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Solving an Optimization Problems by One Dimensional Methods

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Dedication

To the people who I love and respect, my Father, Mother, Wife, brothers, and Sons . A special dedication is to Martyrs of the October Revolution.

Acknowledgments

I would like to express my thanks to my family and then thanks my supervisor Dr. Hussein Abdul Wasi , for standing with me and supporting me to bring this work into big success and thanks to all those who contributed to this work.

list of content

List of Symbols

Symbol	Description
$f(x)$	The Objective Function
$Minf(x)$	The Minimum of Objective Function
$Maxf(x)$	The Maximum of Objective Function
$g(x)$	Equality Constraint
$h(x)$	The Equality Constraint
c	Positive constant
S	The Convex Set
x^*	The Minimizer (Local, Global)
$\nabla f(x^*)$	The Gradient of f
$H_f(x)$	The Hessian of f
dom f	Domain of a function
$f'(x)$	The First Derivative of f for x
Ω	Feasible set

$\{x_k\}$	**The Iterative Sequence**
x_{k+1}	**The Next Iterate**
x_k	**The Current Iterate**
$\lvert \overline{AB} \rvert$	**Vector length**
α_k	**The Step Length**
$J(x)$	**The Jacobean Matrix**
F	**The Feasible Set**
K	**Constant**
$\partial\emptyset$	**Partial derivative**
$\partial\emptyset^2$	**Second Partial derivative**
F_n	**Fibonacci formula**
L_0	**Initial interval**

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Abstract

The problems of optimization are one of the most important topics discussed in the scientific aspects. There are many techniques used to solve optimization problems. In this search, we applied the Fibonacci method, bisection method, Newton-Raphson method and categorical method, also to solve some problems. The solutions were compared and the best method was shown, supported by numerical tables.

Introduction

Optimization has been one of the most fundamental and successful tools in our daily lives. Optimization is an essential mathematical tool that aims to find the best value of variables that provide the minimum value or the maximum for a mathematical function (the objective function). Optimization algorithms are a fundamental and successful tool in mathematical programming to reach a solution, generally with the assistance of a computer. Optimization algorithms start with an initial estimate of the value of the variables and by an iterative technique generates a sequence of improved estimates, or iterates, until an optimal solution is reached. A great algorithm should be efficient, fast, accurate, and robust. It should generate a good approximation of an optimal solution[2]

The problem described :

$$\begin{cases} \text{Maximize or minimize} : \text{Objective function,} \\ \text{Subject to} : \text{Constraints} \end{cases}$$

This format is sufficiently general to include all optimization problems (most of life's problems too for that matter). Since we are interested in mathematical methods for solving such problems, it is necessary that the statement be reduced to symbolic form. for example :

$$\begin{cases} Maximize : F(x_1, x_2, x_3) \\ \text{Subject to} : g(x_1, x_2, x_3) = 0 \end{cases}$$

the above statement reads as follows: Maximize some function F, of x_1; x_2; x_3 by setting x_1,x_2 and x_3 subject to the requirement that another function g of x_1; x_2 and x_3, takes on the value zero.[4]

The general optimization problem form

$$\begin{cases} minimize \quad f(x) \quad \text{Objective function} \\ subject \text{ to } g(x) = 0 \text{ Equality Constraints ,} \\ h(x) \geq 0 \quad \text{Inequality Constraints ,} \end{cases}$$

The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, *optimization* can be defined as the process of finding the conditions that give the maximum or minimum value of a function. It can be seen from Fig.1.[6]. that if a point x^*corresponds to the minimum value of function $f\ (x)$, the same point also corresponds to the maximum value of the negative of the function, $-f\ (x)$. Thus without loss of generality, optimization can be taken to mean minimization since the maximum of a function can be found by seeking the minimum of the negative of the same function. In addition, the following operations on the objective function will not change the optimum solution x^* (see Fig. 2):[3]

1. Multiplication (or division) of f (x) by a positive constant c.

2. Addition (or subtraction) of a positive constant c to (or from) f (x).

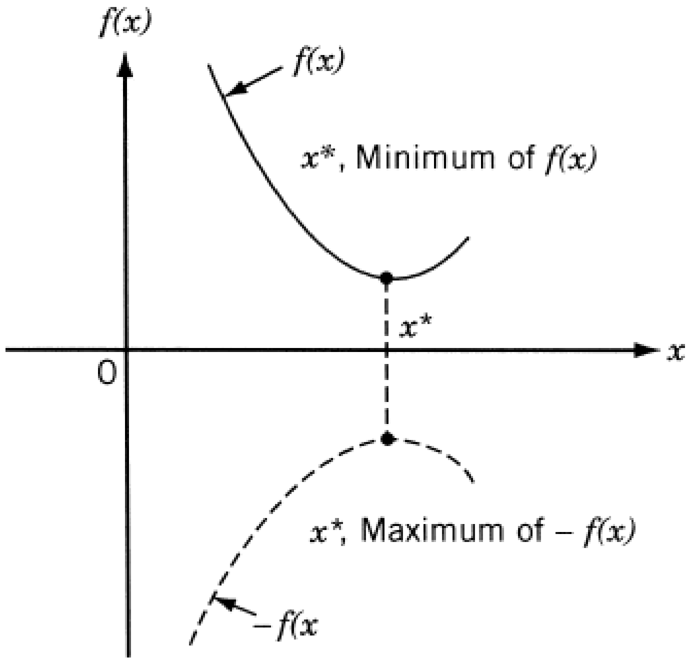


Figure 1 : Minimum of $f\ (x)$ is same as maximum of $-f\ (x)$.

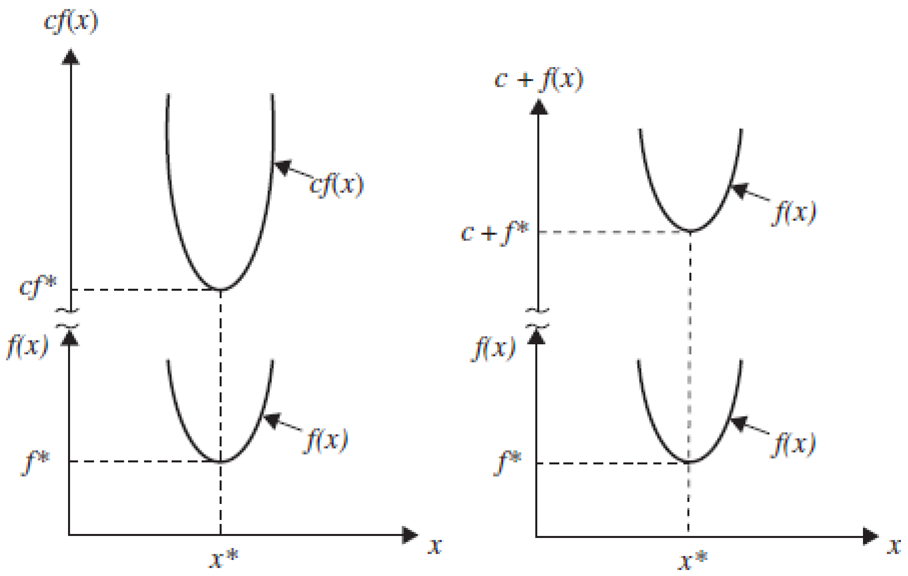


Figure 2 : Optimum solution of $cf(x)$ or $c + f(x)$ same as that of $f(x)$.

Optimization, one of the most interesting topics in the field of Machine learning. Most of the problems we encounter in our daily life are solved using numerical optimization methods. Here in this blog, let us look at some basic numerical optimization algorithms extensively in finding local optima of any given function (which works best with convex functions). Let's start with simple convex functions, where local and global minima are one and the same, and move towards highly non-linear functions with multiple local and global minima's. Entire optimization revolves around basic concepts of linear algebra and differential calculus. Recent updates in deep learning have created huge interest in the field of numerical and stochastic optimization algorithms, to provide theoretical backing for amazing qualitative results shown by deep learning networks. In these kinds of learning algorithms, there doesn't exist any explicitly known function for optimization, but we only have access to the 0th and 1st order oracles. Oracles are the black boxes which return function value (0th order), gradient (1st order) or Hessian (2nd order) of the function at any given point. This blog provides the basic theoretical and numerical understanding of unconstrained and constrained optimization functions and also includes a python implementation of them.[2]

In (1944), Levenberg [7] was the author of the first research in this field, working on adding multiple identities to the Hessian matrix as a stabilization process in the least-squared non-linear problem-solving program.

In (1983), More's survey [8] introduced a remarkable development "review paper in the algorithm of the trust region recently, and there are some huge studies on the algorithm of the trust region.

Many researchers have contributed to the development and modification of the projection method.

In (2003), Wangk Y. J., Xiu N. H., Zhang J. Z. [9] Use the modified Gradient Method for solving nonlinear systems of equations.

In (2005) Burden, R.L, Faires, J.D [10], they showed solution of nonlinear Equations.

In (2010), Yan Q. R., Peng X. Z., Li D. H [11], they showed the Global convergence of their technique for solving nonlinear equations for large scale.

In 2020, Tsukamoto, H., & Chung, S. J. Introduced the convex constraint of nonlinear systems was introduced by Liu and Li in (2015) [12].

CHAPTER ONE

Basic Definitions

1.1 Basic Facts Related to Linear Programming

Definition *(1.1.1)* Objective Function [19]

Objective function is prominently used to represent and solve the optimization problems of linear programming. The **objective function** is of the form Z = ax + by, where x, y are the decision variables. The function Z = ax + by is to be maximized or minimized to find the optimal solution. Here the objective function is governed by the constraint s x > 0, y > 0. The optimization problems which needs to maximize the profit, minimize the cost, or minimize the use of resources, makes use of an objective function.

Definition (1.1.2) Vector[19]

The quantity that has a magnitude and direction is said to be a vector, let AB is a directed Straight segment which denoted as $\overline{AB}$ or$\bar{a}$, we can read as vector $\overline{AB}$ or $\bar{a}$. Note that, the point A is called initial point from the vector$\overline{AB}$, similarly the point B is named terminal point. The distance between two points initial and terminal is called the length (or magnitude) of the vector, which symbolizes as: | $\overline{AB}$|, or | $\bar{a}$ |.

Definition (1. 1.3) Convex Sets [4] .

Let $S \subseteq \mathbb{R}^n$. If the line segment between any two points in S lies in S, i.e $\lambda x_1 + (1-\lambda)x_2 \in S,\ \forall x_1, x_2 \in S, \forall\lambda \in [0,1]$ then S is said to be convex as shown in (Figure 3) . It can be shown that a set $S \subseteq \mathbb{R}^n$ is convex if and only if for any $x_1, \dots, x_n \in S$, the convex combination $\sum_{i=1}^{n} \lambda_i x_i$ Where $\sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, i = 1, \dots, n$, belongs to S.

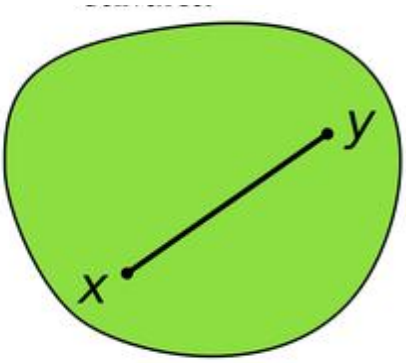


Figure 3: Convex Set

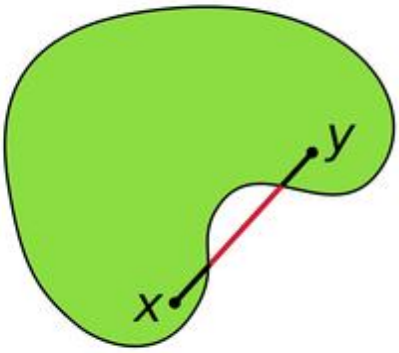


Figure 4 : Non–Convex Set.

Definition(1.1. 4) Convex Functions [7].

Let $S \subseteq R^n$ be a nonempty convex set. If $f : S \to R$ satisfies:

$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) \quad f(x_2), \forall x_1, x_2 \in S, \forall \lambda \in [0,1],$$

then f is said to be a convex function on S. If the above inequality is true as a strict inequality for all $x_1 \neq x_2$ and for all $\lambda \in (0,1)$, then f is called a strictly convex function on S as shown in (Figure 5).

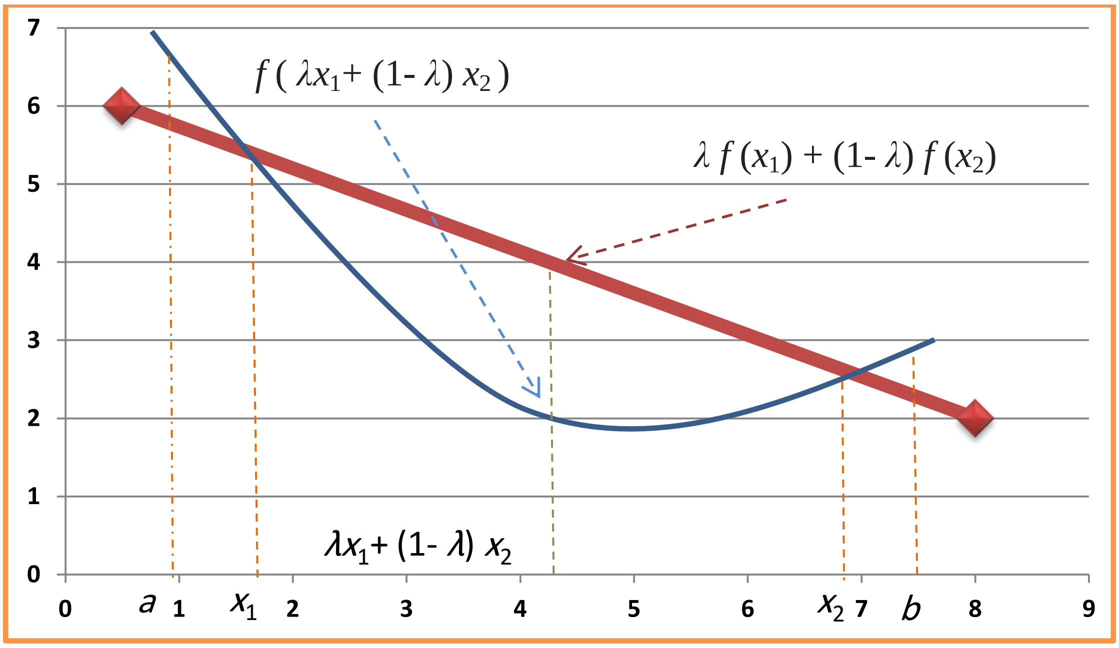


Figure 5 : Convex Function

- f is **concave** if $-f$ is convex
- f is **strictly convex** if $dom\ f$ is the convex and:

$$f[\lambda x + (1-\lambda)\, y] < \lambda\, f(x) + (1-\lambda)\, f(y)\,,$$

For all$x, y \in dom\ f$,$x \neq y$ $0 < \lambda < 1$.

For example: e^{ax} , is convex function, for any$a \in R$.

Definition (1 .1.5) Domain of a function [10]

the domain of a function is the set of inputs accepted by the function. It is sometimes denoted by $dom\ (f)$, where *f* is the function .

Definition (1.1. 6) Feasible Sets [1]

The feasible set is the set of all vectors x in which the value of x determined between the constrained $g_i(x) = 0$ and $h_i(x) \leq 0, \forall i \in \mathbb{N}$.

Definition (1. 1.7) a Feasible Region [2]

Is an area defined by a set of coordinates that satisfy a system of inequalities. The region satisfies all restrictions imposed by a linear programming scenario. The concept is an optimization technique. For example, a planner can use linear programming to determine the best value obtainable under conditions dictated by several linear equations that relate to a real-life problem .

Definition (1.1. 8) Continuous Function [11]

The function *f* is a continuous at a number *a* if $\lim_{x\to a} f(x) = f(a)$ (i.e. we can make the value of *f*(*x*) as close as we like to *f*(*a*) by taking x sufficiently close to a). A function $f : A \to R$ is continuous on a set $B \subseteq A$ if it is continuous at every point in B, and continuous if it is continuous at every point of its domain A.

Definition (1. 1.9) Differentiable Function [11]

A function f is called differentiable at a point x if it has a derivative there, in other words : $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exist.

The domain of f' is the set of points $c \in (a , b)$ for which this limit exists. If the limit exists for every $c \in (a , b)$ then we say that f is differentiable on (a , b).

Definition (1. 1.10) Smooth Function[18]

A function which has continuous derivatives over some domain is said to be a smooth function. Hence we said the function f can be considered smooth over a limited interval such as (a, b) or [a, b].

Definition (1. 1.11) Nonsmooth Function[2]

The function f is called non-smooth when it's discontinuous and non-differentiable.

Definition (1.1. 12) Convergent Sequence [3]

A number $x^* \in R$ is said to be limit for the sequence $\{x_k\}$ if for any $\epsilon > 0$, there existsa constant$K$ such that for every $k > K$ and $|x^* - x_k| < \epsilon$; this implies , x_k situated between x^* - ϵ and x^* + ϵ for every$k > K$. In this case $x^* = \lim_{k\to\infty} x_k$ or $x_k \to x^*$, a sequence that has a limit is said to be a convergent sequence.

Definition (1. 1.13) local Minimizer [4]

Let F be a feasible set, we called $x^* \in F$ is local minimizer if there is a neighborhood C of $x^* s.$ t $f(x^*) \leq f(x)$,$\forall$ $x \in C$. We called $x^*$is strict local maximizer if $f(x^*) < f(x)$. $\forall x \in C, x \neq x^*$.

Definition (1.1.14) Nonlinear Systems [5]

Let

$$f_1(x) = 0$$

$$f_2(x) = 0$$

$$\vdots$$

$$f_n(x) = 0,$$

Be a set of equations, where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. Then this set of equations is called a system of nonlinear equations if f_i is a nonlinear real function for each $(i = 1,2,3 \dots, n)$. The general form of the above system can be expressed as follows: $F(x) = \vec{0}$, where $F: \mathbb{R}^n \to \mathbb{R}^n$. A vector x^* which$F(x) = \vec{0}$. holds called a solution or root of the nonlinear equations

Definition (1.1.15) Jacobian Matrix [5] Given a set $y = f(x)$ of n equations in n variables $x_1, \dots \dots \dots, x_n$, written explicitly as$y = \begin{bmatrix} f_1(x) \\ f_2(x) \\ . \\ . \\ f_n(x) \end{bmatrix}$, or more explicitly $\begin{cases} y_1 = f_1(x) \\ y_2 = f_2(x) \\ \quad . \\ \quad . \\ y_n = f_n(x) \end{cases}$, the Jacobian matrix, sometimes simply called "the Jacobian", is defined by

$$J = (x_1, \dots \dots \dots, x_n) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}.$$

Definition (1. 1.16) Hessian Matrix [6]

The Jacobian of partial derivatives $\frac{\partial \emptyset}{\partial x_1}, \frac{\partial \emptyset}{\partial x_2}, \dots, \frac{\partial \emptyset}{\partial x_n}$, of the function $\emptyset(x_1, x_2, \dots \dots, x_n)$, with respect to the variables$x_1, x_2, \dots, x_n$, is said to be the Hessian matrix H of the function $\emptyset$, so that H =

$$\begin{bmatrix} \frac{\partial^2 \emptyset}{\partial {x_1}^2} & \cdots & \frac{\partial^2 \emptyset}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \emptyset}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 \emptyset}{\partial {x_n}^2} \end{bmatrix}.$$

Definition (1. 1.17) Optimization Problems [7]

The following optimization problem:

minimize $f\ (x)$,

subject to $x \in\ \Omega$,

The function $f: R^n \to R$ is the real value objective function.

With the vector $= \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$, is an $n-$ vector of decision variables $x_1, x_2, \ldots\ldots, x_n$. The set $\Omega\ \subseteq\ R^n$ is called a feasible set. every $x\ \in\ \Omega$ is said to be a feasible solution, this problem finds a vector x of all the possible vectors for the set Ω, Which will get us to the minimum value of the objective function $f\ (x)$, then a vector is said to be the minimizer of f over Ω. There are optimization problems that involve to maximize for objective function $f\ (x)$, which is the same to the minimization of $-f\ (x)$. hence, the optimization problems can be classify into two types:

A. Unconstrained Optimization Problems[16]

$if\ \boldsymbol{\Omega} = \boldsymbol{R^n}$ Then the optimization problem: minimize $f\ (x)$, subject to $x\ \in\ \Omega$, is named as *unconstrained optimization problem*.

B.Constrained Optimization Problems[17]

If $\boldsymbol{\Omega}\ \subseteq \boldsymbol{R^n}$ then the optimization problem: minimize $f\ (x)$, subject to $x\ \in\ \Omega$, where x is limited to satisfy for some conditions, then the problem is named as *constrained optimization problem*.

1.2 Optimality Conditions [14]

After an optimization algorithm has been applied to the model, we must be able to recognize whether it has succeeded in its task of finding a solution. There are elegant mathematical expressions known as optimality conditions for checking that the current set of variables is indeed the solution of the problem. If the optimality conditions are not satisfied, they may give useful information on how the current estimate of the solution can be improved. If a point satisfies the third order sufficient condition, then we have a guaranty that this point is a local minimizer.

1.2.1.First-Order Necessary Condition .[9]

If $x^*$is a local minimizer and f is continuously differentiable in an open neighborhood of x^*then $\nabla f(x^*) = 0$.

1.2.2.Second-Order Necessary Condition . [9]

If x^* is a local minimizer of f and $\nabla^2 f$ is continuous in an open neighborhood of x^* , then $\nabla f\ (x^*)\ =\ 0$ and $\nabla^2 f\ (x^*)$ is positive semidefinite.

1.2.3.Second-Order Sufficient Condition . [9]

Suppose that $\nabla^2 f$ is continuous in an open neighborhood of x^* and that $\nabla f\ (x^*) = 0$ and $\nabla^2 f\ (x^*)$ is positive semidefinite. Then $x^*$is a strict local minimizer of f .

1. 3 Classification of Optimization Problems [16]

The general optimization problems can be classified as shown in Figure 6.We have discussed the objective classification (single or multiple) and the objective type (maximization or minimization) in an earlier section. In case of multiple objectives, the objectives usually contradict each other. If they do not, the multiple objectives can be converted into a single objective problem .The problem classification (in the next page) indicates whether the problem contains constraints or not. Some people believe that there are no unconstrained optimization problems in the real world, as these all will have either constraint functions or variable bounds (upper or lower) or both .The study of unconstrained problems is very important since many optimization algorithms solve constrained problems by converting them into an unconstrained or a sequence of unconstrained problems. In addition, several unconstrained optimization techniques can be extended in a natural way to provide and motivate solution procedures for constrained problems. Now working on an Optimization, Modeling (A Practical Approach)

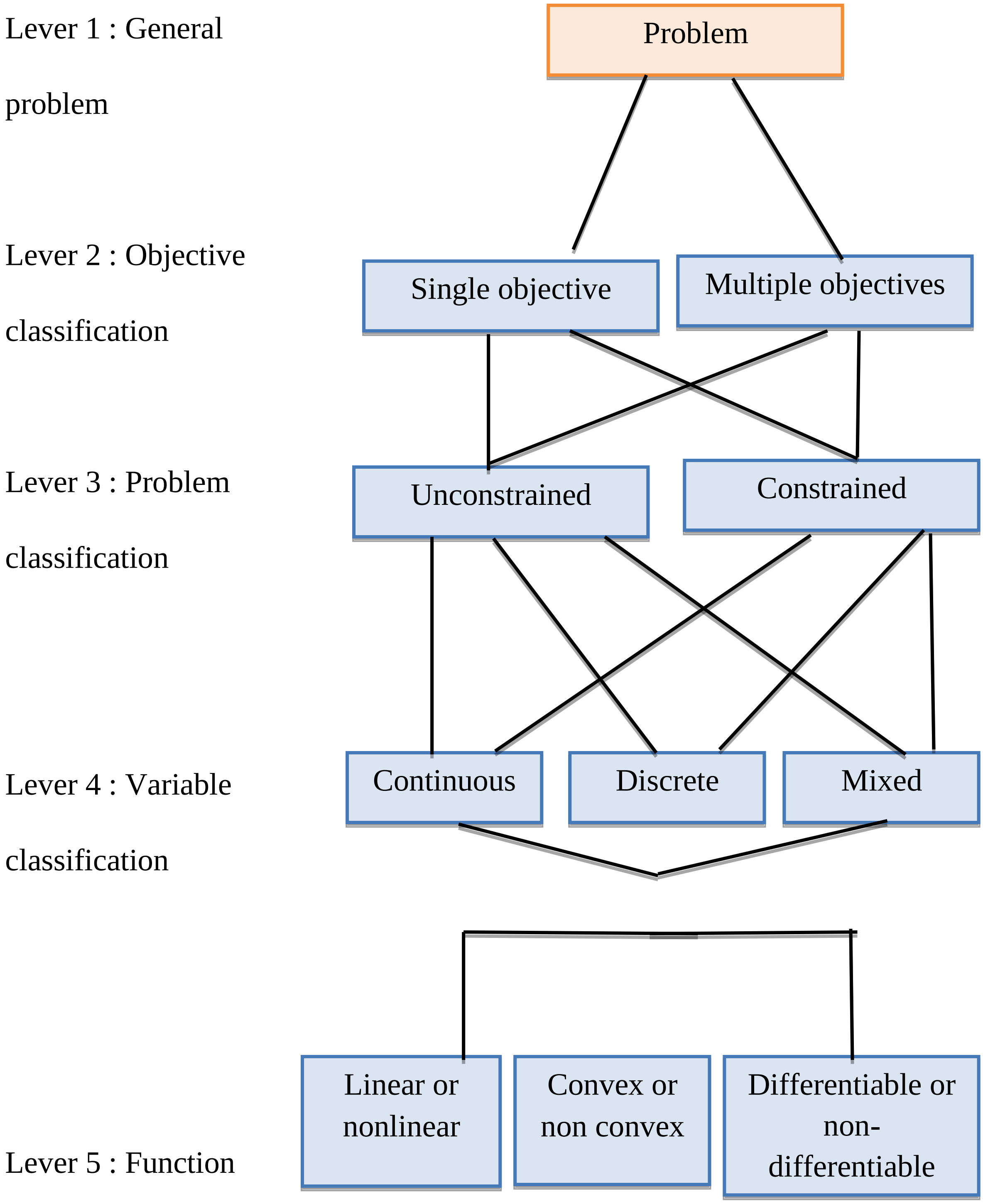


Figure 6 : Classification of Optimization Problems

CHAPTER TWO

One-Dimensional Optimization Methods

2.1 Introduction [1]

In this chapter, we are interested in the problem of minimizing an objective function $f : R \rightarrow R$ (i.e., a one-dimensional problem). The approach is to use an iterative search algorithm, also called a line-search method. One dimensional search methods are of interest for the following reasons. First, they are special cases of search methods used in multivariable problems. Second, they are used as part of general multivariable algorithms . In an iterative algorithm, we start with an initial candidate solution x_0 and generate a sequence of *iterates* x_1 , x_2 , ... For each iteration $k = 0 , 1 , 2 , . . . ,$ the next point x_{k+1} depends on x_k and the objective function f. The algorithm may use only the value of f at specific points, or perhaps its first derivative f', or even its second derivative f'' . In this chapter, we study some algorithms:

• Fibonacci search method,

• Bisection method,

• Newton– Raphson method,

• Secant method.

Note that the first three methods work with the comparison of function values, while quadratic interpolation method interpolates the objective function *f* on [*a, b*] by apolynomial that has the same function value or derivative value at a number of points on [*a, b*], and the rest three methods require derivative information. Suppose a un imodal function *f (x)* is defined on the interval [*a, b*] as depicted in **Figure 7** and we want to find a point x^* at which the given function is minimized, then we consider another two points x_1 and x_2 which lies in the interval [a, b] such that $a < x_1 < x_2 < b$, and hold the following properties to reduce the interval of uncertainty:

1. If $f(x_1) > f(x_2)$ then the minimum x^* does not lie in $[a, x_1]$.

2. If $f(x_1) < f(x_2)$ then the minimum x^* lies either in the interval $[a, x_1]$ or in $[x_1, x_2]$, but does not lie in $[x_2, b]$.[1]

3. If $f(x_1) = f(x_2)$ then the minimum x^*lies in $[x_1, x_2]$ but does not lie in $[a, x_1]$ and $[x_2, b]$. In this chapter, we will discuss solving methods : Fibonacci search method, Bisection method, Newton– Raphson method and Secant method.

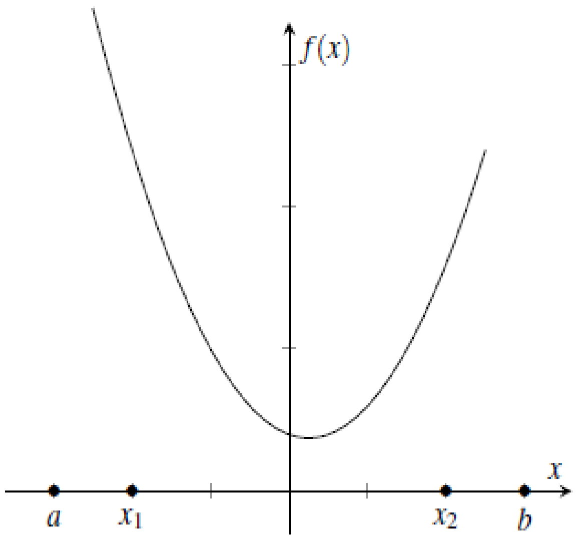


Figure 7 : A un imodal function with function values at two distinct point

2.2 Fibonacci Search Method [1]

This search technique determines the smallest possible interval of uncertainty, in which a minimum lies after completing n experiments .We have seen that the interval halving method reduces the interval size to one-half of its size in every iteration. Fibonacci search method, on the other hand, reduces the interval size by a different factor at each iteration. The reduction follows a sequence number satisfying the following Fibonacci formula :

$F_n = F_{n-1} + F_{n-2}$,

where $n = 2, 3, \ldots$ with

$F_0 = F_1 = 1$

Some numbers in the Fibonacci sequence are given in Table 1 . Note that the Fibonacci method can also be applied even if the function is not continuous. Consider a un imodal function *f* of one variable and define the interval [*a*, *b*].We choose points a and *b* in such a way that the minimizer x^* of *f* may be achieved in the few function.

N	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$\boldsymbol{F_n}$	*1*	*1*	*2*	*3*	*5*	*8*	*13*	*21*	*34*	*55*	*89*

Table 1 Fibonacci numbers[14]

evaluations. Let L_0 be the initial interval of uncertainty defined by $a \leq x \leq b$ and n be the number of experiments to be performed. We present

$$\boldsymbol{L_2^* = \frac{F_{n-2}}{F_n} L_0} \quad \dots \quad (\mathbf{1})$$

to determine x_1 and x_2 for the first two experiments. These points are placed at a distance of $L_2^*$from each end of L_0, which can be shown in Figure **8** and expressed in the following form:

$$x_1 = a + \boldsymbol{L_2^*} = a + \frac{\boldsymbol{F_{n-2}}}{\boldsymbol{F_n}} \boldsymbol{L_0} \quad \dots \quad (\mathbf{2})$$

And

$$x_2 = b - \boldsymbol{L_2^*} = b - \frac{\boldsymbol{F_{n-2}}}{\boldsymbol{F_n}} \boldsymbol{L_0} \quad \dots (\mathbf{3})$$

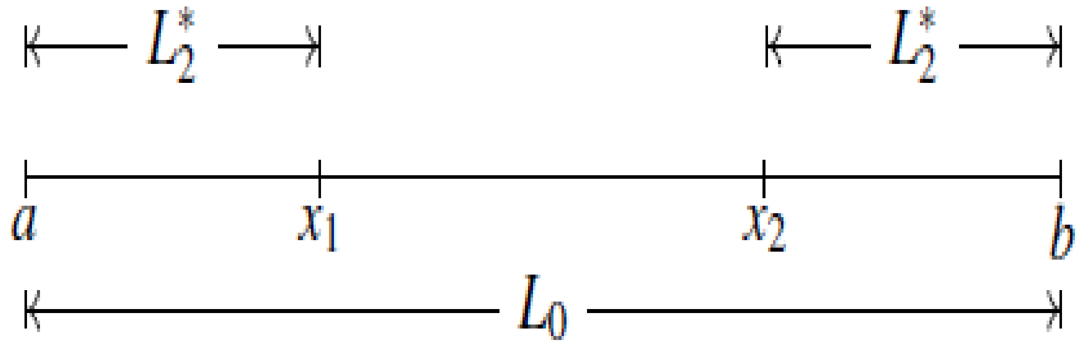


Figure 8 : Fibonacci search in un certanity interval L_0

Example (2.1) [3] : Apply the Fibonacci search method to minimize $f\ (x)\ =\ x^2\ +\ 2x$ on the interval [−3 , 4]. Obtain the optimal value within 5% exact value.

Sol. We have

$$\frac{\text{Length of final interval of uncertainty}}{2\times \text{Length of initial interval of uncertainty}} \leq \frac{5}{100}$$

that is,

$$\frac{L_n}{2} \leq \frac{1}{20} L_0$$

which implies

$$\frac{L_n}{L_0} = \frac{1}{F_n} \leq \frac{1}{10}$$

that is, $F_n \geq 10$

We have the following sequence of Fibonacci numbers:

$$F_0 = 1 , F_1 = 1 , F_2 = 2 , F_3 = 3 , F_4 = 5 , F_5 = 8 , F_6 = 13$$

Thus, $n \geq 6$ will satisfy the above inequality. Therefore, six number of experiments are to be performed to obtain the desire percentage of accuracy. We get $L_0 = 4 - (-3) = 7$, and $n = 6$. To achieve the values of x_1 and x_2, we first find

$$L_2^* = \frac{F_{n-2}}{F_n} L_0 = \frac{F_4}{F_6} L_0 = \frac{5}{13} \times 7 = 2.6923.$$

We obtain $x_1 = a + L_2^* = -3 + 2.6923 = -0.3077$,

$x_2 = b - L_2^* = 4 - 2.6923 = 1.3077$

with $f(x_1) = -0.5207$ *and* $f(x_2) = 4.32$. Since $f(x_1) < f(x_2)$, therefore we discard the interval $(x_2, 4]$.

The new level of uncertainty is $L_2 = [-3, x_2] = [-3,\ 1.3077]$ which is shown in Fig. 9. We find

$L_2 = L_0 - L_2^* = 7 - 2.26923 = 4.73077.$

which gives same value. To obtain x_3, we first find

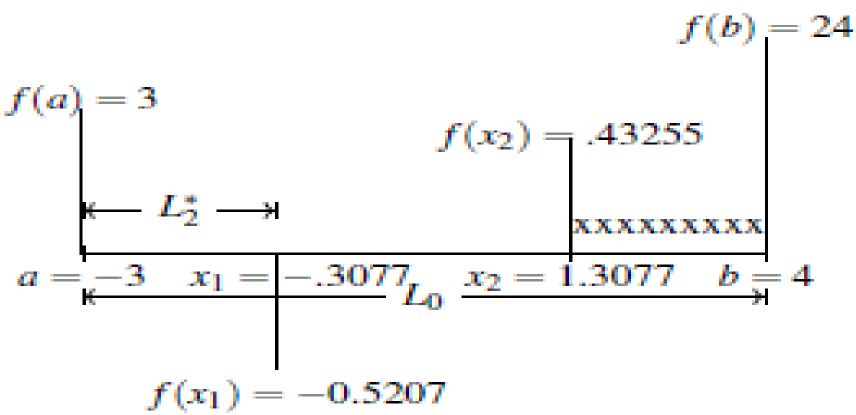


Figure 9 : *Discard* $(x_2, 4]$ *and obtain* $[-3, x_2]$

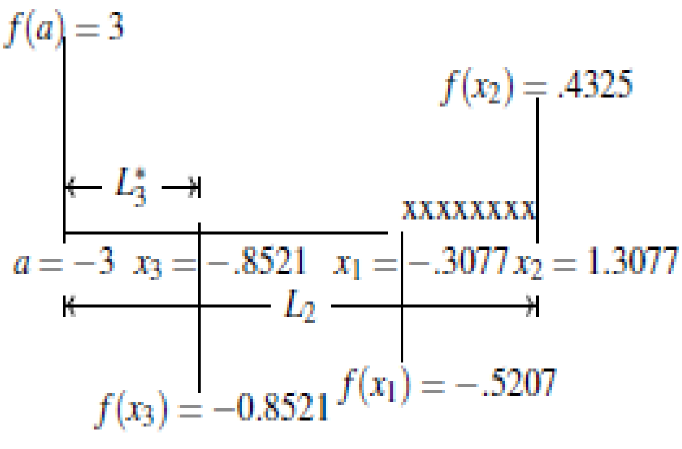


Figure 10 : Discard $(x_1, x_2]$ and obtain $[a, x_1]$

$$L_3^* = \frac{F_{n-3}}{F_n} L_0 = \frac{F_3}{F_6} \times 7 = \frac{3}{13} \times 7 = 1.6154\,.$$

Thus,

$$x_3 = -3 + L_3^* = -3 + 1.6154 = -1.3846,$$

with $f\ (x_3) = -0.8521$ and given that $f\ (x_1) = -0.3077$.

$Since\ f\ (x_3) < f(x_1)$, therefore we discard $(x_1, x_2]$. and obtain new interval $L_3 = [-3, -0.3077]$ which is depicted in Fig. 10. We find

$L_3 = L_2 - L_3^* = 4.73077 - 1.6154 = 3.11537$.

To obtain x_4, we find

$$L_4^* = \frac{F_{n-4}}{F_n} L_0 = \frac{F_2}{F_6} \times 7 = \frac{2}{13} \times 7 = 1.0769$$

Thus, we compute x_4 as:

$x_4 = -3 + L_4^* = -1.9231$,

with $f\ (x_4) = -0.1479$, and we already have $f\ (x_3) = -0.8521$. Since $f\ (x_3) < f\ (x_4)$, thus we discard $[-3, x_4)$ and new interval is $L_4 = [-1.9231, -0.3077]$ which is shown in Fig. 11. We now find

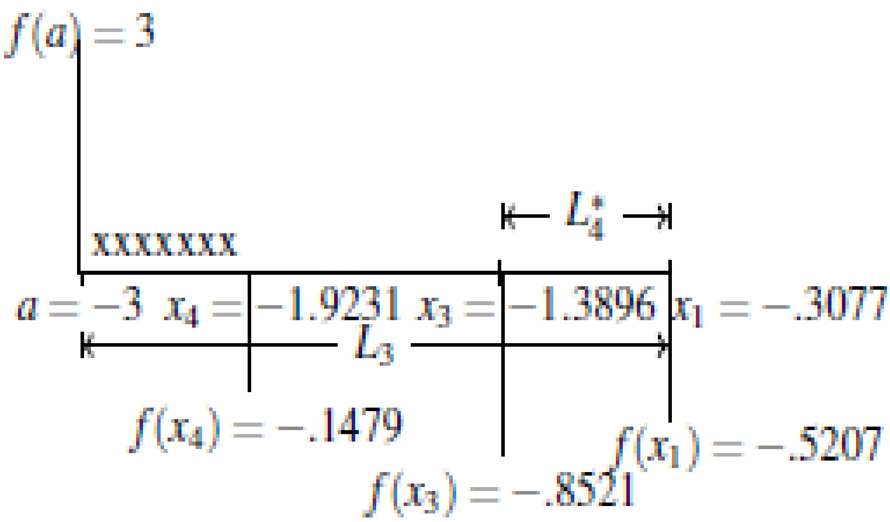


Figure 11 : Discard $[-3, x_4)$

$$L_5^* = \frac{F_{n-5}}{F_n} L_0 = \frac{F_1}{F_6} \times 7 = \frac{1}{13} \times 7 = 0\ .5384\ .$$

so that we can compute

$x_5 = -0.3077 + 0.5385 = -0.8462,$

with corresponding value $f(x_5) = -0.9763$, and we already have $f(x_3) = -0.8521$. Since $f(x_5) < f(x_3)$, thus discard $[x_4, x_3)$, and obtain new interval $L_5 = [-1.3846, -0.3077]$ which is depicted in Fig. 12. To find x_6, we have

$$L_6^* = \frac{F_{n-6}}{F_n} L_0 = \frac{F_0}{F_6} \times 7 = \frac{1}{13} \times 7 = 0.5384$$

We compute $x_6 = x_3 + L_6^* = -0.8461,$

with corresponding value $f(x_6) = -0.9761$, and we have $f(x_5) = -0.8462$. Since $f(x_5) < f(x_6)$, thus discard $(x_6, x_1]$ and new interval is $L_6 = [x_3, x_6] = [-1.3846, -0.8461]$ which is depicted in Fig. 13. The middle point of L_6 is the minimum point $x^* = 0.4385$ and the minimum function value is $f(x^*) = 1.0692.$

Figure 12 : Discard $[-3, x_4)$

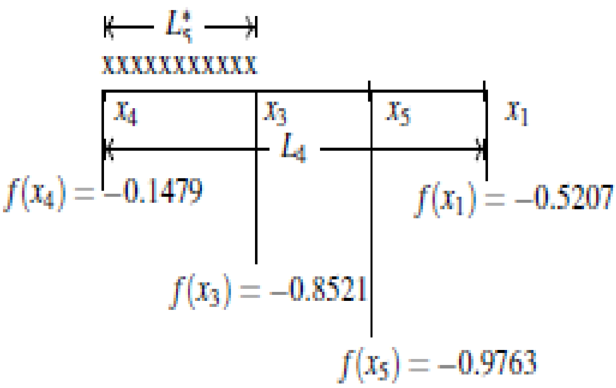
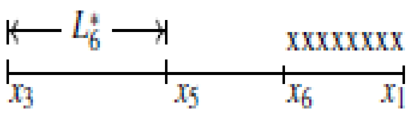


Figure 13: Discard $[-3, x_4)$



Note 5.1 Note that $\frac{L_n}{L_0} = \frac{0.5385}{7} = 0.0769 = \frac{1}{F_n} = \frac{1}{F_6} = \frac{1}{13}$

2.3 Bisection Method[15]

In bisection method we reduce begin with an interval so that 0 ∈[a,b] and divide the interval in two halves ,i.e. $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$. A next search interval is chosen by comparing and finding which one has zero. This is done by evaluating the sign. The algorithm for this is given as follows: Choose a, b so that f(a) f(b) < 0 , which is depicted in fig . 14

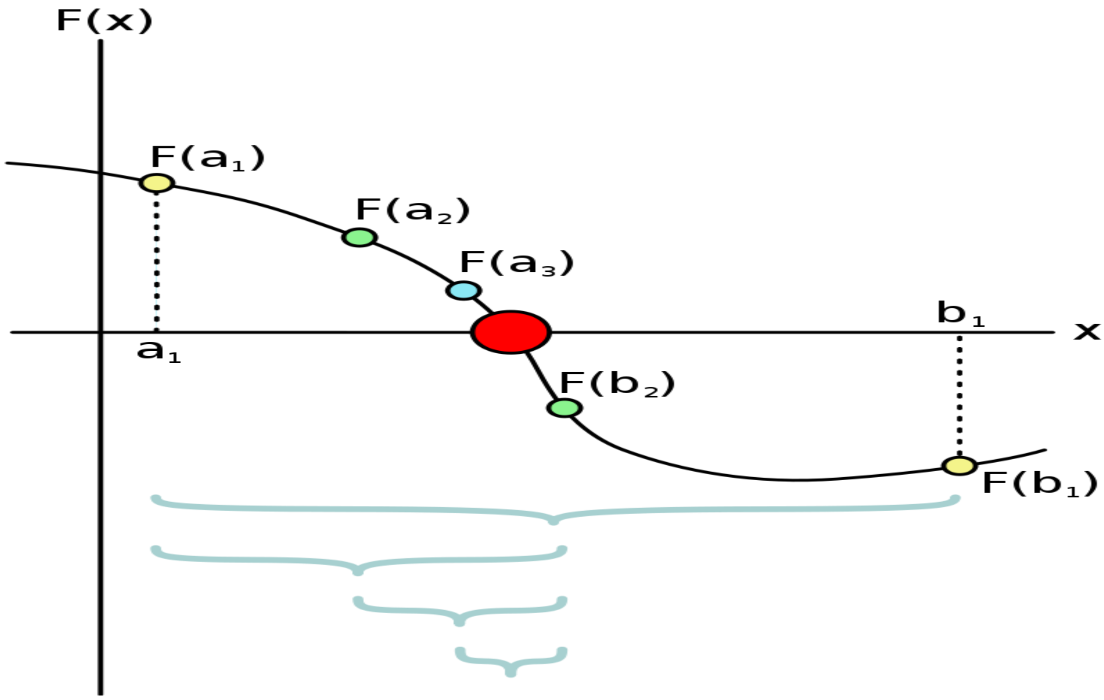


Figure 14 : A few steps of the bisection method applied over the starting range [a_1;b_1]. The bigger red dot is the root of the function.

2.3.1 Bisection Algorithm [12].

1 $m = \frac{a+b}{2}$

2 $f(m) = 0$ Stop and return.

3 $f(m)f(a) < 0; b = m$

4 $f(m)f(b) < 0; a = m$

5 $|f\ (r_n\ )| < \epsilon$ stop and return.

can be summed up advantages of Bisection Method as follows[8] :

- Guaranteed convergence. The bracketing approach is known as the bisection method, and it is always convergent.
- Errors can be managed. Increasing the number of iterations in the bisection method always results in a more accurate root.
- Doesn't demand complicated calculations. There are no complicated calculations required when using the bisection method. To use the bisection method, we only need to take the average of two values.
- Error bound is guaranteed. There is a guaranteed error bound in this technique, and it reduces with each repetition. Each cycle reduces the error bound by 12 per cent.
- The bisection method is simple and straightforward to programme on a computer.
- In the case of several roots, the bisection procedure is quick.

can be summed up disadvantages of Bisection Method as follows [8]:

- Although the Bisection Method convergence is guaranteed , it is often slow .
- Choosing a guess that is close to the root may necessitate numerous iterations to converge
- Some equations roots cannot be found . Because there are no bracketing values .
- Its rate of convergence is linear .
- It is incapable of determining complex roots .
- If the guess interval contains discontinuities , it cannot be used .
- It cannot be applied over an interval where the function returns values of the same sign .

Example (*2.2)* [4]: Converging $f\,(\,x\,)\; = \; x^2 - \; x - 1$ with $[a, b] = [1\,, 2]$ and $\epsilon = 0.003$.

Sol.

$f\,(a) \;=\; f\,(\,1\,) \;=\; (\,1\,)^2 - \;1 - 1 \;=\; -\,1$

$f\,(b) \;=\; f\,(\,2\,) \;=\; (\,2\,)^2 - \;2 - 1 \;=\; 1$

$r \;=\; \frac{a+b}{2} \;=\; \frac{1+2}{2} \;=\; \frac{3}{2} = 1.5$

$f\,(\,r\,) \;=\; f\,(\,1.5\,) \;=\; (\,1.5\,)^2 - \;1.5 - 1 \;=\; -\,0.25$

Number	a	b	$r = \dfrac{a+b}{2}$	$f(a)$	$f(b)$	$f(r)$
1	1	2	1.5	-1	1	$-\,0.25$
2	1.5	2	1.75	$-\,0.25$	1	0.3125
3	1.5	1.75	1.625	$-\,0.25$	0.3125	0.0156
4	1.5	1.625	1.5625	$-\,0\,.25$	0.015625	$-\,0.121$
5	1.5625	1.625	1.59375	$-\,0\,.121$	0.015625	$-0\,.0537$
6	1.59375	1.625	1.6093	-0.0537	0.015625	$-\,0.0194$
7	1.6093	1.625	$\mathbf{1.617}$	-0.0194	0.015625	$-\,0\,.002$

Table 2 Following Iterations Bisection Method

$|-0.002| = 0.002 \; < \; \epsilon \; = 0.003$

$\mathit{The\ root\ of\ the\ equation\ is}\ \boldsymbol{r_7 = 1.617}$

2.4 Newton– Raphson Method [1]

In Newton's method does a linear approximation of the function and finding the x -intercept of that approximation, thereby improving the performance of the bisection method. Linear approximation can be done by using Taylor's series.

$$f'(x_{k+1}) = f(x_k) + f'(x_k)(x_{k+1} - x_k)$$
$$f'(x_{k+1}) = 0$$
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad \dots \; (\mathbf{4})$$

can be summed up advantages of Newton - Raphson Method as follows [9]:

- It is best method to solve the non-linear equations.
- It can also be used to solve the system of non-linear equations, non-linear differential and non-linear integral equations.
- The order of convergence is quadric i.e. of second order which makes this method fast as compared to other methods.
- It is very easy to implement on computer.

can be summed up disadvantages of Newton- Raphson Method as follows [9]:

- This method becomes complicated if the derivative of the function f(x) is not simple.
- This method requires a great and sensitive attention regarding the choice of its approximation.

In each iteration, we have to evaluate two quantities f(x) and f'(x) for some x.

Example (2.3) [5] : Apply Newton–Raphson method to find the minimizer of

$f(x) = x^2 - 4\,sinx$ with a starting a point $x_0 = 3$.

Suppose that the initial value is $x_0 = 3$., and the tolerance value is $\varepsilon = 0.0005$, in the sense that we stop when $|x_{k+1} - x_k| < \varepsilon$.

Sol.

$f'(x) = 2x - 4\cos x$,

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 3 - \frac{8.4355}{9.9600} = 2.1531$$

$$x_2 = 2.1531 - \frac{1.2948}{6.5058} = 1.9540$$

$$x_3 = 1.9540 - \frac{0.1082}{5.4035} = 1.934$$

$$x_4 = 1.934 - \frac{0.0013}{5.2890} = 1.9338$$

$$|x_4 - x_3| = |1.9338 - 1.934| = |-0.0002| = 0.0002 < \epsilon$$

The root of the equation is $\boldsymbol{x_4 = 1.9338}$

Number	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
0	$x_{0=}$ 3	8.4355	9.9600	$x_{1=}$ 2.1531
1	$x_{1=}$ 2.1531	1.2948	6.5058	$x_{2=}$ 1.9540
2	$x_{2=}$ 1.9540	0.1082	5.4035	$x_{3=}$ 1.934
3	$x_{3=}$ 1.934	0.0013	5.2890	$\boldsymbol{x_{4=}}$ **1.9338**

Table 3 Following Iterations Newton–Raphson method

2. 5 Secant Method [20] .

When f' is expensive or cumbersome to calculate, one can use secant's method to approximate the derivative. **[5]**

The derivation of this method comes by replacing first derivative in the newton's method by its approximation (finite differentiation),

i.e $f'(x_k) = \frac{f_k - f_{k-1}}{x_k - x_{k-1}}$, where $f_k = f(x_k)$

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k \quad .$$

Just like Newton's method the secant's method to find the minimum is given by:

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f'_k - f'_{k-1}} f'_k \quad \dots (\mathbf{5})$$

Convergence of secant method is super-linear with $\beta = 1.618$. The table below shows the secant method convergence for $f(\mathrm{x}) = \frac{x}{1+x^2}$ with $x_0 = -0.6$ and $x_1 = 0.75$ as initial points.

Sol.

Number	x_{k-1}	x_k	$f(x_{k-1})$	$f(x_k)$
1	-0.6	0.75	-0.441176	0.48
2	0.75	0.046552	0.48	0.046451
3	0.046552	-0.028817	0.046451	-0.028793
4	-0.028817	0.000024	-0.028793	0.000024

Table 4 Following Iterations Secant Method

Can be summed up advantages of secant method as follows [10]:

1. It converges at faster than a linear rate, so that it is more rapidly convergent than the bisection method.
2. It does not require use of the derivative of the function, something that is not available in a number of applications.
3. It requires only one function evaluation per iteration, as compared with Newton’s method which requires two.

Can be summed up disadvantages of secant method as follows [13] :

1. It may not converge.

2. There is no guaranteed error bound for the computed iterates.

3. It is likely to have difficulty if f 0(α) = 0. This means the x-axis is tangent to the graph of y = f (x) at x = α.

4. Newton's method generalizes more easily to new methods for solving simultaneous systems of nonlinear equations.

CHAPTER THREE

Comparison and Conclusion

3.1 Introduction

In this chapter three,comparison and conclusion we wianll compare the solution of one example between the three methods , Bisection method, Newton– Raphson method and Secant method . We will find out which method is the best solution.

Example *(3.1)* :

Converging $f\ (x) = x^2 - 3$ with $[a, b] = [-1\ , 2]$ and $\epsilon = 0.01$. Apply Bisection Method , Newton– Raphson method and Secant method to find the minimize.

Sol.

Solution in a Bisection Method

$f\ (a) = f\ (-1) = (-1)^2 - 3 = -2$

$f\ (b) = f\ (2) = (2)^2 - 3 = 1$

$r = \frac{a+b}{2} = \frac{-1+2}{2} = \frac{1}{2} = 0.5$

$f\ (r) = f\ (0.5) = (0.5)^2 - 3 = -2.75$

Number	a	b	$r = \frac{a+b}{2}$	$f(a)$	$f(a)$	$f(r)$
1	-1	2	0.5	-2	1	-2.75
2	0.5	2	1.25	-2.75	1	-1.4375
3	1.25	2	1.625	-1.4375	1	-0.359375
4	1.625	2	1.8125	-0.359375	1	0.285156
5	1.625	1.8125	1.71875	-0.359375	0.285	-0.045898
6	1.71875	1.8125	1.765625	-0.045898	0.285	0.1174316
7	1.71875	1.765625	1.7421875	-0.045898	1.35181	0.035217
8	1.71875	1.7421875	**1.730469**	-0.045898	1.2930	-0.005477

Table 5 Following Iterations Bisection Method

$|-0.005477| = 0.005477 < \epsilon = 0.01$

The root of the equation is $r_8 = \mathbf{1.730469}$

Solution in a Newton– Raphson method.

$f'(x) = 2x$

$x_0 = \frac{a+b}{2} = \frac{-1+2}{2} = \frac{1}{2} = 0.5$

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_1 = 0.5 - \frac{(-2.75)}{1} = 0.5 + 2.75 = 3.25$

$x_2 = 3.25 - \frac{7.5625}{6.5} = 2.08654$

$x_3 = 2.08654 - \frac{1.35365}{4.17308} = 1.76216$

$x_4 = 1.76216 - \frac{0.1052}{3.52432} = 1.73231$

$x_5 = 1.73231 - \frac{0.000898}{3.46462} = 1.73205$

$|x_5 - x_4| = |1.73205 - 1.73231| = |-0.00026| = 0.00026 < \epsilon$

The root of the equation is $\boldsymbol{x_5 = 1.73205}$

Number	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
0	$x_{0=}$ 0.5	-2.75	1	$x_{1=}$ 3.25
1	$x_{1=}$ 3.25	7.5625	6.5	$x_{2=}$ 2.08654
2	$x_{2=}$ 2.08654	1.35365	4.17308	$x_{3=}$ 1.76216
3	$x_{3=}$ 1.76216	0.1052	3.52432	$x_{4=}$ 1.73231
4	$x_{4=}$ 1.73231	0.000898	3.46462	$\boldsymbol{x_{5=}}$ **1.73205**

Table 6 Following Iterations Newton– Raphson method

Solution by secant method

$$w = b - f(b)\frac{b-a}{f(b)-f(a)}$$

when n = 1

$$w = 2 - (1)\frac{2+1}{1+2} = 2 - \frac{3}{3} = 2 - 1 = 1$$

when n = 2

$$w = 1.66667 - (-0.2222)\frac{1.66667-1}{-0.2222+2}$$

$$w = 1.66667 + 0.2222\frac{0.66667}{1.7778}$$

$$w = 1.66667 + 0.2222\ (0.375997)$$

$$w = 1.66667 + 0.08332 = 1.74999$$

when n = 3

$$w = 1.74999 - 0.06247\frac{1.74999-1.66667}{0.06247+0.2222}$$

$$w = 1.74999 - 0.06247\frac{0.08332}{0.28467}$$

$$w = 1.74999 - 0.06247\ (0.29269)$$

$$w = 1.74999 - 0.01828 = 1.73171$$

Number	a	b	$f(a)$	$f(b)$	w	$f(w)$
1	-1	2	-2	1	1	-2
2	2	1	1	-2	1.66667	-0.2222
3	1	1.66667	-2	-0.2222	1.74999	0.06247
4	1.66667	1.74999	-0.2222	0.06247	**1.73171**	-0.00118

Table 7 Following Iterations secant method

$$|-0.00118| = 0.00118 < \epsilon = 0.01$$

The root of the equation is **1.73171**

3.2Conclusion

Take one example and compare the three methods ,Bisection method, Newton– Raphson method and Secant method .We found that the best way is the way Newton– Raphson method. Because it has the lowest error rate , then you come Secant method and then you come Bisection Method .

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الملخص

تعتبر مشاكل التحسين من اهم المواضيع المتداولة في الجوانب العلمية . هناك الكثير من التقنيات المتبعة لحل مشاكل التحسين . في هذا البحث ، طبقنا طريقة البحث فيبوناتشي ، طريقة التنصيف ، طريقة نيوتن - رافسون والطريقة القاطعة ، أيضًا لحل بعض المشاكل . تمت المقارنة بين الحلول وتم تبيان الطريقة الافضل معززة بالجداول العددية .

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
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كلية التربية للعلوم الصرفة
قسم الرياضيات

حل مسائل الامثلية من خلال طرق ذات بعد واحد

بحث مقدم الى
مجلس كلية التربية للعلوم الصرفة - جامعة بابل
كجزء من متطلبات نيل شهادة
الدبلوم العالي تربية / الرياضيات

من قبل
احمد عبد الحسن جابرمحسن

بإشراف
د. حسين عبد الوصي حسين

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