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Using linear programming techniques to find optimal solution for optimization problems

A Research

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﴿ بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ ﴾

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Dedication

I dedicate this humble effort to:

- My dear teachers

- My father (ALLah have mercy on him), and my dear mother

- my dear family

Acknowledgments

After thanks for Allah.....express my thanks, gratitude, andsincere appreciation to everyone who helped me in accomplishing this work, especially my supervisor, Dr.(Hussein Abdul wasi), who gave me his paternal care and did not deprive me from his valuable advices and guidance until this research was completed, praise be to Allah.

Abstract

Optimization problems are among the most important problems for which scientists have made great efforts to find the optimal solution. This research aims to present a study in the treatment and solution of this type of problem. In this study, the researcher dealt with the linear programming method, which is one of the most prominent and most common scientific methods in addressing optimization issues. In the beginning, the research presented a definition of the concept of linear programming and then a statement of the formulas and components of the linear model, and then clarifying how to convert the assumed problem into a linear model through an applied example from daily life. After that, the researcher discussed the two most important ways of solving linear programming problems, namely: The simplex and the graph method) are explained in detail with the resources, advantages and characteristics of each one. In the end, the research included a comparison between the two methods, through which it became clear that the simplex method is better than the graph method in terms of the number of steps and in terms of efficiency in reaching the optimal solution.

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Introduction

In our daily life , we always choose the best possible solutions for several problems. Sometimes we encounter problems of a special kind, which are maximum and minimum problems. In these problems we seek to achieve the greatest possible or the smallest possible, and they are called (optimization problems), such as the greatest profit and the lowest cost.

In mathematics , the study of maximum and minimum problems began a very long time ago . There has been increasing interest in the problem of minimizing functions of n variables numerically . There were no uniform ways to find the maxima or minima of problems . The first general methods of investigation and solution of extremal problems were created about 300 years ago , at the time of the formation of mathematical analysis . Then , it became clear that certain special optimization problems play a crucial role in the natural sciences . Specifically , it was found that many laws of nature can be derived from optimization methods [1].

After World War II, Scientists have been able to develop and innovate many mathematical methods that address this type of problem, and the most important of these methods is programming techniques. Linear programming is a basic and important method in solving optimization problems, and it is part of mathematical programming that includes linear and nonlinear programming, and mathematical programming is part of operations research.

Although there are multiple techniques because of the variety of optimization problems. In this research we will be limited to studying the programming techniques of a linear character, because it is the most common method In finding the optimal solution from among a set of alternatives offered to solve optimization problems in all areas.

In this context, we will answer the following questions in this research:

-What is the mechanism of action adopted in the linear programming system?

-What is the ability and feasibility of linear programming techniques in addressing optimization problems?

-What are the best methods of linear programming to find the optimal solution to problems of a linear nature?

Research comes from the great role that the process of reaching the optimal solution plays in addressing economic, administrative, military and service problems and other problems that require taking a set of critical decisions by decision makers based on scientific foundations and ready-made programs away from the principle of intuition or guesswork.

As for the research plan, it consisted of four chapters, as follows:

The first chapter deals with general frameworks on the concept of examples, their problems, their classification, and some basic definitions that we need in subsequent chapters. The second chapter included a detailed study of the concept, construction, formulation and conditions of the linear programming model. As for the third chapter, the researcher presented two methods of solving the linear programming model namely (simplex and graph), which were explained in detail by explaining the steps of the solution and enhancing the explanation with sufficient examples.

Finally, in the fourth chapter, a comparison was made between the two methods in order to clarify the similarities and differences between them and to note which of the two methods is more efficient in reaching the optimal solution.

CHAPTER ONE

BASIC CONSCEPTS

1.1. Introduction about optimization.

For almost all the human activities there is a desire to deliver the most with the least. For example in the business point of view maximum profit is desired from least investment; maximum number of crop yield is desired with minimum investment on fertilizers; maximizing the strength, longevity, efficiency, utilization with minimum initial investment and operational cost of various household as well as industrial equipments and machineries. To set a record in a race, for example, the aim is to do the fastest (shortest time).

The concept of optimization has great significance in both human affairs and the laws of nature which is the inherent characteristic to achieve the best or most favorable (minimum or maximum) from a given situation In addition, as the element of design is present in all fields of human activity, all aspects of optimization can be viewed and studied as design optimization without any loss of generality. This makes it clear that the study of design optimization can help not only in the human activity of creating optimum design of products, processes and systems, but also in the understanding and analysis of mathematical/physical phenomenon and in the solution of mathematical problems. The constraints are inherent part if the real world problems and they have to be satisfied to ensure the acceptability of the solution. There are always numerous requirements and constraints imposed on the designs of components, products, processes or systems in real-life engineering practice, just as in all other fields of design activity. Therefore ‹creating a feasible design under all these diverse requirements/constraints is already a difficult task, and to ensure that the feasible design created is also ‘the best’ is even more difficult. [2]

1.2. Concept of Optimization

The basic concept of optimization means maximizing profit, benefit, revenue, or minimizing costs, depreciation, and time for a specific function known as the "objective function" under certain constraints [3].

1.3. Optimization Problems

Problems that seek to maximize or minimize a mathematical function of a number of variables, subject to certain constraints, form a unique class of problems, which may be called optimization problems. Many real-world and theoretical problems can be modelled in this general framework.A common term optimize is usually used to replace the terms maximize or minimize. The mathematical function that is to be optimized is known as the objective function, containing usually several variables. An objective function can be a function of a single variable for some practical problems ؛however, a single variable function may not challenge from an optimization point of view. Optimization problems may involve more than one objective function and are known as multi-objective optimization problems.Depending on the nature of the problem, the variables in the model may be real or integer (pure integer or binary integer) or a mix of both. The optimization problem could be either constrained or unconstrained. In the constraint part of a mathematical model, the left-hand side of the constraint function (or a single variable) is separated from the right-hand-side value by one of the three signs: (1) equal to (=), (2) less than or equal to (≤), or greater than or equal to (≥).

The functions either objective or constraints, may be from either the linear or nonlinear domain. As per the function properties, they could follow any pattern such as continuous or noncontinuous, differentiable or nondifferentiable, convex or nonconvex, or unimodal or multimodal [4].

1.4. Classification of Optimization Problem

Based on the advanced, we can divide the optimization problems as follows:

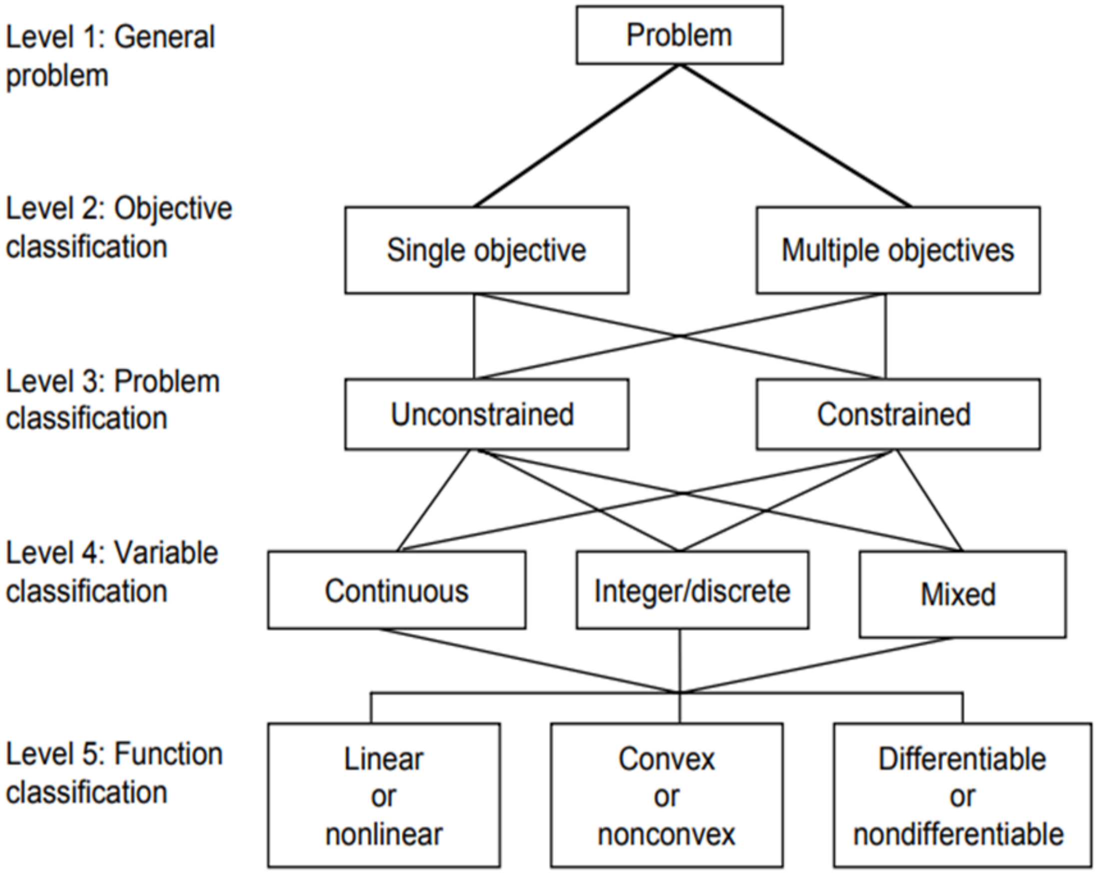


Figure1: Classification of Optimization Problem

1.5. Mathematical model of the optimization problem

All the optimal design problems can be expressed in astandard general form stated as follows:[4]

$$\begin{cases} \text{minimize} \quad f(x) \quad \text{Object function} \\ \text{Subject to } g(x) = 0 \quad \text{Equality Constraints} \\ \qquad\qquad h(x) \geq 0 \; Inequality\ Constraints \end{cases}$$

1.6. Optimization Algorithm

Optimization algorithms execute as equence of iterations to solve an optimization problem. The algorithm starts with an initial guess point and creates a sequence of points x (k) that converges to an optimal point to satisfy certain optimality conditions .The rate at which the iterations approach the optimal point is called the convergence rate of the algorithm. The algorithm converges faster if it takes a short time to obtain the optimal solution.[5]

1.7. Optimality Conditions

After an optimization algorithm has been applied to the model, we must be able to recognize whether it has succeeded in its task of finding a solution. There are elegant mathematical expressions known as optimality conditions for checking that the current set of variables is indeed the solution of the problem. If the optimality conditions are not satisfied, they may give useful information on how the current estimate of the solution can be improved. If a point satisfies the third order sufficient condition, then we have a guaranty that this point is a local minimizer.

1.7.1. First-Order Necessary Condition

If x^* is a local minimizer and f is continuously differentiable in an open neighborhood of x^*, then $\nabla f(x^*) = 0$.

1.7.2. Second-Order Necessary Condition

If x^* is a local minimizer of f and $\nabla^2 f$ is continuous in an open neighborhood of x^*, then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive semidefinite.

1.7.3. Second-Order Sufficient Condition

Presume that $\nabla^2 f$ is continuous in an open neighborhood of $x^*$and that $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite. Then x^* is a strict local minimizer of f.[2]

1.8 Basic Definitions

1. Constrained Optimization

Is the process of optimizing an objective function with respect to some variables in the presence of constraints on those variables.[4]

2. Unconstrained Optimization

It is the process of improving an objective function without including controls or restrictions on its variables.[1]

3. Objective Function

A linear function z = px + qy (p and q are constants) which has to be maximised or minimised, is called aobjective function.[5]

4. Constraints

The linear inequalities or equations or restrictions on the variables of the linear programming problem are called constraints. The conditions $x \geq 0$, $y \geq 0$ are called non-negative restrictions.[3]

5. Feasible Region

The common region determined by all the constraints including non -negative constraints x, y >0 of a linear programming problem is called the feasible region for the problem. The region other than the feasible region is called an infeasible region. The feasible region is always a convex polygon.[4]

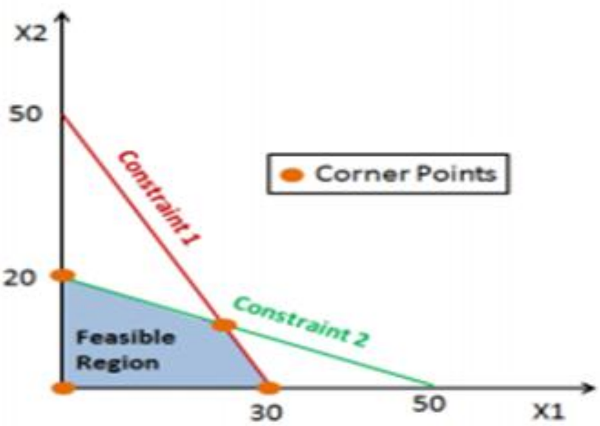


Figure2: Feasible Region

6. Bounded and Unbounded Region

A feasible region of a system of linear inequalities is said to be bounded, if it can be enclosed within a circle. Otherwise, it is called unbounded.[3]

7. Feasible Solutions

Points within and on the boundary of the feasible region represent feasible solutions of the constraints. Any point outside the feasible region is called an infeasible solution.[2]

8. Optimal Feasible Solution

Any point in the feasible region that gives the optimal value of the objective function is called the optimal feasible solution.[3]

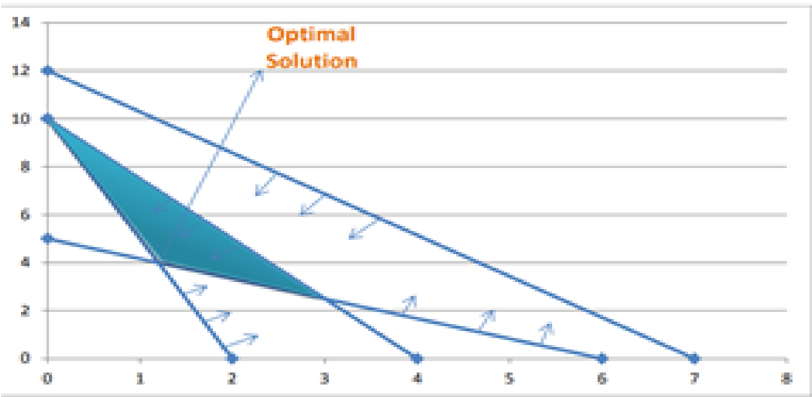


Figure3: Optimal Solutio.

9. Convex Set

A set Ω in n-dimensional Rn is said to be a convex set if any two points x and y in Ω, the line segment joining the two points is also in Ω [6].

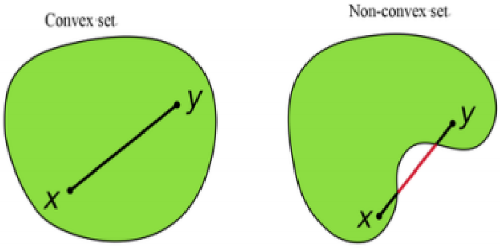


Figure4: Convex Set

10. Corner points

For any convex set S, the point S∈s s is a Corner point if the condition is met: for any line segment that lies entirely within S and contains the point S, the point is a terminal of this line segment.[6]

11. Linear Function

'is a function that represents a straight line on the coordinate plane.

For example, $y = 3x - 2$ represents a straight line on a coordinate plane and hence it represents a linear function. Since y can be replaced with f(x), this function can be written as $f(x) = 3x - 2$.[2]

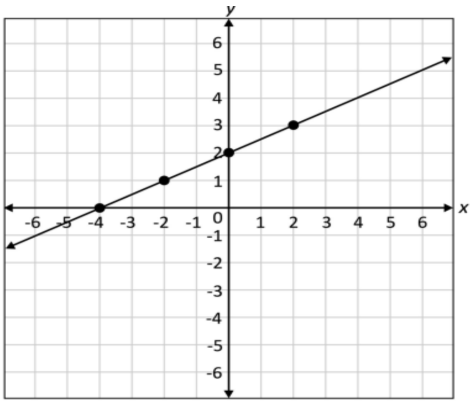


Figure5: linear function

12. Algorithm

An algorithm is a collection of sequential instructions that is used to solve com putational problems. The method involved in the algorithm is either sequential or iterative. The instructions used in the algorithm should be well-defined so that these can be implemented as a computer program. A computer program is a collection of sequential instructions written by a computer programmer in any programming language that performs a specific task when executed by a computer [6].

13. Mathematical Model

A group of interrelated and interrelated variables and factors that express a specific problem or situation, and are linked to each other through a number of relationships (equations or inequalities) according to certain formulas aimed at clarifying the nature of the problem in question with an indication of the specifications of its internal and external variables, can be represented as follows[7]:

$$Maximize\ f\ (x)$$

$$Subject\ to\ g_i(x)\ \leq\ gb_i\ , i\ =\ 1, \ldots, m$$

$$h_j(x)\ =\ h\ b_j\ ,\ \ j\ =\ 1, \ldots, p$$

$$x \geq 0$$

-Types of Mathematical Model.[7]

The mathematical model is divided into two parts:

1-Linear mathematical model: It is the model in which the relationship between the variables (X_j) and the value of the objective function (Z) can be represented in a straight lin.

2- Non-linear mathematical model: It is the model in which the relationship between the variables (X_j) and the value of the objective function(Z) can be represented in the form of a curve.

14. Programming

The use of a logical and scientific method in analyzing the problem and treating it.**[**6]

15. Linear Programming Problem

A linear programming problem is one in which we have to find optimal value (maximum or minimum) of a linear function of several variables (called objective function) subject to certain conditions that the variables are non-negative and satisfying by a set of linear inequalities with variables, are sometimes called division variables[8].

CHAPTER TWO

LINEAR PROGRAMMING

(LP)

2.1 Introduction

The science of linear programming is one of the important tools for solving optimization problems, as it deals with the search for the best profit or the lowest cost in problems that contain limited amounts of resources (such as the number of workers, the quantity of materials, the land area ... etc.). Linear programming can be summarized as the issue of determining the maximum or minimum value of a specific function called the objective function within a specific field that is determined through constraints on a finite number of variables so that the objective function and constraints achieve certain properties that will be clarified later, linear programming is widely used in solving military, economic, and industrial problems; This is because most of the problems in different fields can be expressed as linear programming problems, or they can be approximated to linear programming problems. The methods for solving written problems are accurate and available[7]. Now we must give a precise definition of linear programming as follows:

2.2 Defintion linear programming L.P

Linear programming is subfield of optimization. It is a mathematical technique for finding optimal solutions to the problems. Linear Programming deals with the problem of optimizing a linear objective function subject to linear equality and inequality constraints on the decision variables. Linear programming is not a programming language like C++, Java, or Visual Basic, its mathematical model.
So the linear programming method is a good way to find a better solutions of multiple economic problems involving many variables, it depends on a set of independent variables that also be a set of constraints such as cost or profit. Distribution of linear programming model can be used also in the allocation of limited resources where the model is based on the simple idea of content corresponding material scarcity [8].

The linear preprogramming also defined as:

-A mathematical method used to find the optimal solution for how the project uses its resources, and the word linear indicates that the relationships between the variables that make up the studied problem are linear, while the word programming refers to the mathematical technique used in finding the solution.

- It is also known as a scientific mathematical method or method concerned with addressing The problem of allocating limited resources or energies to achieve a specific goal, and this goal is expressed by a linear function that seeks the goal function, and the function is often a profit function, a cost function, a productive capacity function, and others[9].

2.3 The importance of linear programming

Linear programming is one of the most important mathematical programming methods and optimization in general for the following:
1- The importance of applied problems that can be formulated and solved using linear programming - especially after the development of computer science and data and information storage, which led to the use of this method in solving large-sized problems large - scale problems in the petroleum recycling Sector, the banking sector, the industrial production sector, ...etc.
2- Linear programming plays an important role in relation to other programming methods such as nonlinear programming, stochastic programming, or multi-objective programming, where non-linear problems are transformed at one stage or another to solve them into linear programming in most cases [10].

2.4 Formulation of linear programming model

The importance of the linear programming method is due to the importance of the problems that can be solved in general. But not every problem can be solved using the linear programming method, as solving the problem using the linear programming method requires that the following conditions are met:

1-Determining the objective function : It is the desired objective that we want to achieve and the possibility of expressing this objective in the form of a linear function and obtaining a numerical value for it and trying to maximize this value and find the maximum limit for it if the desired objective is a profit or reduce the value and find the small end if the objective is the cost of any access to the lowest possible cost.

The objective function consists of the variables, either the coefficient for each variable is the profit of one unit in the case of maximizing the objective function, or the coefficient is the cost of one unit in the case of reducing the objective function.

2- Determining the constraints : that is, the possibility of expressing the relationship between the decision variables and the available possibilities in the form of linear constraints.It shows what each production unit needs from a resource from the available limited resources in the form of inequalities or linear equations or a mixture of them and they are called structural constraints**.**

3- Non-negative conditions : the decisional variables in the problem under study must be positive or zero and non-negative variables[11].

Example 2.4.1.[11]

One of the industrial production of the three types of products (1, 2, 3) and want to determine the number of units that must be produced per day of each product so that you get the greatest (Maximum) profit possible, and requires the production per unit productivity of the three operations (A, B, C). The following table shows the time (in minutes) required per unit of each product of different processes, as well as the profit derived from the per unit and total time available for the three operations.

Operation	The time required per unit of product In each production process			Total time available (mint/day)
	Prod.1	Prod.2	Prod.3	
A	2	2	3	420
B	5	0	4	440
C	3	6	0	465
Profit Per unit	5	4	7	

Solution:

From the information in the above table, the required mathematical model can be formulated as follows

1- Decisional variables

It is required to produce a number of products of the three types during the time available for the operations (first, second and third) in order to obtain the maximum profit.

Suppose the number of units to be produced from product $X_1 = 1$.

Suppose that the number of units to be produced from product $X_2 = 2$. Suppose that the number of units that will be produced from the product $X_3 = 3$.

The number of units X_1, X_2, X_3 can be negative, and this cannot be logically possible, as the number of units that will be produced is either produced in any amount or not produced at all, so its value is zero, clarifying the non-negative condition: $X_1, X_2, X_3 \geq 0$

2- Constraints

The first constraint (in the first operation) is that the maximum time available for the operation is 420 minutes per day (not necessarily using the full time available). Since one unit of the first product needs 2 minutes to be manufactured in the first process, while one unit of the second product takes 2 minutes, and one unit of the third product takes 3 minutes. Thus, the first constraint can be formulated as follows:

$$2X_1 + 2X_2 + 3X_3 \leq 420$$

The second constraint (in the second process) The maximum time available for the second process is 440 minutes per day, and that one unit of the first product needs 5 minutes to be manufactured in the second process, while one unit of the second product does not require the second production process (0). As for the unit One of the third product needs 4 minutes, so the second entry will be as follows:

$5X_1 + 4X_3 \leq 440$

The third constraint (in the third process) is that the maximum time available for the second process is 456 minutes per day, and that one unit of the first product needs 3 minutes to be manufactured in the third process, and one unit of the second product needs 6 minutes, while one unit of the third product does not require the process The third productivity (0). Therefore, the third constraint would be:

$3X_1 + 6X_2 \leq 465$

3- The Objective Function: The goal of the decision maker in this problem is to achieve the largest possible profit. The objective function takes the following form:

$$Maximize\ Z = 5X_1 + 4X_2 + 7X3$$

It is clear that the mathematical model of the problem fulfills all the conditions of the linear programming model and takes the following form:

$$Maximize\ Z = 5X_1 + 4X_2 + 7X3$$

$$\begin{aligned} \text{S. t} \quad & 2X_1 + 2X_2 + 3X_3 \leq 420 \\ & 5X_1 + 4X_3 \leq 440 \\ & 3X_1 + 6X_2 \leq 465 \\ & X_1, X_2, X_3 \geq 0 \end{aligned}$$

2. 5 The basic assumptions of linear programming

The general mathematical model of linear programming is characterized by a number of assumptions in order to be appropriate and acceptable from a scientific and practical point of view, namely:

- **Certainty**

The values of the parameters (data) are known and areconstant.

- Proportionality

Any function (objective or constraint) is proportional to the level of the activity (with consistent unit of measure).

- Additivity

The total activity is the sum of all individual activities.

- Divisibility

The decision variables could be either real or integer.

- Nonnegativity

Only positive values of variables are allowed [12].

2.6 Forms of Linear programming Models

a-General Form: The linear programming model in general consists of**:**

1-Variables

2-Tags $(\leq, \geq, >, <, =)$

3-Variables Parameters

4-Objective Function(Z)

5-Constraints

6-Non- Negative Constraints

Therefore, the general form of a linear programming model is:

$$\boldsymbol{Max} \text{ or } Min\ Z = \sum_{j=1}^{n} c_j x_j$$

$$S.t \sum_{j=1}^{n} a_{ij} x_j \leq = \geq b_i$$

$$j = 1,2,3 \ldots \ldots \ldots n$$
$$i = 1,2,3, \ldots \ldots . m$$
$$x_j \geq 0$$

Example 2.6.1.[8]

$$\begin{array}{ll} \text{minimize} & 40x_1 + 25x_2 \\ \text{subject to} & 4x_1 + 3x_2 \geq 8 \\ & 5x_1 + 5x_2 \geq 15 \\ & 3x_1 + 6x_2 \geq 16 \\ & x_1, \quad x_2 \geq 0 \end{array}$$

Example 2.6.2.[8]

$$\begin{array}{ll} \text{Maximize} & 3x_1 + 2x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 1 \\ & x_1 \leq 2 \\ & x_1 + x_2 \geq 3 \\ & x_1, \quad x_2 \geq 0 \end{array}$$

b-Canonical Form of Linear Programming Model:

The difference between the Canonical form of the linear programming model and the general form of the linear programming model is as follows:

1-The objective function (Z) in the general form of the linear programming model is either of type (Max) or of type (Min), while it is in the Canonical form of the programming model Linear type (Max) only.

2-The constraints signs in the general form of a linear programming model are (≤,=,≥), while in the canonical form of a linear programming model it is less and equal to (≤) only.

The components of the linear programming model are the same inthe general and Canonical forms, as follows[12]:

$$Max\ Z = \sum_{j=1}^{n} c_j x_j$$

$$S.t \sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

$$j = 1,2,3 \ldots \ldots \ldots n$$
$$i = 1,2,3, \ldots \ldots . m$$
$$x_j \geq 0$$

The Canonical formula is used in some special cases of linear programming models. The general formula can be converted to the Canonical formula using the following rules:

1 - Minimized the objective function can be converted to maximized and vice versa by multiplying the objective function by (-1).

2- The constraint greater than or equal to ≥ can be converted to less than or equal to ≤ by multiplying both sides of the inequality by (-1).

3- The equality constraint can be converted into two constraints, the first is less than or equal to ≤ and the second is greater than or equal to ≥ and then the second is converted using the rule (2) above.

4- The Absolute Value constraint can be converted into two entries of a type less than or equal to ≤ .

C- Standard Form of Linear Programming Model

Standard Form is an important formula used to solve linear programming models with simplified way (simplex), it has the following required conditions.

1 .The objective function can be formed as (Max. or Min.).

2.The form variables are constrained.

3 .Restriction form have equal authority (equations). Any form of restrictions must be converted to equations, inequalities and is done as follows the restriction prevents varying less than or equal to equal authority to add a variable to the left of it This is called Slack Variable and usually takes place by

the symbol (S). If the limitation varying greater than or equal, is converted to equal authority to put the left part from slack variable is as follows the first constraint if different less than or equal to.

$a_{11} X_1 + a_{12} X_2 + \ldots\ldots + a_{1n} X_n \leq b_1$

It is converted to an equal constraint by adding the variable S1 recession to left end as follows:

$a_{11} X_1 + a_{12} X_2 + \ldots\ldots + a_{1n} X_n + S_1 = b_1$

S_1 slack variable here represents the surplus available resource of untapped b_1, for example if the second constraint varying body is greater than or equal to

$a_{21} X_1 + a_{22} X_2 + \ldots\ldots + a_{2n} X_n \geq b_2$

It is converted to an equal constraint by subtracting the left party S_2 recession variable as follows:

$a_{21} X_1 + a_{22} X_2 + \ldots\ldots + a_{2n} X_n - S_2 = b_2$

Variable S_2 recession here represents the increase in the required slack variables b_2 needs also be constrained point no greater than or equal to zero, which, if the linear programming model Constrained optimization formula with greater or equal, the standard formula writing as follows [7].

$$Max.Z = C_1 X_1 + C_2 X_2 + \ldots\ldots + C_n X_n$$

S.t

$$a_{11} X_1 + a_{12} X_2 + \ldots\ldots + a_{1n} X_n + S_1 = b_1$$

$$a_{21} X_1 + a_{22} X_2 + \ldots\ldots + a_{2n} X_n + S_2 = b_2$$

$$a_{m1} X_1 + a_{m2} X_2 + \ldots\ldots + a_{mn} X_n + S_m = b_m$$

$$X_1, X_2, \ldots, X_n, S_1, S_2, \ldots S_m \geq 0$$

If the linear programming model formulation and restrictions with less or equal, the standard formula writing as follows:

$$Min.Z = C_1 X_1 + C_2 X_2 + \dots\dots + C_n X_n$$

S.t $$a_{11} X_1 + a_{12} X_2 + \dots\dots + a_{1n} X_n - S_1 = b_1$$

$$a_{21} X_1 + a_{22} X_2 + \dots\dots + a_{2n} X_n - S_2 = b_2$$

$$a_{m1} X_1 + a_{m2} X_2 + \dots\dots + a_{mn} X_n - S_m = b_m$$

$$X_1, X_2, \dots, X_n, S_1, S_2, \dots . S_m \geq 0$$

This method has the following properties

1.All constraints on the form of the equations .

2. Objective function in case of Maximization or Minimization.

3. Right end of nonnegative constraint and the formula can be represented as linear model standard objective function in the Maximization:

$$\boldsymbol{Max}\ Z = \sum_{j=1}^{n} c_j x_j + 0.S_i$$

$$S.t \sum_{j=1}^{n} a_{ij} x_j + S_i = b_i$$

$$j = 1,2,3 \dots\dots\dots n$$
$$i = 1,2,3, \dots\dots . m$$
$$x_j \geq 0, S_i \geq 0$$

Example 2.6.3.[10]: Transfer the G. L. P model to:

A) Canonical form

B) Standard form

$$MinZ = 2X_1 + 4X_2$$

S.t $$3x_1 - x_2 \leq 8$$

$$-5x_1 + 2x_2 \geq 3$$

$$4x_1 - x_2 = 6$$

$$x_1, x_2 \geq 0$$

Solution

A) Canonical form

1 - The objective function must be of the type (Maximized by multiplying the objective function by (-1) to become $Max - Z = -2X_1 - 4X_2$

2- Constraints: the first constraint remains the same because its sign is less than or equal to, the constraint The second is multiplied by (-1) because its sign is greater than or equal to, the third constraint is transformed into two constraints, one of which is (≤ 6) and the other is (≥ 6), then we multiply it by (-1) for the purpose of converting it to less than or equal to. meaning that the legal formula form is as follows:

$$Max - Z = -2X_1 - 4X_2$$

$$\text{S.t} \quad 3X_1 - X_2 \leq 8$$

$$+5X_1 - 2X_2 \leq -3$$

$$4X_1 - X_2 \leq 6$$

$$-4X_1 + X_2 \leq -6$$

$$X_1, X_2 \geq 0$$

B) Standard Formula

1- The objective function is of type (Min) and as found in question.

2- Constraints: we add to the first constraint ($+S_1$) because its sign is less than or equal to and we transform the sign to (=), the second constraint we add It has ($-S_2$) because its sign is greater than or equal to, and we convert the sign of the constraint to (=), the third constraint remains the same because its sign (=). Thus three Constraints become in the standard form, as follows:

$$MinZ = 2X_1 + 4X_2$$

$$\text{S.t} \quad 3X_1 - X_2 + S_1 = 8$$

$$-5x_1 + 2x_2 - S_2 = 3$$

$$4x_1 - x_2 \qquad = 6$$

$$x_1, x_2, S_1, S_2 \geq 0$$

Example 2.6.4.[10]: Transfer the G. L. P model to:

A) Canonical form

B) Standard form

$$Max\ Z = 6X_1 + 2X_2$$

$$\text{s. t} \quad X_1 + 8X_2 \leq 50$$

$$2X_1 + 7X_2 \geq 30$$

$$X_1, X_2 \geq 0$$

Solution

A) Canonical form

multiply the second constraint by (-1) to convert it into greater or equal to $-2x_1 - 7x_2 \geq -30$ meaning that the legal formula form is as follows:

$$Max\ Z = 6X_1 + 2X_2$$

s. t

$$X_1 + 8X_2 \leq 50$$

$$-2x_1 - 7x_2 \leq -30$$

$$X_1, X_2 \geq 0$$

B) Standard form

As for the standard form, it is as follow:

$$\text{Max Z} = 6x_1 + 2x_2 + S_1 + S_2$$

$$\text{s. t} \qquad x_1 + 8x_2 + S_1 = 50$$

$$2x_1 + 7x_{2\_}S_2 = 30$$

$$x_1, x_1, S_1, S_2 \geq 0$$

2.7 The linear programming model in matrix terms .

The linear programming model can be written as matrices as follows:

$$Max\ or\ Min\ Z = C^T X$$

$$\text{S.t} \qquad AX(\leq, =, \geq)b$$

$$X \geq 0$$

Where :

X : vertical vector (n × 1) whose elements represent the decision Variables Structural .

b: vertical vector (m × 1) whose elements represent the values at the right end of the constraints.

C: row vector (1 × n) whose elements represent the coefficients of the decision variables in The objective function.

A: is a rectangular matrix Its rows are equal to m and the number of its columns is n, i.e. of: A arrangement (m x n) [13].

Example 2.7.1.[11]

Convert the following linear programming model into matrix form

$$\text{Max Z} = X_1 + 2X_2$$

$$\text{s.t} \quad X_1 + 5X_2 \geq 60$$

$$3X_1 + 4X_2 \leq 20$$

$$X_1, X_1 \geq 0$$

Solution

$$c^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 60 \\ 20 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 5 \\ 3 & 4 \end{bmatrix}$$

CHAPTER THREE

Methods for Solving Linear Programming Models

3.1. Introduction

In the previous chapter, we learned how to formulate applied problems in the form of a linear programming model, and in this chapter we will study some ways to solve these problems to find the maximum value if the problem is a maximum value or the minimum value if the problem is a minimum value problem, but after we mention at the beginning the theory Fundamentals of linear programming.

3.2. Fundamental Theorem of Linear Programming Problem

Consider a linear programming problem in standard form:

$$\text{minimize} \quad c^T x$$

$$\text{subject to} \quad Ax = b$$

where, $c \in R^n$, $x \in R^n$, $b \in R^m$, $A \in R^{m\times n}$, $x \geq 0$.

1. If there exists a feasible solution, then there exists a basic feasible solution.

2. If there exists an optimal feasible solution, then there exists an optimal basic feasible solution [14].

3.3. Different Solution

The methods for solving the linear programming model are summarized in two main ways, namely:

1- Graphical Method.

2-The Simplex Method.

As for the other methods, they deal with special cases of linear programming that can be solved by one of the two previous methods, so in this research we will be content with studying these two methods only.

3.4. Graphical Solution Method

The graph method is an easy and simple way to deal with linear programming problems, It addresses problems in which the number of decision variables is no more than two, and which contain a small number of constraint,The graph method is also useful as an introduction to studying other more complex methods and techniques in solving linear programming problems.

Therefore, the main objective of presenting this method in this chapter is to be an introduction to the simplex method, which we will discuss in detail.

3.4.1 Steps for Using the Graphical Method

Step1 : Converting the inequalities into equals (equations) and then plotting these equals.

Step2 : The direction of each inequality is determined (i.e. the area in which each point fulfills the inequality) and this is usually done by substitution on the left side of the inequality with a point in one of the two parts (a certain direction). The inequality point is not achieved, so the direction of the inequality is in the second part in which the point does not fall.

Step3: Determining the area of feasible solutions (if any), which is the area that achieves all the inequality at the same time (or in other words, the area of the feasible solutions is the category of intersection for all the categories that represent the areas of achieving the inequality) and then identifying the corner points of the area of feasible solutions where they represent all An corner point is a feasible basic solution. This can be graphically, and since any corner point represents the intersection point of two or more lines that correspond to the inequalities, it is feasible to determine any corner point by solving the equations of the two lines arising from their intersection of the corner point.

- Corner point definition: The end point (corner) is the point arising from the intersection of two or more lines that correspond to the structural and non-negative constraints.

- **Basic solution definition**: The different points for solving equations corresponding to structural constraints are called Basic Solutions points, and the basic solutions points are divided into two types. Every corner point is a possible basic solution. The second type is called Basic Nonfeasible Solutions Points, which are basic solutions points, but each point does not meet the non-negative conditions

Step4: Calculating the value of the objective function at each possible corner point and determining the optimal solution among the possible basic solutions[15].

3.4.2. Notes

1- Each feasible corner point represents a feasible basic.

2- If there are conflicting constraints, in this case there is no feasible

solution area.

3- Sometimes there are additional restrictions that do not affect the solution, so they are excluded.[16].

Example 3.4.3.[13]

Find the optimal solution for (LP) model by using graphical Method

$$Max. Z = 3X_1 + 5X_2$$

$$\text{S.}t \quad 2X_1 + 3X_2 \leq 30$$

$$5X_1 + 4X_2 \leq 60$$

$$X_1 , X_2 \geq 0$$

Solution

$2X_1 + 3X_2 = 30$ (Constraint1)

If X_1=0, X_2=10 $\longrightarrow$ $P_1(0,10)$

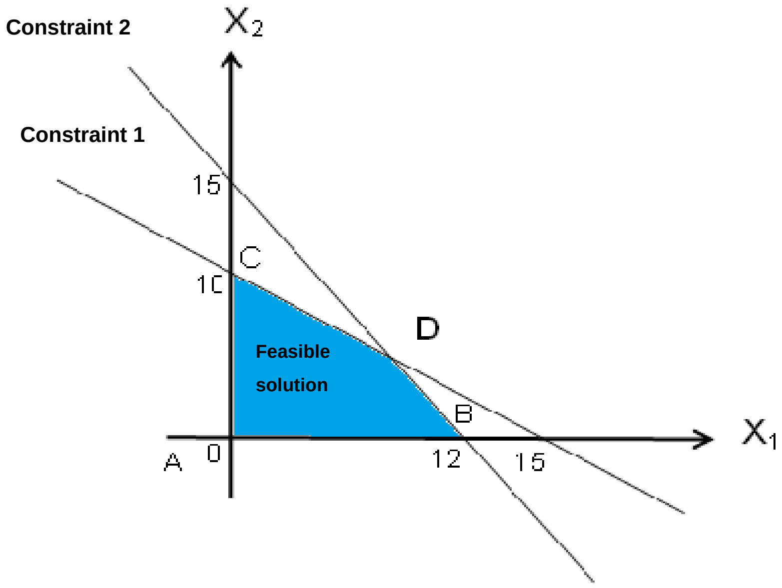
If X_2=0, X_1=15 $\longrightarrow$ $P_2(15,0)$

$5X_1 + 4X_2 = 60$ (Constraint 2)

If X_1=0 ,X_2=15 $\longrightarrow$ $P_1(0,15)$

If X_2=0 ,X_1=12 $\longrightarrow$ $P_2(12,0)$

Therefore, the feasible solution area will be as follows:



From the previous figure, it is clear that the feasible solution area is limited by points(D,C,B,A): A= (0,0) , B= (12,0) , C= (0,10).

To find the coordinates of point d, we intersect the two constraints (the first and the second), that is:

$(2X_1 + 3X_2 = 30)$………. (1) × 5

$(5X_1 + 4X_2 = 60)$………..(2) × 2

$10X_1 + 15X_2 = 150 \ldots\ldots\ldots$ (3)

$\mp 10X_1 \mp 8X_2 = \mp 120$….(4)

$7X_2$= 30 $\longrightarrow$ X_2= 4.3

And by substituting it into the equation number(1)

$2X_1$+3(4.3) = 30 $\longrightarrow$ $2X_1$+12.9= 30

$2X_1$= 17.1

X_1= 8.6 $\longrightarrow$ D (8.6, 4.3)

To find the optimal solution, we make the following table:

border points	X_1	X_2	$Z=3X_1+5X_2$	MaxZ
B(12,0)	12	0	36	50
C(0,10)	0	10	50	
D(8.6,4.3)	8.6	4.3	47.3	

Therefore, the optimal solution for the model is: $X_1 = 0,\ X_2 = 10 \quad Z^*=50$

Example 3.4.4.[12]

Find the optimal solution for (LP) model by using graphical method:

$$Min\, Z = 3X_1 + 2X_2$$

Subject to $4X_1 + 6X_2 \geq 12$

$$8X_1 + 4X_2 \geq 16$$

$$X_1, X_2 \geq 0$$

Solution

$4X_1 + 6X_2 = 12$ (Constraint1)

If $X_1 = 0, X_2 = 2$ $P_1(0,2)$

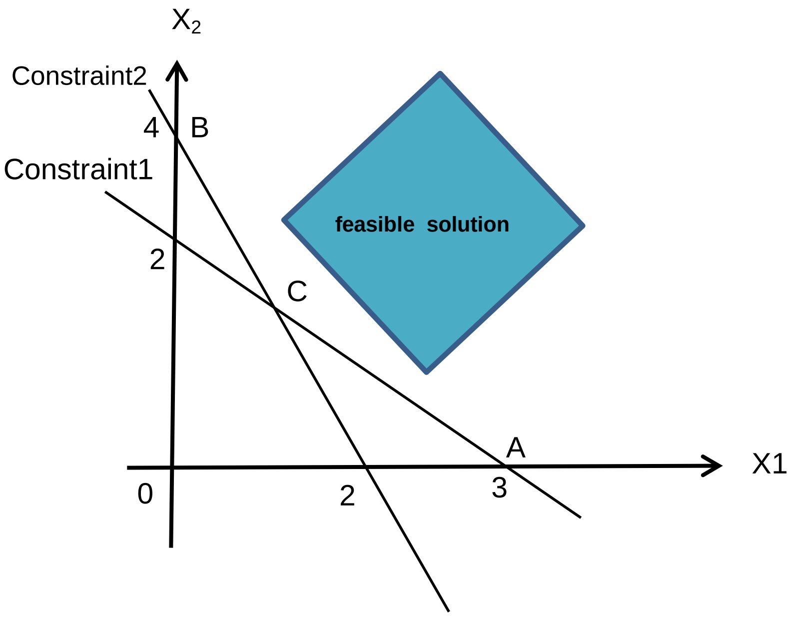
If $X_2 = 0 , X_1 = 3$ $P_2(3,0)$

$8X_1 + 4X_2 = 16$ (Const raint2)

If $X_1 = 0,\ X_2 = 4$ $P_1(0,4)$

If $X_2 = 0,\ X_1 = 2$ $P_2(2,0)$

Therefore, the feasible solution area will be as follows:



From the previous figure, it is clear that the feasible solution area is limited by points(C,B,A): A= (3,0), B= (0,4).

To find the coordinates of point C, we intersect the two constraints (the first and the second), that is:

$$4X_1 + 6X_2 = 12 \ldots\ldots\ldots(1) \times 2$$

$$4X_1 + 6X_2 = 12 \ \ldots\ldots\ldots(2)$$

$$8X_1 + 12X_2 = 24$$

$$\mp 8X_1 \mp 4X_2 = 16$$

$$8X_2 = 8 \longrightarrow X_2 = 1$$

And by substituting it into the equation number(1)

$$4X_1 + 6(1) = 12 \qquad 4X_1 = 6$$

$X_1 = 1.5$ C= (1.5, 1)

To find the optimal solution, we make the following table:

border points	X_1	X_2	$Z=3X_1+2X_2$	MinZ
A(3,0)	3	0	9	6.5
B(0,4)	0	4	8	
D(1.5,1)	1.5	1	6..5	

Therefore, the optimal solution for the model is:

$X_1 = 1.5, X_2 = 1, Z = 6.5$

3. 5 The Simplex method

The simplified method is an advanced mathematical method for solving linear programming (LP) problems, as it deals with problems that contain a large number of variables (two or more variables) It is also better and more accurate than the previous method

We have previously seen that the historical beginnings of the application of the simplex method, go back to the efforts made by the scientist (Dantzig) in 1947, when it became clear to him the inability of the graphical method in dealing with the problems of linear programming (LP), when it contains more than two variables.

The simplified method has been widely used in dealing with linear programming (LP) problems at the present time, as a result of the spread of electronic computers and the development of ready-made software related to this type of problem.

The solution to the Linear Programming Models (LP), according to this method, is found according to three basic and sequential stages, which can be described as follows:

1. The first stage: Finding the initial basic Feasible Solution

2. The second stage: Optimize the first solution to get the best solution.

3. The third stage: Optimize the best solution to get Optimal solution), and this may be done in one stage or several stages[17].

3.6. Solution of (L.P) Model with Maximization Objective Function

To find the optimal solution to the problems of linear programming, according to the simplex method, we follow the following steps:

Step1: Converting the linear programming model(LP) from the legal form to the standard form, after adding the redundant or Slack variables to both the objective function(Z) and the model constraints, taking into account making the objective function(Z) equal (zero).

Step2: Designing a Feasible Solution table, based on all the coefficients of the variables (X_j, S_i) in the model constraints, and the objective function (Z).

Step3: Determining the Entering Variable, based on the largest value with a negative sign in the objective function row (z).

Step4: Determining the Leaving Variable, by dividing the values located on the right side in the column (R.H.S), by the corresponding values of the coefficients in the pivot column and the variable that corresponds to the least positive value of the quotients in the ratio column is the Leaving variable, to be replaced by the Entering variable.

Step5: The column in which the input variable is located is called the Pivot Column.

Step6: The row in which the external variable is located is called the Pivot Row.

Step7: The element that lies under the entering variable, and in front of the Leaving variable, is called the pivot element, in other words, it is the element resulting from the intersection of the column of the entering variable with the row of the Leaving variable.

Step8: The Pivot Equation can be obtained by dividing the values in the row of the Leaving variable by the Pivot Element.

Step9: For the purpose of Optimize the feasible solution, and obtaining the best solution, we follow the following:

a. Find the coefficients of the new objective function (New Z), as follows:

the new coefficients (Z) = coefficients (Z) - coefficient of the variable included in the objective function row * the pivot equation.

b- Finding the new constraints coefficients for the variables (S_i), as follows:

The new (S_i) coefficients The old (S_i) coefficients - the coefficient of the variable included in the row (S_i) * the pivot equation.

Step10: The Solution Optimal for the maximization problem can be obtained when all coefficients of the new objective function (C_j)in the solution table are greater than or equal to zero, i.e. ($C_j \geq 0$), but if the value of At least one of the coefficients (C_j) in the objective function (negative), meaning that ($C_j \leq 0$), this means that the optimal solution has not been reached.

Step11: Repeat the previous steps (3-10) until all coefficients are obtained (C_j) in the objective function (Z), is greater than or equal to zero, meaning that ($C_j \geq 0$), which means, the optimal solution to the problem was obtained.

3.6.1 Slack Variable (S_i)

Variables are added to the objective function and its dealings in the objective function is equal to zero and add to the constraints and be single transactions if limitations of type ≤ if restrictions of the type which is equal to one of the leading coefficient ≥ ask when adding or subtracting S factor which one needs treatment. For limitations to equations by converting the inequality to equality process[18].

3.6.2 solution algorithm diagram

The following figure shows the solution algorithm by simplex method

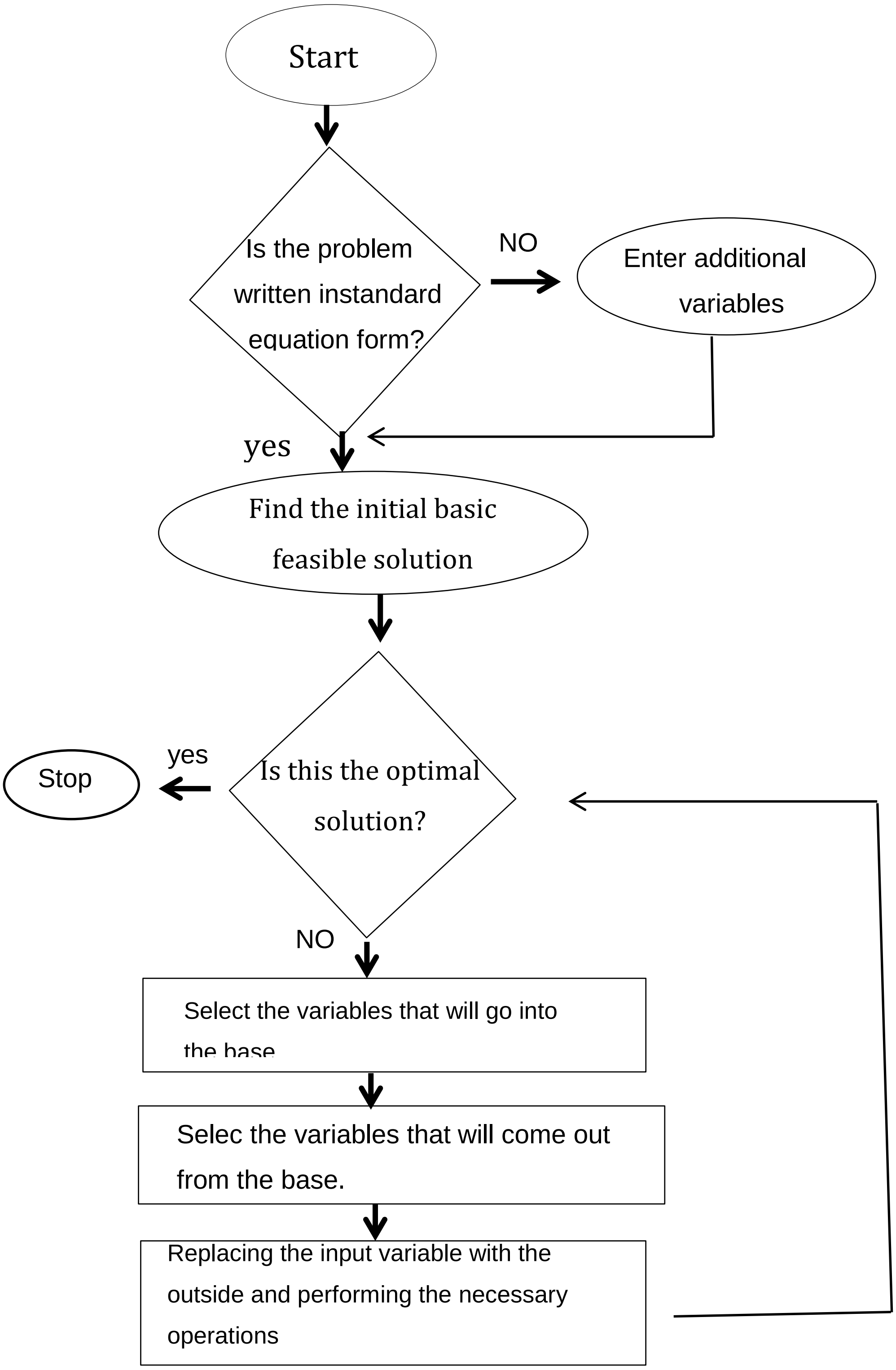


Figure 6: solution algorithm diagram

Example 3.6.3.[9]

Find the optimal solution for (L.P) model using Simplix Method.

$$Max. Z = 30X_1 + 18X_2$$

$$\text{S.t:} \quad X_1 + 2X_2 \leq 200$$

$$3X_1 + 2X_2 \leq 300$$

$$X_1 \leq 150$$

$$X_1, X_2 \geq 0$$

Solution

1- Convert the linear programming (LP) model to the standard form as Follows:

$$Max\ Z - 30X_1 - 18X_2 - OS_1 - 0S_2 - 0S_3 = 0$$

$$\text{S.t:} \quad X_1 + 2X_2 + S1 = 200$$

$$3X_1 + 2X_2 + S_2 = 300$$

$$X_1 + S_3 = 150$$

$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

2- Design the initial solution table as follows:

Table1

Basic Variables	Non- Basic Variables					Constants(b_i) R.H.S	Ratio
	X_1	X_2	S_1	S_2	S_3		
Z	-30	-18	0	0	0	O	-
S_1	1	2	1	0	0	200	200
S_2	3	2	0	1	0	300	100
S_3	1	0	0	0	1	150	150

Pivot Row

Pivot Element

Pivot Column

3- The input variable is (x_1) as it corresponds to the largest value with a negative sign(-30) in the objective function row.

4 - The outside variable is s because it corresponds to the lowest positive value (100) in the ratio column.

Note:

Negative values (-) or undefined (∞) are neglected in the ratio column.

5-The pivot element is the value (3), which can be obtained from the intersection of the pivot column and pivot row.

6- The pivot equation can be obtained by dividing the values of the pivot row by the pivot element (3),that is

$$\text{Pivote quation} = \left[\frac{3}{3},\frac{2}{3},\frac{0}{3},\frac{1}{3},\frac{300}{3}\right] = \left[1,\frac{2}{3},0,\frac{1}{3},0,100\right]$$

7- The new values of the objective function (z) and the variables (S_1, S_2) can be obtained as follows:

$$\text{New (Z)}= [-30, -18, 0,0,0,0]- (-30)^* \left[1,\frac{2}{3},0,\frac{1}{3},0,100\right]$$

= [-30, -18, 0,0,0,0] + [30,20,0,10,3000] = [0,2,0,10,0,3000]

$$\text{New } (S_1) = [1,2,1,0,0,200] - (1)^* \left[1,\frac{2}{3},0,\frac{1}{3},100\right] = \left[0,\frac{4}{3},1,\frac{-1}{3},0,100\right]$$

$$\text{New } (S_3) = [1,0,0,0,1,150] - (1)^* \left[1,\frac{2}{3},0,\frac{1}{3},100\right] = \left[0,-\frac{2}{3},0,\frac{-1}{3},1,50\right]$$

We put the previous results in a second solution table, as follows :

Table2

Basic Variables	Non- Basic Variables					Constants(b_i) R.H.S
	X_1	X_2	S_1	S_2	S_3	
Z^*	0	2	0	10	0	3000
S_1	0	$\frac{4}{3}$	1	$\frac{-1}{3}$	0	100
X_1	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	100
S_3	0	$\frac{-2}{3}$	0	$\frac{-1}{3}$	1	50

Since all the coefficients (C_j) of the new objective function (z) in the above table are greater and equal to zero, that is, ($C_j \geq 0$), then the optimal solution to the problem is : $X_1 = 100, X_2 = 0, Z* = 3000$.

3.7. Solution of (L.P) Model with Minimization Objective Function

The solution of a linear programming model (LP) under the Simplex Method in the case of (Min) the objective function (Z) is reduced, that is, when all the constraints signs are greater than or equal to (≥), or the constraints signs are in the form [equals (=), or greater than or equal to (≥)] in very special cases, it is done by one of the following two methods:

1. The Big M method.

2. The two-phase method.

3.7. 1 The Big M method

3.7.1.1.The Big (M) Frame work

1.Convert the linear programming model of legal form to standard mode after adding Slack variables (S_i) to form the objective function constraints after that requires adding artificial variables (R_i) also objective function constraints.

2. Formulate new objective function (Z) must be for machine variables ((X_j) ،(S_i)) subject to make equal to the value of the function is only.

3. M-design table basic solution is possible depending on all equations variables (X_j, S_i, R_i) in program constraints and objective function (Z).

4.Specify variable inside based on the largest positive value in a row objective function (Z).

5.The adoption of the rest of the preceding steps if veneration (Min) when all equations $C_j$and new objective function in a table optimization This method boils down to finding large M values and extracting them in the way of the simplex table and compensating them with the new table that we have got rid of the large M is an example of how we put the values in a large M[19].

3.7.2. Two-Phase Method

This method more naturalness of big M method in finding the optimal solution to linear programming model in case of decrease (Min) you can get a perfect solution after to make sure that there is a solution to the model by obtaining new objective function value (r) is equal to zero (R = 0) and whether or not it is no solution to form and is the solution under this way on key stages and thus two-phase method framework.

First Phase

1.Convert linear programming model of legal form to standard mode and then adding synthetic variables (R_i) to restrictions form only.

2.New objective function formula (r) rely on synthetic variables R_i that:

r = R_1-R_2-...........-R_n

3.The first solution contains table design based on variable equations (X_j , R_i , S_i) in the form of new objective function constraints(r).

4.Follow the previous steps until you get a value (r = 0), which means there was a solution to the model and associated with that (C_j) for all objective function coefficients (r) .

Second Phase

1.Final basic solution in step (4) from the first stage after excluding synthetic variables R, objective function .

2. The adoption of the objective function (Z) and improve its value to obtain the optimal solution to the problem .

3. If a transactional (C_j) is greater than zero ($C_j > 0$) in a row objective function (Z) be the same steps until you get all the.

4. Transactions (C_j) less than or equal to zero any he ($C_j \leq 0$) which means the optimal Solution obtained for the model [20].

Example 3.7.3.[16] : Find the optimal solution for (L.P) Model using the two-Phase method.

$$Min.\ Z = 2X_1 + X_2$$

$$\text{S.t } X_1 + 3X_2 \geq 30$$
$$4X_1 + 2X_2 \geq 40$$
$$X_1 , X_2 \geq 0$$

Solution

First Phase

1- converting the mathematical model from the legal form to the standard form, as follows:

$$Min\,.\ Z = 2X_1 + X_2 - 0S_1 - 0S_2$$

$$\text{S.t} \qquad X_1 + 3X_2 - S_1 = 30$$

$$4X_1 + 2X_2 - S_2 = 40$$

$$X_1, X_2, S_{1,} S_2 \geq 0$$

2- We add the artificial variables as follows:

$X_1 + 3X_2 - S_1 + R_1 = 30$.............(1)

$4X_1 + 2X_2 - S_2 + R_2 = 40$................(2)

$$X_1, X_2, S_{1,} S_2, R_1, R_2 \geq 0$$

3-formulating a new objective function (r), depending on the values (R_1 - R_2), taking into account making the function equal to a fixed value, since

From equations (1) and (2), you get:

$r = R_1 + R_2 \rightarrow Min$...........(3)

$$R_1 = 30 - X_1 - 3X_2 + S_1$$

$$R_2 = 40 - 4X_2 - 2X_2 + S_2$$

Substitute the value (R_1) and (R_2) given in relation (3), in the new objective function (r), as follows:

$$Min\, r = (30 - X_1 - 3X_2 + S_1) + (40 - 4X_1 - 2X_2 - S_2)$$

$$= 70 - 5X_1 - 5X_2 + S_1 + S_2$$

$$r + 5X_1 + 5X_2 - S_1 - S_2 = 70$$

4-Design a table that includes the initial solution, as follows:

Table1

Basic Variables	Non- Basic Variables						R.H.S	Ratio
	X_1	X_2	S_1	S_2	R_1	R_2		
r	5	5	-1	-1	0	0	70	-
R_1	1	3	-1	-1	1	0	30	30
R_2	4	2	0	0	0	1	40	40

Pivot Row   Pivot Column   Pivot Element

a. The internal variable (X_1), the external variable (R_2), and the pivotal element is (4). The pivotal equation is written as follows

b. Pivot Equation = $\left[1, \frac{1}{2}, 0, \frac{-1}{4}, 0, \frac{1}{4}, 10\right]$

c. The new values of (r) and (R_1), are given as follows:

New(r)=[5,5,-1,-1,0,0,70]-5*$\left[1, \frac{1}{2}, 0, \frac{-1}{4}, 0, \frac{1}{4}, 0\right]$

New(r)=$\left[0, \frac{5}{2}, -1, \frac{1}{4}, 0, -\frac{5}{4}, 20\right]$

New(R_1)=$\left[0, \frac{5}{2}, 1, \frac{1}{4}, 1, \frac{-1}{4}, 30\right]$

We put the above results, in a second table, as follows:

Table2

Basic Variables	Non- Basic Variables						R.H.S	Ratio
	X_1	X_2	S_1	S_2	R_1	R_2		
r	0	$\frac{5}{2}$	-1	$\frac{1}{4}$	0	$-\frac{5}{4}$	20	-
R_1	1	$\frac{5}{2}$	-1	$\frac{1}{4}$	1	$-\frac{1}{4}$	20	8
X_1	4	$\frac{1}{2}$	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	10	20

Pivot Row Pivot Element

Pivot Column

a. The internal variable (X_2), the external variable (R_1), and the pivotal element is $[\frac{5}{2}]$.

b.The pivotal equation is written as follows :

Pivot Equation = $\left[0,1\,,\frac{-2}{5}\,,\frac{1}{10}\,,\,\frac{2}{5}\,,\frac{-1}{10}\,,8\right]$

c. The new values of (r) and (X), are given as follows:

New(r)=$\left[0\,,\frac{5}{2},-1,\frac{1}{4},0,\frac{-5}{4},20\right]$- $\frac{5}{2}$* $\left[0,1,\frac{-2}{5},\frac{1}{10},\frac{2}{5},\frac{-1}{10},8\right]$

=$[0,0,0,0,-1,-1,0]$

New(X_1)= $\left[1\,,\frac{1}{2},0,\frac{-1}{4},0,\frac{-1}{4},10\right]$- $\frac{1}{2}$* $\left[0,1,\frac{-2}{5},\frac{1}{10},\frac{2}{5},\frac{-1}{10},8\right]$

=$\left[1,0,\frac{1}{5},\frac{-3}{10},\frac{-1}{5},\frac{3}{10},6\right]$

We put the above results, in a third table, as follows:

Table3

Basic Variables	Non- Basic Variables						R.H.S	Ratio
	X_1	X_2	S_1	S_2	R_1	R_2		
R	0	0	0	0	-1	- 1	0	-
X_2	0	1	$-\frac{2}{5}$	$\frac{1}{10}$	$\frac{2}{5}$	$-\frac{1}{10}$	8	-
X_1	1	0	$\frac{1}{5}$	$-\frac{3}{10}$	$-\frac{1}{5}$	$\frac{3}{10}$	6	-

d. Since the value of the objective function (r = 0), which is associated with $C_j \leq 0$ which indicates the existence of a solution to the model, and the continuation of the second stage.

Second Phase

1. Adopting the results of the final basic solution contained in the third table from the first stage, after excluding the artificial variables (R_1) and ($R_{2)}$, and the objective function (r) from the table.

2. Adoption of the original objective function (2), which is:

$$Min\,.Z = 2X_1 + X_2 + OS_1 + OS_2$$

And we improve its value, to get the final optimal solution, as follows:

Table4

Basic Variables	Non- Basic Variables				R.H.S
	X_1	X_2	S_1	S_2	
Z	2	1	0	0	0
X_2	0	1	$-\frac{2}{5}$	$\frac{1}{10}$	8
X_1	1	0	$\frac{1}{5}$	$-\frac{3}{10}$	9

After excluding the values of (R_1 and R_2), and the objective function (r) from the last solution table, and adding the original objective function (Z) to the table, we write the constraints based on the results of the final table, as follows:

$X_2 - \frac{2}{5}S_1 + \frac{1}{10}S_2 = 8$ ....(1)

$X_1 + \frac{1}{15}S_1 - \frac{3}{10}S_2 = 6$ ....(2)

From equations (1) and (2), you get the values of , (X_2, X_1), as follows:

$$X_2 = 8 + \frac{2}{5}S_1 - \frac{1}{10}S_2$$

$X_1 = 6 - \frac{1}{15}S_1 - \frac{3}{10}S_2$ ....(3)

We substitute the values of (X_1, X_2) given in relation (3) into the original objective function (2), resulting in:

$Z = 2X_1 + X_2$

$\therefore Z = 2 * \left[6 - \frac{1}{5}S_1 + \frac{3}{10}S_2\right] + \left[8 + \frac{2}{5}S_1 - \frac{1}{10}S_2\right]$

$=12 - \frac{2}{5}S_1 + \frac{6}{10}S_2 + 8 + \frac{2}{5}S_1 - \frac{1}{10}S_2 = 20 + \frac{5}{10}S_2$

$$Z - \frac{1}{2}S_2 = 20$$

We put the final result of the original objective function (2) in the final solution table, as follows:

Table5

Basic Variables	Non- Basic Variables				R.H.S
	X_1	X_2	S_1	S_2	
Z^*	0	0	0	$-\frac{1}{2}$	20
X_2	0	1	$-\frac{2}{5}$	$\frac{1}{10}$	8
X_1	1	0	$\frac{1}{5}$	$-\frac{3}{10}$	6

It is clear from the results of the final solution table, that all coefficients of the objective function (Z), are less than and equal to zero, that is, ($C_j \leq 0$) so the final solution, is as follows :

$X_1 = 6, X_2 = 8, Z^* = 20$

CHAPTER FOUR

COMPARISON

4.1 Introduction

In this chapter, we make a comparison between the simplex method and the graph method in order to stand on the similarities and differences between them, and then note which of the two methods is better and more efficient in reaching the optimal solution, by solving some examples of both methods together.

4.2. Solve examples of linear programming problems using the simplex method and the graph method.

Example 4.2.1.[18]: Solve the following problem:

$$\text{Maximize } Z = f(x, y) = 3x + 2y$$

$$\text{subject to: } \quad 2x + y \leq 18$$

$$2x + 3y \leq 42$$

$$3x + y \leq 24$$

$$x \geq 0, y \geq 0$$

1-The solution using the graph method

Solution

$2x + y = 18$ (L1)

If $X = 0, y = 18 \longrightarrow$ A(0,18)

If $y = 0, X = 9 \longrightarrow$ B(9,0)

$2x + 3y = 42$ (L2)

If $X = 0, y = 14$ ⟶ C(0,14)

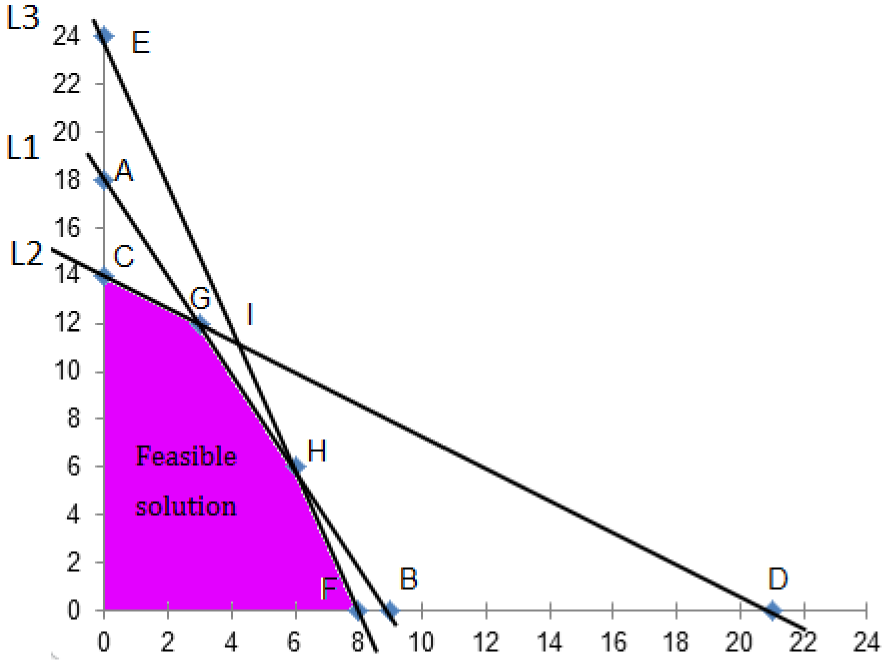
If $y = 0, X = 21$ ⟶ D(21,0)

$3x + y = 24$ (L3)

If $X = 0, y = 24$ ⟶ E(0,24)

If $y = 0, X = 8$ ⟶ F(8,0)

Ttherefore,the feasible solute,the feasible solution area will be as follows



From the previous figure, it is clear that the feasible solution area is limited by points(C,G,H,F,O): C(0,14),F(8,0),O(0,0)

To find the coordinates of point G, we intersect the two constraints (L_1 and L_2)), that is:

$$2x + y = 18 \qquad (1)$$

$$\mp 2X \mp 3y = \mp 42 \qquad (2)$$

$$-2y = -24 \longrightarrow y = 12$$

And by substituting it into the equation number(1)

$$2x + 12 = 18 \longrightarrow X = 3 \qquad G(3,12)$$

In the same way, we find the value of H from the intersection of the two constraints(L_1,L_2) H(6,6)

To find the optimal solution, we make the following table:

Extreme point	Coordinates (x,y)	Objective value (Z)
O	(0,0)	0
C	(0,14)	28
G	(3,12)	33
H	(6,6)	30
F	(8,0)	24

Therefore, the feasible solution area will be as follows:

$Z = 33$ with $x = 3$ and $y = 12$.

Through this solution, it is clear that we have calculated the objective function at each of the corner points of the feasible region in order to find the optimal solution to this problem.

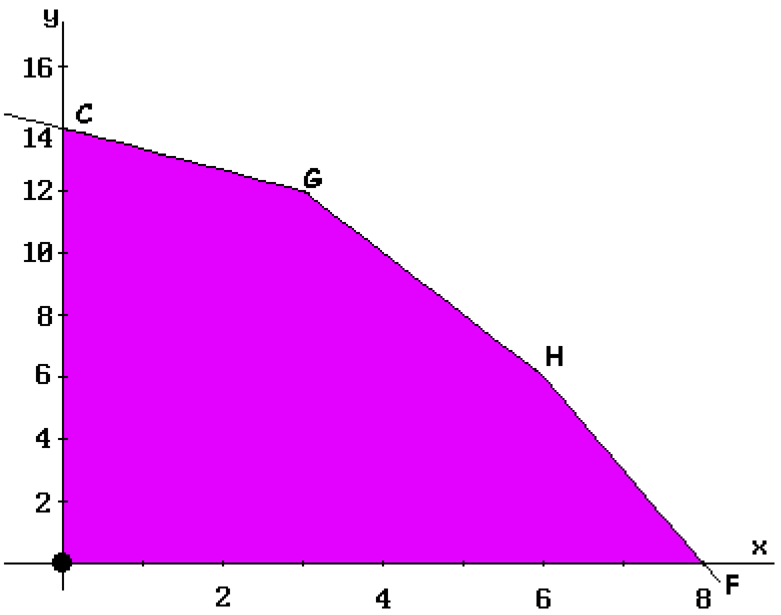
2-Graphical method and Simplex method comparison

Successive constructed tableaux in the Simplex method will provide the value of the objective function at the vertices of the feasible region, adjusting simultaneously, the coefficients of initial and slack variables.

In the initial tableau the value of the objective function at the O-vertex is calculated, the coordinates (0,0) correspond to the value which have the basic variables, being the result 0.

Table1

Bass	X	y	S_1	S_2	S_3	R.H.S
S_1	2	1	1	0	0	18
S_2	2	3	0	1	0	42
S_3	3	1	0	0	1	24
Z	-3	-2	0	0	0	0

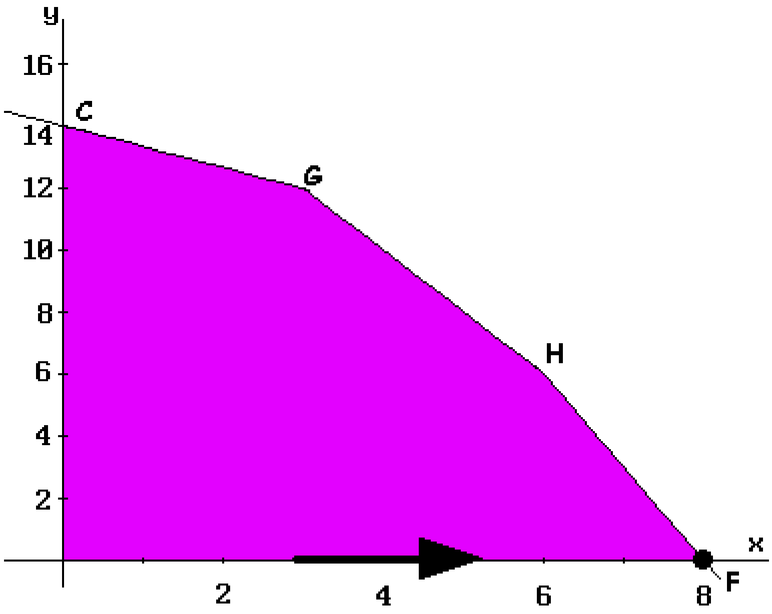


The input base variable in the Simplex method determines towards what new vertex is performed the displacement. In this example, as x enters, the displacement is carried out by the OF-edge to reach the F-vertex, where the Z-function value is calculated. This step occurs in the second iteration of the

Simplex method, as shown in table2. The corresponding value to F-vertex is calculated in it, and $Z = 24$ is the obtained value for the function.

Table2

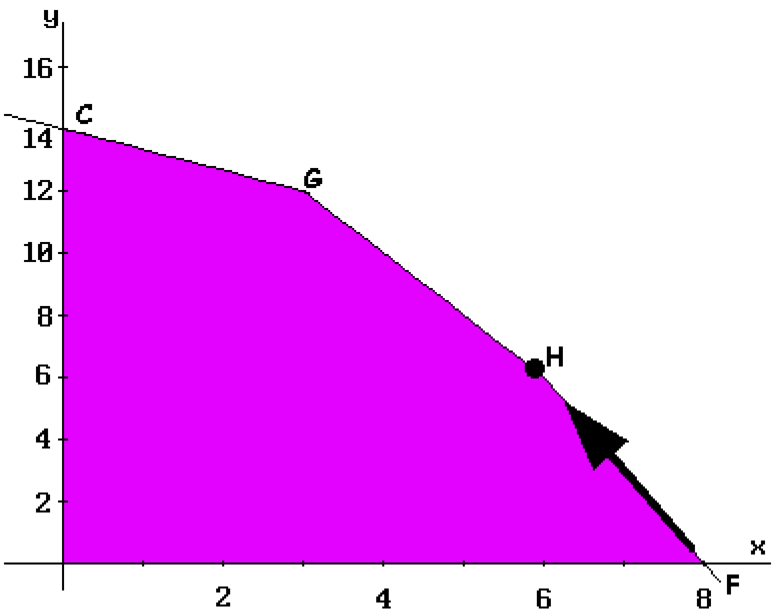
Bass	X	y	S_1	S_2	S_3	R.H.S
S_1	0	1/3	1	0	-2/3	2
S_2	0	7/3	0	1	-2/3	26
X	1	1/3	0	0	1/3	8
Z	0	-1	0	0	1	24



A new displacement by the FH-edge is made, up to H-vertex (data in Table 3). In the third iteration, the value of the function at the H-vertex is calculated to obtain Z = 3

Table 3

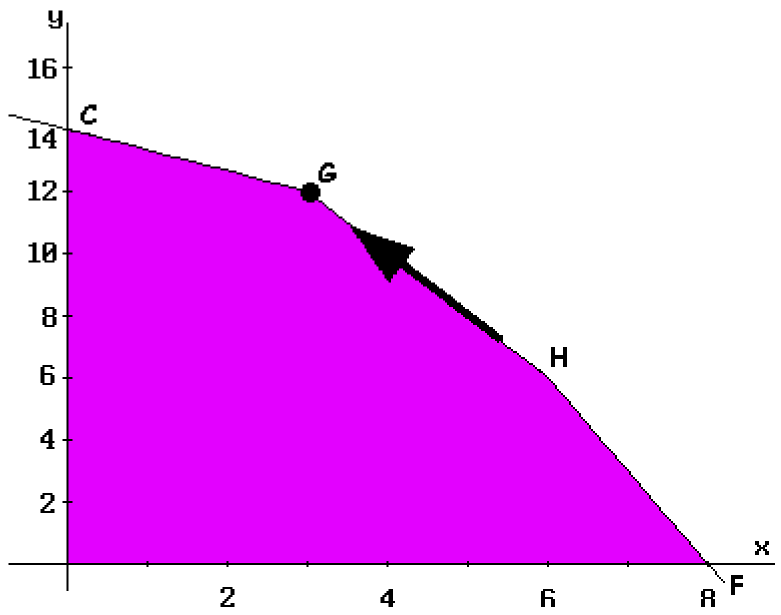
Bass	X	y	S_1	S_2	S_3	R.H.S
y	0	1	3	0	-2	6
S_2	0	0	-7	1	4	12
X	1	0	-1	0	1	6
Z	0	0	3	0	-1	30



The process goes on through the HG-edge up to G-vertex, obtained data are shown in table 4. At this point, the process ends, being able to check that the solution does not improve moving along GC-edge up to C-vertex (the current value of the Z-function is not increased).

table 4

Bass	X	y	S_1	S_2	S_3	R.H.S
y	0	1	-1/2	1/2	0	12
S_3	0	0	-7/4	1/4	1	3
X	1	0	3/4	-1/4	0	3
Z	0	0	5/4	1/4	0	33



The maximum value of the objective function is 33, and it corresponds to the values $x = 3$ and $y = 12$ (G-vertex coordinates).

Example 4.2.2.[11] Solve the following problem:

$$max\ z = x_1 + 2x_2$$

$$\text{s.t} \quad -x_1 + x_2 \leq 1$$

$$-x_1 + 2x_2 \leq 3$$

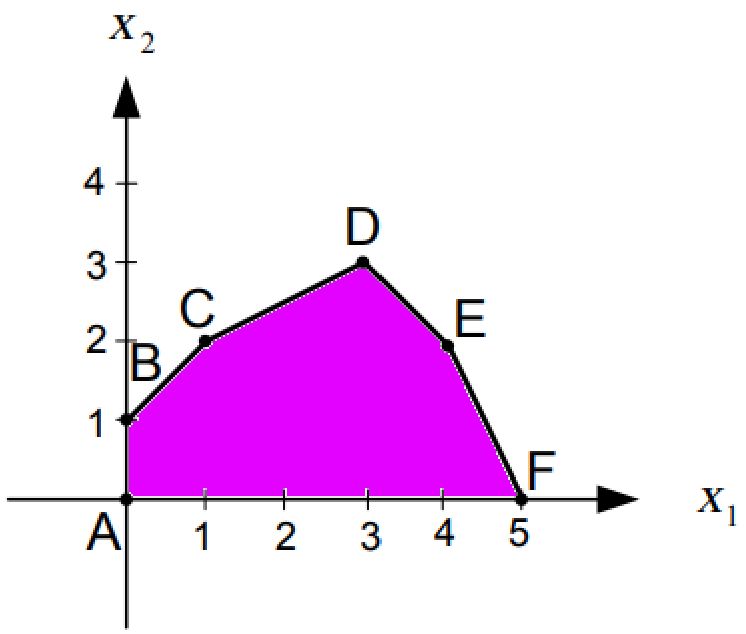
$$x_1 + x_2 \leq 6$$

$$2x_1 + x_2 \leq 10$$

$$X_1, X_2 \geq 0$$

1- The solution using the graph method

After drawing the constraints and finding the points of intersection, the solution area is as follows:



To find the optimal solution, we make the following table:

Extreme point	Coordinates (x,y)	Objective value (z)
A	(0,0)	0
B	(0,1)	2
C	(1,2)	5
D	(3,3)	9
E	(4,2)	8
F	(5,0)	5

Therefore, the optimal solution will be as follows:

Z=9 with $X_1 = 3,\ X_2 = 3$

2 -Solve the example by simplex method

Put the form in standard form:

$$Max\ Z - X_1 - 2X = 0$$

S.t $-X_1 + X_2 + S_1 = 1$

$$-X_1 + 2X_2 + S_2 = 3$$

$$X_1 + X_2 + S_3 = 6$$

$$2X_1 + X_2 + S_4 = 10$$

$$X_i, S_j \geq 0\ \forall\ \ i = 1,2$$

$$j = 1,2,3,4$$

This formula can be expressed in the following table

Table1

Bass	X_1	X_2	S_1	S_2	S_3	S_4	R.H.S
S_1	-1	1	1	0	0	0	1
S_2	-1	2	0	1	0	0	3
S_3	1	1	0	0	1	0	6
S_4	2	1	0	0	0	1	10
Z	-1	-2	0	0	0	0	0

This table represents the initial basic solution and the value of $Z = 0$ and $X_1 = 0$, X_2=0. This solution corresponds to point A(0,0) in the previous solution. Now, after determining the internal and external variables, the solution is as follows:

Table1

Table2

Bass	X_1	X_2	S_1	S_2	S_3	S_4	R.H.S
X_2	-1	1	1	0	0	0	1
S_2	1	0	-2	1	0	0	1
S_3	2	0	-1	0	1	0	5
S_4	3	0	-1	0	0	1	9
Z	-3	0	2	0	0	0	2

We note that this solution agrees with point B(0,1), but it does not represent the optimal solution, So we move to the post table where X_1 is the input variable and S_2 is the outside variable.

Table3

Bass	X_1	X_2	S_1	S_2	S_3	S_4	R.H.S
X_2	0	1	-1	1	0	0	2
X_1	1	0	-2	1	0	0	1
S_3	0	0	3	-2	1	0	3
S_4	0	0	5	-3	0	1	6
Z	0	0	-4	3	0	0	5

This solution corresponds to point C(1,2) which is also not the optimal solution, So we move towards the other point.

Table4

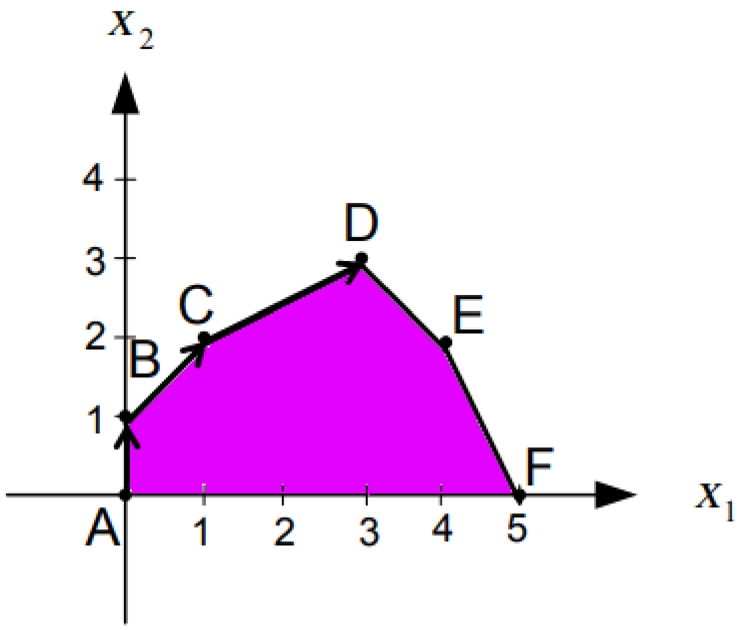
Bass	X_1	X_2	S_1	S_2	S_3	S_4	R.H.S
X_2	0	1	0	$1/3$	$1/3$	0	3
X_1	1	0	0	$-1/3$	$2/3$	0	3
S_1	0	0	1	$-2/3$	$1/3$	0	1
S_4	0	0	0	$1/3$	$-5/3$	0	1
Z	0	0	0	$1/3$	$4/3$	0	9

Thus, we have reached the optimal solution when it is $Z^* = 9, {X_1}^* = 3$

${X_2}^* = 3$. It corresponds to point D at which point the solution stands.

3 - Geometric explanation of the solution

As we have seen in the previous example, we can represent the algorithm for solving the simplex method in this example as follows:



Where we note that the initial solution started from point A and then moved towards point B and stopped at point D without the need to examine the other corner points.

4.3 Discuss the results of the solution

By solving the above two examples with the two methods together and comparing them, we note the following:

In the simplex method is moving from afeasible basic solution point to a better afeasible basic solution point, this may lead to the fact that the number of feasible basic solutions points that are examined is less than the total number offeasible basic solutions points. While the graphicel method, finding the optimal solution is required:

First: identify all the feasible basic solution points, i.e. the feasible corner points (note that in the simplex method, it may not require examining all the basic solution points as we mentioned above).

Secondly, calculating the value of the objective function at each feasible corner point, then choosing the best feasible basic solution point. This procedure is considered an inefficient procedure.

The matter becomes more complicated as the number of constraints and variables increases, For example, if the number of variables equals 20 and the number of structural constraints equals 6, then the number of vertical points in this case equals 38760,where:

$$C_6^{20} = \frac{20!}{6!(20-6)!} = \frac{20\times19\times18\times17\times16\times15}{6\times5\times4\times3\times2\times1} = 3878760$$

That is, the 38760 points must be determined, and then each of these points should be examined to determine its type. If a point is possible, then the value of the objective function is calculated at it - and this is an insufficient procedure, especially as the number of structural constraints and the number of variables increase. But the efficiency of the simplex method lies in the fact that we move from a feasible basic solution point to a feasible basic solution point that is better than the previous one, due to the presence of optimization conditions and this leads to the failure to identify and check all the corner points, which leads to the speed and efficiency of reaching the optimal solution[18].

In other words: In Graphical method is necessary to calculate the value of the objective function at each vertex of feasible region, while the Simplex method ends when the optimum value is found.

4.4 Conclusion

We conclude from all of this that the simplex method is distinguished from the graph method in several things:

1- Graphical method can be used only when two variables are in model ; simplex can handle any dimensions

2- Graphical method must evaluate all corner points (if the comer point method is used) ; simplex checks a lesser number of corners .

3- Simplex method can be automated and computerized .

4- Simplex method involves use of surplus , slack , and artificial variables but provides useful economic data as a by product.

5-The graph method is considered impractical, because it is rare to find a life problem that consists of only two variables.

However, the two methods agree on some things:

1-Both methods find the optimal solution at a comer point .

2- Both methods require a feasible region and the same problem structure , that is , objective function and constraints .

References

[1] Foulds, L.R., **2012**. Optimization techniques: an introduction. Springer Science & Business Media. New York.

[2] Jahn, J., **2007**. Introduction to the theory of nonlinear optimization. Springer Science & Business Media. New York.

[3] Kwon, R.H., **2013**. Introduction to linear optimization and extensions with MATLAB®. CRC Press. New York

[4] Mishra, S.K. and Ram, B., **2019**. Introduction to Unconstrained Optimization with R. Springer Nature.. Singapore.

[5] Nocedal, J. and Wright, S., **2006**. Numerical optimization. Springer Science & Business Media. New York.

[6] Sarker, R.A. and Newton, C.S., **2007**. Optimization modelling: a practical approach. CRC press. New York.

[7] Yang, X.S., **2018**. Optimization techniques and applications with examples. John Wiley & Sons. New York.

[8] Zoutendijk, G. **1970**. Nonlinear programming computational methods, J.Abadie (Ed.), Integer and Nonlinear Programming, North-Holland, Amsterdam, p.p. 37– 86.

[9] Beck, A., **2014**. Introduction to nonlinear optimization: Theory, algorithms, and applications with MATLAB. Society for Industrial and Applied Mathematics. Philadelphia .

[10] Belegundu, A.D. and Chandrupatla, T.R., **2019**. Optimization concepts and applications in engineering. Cambridge University Press. New York.

[11] Bonnans, J.F., Gilbert, J.C., Lemaréchal, C. and Sagastizábal, C.A., **2006**. Numerical optimization: theoretical and practical aspects. Springer Science & Business Media. Berlin Heidelberg .

[12] Butenko, S. and Pardalos, P.M., **2014**. Numerical methods and optimization: An introduction. CRC Press. United Kingdom.

[13] Marco Cavazzuti., 2012. Optimization methods: from theory to desig scientific andtechnological aspects in mechanics. Springer Science & Business Media.

[14] Michael T., 2017. The fundamental theorem of linear programming .

[15] Michael Panik.,2018. Linear programming and resource allocation modeling. John Wiley& Sons.
[16] Best, Michael J., and Klaus Ritter., 1985. Linear programming. Upper Saddle River ،NJ: Prentice Hall.
[17] Edwin K. P. C. and Stanislaw H.,2013. An Introduction To Optimization. John Wiley & Sons: New Jersey
[18] David Luenberger, Yinyu Ye, et al., 1984. Linear and nonlinear programming, volume 2. Springer.
[19] Singiresu Rao., 2019. Engineering optimization: theory and practice. John Wiley & Sons.
[20] Lewis, Catherine.,2008. "Linear programming: Theory and applications." Whitman College Mathematics Department

الخلاصة

تعتبر مشاكل التحسين من أهم المشاكل التي بذل العلماء جهوداً كبيرة في إيجاد الحل الامثل لها. وهذا البحث يهدف إلى تقديم دراسة في معالجة وحل هذا النوع من المشاكل. حيث تناول الباحث في هذه الدراسة اسلوب البرمجة الخطية الذي يعتبر من أبرز الأساليب العلمية وأكثرها شيوعاً في معالجة مسائل التحسين. في البداية تعرض البحث إلى تعريف مفهوم البرمجة الخطية ومن ثم بيان صيغ ومكونات النموذج الخطي، ثم توضيح كيفية تحويل المشكلة المفترضة إلى نموذج خطي من خلال مثال تطبيقي من الحياة اليومية. بعد ذلك ناقش الباحث اهم طريقتين من طرق حل مسائل البرمجة الخطية وهما :(طريقة السمبلكس وطريقة الرسم البياني)، حيث تم شرحهما بشكل مفصل مع بيان موارد ومزايا وخصائص كل واحدة منهما. وفي النهاية تضمن البحث إجراء مقارنة بين الطريقتين اتضح من خلالها أن طريقة السمبلكس أفضل من طريقة الرسم البياني من حيث عدد الخطوات ومن حيث الكفاءة في الوصول إلى الحل الامثل.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

استخدام تقنيات البرمجة الخطية لإيجاد الحل الأمثل لمشاكل التحسين

بحث مقدم

الى مجلس كلية التربية للعلوم الصرفة/ جامعة بابل كجزءِ من متطلبات نيل درجة الدبلوم العالي تربية/ الرياضيات

مقدم من قبل الطالب

حسن عنون هاشم حسين

بإشراف الدكتور

حسين عبد الوصي حسين

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