. [2]

Partial derivatives can be used to optimize an objective function which is a function of several variables subject to a constraint or a set of constraints, given that the functions are differentiable. Mathematically, the constrained optimization problem requires to optimize a continuously differentiable function $f(x_1, x_2, \dots, x_n)$ subject to a set of constraints.

Chapter one

Constrained and Unconstrained Optimization

CHAPTER 1

Constrained and Unconstrained Optimization

1- Linear programming (LP)

Linear programming is a subfield of optimization. Linear programming is a mathematical technique for finding optimal solutions to the problems. Linear Programming deals with the problem of optimizing a linear objective function subject to linear equality and inequality constraints on the decision variables. Linear programming is not a programming language like C++, Java, or Visual Basic, it's mathematical? model. A feasible solution is a solution that satisfies all of the constraints. The feasible set or feasible region is the set of all feasible solutions. Finally, an optimal solution is the feasible solution that produces the best objective function value possible. [3]

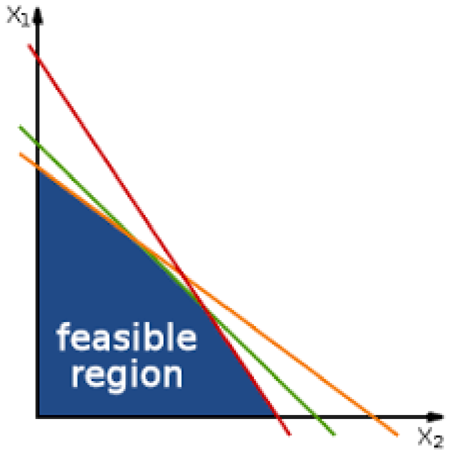


Figure 1-1: Feasible Region

2- Summarize

- Optimization problem is the problem of finding the best solution from all feasible solutions.
- Optimization problems can be divided into two groups depending on whether the variables are continuous or discrete.
- Application: - In mathematics, computer science, economics, or management science, mathematical programming and engineering

3- Definitions and Properties

In this section, we provide definitions, notation, and necessary results of Euclidean space. Also, we present some basic concepts and facts of real mathematical analysis. For more on this material, see, e.g.,

3.1-Vector Norm

In this section, we introduce the Euclidean space, which means a finite dimensional vector space with an inner product $\langle\cdot,\cdot\rangle$ and the Euclidean norm defined by $\|x\| = \sqrt{\langle x, x\rangle}$ for all vectors x in the vector space.[4]

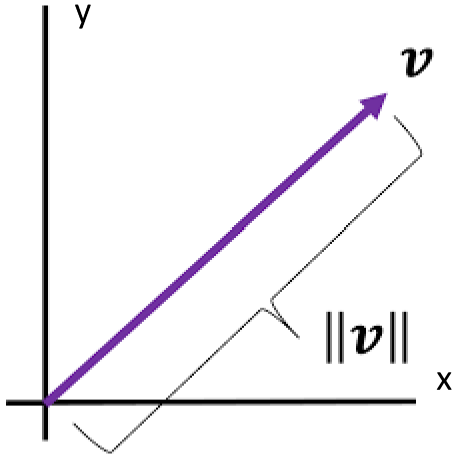


Figure 1- 2: Vector Norm space

Definition 3.1.1: A function $\langle\cdot,\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is an inner product if

(i) $\langle x, x\rangle > 0, \langle x, x\rangle = 0 \Leftrightarrow x = 0$ (positivity).

(ii) $\langle x, y\rangle = \langle y, x\rangle$ (symmetric).

(iii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ (additivity).

(iv) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for all $\alpha \in \mathbb{R}$ (homogeneity).

Example: consider $\langle u, v\rangle = 3u_1v_1 + 2u_2v_2$, find Inner product.

Solution:

1) $\langle v, v\rangle = 3v_1v_1 + 2v_2v_2$

$= 3(v_1)^2 + 2(v_2)^2 > 0$

2) $\langle v, u\rangle = 3v_1u_1 + 2v_2u_2 = \langle u, v\rangle$

3) $\langle u, v+w \rangle = 3u_1(v_1+w_1) + 2u_2(v_2+w_2)$

$= 3u_1v_1 + 3u_1w_1 + 2u_2v_2 + 2u_2w_2$

$= 3u_1v_1 + 2u_2v_2 + 3u_1w_1 + 2u_2w_2$

$= \langle u, v \rangle + \langle u, w \rangle$

4) $c\langle u, v \rangle = c(3u_1v_1 + 2u_2v_2)$

$= 3cu_1v_1 + 2cu_2v_2$

$= \langle cu, v \rangle$

Definition 3.1.2: A function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is said to be a norm, if it satisfies the following properties:

(i) $\|x\| \geq 0, \forall x \in \mathbb{R}^n$; $\|x\| = 0$ if and only if $x = 0$

(ii) $\|\alpha x\| = |\alpha|\|x\|, \forall \alpha \in \mathbb{R}. \forall x \in \mathbb{R}^n$

(iii) $\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in \mathbb{R}^n$

The Euclidean norm is defined by

$$\|x\|_2 = \sqrt{x^T x} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Example: consider $x = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} \to \|x\|_1 = |1| + |-3| + |5| = 9$

$$\to \|x\|_2 = (|1|^2 + |-3|^2 + |5|^2)^{\frac{1}{2}} = \sqrt{1+9+25} = \sqrt{35}$$

- The inner product of any two real n −vectors x and y is defined by

$$\langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

- The Cauchy-Schwarz Inequality: For vectors x and y in $\mathbb{R}^n$, we have

$$\langle x, y \rangle \leq \|x\|\|y\|$$

3.2-Definition of Saddle Points

Saddle points of a multivariable function are those points in its domain where the tangent is parallel to the horizontal axis, but this point tends to be neither a local maximum nor a local minimum. [5]

For a two-variable function $f(x, y)$, its saddle point is defined as
If $Z = f(x, y)$, then the point (x, y, z). is said to be a saddle point if both the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ vanishes, but f does not attain any extremum values (maxima or minima) at (x, y).

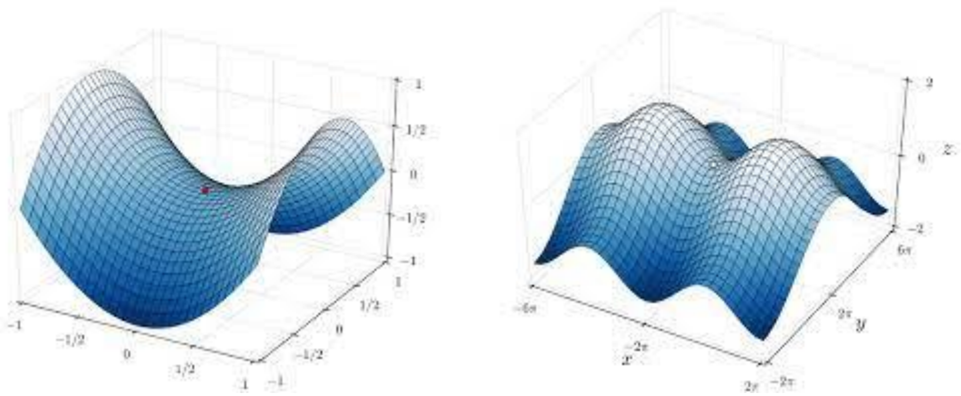


Figure 1-3: Saddle point

Definition 3.2.1: Saddle points in a multivariable function are those critical points where the function attains neither a local maximum value nor a local minimum value. Saddle points mostly occur in multivariable functions. A few single variable functions like $f(x) = x^3$ show a saddle point in its domain. [5]

Definition 3.2.2: Critical points of a function are the points in the domain of the function where either the first **derivative** of the function is equal to zero or the derivative does not exist. If $x = c$ is a critical point for a function f and $f(c)$ exists, then c is called the critical point of f if either $\acute{f}(c) = 0$ *or* $\acute{f}(c)$ does not exist. [5]

3.3-Global and Local maximum (minimum)

Definition 3.3.1: Let $S \subseteq \mathbb{R}^n$ be a nonempty set. If $f: S \to \mathbb{R}$. the **argument of the minimum** is the set of elements in S that achieve the global minimum in S, which is defined by

$$argmin_{x \in S} f(x) = \{x \in S | f(y) \geq f(x),\ \forall\ y \in S\}.$$

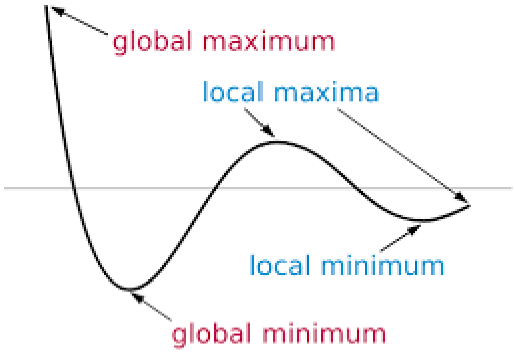


Figure 1- 4: Global and Local maximum (minimum)

Example 1: The graph illustrate $f(x) = (\sin(x - 0 \cdot 5) + \cos x^2) * 2$

The global minimum of $f(x)$ is min $\left(f(x)\right) \approx -2$. While the argmin $(f(x)) \approx 4 \cdot 9$

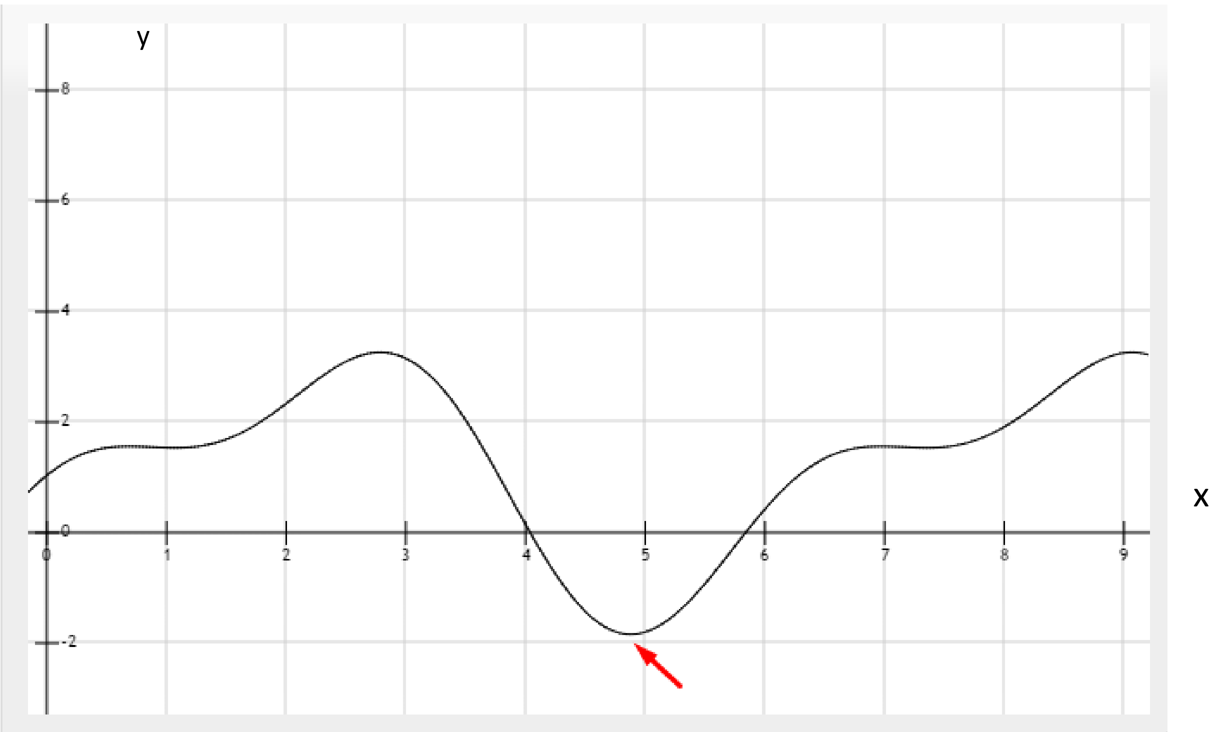


Figure 1- 5: Global Minimum

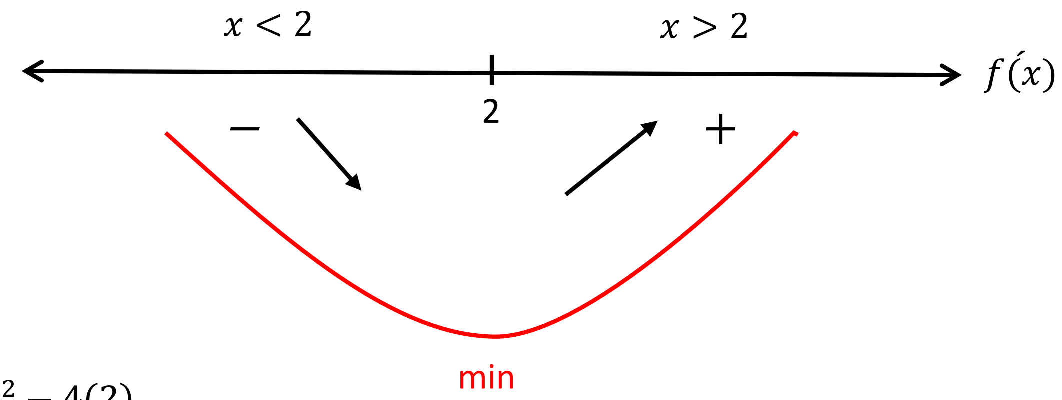
Example 2: consider $f(x) = x^2 - 4x$.

Solution: $f'(x) = 2x - 4(1) = 0$

$\rightarrow 2x = 4$

$\rightarrow x = 2$

$f'(2) = 0$



$f(x) = (2)^2 - 4(2)$

$= 4 - 8$

$= -4$. $(2, -4)$ local min

Example 3: consider $f(x) = 2x^3 + 3x^2 - 12x$.

Solution: $f'(x) = 2(3x^2) + 3(2x) - 12(1)$

$= 6x^2 + 6x - 12$

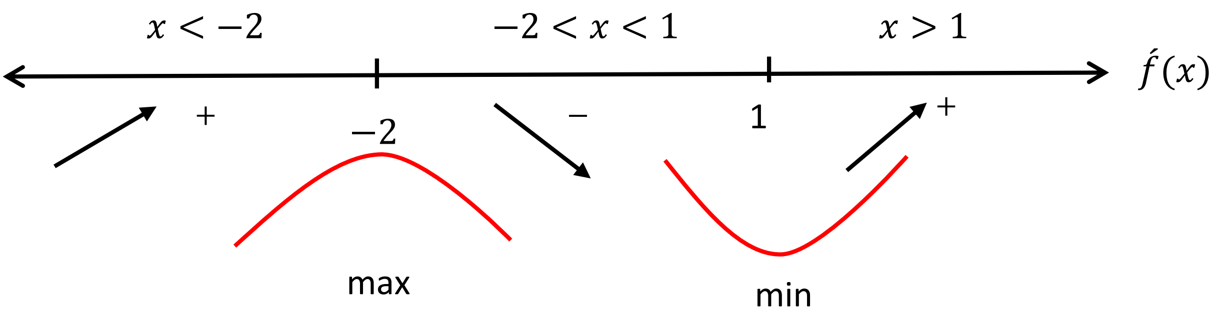
$\because f'(x) = 0 \rightarrow 6x^2 + 6x - 12 = 0$

$6(x^2 + x - 2) = 0$

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0 \rightarrow either\ x + 2 = 0 \rightarrow x = -2$

$or\ x - 1 = 0 \rightarrow x = 1$



$f(1) = 2(1)^3 + 3(1)^2 - 12(1)$

$= 2 + 3 - 12$

$= -7 \quad , \quad (1,-7)\ min$

$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2)$

$= 2(-8) + 3(4) + 24$

$= -16 + 12 + 24$

$= 20 \quad , \quad (-2,20)\ max$

Example 4: consider $f(x) = 3x^4 - 16x^3 + 24x^2$.

Solution: $\acute{f}(x) = 3(4x^3) - 16(3x^2) + 24(2x)$

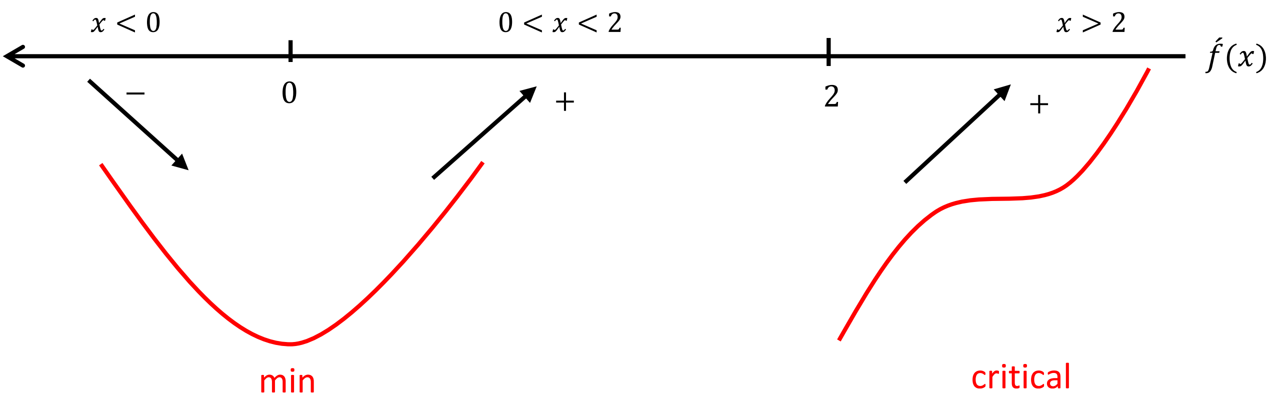
$= 12x^3 - 48x^2 + 48x$

$= 12x(x^2 - 4x + 4)$

$\because \acute{f}(x) = 0 \quad \rightarrow \quad 12x(x^2 - 4x + 4) = 0$

$12x(x-2)^2 = 0$

either $12x = 0 \quad \rightarrow \quad x = 0 \qquad or \ (x-2)^2 = 0 \ \rightarrow x - 2 = 0 \ \rightarrow x = 2$



$f(0) = 0 \quad , \quad (0,0)\ min$

$f(2) = 3(2)^4 - 16(2)^3 + 24(2)^2$

$= 3(16) - 16(8) + 24(4)$

$= 48 - 128 + 96$

$= 16 \quad , \quad (2,16)\ \ critical\ point$

Definition 3.3.2: Let $f: \mathbb{R}^n \to \mathbb{R}$ and $x^* \in \mathbb{R}^n$. Then x^* is called a **local minimizer** of f if there is a scalar $t > 0$ such that $f(x^*) \leq f(x)$ for all $x \in B(x^*, t) = \{x \in \mathbb{R}^n | \|x - x^*\| \leq t\}$.

We call x^* a **strict local minimizer** of f if there is a scalar $t > 0$ such that $f(x^*) < f(x)$ for all $x \neq x^*$ such that $x \in B(x^*, t)$. We call x^* a **global minimizer** of f if $f(x^*) \leq f(x)$ for all $x \in \mathbb{R}^n$. Finally, we say that x^* is a **strict global minimizer** of f if $f(x^*) < f(x)$ for all $x \neq x^*$.[5]

First derivative test

$$\frac{\mathrm{df}(\mathrm{x}^*)}{\mathrm{dx}} = 0$$

Second derivative test

If $\frac{d^2 f(x^*)}{dx^2} < 0$ $then\ f\ has\ local\ \mathrm{max}\ at\ x^*$

If $\frac{d^2 f(x^*)}{dx^2} > 0$ $then\ f\ has\ local\ mim\ at\ x^*$

If $\frac{d^2 f(x^*)}{dx^2} = 0$ $test\ inconclusive$

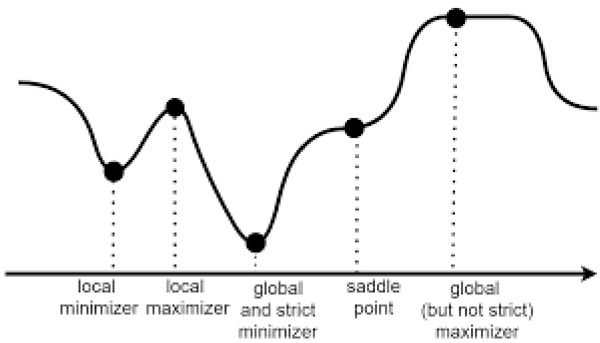


Figure 1-6: strict global minimizer and strict local maximizer

Example: consider $f(x) = -x^2 - \frac{x^3}{3} + \frac{x^4}{4}$.

Solution:

$$\frac{df(x)}{dx} = -2x - x^2 + x^3$$

$x^* solution\ 1 = -1$ $(local\ minimum)$

$x^*\ solution\ 2 = 0$ $(local\ maximum)$

$x^*\ solution\ 3 = 2$ $(local\ minimum)$

$$\frac{d^2 f(x)}{dx^2} = -2 - 2x + 3x^2$$

$$\frac{d^2 f(x^* solution\ 1)}{dx^2} = +3$$

$$\frac{d^2 f(x^*\ solution\ 2)}{dx^2} = -2$$

$$\frac{d^2 f(x^*\ solution\ 3)}{dx^2} = +6$$

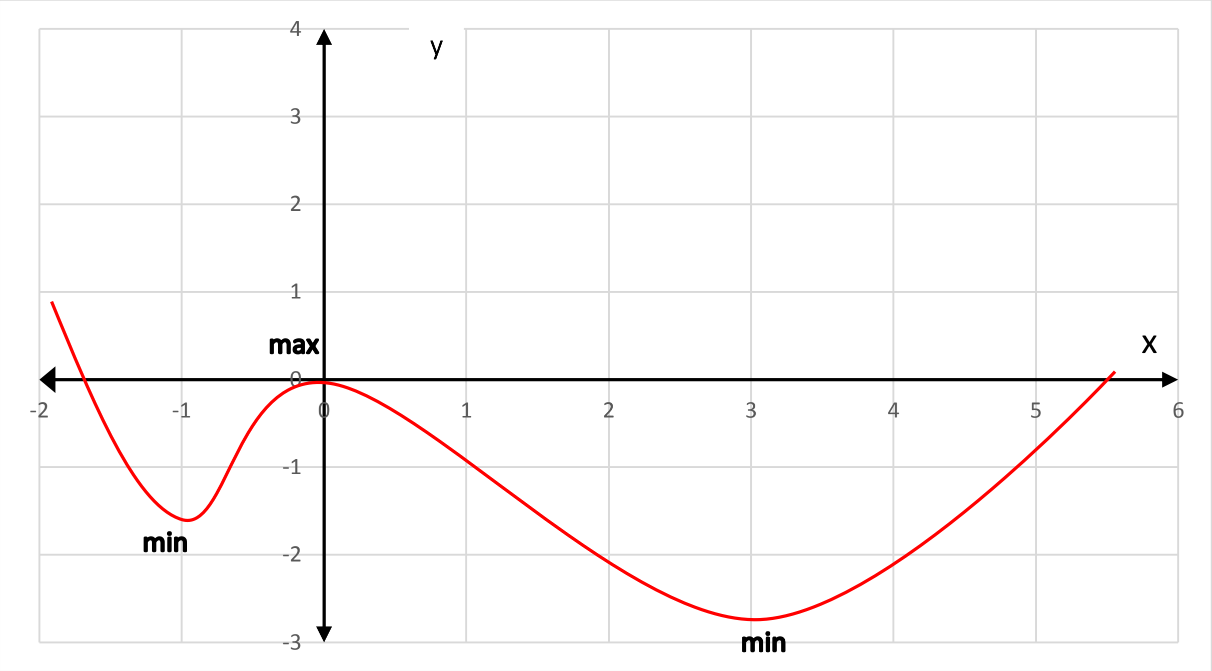


Figure 1-7: Local minimum and local maximum

4-Convex Sets and Functions

Definition 4.1: Let $S \subseteq \mathbb{R}^n$ · If the line segment between any two points in S lies in S, i.e.,

$$\lambda x_1 + (1 - \lambda)x_2 \in S, \ \ \forall\, x_1, x_2 \in S, \forall\, \lambda \in [0,1].$$

then S is said to be **convex**. It can be shown that a set $S \subseteq \mathbb{R}^n$ is convex if and only if for any $x_1, \dots\ , x_n\ \in S$, the convex combination

$$\sum_{i=1}^{n} \lambda_i x_i$$

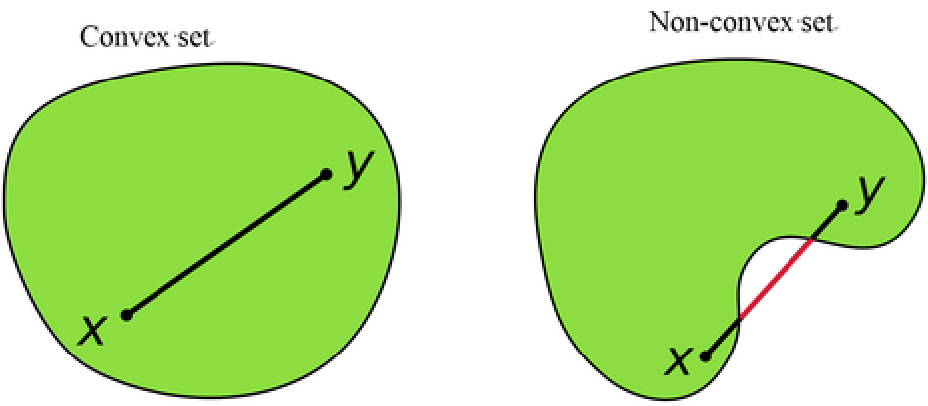


Figure 1-8: Convex set

Where $\sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, i = 1, \dots \ , n.$ belongs to S .[6]

Example 1: Let A_1, A_2 be convex sets in R^n , show that

1. The set $A_1 + A_2$ is convex.
2. The set $A_1 \cap A_2$ is convex.

Solution:

$$1) \because \ A_1 \text{ is convex set } \rightarrow \forall \, x_1, y_1 \in A_1, \forall \, \lambda \in [0,1].$$

$$\rightarrow \lambda x_1 + (1-\lambda) y_1 \in A_1 \dots (1)$$

$$\because \ A_2 \text{ is convex set } \rightarrow \forall \, x_2, y_2 \in A_2 \, , \forall \, \lambda \in [0,1].$$

$$\rightarrow \lambda x_2 + \lambda(1-\lambda) \in A_2 \dots (2)$$

To prove $\lambda x_1 + \lambda x_2 + (1-\lambda) y_1 + (1-\lambda) y_2 \in A_1 + A_2$

$$\lambda \underbrace{(x_1 + x_2)}_{x_i \in A_1 + A_2 . i=1.....n} + (1-\lambda) \underbrace{(y_1 + y_2)}_{y_i \in A_1 + A_2 . i=1.....n} \in A_1 + A_2$$

So $A_1 + A_2$ *is convex set* , $\forall \, x_1, x_2, y_1, y_2 \in A_1 + A_2$

$\forall \, \lambda \, , 1 - \lambda \ \in [0,1].$

There for: $\lambda x + (1-\lambda) y \in A_1 + A_2$.

2) Let $x, y \in A_1 \cap A_2$, $\forall \lambda \in [0,1]$.

$\rightarrow x, y \in A_1 \wedge x, y \in A_2$

$as\ x, y \in A_1, A_1\ is\ convex$

$\rightarrow \lambda x + (1-\lambda)y \in A_1$

Also, as $x, y \in A_2, A_2\ is\ convex$

$\rightarrow \lambda x + (1-\lambda)y \in A_2$

There for: $\lambda x + (1-\lambda)y \in A_1 \cap A_2, \forall \lambda \in [0,1]$.

$A_1 + A_2\ \ is\ convex\ set$

Example 2: Let $A_1, A_2\ and\ A_3$ be convex sets in $R^n, and\ \alpha \in R$, Show that

1. The set $\alpha A_1 = \{Z \in R^n | Z = \alpha x, x \in A_1\}$ is convex.
2. The set $A_2 + A_3 = \{Z \in R^n | Z = x_1 + x_2, x_1 \in A_2,\ x_2 \in A_3\}$

Proof:

1) suppose $a, b \in A_1, \lambda \in [0,1]\ , then$

$a = \alpha x\ \ such\ that\ \ x \in A_1\ \ ,\ \ \alpha \in R$

$b = \alpha y\ \ such\ that\ \ y \in A_1\ \ ,\ \ \alpha \in R$

$\rightarrow \lambda a + (1-\lambda)b = \lambda(\alpha x) + (1-\lambda)(\alpha y)$

$= \alpha \underbrace{(\lambda x + (1-\lambda)y)}_{\in A_1} \in \alpha A_1$

$\therefore\ \alpha A_1$ is convex.

2) suppose $a, b \in A_2 + A_3\ , \lambda \in [0,1]\ , then$

$a = x_1 + x_2\ such\ that\ x_1 \in A_2, x_2 \in A_3$

$b = y_1 + y_2\ such\ that\ y_1 \in A_2, y_2 \in A_3$

$\lambda a + (1-\lambda)b = \lambda(x_1 + x_2) + (1-\lambda)(y_2 + y_3)$

$= \underbrace{\lambda x_1 + (1-\lambda)y_2}_{\in A_2} + \underbrace{\lambda x_2 + (1-\lambda)y_3}_{\in A_3} \in A_2 + A_3$

$\therefore\ A_2 + A_3$ is convex.

Definition 4.2: The **convex hull** of a set $S \subseteq \mathbb{R}^n$ is denoted $con(S)$ and is the smallest convex set that contains S. The convex hull of a set S is the set of all convex combinations of points in S; that is

$$con(S) = \{\sum_{i=1}^{n} \lambda_i x_i \,|x_i \in S, \lambda_i \geq 0, i = 1, \dots, n.\ \sum_{i=1}^{n} \lambda_i = 1\}$$

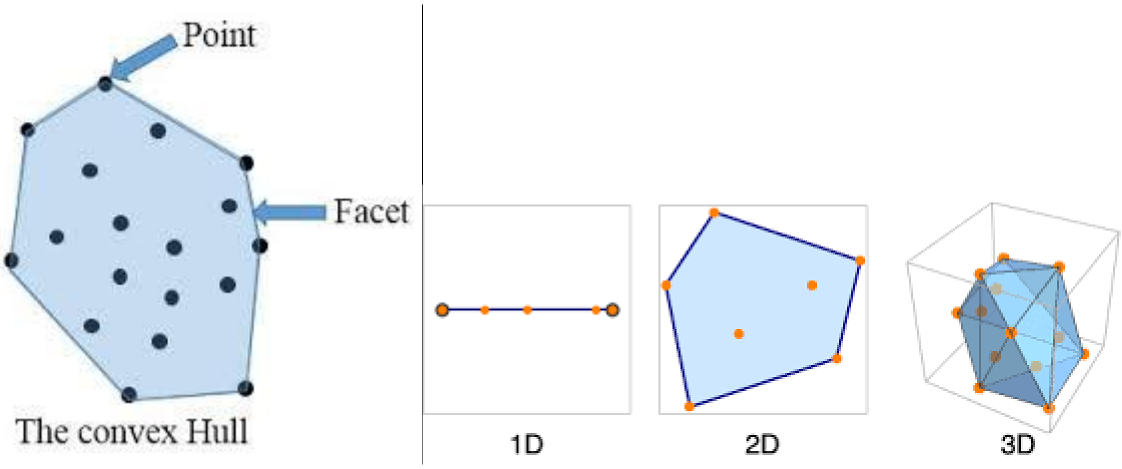


Figure 1-9: convex hull

Definition 4.3: Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set. If $f: S \to \mathbb{R}$ satisfies

$$f(\lambda x + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2), \qquad \forall\, x_1, x_2 \in S, \forall\, \lambda \in [0,1].$$

then f is said to be a **convex function** on S. If the above inequality is true as a strict inequality for all $x_1 \neq x_2$ and for all $\lambda \in (0,1)$, then f is called a **strictly convex function** on S. If there is a constant $b > 0$ such that for all $x_1, x_2 \in S$, and for all $\alpha \in [0,1]$.

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) - \frac{1}{2} b\lambda(1-\lambda)\|x_1 - x_2\|^2.$$

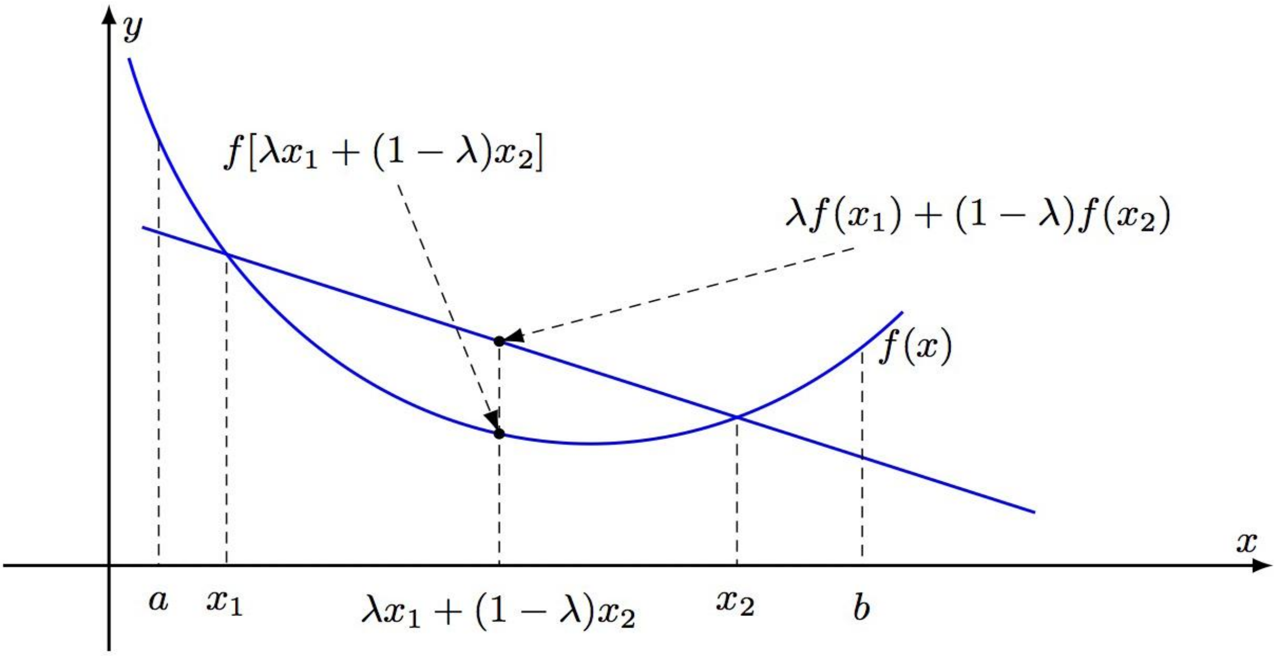


Figure 1-10: Convex function

then f is called **strongly convex function** on S. A function f is concave if $-f$ is convex.[6]

Example 1: If $f_1, f_2, \dots\ , f_n$ are convex function. Show that $f(x) = Max\{f_1(x), f_2(x),\ \dots\ , f_n(x)\}$ is convex?

Solution:

suppose $x, y \in R^n\ and\ \lambda \in [0,1]$.

Since $f_1, f_2, \dots\ , f_n$ are convex functions .

Then $f_1(\lambda x + (1-\lambda)y) \leq \lambda f_1(x) + (1-\lambda)f_1(y)$

$$\leq \lambda f(x) + (1-\lambda)f(y)$$

$$f_2(\lambda x + (1-\lambda)y) \leq \lambda f_2(x) + (1-\lambda)f_2(y)$$

$$\leq \lambda f(x) + (1-\lambda)f(y)$$

$\vdots$

$\vdots$

$$f_n(\lambda x + (1-\lambda)y) \leq f_n(x) + (1-\lambda)f_n(y)$$

$$\leq \lambda f(x) + (1-\lambda)f(y)$$

So, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$

There for; then $f(x) = Max\{f_1(x), f_2(x), \dots \ , f_n(x)\}\ is\ convex.$

Example 2: Choose the correct answer of the following functions:

A) Convex function.

B) Non convex function.

1) $f(x) = 2x$

answer: convex function

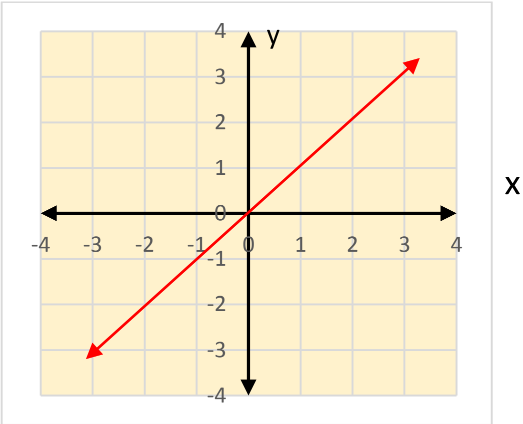


Figure 1-11: Convex function (1)

2) $f(x) = \sin x$

answer: non convex.

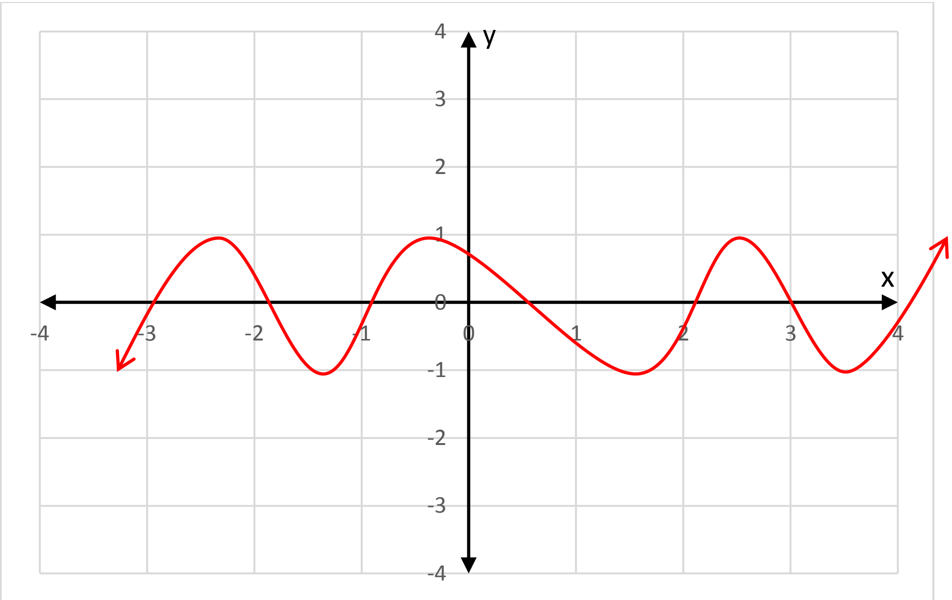


Figure 1-12: Convex function

5-Optimality Conditions for Constrained Optimization

The optimization problem is called the general nonlinear programming problem if we also have some equality (or inequality $h_j(x) \leq 0, j, \ldots\ , I$) constraints, or both of them, and it can be expressed as:

$$\text{minimize } \ f(x)$$
$$\text{subject to } \ g_i(x) = 0, i = 1, \ldots\ , E.$$
$$x \in X$$

where f and g_i defined from $X \subseteq \mathbb{R}^n$ into IR are assumed to be continuously differentiable functions. The feasible set of is denoted by

$$S = \{x \in X | g(x) = 0\}.$$

where g is the function with component functions $g_i, \ldots\ , g_E$. In our study we assume $X = \mathbb{R}^n$.

The **Lagrangian function** $\mathcal{L}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is represented by

$$\mathcal{L}(x, \beta) = f(x) + \beta^T g(x).$$

where $\beta = (\beta_1, \ldots\ , \beta_E)^T$ is called the **Lagrange multiplier** vector. We have the following optimality conditions. [7]

5.1-Lagrangian function method:

Example : Let $\begin{cases} maximize \\ R = f(x,y) = x^2 e^y y = c \\ subject\ to\ B = g(x,y) = x^2 + y^2 = 4 = b \end{cases}$

Solution:

$$L(x, y, \lambda) = R(x, y) - \lambda(B(x, y) - \underbrace{b}_{constant})$$

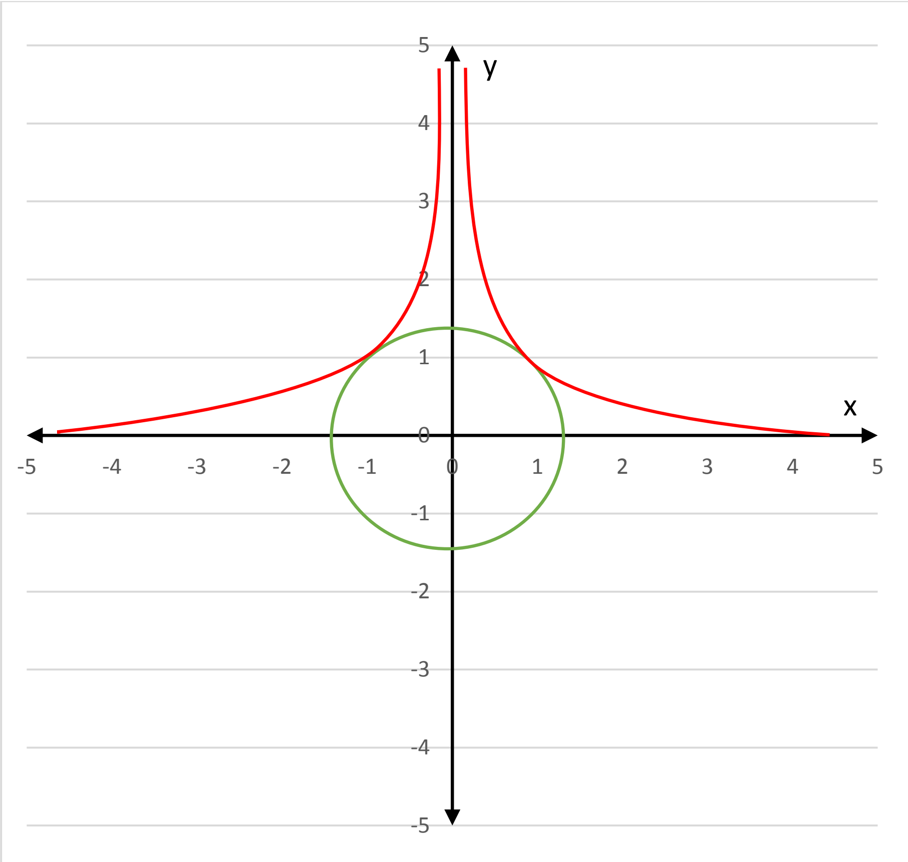


Figure 1-13: Lagrangian function

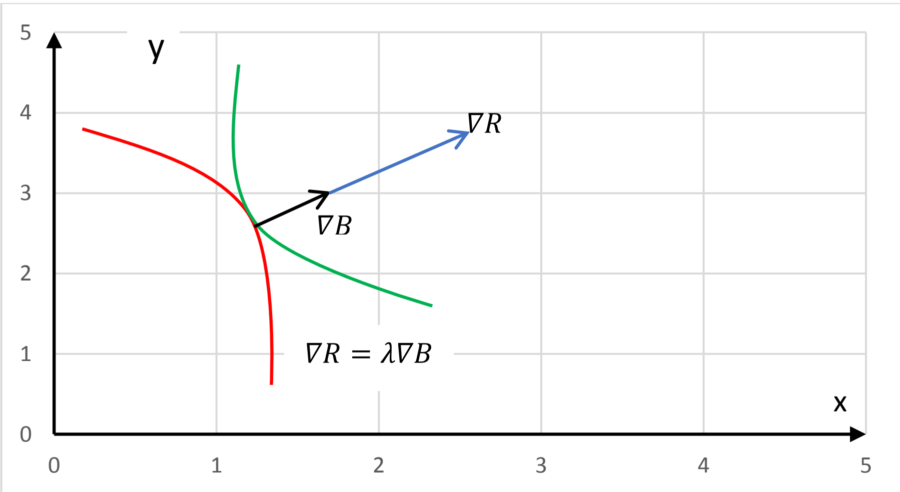


Figure 1-14: Lagrangian function

$$\nabla L = 0$$

$$\begin{bmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial y} \\ \frac{\partial L}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial L}{\partial x} = \frac{\partial R}{\partial x} - \lambda \frac{\partial B}{\partial x} = 0$$

$\frac{\partial L}{\partial y} = \frac{\partial R}{\partial y} - \lambda \frac{\partial B}{\partial y} = 0$

$\frac{\partial L}{\partial \lambda} = 0 - (B(x,y) - b) \ \rightarrow B(x,y) = b$

5.2-Lagrange multipliers method:

$max/min \qquad f(x,y,z)$

subject to $\quad g(x,y,z) = k \ \ (constant)$

from: $\quad L(x,y,z,\lambda) = f(x,y,z,\lambda) - \lambda(g(x,y,z) - k)$

solve: $\quad f_x = 0$ $\qquad$ sub solutions back into $f(x,y,z)$

$f_y = 0$

$f_z = 0$

$f_\lambda = 0$

Example 1: Given $f(x,y,z) = 3x^2 + y^2 - 2z^2$ and $3x + 2y - 8z = -50$ use Lagrange Multipliers to find maximum or minimum values.

Solution:

$$f(x,y,z,\lambda) = 3x^2 + y^2 - 2z^2 - \lambda(3x + 2y - 8z + 50)$$

$$\frac{\partial L}{\partial x} = 0 \ \rightarrow \ \frac{\partial L}{\partial x} = 6x - \lambda(3) \rightarrow 6x - 3\lambda = 0$$

$$6x = 3\lambda \rightarrow x = \frac{3\lambda}{6} \ \rightarrow x = \frac{1}{2}\lambda$$

$$\frac{\partial L}{\partial y} = 0 \rightarrow \frac{\partial L}{\partial y} = 2y - \lambda(2) \rightarrow 2y - 2\lambda = 0$$

$$2y = 2\lambda \rightarrow y = \lambda$$

$$\frac{\partial L}{\partial z} = 0 \rightarrow \frac{\partial L}{\partial z} = -4z - \lambda(-8) \rightarrow -4z + 8\lambda = 0$$

$$-4z = -8\lambda \ \rightarrow z = 2\lambda$$

$$\frac{\partial L}{\partial \lambda} = 0 \ \rightarrow -3x - 2y + 8z - 50 = 0 \ * (-1\,)$$

$$3x + 2y - 8z + 50 = 0 \rightarrow 3\left(\frac{1}{2}\lambda\right) + 2(\lambda) - 8(2\lambda) + 50 = 0$$

$$\frac{3}{2}\lambda - 14\lambda + 50 = 0 \quad] * 2$$

$$\rightarrow 3\lambda - 28\lambda + 100 = 0 \rightarrow -25\lambda = -100 \rightarrow \lambda = 4$$

$$\therefore x = \frac{1}{2}(4) \rightarrow x = 2$$

$$\therefore y = 4$$

$$z = 2(4) \rightarrow z = 8$$

$$f(2,4,8) = 3(2^2) + 4^2 - 2(8^2)$$

$$= 12 + 16 - 128$$

$$= 28 - 128$$

$$= -100 \quad \leftarrow min$$

Example 2: Given $f(x, y, z) = 4x + 2y + 6z \; and \; x^2 + y^2 + z^2 = 14$ use Lagrange Multipliers to find maximum or minimum values.

Solution:

$$L(x, y, z, \lambda) = 4x + 2y + 6z - \lambda(x^2 + y^2 + z^2 - 14)$$

$$\frac{\partial L}{\partial x} = 0 \rightarrow \frac{\partial L}{\partial x} = 4 - \lambda(2x) \rightarrow 4 - 2\lambda x = 0 \rightarrow x = \frac{2}{\lambda}$$

$$\frac{\partial L}{\partial Y} = 0 \rightarrow \frac{\partial L}{\partial Y} = 2 - \lambda(2y) \rightarrow 2 - 2\lambda y = 0 \rightarrow y = \frac{1}{\lambda}$$

$$\frac{\partial L}{\partial z} = 0 \rightarrow \frac{\partial L}{\partial z} = 6 - \lambda(2z) \rightarrow 6 - 2\lambda z = 0 \rightarrow z = \frac{3}{\lambda}$$

$$\left(\frac{2}{\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 = 14 \rightarrow \frac{4}{\lambda^2} + \frac{1}{\lambda^2} + \frac{9}{\lambda^2} = 14 \quad * \; \lambda^2$$

$$4 + 1 + 9 = 14\lambda^2 \quad \rightarrow 14 = 14\lambda^2 \quad \rightarrow \lambda^2 = 1 \quad \rightarrow \lambda = \pm 1$$

If $\lambda = 1 \;\rightarrow\; x = \frac{2}{1} \rightarrow x = 2$

$$y = \frac{1}{1} \rightarrow y = 1$$

$$z = \frac{3}{1} \rightarrow z = 3$$

If $\lambda = -1 \rightarrow x = -2 \ , \ y = -1 \ , \ z = -3$

$$f(2,1,3) = 4(2) + 2(1) + 6(3)$$
$$= 8 + 2 + 18$$
$$= 28 \ \leftarrow \ max$$

$$f(-2,-1,-3) = 4(-2) + 2(-1) + 6(-3)$$
$$= -8 - 2 - 18$$
$$= -28 \ \leftarrow \ min$$

Example 3: Given $f(x, y, z) = xyz \ and \ x^2 + 4y^2 + 16z^2 = 48$ use Lagrange multipliers to Find any maximum or minimum values.

Solution:

$$L(x, y, z, \lambda) = xyz - \lambda(x^2 + 4y^2 + 16z^2 - 48)$$

$$\frac{\partial L}{\partial x} = 0 \ \rightarrow \ \frac{\partial L}{\partial x} = yz - \lambda(2x) \ \rightarrow yz - 2\lambda x = 0 \ \rightarrow \ \lambda = \frac{yz}{2x} \ ... \ (1)$$

$$\frac{\partial L}{\partial y} = 0 \ \rightarrow \ \frac{\partial L}{\partial y} = xz - \lambda(8y) \ \rightarrow \ xz - 8\lambda y = 0 \ \rightarrow \ \lambda = \frac{xz}{8y} \ ... \ (2)$$

$$\frac{\partial L}{\partial z} = 0 \ \rightarrow \ \frac{\partial L}{\partial z} = xy - \lambda(32z) \ \rightarrow \ xy - 32\lambda z = 0 \ \rightarrow \ \lambda = \frac{xy}{32z} \ ... \ (3)$$

From eq.(1) and eq.(2) we get: $\frac{xz}{2x} = \frac{xz}{8y} \ \rightarrow 8y^2 = 2x^2 \ \rightarrow 4y^2 = x^2 \ ... \ (4)$

From eq.(2) and eq.(3) we get: $\frac{xz}{8y} = \frac{xy}{32z} \rightarrow 8y^2 32z^2 \ \rightarrow \ 4y^2 = 16z^2 \ ... \ (5)$

Make up for (4) and (5) in constraint we get: $4y^2 + 4y^2 + 4y^2 = 48$

$$12y^2 = 48 \ \rightarrow \ y^2 = 4 \ \rightarrow \ \sqrt{y} = \sqrt{4} \rightarrow y = \pm 2$$

in eq.(4) we get: $4(4) = x^2 \rightarrow x^2 = 16 \rightarrow \sqrt{x^2} = \sqrt{16} \rightarrow x = \pm 4$

in eq.(5) we get: $4(4) = 16z^2 \rightarrow z^2 = 1 \rightarrow \sqrt{z^2} = \sqrt{1} \rightarrow z = \pm 1$

$f(4,2,1) = 4(2)(1) = 8 \leftarrow max$

$f(-4,-2,-1) = -4(-2)(-1) = -8 \leftarrow min$

Example 4: Given $f(x,y,z) = 4x + 6y - 2z$, $2x + y + 5z = 1$ *and* $y^2 + 2z^2 = 22$. use Lagrange maximum or minimum values.

Solution:

$g(x,y,z) = k$, $h(x,y,z) = c$

$L(x,y,z,\lambda_1,\lambda_2) = 4x + 6y - 2z - \lambda_1(2x + y + 5z - 1) - \lambda_2(y^2 + 2z^2 - 22)$

$\frac{\partial L}{\partial x} = 0 \rightarrow \frac{\partial L}{\partial x} = 4 - \lambda_1(2) - \lambda_2(0) \rightarrow 4 = 2\lambda_1 \rightarrow \lambda_1 = 2$

$\frac{\partial L}{\partial y} =\rightarrow \frac{\partial L}{\partial y} = 6 - \lambda_1(1) - \lambda_2(2y) \rightarrow 6 - (2)(1) - \lambda_2\,(2y) = 0 \rightarrow 6 - 2 - 2\lambda_2 = 0$

$4 - 2\lambda_2 y = 0 \rightarrow y = \frac{2}{\lambda_2}$

$\frac{\partial L}{\partial z} = 0 \rightarrow \frac{\partial L}{\partial z} = -2 - \lambda_1(5) - \lambda_2(4z) \rightarrow -2 - (2)(5) - \lambda_2(4z) = 0$

$-2 - (2)(5) - \lambda_2(4z) = 0 \rightarrow -2 - 10 - \lambda_2(4z) = 0 \rightarrow -4\lambda_2 z = 12 \rightarrow z = -\frac{12}{4\lambda_2}$

$\rightarrow z = -\frac{3}{\lambda_2}$

We substitute a value y and z in the second entry: $\left(\frac{2}{\lambda_2}\right)^2 + 2\left(\frac{-3}{\lambda_2}\right)^2 = 22$

$\frac{4}{\lambda_2^2} + \frac{18}{\lambda_2^2} = 22$

$\frac{22}{\lambda_2^2} = 1 \rightarrow \sqrt{\lambda_2^2} = \sqrt{1} \rightarrow \lambda_2 = \pm 1$

If $\lambda_2 = 1 \rightarrow y = 2\,, z = -3$

We substitute a value y and z in the first entry: $2x + 2 + 5(-3) = 1$

$$2x + 2 - 15 = 1$$

$$2x = 14 \ \rightarrow x = 7$$

If $\lambda_2 = -1 \ \ \rightarrow y = -2 \ , \ z = 3$

We substitute a value y and z in the first entry: $2x + (-2) + 5(3) = 1$

$$2x - 2 + 15 = 1$$

$$2x + 13 = 1$$

$$2x = -12 \ \rightarrow x = -6$$

$$f(7,2,-3) = 4(7) + 6(2) - 2(-3)$$

$$= 28 + 12 + 6$$

$$= 46 \ \leftarrow max$$

$$f(-6,-2,3) = 4(-6) + 6(-2) - 2(3)$$

$$= \ -24 - 12 - 6$$

$$= \ -42 \ \leftarrow min$$

6-Optimality Conditions for Unconstrained Optimization

In this section, we consider the unconstrained optimization problem. If $X = \mathbb{R}^n$i.e., minimize f without constraints, it can be expressed as:

$$minimize_{x \in \mathbb{R}^n} f(x)$$

- If f is continuously differentiable, then a necessary condition for $x^* \in \mathbb{R}^n$ to be a solution of problem is

$$\nabla f(x^n) = 0$$

- If f is twice continuously differentiable, then a necessary condition for $x^* \in \mathbb{R}^n$ to be a solution of problem is

$$\nabla f(x^*) = 0, \nabla^2 f(x^*) \geq 0$$

- The sufficient conditions for $x^* \in \mathbb{R}^n$ to be a local solution of problem are

$$\nabla f(x^*) = 0, \nabla^2 f(x^*) > 0. \qquad [8]$$

Application of optimization

Example 1: Maximize utility $u = f(x, y) = xy$ subject to the constraint $g(x, y) = x + 4y = 240$. Here the price of per unit x is 1, the price of y is 4 and the budget available to buy x and y is 240. Solve the problem using the **geometric approach**.

Solution:

the optimization problem is:

Objective function: maximize $u(x, y) = xy$

Subject to: $g(x, y) = x + 4y = 240$

Step1: $-\frac{f_x}{f_y} = -\frac{y}{x}$ (Slope of the curve)

Step2: $-\frac{g_x}{g_y} = -\frac{1}{4}$ (Slope of the line)

Step3: $-\frac{f_x}{f_y} = -\frac{g_x}{g_y}$ (Utility maximization requires the slope of the curve to be equal to the slope of the line) **.**

$$-\frac{y}{x} = -\frac{1}{4}$$

$$x = 4y$$

Step4: From step 3, use the relation between x and y in the constraint function to get the critical values.

$$x = 4y = 240$$

$$4y + 4y = 240$$

$$8y = 240$$

$$y = 30$$

Using $y = 30$ in the relation $x = 4y$. We get $x = 4 \times 30 = 120$ Utility may be maximized at $(120,30)$.

Example 2: suppose a consumer consumes two goods, x *and* y and has utility function $u(x, y) = xy$. He has a budget of $ 400. The price of x *is* $p_x = 10$ and the price of y *is* $p_y = 20$. Find his optimal consumption bundle using the **Lagrange method.**

Solution:

the optimization problem is:

Objective function: maximize $u(x, y) = xy$

Subject to : $g(x, y) = 10x + 20y = 400$.

This is a problem of constrained optimization.

Form the Lagrange function:

$$L(x, y, \mu) = f(x, y) - \mu(g(x, y) - k)$$

$$L(x, y, \mu) = xy - \mu(10x + 20y - 400)$$

Set each first order partial derivative equal to zero:

$$\frac{\partial L}{\partial x} = y - 10\mu = 0 \quad \dots (1)$$

$$\frac{\partial L}{\partial y} = x - 20\mu = 0 \quad \dots (2)$$

$$\frac{\partial L}{\partial \mu} = -(10x + 20y - 400) = 0 \quad \dots (3)$$

From equation (1) and (2) we find:

$$x = 2y$$

Use $x = 2y$ in equation (3) to get:

$$10x + 20y = 400$$

$$40y = 400$$

$$y = 10$$

$$x = 2y = 20$$

See the graph below.

Example 3: The effects of a change in price

Suppose a consumer consumes two goods, $x\ and\ y$ and has the utility function $U(x, y) = xy$. He has a budget of $ 400. The price of $x\ is\ p_x = 10$ and the price of $y\ is\ p_y = 20$.

a) Find his optimal consumption bundle.
b) What happens when the price of $x\ falls\ to\ p_x = 5$, other factors remaining constant?

Solution:

a) when $p_x = 10$, the optimal bundle $(x, y)\ is\ (20,10)$ (see Example 2)

b) when the price of $x\ falls\ to\ p_x = 5$,

New budget equation is: $5x + 20y = 400$

Slope of the indifference curve $= -\frac{x}{y}$

Slope of the budget line $= -\frac{5}{20} = -\frac{1}{4}$

Optimality requires that the slope of the indifference curve equals to the slope of the budget line:

$$-\frac{y}{x} = -\frac{1}{4}$$

$$x = 4y$$

Use the relation $x = 4y$ in the new budget equation:

$$5x + 20y = 400$$

$$5(4y) + 20y = 400$$

$$20y + 20y = 400$$

$$y = 10$$

$$x = 4y = 4(10) = 40$$

When the price of x falls while all other factors remain constant, in this typical example, the consumption of x increases.

Example 4: The effects of a change in income

Suppose a consumer consumes two goods, x and y and has utility function $U(x, y) = xy$.

He has a budget of $ 400. The price of x is $p_x = \$ 10$ and the price of y is $p_y = \$ 20$.

a) Find his optimal consumption bundle.
b) What happens when the income rises to $B = 800$, other factors remaining constant?

Solution:

a) When $p_x = \$ 10$, $p_y = \$ 20$ and $B = 400$, the optimal bundle is $(20,10)$.

   (see example 2)

b) When the income increases to 800 while other factors remain constant,

   New budget equation: $10x + 20y = 800$

   Slope of the indifference curve $= -\frac{y}{x}$

Slope of the budget line $= -\frac{10}{20} = -\frac{1}{2}$

Optimality requires that slope of the indifference curve equals the slope of the budget line:

$$-\frac{y}{x} = -\frac{1}{2}$$

$$x = 2y$$

Use the relation, $x = 2y$ in the new budget equation:

$$10x + 20y = 800$$

$$10(2y) + 20y = 800$$

$$20y + 20y = 800$$

$$y = 20$$

$$x = 2y = 2(20) = 40$$

When consumer's income increases, while other factors remain constant, for typical goods discussed in this example, consumption of both goods increases.

7-Unconstrained Optimization

In this section we address the problem of maximizing (minimizing) a function in the case when there are no constraints on its arguments. This is not a very interesting case for economics, which typically deals with problems where resources are constrained, but represents a natural starting point to solving the more economically relevant constrained optimization problems.

In this section, we consider the unconstrained optimization problem. If $X = \mathbb{R}^n$, I.e, minimize f

without constraints, it can be expressed as:

$$minimize\ f(x)$$
$$x \in \mathbb{R}^n$$

- If f is continuously differentiable, then a necessary condition for $x^* \in \mathbb{R}^n$ to be
  a solution of problem is
  $$\nabla f(x^*) = 0$$

- If f is twice continuously differentiable, then a necessary conditions for $x^* \in \mathbb{R}^n$
  a solution of problem is
  $$\nabla f(x^*) = 0,\ \nabla^2 f(x^*) \geq 0$$

- The sufficient conditions for $x^* \in \mathbb{R}^n$ to be a local solution of problem are
  $$\nabla f(x^*) = 0, \nabla^2 f(x^*) > 0$$

Theorem 8.1:(First-Order Necessary Condition):

Let $f: \mathbb{R}^n \to \mathbb{R}$ be differentiable . If x^* is a local minimizer of f , then $\nabla f(x^*) = 0$.

Theorem 8.2:(Second-Order Necessary Condition):

Let $f: \mathbb{R}^n \to \mathbb{R}$ be twice differentiable. If x^* is a local minimizer of f, then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive semidefinite.

Theorem 8.3:(Second-Order Sufficient Condition):

Let $f: \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. If $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite, then x^* is a strict local minimizer.

Definition 8.4: Let $f: \mathbb{R}^n \to \mathbb{R}$ be differentiable. A point $x^* \in \mathbb{R}^n$ is said to be a stationary point of f if $\nabla f(x^*) = 0$.

To find local extrema:

1. Compute f_x and f_y
2. Solve $f_x = 0 = f_y$
3. Use 2nd derivative test to classify critical points

Definition 8.5: Let $f = \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$. Then the **partial derivative** of f at x with respect to x_i is defined as

$$\frac{\partial f(x)}{\partial x_i} = \lim_{t \to 0} \frac{f(x + te_i - f(x)}{t}.$$

Where e_i is ith unit vector. The gradient of f at x is defined as the column vector

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_i} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}.$$

The **Hessian matrix** is defined as the $n \times n$ symmetric matrix

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}.$$

The directional derivative of the function f at x in the direction d given by

$$\acute{f}(x.d) = \lim_{x \to 0^+} \frac{f(x+td)-f(x)}{t}.$$

We say the function f is differentiable at x and only if the gradient $\nabla f(x)$ exists and satisfies

$\langle \nabla f(x), d \rangle = \acute{f}(x, d), \forall\, d \in \mathbb{R}^n$. Moreover, we say the function f is **differentiable over a subset** S of $\mathbb{R}^n$ if it is differentiable at every $x \in S.$ and f is **continuously differentiable over** S, if

$$\lim_{d \to 0} \frac{f(x+d)-f(x)-\langle \nabla f(x), d \rangle}{\|d\|} = 0, \ \ \forall x \in S\ ,$$

Where $\|\cdot\|$ is an arbitrary vector norm.

Definition 8.6: A function $f = \mathbb{R}^n \to \mathbb{R}^m$. with component functions $f_1, \dots, f_m$ is called differentiable if each component is differentiable. The gradient matrix of f, denoted $\nabla f(x)$. is
the $n \times m$ matrix whose ith column is the gradient $\nabla_i f(x)$ of f_i:

$$\nabla f(x) = [\nabla f_1(x) \cdots \nabla f_m(x)].$$

Then the **Jacobian** of f at x is defined as

$$D(x) = [\nabla f(x)]^T = \begin{bmatrix} \nabla f_1(x)^T \\ \vdots \\ \nabla f_m(x)^T \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{bmatrix}. \quad [9]$$

Example 1:find and classify the critical points of $f(x, y) = 2xy + 2x - x^2 - 2y^2$

Solution:

$$f_x = \frac{\partial f}{\partial x} = 2y + 2 - 2x \quad \to \quad 2y + 2 - 2x = 0 \dots (1)$$

$$f_y = \frac{\partial f}{\partial y} = 2x - 4y \quad \to \quad 2x - 4y = 0 \dots (2)$$

From(1): $2y = 2x - 2$

$$y = x - 1 \dots (3) \;\; in \;\; (2)$$

$2x - 4(x - 1) = 0$

$2x - 4x + 4 = 0$

$-2x = -4$

$\therefore x = -2 \;\; in \;\; (3)$

$y = 2 - 1$

$\therefore y = 1$

$\therefore$ stationary point is (2,1)

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = -2 < 0 \quad , \; f_{yy} = \frac{\partial^2 f}{\partial y^2} = -4 < 0$$

Cross partial derivative:

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 2 \, , f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = 2 \quad \rightarrow \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = 2$$

∴ Hessian matrix:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \rightarrow \; H = \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$$

$\therefore \det(H) = 8 - 4 = 4 > 0$

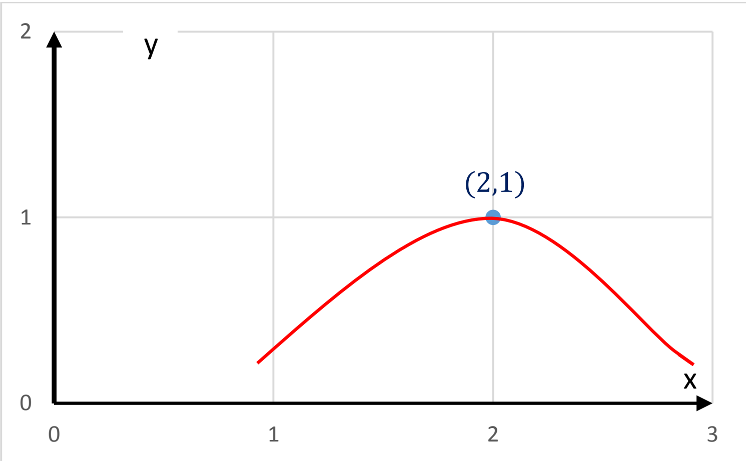


Figure 1-15: plot the example

∴ (2,1) is Local Maximum.

Example 2: find and classify the critical points of $f(x, y) = x^4 - 2x^2 - y^2 e^x$

Solution:

$f_x(x, y) = 4x^3 - 4x - y^2 e^x$

$f_y(x, y) = -2ye^x$

$f_x \, , f_y \; exist \; every \; where \; solve\colon f_x = f_y = 0$

$f_y(x, y) = 0 \; \rightarrow \; -2y \underbrace{e^x}_{>0} = 0 \;\; \rightarrow y = 0$

$f_x(x, y) = 0 \; \rightarrow 4x^{\,3} - 4x = 0 \; \rightarrow 4x(x^2 - 1) = 0$

$$\rightarrow 4x(x-1)(x+1) = 0$$

$$\rightarrow x = 0\,, \pm 1$$

Critical points: $(-1,0), (0,0)\,, (1,0)$

$f_{xx} = 12x^2 - 4 - y^2e^x$

$f_{yy} = -2e^x$

$f_{xy} = f_{yx} = -2ye^x$

Derivatives	$(-1,0)$	$(0,0)$	$(1,0)$
$f_{xx} = 12x^2 - 4 - y^2e^x$	8	-4	8
$f_{yy} = -2e^x$	$-2e^{-1}$	-2	$-2e^1$
$f_{xy} = f_{yx} = -2ye^x$	0	0	0
$D = f_{xx}f_{yy} - (f_{xy})^2$	$-16e^{-1} < 0$	$8 > 0$	$-16e^1 < 0$
Classification	Saddle point	Local max	Saddle point

Table 1-1: classify points

The graph of $f(x,y) = x^4 - 2x^2 - y^2e^x$

Saddle points: $(\pm 1, 0)$ Local maximum: $(0,0)$

Example 3: Find and classify the critical points of $f(x,y) = (x-y)(1-xy)$

Solution:

$f(x,y) = x - y - x^2y + xy^2$

$f_x = 1 - 2xy + y^2 = 0$

$f_y = -1 - x^2 + 2xy = 0$

$y^2 - x^2 = 0 \quad \rightarrow y = \pm x$

If $y = x \;\rightarrow\; f_x = 1 - 2x^2 + x^2 = 0 \;\rightarrow 1 - x^2 = 0 \;\rightarrow x = \pm 1$

$(1,1), (-1,-1)$

If $y = -x \ \rightarrow \ f_x = 1 + 2x^2 + x^2 = 1 + 3x^2 \ \notin \ \mathbb{R}$

Derivatives	$(1,1)$	$(-1,-1)$
$f_{xx} = -2y$	-2	2
$f_{xy} = -2x + 2y$	0	0
$f_{yy} = 2x$	2	-2
	$AC - B^2 = (-2)(2) - 0^2$ $= -4 < 0$	

Table 1-2: classify points

$\therefore (1,1), (-1,-1)$ Both saddles

Example 4: For the following function, find the critical points:

$$y = 2x^3 - 0 \cdot 5x^2 + 2$$

Determine whether the critical points are local maximums, minimums or saddle points.

Solution:

$$\frac{dy}{dx} = 6x^2 - x = 0 \ .$$

There are two critical values: $x = 0 \, . \, ^1/_6$.

The second order condition is: $\frac{d^2y}{dx^2} = 12x - 1$

Evaluate at each critical value: $\left.\frac{d^2y}{dx^2}\right|_{x=0} = -1 \ < 0 \ \ and \ \ \left.\frac{d^2y}{dx^2}\right|_{x=^1/_6} = 1 > 0$.

So $x = 0$ is a max and $x = {}^1/_6$ is a min.

Chapter two

Solving Linear Programming Problems Graphically

CHAPTER 2

Solving Linear Programming Problems Graphically

constraints—things that limit you in your goal to get to your destination in as little time as possible.

1-solving linear programming problem graphically Method

A linear programming problem involves constraints that contain inequalities. An **inequality** is denoted with familiar symbols, $<, >, \leq$ *and* $\geq$. Due to difficulties with strict inequalities ($<$ *and* $>$). We will only focus on $\leq$ *and* $\geq$.

In order to have a linear programming problem, we must have:
Inequality constraints

- An **objective function**, that is, a function whose value we either want to be as large as possible (want to maximize it) or as small as possible (want to minimize it). [10]

The problem described:

$$\begin{cases} \text{Maximize or minimize}: \ \text{Objective function.} \\ \text{Subject to}: \ \text{Constraints.} \end{cases}$$

This format is sufficiently general to include all optimization problems (most of life's problems too for that matter). Since we are interested in mathematical methods for solving such problems, it is necessary that the statement be reduced to symbolic form.
for example:

$$\begin{cases} M\text{aximize: } \mathrm{F}(x_1, x_2, x_3) \\ \text{subject to: } g(x_1, x_2, x_3) = 0 \end{cases}$$

the above statement reads as follows: Maximize some function $F, of\ x_1, x_2, x_3$ by setting $x_1, x_2\ and\ x_3$ subject to the requirement that another function $g\ of\ x_1, x_2\ and\ x_3$ takes on the value zero.

The general optimization problem form:

$$\begin{cases} minimize \;\; f(x) \;\; Objective\ function \\ subject\ to \;\; g(x) = 0 \;\; Equality\ constraints. \\ h(x) \geq 0 \;\; Inequality\ constranits \end{cases}$$

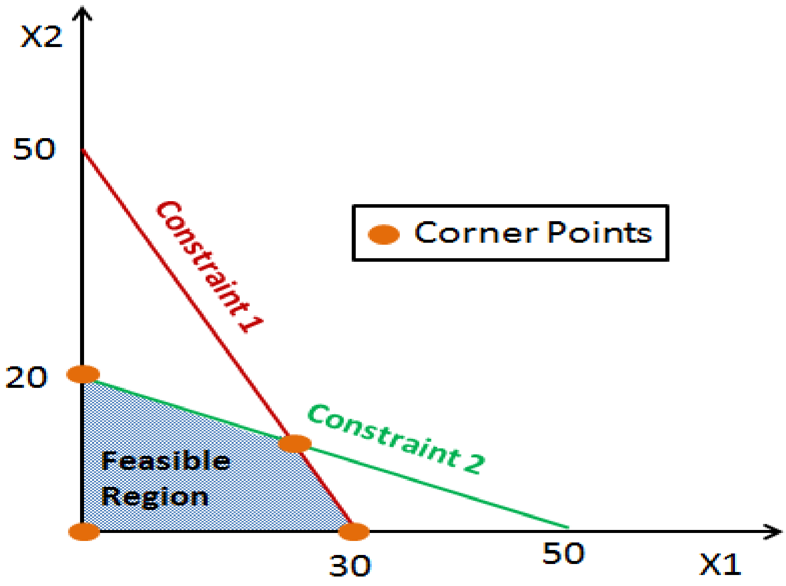


Figure 2-1: The feasible region is the set of all points whose coordinates satisfy the constraints of a problem.

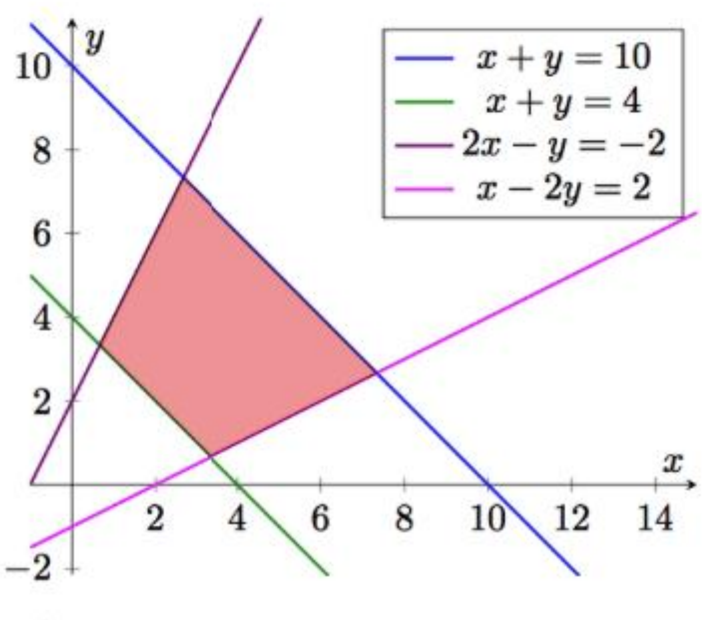


Figure 2-2: The feasible region of four constraints.

Example 1: consider the following simple linear programming problem:

$$\begin{cases} maximize \;\; 3x + 5y \\ subject\ to \;\; x + y \le 4 \\ \qquad\qquad x + 3y \le 6 \\ \qquad\qquad x \ge 0\,,\;\; y \ge 0 \end{cases}$$

Solution:

$$x + y = 4\,,\;\; let\ x = 0 \;\; \rightarrow y = 4 \;\; .(0,4)$$

$$let\ y = 0 \;\; \rightarrow x = 4 \;\; .(4,0)$$

$$x + 3y = 6\,, when\ x = 0 \rightarrow y = 2\,, (0,2)$$

$$when\ y = 0 \rightarrow x = 6\,, (6,0)$$

$x + y = 4 \;\; \dots \; (1)$

$x + 3y = 6 \dots (2)$

$\;\; x + y = 4$

$\mp x \mp 3y = \mp 6$

$-2y = -2 \;\; \rightarrow y = 1 \quad$ in (1)

$x + 1 = 4 \;\; \rightarrow x = 3 \;\; , (3,1)$

Feasible region is the set of points defined by the constraint:

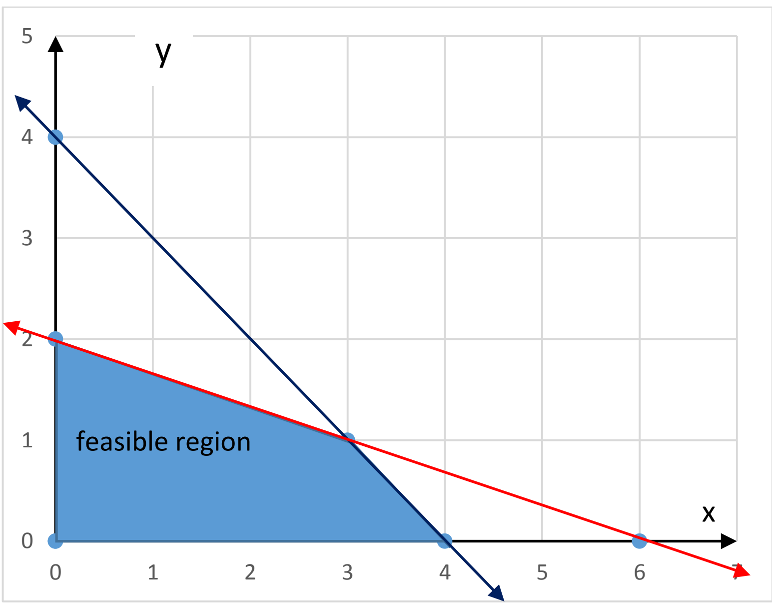


Figure 2-3: plot the example

The feasible points are (substitute in the objective point):

- $(0,0) \to 3(0) + 5(0) = 0$
- $(0,2) \to 3(0) + 5(2) = 10$
- $(4,0) \to 3(4) + 5(0) = 12$
- $(3,1) \to 3(3) + 5(1) = 14$

Therefore the maximize objective value is 14 at the point (3,1)

Example 2: solve the problem graphically:

$$\begin{cases} \text{minimize} \quad 200x + 500y \\ \quad subject\ to\ \ x + 2y \geq 10 \\ \qquad\qquad\qquad 3x + 4y \leq 24 \\ \quad x \geq 0\,, y \geq 0 \end{cases}$$

Solution:

$$x + 2y = 0\,, let\ x = 0 \to y = 5\,, \quad (0,5)$$

$$let\ y = 0 \to x = 10\,, (10,0)$$

$$3x + 4y = 24\,, when\ x = 0 \to y = 6\,. (0,6)$$

$$when\ y = 0 \ \to x = 8\,. (8,0)$$

$x + 2y = 10 \ \ldots \ (1)$

$3x + 4y = 24 \ \ldots (2)$

From(1): $x = 10 - 2y \ldots (3)\ in\ (2)$

$3(10 - 2y) + 4y = 24$

$30 - 6y + 4y = 24$

$-2y = -6 \ \ \to y = 3 \ \ in\ (3)$

$x = 10 - 2(3) \ \to x = 4 \ \ , (4,3)$

The feasible points are (substitute in the objective point):

- $(0,0) \to 200(0) + 500(0) = 0$
- $(0,5) \to 200(0) + 500(5) = 2500$

- $(8,0) \to 200(8) + 500(0) = 1600$
- $(4,3) \to 200(4) + 500(3) = 9500$

Feasible region is the set of points defined by the constraint:

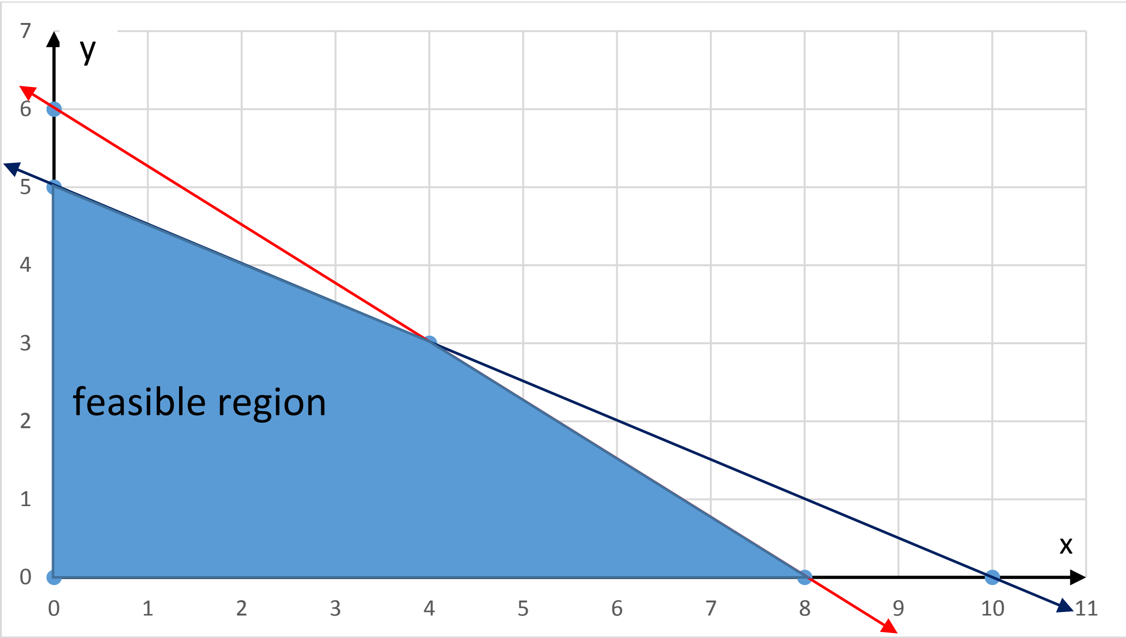


Figure 2-4: plot the example

Therefore the minimum objective value is zero at the point (0,0).

Example 3: solve the problem graphically:

$$\begin{cases} \quad minimize \;\; 3x + 9y \\ subject\ to\ x + 3y \leq 60 \\ \qquad\qquad x + y \geq 10 \\ \qquad\qquad x \leq y \\ \qquad\quad x \geq 0\,,\ y \geq 0 \end{cases}$$

Solution:

$$x + 3y = 60 \quad , \; let\ x = 0 \to y = 20 \;\; , \; (0,20)$$

$$let\;\; y = 0 \to x = 60 \;\; , \;\; (60,0)$$

$$x + y = 10 \quad , when\ x = 0 \to y = 10 \;\; , \; (0,10)$$

$$when\ \ y = 0 \rightarrow x = 10\ \ ,\ \ (10,0)$$

$x + 3y = 60\ \ \ldots\ \ (1)$

$\mp x \mp y = \mp 10\ \ \ldots\ \ (2)$

$2y = 50\ \ \ \rightarrow y = 25\ \ \ in\ (2)$

$x + 25 = 10\ \ \rightarrow\ \ x = -15\ \ .\ (-15,25)$

$x + 3y = 60\ \ldots\ \ (1)$

$x = y\ \ \ \ \ \ \ \ \ \ \ldots\ \ (3)\ \ in\ (1)$

$y + 3y = 60\ \ \rightarrow 4y = 60\ \ \rightarrow y = 15\ \ in\ (3)$

$x = 15\ \ \ .\ (15,15)$

$x + y = 10\ \ldots\ \ (2)$

$x = y\ \ \ \ \ \ \ \ \ \ \ldots\ \ (3)\ \ in\ (2)$

$y + y = 10\ \rightarrow 2y = 10\ \ \rightarrow y = 5\ \ in\ (2)$

$x = 5\ \ ,\ \ (5,5)$

The feasible points are (substitute in the objective point):

- $(0,10) \rightarrow 3(0) + 9(10) = 90$
- $(0,20) \rightarrow 3(0) + 9(20) = 180$
- $(5,5) \rightarrow\ \ 3(5) + 9(5) = 60$
- $(15,15) \rightarrow 3(15) + 9(15) = 180$

Feasible region is the set of points defined by the constraint:

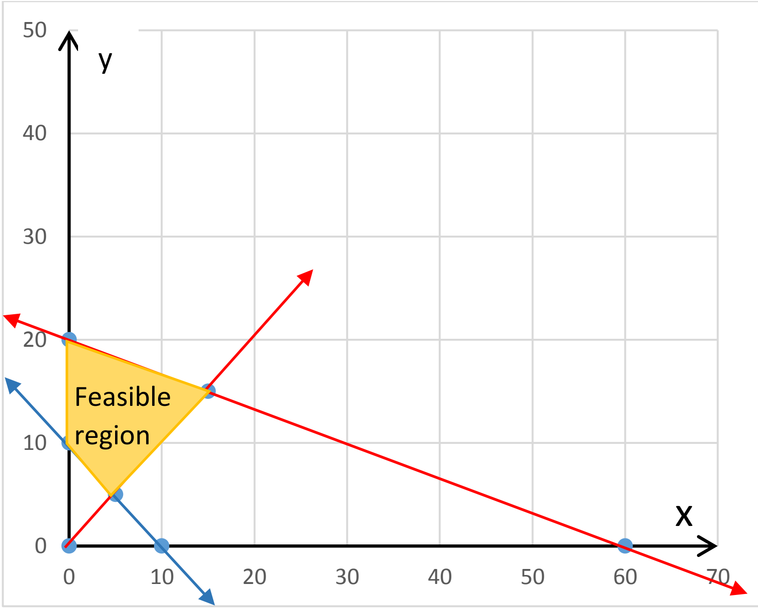


Figure 2-5: plot the example

Therefore the minimize objective value is (60) at the point (5,5)

2-Solving a Linear Programming Problem Graphical Method

1. Define the variables to be optimized. The question asked is a good indicator as to what these will be.
2. Write the objective function in words, then convert to mathematical equations.
3. Write the constraints in words, then convert to mathematical inequalities.
4. Graph the constraints as equations.
5. Shade feasible regions by taking into account the inequality sign and its direction. If,

a) A vertical line

≤, then shade to the left

≥, then shade to the right

b) A horizontal line

≤, then shade below

≥, then shade above

c) A line with a non-zero, defined slope

≤, shade below

≥, shade above

6. Identify the corner points by solving systems of linear equations whose intersection represents a corner point.
7. Test all corner points in the objective function. The "winning" point is the point that optimizes the objective function (biggest if maximizing, smallest if minimizing)

There is one instance in which we must take great caution. First, consider the (true) inequality,

5 > 3

Suppose we were to divide both sides by –1. Would it still be true to say the following?

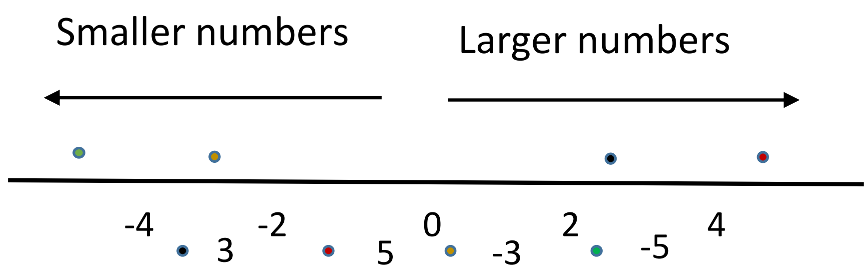
$$\frac{5}{-1} > \frac{3}{-1}$$

$$-5 > -3$$

Clearly, –5 is not larger than –3! To keep the statement true, we should change the direction of the inequality sign so that,

$$-5 < -3$$

We can see by the number line below that the two sets of numbers are symmetric about 0, except that the way in which we describe size is opposite. This justifies that we should also use the opposite sign when we reflect values to the other side of 0.



Changing the Inequality Sign

When multiplying/dividing any inequality by –1, the direction of the inequality should change

Example 1: An airline offers coach and first-class tickets. For the airline to be profitable, it must sell a minimum of 25 first-class tickets and a minimum of 40 coach tickets. The company makes a profit of $225 for each coach ticket and $200 for each first-class ticket. At most, the plane has a capacity of 150 travelers. How many of each ticket should be sold in order to maximize profits?

Solution:

The first step is to identify the unknown quantities. We are asked to find the number of each ticket that should be sold. Since there are coach and first-class tickets, we identify those as the unknowns. Let,

x = # of coach tickets

y = # of first-class tickets

Next, we need to identify the objective function. The question often helps us identify the objective function. Since the goal is to maximize profits, our objective is identified.

Profit for coach tickets is $225. If

x coach tickets are sold, the total profit for these tickets is $225x$.

Profit for first-class tickets is $200. Similarly, if

y first class tickets are sold, the total profit for these tickets is 200y.

The total profit, P, is

$p = 225x + 200y$

We want to make the value of

as large as possible, provided the constraints are met. In this case, we have the following constraints:

- Sell at least 25 first-class tickets
- Sell at least 40 coach tickets
- No more than 150 tickets can be sold (no more than 150 people can fit on the plane)

We need to quantify these.

- At least 25 first-class tickets mean that 25 or more should be sold. That is, $y \geq 25$
- At least 40 coach tickets mean that 40 or more should be sold. That is, $x \geq 40$
- The sum of first-class and coach tickets should be 150 or fewer. That is, $x + y \leq 150$

Thus, the objective function along with the three mathematical constraints is:

Objective function: $\mathrm{p} = 225x + 200y$

Constraint: $y \geq 25; x \geq 40; x + y \leq 150$

We will work to think about these constraints graphically and return to the objective function afterwards. We will thus deal with the following graph:

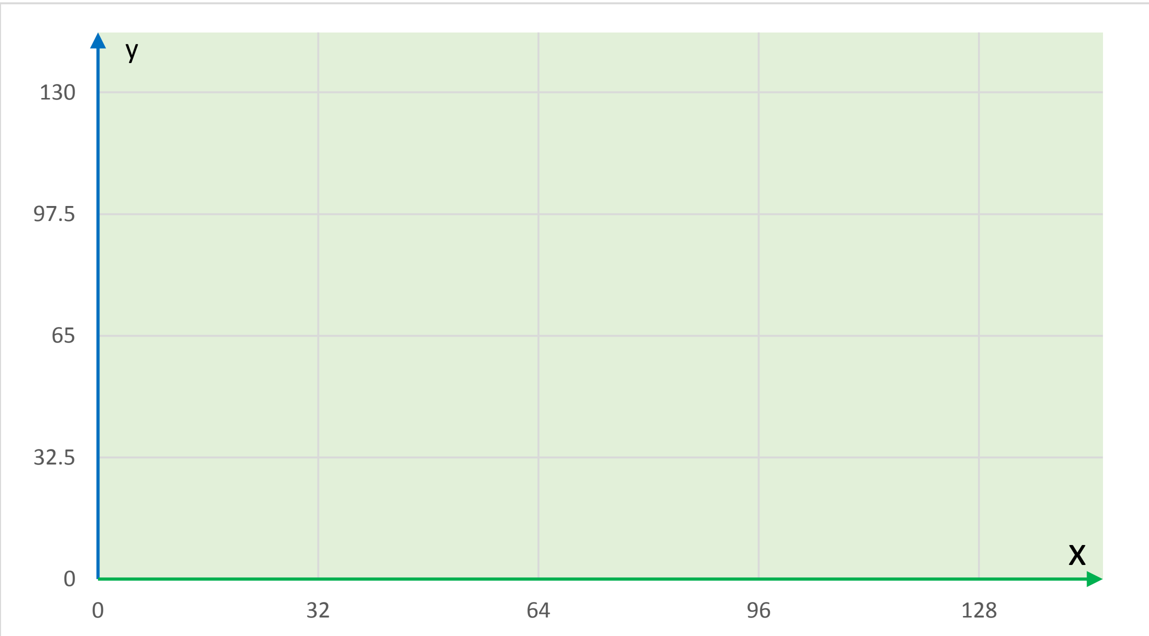


Figure 2-6

Note that we are only interested in the first quadrant, since we cannot have negative tickets.

We will first plot each of the inequalities as equations, and then worry about the inequality signs. That is, first plot,

$x = 25$

$y = 40$

$x + y = 150$

The first two equations are horizontal and vertical lines, respectively. To plot $x + y = 150$, it is preferable to find the horizontal and vertical intercepts.

To find the vertical intercept, we let

$x = 0$

$y = 150$

Giving us the point (0,150)

To find the horizontal intercept, we let

$\mathrm{y} = 0$

$x = 150$

Giving us the point (150,0)

Plotting all three equations gives:

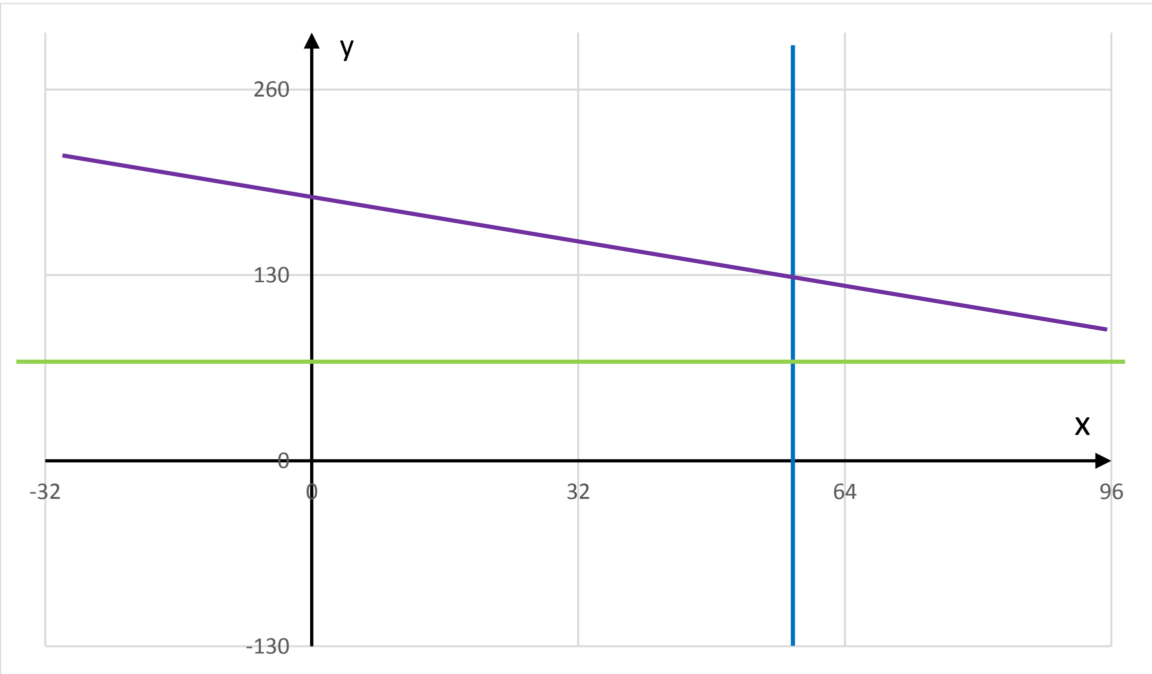


Figure 2-7

Our next task is to take into account the inequalities.

We first ask, when is $y \geq 25$? Since this is a horizontal line running through a *y*-value of 25, anything above this line represents a value greater than 25. We denote this by shading above the line:

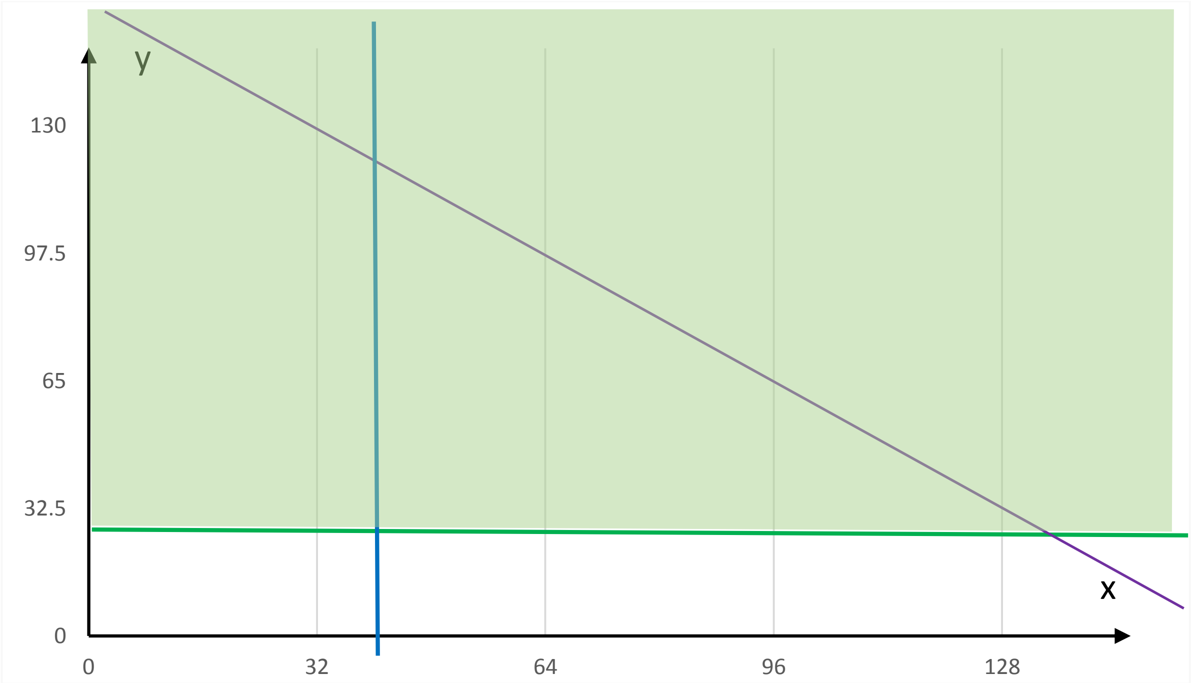


Figure 2-8

This tells us that any point in the green shaded region satisfies the constraint that

$y \geq 25$

Next, we deal with

$x \geq 40$. We ask, when is the *x*-value larger than 40? Values to the left are smaller than 40, so we must shade to the right to get values larger than 40:

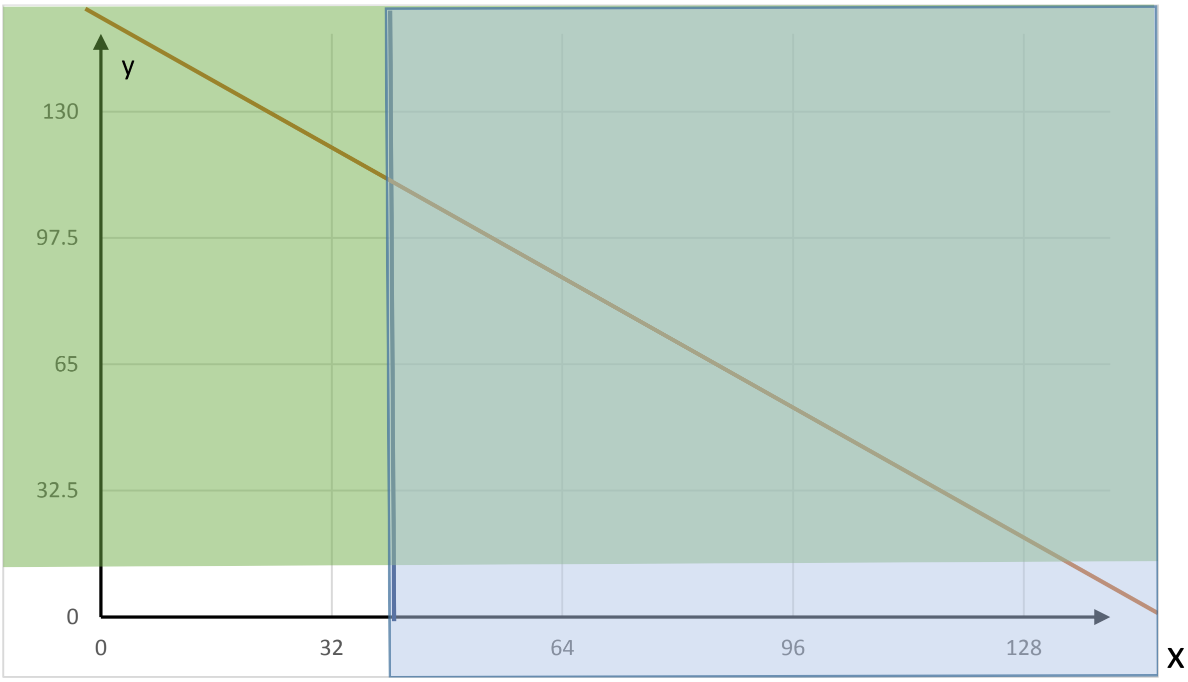


Figure 2-9

The blue area satisfies the second constraint, but since we must satisfy all constraints, only the region that is green and blue will suffice.

We have one more constraint to consider:

$x + y \geq 150$. We have two options, either shade below or shade above. To help us better see that we will, in fact, need to shade below the line, let us consider an ordered pair in both regions. Selecting an ordered pair above the line, such as (64, 130) gives:

$64 + 65 \geq 150$

Which is a false statement since 64 + 130 = 194, a value larger than 150.

According to the graph, the point (64, 65) is one that falls below the graph. Putting this pair in yields the statement:

$64 + 65 \geq 150$

Which is a true statement since 64+65 is 129, a value smaller than 150.

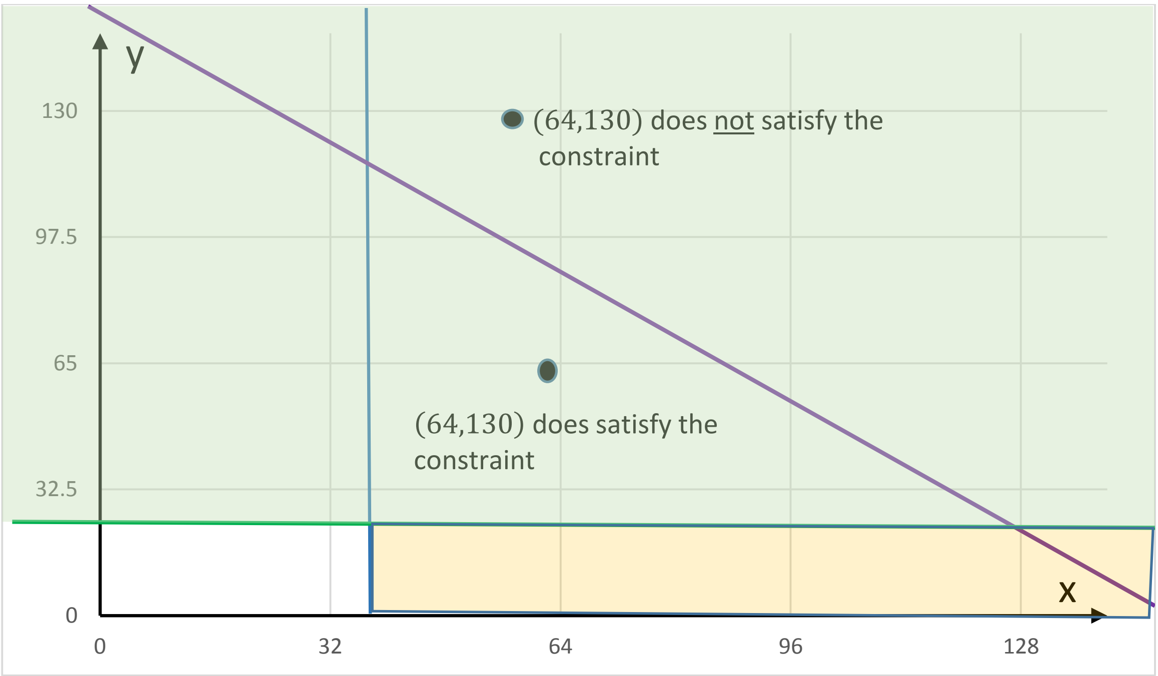


Figure 2-10

Therefore, we shade below the line:

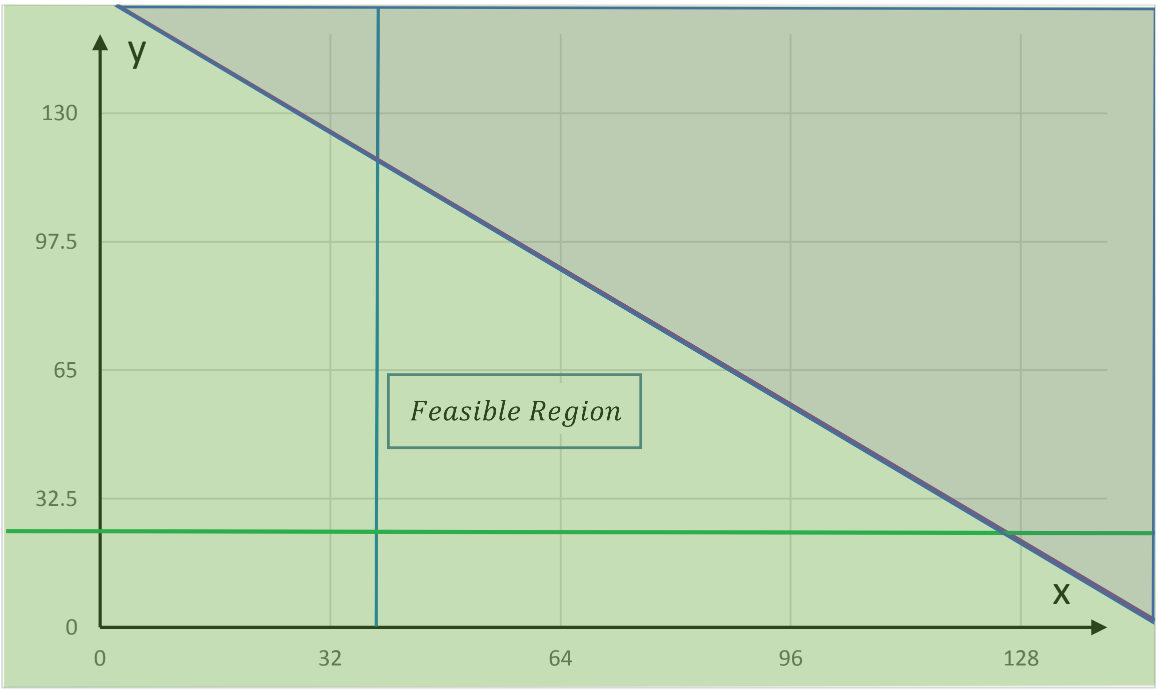


Figure 2-11

The region in which the green, blue, and purple shadings intersect satisfies all three constraints. This region is known as the **feasible regions**, since this set of points is feasible, given all constraints. We can verify that a point chosen in this region satisfies all three constraints. For example, choosing (64, 65) gives:

$64 \geq 40$ TRUE

$65 \geq 25$ TRUE

$64 + 65 \geq 150$ TRUE

This gets us to a great point, but still does not answer the question:

which point maximizes profit? Fortunately, there is a theorem discovered by mathematicians that allows us to answer this question.

First off, we define a new term: a **corner point** is a point that falls along the corner of a feasible region. In our situation, we have three corner points, shown on the graph as the solid black dots:

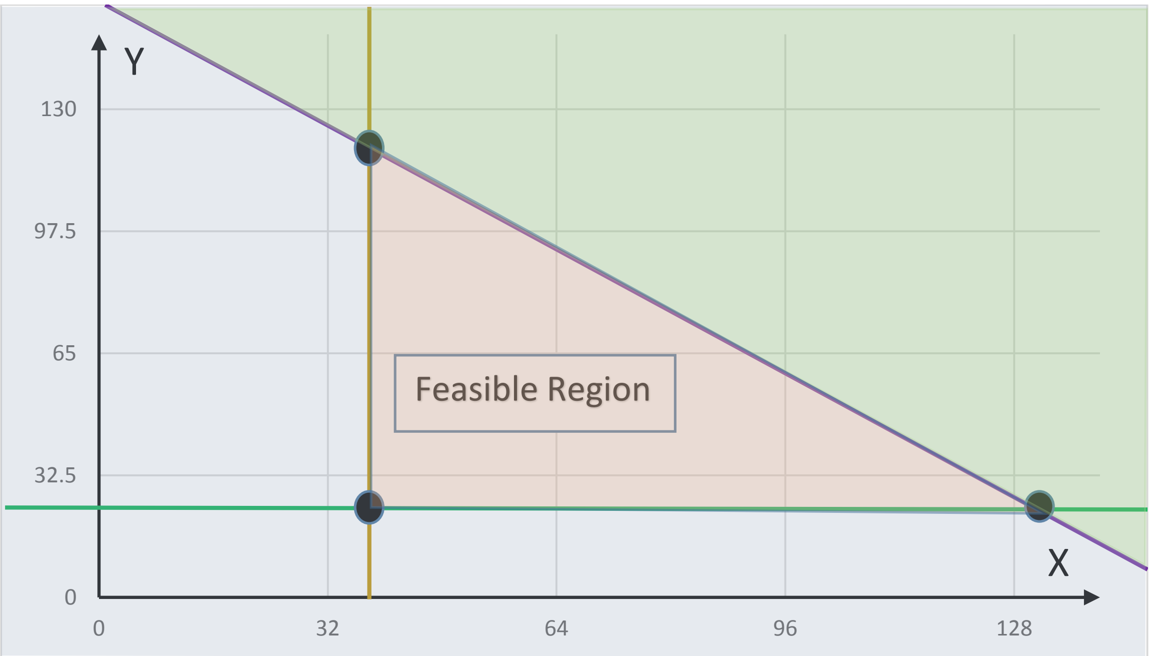


Figure 2-12

The objective function along with the three corner points above forms a **bounded** linear programming problem. That is, imagine you are looking at three fence posts connected by fencing (black point and lines, respectively). If you were to put your dog in the middle, you could be sure it would not escape (assuming the fence is tall enough). If this is the case, then you have a bounded linear programming problem. If the dog could walk infinitely in any one direction, then the problem is unbounded.

3-Fundamental Theorem of Linear Programming

1.If a solution exists to a bounded linear programming problem, then it occurs at one of the corner points.

2.If a feasible region is unbounded, then a maximum value for the objective function does not exist.

3.If a feasible region is unbounded, and the objective function has only positive coefficients, then a minimum value exists.

This means we have to choose among three corner points. To verify the "winner," we must see which of these three points maximizes the objective function. To find the corner points as ordered pairs, we must solve three systems of two equations each:

System 1

$x = 40$

$x + y = 150$

System 2

$x = 40$

$y = 25$

System 3

$\mathrm{y} = 25$

$x + y = 150$

We could decide to solve by using matrix equations, but these equations are all simple enough to solve by hand:

System 1

$(40) + \mathrm{y} = 150$

$y = 110$

Point: (40,110)

System 2

Point already given

Point: (40,25)

System 3

$x + 25 = 110$

$x = 125$

Point:(125,25)

We test each of these three points in the objective function:

Point	profit
(40,110)	225(40) + 200(110) =$ 31,000
(40, 25)	225(40) + 200(25) =$ 14,000
(125,25)	225(125) + 200(25) =$ 33,125

Table 2-1: optimal solution

The third point, (125,25) maximizes profit. Therefore, we conclude that the airline should sell 125 coach tickets and 25 first-class tickets in order to maximize profits.

Example 2: A health-food business would like to create a high-potassium blend of dried fruit in the form of a box of 10 fruit bars. It decides to use dried apricots, which have 407 mg of potassium per serving, and dried dates, which have 271 mg of potassium per serving. The company can purchase its fruit in bulk for a reasonable price. Dried apricots cost $9.99/lb. (about 3 servings) and dried dates cost $7.99/lb. (about 4 servings). The company would like the box of bars to have at least the recommended daily potassium intake of about 4700 mg, but would like to keep it under twice the recommended daily intake. In order to minimize cost, how many servings of each dried fruit should go into the box of bars?

Solution:

We begin by defining the variables. Let,

$x = \#$ of servings of dried apricots

$y = \#$ of servings of dried dates

We next work on the objective function.

For apricots, there are 3 servings in one pound. This means that the cost per serving is $9.99/3 = $3.33. The cost for *x* servings would thus be 3.33*x*.

For dates, there are 4 servings per pound. This means that the cost per serving is $7.99/4 $2.00. The cost for *y* servings would thus be 2.00*y*.

The total cost for apricots and dates would be

$C = 3.33x + 2.00y$

We have two major constraints (in addition to the constraints that

$x \geq 0$

and $y \geq 0$. given that negative servings cannot be used):

- Product must contain at least 4700mg of potassium
- Product should contain no more than 4700 × 2 = 9400mg of potassium

Mathematically,

- There are 407*x* mg of potassium in *x* servings of apricots and 271*y* mg of potassium in *y* servings of dates. The sum should be greater than or equal to 4700mg of potassium, or $407x + 271y \geq 4700$
- The same sum should be less than or equal to 9400 mg of potassium, or $407x + 271y \leq 9400$

Thus we have,

Objective Function: *C* = 3.33*x* + 2.00*y* Subject **to Constraints:**

$407x + 271y \geq 4700$

$407x + 271y \leq 9400$

$x \geq 0$

$y \geq 0$

We graph the constraints as equations:

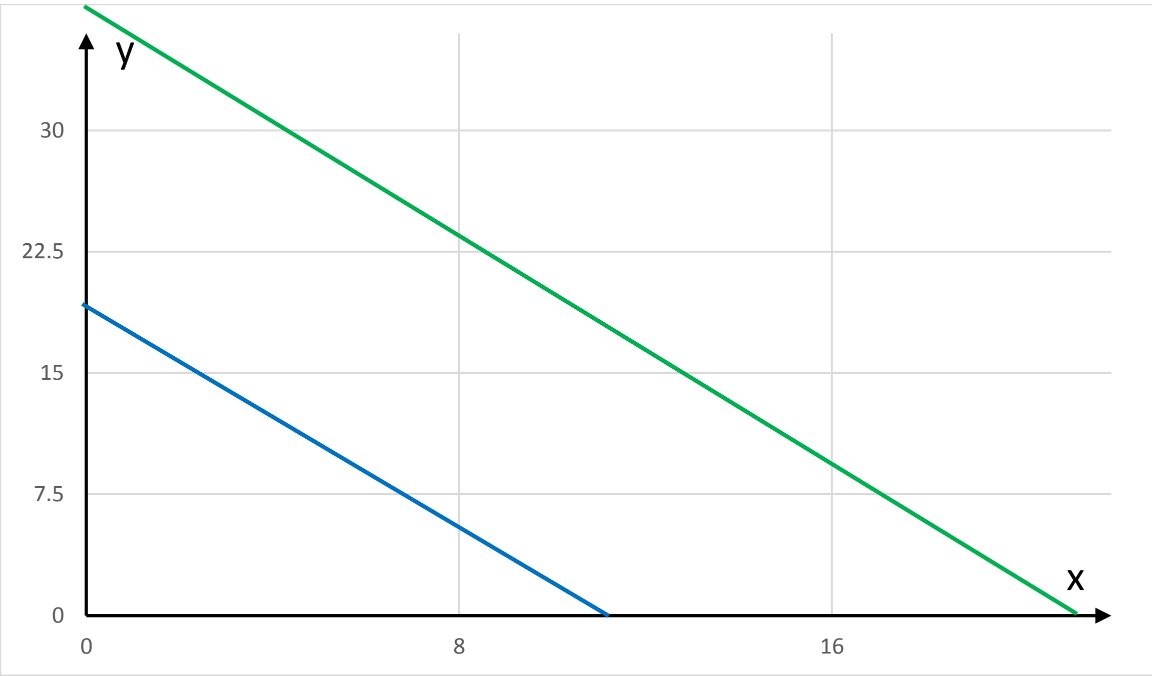


Figure 2-13

Since the first inequality has ≥, we must shade above and, since the second inequality has ≤, we must shade below (This idea can be confirmed by selecting points above and below each line in order to verify.):

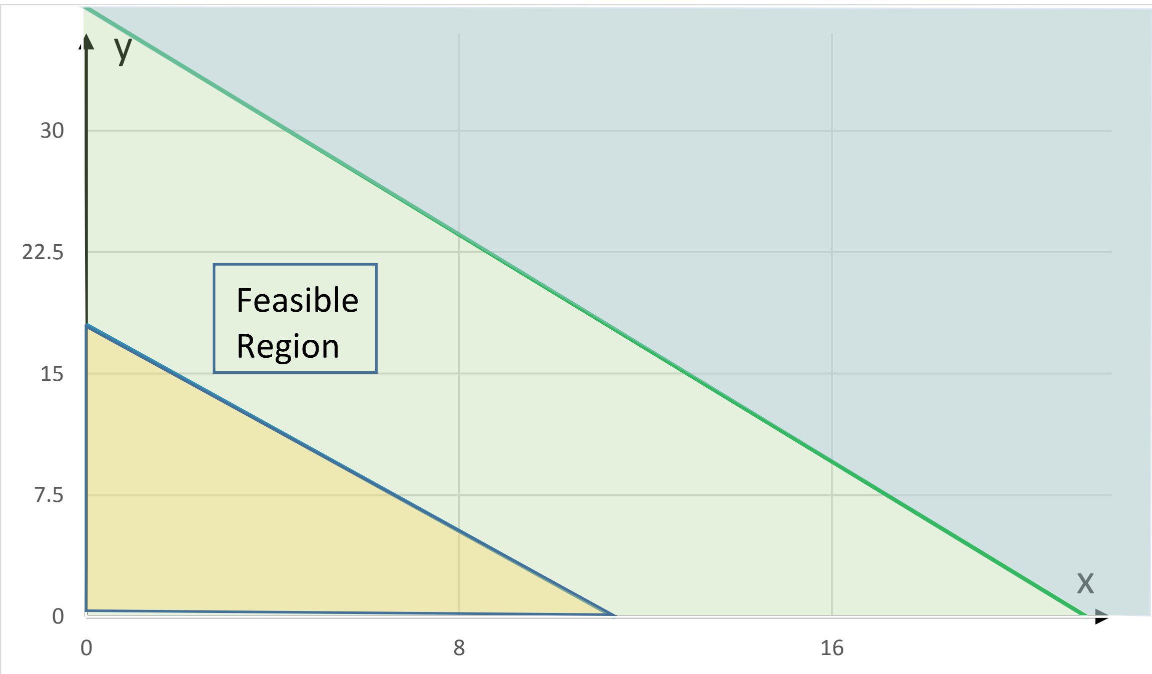


Figure 2-14

The feasible region is the green and blue shaded section between the two lines. We see that there are four corner points that form an upside-down trapezoid, as shown in the graph below:

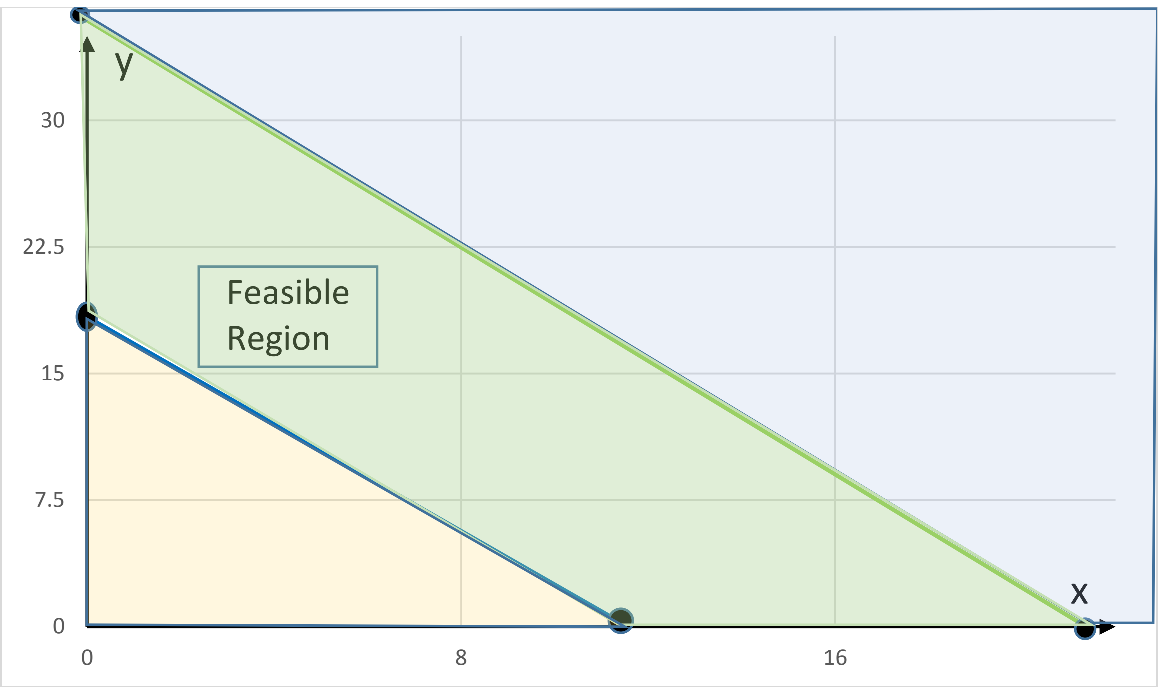


Figure 2-15

We must solve the following systems to find the corner points (bottom-to-top, left-to-right)

System 1

$x = 0$

$407x + 271y = 4700$

Solution:

$0 + 271\text{y} = 4700$

$y \approx 17.3$

point:$(0. 17.3)$

System 2

$x = 0$

$407x + 271y = 9400$

Solution:

$0 + 271\text{y} = 9400$

$\text{y} \approx 34.7$

point:$(0, 34.7)$

System 3

$\text{y} = 0$

$407x + 271y = 4700$

Solution:

$407x + 0 = 4700$

$x \approx 11.5$

Point(11.5,0)

System 4

$y = 0$

$407x + 271y = 9400$

Solution:

$407x + 0 = 9400$

$x \approx 23.1$

Point: (23.1,0)

Again, we could solve by using matrix equations, but the systems are straightforward to solve by substitution. Since the problem is bounded, we now check to see which one minimizes cost:

Point	cost
$(0, 17.3)$	$3.33(0) + 2.00(17.3) = \$\,34.60$
$(0, 34.7)$	$3.33(0) + 2.00(34.7) = \$\,69.40$
$(11.5,0)$	$3.33(11.5) + 2.00(0) = \$\,38.30$
$(23.1,0)$	$3.33(23.1) + 2.00(0) = \$\,76.92$

Table 2-2: optimal solution

The cheapest route for the company will be to create bars that contain no dried apricots and 17.3 servings of dried dates.

It is interesting to note that each of the corner points corresponds to either a horizontal or vertical intercept.

Why are we seeing what we're seeing? This is truly a case of real-world product creation! Of course, it doesn't make sense to increase the daily intake for the box, since this would mean increasing the amount of dried fruit, hence increasing cost. Since the cost of dried dates is cheaper ($2.00 per serving) and since for the price of one serving of apricots ($3.33 per serving) we can pay:

$$\frac{407mg}{\$\,3.33} \approx 122.2$$

mg per dollar for apricots, and

$$\frac{271mg}{\$\,2.00} \approx 135.5$$

mg per dollar for dates

It makes complete sense to buy dates, since the same dollar amount yields a higher content of potassium.

The question still remains: is it desirable to require a larger quantity of dates for a smaller price, or is it more desirable to require a smaller quantity of apricots for a larger price? This indeed depends on the constraints. The company might want to consider the amount of packaging/processing/etc. required in both instances. Perhaps the manufacturing and packaging costs could add constraints that alter the decision-making process. A similar problem will be left as a homework exercise for the reader to think about.

As a mathematical note, what we are seeing occurs as a result of having constraint lines that are parallel.

There are two terms we should be familiar with when dealing with inequalities: **bounded** and **unbounded**. A feasible region is said to be bounded if the constraints enclose the feasible region.

That is, if the shading does not continue to cover the entire plane, we are dealing with a bounded linear programming problem.

Both examples thus far have been examples of bounded linear programming problems, since the first feasible region was in the shape of a triangle and the second in the shape of a trapezoid.

If the feasible region cannot be enclosed among the lines formed by constraints, it is said to be unbounded. An example of an unbounded linear programming problem would be:

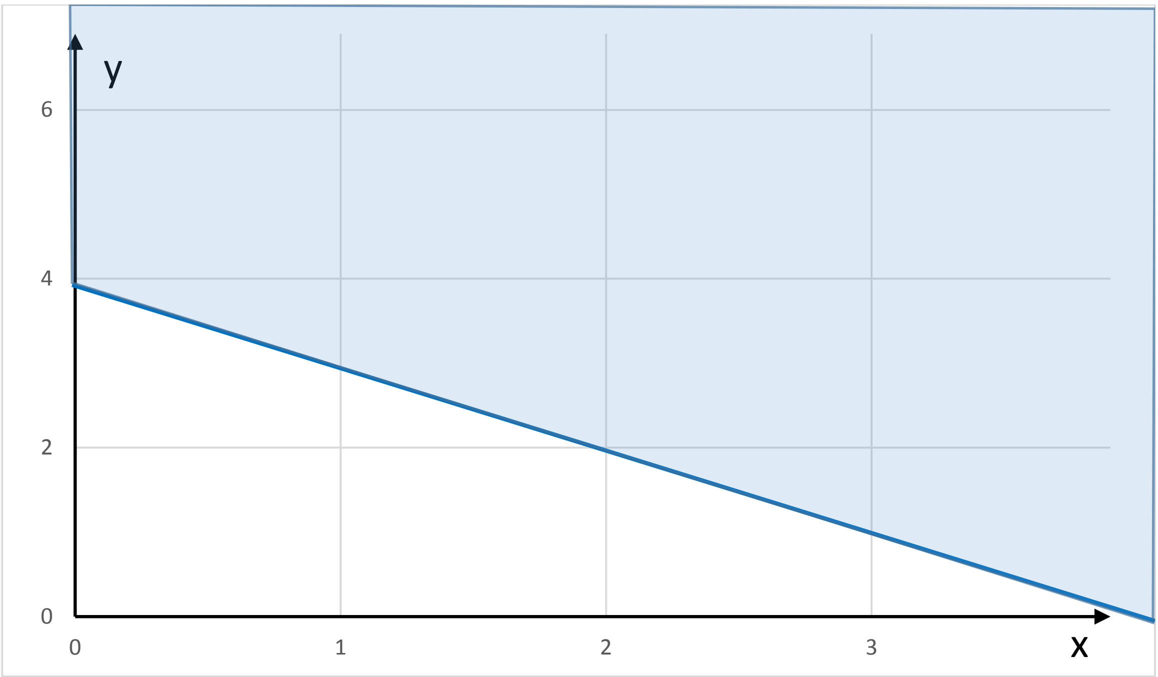


Figure 2-16

Example 3: A human resources office is working to implement an increase in starting salaries for new administrative secretaries and faculty at a community college. An administrative secretary starts at $28,000 and new faculty receive $40,000. The college would like to determine the percentage increase to allocate to each group, given that the college will be hiring 8 secretaries and 7 faculties in the upcoming academic year. The college has at most $5,000 to put towards raises. What should the percentage increase be for each group?

Solution:

Our goal is to determine the percentage increase for administrative secretaries and faculty, so let

$x =$ percentage increase for secretaries

$y =$ percentage increase for faculty

The college would like to minimize its total expenditures, so the objective function must include the total amount of money outflows. Since the new secretaries will require a total budget of

$28,000 × 8 = $224,000 and the faculty a total budget of $40,000 × 7 = $280,000, the total cost will be the raise percentage for each group, multiplied by the total salaries:

$C = 224x + 280y$

There is one constraint given, which is that the total raises must be $5,000 or less. That is,

$224x + 280y \leq 5$

Of course, the college does not want to reduce the salaries, so

$x \geq 0 \; and \; y \geq 0$

To visualize the situation, we graph the constraint as an equation. To help us find points, we first find the intercepts:

Horizontal Intercept: $224(0) + 280y = 5$

$y \approx \cdot 018$

Vertical Intercept: $224x + 280(0) = 5$

$x \approx \cdot 022$

We then plot the points and connect them with a straight line:

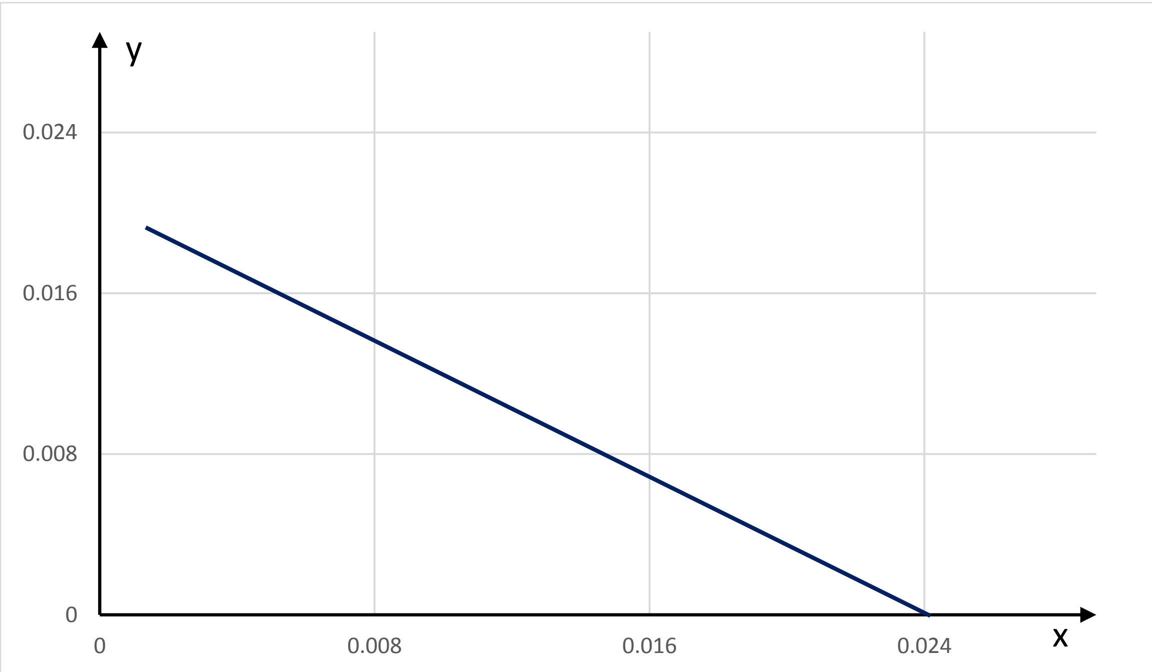


Figure 2-17

Since the inequality sign is ≤, we shade below the line:

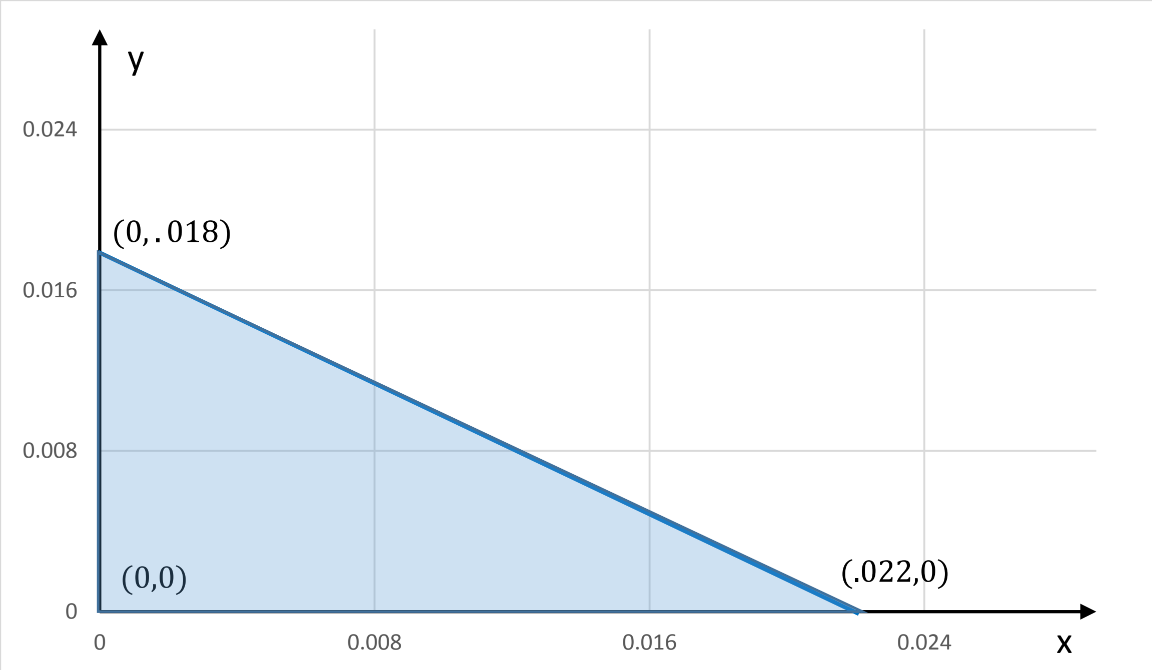


Figure 2-18

This gives us three corner points, as shown above. We test each to verify which of the pairs of percentages gives the minimum cost:

Point	cost
(0,0)	224(0) +280(0) = $ 0
(0,.018)	224(0) +280(.018) = $ 5.04
(.020,0)	224(.020) +280(0) = $ 4.48

Table 2-3: optimal solution

Clearly, the first option gives the smallest cost; however, this combination tells us to give a 0% raise to both groups, which, of course, is not practical, since the company's goal was to give a raise to each group.

Why did this happen, and what should we do to fix it? Well, when we think about the constraint of spending $5,000 or less and hoping to make expenditures as small as possible, wouldn't it make sense to say, "don't spend anything!"? This outcome will occur anytime we are minimizing, have constraints with the le inequality sign, and when the origin is included in the feasible region. To fix the problem, the company should make additional specifications, such as, what is the minimum percentage raise to give to each group? Is it desirable for one of the raises to be larger than the other? These are questions the analyst should discuss with human resources and administration.

Conclusion

- The problem of linear programming for numerical improvement was verified through a basic mathematical tool that aims to obtain the best value for the variables that provide the minimum or the maximum of the mathematical function (objective function).

- The most important concepts have been addressed with the graphical method, and graphically method.
- It was concluded that the graph solution method is the best mathematical equation and results to show how to get the restrictions that limit your goal to get to your desired goal faster and in the least possible time.

REFERENCES

[1]. Moré, Jorge J., Burton S. Garbow, and Kenneth E. Hillstrom. "Testing unconstrained optimization software." *ACM Transactions on Mathematical Software (TOMS)* 7.1 (1981): 17-41.

[2]. Platt, John, and Alan Barr. "Constrained differential optimization." *Neural Information Processing Systems*. 1987.

[3]. Zhang, Jianzhong, and Zhenhong Liu. "Calculating some inverse linear programming problems." *Journal of Computational and Applied Mathematics* 72.2 (1996): 261-273.

[4]. Kiendl, H., J. Adamy, and P. Stelzner. "Vector norms as Lyapunov functions for linear systems." *IEEE Transactions on Automatic Control* 37.6 (1992): 839-842.

[5]. Mokhtari, Aryan, Asuman Ozdaglar, and Ali Jadbabaie. "Escaping saddle points in constrained optimization." *Advances in Neural Information Processing Systems* 31 (2018).

[6]. Adilov, Gabil, and Aleksandr M. Rubinov. "$\mathbb{B}$-convex sets and functions." *Numerical Functional Analysis and Optimization* 27.3-4 (2006): 237-257.

[7]. Andreani, Roberto, José Mario Martínez, and Benar Fux Svaiter. "A new sequential optimality condition for constrained optimization and algorithmic consequences." *SIAM Journal on Optimization* 20.6 (2010): 3533-3554.

[8]. Bednařík, Dušan, and Karel Pastor. "On second-order conditions in unconstrained optimization." *Mathematical programming* 113.2 (2008): 283-298.

[9]. Zhmud, V., L. Dimitrov, and O. Yadrishnikov. "Calculation of regulators for the problems of mechatronics by means of the numerical optimization method." *2014 12th International Conference on Actual Problems of Electronics Instrument Engineering (APEIE)*. IEEE, 2014.

[10]. Komodakis, Nikos, and Georgios Tziritas. "Approximate labeling via graph cuts based on linear programming." *IEEE transactions on pattern analysis and machine intelligence* 29.8 (2007): 1436-1453.

الخلاصة

يتناول هذا البحث حل المسائل المقيدة وغير المقيدة للتحسين العددي. تم التحقق من مشكلة البرمجة الخطية للتحسين العددي من خلال أداة رياضية أساسية تهدف إلى الحصول على أفضل قيمة للمتغيرات التي توفر الحد الأدنى أو الحد الأقصى للدالة الرياضية (دالة الهدف)، والتي تشمل جميع مشاكل التحسين للعثور على الأمثل. حلول مشاكل البرمجة الخطية. تم تناول أهم المفاهيم بالطريقة الرسومية. الأساسيات التي تلبي الاحتياجات الخاصة للعمل واستخدام العديد من الطرق ومنها المقيد وغير المقيّد للوصول إلى عرض المشكلة من خلال الرسم التخطيطي والحصول على الحل الأمثل للنتائج. تم استنتاج أن طريقة حل الرسم البياني هي أفضل معادلة رياضية ونتائج لإظهار كيفية الحصول على القيود التي تحد من هدفك للوصول إلى هدفك المطلوب بشكل أسرع وفي أقل وقت ممكن.

جمهورية العراق
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جامعة بابل
كلية التربية للعلوم الصرفة
قسم الرياضيات

حل المسائل المقيدة وغير المقيدة للتحسين العددي

بحث مقدم

الى مجلس قسم الرياضيات كلية التربية للعلوم الصرفة جامعة بابل كجزء
من متطلبات نيل درجة الدبلوم العالي تربية / الرياضيات

مقدم من قبل

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اشراف

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2022 م

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