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Semi-Extending Semimodules

A Thesis
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Contents

Abstract

In this work, some generalizations of semi-essential semimodule were introduced. We developed these results by setting appropriate conditions, and defining new properties, relating to our concept, e.g. (fully prime semimodule, fully essential semimodule and semi-complement subsemimodule) such that: if for each nonzero proper subsemimodule of $\mathcal{T}$-semimodule $\mathcal{W}$ is prime, then $\mathcal{W}$ is called fully prime. If every semi-essential subsemimodule of $\mathcal{T}$-semimodule $\mathcal{W}$ is essential then $\mathcal{W}$ is called fully essential. A prime subsemimodule $\mathcal{H}$ of $\mathcal{W}$ is called semi-relative intersection complement (briefly, semi-complement) of subsemimodule $\mathcal{V}$ in $\mathcal{W}$, if $\mathcal{V} \cap \mathcal{H} = 0$ and whenever $\mathcal{V} \cap \mathcal{B} = 0$ with $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$ such that $\mathcal{H} \subseteq \mathcal{B}$, then $\mathcal{H} = \mathcal{B}$.

Also, we studied the concept of semi-extending on semimodules where (a $\mathcal{T}$-semimodule $\mathcal{W}$ is called semi-extending (briefly, SCS-semimodule) if for each $\mathcal{V} \leq \mathcal{W}$, then $\mathcal{V} \leq_{sem} \mathcal{H}$, with $\mathcal{H} \leq^{\oplus} \mathcal{W}$). Then, the concept of prime-extending semimodule was studied where (a $\mathcal{T}$-semimodule $\mathcal{W}$ is said to be prime-extending semimodule (**PE-semimodule)** if for any $0 \neq \mathcal{V} \leq \mathcal{W}$, then $\mathcal{V} \leq_{sem} \mathcal{B}$, with $\mathcal{B}$ is a prime direct summand in $\mathcal{W}$).

Finally, if $\mathcal{A}$ is a proper subsemimodule of a $\mathcal{T}$-semimodule $\mathcal{W}$, then $\mathcal{A}$ is called semi-prime if $t \in \mathcal{T}, w \in \mathcal{W}$ and $t^2 w \in \mathcal{A}$ then $tw \in \mathcal{A}$. A nonzero subsemimodule $\mathcal{V}$ of $\mathcal{T}$-semimodule $\mathcal{W}$ is said to be J-essential in $\mathcal{W}$ if $\mathcal{V} \cap \mathcal{A} \neq 0$ for all $0 \neq \mathcal{A}$ is semi-prime subsemimodule in $\mathcal{W}$ and (a $\mathcal{T}$-semimodule $\mathcal{W}$ is said to be J-extending, if for each $\mathcal{V} \leq \mathcal{W}$, then $\mathcal{V} \leq_{J.es} \mathcal{H}$, with $\mathcal{H} \leq^{\oplus} \mathcal{W}$). A semiring $\mathcal{T}$ is said to be J-extending if $\mathcal{W}$ is a J-extending $\mathcal{T}$-semimodule. A nonzero subsemimodule $\mathcal{N}$ of $\mathcal{W}$ is called

J-closed in $\mathcal{W}$ (briefly $\mathcal{N} \leq_{Jc} \mathcal{W}$) if $\mathcal{N}$ has no proper J-essential extension in $\mathcal{W}$. Some interesting results and properties of the new concepts were obtained.

List of symbols and abbreviations

Symbol, abbreviation	The meaning
$\mathcal{T}$	Semiring
$\mathcal{W}$	left $\mathcal{T}$-semimodule
$\mathbb{N}$	The set of all natural numbers
$\mathbb{N}_n$	The set of naturals modulo n
$\mathcal{T}\backslash E$	all elements belong to $\mathcal{T}$ but not to E
$\cap$	intersection
$\in$	belong to
$\subseteq$	Subset
$\leq$	Subsemimodule or ideal of semiring
$\nleq$	not subsemimodule
$\lneqq$	Proper subsemimodule
$\longrightarrow$	Thus, implication
$(\Longrightarrow)$	Proof the first direction
$(\Longleftarrow)$	Proof the second direction

$[\mathcal{B}:\mathcal{W}]$	$\{t \in \mathcal{T} \mid t\mathcal{W} \subseteq \mathcal{B}\}$
(s-P) $\mathcal{T}$-semimodule	all prime subsemimodules are subtractive
$rad(\mathcal{W})$	Intersection all prime subsemimodules in $\mathcal{W}$
$ann_{\mathcal{T}}(\mathcal{W})$	Left annihilator of $\mathcal{W}$
$End(\mathcal{W})$	A set all endmorphism on $\mathcal{W}$
$E(\mathcal{W})$	injective hull of $\mathcal{W}$
$\leq^{\oplus}$	direct summand
$\leq_e$	Essential subsemimodule
$\leq_{sem}$	Semi-essential subsemimodule
$\leq_{J.es}$	J-essential subsemimodule
$\leq_c$	Closed subsemimodule
$\leq_{Stc}$	St-closed subsemimodule
$\leq_{Jc}$	J-closed subsemimodule
CS-semimodule	Extending semimodule
SCS-semimodule	Semi-extending semimodule
PE-semimodule	Prime-extending semimodule
□	end of the proof

Introduction

The modules are one of the most important concepts of algebra. Among these areas is the extending of modules and the demonstration of new results that have different applications. Extending modules have been extensively studied by N. V. Dung, D. V. Huyn, P. F. Smith and R. Wisbauer [15, 1994], as well as in earlier book by S. H. Mohammed and B. J. Muller [24, 1990].

In [17, 1999], by Golan, the semimodule, which is a class broader than the class of modules, was studied. When studying semimodules on semirings which are wider than modules on rings, we found that it has a long history in terms of building classes, which are important generalizations of loops, as because of their applications in various fields in the computer science, physics...etc. and at the same time we notice important differences between them. Among these differences are the concepts of prime subsemimodule and semi-essential subsemimodules of semimodule.

A semiring is a nonempty set together with two operations, addition and multiplication under some conditions (see, Definition (1.1.1), where a monoid is a semigroup with identity. As an example, the set of natural number is a commutative semiring under usual addition and multiplication, however it is not a ring. A semimodule over semiring is defined similarly as in module over ring.

Some of the interesting topics semimodules are the prime subsemimodules, the essential subsemimodules, prime extending semimodule, semi-essential subsemimodule, semi-extending semimodule and extending semimodule.

Let $\mathcal{T}$ be a semiring, and $\mathcal{W}$ a left $\mathcal{T}$-semimodule. A subset $\mathcal{N}$ of $\mathcal{W}$ is subsemimodule if it is closed under addition and scalar multiplication [17, 1999 p.150]. "A proper submodule $\mathcal{B}$ of $\mathcal{R}$-module $\mathcal{M}$ is prime if every $r \in \mathcal{R}, m \in \mathcal{M}$ such that $rm \in \mathcal{B}$, then either $m \in \mathcal{B}$ or $r \in [\mathcal{B}: \mathcal{M}]$" [15, 1994]. In [21, 1982, p.106], Kasch defined essential submodule, where a nonzero submodule $\mathcal{V}$ in $\mathcal{R}$-module $\mathcal{M}$ is essential if $\mathcal{V} \cap \mathcal{H} \neq 0$, with $\mathcal{H}$ is a nonzero submodule in $\mathcal{M}$. In [23, 2009] Mijbass and Abdullah defined semi-essential submodule, where a nonzero submodule $\mathcal{V}$ in $\mathcal{R}$-module $\mathcal{M}$ is semi-essential if $\mathcal{V} \cap \mathcal{H} \neq 0$, with $\mathcal{H}$ is a nonzero prime submodule in $\mathcal{M}$. In [16, 2011], Fall etal., defined semi-essential subsemimodule, "(let $\mathcal{W}$ be a left $\mathcal{T}$-semimodule and $\mathcal{V}$ is a subsemimodule of $\mathcal{W}$, then $\mathcal{V}$ is said to be semi-essential in $\mathcal{W}$ if for any subsemimodule $\mathcal{H}$ of $\mathcal{W}$, $\mathcal{H} \cap \mathcal{V} = 0 \Longrightarrow \mathcal{H} = 0$)". While, Pawar in [26, 2013] used the above definition for essential subsemimodule, and in [27, 2015] defined semi-essential subsemimodule as follows: "(let $\mathcal{V}$ be a subsemimodule of $\mathcal{T}$-semimodule $\mathcal{W}$, then $\mathcal{V}$ is said to be semi-essential if $\mathcal{V} \cap \mathcal{B} \neq 0$, with $\mathcal{B}$ is a non-zero prime subsemimodule in $\mathcal{W}$)". In this work, Pawar's definition for semi-essential subsemimodule will be taken.

The prime-extending modules was studied in [20, 2011] by Ibrahiem, where: "an $\mathcal{R}$-module $\mathcal{M}$ is called prime-extending module if any nonzero proper submodule $\mathcal{V}$ in $\mathcal{M}$ is essential in a prime direct summand of $\mathcal{M}$". These concepts will be introduced for a semimodule and some outputs will be studied on a prime-extending semimodule.

The issue of extending semimodule was also studied by Alhashemi and Alhossaini, in [7, 2021], "a $\mathcal{T}$-semimodule $\mathcal{W}$ is called extending (CS-semimodule) if any subsemimodule is essential in a direct summand of $\mathcal{W}$". In [5, 2015], the concept of St-closed was defined on a submodule as follow: "let $\mathcal{M}$ be an $\mathcal{R}$-module and $\mathcal{N}$ a submodule of $\mathcal{M}$, then $\mathcal{N}$ is called St-closed if it has no proper semi-essential extensions in $\mathcal{M}$". Also, in [6, 2015], semi-extending modules have been studied, where: "a left $\mathcal{T}$-module $\mathcal{W}$ is called semi-extending if each submodule is semi-essential in a direct summand".

Through what was mentioned above the researcher chose the topic of a semi-extending semimodule to transfer some of the results and properties that were studied on modules to semimodules. The researcher also wanted in his work to show the relationship between the semi-extending semimodule and the positions studied by the researchers in [20, 2011].

Throughout this work, $\mathcal{T}$ denotes a semiring with identity and $\mathcal{W}$ is a unitary left $\mathcal{T}$-semimodule. This work consists of three chapters.

The first chapter contains three sections. In first the section, some definitions that we need in this work are mentioned. In section two, some results about the semi-essential subsemimodule were studied, some of those results are, if $\mathcal{W}$ is a faithful multiplication $\mathcal{T}$-semimodule, then E is a prime ideal of $\mathcal{T}$ with $\mathcal{W} \neq E\mathcal{W}$ if and only if $E\mathcal{W}$ is a prime subsemimodule of $\mathcal{W}$. Also, if $\mathcal{W}$ is a faithful multiplication $\mathcal{T}$-semimodule, then $\mathcal{V} \leq_{sem} \mathcal{W}$ if and only if $[\mathcal{V}: y] \leq_{sem} \mathcal{T}$ for any $y \in \mathcal{W}$. In the last section, some new concepts were studied as (fully prime semimodule, fully essential semimodule and semi-complement subsemimodule). Also, some examples and results have been studied about the concepts just mentioned. Among these results: (Proposition 1.3.18): let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a distributive and fully prime $\mathcal{T}$-semimodule, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are subsemimodules of $\mathcal{W}$, $< 0 > \neq \mathcal{N}_i \leq \mathcal{W}_i$ $(i = 1, 2)$, then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$ if and only if $\mathcal{N}_i \leq_{sem} \mathcal{W}_i, (i = 1, 2)$, and (Proposition 1.3.19): let $\mathcal{W}$ be an (s-P) $\mathcal{T}$-semimodule, $0 \neq \mathcal{V} \leq \mathcal{W}$ and $\mathcal{H}$ is a nonzero prime subsemimodule in $\mathcal{W}$. Then $\mathcal{H}$ is semi- complement of $\mathcal{V}$ in $\mathcal{W}$ if and only if $\frac{(\mathcal{V}\oplus\mathcal{H})}{\mathcal{H}} \leq_{sem} \frac{\mathcal{W}}{\mathcal{H}}$.

The second chapter is divided into two sections. In the first section, a semi-extending semimodule is studied, where: (a $\mathcal{T}$-semimodule $\mathcal{W}$ is called semi-extending (briefly, SCS-semimodule) if for each $\mathcal{V} \leq \mathcal{W}$, then $\mathcal{V} \leq_{sem} \mathcal{H}$, with $\mathcal{H} \leq^{\oplus} \mathcal{W}$) and some results on the topic that were mentioned recently. Among these results (Proposition 2.1.11), where: let $\mathcal{W} = \mathcal{V} \oplus \mathcal{H}$ be an SCS-semimodule. If for each semi-essential extension of $\mathcal{B} \oplus \mathcal{H}$ is a fully essential semimodule, where $\mathcal{B} \leq_{Stc} \mathcal{V}$, then $\mathcal{V}$ is an SCS-semimodule, and (Proposition 2.1.19), where: let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be an (s-P) $\mathcal{T}$-semimodule and $\mathcal{W}_1, \mathcal{W}_2$ are prime subsemimodules in $\mathcal{W}$, such that $\mathcal{W}_1$ and $\mathcal{W}_2$ are SCS-semimodules then, $\mathcal{W}$ is an SCS-semimodule if and only if for any St-closed $\mathcal{H}$ of $\mathcal{W}$ with $\mathcal{H} \cap \mathcal{W}_1 = 0$ or $\mathcal{H} \cap \mathcal{W}_2 = 0$ is a direct summand in $\mathcal{W}$. As for the second section, we will study another concept that, we called the prime-extending semimodule (**PE**-semimodule), with some of the relationships that link it with SCS-semimodules. As:

- $\mathcal{W}$ is a **PE**-semimodule if and only if $\mathcal{T}$ is a **PE**-semimodule.
- A $\mathcal{T}$-semimodule $\mathcal{W}$ is a **PE**-semimodule if and only if $\mathcal{W}$ is an SCS-semimodule, provided for any direct summand of $\mathcal{W}$ is prime.

Finally, the third chapter, is divided into four sections. The first section consists the definition of the concept of semi-prime subsemimodule with some results. In the second section, the concept of J-essential was defined, which is a generalization of the concept of essential, where the researcher stated that every essential subsemimodule is J-essential, but the converse is not true in general, see (3.2.2 (2),(3)). Also, the researcher explained that every J-essential is semi-essential, but the converse is not true in general.

See (3.2.2 (4),(5)), and some results were studied, the most important of which (Proposition 3.2.10) where: if $\mathcal{W}$ is a multiplication faithful $\mathcal{T}$-semimodule, $0 \neq \mathcal{V}$ is a semi-prime subsemimodule in $\mathcal{W}$ and $\mathcal{V}$ is not minimal semi-prime subsemimodule then $\mathcal{V} \leq_{J.es} \mathcal{W}$. In the third section, the concept of J-extending semimodule is defined with some notes and examples, and the properties and results have been studied, including (Proposition 3.3.7) where: let $\mathcal{W}$ be a fully prime $\mathcal{T}$-semimodule, $\mathcal{W}$ is a J-Extending semimodule if and only if $\mathcal{W}$ is an SCS-semimodule. In the last section, the concept of J-closed subsemimodule was studied with some properties and examples. See, (Theorem 3.4.6): a $\mathcal{T}$-semimodule $\mathcal{W}$ is a J-Extending semimodule if and only if any J-closed subsemimodule in $\mathcal{W}$ is a direct summand of $\mathcal{W}$.

Chapter one

Fully Prime Semimodules, Fully Essential Semimodules and Semi-Complement Subsemimodules

Fully Prime Semimodules, Fully Essential Semimodules and Semi-Complement Subsemimodules

This chapter consists of three sections, the first section contains some definitions that we need in this work (preliminaries). In section two, some new results are obtained via adding the necessary conditions. Finally, in the last section, some new concepts were studied as (fully prime semimodule, fully essential semimodule and semi-complement subsemimodule).

1.1. Preliminaries

Definition 1.1.1:[17, p. 1] A semiring is a set $\mathcal{T}$ together with two binary operations called addition $(+)$ and multiplication $(\cdot)$ such that $(\mathcal{T},+)$ is a commutative monoid with identity element $0_{\mathcal{T}}$, $(\mathcal{T},\ \cdot)$ is a monoid with identity element 1, multiplication distributes over addition from either side and 0 is multiplicative absorbing, that is, $a \cdot 0 = 0 \cdot a = 0$ for each $a \in \mathcal{T}$.

Definition 1.1.2:[17, p. 65] Let I be a proper subset of a semiring $\mathcal{T}$, I is called left ideal (or right ideal) in $\mathcal{T}$ if satisfying the following conditions:

1. If $a, b \in I$ then $a + b \in I$;
2. If $a \in I$ and $t \in \mathcal{T}$ then ta (or at) $\in I$.

Also, I is **ideal** of $\mathcal{T}$ if a subset which is both a left ideal and right ideal of $\mathcal{T}$.

Definition 1.1.3:[17, p. 149] Let $\mathcal{T}$ be a semiring. A left $\mathcal{T}$-semimodule $\mathcal{W}$ is a commutative monoid $(\mathcal{W},+)$ for which we have a function $\mathcal{T} \times \mathcal{W} \to \mathcal{W}$ defined by $(t, w) \mapsto tw$ $(t \epsilon \mathcal{T},\ w \epsilon \mathcal{W})$ such that for all $t, t' \epsilon \mathcal{T}$ and $w, v\ \epsilon \mathcal{W}$, the following conditions are satisfied:

i. $t(w + v) = tw + tv$.

ii. $(t + t')w = tw + t'w$.

iii. $(t.t')w = t(t'w)$.

iv. $0_{\mathcal{T}}w = 0 = t0_{\mathcal{W}}$.

When $1w = w$ holds for each $w \epsilon \mathcal{W}$ then a left $\mathcal{T}$-semimodule $\mathcal{W}$ is said to be unitary.

Definition 1.1.4:[17, p. 150] If $\mathcal{V}$ is a nonempty subset of a left $\mathcal{T}$-semimodule $\mathcal{W}$, then $\mathcal{V}$ is said to be subsemimodule of $\mathcal{W}$ if it is closed under addition and scalar multiplication (briefly, $\mathcal{V} \leq \mathcal{W}$)

Definition 1.1.5:[17, p. 154] A subsemimodule $\mathcal{V}$ of a $\mathcal{T}$-semimodule $\mathcal{W}$ is said to be subtractive if each of $w, w' \epsilon \mathcal{W}$ with $w, w + w' \epsilon \mathcal{V}$ implies $w' \epsilon \mathcal{V}$. A $\mathcal{T}$-semimodule $\mathcal{W}$ is said to be subtractive if all its subsemimodules are subtractive.

Definition 1.1.6:[29] A $\mathcal{T}$-semimodule $\mathcal{W}$ is called semisubtractive if for every w, w' in $\mathcal{W}$, there exists h in $\mathcal{W}$ such that $w = w' + h$ or $w + h = w'$.

Definition 1.1.7:[17, p. 172] A $\mathcal{T}$-semimodule $\mathcal{W}$ is additively cancellative if for all w, h, k in $\mathcal{W}$ with $w + h = w + k$ implies $h = k$.

Definition 1.1.8:[17, p. 184] Let $\mathcal{V}$ and $\mathcal{H}$ be subsemimodules of a $\mathcal{T}$-semimodule $\mathcal{W}$, $\mathcal{W}$ is said to be a direct sum of $\mathcal{V}$ and $\mathcal{H}$, denoted by $\mathcal{W} = \mathcal{V} \oplus \mathcal{H}$, if each $w \epsilon \mathcal{W}$ uniquely written as $w = v + h$ where $v \epsilon \mathcal{V}$ and $h \epsilon \mathcal{H}$. In this case $\mathcal{V}$ (similarly $\mathcal{H}$) is called a direct summand of $\mathcal{W}$.

Recall that, the following was appeared in modules [19, p. 341], we will give analog property for semimodules.

Definition 1.1.9: A $\mathcal{T}$-semimodule $\mathcal{W}$ is called distributive if for any subsemimodules $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ of $\mathcal{W}$ then $\mathcal{X} \cap (\mathcal{Y} + \mathcal{Z}) = \mathcal{X} \cap \mathcal{Y} + \mathcal{X} \cap \mathcal{Z}$.

Definition 1.1.10:[26] Let $0 \neq \mathcal{V}$ be a subsemimodule of a $\mathcal{T}$-semimodule $\mathcal{W}$. Then $\mathcal{V}$ is said to be essential in $\mathcal{W}$ if for each subsemimodule $\mathcal{H}$ of $\mathcal{W}$, $\mathcal{V} \cap \mathcal{H} = 0$ implies $\mathcal{H} = 0$ (briefly, $\mathcal{V} \leq_e \mathcal{W}$). In this case $\mathcal{W}$ is said to be an essential extension of $\mathcal{V}$.

Definition 1.1.11: [1, p.30] A $\mathcal{T}$-semimodule $\mathcal{W}$ is called uniform if each subsemimodule $\mathcal{V}$ of $\mathcal{W}$ is an essential subsemimodule in $\mathcal{W}$.

Remark 1.1.12: Let $0 \neq \mathcal{V}$ be a subsemimodule of a $\mathcal{T}$-semimodule $\mathcal{W}$. Then, $\mathcal{V} \leq_e \mathcal{W}$ if and only if every $0 \neq a \in \mathcal{W}$ there exists $t \in \mathcal{T}$ such that $0 \neq ta \in \mathcal{V}$.

Proof:

($\Longrightarrow$) It is clear.

($\Longleftarrow$) Let $0 \neq \mathcal{H} \leq \mathcal{W}$, then there exists $0 \neq a \in \mathcal{H} \leq \mathcal{W}$ thus $a \in \mathcal{W}$, by assumption then there exists $t \in \mathcal{T}$ such that $0 \neq ta \in \mathcal{V}$. But $0 \neq ta \in \mathcal{H}$, thus $\mathcal{V} \cap \mathcal{H} \neq 0$. Therefore, $\mathcal{V} \leq_e \mathcal{W}$. □

Definition 1.1.13:[9] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule and $\mathcal{U}$,$\mathcal{V}$ are subsemimodules of $\mathcal{W}$. Then $\mathcal{U}$ is said to be intersection complement of $\mathcal{V}$ if $\mathcal{U} \cap \mathcal{V} = 0$ and $\mathcal{U}$ is maximal in the set of all subsemimodules of $\mathcal{W}$ that have zero intersection with $\mathcal{V}$.

$\mathcal{U}$ and $\mathcal{V}$ are said to be mutually complement if they are intersection complement of each other.

Definition 1.1.14:[29] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule and $w \in \mathcal{W}$. The left annihilator of w is defined by $ann_{\mathcal{T}}(w) = \{t \in \mathcal{T} \mid tw = 0\}$.

Note: $ann(w)$ is a left ideal of $\mathcal{T}$. If $\mathcal{V}$ is a subsemimodule of $\mathcal{W}$, then $ann_{\mathcal{T}}(\mathcal{V}) = \{ t \in \mathcal{T} \mid tw = 0, \text{ for all } w \in \mathcal{V}\}$.

Definition 1.1.15:[31] If $\mathcal{T}$ is a semiring and $\mathcal{W}$ be a $\mathcal{T}$-semimodule. A proper subsemimodule $\mathcal{B}$ of $\mathcal{W}$ is called prime if for any $w \in \mathcal{W}, t \in \mathcal{T}$ and $tw \in \mathcal{B}$ then $w \in \mathcal{B}$ or $t \in [\mathcal{B}:\mathcal{W}]$, where $[\mathcal{B}:\mathcal{W}] = \{t \in \mathcal{T} | t\mathcal{W} \subseteq \mathcal{B}\}$.

Note:

1. Let $\mathcal{T}$ be a semiring and E is ideal of $\mathcal{T}$, then E is a prime ideal in $\mathcal{T}$ if each A, B ideals of $\mathcal{T}, A.B \subseteq E$ implies either $A \subseteq E$ or $B \subseteq E$, [17, p.85]
2. Assume that E is a prime ideal of semiring $\mathcal{T}$ such that $ab \in E$ implies $ba \in E$. If $ab \in E$ then either $a \in E$ or $b \in E$, $\forall\, a, b \in \mathcal{T}$, [17, p.86].

In particular, An ideal E of $\mathcal{T}$ is called prime if $a, b \in \mathcal{T}$ and $a.b \in E$ then either $a \in E$ or $b \in E$.

Recall that, in semimodules to define the quotient $\mathcal{W}/\mathcal{V}$, $\mathcal{V}$ must be a subtractive subsemimodule of $\mathcal{W}$. [17, p.165].

The definition of quotient semimodule is according to Golan see [17]

Now, if $\mathcal{B}$ is a subtractive subsemimodule of a $\mathcal{T}$-semimodule $\mathcal{W}$, then

$$ann_{\mathcal{T}}(\mathcal{W}/\mathcal{B}) = \{t \in \mathcal{T} | t(\mathcal{W}/\mathcal{B}) = 0\} = \{t \in \mathcal{T} | t\mathcal{W}/\mathcal{B} = 0\}$$
$$= \{t \in \mathcal{T} | t\mathcal{W} \subseteq \mathcal{B}\} = [\mathcal{B}:\mathcal{W}]$$

Definition 1.1.16:[27] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $0 \neq \mathcal{V} \leq \mathcal{W}$, then $\mathcal{V}$ is called a semi-essential subsemimodule in $\mathcal{W}$ if for every nonzero prime subsemimodule $\mathcal{B}$ of $\mathcal{W}$ then $\mathcal{V} \cap \mathcal{B} \neq 0$, i.e. if $\mathcal{V} \cap \mathcal{B} = 0$ then $\mathcal{B} = 0$ (briefly, $\mathcal{V} \leq_{sem} \mathcal{W}$).

Note: Every essential subsemimodule of a semimodule is semi-essential. But the converse is not true in general. For example: We will consider the

$\mathbb{N}$-semimodule $\mathcal{W} = \mathbb{N}_{12}$ (the semimodule of natural numbers modulo 12). In this semimodule, there are six subsemimodules which are $\mathcal{V}_1 =< \bar{0} >$ $, \mathcal{V}_2 =< \bar{2} >, \mathcal{V}_3 =< \bar{3} >, \mathcal{V}_4 =< \bar{4} >, \mathcal{V}_5 =< \bar{6} >,$ and $\mathcal{V}_6 = \mathcal{W}$. And there are two prime subsemimodules which are $\mathcal{V}_2$ and $\mathcal{V}_3$. Now, $\mathcal{V}_5 \leq_{sem} \mathcal{W} = \mathbb{N}_{12}$, since $\left[\begin{cases} 0 \neq \mathcal{V}_5 \cap \mathcal{V}_2 = \mathcal{V}_5 \\ 0 \neq \mathcal{V}_5 \cap \mathcal{V}_3 = \mathcal{V}_5 \end{cases}\right]$. But $\mathcal{V}_5$ is not essential subsemimodule of $\mathbb{N}_{12}$, since $\mathcal{V}_5 \cap \mathcal{V}_4 =< \bar{0} >$. □

Definition 1.1.17:[13] A $\mathcal{T}$-semimodule $\mathcal{W}$ is called multiplication if each subsemimodule of $\mathcal{W}$ is of the form $I\mathcal{W}$, for some ideal I of $\mathcal{T}$.

Note that, If $\mathcal{W}$ is a multiplication $\mathcal{T}$-semimodule then $\mathcal{V} = [\mathcal{V}: \mathcal{W}]\mathcal{W}, \forall\, \mathcal{V} \leq \mathcal{W}$.

$(\Longrightarrow)[\mathcal{V}: \mathcal{W}]\mathcal{W} \subseteq \mathcal{V}$ in general, true.

$(\Longleftarrow)$ Since $\mathcal{W}$ is multiplication then $\mathcal{V} = J\mathcal{W}$ for some $J \leq \mathcal{T}$ then $J \subseteq [\mathcal{V}: \mathcal{W}] \Longrightarrow \mathcal{V} = J\mathcal{W} \subseteq [\mathcal{V}: \mathcal{W}]\mathcal{W}$. Therefore $\mathcal{V} = [\mathcal{V}: \mathcal{W}]\mathcal{W}$. □

Definition 1.1.18: [17, p.153] A $\mathcal{T}$-semimodule $\mathcal{W}$ is called finitely generated by a subset $\mathcal{A}$ of $\mathcal{W}$ if $\mathcal{A}$ is finite and $\mathcal{W}$ is the intersection of all subsemimodules of $\mathcal{W}$ containing $\mathcal{A}$. Note:

$\mathcal{T}\mathcal{A} = \{t_1 a_1 + t_2 a_2 + \cdots\cdots\cdots + t_n a_n \,|\, t_i \in \mathcal{T} \text{ and } a_i \in \mathcal{A}, i = 1,2, \ldots n\}$.

Definition 1.1.19: [17, p.156] If $\mathcal{T}$ is a semiring, $\mathcal{W}, \mathcal{W}'$ be $\mathcal{T}$-semimodules, then a function $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ is called a homomorphism of $\mathcal{T}$-semimodules if the following conditions are satisfied:

1) $\Psi(w_1 + w_2) = \Psi(w_1) + \Psi(w_2)$ for any $w_1, w_2 \in \mathcal{W}$
2) $\Psi(tw) = t\Psi(w)$ for any $w \in \mathcal{W}$ and $t \in \mathcal{T}$.

Definition 1.1.20:[27] Let $\mathcal{W}, \mathcal{W}'$ be two $\mathcal{T}$-semimodules. A $\mathcal{T}$-homomorphism $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ is called semi-essential if $\Psi(\mathcal{W})$ is a semi-essential subsemimodule of $\mathcal{W}'$.

Definition 1.1.21: A $\mathcal{T}$-semimodule $\mathcal{W}$ is faithful if for any nonzero $t \in \mathcal{T}$ there is an element $w \in \mathcal{W}$ such that $tw \neq 0$.

Definition 1.1.22:[12] A $\mathcal{T}$-semimodule $\mathcal{W}$ is called torsion-free if every $t \in \mathcal{T}, w \in \mathcal{W}$ and $tw = 0$ implies either $w = 0$ or $t = 0$.

Note: Every torsion-free is faithful.

Definition 1.1.23:[2] Let $\mathcal{T}$ be a semiring and $\mathcal{W}$ is a left $\mathcal{T}$-semimodule. $\mathcal{W}$ is called simple if for every $\mathcal{V} \leq \mathcal{W}$ then either $\mathcal{V} =< 0 >$ or $\mathcal{V} = \mathcal{W}$. For example: The $\mathbb{N}$-semimodule $\mathbb{N}_2$ is simple.

Definition 1.1.24:[8] Let $\mathcal{T}$ be a semiring and $\mathcal{W}$ is a left $\mathcal{T}$-semimodule. $\mathcal{W}$ is called semisimple if for each $\mathcal{U} \leq \mathcal{W}$, then $\mathcal{U}$ is direct summand in $\mathcal{W}$. For example: The $\mathbb{N}$-semimodule $\mathbb{N}_6$ is semisimple.

Definition 1.1.25:[17, p.185] A $\mathcal{T}$-semimodule. $\mathcal{W}$ is said to be indecomposable if there are no $0 \neq \mathcal{V}_1, \mathcal{V}_2 \leq \mathcal{W}$ satisfying $\mathcal{W} = \mathcal{V}_1 \oplus \mathcal{V}_2$. For example: The $\mathbb{N}$-semimodule $\mathbb{N}_8$ is indecomposable.

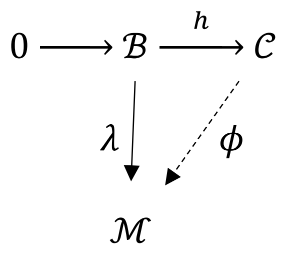
Definition 1.1.26:[9] A subsemimodule $\mathcal{V}$ of $\mathcal{T}$-semimodule $\mathcal{W}$ is called closed (briefly $\mathcal{V} \leq_c \mathcal{W}$) if it has no proper essential extensions.

Definition 1.1.27: A subsemimodule $\mathcal{V}$ of $\mathcal{T}$-semimodule $\mathcal{W}$ is called St-closed (briefly $\mathcal{V} \leq_{Stc} \mathcal{W}$) if it has no proper semi-essential extensions.

Remark: Every St-closed subsemimodule is closed. But the converse is not true in general. For example: In the $\mathbb{N}$-semimodule $\mathbb{N}_{24}$ we note $< 3 >$ is a closed subsemimodule in $\mathbb{N}_{24}$, but it is not St-closed.

Definition 1.1.28: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule and E be ideal of $\mathcal{T}$, $\mathcal{W}$ is said to be E-cyclie provided there is $s \in \mathcal{T} \backslash E$ and $w \in \mathcal{W}$ such that $s\mathcal{W} \subseteq \mathcal{T}w$.

Definition 1.1.29:[10] A $\mathcal{T}$-semimodule, $\mathcal{W}$ is called injective if every $\mathcal{T}$-monomorphism $h: \mathcal{B} \longrightarrow \mathcal{C}$ and for every $\mathcal{T}$-homomorphism $\lambda: \mathcal{B} \longrightarrow \mathcal{M}$, there is a $\mathcal{T}$-homomorphism $\phi: \mathcal{C} \longrightarrow \mathcal{M}$ such that $\phi h = \lambda$. The following figure shows the relationship:



Definition 1.1.30:[29] Let $f: \mathcal{W} \longrightarrow \mathcal{W}'$ be a homomorphism with $\mathcal{W}, \mathcal{W}'$ are $\mathcal{T}$-semimodules, then

- $ker(f) = \{w \in \mathcal{W} | f(w) = 0\}$.
- f is called k-regular if $f(w_1) = f(w_2) \longrightarrow w_1 + i = w_2 + j$ for some $i, j \in ker(f)$, and $w_1, w_2 \in \mathcal{W}$.

  Note: If f is one-to-one then f is k-regular.

Definition 1.1.31:[9] A $\mathcal{T}$-semimodule $\mathcal{W}$ is called π-injective if for any two subsemimodule $\mathcal{H}_1, \mathcal{H}_2$ with $\mathcal{H}_1 \cap \mathcal{H}_2 =< 0 >$, there exists $\phi, \Psi \in End(\mathcal{W})$ such that: ϕ, Ψ are idempotent, $\phi + \Psi = 1_{\mathcal{W}}$, $\mathcal{H}_1 \subseteq ker\phi, \mathcal{H}_2 \subseteq ker\Psi$.

Definition 1.1.32:[22] If E is an injective $\mathcal{T}$-semimodule, and it is a minimal injective extension of the $\mathcal{T}$-semimodule $\mathcal{W}$, then E is said to be an injective hull of $\mathcal{W}$ denoted by $E(\mathcal{W})$.

It is commonly known, however, that if $\mathcal{T}$ is a semiring, the injective hull of $\mathcal{T}$-semimodule always exists. But, Golan ([17], Proposition 17.21, p. 198) demonstrated that injective hulls of non-zero $\mathcal{T}$- semimodules do not need to exist for every semiring $\mathcal{T}$. Every semimodule over an additively idempotent semiring has an injective hull, as Wang [30] shown.

For further information on an injective hull of semimodules over semirings see [22]. For example a semimodule $(\mathbb{N},+)$ over $(\mathbb{N},+,.)$ without injective hull [17].

Zorn's Lemma:[21, p. 25] If $\mathcal{V}$ is an ordered set and for each totally ordered subset in $\mathcal{V}$ has an upper bound in $\mathcal{V}$, thus $\mathcal{V}$ has a maximal element.

Semi-Modular Law:[7] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$ are subsemimodules of $\mathcal{W}$. If $\mathcal{C} \leq \mathcal{B}$ and $\mathcal{B}$ is subtractive then $\mathcal{B} \cap (\mathcal{A}+\mathcal{C}) = (\mathcal{B} \cap \mathcal{A}) + \mathcal{C}$.

1.2. Some properties of semi-essential subsemimodules

Proposition 1.2.1:[27] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule and $\mathcal{V}_1,\mathcal{V}_2$ are subsemimodules of $\mathcal{W}$ such that $\mathcal{V}_1 \leq \mathcal{V}_2$. If $\mathcal{V}_1 \leq_{sem} \mathcal{W}$, then $\mathcal{V}_2 \leq_{sem} \mathcal{W}$.

Corollary 1.2.2:[27] Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be subsemimodules of a $\mathcal{T}$-semimodule $\mathcal{W}$. If $(\mathcal{V}_1 \cap \mathcal{V}_2) \leq_{sem} \mathcal{W}$, then $\mathcal{V}_i \leq_{sem} \mathcal{W}$, $(i = 1,2)$.

Note that, the opposite direction in Corollary 1.2.2 is not achieved, and the following example illustrates this.

Example 1.2.3: Consider the $\mathbb{N}$-semimodule $\mathcal{W} = \mathbb{N}_{30}$ (the semimodule of natural numbers modulo 30). In semimodule $\mathbb{N}_{30}$, there are eight subsemimodules which are $\mathcal{V}_1 =< \overline{0} >, \mathcal{V}_2 =< \overline{2} >, \mathcal{V}_3 =< \overline{3} >, \mathcal{V}_4 =< \overline{5} >, \mathcal{V}_5 =< \overline{6} >, \mathcal{V}_6 =< \overline{10} >, \mathcal{V}_7 =< \overline{15} >$ and $\mathcal{V}_8 = \mathcal{W}$. And there are three prime subsemimodules which are $\mathcal{V}_2, \mathcal{V}_3,$ and $\mathcal{V}_4$. Now, $\mathcal{V}_2 \leq_{sem} \mathcal{W} = \mathbb{N}_{30}$, since $\left[\begin{cases} 0 \neq \mathcal{V}_2 \cap \mathcal{V}_3 = \mathcal{V}_5 \\ 0 \neq \mathcal{V}_2 \cap \mathcal{V}_4 = \mathcal{V}_6 \end{cases}\right]$, also $\mathcal{V}_3 \leq_{sem} \mathcal{W} = \mathbb{N}_{30}$, since $\begin{bmatrix} \begin{cases} 0 \neq \mathcal{V}_3 \cap \mathcal{V}_2 = \mathcal{V}_5 \\ 0 \neq \mathcal{V}_3 \cap \mathcal{V}_4 = \mathcal{V}_7 \end{cases} \end{bmatrix}$. But $\mathcal{V}_2 \cap \mathcal{V}_3 = \mathcal{V}_5$ is not semi-essential subsemimodule, since $\mathcal{V}_5 \cap \mathcal{V}_4 =< \overline{0} >$. □

Proposition 1.2.4: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, and $\mathcal{V}_i \leq_{sem} \mathcal{W}$ for $i = 1,2$. If $(\mathcal{V}_i \cap \mathcal{B})$ for $(i = 1,2)$ is a prime subsemimodule in $\mathcal{W}$ for every prime subsemimodules $\mathcal{B}$ in $\mathcal{W}$, then $(\mathcal{V}_1 \cap \mathcal{V}_2) \leq_{sem} \mathcal{W}$.

Proof: The proof is similar to a proposition in modules, [4, p. 183]. □

Lemma 1.2.5: If $\mathcal{B}$ is a prime subsemimodule of a $\mathcal{T}$-semimodule $\mathcal{W}$ and $\mathcal{V} \leq \mathcal{W}$ such that $\mathcal{V} \nleq \mathcal{B}$. Then $(\mathcal{V} \cap \mathcal{B})$ is a prime subsemimodule of $\mathcal{V}$.

Proof: Let $v \in \mathcal{V} \leq \mathcal{W}$ and $t \in \mathcal{T}$ such that $tv \in (\mathcal{V} \cap \mathcal{B})$, then $tv \in \mathcal{V}$ and $tv \in \mathcal{B}$. Since $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$, then $v \in \mathcal{B}$ or $t\mathcal{W} \subseteq \mathcal{B}$, but $t\mathcal{V} \subseteq t\mathcal{W}$ then $t\mathcal{V} \subseteq \mathcal{B}$. Now, either ($v \in \mathcal{V}$ and $v \in \mathcal{B}$) or ($t\mathcal{V} \subseteq \mathcal{V}$ and $t\mathcal{V} \subseteq \mathcal{B}$), consequently $v \in (\mathcal{V} \cap \mathcal{B})$ or $t\mathcal{V} \subseteq (\mathcal{V} \cap \mathcal{B})$, hence $(\mathcal{V} \cap \mathcal{B})$ is a prime subsemimodule of $\mathcal{V}$. □

Corollary 1.2.6: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule such that $\mathcal{V}_i \leq_{sem} \mathcal{W}$ for $i = 1, 2$. If $\mathcal{B}$ is a prime subsemimodule of $\mathcal{W}$ with $\mathcal{V}_i \nleq \mathcal{B}$ for some $i = 1, 2$, then $\mathcal{V}_1 \cap \mathcal{V}_2$ is also semi-essential subsemimodule of $\mathcal{W}$.

Proof: It follows by (Proposition 1.2.4 and Lemma 1.2.5). □

Proposition 1.2.7: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $\mathcal{V}_1, \mathcal{V}_2$ be semi-essential subsemimodules of $\mathcal{W}$ such that $\mathcal{V}_1 \cap \mathcal{V}_2 \neq< 0 >$ and all prime subsemimodules in $\mathcal{V}_1$ are prime subsemimodules in $\mathcal{W}$, then $(\mathcal{V}_1 \cap \mathcal{V}_2)$ is a semi-essential subsemimodule of $\mathcal{W}$.

Proof: The proof is similar to a proposition in modules, [4, p. 182]. □

Proposition 1.2.8: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $\mathcal{V}_i \leq \mathcal{W}, (i = 1,2,3)$ with $\mathcal{V}_1 \leq \mathcal{V}_2 \leq \mathcal{V}_3$. If $\mathcal{V}_1 \leq_{sem} \mathcal{V}_2$ and $\mathcal{V}_2 \leq_{sem} \mathcal{V}_3$ then $\mathcal{V}_1 \leq_{sem} \mathcal{V}_3$.

Proof: The proof is similar to a proposition in modules, [4, Proposition 1.5]. □

Note that, the opposite direction in (Proposition 1.2.8) is not achieved, and the following example illustrates this.

Example 1.2.9: Consider the $\mathbb{Z}$-semimodule $\mathcal{W} = \mathbb{Z}_8 \oplus \mathbb{Z}_2$. In this semimodule, there are eleven subsemimodules which are $\mathcal{V}_1 =< (\bar{0},\bar{0}) >$ $,\mathcal{V}_2 =< (\bar{1},\bar{0}) >, \mathcal{V}_3 =< (\bar{0},\bar{1}) >, \mathcal{V}_4 =< (\bar{1},\bar{1}) >, \mathcal{V}_5 =< (\bar{2},\bar{0}) >, \mathcal{V}_6 =$ $< (\bar{2},\bar{1}) >, \mathcal{V}_7 =< (\bar{4},\bar{0}) >, \mathcal{V}_8 =< (\bar{4},\bar{1}) >, \mathcal{V}_9 =< (\bar{0},\bar{1}), (\bar{4},\bar{0}) >$ $,\mathcal{V}_{10} =< (\bar{2},\bar{0}), (\bar{4},\bar{1}) >$ and $\mathcal{V}_{11} = \mathcal{W}$. There are four prime subsemimodules of $\mathcal{W}$ which are $\mathcal{V}_2, \mathcal{V}_4, \mathcal{V}_5$ and $\mathcal{V}_{10}$, and three prime subsemimodules of the subsemimodule $\mathcal{V}_9$ which are $\mathcal{V}_3, \mathcal{V}_7$ and $\mathcal{V}_8$. Now, $\mathcal{V}_7 \leq \mathcal{V}_9 \leq \mathcal{W}$ and $\mathcal{V}_7 \leq_{sem} \mathcal{W}$, but $\mathcal{V}_7$ is not semi-essential in $\mathcal{V}_9$. □

In fact, the above-mentioned example is validated on modules, and it was studied for the first time on modules in [4].

Lemma 1.2.10: If $\mathcal{W}$ is a faithful multiplication $\mathcal{T}$-semimodule and $\mathcal{V} \leq \mathcal{W}$. Then $\mathcal{V} \leq_e \mathcal{W}$ if and only if $[\mathcal{V}: y] \leq_e \mathcal{T}$ for any $y \in \mathcal{W}$.

Proof: Since $\mathcal{W}$ is a multiplication $\mathcal{T}$-semimodule, then $\mathcal{V} = I\mathcal{W}$ for some ideal I in $\mathcal{T}$ and $I \subseteq [\mathcal{V}: y]$ for all $y \in \mathcal{W}$. Now, assume that $[\mathcal{V}: y] \cap J = 0$ for some ideal J of $\mathcal{T}$. Thus $I \cap J = 0$, so $(I \cap J)\mathcal{W} =< 0 >$ then $\mathcal{V} \cap J\mathcal{W} =< 0 >$ (for, let $x \in I\mathcal{W} \cap J\mathcal{W}$ then there is $a \in I, b \in J$ and $w_1, w_2 \in \mathcal{W}$ such that $x = aw_1 = bw_2$, thus $a.b \in (I \cap J) = 0$. That is, either $a = 0 \longrightarrow aw_1 = 0$ or $b = 0 \longrightarrow bw_2 = 0$, therefore $x = 0$). Since $\mathcal{V} \leq_e \mathcal{W}$, it follows $J\mathcal{W} =< 0 >$, but $\mathcal{W}$ is a faithful $\mathcal{T}$-semimodule, then $J = 0$, therefore $[\mathcal{V}: y] \leq_e \mathcal{T}$.

Conversely if $[\mathcal{V}: y] \leq_e \mathcal{T}$, for all $y \in \mathcal{W}$, $\mathcal{V} = I\mathcal{W}$ for some I ideal of $\mathcal{T}$ and $\mathcal{V} \cap \mathcal{N} =< 0 >$ for some subsemimodule $\mathcal{N}$ of $\mathcal{W}$, then $\mathcal{N} = J\mathcal{W}$ for some $J \leq \mathcal{T}$ then $I\mathcal{W} \cap J\mathcal{W} =< 0 >$, thus $(I \cap J)\mathcal{W} =< 0 >$, but $\mathcal{W}$ is faithful $\mathcal{T}$-semimodule, then $I \cap J = 0$.

- If $[\mathcal{V}: y] \cap J \neq 0$ for all $y \in \mathcal{W}$, then for all $y \in \mathcal{W}$, there is $t \in J$ such that $0 \neq ty \in \mathcal{V}$, $\mathcal{V} \leq_e \mathcal{W}$.
- If $[\mathcal{V}: y] \cap J = 0$ for some $y \in \mathcal{W}$, then $J = 0$, hence $\mathcal{N} =< 0 >$, consequently $\mathcal{V} \leq_e \mathcal{W}$. □

<u>Lemma 1.2.11</u>: Let E be a maximal ideal of $\mathcal{T}$, with $\mathcal{T}$ is a commutative semiring then $\mathcal{T}\backslash E$ is closed under multiplication.

<u>Proof</u>: Let $s_1, s_2 \in \mathcal{T}\backslash E$, then $\mathcal{T}s_1 + E = \mathcal{T}$ thus there is $t \in \mathcal{T}, e \in E$ such that $ts_1 + e = 1$, consequently $ts_1s_2 + es_2 = s_2$. If $s_1s_2 \in E$, then $s_2 \in E$, hence $s_1s_2 \notin E$. That is $s_1s_2 \in \mathcal{T}\backslash E$. □

<u>Lemma 1.2.12</u>: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, with $\mathcal{T}$ is a commutative semiring and E a maximal ideal of $\mathcal{T}$, then, $T_E(\mathcal{W}) = ann_{\mathcal{W}}(\mathcal{T}\backslash E) = \{w \in \mathcal{W} | sw = 0 \text{ for some } s \in \mathcal{T}\backslash E\}$ is a subtractive subsemimodule of $\mathcal{W}$.

<u>Proof</u>: Since $0 \in \mathcal{W}$, $1.0 = 0$ and $1 \notin E$ then $T_E(\mathcal{W}) \neq \phi$. Let $w_1, w_2 \in T_E(\mathcal{W})$, then $s_1w_1 = 0$ and $s_2w_2 = 0$ for some $s_1, s_2 \in \mathcal{T}\backslash E$, hence $s_1s_2(w_1 + w_2) = 0$ and $s_1s_2 \in \mathcal{T}\backslash E$ by (Lemma 1.2.11), then $(w_1 + w_2) \in T_E(\mathcal{W})$. Let $w \in \mathcal{T}_E(\mathcal{W})$ and $t \in \mathcal{T}$, then $sw = 0$ for some $s \notin E$, hence $s(tw) = 0$, and so $tw \in T_E(\mathcal{W})$. Now, if $w_1, w_1 + w_2 \in T_E(\mathcal{W})$, then $s_1w_1 = 0$ and $s_2(w_1 + w_2) = 0$, thus $s_2s_1w_1 = 0$ and $s_1s_2w_1 + s_1s_2w_2 = 0$ then $s_1s_2w_2 = 0$. Therefore, $w_2 \in T_E(\mathcal{W})$. □

<u>Lemma 1.2.13</u>: Let $\mathcal{T}$ be a semisubtractive commutative semiring with identity. Then a $\mathcal{T}$-semimodule $\mathcal{W}$ is cancellative multiplication semimodule if and only if for each maximal ideal E of $\mathcal{T}$ either $\mathcal{W} = T_E(\mathcal{W})$ or $\mathcal{W}$ is E-cyclic.

Proof:

(⟹) Assume that $\mathcal{W}$ is cancellative multiplication and E is a maximal ideal of $\mathcal{T}$. Suppose that $E\mathcal{W} = \mathcal{W}$. Let $w \in \mathcal{W}$, then $\mathcal{T}w = \mathcal{A}\mathcal{W}$ for some ideal $\mathcal{A}$ of $\mathcal{T}$. Hence $\mathcal{T}w = \mathcal{A}\mathcal{W} = \mathcal{A}E\mathcal{W} = E\mathcal{A}\mathcal{W} = Ew$ thus $w = ew$ for some $e \in E$. (since $\mathcal{T}$ is semisubtractive), we have either $1 = s + e$ or $1 + s = e$ for some $s \in \mathcal{T}$ (then $s \in \mathcal{T} \setminus E$) $\longrightarrow$

$$\begin{cases} \text{either } sw + ew = ew \longrightarrow sw = 0 \\ \text{or} \quad w = w + sw \longrightarrow sw = 0 \end{cases}, \longrightarrow w \in T_E(\mathcal{W}) \longrightarrow \mathcal{W} = T_E(\mathcal{W}).$$

If $E\mathcal{W} \neq \mathcal{W}$, there exists $x \in \mathcal{W}$ and $x \notin E\mathcal{W}$, then there is $I \leq \mathcal{T}$ such that $\mathcal{T}x = I\mathcal{W}$. Clearly $I \nleq E$, hence there exists $s \in I$ and $s \notin E$ thus $s\mathcal{W} \subseteq \mathcal{T}x$, therefore $\mathcal{W}$ is E-cyclic.

(⟸) by [25, Theorem 3]. □

Lemma 1.2.14: Let E be a prime ideal of a commutative semiring $\mathcal{T}$, with $\mathcal{T}$ is semisubtractive and $\mathcal{W}$ is cancellative faithful multiplication $\mathcal{T}$-semimodule. Let $a \in \mathcal{T}, x \in \mathcal{W}$ satisfying $ax \in E\mathcal{W}$, then $a \in E$ or $x \in E\mathcal{W}$.

Proof: Assume that $a \notin E$. Let $K = \{t \in \mathcal{T} | tx \in E\mathcal{W}\}$. If $K = \mathcal{T}$, then $x \in E\mathcal{W}$. If not, then there exists a maximal ideal Q of $\mathcal{T}$ such that $K \subseteq Q$. Clearly $x \notin T_Q(\mathcal{W})$ (since if $sx = 0 \in E\mathcal{W} \longrightarrow s \in K \subseteq Q$). By (Lemma 1.2.13) then $\mathcal{W}$ is Q-cyclic, that is there exists $w \in \mathcal{W}, q \notin Q$ such that $q\mathcal{W} \subseteq \mathcal{T}w$. In particular $qx = sw$, and $qax = ew$ for some $s \in \mathcal{T}$ and $e \in E$. Thus $saw = ew \longrightarrow \exists\, h \in ann_{\mathcal{T}}(w)$ such that $sa + h = e$ or $sa = e + h$. But $\mathcal{W}$ is faithful, hence $h = 0$, so $sa = e \in E \longrightarrow s \in E$ (since E is prime ideal and $a \notin E$). Then $qx = sw \in E\mathcal{W} \longrightarrow q \in K \subseteq Q$, a contradiction. It follows that $K = \mathcal{T}$ and $x \in E\mathcal{W}$, as required. □

Corollary 1.2.15: Let $\mathcal{W}$ be a faithful cancellative multiplication $\mathcal{T}$-semimodule and $\mathcal{T}$ is a semisubtractive commutative, then E is a prime ideal of $\mathcal{T}$ with $\mathcal{W} \neq E\mathcal{W}$ if and only if $E\mathcal{W}$ is a prime subsemimodule of $\mathcal{W}$.

Proof: Since $\mathcal{W}$ is a multiplication $\mathcal{T}$-semimodule and $\mathcal{W} \neq E\mathcal{W}$ then there is $\mathcal{V} \leq \mathcal{W}$ such that $\mathcal{V} = E\mathcal{W}$.

($\Longrightarrow$) Assume that E is a prime ideal of $\mathcal{T}$, to prove $\mathcal{V} = E\mathcal{W}$ is a prime subsemimodule of $\mathcal{W}$. Let $t \in \mathcal{T}, w \in \mathcal{W}$ and $tw \in \mathcal{V} = E\mathcal{W}$, by (Lemma 1.2.14) then $t \in E$ or $w \in E\mathcal{W} = \mathcal{V}$, but E is a prime ideal of $\mathcal{T}$, then $t.s \in E$ for any $s \in \mathcal{T} \Longrightarrow tw \in E\mathcal{W}$ for any $w \in \mathcal{W}$, therefore $t\mathcal{W} \subseteq E\mathcal{W} = \mathcal{V} \Longrightarrow E\mathcal{W} = \mathcal{V}$ is a prime subsemimodule of $\mathcal{W}$.

($\Longleftarrow$) Suppose that, $\mathcal{V} = E\mathcal{W}$ is a prime subsemimodule of $\mathcal{W}$, to prove E is a prime ideal of $\mathcal{T}$. Let $t_1, t_2 \in \mathcal{T}$ and $t_1.t_2 \in E$, let $w \in \mathcal{W}$ then $(t_1.t_2)w \in E\mathcal{W} = \mathcal{V}$ implies $t_1(t_2 w) \in E\mathcal{W} = \mathcal{V}$, by (Lemma 1.2.14) then $t_1 \in E$ or $t_2 w \in E\mathcal{W} = \mathcal{V}$. But $\mathcal{V}$ is a prime subsemimodule in $\mathcal{W}$, then $w \in \mathcal{V}$ or $t_2 \in [\mathcal{V}: \mathcal{W}]$. Thus, E is a prime ideal of $\mathcal{T}$. □

Theorem 1.2.16: If $\mathcal{W}$ is a faithful cancellative multiplication $\mathcal{T}$-semimodule and $\mathcal{T}$ is a semisubtractive commutative semiring with identity. Then $\mathcal{V} \leq_{sem} \mathcal{W}$ if and only if $[\mathcal{V}: y] \leq_{sem} \mathcal{T}$ for any $y \in \mathcal{W}$.

Proof:

($\Longrightarrow$) Let $\mathcal{V} \leq_{sem} \mathcal{W},\ \mathcal{V} = I\mathcal{W}$, then $I \subseteq [\mathcal{V}: y]$ for each $y \in \mathcal{W}$. To prove $[\mathcal{V}: y] \leq_{sem} \mathcal{T}$.

Suppose that, $[\mathcal{V}: y] \cap E = 0$ for some prime ideal E of $\mathcal{T}$, then $I \cap E = 0$ (since $I \subseteq [\mathcal{V}: y]$), then $(I \cap E)\mathcal{W} =< 0 >$. Thus $\mathcal{V} \cap E\mathcal{W} =< 0 >$ (for, let $x \in I\mathcal{W} \cap E\mathcal{W}$ then there is $a \in I, b \in E$ and $w_1, w_2 \in \mathcal{W}$ such that $x = aw_1 = bw_2$, thus $a.b \in (I \cap E) = 0$. That is, either $a = 0 \longrightarrow aw_1 = 0$ or $b = 0 \longrightarrow bw_2 = 0$, therefore $x = 0$). However, by (Corollary

1.2.15), then $E\mathcal{W}$ is a prime subsemimodule of $\mathcal{W}$. Hence $E\mathcal{W} =< 0 >$, since $\mathcal{W}$ is faithful $\mathcal{T}$-semimodule, thus $E = 0$. Therefore, $[\mathcal{V}: y] \leq_{sem} \mathcal{T}$.

($\Leftarrow$) If $[\mathcal{V}: y] \leq_{sem} \mathcal{T}\ \forall\ y \in \mathcal{W}, \mathcal{V} = I\mathcal{W}$ and $\mathcal{V} \cap \mathcal{B} =< 0 >$ for some prime subsemimodule $\mathcal{B}$ of $\mathcal{W}$, then $\mathcal{B} = E\mathcal{W}$ for some prime ideal E of $\mathcal{T}$. Now, $\mathcal{V} \cap \mathcal{B} =< 0 >$ then $I\mathcal{W} \cap E\mathcal{W} =< 0 >$, implies $(I \cap E)\mathcal{W} =< 0 >$, since $\mathcal{W}$ is faithful $\mathcal{T}$-semimodule, thus $I \cap E = 0$.

- If $[\mathcal{V}: y] \cap E = 0$ for some $y \in \mathcal{W}$, then $E = 0$ (since $[\mathcal{V}: y] \leq_{sem} \mathcal{T}$).
- If $[\mathcal{V}: y] \cap E \neq 0$ for all $y \in \mathcal{W}$, this mean $\forall\ y \in \mathcal{W},\ \exists\ t \in E$ such that $0 \neq ty \in \mathcal{V}$, then $\mathcal{V} \leq_{sem} \mathcal{W}$.

In any case $\mathcal{V} \leq_{sem} \mathcal{W}$. □

Although the above theorem was previously studied in [18], the proof was written, because the proof is different, so it was proved for the sake of integration without going out to other concepts that we did not touch.

Lemma 1.2.17:[27] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $\mathcal{V} \leq \mathcal{W}$ and $\mathcal{B}$ is a prime subsemimodule of $\mathcal{W}$. If $(\mathcal{V} \cap \mathcal{B}: y) = ann(\mathcal{W})$, each $y \in \mathcal{W}$ and $y \notin (\mathcal{V} \cap \mathcal{B})$, then $(\mathcal{V} \cap \mathcal{B})$ is a prime subsemimodule of $\mathcal{W}$.

The intersection of two semi-essential subsemimodules is not necessarily semi-essential, we have shown this in (Example 1.2.3). But, by adding the condition in the following proposition, it becomes semi-essential.

Proposition 1.2.18: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $\mathcal{V}_1, \mathcal{V}_2$ semi-essential subsemimodules of $\mathcal{W}$, if $(\mathcal{V}_i \cap \mathcal{B}: y) = ann(\mathcal{W}), i = 1,2$ for each prime subsemimodule $\mathcal{B}$ of $\mathcal{W}$, each $y \in \mathcal{W}$ and $y \notin (\mathcal{V}_i \cap \mathcal{B})$ for some $i = 1, 2$, then $(\mathcal{V}_1 \cap \mathcal{V}_2)$ is a semi-essential subsemimodule of $\mathcal{W}$.

Proof: Let $\mathcal{B}$ be a prime subsemimodule of $\mathcal{W}$ such that $(\mathcal{V}_1 \cap \mathcal{V}_2) \cap \mathcal{B} =< 0 >$, implies $\mathcal{V}_2 \cap (\mathcal{V}_1 \cap \mathcal{B}) =< 0 >$. Then by (Lemma 1.2.17) $\mathcal{V}_1 \cap \mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$, since $\mathcal{V}_2 \leq_{sem} \mathcal{W}$, then $\mathcal{V}_1 \cap \mathcal{B} =< 0 >$, but $\mathcal{V}_1 \leq_{sem} \mathcal{W}$ then $\mathcal{B} =< 0 >$, therefore $(\mathcal{V}_1 \cap \mathcal{V}_2) \leq_{sem} \mathcal{W}$. □

Lemma 1.2.19: Let $\mathcal{W}, \mathcal{W}'$ be a $\mathcal{T}$-semimodule, $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ be a $\mathcal{T}$-homomorphism. If $\mathcal{B}'$ is a prime subsemimodule of $\mathcal{W}'$ then $\Psi^{-1}(\mathcal{B}')$ is a prime subsemimodule of $\mathcal{W}$.

Proof: Let $t \in \mathcal{T}$, $w \in \mathcal{W}$ such that $tw \in \Psi^{-1}(\mathcal{B}')$ then $\Psi(tw) \in \mathcal{B}'$, so $t\Psi(w) \in \mathcal{B}'$. But $\mathcal{B}'$ is a prime subsemimodule in $\mathcal{W}'$ then either $\Psi(w) \in \mathcal{B}'$ or $t\mathcal{W}' \subseteq \mathcal{B}'$, therefore $w \in \Psi^{-1}(\mathcal{B}')$ or $\Psi^{-1}(t\mathcal{W}') = t\mathcal{W} \subseteq \Psi^{-1}(\mathcal{B}')$, consequently $\Psi^{-1}(\mathcal{B}')$ is a prime subsemimodule of $\mathcal{W}$. □

Note: If $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$, then $\Psi(\mathcal{B})$ is not necessary be a prime subsemimodule in $\mathcal{W}'$. For example: $\Psi: \mathbb{N}_{30} \longrightarrow \mathbb{N}_{30}$ define by $\Psi(\bar{x}) = 3\bar{x}$ for all $x \in \mathbb{N}_{30}$. Put $\mathcal{B} =< \bar{2} >$ is a prime subsemimodule of $\mathbb{N}_{30}$. But, $\Psi(\mathcal{B}) =< \bar{6} >$ is not prime.

In the following result, we will show that if $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$, then $\Psi(\mathcal{B})$ is also prime subsemimodule in $\mathcal{W}'$ under a given condition.

Lemma 1.2.20: Let $\mathcal{W}, \mathcal{W}'$ be $\mathcal{T}$-semimodules, $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ is k-regular epimorphism and $\mathcal{B}$ is a prime subtractive subsemimodule of $\mathcal{W}$ such that $ker(\Psi) \leq \mathcal{B}$, then $\Psi(\mathcal{B})$ is a prime subsemimodule of $\mathcal{W}'$.

Proof: Let $t \in \mathcal{T}$, $w' \in \mathcal{W}'$ such that $tw' \in \Psi(\mathcal{B})$, since Ψ is onto then $w' = \Psi(w)$ for some $w \in \mathcal{W}$, thus $tw' = t\Psi(w) = \Psi(tw) \in \Psi(\mathcal{B})$, thus $\Psi(tw) = \Psi(b)$ for some $b \in \mathcal{B}$. Since Ψ is k-regular then $tw + k_1 = b + k_2$ for some $k_1, k_2 \in ker(\Psi)$, but $ker(\Psi) \leq \mathcal{B}$ and $\mathcal{B}$ is subtractive, thus $tw \in \mathcal{B}$. Since $\mathcal{B}$ is a prime subsemimodule of $\mathcal{W}$, then either $w \in \mathcal{B}$ or $t\mathcal{W} \subseteq \mathcal{B}$ thus $w' = \Psi(w) \in \Psi(\mathcal{B})$ or $t\mathcal{W}' \subseteq \Psi(\mathcal{B})$. Therefore, $\Psi(\mathcal{B})$ is a prime subsemimodule of $\mathcal{W}'$. □

Remark 1.2.21: If $\phi: \mathcal{W} \longrightarrow \frac{\mathcal{W}}{\mathcal{N}}$ is the natural epimorphism where $\mathcal{N}$ is subtractive then ϕ is k-regular.

Proof: Let $w_1, w_2 \in \mathcal{W}$ such that $\phi(w_1) = \phi(w_2)$, since $\phi(w_1) \in w_1 + \mathcal{N}, \phi(w_2) \in w_2 + \mathcal{N}$ then there is $n_1, n_2 \in \mathcal{N}$ such that $\phi(w_1) = w_1 + n_1$ and $\phi(w_2) = w_2 + n_2$. Hence $w_1 + n_1 = w_2 + n_2$, since ϕ is natural epimorphism then $kef(\phi) = \mathcal{N}$, thus $w_1 + n_1 = w_2 + n_2$ for some $n_1, n_2 \in kef(\phi)$. Therefore, ϕ is k-regular. □

Corollary 1.2.22: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $\mathcal{V} \leq \mathcal{B} \leq \mathcal{W}$, where $\mathcal{V}$ is subtractive, and $\phi: \mathcal{W} \longrightarrow \frac{\mathcal{W}}{\mathcal{V}}$ be the natural map, then $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$ if and only if $\frac{\mathcal{B}}{\mathcal{V}}$ is a prime subsemimodule in $\frac{\mathcal{W}}{\mathcal{V}}$.

Proof: Let $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$, by (Remark 1.2.21) then ϕ is k-regular and by (Lemma 1.2.20) then $\frac{\mathcal{B}}{\mathcal{V}}$ is a prime subsemimodule in $\frac{\mathcal{W}}{\mathcal{V}}$.

Conversely, by (Lemma 1.2.19) then $\phi^{-1}\left(\frac{\mathcal{B}}{\mathcal{V}}\right) = \mathcal{B}$ (where ϕ is the natural map of $\mathcal{W}$ onto $\frac{\mathcal{W}}{\mathcal{V}}$) is a prime subsemimodule of $\mathcal{W}$. □

Proposition 1.2.23: Let $\mathcal{W}, \mathcal{W}'$ be $\mathcal{T}$-semimodules, $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ be an isomorphism. If $\mathcal{V} \leq_{sem} \mathcal{W}$, then $\Psi(\mathcal{V}) \leq_{sem} \mathcal{W}'$.

Proof: The proof is similar to a proposition in modules, [4, p. 181]. □

Lemma 1.2.24: Let $\mathcal{W}$ be a semisubtractive, $\mathcal{W}'$ cancellative $\mathcal{T}$-semimodules and $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ be a homomorphism of semimodules. If $\mathcal{B}$ is a subtractive subsemimodule of $\mathcal{W}$ such that $ker\Psi \subseteq \mathcal{B}$, then $\Psi^{-1}\big(\Psi(\mathcal{B})\big) = \mathcal{B}$.

Proof: $\mathcal{B} \subseteq \Psi^{-1}(\Psi(\mathcal{B}))$ in general, true.

Let $w \in \Psi^{-1}(\Psi(\mathcal{B}))$, then $\Psi(w) \in \Psi(\mathcal{B})$, that is $\Psi(w) = \Psi(b)$ for some $b \in \mathcal{B}$, since $\mathcal{W}$ is semisubtractive, then there is $h \in \mathcal{W}$ such that $w + h = b$ or $w = b + h$, in any case $h \in ker\Psi$ (since, $\mathcal{W}'$ is cancellative), hence $h \in \mathcal{B}$ (By hypothesis), so $w = b + h \longrightarrow w \in \mathcal{B}, w + h = b \longrightarrow w \in \mathcal{B}$ (since $\mathcal{B}$ is a subtractive). Therefore $\Psi^{-1}(\Psi(\mathcal{B})) = \mathcal{B}$. □

Note:

1. If $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ such that Ψ is onto then $[\mathcal{B}: \mathcal{W}] \subseteq [\Psi(\mathcal{B}): \mathcal{W}']$.
2. A $\mathcal{T}$-semimodule $\mathcal{W}$ is called (s-P) $\mathcal{T}$-semimodule if any prime subsemimodule of $\mathcal{W}$ is subtractive in $\mathcal{W}$.

Lemma 1.2.25: Let $\mathcal{W}$ be an (s-P) semisubtractive, $\mathcal{W}'$ cancellative $\mathcal{T}$-semimodules and $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ an epimorphism of semimodule. If $\mathcal{B}$ is a prime subsemimodule of $\mathcal{W}$ such that $ker\Psi \subseteq \mathcal{B}$, then $\Psi(\mathcal{B})$ is a prime subsemimodule of $\mathcal{W}'$.

Proof: Let $t \in \mathcal{T}, w' \in \mathcal{W}'$ and $tw' \in \Psi(\mathcal{B})$, since Ψ is epimorphism, then there is $w \in \mathcal{W}$ such that $\Psi(w) = w'$, thus $t\Psi(w) \in \Psi(\mathcal{B})$ implies $\Psi(tw) \in \Psi(\mathcal{B}) \longrightarrow tw \in \Psi^{-1}(\Psi(\mathcal{B}))$, by (Lemma 1.2.24) then $tw \in \mathcal{B}$. Since $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$, then either $w \in \mathcal{B}$ or $t \in [\mathcal{B}: \mathcal{W}]$, thus either $w' \in \Psi(\mathcal{B})$ or $t \in [\Psi(\mathcal{B}): \mathcal{W}']$. Therefore, $\Psi(\mathcal{B})$ is a prime subsemimodule of $\mathcal{W}'$. □

Proposition 1.2.26: Let $\mathcal{W}$ be an (s-P) semisubtractive, $\mathcal{W}'$ cancellative $\mathcal{T}$-semimodules and $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ be a $\mathcal{T}$-epimorphism such that $ker(\Psi) \subseteq rad(\mathcal{W})$. If $\mathcal{H} \leq_{sem} \mathcal{W}'$, then $\Psi^{-1}(\mathcal{H}) \leq_{sem} \mathcal{W}$.

Proof: Let $\mathcal{B}$ be a prime subsemimodule of $\mathcal{W}$ such that $\Psi^{-1}(\mathcal{H}) \cap \mathcal{B} =< 0 >$, by (Lemma 1.2.24) then $\Psi^{-1}(\mathcal{H}) \cap \Psi^{-1}(\Psi(\mathcal{B})) =< 0 >$ implies $\Psi^{-1}(\mathcal{H} \cap \Psi(\mathcal{B})) =< 0 >$, thus $\mathcal{H} \cap \Psi(\mathcal{B}) =< 0 >$, since $ker(\Psi) \subseteq rad(\mathcal{W}) \subseteq \mathcal{B}$, for every prime subsemimodule $\mathcal{B}$ of $\mathcal{W}$ and by (Lemma 1.2.25) then, $\Psi(\mathcal{B})$ is a prime subsemimodule of $\mathcal{W}'$. However, $\mathcal{H} \leq_{sem} \mathcal{W}'$ then $\Psi(\mathcal{B}) = 0$. Thus $\mathcal{B} \subseteq ker(\Psi) \subseteq \Psi^{-1}(\mathcal{H})$, hence $\mathcal{B} = \Psi^{-1}(\mathcal{H}) \cap \mathcal{B} =< 0 >$. Therefore, $\Psi^{-1}(\mathcal{H}) \leq_{sem} \mathcal{W}$. □

Remark 1.2.27: If $\mathcal{B}$ is a prime subsemimodule of $\mathcal{T}$-semimodule $\mathcal{W}$, and $\mathcal{B} \lneqq \mathcal{B}' \leq \mathcal{W}$, then $\mathcal{B}$ is a prime subsemimodule of $\mathcal{B}'$.

Proposition 1.2.28: Let $\mathcal{W}$ be a finitely generated faithful cancellative and multiplication $\mathcal{T}$-semimodule, $\mathcal{T}$ is a semisubtractive commutative semiring with identity then $I \leq_{sem} J$ if and only if $I\mathcal{W} \leq_{sem} J\mathcal{W}$ for every two ideals I and J of $\mathcal{T}$.

Proof:

($\Longrightarrow$) Assume that, $I \leq_{sem} J$, to prove $I\mathcal{W} \leq_{sem} J\mathcal{W}$. Let $\mathcal{B}$ be a prime subsemimodule in $J\mathcal{W}$ such that $I\mathcal{W} \cap \mathcal{B} =< 0 >$. Since $\mathcal{W}$ is a multiplication $\mathcal{T}$-semimodule, then there exists a prime ideal E of $\mathcal{T}$ such that $\mathcal{B} = E\mathcal{W}$. Now, $< 0 >= I\mathcal{W} \cap E\mathcal{W} \supseteq (I \cap E)\mathcal{W}$ thus $(I \cap E)\mathcal{W} =< 0 >$, but $\mathcal{W}$ is a faithful $\mathcal{T}$-semimodule, then $(I \cap E) = 0$. Since $\mathcal{B} = E\mathcal{W} \lneqq J\mathcal{W}$ and $\mathcal{W}$ is a finitely generated $\mathcal{T}$-semimodule, then $E < J$ and since E is a prime in $\mathcal{T}$ by (Remark 1.2.27), then E is a prime in J. However, $I \leq_{sem} J$, thus $E = 0$. Therefore, $I\mathcal{W} \leq_{sem} J\mathcal{W}$.

($\Longleftarrow$) Assume that, $I\mathcal{W} \leq_{sem} J\mathcal{W}$, to prove $I \leq_{sem} J$. Let E be a prime ideal of J such that $(I \cap E) = 0$, then $(I \cap E)\mathcal{W} = 0.\mathcal{W}$, see proof (Theorem 1.2.16), then $I\mathcal{W} \cap E\mathcal{W} =< 0 >$. Since E is a prime ideal of J and by (Corollary 1.2.15), then $E\mathcal{W}$ is a prime subsemimodule in $J\mathcal{W}$, but

$I\mathcal{W} \leq_{sem} J\mathcal{W}$, then $E\mathcal{W} =< 0 >$ and since $\mathcal{W}$ is a faithful $\mathcal{T}$-semimodule, then $E = 0$. Therefore, $I \leq_{sem} J$. □

1.3. Fully prime semimodule, fully essential semimodule and semi-relative complement

The main objective of this section is to generalize the definition of each (semi-uniform, fully prime, fully essential and semi-complement) on semimodules after studying the above-mentioned definitions in modules. See [3],[14],[23].

Definition 1.3.1: A $\mathcal{T}$-semimodule $\mathcal{W}$ is called semi-uniform if any subsemimodule $\mathcal{V}$ of $\mathcal{W}$ is a semi-essential subsemimodule in $\mathcal{W}$.

Example 1.3.2:

1. Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{36}$ (the semimodule of naturals modulo 36). In semimodule $\mathbb{N}_{36}$, there are nine subsemimodules, which are $\mathcal{V}_1 =< \overline{0} >, \mathcal{V}_2 =< \overline{2} >, \mathcal{V}_3 =< \overline{3} >, \mathcal{V}_4 =< \overline{4} >, \mathcal{V}_5 =< \overline{6} >, \mathcal{V}_6 =< \overline{9} >, \mathcal{V}_7 =< \overline{12} >, \mathcal{V}_8 =< \overline{18} >$ and $\mathcal{V}_9 = \mathbb{N}_{36}$. And there are two prime subsemimodules which are $\mathcal{V}_2$, and $\mathcal{V}_3$. Now, $\mathcal{V}_i \cap \mathcal{V}_2 \neq 0$ and $\mathcal{V}_i \cap \mathcal{V}_3 \neq 0$, for $i = 2,3, \dots ,9$ then $\mathcal{V}_i \leq_{sem} \mathbb{N}_{36}$. Therefore, $\mathbb{N}$-semimodule $\mathbb{N}_{36}$ is semi-uniform. While it is not uniform ($\mathcal{V}_4 \cap \mathcal{V}_6 = 0$).
2. Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{24}$ (the semimodule of naturals modulo 24). In semimodule $\mathbb{N}_{24}$, there are eight subsemimodules which are $\mathcal{V}_1 =< \overline{0} >, \mathcal{V}_2 =< \overline{2} >, \mathcal{V}_3 =< \overline{3} >, \mathcal{V}_4 =< \overline{4} >, \mathcal{V}_5 =< \overline{6} >, \mathcal{V}_6 =< \overline{8} >, \mathcal{V}_7 =< \overline{12} >$ and $\mathcal{V}_8 = \mathbb{N}_{24}$. And there are two prime subsemimodules which are $\mathcal{V}_2$, and $\mathcal{V}_3$. Now, $\mathcal{V}_8 \cap \mathcal{V}_3 = 0$ then $\mathcal{V}_8$ is not semi-essential in $\mathbb{N}_{24}$. Therefore $\mathbb{N}$-semimodule $\mathbb{N}_{24}$ is not semi-uniform.

Definition 1.3.3: A $\mathcal{T}$-semimodule $\mathcal{W}$ is called fully prime if for each nonzero proper subsemimodule $\mathcal{V}$ of $\mathcal{W}$ is a prime subsemimodule in $\mathcal{W}$.

Example 1.3.4:

1. Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{15}$ (the semimodule of naturals modulo 15). In semimodule $\mathbb{N}_{15}$, there are two a nonzero proper subsemimodules of $\mathbb{N}_{15}$ which are $\mathbb{N}_1 =< \overline{3} >$ and $\mathcal{V}_2 =< \overline{5} >$. As, they are prime subsemimodules in $\mathbb{N}_{15}$. Therefore $\mathbb{N}$-semimodule $\mathbb{N}_{15}$ is fully prime semimodule.
2. Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{24}$ (the semimodule of naturals modulo 24). See (Example 1.3.2 (2)), note, $\mathcal{V}_6 =< \overline{8} >$ is not prime subsemimodule in $\mathbb{N}_{24}$. Therefore $\mathbb{N}$-semimodule $\mathbb{N}_{24}$ is not fully prime.

Definition 1.3.5: A nonzero $\mathcal{T}$-semimodule $\mathcal{W}$ is called fully essential, if every nonzero semi-essential subsemimodule of $\mathcal{W}$ is an essential subsemimodule in $\mathcal{W}$.

Example 1.3.6:

1. Consider the $\mathbb{N}$-semimodule $\mathbb{N}_8$ (the semimodule of naturals modulo 8). In semimodule $\mathbb{N}_8$, there are two a nonzero proper subsemimodules of $\mathbb{N}_8$ which are $\mathcal{V}_1 =< \overline{2} >$ and $\mathcal{V}_2 =< \overline{4} >$. Now, $\mathcal{V}_1 =< \overline{2} > \leq_{sem} \mathbb{N}_8 \longrightarrow < \overline{2} > \leq_e \mathbb{N}_8$ And $\mathcal{V}_2 =< \overline{4} > \leq_{sem} \mathbb{N}_8 \longrightarrow < \overline{4} > \leq_e \mathbb{N}_8$. Therefore $\mathbb{N}$-semimodule $\mathbb{N}_8$ is fully essential semimodule.
2. Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{24}$ (the semimodule of naturals modulo 24). See (Example 1.3.2 (2)), note, $\mathcal{V}_5 =< \overline{6} >$ is semi-essential of $\mathbb{N}_{24}$. But, $\mathcal{V}_5 =< \overline{6} >$ is not essential subsemimodule in

$\mathbb{N}_{24}$, (since, $<\overline{6}>\cap<\overline{8}>=0$. Therefore $\mathbb{N}$-semimodule $\mathbb{N}_{24}$ is not fully essential.

Definition 1.3.7: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule and $\mathcal{V} \leq \mathcal{W}$. A prime subsemimodule $\mathcal{H}$ of $\mathcal{W}$ is called semi-relative intersection complement (shortly semi complement) of $\mathcal{V}$ in $\mathcal{W}$ if $\mathcal{V} \cap \mathcal{H} = 0$ and whenever $\mathcal{V} \cap \mathcal{B} = 0$ with $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$ such that $\mathcal{H} \subseteq \mathcal{B}$, then $\mathcal{H} = \mathcal{B}$.

Example 1.3.8: Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{24}$ (the semimodule of naturals modulo 24). See (Example 1.3.2 (2)), note, $\mathcal{V}_3 =<\overline{3}>$ is semi complement of $\mathcal{V}_6 =<\overline{8}>$ in $\mathbb{N}_{24}$, (since $<\overline{3}>$ is a prime subsemimodule in $\mathbb{N}_{24}$ and $<\overline{8}>\cap<\overline{3}>=0$).

Remark 1.3.9: If $\mathcal{W}$ is a fully prime $\mathcal{T}$-semimodule and $<0>\neq \mathcal{V} \leq \mathcal{H} \leq \mathcal{W}$, then $\mathcal{V}$ is semi-essential subsemimodule of $\mathcal{H}$ if and only if it is essential subsemimodule of $\mathcal{H}$.

Remark 1.3.10: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule. Then $\mathcal{W}$ is a uniform $\mathcal{T}$-semimodule if and only if $\mathcal{W}$ is a semi-uniform and fully essential $\mathcal{T}$-semimodule.

Proof: The proof is similar to a proposition in modules, see [4, p. 182]. □

Proposition 1.3.11: If $\mathcal{W}$ is a finitely generated faithful cancellative and multiplication $\mathcal{T}$-semimodule, with $\mathcal{T}$ is semisubtractive. Then $\mathcal{W}$ is a semi-uniform $\mathcal{T}$-semimodule if and only if $\mathcal{T}$ is a semi-uniform semiring.

Proof: By using (Proposition 1.2.28). □

Corollary 1.3.12: If $\mathcal{W}$ is a fully prime $\mathcal{T}$-semimodule. Then $\mathcal{W}$ is a uniform $\mathcal{T}$-semimodule if and only if $\mathcal{W}$ is a semi-uniform $\mathcal{T}$-semimodule.

Proof: The proof is similar to a corollary in modules, [4, p. 182]. □

Proposition 1.3.13: Let $\mathcal{W}$ be a nonzero faithful cancellative and multiplication $\mathcal{T}$-semimodule, $\mathcal{T}$ is a semisubtractive commutative semiring with identity. Then $\mathcal{W}$ is fully essential $\mathcal{T}$-semimodule if and only if $\mathcal{T}$ is fully essential semiring.

Proof:

($\Longrightarrow$) Assume that, $\mathcal{W}$ is fully essential $\mathcal{T}$-semimodule. Let $I \leq_{sem} \mathcal{T}$, since $\mathcal{W}$ is a multiplication $\mathcal{T}$-semimodule, then there is $\mathcal{V} \leq \mathcal{W}$ such that $\mathcal{V} = I\mathcal{W}$ and $I \subseteq [\mathcal{V}: y]$ for each $y \in \mathcal{W}$, thus, $[\mathcal{V}: y] \leq_{sem} \mathcal{T}$ $\forall\, y \in \mathcal{W}$, by (Theorem 1.2.16) then: $\mathcal{V} \leq_{sem} \mathcal{W}$, by assumption, then $\mathcal{V} \leq_e \mathcal{W}$ and by (Lemma 1.2.10) then $[\mathcal{V}: y] \leq_e \mathcal{T}$ $\forall\, y \in \mathcal{W}$. If $J \cap I = 0$, where $J \leq \mathcal{T}$, either $J \cap [\mathcal{V}: y] = 0$ for some $y \in \mathcal{W}$, then $J = 0$, or $J \cap [\mathcal{V}: y] \neq 0$ for any $y \in \mathcal{W}$, that is for all $y \in \mathcal{W}$, there is $t \in J$ such that $0 \neq ty \in \mathcal{V} = I\mathcal{W}$, hence $J \cap I \neq 0$, not possible. Therefore $\mathcal{T}$ is a fully essential semiring.

($\Longleftarrow$) Assume that, $\mathcal{T}$ is a fully essential semiring. Let $\mathcal{V} \leq_{sem} \mathcal{W}$, then: $[\mathcal{V}: y] \leq_{sem} \mathcal{T}$ for all $y \in \mathcal{W}$ then by assumption $[\mathcal{V}: y] \leq_e \mathcal{T}$, $\forall\, y \in \mathcal{W}$, by (Lemma 1.2.10), then $\mathcal{V} \leq_e \mathcal{W}$. Therefore, $\mathcal{W}$ is a fully essential $\mathcal{T}$-semimodule. □

Proposition 1.3.14: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a $\mathcal{T}$-semimodule, where $\mathcal{W}_i \leq \mathcal{W}$ and $< 0 > \neq \mathcal{V}_i \leq \mathcal{W}_i$, for $(i = 1, 2)$. If $\mathcal{V}_i \leq_{sem} \mathcal{W}_i$, $(i = 1, 2)$ then $(\mathcal{V}_1 \oplus \mathcal{V}_2) \leq_{sem} \mathcal{W}$.

Proof: Let $0 \neq \mathcal{B}$ be a prime subsemimodule in $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ and λ_i is natural projection function of $\mathcal{W}$ on to $\mathcal{W}_i$, $(i = 1, 2)$. If $\mathcal{B} \cap \mathcal{W}_1 \neq 0$, by (Lemma 1.2.5), then $\mathcal{B} \cap \mathcal{W}_1$ is a prime subsemimodule in $\mathcal{W}_1$. Since $\mathcal{V}_1 \leq_{sem} \mathcal{W}_1$, then $\mathcal{V}_1 \cap (\mathcal{B} \cap \mathcal{W}_1) \neq 0$, consequently there exists $0 \neq b \in \mathcal{V}_1 \cap \mathcal{B} \cap \mathcal{W}_1$, then $0 \neq b \in \mathcal{V}_1 \cap \mathcal{B}$. Thus $0 \neq b \in (\mathcal{V}_1 \oplus \mathcal{V}_2) \cap \mathcal{B}$.

Therefore, $(\mathcal{V}_1 \oplus \mathcal{V}_2) \cap \mathcal{B} \neq 0$. Similarly, if $\mathcal{B} \cap \mathcal{W}_2 \neq 0$, then $(\mathcal{V}_1 \oplus \mathcal{V}_2) \cap \mathcal{B} \neq 0$.

Assume that, $\mathcal{B} \cap \mathcal{W}_1 = \mathcal{B} \cap \mathcal{W}_2 = 0$, thus 0 is a prime subsemimodule in $\mathcal{W}_i$, $(i = 1, 2)$. Then any subsemimodule of $\mathcal{W}_i$ is a prime subsemimodule by (Remark 1.2.27) $(i = 1, 2)$. Then $\lambda_i(\mathcal{B})$ is a prime subsemimodule in $\mathcal{W}_i$, $i = 1,2$, since $\mathcal{V}_i \leq_{sem} \mathcal{W}_i$ then $\mathcal{V}_i \cap \lambda_i(\mathcal{B}) \neq 0$ for $(i = 1, 2)$.

Let $0 \neq b = \lambda_1(b) + \lambda_2(b) \in \mathcal{B}$, if $\lambda_2(b) = 0$ then $0 \neq b = \lambda_1(b) \in \lambda_1(\mathcal{B}) \cap \mathcal{W}_1 \subseteq \mathcal{B} \cap \mathcal{W}_1 = 0$ contradiction. Then $\lambda_2(b) \neq 0$, also $\lambda_1(b) \neq 0$, Hence $0 \neq b = \lambda_1(b) + \lambda_2(b) \in (\lambda_1(\mathcal{B}) \cap \mathcal{V}_1) \oplus (\lambda_2(\mathcal{B}) \cap \mathcal{V}_2) \subseteq \mathcal{V}_1 \oplus \mathcal{V}_2$. That is, $0 \neq b \in (\mathcal{V}_1 \oplus \mathcal{V}_2) \cap \mathcal{B}$. Therefore, $(\mathcal{V}_1 \oplus \mathcal{V}_2) \leq_{sem} \mathcal{W}$. □

Note that the opposite direction of (Proposition 1.3.14) is not achieved, and the following (Example 1.3.15) illustrates this. However, within certain conditions, the opposite direction of the (Proposition 1.3.14) becomes true, and this is what we will show in the Proposition 1.3.16, Proposition 1.3.17 and Proposition 1.3.18.

Example 1.3.15: Consider the $\mathbb{N}$-semimodule $\mathcal{W} = \mathbb{N}_4 \oplus \mathbb{N}_6$. Let $\mathcal{V}_1 =< \overline{2} > \leq \mathbb{N}_4, \mathcal{V}_2 =< \overline{2} > \leq \mathbb{N}_6$, then,

$(\mathcal{V}_1 \oplus \mathcal{V}_2) = (< \overline{2} > \oplus < \overline{2} >) \leq_{sem} (\mathbb{N}_4 \oplus \mathbb{N}_6) = (\mathcal{W}_1 \oplus \mathcal{W}_2)$ (since $((< \overline{2} > \oplus < \overline{2} >) \cap \mathcal{B} \neq 0))$ for any $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W} = \mathbb{N}_4 \oplus \mathbb{N}_6$. However, $< \overline{2} >$ is not semi-essential subsemimodule in $\mathbb{N}_6$ (since $< \overline{2} > \cap < \overline{3} > = < \overline{0} >$). □

Proposition 1.3.16: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a $\mathcal{T}$-semimodule, where $\mathcal{W}_i \leq \mathcal{W}, i = 1,2$, $< 0 > \neq \mathcal{V}_1 \leq \mathcal{W}_1$ and $< 0 > \neq \mathcal{V}_2 \leq \mathcal{W}_2$, if $(\mathcal{V}_1 \oplus \mathcal{V}_2) \leq_{sem} \mathcal{W}$ then $\mathcal{V}_1 \leq_{sem} \mathcal{W}_1$, provided that every prime subsemimodule in $\mathcal{W}_1$ is a prime subsemimodule of $\mathcal{W}$.

Proof: The proof is similar to a proposition in modules, [4, p. 182]. □

Proposition 1.3.17: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a distributive and fully essential $\mathcal{T}$-semimodule, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are subsemimodules of $\mathcal{W}$, $< 0 >\neq \mathcal{N}_i \leq \mathcal{W}_i , i = 1, 2$, then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$ if and only if $\mathcal{N}_i \leq_{sem} \mathcal{W}_i, i = 1, 2$.

Proof:

($\Longrightarrow$) Assume that, $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$, to prove $\mathcal{N}_i \leq_{sem} \mathcal{W}_i , i = 1, 2$.

Since $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$ and $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ is a fully essential $\mathcal{T}$-semimodule, then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_e \mathcal{W}$. Let $< 0 >\neq \mathcal{H}_1 \leq \mathcal{W}_1 \leq \mathcal{W}$, then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \cap \mathcal{H}_1 \neq< 0 >$, implies $(\mathcal{N}_1 \cap \mathcal{H}_1) + (\mathcal{N}_2 \cap \mathcal{H}_1) \neq< 0 >$. However, $(\mathcal{N}_2 \cap \mathcal{H}_1) =< 0 >$, conesquntly $(\mathcal{N}_1 \cap \mathcal{H}_1) \neq< 0 >$, then $\mathcal{N}_1 \leq_e \mathcal{W}_1$. Therefore, $\mathcal{N}_1 \leq_{sem} \mathcal{W}_1$. Similarly, $\mathcal{N}_2 \leq_{sem} \mathcal{W}_2$.

($\Longleftarrow$) Assume that, $\mathcal{N}_i \leq_{sem} \mathcal{W}_i, (i = 1, 2)$, to prove $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$.

By (Proposition 1.3.14), then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$. □

Proposition 1.3.18: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a distributive and fully prime $\mathcal{T}$-semimodule, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are subsemimodules of $\mathcal{W}$, $< 0 >\neq \mathcal{N}_i \leq \mathcal{W}_i \ (i = 1, 2)$, then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$ if and only if $\mathcal{N}_i \leq_{sem} \mathcal{W}_i, (i = 1, 2)$.

Proof:

($\Longrightarrow$) Assume that, $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$, to prove $\mathcal{N}_i \leq_{sem} \mathcal{W}_i \ (i = 1, 2)$.

Let $\mathcal{P}_1$ be a prime subsemimodule of $\mathcal{W}_1$ such that $\mathcal{N}_1 \cap \mathcal{P}_1 =< 0 >$, since $\mathcal{W}$ is fully prime and $\mathcal{P}_1 \leq \mathcal{W}_1 \leq \mathcal{W}$, then $\mathcal{P}_1$ is a prime subsemimodule in $\mathcal{W}$. Since $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$, thus $\mathcal{W}_1 \cap \mathcal{W}_2 =< 0 >$, implies $\mathcal{P}_1 \cap \mathcal{N}_2 =< 0 >$ for each $\mathcal{N}_2 \leq \mathcal{W}_2$. Now,

$(\mathcal{N}_1 \cap \mathcal{P}_1) \oplus (\mathcal{N}_2 \cap \mathcal{P}_1) =< 0 >$, since $\mathcal{W}$ is distributive $\mathcal{T}$-semimodule thus $(\mathcal{N}_1 \oplus \mathcal{N}_2) \cap \mathcal{P}_1 =< 0 >$, but $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$. Therefore, $\mathcal{P}_1 =< 0 >$, consequently $\mathcal{N}_1 \leq_{sem} \mathcal{W}_1$. Similarly, $\mathcal{N}_2 \leq_{sem} \mathcal{W}_2$.

($\Leftarrow$) Assume that, $\mathcal{N}_i \leq_{sem} \mathcal{W}_i, (i = 1, 2)$, to prove $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$.

By (Proposition 1.3.14) then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$. □

Proposition 1.3.19: Let $\mathcal{W}$ be an (s-P) $\mathcal{T}$-semimodule, $0 \neq \mathcal{V} \leq \mathcal{W}$ and $\mathcal{H}$ is a nonzero prime subsemimodule in $\mathcal{W}$. Then $\mathcal{H}$ is a semi- complement of $\mathcal{V}$ in $\mathcal{W}$ if and only if $\frac{(\mathcal{V} \oplus \mathcal{H})}{\mathcal{H}} \leq_{sem} \frac{\mathcal{W}}{\mathcal{H}}$.

Proof: Let $h: \mathcal{W} \longrightarrow \frac{\mathcal{W}}{\mathcal{H}}$ be the natural map.

($\Rightarrow$) Assume that, $\mathcal{H}$ is a semi-relative intersection complement of $\mathcal{V}$ in $\mathcal{W}$. Let $\frac{\mathcal{B}}{\mathcal{H}}$ be a prime subsemimodule of $\frac{\mathcal{W}}{\mathcal{H}}$ such that $\frac{(\mathcal{V} \oplus \mathcal{H})}{\mathcal{H}} \cap \frac{\mathcal{B}}{\mathcal{H}} =< 0 >$, by (Corollary 1.2.22), then $h^{-1}(\frac{\mathcal{B}}{\mathcal{H}})$ is a prime subsemimodule in $\mathcal{W}$. Then $\mathcal{B} = h^{-1}(\frac{\mathcal{B}}{\mathcal{H}})$ is prime in $\mathcal{W}$, thus $\frac{(\mathcal{V} \oplus \mathcal{H})}{\mathcal{H}} \cap \frac{\mathcal{B}}{\mathcal{H}} =< 0 >$, then $\frac{(\mathcal{V} \oplus \mathcal{H}) \cap \mathcal{B}}{\mathcal{H}} =< 0 >$,thus $(\mathcal{V} \oplus \mathcal{H}) \cap \mathcal{B} = \mathcal{H}$. Therefore $(\mathcal{V} \cap \mathcal{B}) \oplus \mathcal{H} = \mathcal{H}$, then $(\mathcal{V} \cap \mathcal{B}) \leq \mathcal{H}$. And since $(\mathcal{V} \cap \mathcal{B}) \leq \mathcal{V}$, then $(\mathcal{V} \cap \mathcal{B}) \leq (\mathcal{V} \cap \mathcal{H})$. Since $\mathcal{H}$ is a semi-relative intersection complement of $\mathcal{V}$, then $\mathcal{V} \cap \mathcal{B} =< 0 >$. However, $\mathcal{V} \cap \mathcal{H} =< 0 >$, and $\mathcal{H} \subseteq \mathcal{B}$, then $\mathcal{H} = \mathcal{B}$, consequently $\frac{\mathcal{B}}{\mathcal{H}} =< 0 >$. Therefore $\frac{(\mathcal{V} \oplus \mathcal{H})}{\mathcal{H}} \leq_{sem} \frac{\mathcal{W}}{\mathcal{H}}$.

($\Leftarrow$) Assume that $\frac{(\mathcal{V} \oplus \mathcal{H})}{\mathcal{H}} \leq_{sem} \frac{\mathcal{W}}{\mathcal{H}}$. Let $\mathcal{B}$ be a prime subsemimodule of $\mathcal{W}$ such that $\mathcal{H} \subseteq \mathcal{B}$ and $\mathcal{V} \cap \mathcal{B} =< 0 >$. Suppose that $b \in (\mathcal{V} \oplus \mathcal{H}) \cap \mathcal{B}$, thus $b = v + h$, where $v \in \mathcal{V}, h \in \mathcal{H}$ *and* $b \in \mathcal{B}$. Since $\mathcal{H} \subseteq \mathcal{B}$, then $h \in \mathcal{B}$, but $\mathcal{B}$ is subtractive subsemimodule of $\mathcal{W}$ and $b = (v + h) \in \mathcal{B}$, then $v \in \mathcal{B}$, thus $v \in \mathcal{V} \cap \mathcal{B}$, therefore $v = 0$, implies $b = h$, consequently $(\mathcal{V} \oplus \mathcal{H}) \cap \mathcal{B} = \mathcal{H}$. It follows that $\left(\frac{(\mathcal{V}\oplus\mathcal{H})}{\mathcal{H}}\right) \cap \frac{\mathcal{B}}{\mathcal{H}} =< 0 >$. However, by (Corollary 1.2.22), $\frac{\mathcal{B}}{\mathcal{H}}$ is a prime subsemimodule of $\frac{\mathcal{W}}{\mathcal{H}}$ and $\frac{(\mathcal{V}\oplus\mathcal{H})}{\mathcal{H}} \leq_{sem} \frac{\mathcal{W}}{\mathcal{H}}$, then $\frac{\mathcal{B}}{\mathcal{H}} =< 0 >$, implies $\mathcal{B} = \mathcal{H}$. Therefore, $\mathcal{H}$ is a semi-relative intersection complement of $\mathcal{V}$ in $\mathcal{W}$. □

Proposition 1.3.20: Let $\mathcal{W}$ be an (s-P) $\mathcal{T}$-semimodule, $< 0 >\neq \mathcal{V} \leq \mathcal{W}$. If $\mathcal{U}$ is a semi-complement of $\mathcal{V}$ in $\mathcal{W}$, then $(\mathcal{V} \oplus \mathcal{U}) \leq_{sem} \mathcal{W}$.

Proof: Let $h: \mathcal{W} \longrightarrow \frac{\mathcal{W}}{\mathcal{U}}$ be the natural map. Since $\mathcal{U}$ is a semi-complement of $\mathcal{V}$ in $\mathcal{W}$, then by (Proposition 1.3.19) $\frac{(\mathcal{V}\oplus\mathcal{U})}{\mathcal{U}} \leq_{sem} \frac{\mathcal{W}}{\mathcal{U}}$ and so by (Proposition 1.2.26), $h^{-1}\left(\frac{\mathcal{V}\oplus\mathcal{U}}{\mathcal{U}}\right) \leq_{sem} \mathcal{W}$, implies $(\mathcal{V}\oplus\mathcal{U}) \leq_{sem} \mathcal{W}$. □

Chapter two

Semi-Extending and Prime-Extending on Semimodules

Semi-Extending and Prime-Extending on Semimodules

The idea of semi-extending in modules has been studied by some researchers. Recall Ahmed and Abbas in [6] that, a left $\mathcal{T}$-module $\mathcal{W}$ is called semi-extending if for each submodule $\mathcal{V}$ in $\mathcal{W}$ then $\mathcal{V} \leq_{sem} \mathcal{H}$, with $\mathcal{H}$ is a direct summand in $\mathcal{W}$. The issue of extending semimodules was also studied by Alhashemi, where in [7], a $\mathcal{T}$-semimodule $\mathcal{W}$ is called extending (CS-semimodule) if for any subsemimodule $\mathcal{N}$ of $\mathcal{W}$ then $\mathcal{N} \leq_e \mathcal{H}$, with $\mathcal{H}$ is a direct summand in $\mathcal{W}$. π-injective and quasi-continuous are equivalent, while for semimodules quasi-continuous implies π-injective in the class of semimodules with injective hull. The converse is true with more conditions. (See [7]). In [9] Injectivity implies π-injective (or quasi-continuous).

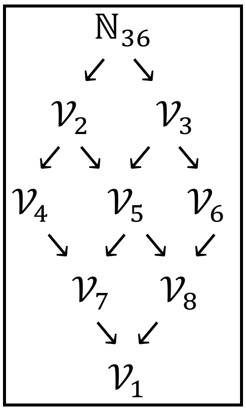
2.1. Semi-extending (SCS-semimodules)

Definition 2.1.1: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule. $\mathcal{W}$ is called semi-extending (briefly **SCS-semimodule**) if for each $\mathcal{V} \leq \mathcal{W}$, then $\mathcal{V} \leq_{sem} \mathcal{H}$, with $\mathcal{H} \leq^{\oplus} \mathcal{W}$. A semiring $\mathcal{T}$ is called semi-extending, if $\mathcal{T}$ is SCS-semimodule.

Remarks and Examples 2.1.2:

1. Every semisimple semimodule is an SCS-semimodule, (since, every subsemimodule of semisimple semimodule is a direct summand.
   e.g. The $\mathbb{N}$-semimodule $\mathbb{N}_6$ is an SCS-semimodule, (since, $\mathbb{N}_6$ is simisimple).
2. Every CS-semimodule is SCS-semimodule, (since, if $\mathcal{V} \leq_e \mathcal{W}$, then $\mathcal{V} \leq_{sem} \mathcal{W}$, for all $\mathcal{V} \leq \mathcal{W}$).
3. Any uniform semimodule is an SCS-semimodule.
4. The converse of (Remark 2.1.2 (3)) is not true in general. Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{36}$, see (Example 1.3.2 (1)). Note that, there are

four direct summands in $\mathbb{N}_{36}$ which are $\mathcal{V}_1, \mathcal{V}_4, \mathcal{V}_6$ and $\mathcal{V}_9$. $\mathcal{V}_2$ and $\mathcal{V}_3$ are a prime subsemimodules in $\mathcal{V}_9$. $\mathcal{V}_7$ is a prime subsemimodule in $\mathcal{V}_4$. $\mathcal{V}_8$ is a prime subsemimodule in $\mathcal{V}_6$. See the diagram below.



Now, since $\mathcal{V}_4$ and $\mathcal{V}_6$ are direct summands, $\mathcal{V}_2, \mathcal{V}_3$ and $\mathcal{V}_5$ are semi-essential in $\mathbb{N}_{36}$. It is enough to examin $\mathcal{V}_7$ and $\mathcal{V}_8$. Since $\mathcal{V}_7 \cap \mathcal{V}_7 = \mathcal{V}_7 \neq < 0 > \Longrightarrow \mathcal{V}_7 \leq_{sem} \mathcal{V}_4 \leq^{\oplus} \mathbb{N}_{36}$. Also, since $\mathcal{V}_8 \cap \mathcal{V}_8 = \mathcal{V}_8 \neq < 0 >$, thus $\mathcal{V}_8 \leq_{sem} \mathcal{V}_6 \leq^{\oplus} \mathbb{N}_{36}$. Therefore, the $\mathbb{N}$-semimodule $\mathbb{N}_{36}$ is SCS-semimodule. However, $\mathcal{V}_8 \cap \mathcal{V}_7 =< \overline{18} > \cap < \overline{12} > = < \overline{0} >$, then $\mathcal{V}_8 =< \overline{18} >$ is not essential subsemimodule in $\mathbb{N}_{36}$. Therefore the $\mathbb{N}$-semimodule $\mathbb{N}_{36}$ is not uniform.

5. Any semi-uniform semimodule is SCS-semimodule.
   Let $\mathcal{V} \leq \mathcal{W}$ where $\mathcal{W}$ be a $\mathcal{T}$-semimodule. If $\mathcal{V} =< 0 >$, then $\mathcal{V} \leq_{sem} < 0 >$ and $< 0 > \leq^{\oplus} \mathcal{W}$. If $\mathcal{V} \neq < 0 >$, since $\mathcal{W}$ is semi-uniform, then $\mathcal{V} \leq_{sem} \mathcal{W}$ and $\mathcal{W} \leq^{\oplus} \mathcal{W}$. □
6. The converse of (Remark 2.1.2 (5)) is not true in general. Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{24}$, see (Example 1.3.2 (2)). Note that, there are four direct summands in $\mathbb{N}_{24}$ and they $\mathcal{V}_1, \mathcal{V}_3, \mathcal{V}_6$ and $\mathcal{V}_8$. $\mathcal{V}_2$ and $\mathcal{V}_3$ are prime subsemimodules in $\mathcal{V}_8$. Also $\mathcal{V}_5$ is a prime subsemimodule in $\mathcal{V}_3$. See the diagram below.

$\mathbb{N}_{24}$
$\swarrow \searrow$
$\mathcal{V}_2 \quad \mathcal{V}_3$
$\downarrow \searrow \quad \downarrow$
$\mathcal{V}_4 \quad \mathcal{V}_5$
$\downarrow \searrow \quad \downarrow$
$\mathcal{V}_6 \quad \mathcal{V}_7$
$\searrow \swarrow$
$\mathcal{V}_1$

Now, since $\mathcal{V}_3$ and $\mathcal{V}_8$ are direct summands, $\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5$ and $\mathcal{V}_7$ are semi-essential in $\mathbb{N}_{24}$. Therefore, the $\mathbb{N}$-semimodule $\mathbb{N}_{24}$ is SCS-semimodule. But $\mathcal{V}_6 \cap \mathcal{V}_3 =< \overline{8} > \cap < \overline{3} > = < \overline{0} >$, then $\mathcal{V}_6 =< \overline{8} >$ is not semi-essential subsemimodule in $\mathbb{N}_{24}$. Therefore the $\mathbb{N}$-semimodule $\mathbb{N}_{24}$ is not semi-uniform.

Proposition 2.1.3: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule. If $\mathcal{W}$ is an SCS-semimodule and indecomposable then $\mathcal{W}$ is semi-uniform

Proof: Let $0 \neq \mathcal{V}$ be a subsemimodule in $\mathcal{W}$, since $\mathcal{W}$ is an SCS-semimodule, thus $\mathcal{V} \leq_{sem} \mathcal{B}$ with $\mathcal{B} \leq^{\oplus} \mathcal{W}$. But $\mathcal{W}$ is an indecomposable semimodule then $\mathcal{B} =< 0 >$ or $\mathcal{B} = \mathcal{W}$ and since $\mathcal{V} \neq < 0 >$, then $\mathcal{B} = \mathcal{W}$. Therefore, $\mathcal{W}$ is a semi-uniform semimodule. □

Corollary 2.1.4: Let $\mathcal{W}$ be a fully essential $\mathcal{T}$-semimodule. If $\mathcal{W}$ is an SCS-semimodule and indecomposable, then $\mathcal{W}$ is a uniform semimodule.

Proof: By (Proposition 2.1.3), then $\mathcal{W}$ is a semi-uniform semimodule. But $\mathcal{W}$ is fully essential semimodule. Therefore, $\mathcal{W}$ is a uniform semimodule. □

Lemma 2.1.5:

a. Let $\mathcal{N} \leq \mathcal{W}$ and $\{\mathcal{H}_\alpha\}_{\alpha \in \Lambda}$ a chain of subsemimodules of $\mathcal{W}$ such that $\mathcal{N} \leq_{sem} \mathcal{H}_\alpha$, for every $\alpha \in \Lambda$. If $\mathcal{H} = \bigcup_{\alpha \in \Lambda} \mathcal{H}_\alpha$ then $\mathcal{N} \leq_{sem} \mathcal{H}$.

b. Let $0 \neq \mathcal{N} \leq \mathcal{W}$ with $\mathcal{W}$ is a $\mathcal{T}$-semimodule. Thus there is $\mathcal{H} \leq_{Stc} \mathcal{W}$ with $\mathcal{N} \leq_{sem} \mathcal{H}$.

Proof (a):

Let $0 \neq \mathcal{B} \leq \mathcal{W}$ be a prime in $\mathcal{H}$, then $\mathcal{B} \cap \mathcal{H}_\alpha \neq 0$ for some $\alpha \in \Lambda$, by (Lemma 1.2.5) then $\mathcal{B} \cap \mathcal{H}_\alpha$ is a prime subsemimodule in $\mathcal{H}_\alpha$, since $\mathcal{N} \leq_{sem} \mathcal{H}_\alpha$, then $\mathcal{N} \cap (\mathcal{B} \cap \mathcal{H}_\alpha) \neq 0$, so $\mathcal{N} \cap \mathcal{B} \neq 0$. Therefore, $\mathcal{N} \leq_{sem} \mathcal{H}$. □

Proof (b):

Let $\Omega = \{\mathcal{C} \leq \mathcal{W} | \mathcal{N} \leq_{sem} \mathcal{C}\}$. Ω is not empty (since, $\mathcal{N} \leq_{sem} \mathcal{N}, \mathcal{N} \in \Omega$) and Ω is partially ordered by inclusion relation.

Let $\Gamma = \{\mathcal{H}_\alpha\}_{\alpha \in \Lambda}$ be any chain in Ω, then by (Lemma 2.1.5 (a)) $\mathcal{H} = \bigcup_{\alpha \in \Lambda} \mathcal{H}_\alpha$ is upper bound of Γ in Ω. By (Zorn's Lemma) Ω has a maximal element say $\mathcal{H}$. It is clear that $\mathcal{N} \leq_{sem} \mathcal{H}$ (since $\mathcal{H} \in \Omega$). Now, if $\mathcal{H} \leq_{sem} \mathcal{H}' \leq \mathcal{W}$, then $\mathcal{H}' \in \Omega$ by (Proposition 1.2.8) $\mathcal{N} \leq_{sem} \mathcal{H}'$, by maximality of $\mathcal{H}$, it follows $\mathcal{H} = \mathcal{H}'$. Therefore, $\mathcal{H} \leq_{Stc} \mathcal{W}$. □

The following theorem has been proved on modules [6], and we will give a different proof due its importance for it for semimodules.

Note: If $\mathcal{H}$ is a direct summand of $\mathcal{T}$-semimodule $\mathcal{W}$ then it is not necessary $\mathcal{H} \leq_{Stc} \mathcal{W}$. For example: Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{12}$, note that $< \bar{4} > \oplus < \bar{3} > = \mathbb{N}_{12}$ and $< \bar{4} >$ is not semi-essential subsemimodule in $\mathbb{N}_{24}$. Therefore, $< \bar{4} > \leq_{Stc} \mathbb{N}_{12}$. But $< \bar{3} > \leq_{sem} \mathbb{N}_{12}$. Therefore, $< \bar{3} >$ is not St-closed in $\mathbb{N}_{12}$.

Theorem 2.1.6: A $\mathcal{T}$-semimodule $\mathcal{W}$ is an SCS-semimodule if and only if any $\mathcal{V} \leq_{Stc} \mathcal{W}$ then $\mathcal{V}$ is a direct summand in $\mathcal{W}$.

Proof:

($\Longrightarrow$) Suppose that $\mathcal{W}$ is an SCS-semimodule. Let $\mathcal{V} \leq_{Stc} \mathcal{W}$, since $\mathcal{W}$ is an SCS-semimodule, then $\mathcal{V} \leq_{sem} \mathcal{B}$ with $\mathcal{B} \leq^{\oplus} \mathcal{W}$, but $\mathcal{V} \leq_{Stc} \mathcal{W}$, then $\mathcal{V} = \mathcal{B}$. Therefore, $\mathcal{V} \leq^{\oplus} \mathcal{W}$.

($\Longleftarrow$) Suppose that every St-closed subsemimodule in $\mathcal{W}$ is a direct summand in $\mathcal{W}$. Let $\mathcal{V} \leq \mathcal{W}$, if $\mathcal{V} =< \overline{0} >$ then $\mathcal{V} \leq_{sem}< \overline{0} >, < \overline{0} > \leq^{\oplus} \mathcal{W}$. If $\mathcal{V} \neq< \overline{0} >$ then by (Lemma 2.1.5 (b)) there is $\mathcal{A} \leq_{Stc} \mathcal{W}$ such that $\mathcal{V} \leq_{sem} \mathcal{A}$. But every St-closed subsemimodule in $\mathcal{W}$ is a direct summand in $\mathcal{W}$, thus $\mathcal{A} \leq^{\oplus} \mathcal{W}$. Therefore, $\mathcal{W}$ is an SCS-semimodule. □

<u>Lemma 2.1.7</u>: If $\mathcal{N} \leq^{\oplus} \mathcal{W}$ where $\mathcal{W}$ is a $\mathcal{T}$-semimodule and $\mathcal{N} \leq \mathcal{H}$, then $\mathcal{N} \leq^{\oplus} \mathcal{H}$, provided that $\mathcal{H}$ is a subtractive subsemimodule of $\mathcal{W}$.

<u>Proof</u>: Since $\mathcal{N} \leq^{\oplus} \mathcal{W}$ then there is $\mathcal{N}' \leq \mathcal{W}$ such that $\mathcal{W} = \mathcal{N} \oplus \mathcal{N}'$. Since $\mathcal{N} \leq \mathcal{H}$ and $\mathcal{H}$ is a subtractive, then by (semi-Modular Law) $\mathcal{H} = \mathcal{N} + (\mathcal{H} \cap \mathcal{N}')$. It remains to prove the unique representable. Let $h \in \mathcal{H}$, then there is unique elements n, n' such that $h = n + n'$, $n \in \mathcal{N}, n' \in \mathcal{N}'$, since $n \in \mathcal{N} \leq \mathcal{H}, h \in \mathcal{H}$, since $\mathcal{H}$ is a subtractive, then $n' \in \mathcal{H} \longrightarrow n' \in \mathcal{N}' \cap \mathcal{H}$. Thus $\mathcal{H} = \mathcal{N} \oplus (\mathcal{H} \cap \mathcal{N}')$. Therefore, $\mathcal{N} \leq^{\oplus} \mathcal{H}$. □

<u>Corollary 2.1.8</u>: [17, Proposition 16.7, P.185] Let $\mathcal{W}$ be a left $\mathcal{T}$-semimodule. If $\mathcal{H}_1, \mathcal{H}_2$ are direct summands in $\mathcal{W}$ and $\mathcal{H}_1 \leq \mathcal{H}_2$ then, $\mathcal{H}_1 \leq^{\oplus} \mathcal{H}_2$.

<u>Proposition 2.1.9</u>: Let $\mathcal{V}$ and $\mathcal{H}$ be subtractive subsemimodules of $\mathcal{W}$, where $\mathcal{W}$ is a semi-extending $\mathcal{T}$-semimodule. If $(\mathcal{V} \cap \mathcal{H})$ is an St-closed in $\mathcal{W}$, then $(\mathcal{V} \cap \mathcal{H}) \leq^{\oplus} \mathcal{V}$ and $(\mathcal{V} \cap \mathcal{H}) \leq^{\oplus} \mathcal{H}$.

<u>Proof</u>: Follows from (Theorem 2.1.6 and Lemma 2.1.7). □

Lemma 2.1.10: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a fully essential $\mathcal{T}$-semimodule where $\mathcal{W}_i \leq \mathcal{W}, (i = 1,2)$. If $< 0 >\neq \mathcal{N}_i \leq_{Stc} \mathcal{W}_i, (i = 1, 2)$, then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{Stc} \mathcal{W}$.

Proof: Let $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{H} \leq \mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$. Since $\mathcal{W}$ is fully essential then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_e \mathcal{H}$. Now, let $0 \neq h \in (\mathcal{H} \cap \mathcal{W}_1)$, then there is $t \in \mathcal{T}$ such that $0 \neq th = n_1 + n_2$ but $h = (w_1 + 0) \in \mathcal{W}_1$ implies $n_1 + n_2 = tw_1 + 0$, then $n_1 = tw_1$ and $n_2 = 0$. Hence $th = n_1 \in \mathcal{N}_1$ that is, $\mathcal{N}_1 \leq_e (\mathcal{H} \cap \mathcal{W}_1)$, not possible since $\mathcal{N}_1 \leq_{Stc} \mathcal{W}_1$. Therefore $\mathcal{H} \cap \mathcal{W}_1 = 0$. But $\mathcal{N}_1 \subseteq (\mathcal{N}_1 \oplus \mathcal{N}_2) \subseteq \mathcal{H}$, implies $\mathcal{N}_1 \subseteq \mathcal{H} \cap \mathcal{W}_1$, a contradiction. Therefore $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{Stc} \mathcal{W}$. □

Proposition 2.1.11: Let $\mathcal{W} = \mathcal{V} \oplus \mathcal{H}$ be an SCS-semimodule. If for each semi-essential extension of $\mathcal{B} \oplus \mathcal{H}$ is a fully essential semimodule, where $\mathcal{B} \leq_{Stc} \mathcal{V}$, then $\mathcal{V}$ is an SCS-semimodule.

Proof: Suppose that, $\mathcal{B} \oplus \mathcal{H}$ is a fully essential semimodule. Let $0 \neq \mathcal{B} \leq_{Stc} \mathcal{V}$. Since $\mathcal{H} \leq_{Stc} \mathcal{H}$ and $\mathcal{B} \leq_{Stc} \mathcal{V}$, by (Lemma 2.1.10) then, $(\mathcal{B} \oplus \mathcal{H}) \leq_{Stc} \mathcal{W}$, hence by (Theorem 2.1.6), $(\mathcal{B} \oplus \mathcal{H}) \leq^{\oplus} \mathcal{W}$. Thus $\mathcal{W} = (\mathcal{B} \oplus \mathcal{H}) \oplus \mathcal{A} = \mathcal{B} \oplus (\mathcal{H} \oplus \mathcal{A})$, where $\mathcal{A} \leq \mathcal{W}$, thus $\mathcal{B} \leq^{\oplus} \mathcal{W}$. But $\mathcal{B} \leq \mathcal{V}$, by (Proposition 2.1.9), then $\mathcal{B} \leq^{\oplus} \mathcal{V}$. Therefore $\mathcal{V}$ is an SCS-semimodule. □

Proposition 2.1.12: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule with injective hull $E(\mathcal{W})$ and $(\mathcal{W} \cap \mathcal{B})$ is an St-closed subsemimodule of $\mathcal{W}$ for every direct summand $\mathcal{B}$ in $E(\mathcal{W})$. Then, $\mathcal{W}$ is an SCS-semimodule if and only if $(\mathcal{W} \cap \mathcal{B}) \leq^{\oplus} \mathcal{W}$.

Proof:

Let $\mathcal{B} \leq^{\oplus} E(\mathcal{W})$, by hypotheses, then $(\mathcal{W} \cap \mathcal{B}) \leq_{Stc} \mathcal{W}$ since $\mathcal{W}$ is SCS-semimodule, therefore $(\mathcal{W} \cap \mathcal{B}) \leq^{\oplus} \mathcal{W}$.

Conversely, suppose that the stated condition hold. Let $\mathcal{N} \leq \mathcal{W}$, then $\mathcal{N} \leq_e E(\mathcal{N}) \leq^{\oplus} E(\mathcal{W})$. By hypotheses then $\mathcal{N} \leq_e \mathcal{W} \cap E(\mathcal{N}) \leq^{\oplus} \mathcal{W}$. Thus $\mathcal{N} \leq_{sem} \mathcal{W} \cap E(\mathcal{N}) \leq^{\oplus} \mathcal{W}$. Therefore $\mathcal{W}$ is SCS-semimodule. □

Proposition 2.1.13: If $\mathcal{W}$ is a fully essential $\mathcal{T}$-semimodule. Then, $\mathcal{W}$ is an SCS-semimodule if and only if $\mathcal{W}$ is a CS-semimodule.

Proof:

(⟹) Suppose that, $\mathcal{W}$ is an SCS-semimodule. To prove $\mathcal{W}$ is a CS-semimodule. Let $\mathcal{N} \leq \mathcal{W}$. Since $\mathcal{W}$ is an SCS-semimodule then $\mathcal{N}$ is a semi-essential in direct summand of $\mathcal{W}$, but $\mathcal{W}$ is fully essential. Thus $\mathcal{N} \leq_e \mathcal{H}$, with $\mathcal{H}$ is a direct summand in $\mathcal{W}$. Therefore, $\mathcal{W}$ is an CS-semimodule.

(⟸) Clear. □

The following result show that the transitive property for St-closed subsemimodule under a given condition.

Lemma 2.1.14: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule. If $\mathcal{N} \leq_{Stc} \mathcal{H}$ and $\mathcal{H} \leq_{Stc} \mathcal{W}$ then $\mathcal{N} \leq_{Stc} \mathcal{W}$ provided that $\mathcal{H}$ contained in (or containing) any semi-essential extension of $\mathcal{N}$.

Proof: The proof of this Lemma is similar to a prove in modules, [5, Proposition 1.5]. □

In fact, the transitive property is also true if we replace the above condition with the condition $\mathcal{W}$ is a fully prime semimodule.

Lemma 2.1.15:[7, Proposition 2.3] A $\mathcal{T}$-semimodule $\mathcal{W}$ is a CS-semimodule if and only if every closed subsemimodule $\mathcal{V}$ of $\mathcal{W}$ is a direct summand in $\mathcal{W}$.

Proposition 2.1.16: Let $\mathcal{W}$ be a semisubtractive cancellative $\mathcal{T}$-semimodule. If $\mathcal{W}$ is π-injective, then $\mathcal{W}$ is CS-semimodule.

Proof: Let $\mathcal{V} \leq_c \mathcal{W}$, then $\mathcal{V}$ is a complement of some subsemimodule $\mathcal{H}$ of $\mathcal{W}$, that is $\mathcal{V} \cap \mathcal{H} = 0$. Since $\mathcal{W}$ is π-injective, then there exists α, β idempotent in $End(\mathcal{W})$ such that $\alpha + \beta = 1_{\mathcal{W}}$ and $\mathcal{V} \leq ker\alpha, \mathcal{H} \leq ker\beta$. If $w \in ker\alpha \cap ker\beta$, then $\alpha(w) = 0$ and $\beta(w) = 0$, but $w = \alpha(w) + \beta(w)$, thus $w = 0 + 0 = 0$, so $ker\alpha \cap ker\beta = 0$. On the other hand, for any $w \in \mathcal{W}$, we have $w = \alpha(w) + \beta(w)$, then $\alpha(w) = \alpha(w) + \alpha\big(\beta(w)\big)$ implies $\beta(w) \in ker\alpha$ (since, $\mathcal{W}$ is cancellative). Similarly $\alpha(w) \in ker\beta$, hence $w \in ker\alpha + ker\beta$, that is, $\mathcal{W} = ker\alpha + ker\beta$.

Finally, we prove that $ker\alpha + ker\beta = ker\alpha \oplus ker\beta$. Let $w_1 + w_2 = w_1' + w_2'$, where $w_1, w_1' \in ker\alpha$ and $w_2, w_2' \in ker\beta$ then $\alpha(w_2) + \alpha(w_2') \ldots\ldots\ldots (1)$. By semisubtractive property either $w_2 + a = w_2'$ or $w_2 = w_2' + a$ for some $a \in \mathcal{W}$. Since $ker\beta$ is subtractive, then $a \in ker\beta$ on the other hand, $\alpha(a) = 0$ (in the two cases), by (1) and cancellative then $a \in ker\alpha \cap ker\beta = 0$, hence $w_2 = w_2'$. Similarly we get $w_1 = w_1'$, which lead to unique representation. Therefore, $\mathcal{W} = ker\alpha \oplus ker\beta$. Now, $\mathcal{V} \subseteq ker\alpha$ and $ker\alpha \cap \mathcal{H} = 0$ implies $\mathcal{V} = ker\alpha$ (since $\mathcal{V}$ is complement of $\mathcal{H}$). Hence, $\mathcal{V} \leq^{\oplus} \mathcal{W}$ and by (Lemma 2.1.15), then $\mathcal{W}$ is a CS-semimodule. □

Corollary 2.1.17: Let $\mathcal{W}$ be a semisubtractive cancellative $\mathcal{T}$-semimodule. Then, if $\mathcal{W}$ is π-injective then $\mathcal{W}$ is SCS-semimodule.

Proof: Follows from (Proposition 2.1.16 and Remark 2.1.2 (2)). □

Proposition 2.1.18: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be an (s-P) $\mathcal{T}$-semimodule and $\mathcal{W}_1, \mathcal{W}_2$ are prime subsemimodules in $\mathcal{W}$. Then, $\mathcal{W}$ is an SCS-semimodule if and only if any semi-complement of $\mathcal{W}_i$ $(i = 1, 2)$ is an SCS-semimodule and direct summand in $\mathcal{W}$.

Proof:

($\Longrightarrow$) Assume that, $\mathcal{W}$ is an SCS-semimodule. Let $\mathcal{H}$ be a semi-complement of $\mathcal{W}_1$ in $\mathcal{W}$. Since $\mathcal{W}$ is an SCS-semimodule, then $\mathcal{H} \leq_{Stc} \mathcal{W}$ and by (Theorem 2.1.6) $\mathcal{H} \leq^{\oplus} \mathcal{W}$. Now, let $\mathcal{V} \leq_{Stc} \mathcal{H}$, by (Lemma 2.1.14) then $\mathcal{V} \leq_{Stc} \mathcal{W}$ and $\mathcal{V} \cap \mathcal{W}_2 = 0$. Also, $\mathcal{W}$ is an SCS-semimodule, then $\mathcal{V} \leq^{\oplus} \mathcal{W} \longrightarrow \mathcal{W} = \mathcal{V} \oplus \mathcal{V}'$ for some $\mathcal{V}' \leq \mathcal{W}$, by (Semi-Modular Law) then $\mathcal{H} = \mathcal{V} \oplus (\mathcal{V}' \cap \mathcal{H})$, thus $\mathcal{V} \leq^{\oplus} \mathcal{H}$. Therefore $\mathcal{H}$ is an SCS-semimodule.

($\Longleftarrow$) Assume that, any semi-complement of $\mathcal{W}_i$, $i = 1, 2$ is SCS-semimodule and direct summand in $\mathcal{W}$. Let $\mathcal{H} \leq_{Stc} \mathcal{W}$, by (Lemma 2.1.5 (b)) there is $\mathcal{V} \leq_{Stc} \mathcal{H}$ such that $(\mathcal{H} \cap \mathcal{W}_1) \leq_{sem} \mathcal{V}$. Since $\mathcal{W}_2$ is prime subsemimodule in $\mathcal{W}$, then $\mathcal{V} \cap \mathcal{W}_2$ is prime subsemimodule in $\mathcal{V}$, thus $\mathcal{V} \cap \mathcal{W}_2 = 0$, (since, $(\mathcal{H} \cap \mathcal{W}_1) \leq_{sem} \mathcal{V}$ and $\mathcal{W}_1 \cap \mathcal{W}_2 = 0$). By (Zorn's Lemma), then there is a semi-complement $\mathcal{M}$ of $\mathcal{W}_2$ in $\mathcal{W}$, with $\mathcal{V} \leq \mathcal{M}$, by (Lemma 2.1.14), then $\mathcal{V} \leq_{Stc} \mathcal{W}$, hence $\mathcal{V} \leq_{Stc} \mathcal{M}$. However, $\mathcal{M}$ is semi-complement of $\mathcal{W}_2$, then by assumption $\mathcal{M}$ is SCS-semimodule. Hence $\mathcal{V} \leq^{\oplus} \mathcal{W}$ (since, Theorem 2.1.6), thus $\mathcal{W} = \mathcal{V} \oplus \mathcal{V}'$ for some $\mathcal{V}' \leq \mathcal{W}$, by (Semi-Modular Law) $\mathcal{H} = \mathcal{V} \oplus (\mathcal{H} \cap \mathcal{V}')$, since $(\mathcal{H} \cap \mathcal{V}') \leq_{Stc} \mathcal{W}$ and $(\mathcal{H} \cap \mathcal{V}') \cap \mathcal{W}_1 = 0$ then $(\mathcal{H} \cap \mathcal{V}') \leq^{\oplus} \mathcal{W}$ and also for $\mathcal{V}', \mathcal{V}' = (\mathcal{H} \cap \mathcal{V}') \oplus \mathcal{V}''$ for some $\mathcal{V}'' \leq \mathcal{W}$, so $\mathcal{W} = \mathcal{H} \oplus \mathcal{V}''$. Therefore, $\mathcal{H} \leq^{\oplus} \mathcal{W}$. By (Theorem 2.1.6), then $\mathcal{W}$ is an SCS-semimodule. □

Proposition 2.1.19: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be an (s-P) $\mathcal{T}$-semimodule and $\mathcal{W}_1, \mathcal{W}_2$ are prime subsemimodules in $\mathcal{W}$, such that $\mathcal{W}_1$ and $\mathcal{W}_2$ are SCS-semimodule. Then, $\mathcal{W}$ is SCS-semimodule if and only if for any St-closed $\mathcal{H}$ of $\mathcal{W}$ with $\mathcal{H} \cap \mathcal{W}_1 = 0$ or $\mathcal{H} \cap \mathcal{W}_2 = 0$ is a direct summand in $\mathcal{W}$.

Proof:

($\Longrightarrow$) It is proved by (Theorem 2.1.6).

($\Longleftarrow$) Let $\mathcal{V} \leq_{Stc} \mathcal{W}$, then either $\mathcal{V} \cap \mathcal{W}_1 = 0$, by assumption, then $\mathcal{V} \leq^{\oplus} \mathcal{W}$. Or $\mathcal{V} \cap \mathcal{W}_1 \neq 0$, then by (Lemma 2.1.5 (b)) there is $\mathcal{B} \leq \mathcal{V}$ such that $(\mathcal{V} \cap \mathcal{W}_1) \leq_{sem} \mathcal{B} \leq_{Stc} \mathcal{V}$. Since $\mathcal{W}_2$ is prime subsemimodule in $\mathcal{W}$ then $\mathcal{B} \cap \mathcal{W}_2$ is prime subsemimodule in $\mathcal{B}$ thus $\mathcal{B} \cap \mathcal{W}_2 = 0$, (since, $(\mathcal{V} \cap \mathcal{W}_1) \leq_{sem} \mathcal{B}$ and $\mathcal{W}_1 \cap \mathcal{W}_2 = 0$). Note that by (Lemma 2.1.14), then $\mathcal{B} \leq_{Stc} \mathcal{W}$, by assumption then $\mathcal{B} \leq^{\oplus} \mathcal{W}$, thus $\mathcal{W} = \mathcal{B} \oplus \mathcal{B}'$ for some $\mathcal{B}' \leq \mathcal{W}$, by (Semi-Modular Law), then $\mathcal{V} = \mathcal{B} \oplus (\mathcal{V} \cap \mathcal{B}')$. But $(\mathcal{V} \cap \mathcal{B}') \leq_{Stc} \mathcal{W}$, then $(\mathcal{V} \cap \mathcal{B}') \cap \mathcal{W}_1 = 0$ (since, $\mathcal{V} \cap \mathcal{W}_1 \subseteq \mathcal{B}$ and $\mathcal{B} \cap \mathcal{B}' = 0$), also by assumption $(\mathcal{V} \cap \mathcal{B}') \leq^{\oplus} \mathcal{B}'$. Then $\mathcal{B}' = (\mathcal{V} \cap \mathcal{B}') \oplus \mathcal{B}''$ for some $\mathcal{B}'' \leq \mathcal{W}$, so $\mathcal{W} = \mathcal{B} \oplus (\mathcal{V} \cap \mathcal{B}') \oplus \mathcal{B}'' = \mathcal{V} \oplus \mathcal{B}'$. Therefore, $\mathcal{V} \leq^{\oplus} \mathcal{W}$ and $\mathcal{W}$ is an SCS-semimodule. □

2.2. Prime-extending semimodules (PE-Semimodules)

The concept of prime-extending modules is studied, by Ibrahiem in [20], where (A $\mathcal{T}$-module $\mathcal{W}$ is said to be prime-extending module if for any $0 \neq \mathcal{V} \lneqq \mathcal{W}$, then $\mathcal{V} \leq_e \mathcal{B}$, with $\mathcal{B}$ is a prime direct summand in $\mathcal{W}$).

Definition 2.2.1: A $\mathcal{T}$-semimodule $\mathcal{W}$ is called a prime-extending semimodule (**PE-**Semimodule**)** if for any $0 \neq \mathcal{V} \leq \mathcal{W}$, then $\mathcal{V} \leq_{sem} \mathcal{B}$, with $\mathcal{B}$ is a prime direct summand in $\mathcal{W}$. $\mathcal{T}$ is called **PE-**Semimodule if $\mathcal{W}$ is a prime-extending $\mathcal{T}$-semimodule.

Remarks and Examples 2.2.2:

1. $\mathbb{N}_6$ as $\mathbb{N}$-semimodule is a **PE-**semimodule (since, there are only two nonzero proper subsemimodules $< \bar{2} >$ and $< \bar{3} >$ which are both prime and summand.

2. A semisimple semimodule $\mathcal{W}$ is not necessary **PE**-semimodule, for example: consider the $\mathbb{N}$-semimodule $\mathbb{N}_{30}$, note there are six a nonzero proper subsemimodules of $\mathbb{N}_{30}$ which are $\mathcal{V}_1 =< \bar{2} >, \mathcal{V}_2 = < \bar{3} >, \mathcal{V}_3 =< \bar{5} >, \mathcal{V}_4 =< \bar{6} >, \mathcal{V}_5 =< \overline{10} >$ and $\mathcal{V}_6 =< \overline{15} >$.

   And they are direct summands in $\mathbb{N}_{30}$ such that $\mathbb{N}_{30} = (\mathcal{V}_1 \oplus \mathcal{V}_6) = (\mathcal{V}_2 \oplus \mathcal{V}_5) = (\mathcal{V}_3 \oplus \mathcal{V}_4)$. Then $\mathbb{N}_{30}$ is semisimple $\mathbb{N}$-semimodule. However, note that, $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{V}_3$ are prime subsemimodules in $\mathbb{N}_{30}$. $\mathcal{V}_4$ and $\mathcal{V}_5$ are prime subsemimodules in $\mathcal{V}_1$. $\mathcal{V}_4$ and $\mathcal{V}_6$ are prime subsemimodules in $\mathcal{V}_2$. And $\mathcal{V}_5$ and $\mathcal{V}_6$ are prime subsemimodules in $\mathcal{V}_3$. $\mathcal{V}_4$ is not prime in $\mathbb{N}_{30}$. Also, $\mathcal{V}_4 \cap \mathcal{V}_6 = 0 \Longrightarrow \mathcal{V}_4$ is not semi-essential in $\mathcal{V}_1$. Therefore, the $\mathbb{N}$-semimodule $\mathbb{N}_{30}$ is not **PE**-semimodule.
3. Every **PE**-semimodule is SCS-semimodule, but the convers is not true in general.

   The only summand of $\mathbb{N}_{12}$ containing $< \bar{4} >$ is $< \bar{4} >$ itself, which is not a prime summand.
4. Any semi-uniform semimodule is not **PE**-semimodule. Since the only summand containing a nonzero subsemimodule of $\mathcal{W}$ is $\mathcal{W}$ itself which is not prime.
5. Any uniform semimodule is not **PE**-semimodule.

Lemma 2.2.3: Let $\mathcal{W}$ be a faithful cancellative multiplication $\mathcal{T}$-semimodule, with $\mathcal{T}$ is a semisubtractive commutative. If $\mathcal{T}$ is a fully prime semiring then $\mathcal{W}$ is a fully prime semimodule.

Proof: Let $0 \neq \mathcal{V} \leq \mathcal{W}$, since $\mathcal{W}$ is a multiplication $\mathcal{T}$-semimodule, thus $\mathcal{V} = E\mathcal{W}$ for some ideal E of $\mathcal{T}$. But $\mathcal{T}$ is a fully prime semiring thus E is a prime ideal in $\mathcal{T}$. By (Corollary 1.2.15) we have $E\mathcal{W} = \mathcal{V}$ is a prime subsemimodule of $\mathcal{W}$. Therefore, $\mathcal{W}$ is a fully prime semimodule. □

Proposition 2.2.4: Let $\mathcal{W}$ be a faithful multiplication $\mathcal{T}$-semimodule. if $\mathcal{T}$ is a fully prime semiring. Then every semisimple $\mathcal{W}$, which is not simple is **PE**-semimodule.

Proof: Suppose that, $\mathcal{W}$ is a semisimple $\mathcal{T}$-semimodule and $0 \neq \mathcal{V} \lneqq \mathcal{W}$. Then $\mathcal{V} \leq^{\oplus} \mathcal{W}$. However, $\mathcal{T}$ be a fully prime semiring. By (Lemma 2.2.3), we have $\mathcal{V}$ is a prime subsemimodule in $\mathcal{W}$, thus $\mathcal{V} \leq_{sem} \mathcal{V} \leq^{\oplus} \mathcal{W}$. Therefore $\mathcal{W}$ is **PE**-semimodule. □

Proposition 2.2.5: A $\mathcal{T}$-semimodule $\mathcal{W}$ is a **PE**-semimodule if and only if any St-closed subsemimodule in $\mathcal{W}$ is a prime direct summand.

Proof:

(⟹) Suppose that $\mathcal{W}$ is **PE**-semimodule. Let $0 \neq \mathcal{V} \leq_{Stc} \mathcal{W}$. Since $\mathcal{W}$ is **PE**-semimodule, then there is a prime direct summand $\mathcal{B}$ of $\mathcal{W}$ and $\mathcal{V} \leq_{sem} \mathcal{B}$. But $\mathcal{V}$ has no proper semi-essential extension, so $\mathcal{B} = \mathcal{V}$. Therefore $\mathcal{V}$ is a prime direct summand.

(⟸) Asuume that any St-closed subsemimodule in $\mathcal{W}$ is a prime direct summand. Let $0 \neq \mathcal{N} \leq \mathcal{W}$, then by (Lemma 2.1.5 (b)) there is $\mathcal{H} \leq_{Stc} \mathcal{W}$ with $\mathcal{N} \leq_{sem} \mathcal{H}$. By assumption, $\mathcal{H}$ is prime direct summand of $\mathcal{W}$. Therefore $\mathcal{N}$ is contained is a prime direct summand, that is $\mathcal{W}$ is a **PE**-semimodule. □

In the following, we will show the relationship between semiring and semimodules with respect for **PE**-semimodule.

Proposition 2.2.6: Let $\mathcal{W}$ be a faithful cancellative multiplication and finitely generated $\mathcal{T}$-semimodule with $\mathcal{T}$ is a semisubtractive semiring . Then, $\mathcal{T}$ is a **PE**-semimodule if and only if $\mathcal{W}$ is a **PE**-semimodule.

Proof:

Suppose that, $\mathcal{T}$ is a **PE**-semimodule, and let $0 \neq \mathcal{N} \leq \mathcal{W}$. Since $\mathcal{W}$ is multiplication $\mathcal{T}$-semimodule then $\mathcal{N} = J\mathcal{W}$ for some ideal J of $\mathcal{W}$. However, $\mathcal{T}$ is a **PE-**semimodule then there is a prime ideal direct summand E in $\mathcal{T}$ such that $J \leq_{sem} E \leq^{\oplus} \mathcal{T}$. Let $\mathcal{B} = E\mathcal{W}$, by (Proposition 1.2.28), then $J\mathcal{W} \leq_{sem} E\mathcal{W} \Longrightarrow \mathcal{N} \leq_{sem} \mathcal{B}$. So, by (Corollary 1.2.15), then $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$. Now, since $E \leq^{\oplus} \mathcal{T}$, thus there is ideal I of $\mathcal{T}$ such that $E \oplus I = \mathcal{T}$, implies $(E \oplus I)\mathcal{W} = \mathcal{T}\mathcal{W} \Longrightarrow \mathcal{B} \oplus I\mathcal{W} = \mathcal{W}$, thus $\mathcal{B} \leq^{\oplus} \mathcal{W}$, then $\mathcal{N} \leq_{sem} \mathcal{B}$ with $\mathcal{B}$ is a prime direct summand in $\mathcal{W}$. Therefore, $\mathcal{W}$ is **PE-**semimodule.

Conversely, let J be an ideal of $\mathcal{T}$, then $J\mathcal{W} \leq \mathcal{W}$. But $\mathcal{W}$ is **PE-**semimodule, then there is a prime direct summand $\mathcal{B}$ in $\mathcal{W}$ such that $J\mathcal{W} \leq_{sem} \mathcal{B}$. Hence $\mathcal{B} = E\mathcal{W}$ for some prime ideal E in $\mathcal{T}$. Thus $J\mathcal{W} \leq_{sem} E\mathcal{W}$, by (Proposition 1.2.28) then $J \leq_{sem} E$. Now, since $\mathcal{T}\mathcal{W} = \mathcal{W}$ and $\mathcal{B} \leq^{\oplus} \mathcal{W}$, thus there is $\mathcal{H} \leq^{\oplus} \mathcal{W}$ that is $\mathcal{B} \oplus \mathcal{H} = \mathcal{W}$ and $\mathcal{H} = I\mathcal{W}$ for some ideal I in $\mathcal{T}$. Thus $E\mathcal{W} \oplus I\mathcal{W} = \mathcal{T}\mathcal{W}$, implies $(E \oplus I)\mathcal{W} = \mathcal{T}\mathcal{W} \Longrightarrow E \oplus I = \mathcal{T}$, then E is a prime direct summand in $\mathcal{T}$, thus $J \leq_{sem} E \leq^{\oplus} \mathcal{T}$, where E is a prime in $\mathcal{T}$. Therefore $\mathcal{T}$ is a **PE-**semimodule. □

We mentioned earlier in (Remark 2.2.2 (3)) that every **PE-**semimodule is SCS-semimodule, but the converse is not true in general. In the following (Proposition 2.2.7), we guarantee the appropriate condition for the converse direction to be true.

Proposition 2.2.7: A $\mathcal{T}$-semimodule $\mathcal{W}$ is a **PE**-semimodule if and only if $\mathcal{W}$ is an SCS-semimodule, provided for any direct summand of $\mathcal{W}$ is a prime.

Proof:

($\Longrightarrow$) Clear.

($\Longleftarrow$) Let $0 \neq \mathcal{V}$ be a proper subsemimodule in $\mathcal{W}$. Since $\mathcal{W}$ is an SCS-semimodule, thus there is $\mathcal{B} \leq^{\oplus} \mathcal{W}$ such that $\mathcal{V} \leq_{sem} \mathcal{B}$. However, any direct summand in $\mathcal{W}$ is a prime, thus $\mathcal{B}$ is a prime direct summand in $\mathcal{W}$. Therefore, $\mathcal{W}$ is **PE-**semimodule. □

Corollary 2.2.8: Let $\mathcal{W}$ be a fully prime $\mathcal{T}$-semimodule $\mathcal{W}$. Then, $\mathcal{W}$ is a **PE-**semimodule if and only if $\mathcal{W}$ is an SCS-semimodule.

Proof: clear. □

Chapter three

J-Extending Semimodules

J-Extending semimodules

In this chapter, we will define new concepts and show some results and properties of these classes.

3.1. Semi-prime subsemimodules

The main objective of this section is to generalize the definition of each (semi-prime ideal, semi-prime subsemimodules) after studying the above-mentioned definitions in modules [11],[17] and [28].

- Let $\mathcal{T}$ be a commutative semiring and I be ideal of $\mathcal{T}$. Then, I is called semi-prime if $t^2 \in I$, then $t \in I$, for any $t \in \mathcal{T}$ [28].

Definition 3.1.1: Let $\mathcal{A}$ be a proper subsemimodule of a $\mathcal{T}$-semimodule $\mathcal{W}$. $\mathcal{A}$ is said to semi-prime if $t \in \mathcal{T}, x \in \mathcal{W}$ and $t^2x \in \mathcal{A}$ then $tx \in \mathcal{A}$.

Remarks and Examples 3.1.2:

1. Every prime subsemimodule is a semi-prime subsemimodule. (For if $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$ and $t^2x \in \mathcal{B}$ where $t \in \mathcal{T}, x \in \mathcal{W}$, then $t.tx \in \mathcal{B}$ thus either $tx \in \mathcal{B}$ or $t\mathcal{W} \subseteq \mathcal{B} \longrightarrow tx \in \mathcal{B}$. Therefore, $\mathcal{B}$ is a semi-prime subsemimodule in $\mathcal{W}$).
2. If $\mathcal{A}$ is a semi-prime subsemimodule in $\mathcal{W}$. Then $\mathcal{A}$ is not necessary prime subsemimodule, consider the example: The $\mathbb{N}$-semimodule $\mathbb{N}_{12}$, note that $< \bar{6} >$ is semi-prime subsemimodule but not prime in $\mathbb{N}_{12}$.
3. If $\mathcal{A}$ is a direct summand subsemimodule of $\mathcal{W}$, then $\mathcal{A}$ is not necessary semi-prime, for example: Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{36}$, see example 1.3.2.1: Note that: there are two non-zero proper direct summands of $\mathbb{N}_{36}$, they are $\mathcal{V}_4 =< \bar{4} >$ and $\mathcal{V}_6 =< \bar{9} >$. So, $\mathcal{V}_4$ is not semi-prime of $\mathbb{N}_{36}$. Also $\mathcal{V}_6$ is not semi-prime of $\mathbb{N}_{36}$. □

4. If $\mathcal{W}$ is a torsion-free $\mathcal{T}$-semimodule and $\mathcal{A} \leq^{\oplus} \mathcal{W}$, then $\mathcal{A}$ is a semi-prime subsemimodule of $\mathcal{W}$.

   Let $0 \neq \mathcal{A}$ be a proper direct summand subsemimodule in $\mathcal{W}$, hence $\mathcal{W} = \mathcal{A} \oplus \mathcal{A}'$ for some $\mathcal{A}' \leq \mathcal{W}$. Now, let $t \in \mathcal{T}, w \in \mathcal{W}$ and $t^2 w \in \mathcal{A}$ then $tw = a + a'$ for some $a \in \mathcal{A}$ and $a' \in \mathcal{A}'$. So, $t^2 w = ta + ta'$, where $t^2 w, ta \in \mathcal{A}$. Since $\mathcal{A}$ is a direct summand, hence it is subtractive and $ta' \in \mathcal{A}$, i.e., $ta' \in (\mathcal{A} \cap \mathcal{A}') = 0$, then $t^2 w = ta$. However, $\mathcal{W}$ is torsion-free, so $tw = a \in \mathcal{A}$. Therefore, $\mathcal{A}$ is a semi-prime subsemimodule in $\mathcal{W}$. □

5. If $\mathcal{B}$ is a semi-prime subsemimodule of $\mathcal{T}$-semimodule $\mathcal{W}$, and $\mathcal{B} \subsetneqq \mathcal{B}' \leq \mathcal{W}$, then $\mathcal{B}$ is a semi-prime subsemimodule of $\mathcal{B}'$.

Remark 3.1.3: If $\mathcal{W}$ is a simple $\mathcal{T}$-semimodule, then $ann_{\mathcal{T}}\mathcal{W}$ is a prime ideal of $\mathcal{T}$.

Proof: Let $st \in ann_{\mathcal{T}}\mathcal{W}$, and $0 \neq w \in \mathcal{W}$, then $stw = 0$. If $tw \neq 0$ for some $w \in \mathcal{W}$, then tw generates $\mathcal{W}$ and $s(tw) = 0$ that is, $s \in ann_{\mathcal{T}}\mathcal{W}$. Therefore, either $s \in ann_{\mathcal{T}}\mathcal{W}$ or $t \in ann_{\mathcal{T}}\mathcal{W}$, i.e. $ann_{\mathcal{T}}\mathcal{W}$ is prime. □

Corollary 3.1.4: If $\mathcal{W}$ is a simple $\mathcal{T}$-semimodule, then $ann_{\mathcal{T}}(\mathcal{W})$ is a semi-prime ideal of $\mathcal{T}$.

Proof: It is clear by (Remark 3.1.3). □

Proposition 3.1.5: If $\mathcal{W}$ is a semisimple $\mathcal{T}$-semimodule, and $t^2 w = 0$ for some $t \in \mathcal{T}$ and $w \in \mathcal{W}$, then $tw = 0$.

Proof: Since $\mathcal{W}$ is a semisimple, then it is a direct sum of its simple subsemimodules. Assume that $t^2 w = 0, t \in \mathcal{T}$ and $0 \neq w \in \mathcal{W}$. Then $w = a_1 + a_2 + \dots\dots\dots + a_n$, where $0 \neq a_i \in \mathcal{A}_i$ and $\mathcal{A}_i$ are simple subsemimodules of $\mathcal{W}$. Hence $t^2 w = 0$ implies $t^2 a_i = 0$ for every $i \in \{1, 2, \dots\dots, n\}$ but each a_i generate $\mathcal{A}_i$ (which is simple), so $t^2 \in$

$ann_{\mathcal{T}}\mathcal{A}_i, i = 1, 2, \dots \dots, n$. By (Corollary 3.1.4), then $t \in ann_{\mathcal{T}}\mathcal{A}_i$ $(i = 1, 2, \dots \dots, n)$ hence $t w = 0$. □

Proposition 3.1.6: Any proper subsemimodule of a semisimple $\mathcal{T}$-semimodule is semi-prime.

Proof:

Let $\mathcal{W}$ be a semisimple $\mathcal{T}$-semimodule and $\mathcal{A} \leq \mathcal{W}$. Then $\mathcal{W} = \mathcal{A} \oplus \mathcal{A}'$ where $\mathcal{A}'$ is a semisimple subsemimodule of $\mathcal{W}$.

Assume that $t^2 w \in \mathcal{A}$ for some $t \in \mathcal{T}$ and $w \in \mathcal{W}$. Let $w = a + a'$, where $a \in \mathcal{A}$ and $a' \in \mathcal{A}'$. Then $t^2 w = t^2 a + t^2 a'$, hence $t^2 a' = 0$ (since $t^2 w \in \mathcal{A}$). By (Proposition 3.1.5) $ta' = 0$ (since, $\mathcal{A}'$ is semisimple). Therefore $tw = ta \in \mathcal{A}$, and $\mathcal{A}$ is semi-prime. □

Proposition 3.1.7: If $\mathcal{A}$ is a semi-prime subsemimodule of a $\mathcal{T}$-semimodule $\mathcal{W}$ and $\mathcal{N} \leq \mathcal{W}$ such that $\mathcal{N} \nleq \mathcal{A}$ then $(\mathcal{N} \cap \mathcal{A})$ is a semi-prime subsemimodule in $\mathcal{N}$.

Proof: Let $n \in \mathcal{N} \leq \mathcal{W}$ and $t \in \mathcal{T}$ such that $t^2 n \in (\mathcal{N} \cap \mathcal{A})$, then $t^2 n \in \mathcal{N}$ and $t^2 n \in \mathcal{A}$. Since $\mathcal{A}$ is a semi-prime subsemimodule in $\mathcal{W}$, then $tn \in \mathcal{A}$, but $tn \in \mathcal{N}$, hence $tn \in (\mathcal{N} \cap \mathcal{A})$. Therefore, $(\mathcal{N} \cap \mathcal{A})$ is a semi-prime subsemimodule of $\mathcal{N}$. □

Lemma 3.1.8: Let E_1, E_2 be a semi-prime ideals of semiring $\mathcal{T}$. If $E_1 \cap E_2 \neq 0$ then $E_1 \cap E_2$ is a semi-prime ideal in $\mathcal{T}$.

Proof:

Let $t \in \mathcal{T}$ and $t^2 \in E_1 \cap E_2$ then $t^2 \in E_1$ and $t^2 \in E_2$, since E_1, E_2 are semi-prime ideals in $\mathcal{T}$ then $t \in E_1$ and $t \in E_2$, implies $t \in E_1 \cap E_2$. Therefore, $E_1 \cap E_2$ is a semi-prime ideal in $\mathcal{T}$. □

Lemma 3.1.9: Let $\mathcal{A}_1, \mathcal{A}_2$ be semi-prime subsemimodules of $\mathcal{T}$-semimodule $\mathcal{W}$. If $\mathcal{A}_1 \cap \mathcal{A}_2 \neq 0$ then $\mathcal{A}_1 \cap \mathcal{A}_2$ is a semi-prime subsemimodule in $\mathcal{W}$.

Proof: Let $t \in \mathcal{T}, w \in \mathcal{W}$ and $t^2 w \in \mathcal{A}_1 \cap \mathcal{A}_2$, then $t^2 w \in \mathcal{A}_1$ and $t^2 w \in \mathcal{A}_2$, since $\mathcal{A}_1, \mathcal{A}_2$ are semi-prime subsemimodules in $\mathcal{W}$, then $tw \in \mathcal{A}_1$ and $tw \in \mathcal{A}_2$, thus $tw \in \mathcal{A}_1 \cap \mathcal{A}_2$. Hence, $\mathcal{A}_1 \cap \mathcal{A}_2$ is semi-prime subsemimodule in $\mathcal{W}$. □

Note: If $\mathcal{A}$ is a semi-prime subsemimodule in $\mathcal{W}$, then $\phi(\mathcal{A})$ is not necessary be a semi-prime subsemimodule in $\mathcal{W}'$. For example: $\phi: \mathbb{N}_{40} \longrightarrow \mathbb{N}_{40}$ define by $\phi(\bar{x}) = 2\bar{x}$ for all $x \in \mathbb{N}_{40}$. Put $\mathcal{A} =< \overline{10} >$ is a semi-prime subsemimodule of $\mathbb{N}_{40}$. But, $\phi(\mathcal{A}) =< \overline{20} >$ is not semi-prime.

In the following result, we will show that if $\mathcal{A}$ is a semi-prime subsemimodule in $\mathcal{W}$, then $\phi(\mathcal{A})$ is also semi-prime subsemimodule in $\mathcal{W}'$ under a given condition.

Lemma 3.1.10: Let $\mathcal{W}, \mathcal{W}'$ be a $\mathcal{T}$-semimodule, $\phi: \mathcal{W} \longrightarrow \mathcal{W}'$ be a isomorphism. If $\mathcal{A}$ is a semi-prime subsemimodule of $\mathcal{W}$ then $\phi(\mathcal{A})$ is a semi-prime subsemimodule of $\mathcal{W}'$.

Proof: Let $t \in \mathcal{T}$, $w' \in \mathcal{W}'$, such that $t^2 w' \in \phi(\mathcal{A})$. Since ϕ is epimorphism then there is $w \in \mathcal{W}$ such that $\phi(w) = w'$, thus $t^2 \phi(w) \in \phi(\mathcal{A})$, implies $\phi(t^2 w) \in \phi(\mathcal{A})$. Since ϕ is monomorphism, then $t^2 w \in \mathcal{A}$, but $\mathcal{A}$ is semi-prime subsemimodule of $\mathcal{W}$ then $tw \in \mathcal{A}$, thus $\phi(tw) \in \phi(\mathcal{A})$, hence $t\phi(w) = tw' \in \phi(\mathcal{A})$. Therefore, $\phi(\mathcal{A})$ is a semi-prime subsemimodule in $\mathcal{W}'$. □

Lemma 3.1.11: Let $\mathcal{W}, \mathcal{W}'$ be a $\mathcal{T}$-semimodule, $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ is a epimorphism. If $\mathcal{A}'$ is a semi-prime subsemimodule of $\mathcal{W}'$, then $\Psi^{-1}(\mathcal{A}')$ is a semi-prime subsemimodule of $\mathcal{W}$.

Proof: Let $t \in \mathcal{T}$, $w \in \mathcal{W}$ such that $t^2 w \in \Psi^{-1}(\mathcal{A}')$ then $\Psi(t^2 w) \in \mathcal{A}'$, since Ψ is a $\mathcal{T}$-epimorphism then $t^2\Psi(w) \in \mathcal{A}'$. Howevwe, $\mathcal{A}'$ is a semi-prime subsemimodule in $\mathcal{W}'$, then $t\Psi(w) \in \mathcal{A}'$. Therefore, $tw \in \Psi^{-1}(\mathcal{A}')$, consequently $\Psi^{-1}(\mathcal{A}')$ is a semi-prime subsemimodule in $\mathcal{W}$. □

Note: If Ψ is not $\mathcal{T}$-epimorphism and $\mathcal{A}'$ is a semi-prime subsemimodule in $\mathcal{W}'$, then $\Psi^{-1}(\mathcal{A}')$ is not necessary be a semi-prime subsemimodule in $\mathcal{W}$. For example: $\Psi: \mathbb{N}_2 \longrightarrow \mathbb{N}_4$ define by $\Psi(\bar{x}) = 2\bar{x}$ for all $x \in \mathbb{N}_4$. Put $\mathcal{A}' =< \bar{2} >$ is a semi-prime subsemimodule of $\mathbb{N}_4$. But, $\Psi^{-1}(\mathcal{A}') = \mathbb{N}_2$ is not semi-prime.

3.2. J-essential semimodules

In this section, we generalized the concept of the essential to another concept that lie between essential and semi-essential, and we studied some of its properties

Definition 3.2.1: Let $0 \neq \mathcal{V}$ be a subsemimodule of a $\mathcal{T}$-semimodule $\mathcal{W}$. $\mathcal{V}$ is called J-essential in $\mathcal{W}$ (briefly $\mathcal{V} \leq_{J.es} \mathcal{W}$) if $\mathcal{V} \cap \mathcal{A} \neq 0$, with $0 \neq \mathcal{A}$ is semi-prime subsemimodule in $\mathcal{W}$, i.e. If $\mathcal{V} \cap \mathcal{A} = 0$, with $\mathcal{A}$ is semi-prime subsemimodule in $\mathcal{W}$ then $\mathcal{A} = 0$.

Note: A nonzero ideal I of a semiring $\mathcal{T}$ is called J-essential (briefly $I \leq_{J.es} \mathcal{T}$) if $I \cap K \neq 0$, with $0 \neq K$ is semi-prime ideal in $\mathcal{T}$.

Remarks and Examples 3.2.2:

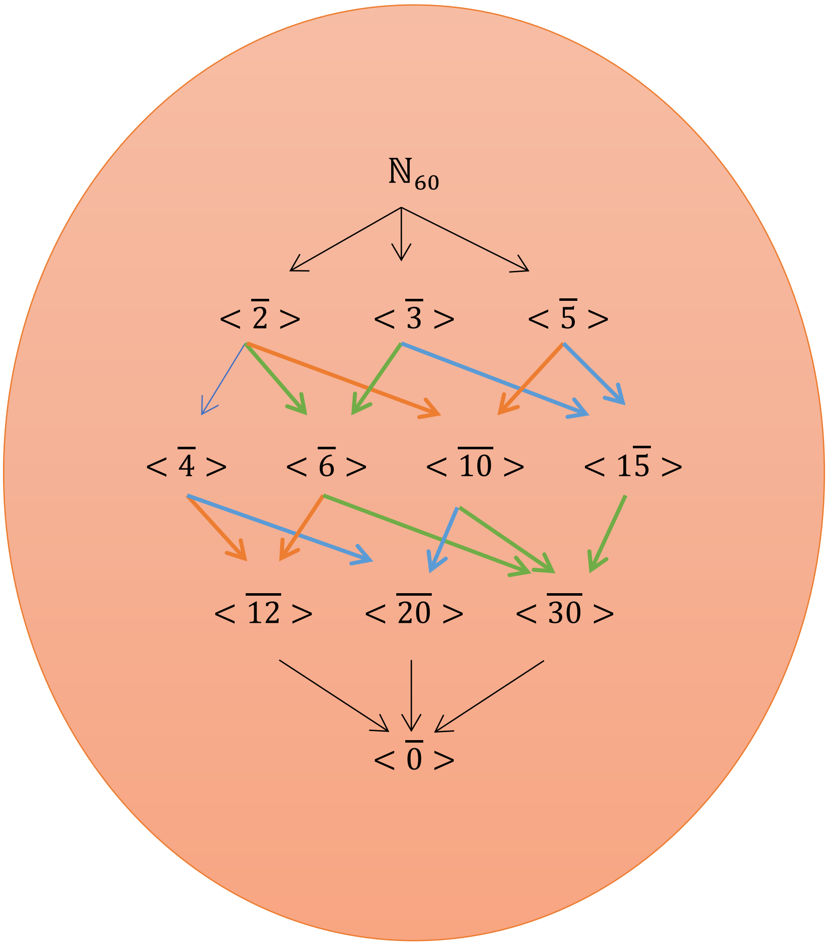
1. Every $\mathcal{T}$-semimodule $\mathcal{W}$ is J-essential in itself.
2. Every essential subsemimodule is J-essential subsemimodule.
3. If $\mathcal{V} \leq_{J.es} \mathcal{W}$ then it is not necessary $\mathcal{V} \leq_{e} \mathcal{W}$. For example: Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{20}$, which, it is has three non-zero semi-prime subsemimodules $\mathcal{V}_1 =< \bar{2} >, \mathcal{V}_2 =< \bar{5} >$ and $\mathcal{V}_3 =<$

$\overline{10}>$. Now, since $<\overline{5}>\cap<\overline{4}>=0$ and $<\overline{4}>\neq 0$ then $<\overline{5}>$ is not essential subsemimodule in $\mathbb{N}_{20}$. But $\begin{cases} <\overline{5}>\cap \mathcal{V}_1 \neq 0 \\ <\overline{5}>\cap \mathcal{V}_2 \neq 0 \\ <\overline{5}>\cap \mathcal{V}_3 \neq 0 \end{cases}$, therefore $<\overline{5}>\leq_{J.es} \mathbb{N}_{20}$.

4. Every J-essential subsemimodule is semi-essential subsemimodule.

   Since $\mathcal{V} \leq_{J.es} \mathcal{W}$ and by (Remark 3.1.2 (1)) all prime subsemimodule is semi-prime then $\mathcal{V} \cap \mathcal{A} \neq 0$ for any $\mathcal{A}$ is a prime subsemimodule of $\mathcal{W}$, thus $\mathcal{V} \leq_{sem} \mathcal{W}$. □

5. If $\mathcal{V} \leq_{sem} \mathcal{W}$ then it is not necessary $\mathcal{V} \leq_{J.es} \mathcal{W}$. For example: Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{60}$, which, it is has twelve subsemimodules which are: as in the diagram below.



Now, there are three non-zero prime subsemimodules $\mathcal{V}_1 =<\overline{2}>$, $\mathcal{V}_2 =<\overline{3}>$ and $\mathcal{V}_3 =<\overline{5}>$, and there are seven non-zero semi-prime subsemimodules $\mathcal{V}_1 =< \overline{2} >, \mathcal{V}_2 =< \overline{3} >, \mathcal{V}_3 =< \overline{5} >, \mathcal{V}_4 =< \overline{6} >, \mathcal{V}_5 =< \overline{10} >, \mathcal{V}_6 =< \overline{15} >$ and $\mathcal{V}_7 =< \overline{30} >$. Now, since $< \overline{4} > \cap \mathcal{V}_6 = 0$ and $\mathcal{V}_6 =< \overline{15} > \neq 0$ then $< \overline{4} >$ is not J-essential.

But $\begin{cases} < \overline{4} > \cap \mathcal{V}_1 \neq 0 \\ < \overline{4} > \cap \mathcal{V}_2 \neq 0 \\ < \overline{4} > \cap \mathcal{V}_3 \neq 0 \end{cases}$, then $< \overline{4} > \leq_{sem} \mathbb{N}_{60}$. □

<u>Proposition 3.2.3</u>: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule. If $\mathcal{V}_1 \leq \mathcal{V}_2 \leq \mathcal{W}$ and $\mathcal{V}_1 \leq_{J.es} \mathcal{W}$ then $\mathcal{V}_2 \leq_{J.es} \mathcal{W}$.

<u>Proof</u>: Suppose that $\mathcal{V}_1 \leq_{J.es} \mathcal{W}$. Let $\mathcal{A}$ be a semi-prime subsemimodule in $\mathcal{W}$ such that $\mathcal{V}_2 \cap \mathcal{A} = 0$, since $\mathcal{V}_1 \leq \mathcal{V}_2$ then $\mathcal{V}_1 \cap \mathcal{A} \leq \mathcal{V}_2 \cap \mathcal{A} = 0$, hence $\mathcal{V}_1 \cap \mathcal{A} = 0$. However, $\mathcal{V}_1 \leq_{J.es} \mathcal{W}$ then $\mathcal{A} = 0$. Therefore, $\mathcal{V}_2 \leq_{J.es} \mathcal{W}$. □

<u>Proposition 3.2.4</u>: Let $\mathcal{V}_i$ be a semi-prime subsemimodules of $\mathcal{T}$-semimodule $\mathcal{W}$, $(i = 1, 2, \ldots, n)$. If $\mathcal{V}_i \leq_{J.es} \mathcal{W}, i = 1, 2, \ldots, n,$ then $\bigcap_{i=1}^{n} \mathcal{V}_i \leq_{J.es} \mathcal{W}$.

<u>Proof</u>: To prove that on two subsemimodules of $\mathcal{W}$. Let $\mathcal{A}$ be a semi-prime subsemimodule in $\mathcal{W}$, then $(\mathcal{V}_1 \cap \mathcal{V}_2) \cap \mathcal{A} = \mathcal{V}_1 \cap (\mathcal{V}_2 \cap \mathcal{A})$. Since $\mathcal{V}_2 \leq_{J.es} \mathcal{W}$ then $\mathcal{V}_2 \cap \mathcal{A} \neq 0$. By (Lemma 3.1.9), then $\mathcal{V}_2 \cap \mathcal{A}$ is semi-prime in $\mathcal{W}$. However, $\mathcal{V}_1 \leq_{J.es} \mathcal{W}$, thus $\mathcal{V}_1 \cap (\mathcal{V}_2 \cap \mathcal{A}) \neq 0$. Thus, $\mathcal{V}_1 \cap \mathcal{V}_2 \leq_{J.es} \mathcal{W}$. Therefore, by induction $\bigcap_{i=1}^{n} \mathcal{V}_i \leq_{J.es} \mathcal{W}$. □

<u>Proposition 3.2.5</u>: Let $\mathcal{V}_1, \mathcal{V}_2 \leq \mathcal{W}$. Then $(\mathcal{V}_1 \cap \mathcal{V}_2) \leq_{J.es} \mathcal{W}$ if and only if $\mathcal{V}_i \leq_{J.es} \mathcal{W}, i = 1, 2$, provided $\mathcal{V}_2 \cap \mathcal{A}$ is semi-prime in $\mathcal{W}$ for each $\mathcal{A}$ is semi-prime subsemimodule in $\mathcal{W}$.

<u>Proof</u>: Since $(\mathcal{V}_1 \cap \mathcal{V}_2) \leq \mathcal{V}_i, i = 1, 2$ and $(\mathcal{V}_1 \cap \mathcal{V}_2) \leq_{J.es} \mathcal{W}$, then by (Proposition 3.2.3) $\mathcal{V}_i \leq_{J.es} \mathcal{W}, i = 1, 2$.

Conversely, let $\mathcal{A}$ be a semi-prime subsemimodule in $\mathcal{W}$ such that $(\mathcal{V}_1 \cap \mathcal{V}_2) \cap \mathcal{A} = 0$, implies $\mathcal{V}_1 \cap (\mathcal{V}_2 \cap \mathcal{A}) = 0$. Since $\mathcal{V}_2 \cap \mathcal{A}$ is semi-prime in $\mathcal{W}$ and $\mathcal{V}_1 \leq_{J.es} \mathcal{W}$ then $\mathcal{V}_2 \cap \mathcal{A} = 0$. However, $\mathcal{V}_2 \leq_{J.es} \mathcal{W}$, thus $\mathcal{A} = 0$. Therefore, $(\mathcal{V}_1 \cap \mathcal{V}_2) \leq_{J.es} \mathcal{W}$. □

Proposition 3.2.6: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $\mathcal{V}_1, \mathcal{V}_2$ be J-essential subsemimodules of $\mathcal{W}$ such that $\mathcal{V}_1 \cap \mathcal{V}_2 \neq 0$ and all semi-prime subsemimodules in $\mathcal{V}_1$ are semi-prime subsemimodules in $\mathcal{W}$, then $(\mathcal{V}_1 \cap \mathcal{V}_2) \leq_{J.es} \mathcal{W}$.

Proof: Let $\mathcal{B}$ be a semi-prime subsemimodule of $\mathcal{W}$ such that $(\mathcal{V}_1 \cap \mathcal{V}_2) \cap \mathcal{B} = 0$, implies $(\mathcal{V}_2 \cap \mathcal{V}_1) \cap \mathcal{B} = 0$, thus $\mathcal{V}_2 \cap (\mathcal{V}_1 \cap \mathcal{B}) = 0$. By (Proposition 3.1.7), then $\mathcal{V}_1 \cap \mathcal{B}$ is a semi-prime subsemimodule in $\mathcal{V}_1$, by assumption, then $(\mathcal{V}_1 \cap \mathcal{B})$ is a semi-prime subsemimodule in $\mathcal{W}$ and $\mathcal{V}_2 \leq_{J.es} \mathcal{W}$ then $\mathcal{V}_1 \cap \mathcal{B} = 0$. However, $\mathcal{V}_1 \leq_{J.es} \mathcal{W}$, then $\mathcal{B} = 0$. Therefore $(\mathcal{V}_1 \cap \mathcal{V}_2) \leq_{J.es} \mathcal{W}$. □

Proposition 3.2.7: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $0 \neq \mathcal{N} \leq \mathcal{W}$, then $\mathcal{N} \leq_{J.es} \mathcal{W}$ if and only if $\mathcal{N} \leq_{sem} \mathcal{W}$, provided every semi-prime subsemimodule in $\mathcal{W}$ is prime.

Proof:

($\Longrightarrow$) It is clear.

($\Longleftarrow$) Assume that, $\mathcal{N} \leq_{sem} \mathcal{W}$. Let $\mathcal{A}$ be a semi-prime subsemimodule in $\mathcal{W}$ such that $\mathcal{N} \cap \mathcal{A} = 0$, (by assumption), then $\mathcal{A}$ is prime subsemimodule in $\mathcal{W}$, since $\mathcal{N} \leq_{sem} \mathcal{W}$ then $\mathcal{A} = 0$. Therefore, $\mathcal{N} \leq_{J.es} \mathcal{W}$. □

Proposition 3.2.8: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $\mathcal{V}_i \leq \mathcal{W}, (i = 1,2,3)$ with $\mathcal{V}_1 \leq \mathcal{V}_2 \leq \mathcal{V}_3$. If $\mathcal{V}_1 \leq_{J.es} \mathcal{V}_2$ and $\mathcal{V}_2 \leq_{J.es} \mathcal{V}_3$ then $\mathcal{V}_1 \leq_{J.es} \mathcal{V}_3$.

Proof:

Let $\mathcal{A}$ be a semi-prime subsemimodule of $\mathcal{V}_3$ such that $\mathcal{V}_1 \cap \mathcal{A} = 0$, to prove $\mathcal{A} = 0$. Since $\mathcal{V}_1 \leq \mathcal{V}_2$ then $0 = \mathcal{V}_1 \cap \mathcal{A} = (\mathcal{V}_1 \cap \mathcal{A}) \cap \mathcal{V}_2$ implies $\mathcal{V}_1 \cap (\mathcal{A} \cap \mathcal{V}_2) = 0$. But by (Proposition 3.1.7) then $(\mathcal{A} \cap \mathcal{V}_2)$ is a semi-prime subsemimodule of $\mathcal{V}_2$ and $\mathcal{V}_1 \leq_{J.es} \mathcal{V}_2$, thus $\mathcal{V}_2 \cap \mathcal{A} = 0$. However, $\mathcal{V}_2 \leq_{J.es} \mathcal{V}_3$ then $\mathcal{A} = 0$, therefore $\mathcal{V}_1 \leq_{J.es} \mathcal{V}_3$. □

In the following, we show that the converse direction to the previous (Proposition 3.2.8) is not true, in general.

Example 3.2.9:

We will consider the ℕ-semimodule $\mathcal{W} = \mathbb{Z}_8 \oplus \mathbb{Z}_2$, see (Example 1.2.9). Now, there are four semi-prime subsemimodules of $\mathcal{W}$ which are $\mathcal{V}_2, \mathcal{V}_4, \mathcal{V}_5$ and $\mathcal{V}_{10}$, and three semi-prime subsemimodules of the subsemimodule $\mathcal{V}_9$ which are $\mathcal{V}_3, \mathcal{V}_7$ and $\mathcal{V}_8$. Now, $\mathcal{V}_7 \leq \mathcal{V}_9 \leq \mathcal{W}$ and $\mathcal{V}_7 \leq_{J.es} \mathcal{W}$, but $\mathcal{V}_7$ is not J-essential in $\mathcal{V}_9$. □

Proposition 3.2.10: If $\mathcal{W}$ is a multiplication faithful $\mathcal{T}$-semimodule, $0 \neq \mathcal{V}$ is a semi-prime subsemimodule in $\mathcal{W}$ and $\mathcal{V}$ is not minimal semi-prime subsemimodule then $\mathcal{V} \leq_{J.es} \mathcal{W}$.

Proof:

Let $\mathcal{A}$ be a non-zero semi-prime subsemimodule in $\mathcal{W}$ such that $\mathcal{V} \cap \mathcal{A} = 0$. Since $\mathcal{V}$ is a semi-prime subsemimodule in $\mathcal{W}$ and $\mathcal{W}$ is multiplication, then there exists semi-prime ideals $I, \mathcal{S}$ of $\mathcal{T}$ such that $\mathcal{V} = I\mathcal{W}$ and $\mathcal{A} = \mathcal{S}\mathcal{W}$. However, $\mathcal{V}$ is not minimal semi-prime, then there is a semi-prime subsemimodule $\mathcal{H}$ in $\mathcal{W}$ and $\mathcal{H} \subseteq \mathcal{V}$. Therefore, there is a semi-prime ideal nonzero K in $\mathcal{T}$ such that $\mathcal{H} = K\mathcal{W} \neq \mathcal{W}$. Now, $(I \cap \mathcal{S})\mathcal{W} \subseteq I\mathcal{W} \cap \mathcal{S}\mathcal{W} = \mathcal{V} \cap \mathcal{A} = 0$, but $\mathcal{W}$ is a faithful, then $I \cap \mathcal{S} = 0$, hence $I \cap \mathcal{S} \subseteq K$. Thus, either $I \subseteq K$ or $\mathcal{S} \subseteq K$. If $I \subseteq K$, then $\mathcal{V} = I\mathcal{W} \subseteq$

$K\mathcal{W} = \mathcal{H}$, contradiction. If $\mathcal{S} \subseteq K$, then $\mathcal{S}\mathcal{W} \subseteq K\mathcal{W}$. Thus that, $\mathcal{A} \subseteq \mathcal{H} \subseteq \mathcal{V}$. Hence $0 = \mathcal{V} \cap \mathcal{A} = \mathcal{A}$, a contradiction. This prove $\mathcal{V} \cap \mathcal{A} \neq 0$. Therefore, $\mathcal{V} \leq_{J.es} \mathcal{W}$. □

Note: If $\mathcal{V}$ is a J-essential subsemimodule in $\mathcal{W}$, then $\Psi(\mathcal{V})$ is not necessary be a J-essential subsemimodule in $\mathcal{W}'$. For example: $\Psi: \mathbb{N}_{24} \longrightarrow \mathbb{N}_{24}$ define by $\Psi(\bar{x}) = 2\bar{x}$ for all $x \in \mathbb{N}_{24}$. We have $\mathcal{V} =< \bar{4} >$ is a J-essential subsemimodule of $\mathbb{N}_{24}$. But, $\Psi(\mathcal{V}) =< \bar{8} >$ is not J-essential.

In the following result, we will show that if $\mathcal{V}$ is a J-essential subsemimodule in $\mathcal{W}$, then $\Psi(\mathcal{V})$ is also J-essential subsemimodule in $\mathcal{W}'$ under a given condition.

Proposition 3.2.11: Let $\mathcal{W}, \mathcal{W}'$ be $\mathcal{T}$-semimodules, $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ be an isomorphism. If $\mathcal{V} \leq_{J.es} \mathcal{W}$, then $\Psi(\mathcal{V}) \leq_{J.es} \mathcal{W}'$.

Proof: Let $0 \neq \mathcal{B}'$ be a semi-prime subsemimodule of $\mathcal{W}'$. Since Ψ is an epimorphism then by (Lemma 3.1.11), $0 \neq \Psi^{-1}(\mathcal{B}')$ is a semi-prime subsemimodule of $\mathcal{W}$, but $\mathcal{V} \leq_{J.es} \mathcal{W}$, then $\mathcal{V} \cap \Psi^{-1}(\mathcal{B}') \neq 0$. Since Ψ is a monomorphism, then $\Psi(\mathcal{V} \cap \Psi^{-1}(\mathcal{B}')) \neq 0$, thus, $\Psi(\mathcal{V}) \cap \mathcal{B}' \neq 0$ for each $\mathcal{B}'$ in $\mathcal{W}'$. Therefore, $\Psi(\mathcal{V}) \leq_{J.es} \mathcal{W}'$. □

Proposition 3.2.12: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a $\mathcal{T}$-semimodule, where $\mathcal{W}_i \leq \mathcal{W}$, $(i = 1,2)$, $< 0 > \neq \mathcal{V}_i \leq \mathcal{W}_i$, $(i = 1, 2)$. If $\mathcal{V}_i \leq_{J.es} \mathcal{W}_i$, $(i = 1, 2)$ then $(\mathcal{V}_1 \oplus \mathcal{V}_2) \leq_{J.es} \mathcal{W}$.

Proof:

Let $0 \neq \mathcal{B}$ be a semi-prime subsemimodule in $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$, and λ_i is a natural projection map of $\mathcal{W}$ on $\mathcal{W}_i$, $(i = 1, 2)$. If $\mathcal{B} \cap \mathcal{W}_1 \neq 0$, by (Proposition 3.1.7), $\mathcal{B} \cap \mathcal{W}_1$ is a semi-prime subsemimodule in $\mathcal{W}_1$. Since $\mathcal{V}_1 \leq_{J.es} \mathcal{W}_1$, then $\mathcal{V}_1 \cap (\mathcal{B} \cap \mathcal{W}_1) \neq 0$, consequently, there is $0 \neq b \in \mathcal{V}_1 \cap \mathcal{B} \cap \mathcal{W}_1$, then $0 \neq b \in \mathcal{V}_1 \cap \mathcal{B}$. Thus $0 \neq b \in (\mathcal{V}_1 \oplus \mathcal{V}_2) \cap \mathcal{B}$.

Therefor, $(\mathcal{V}_1 \oplus \mathcal{V}_2) \cap \mathcal{B} \neq 0$. Similarly, if $\mathcal{B} \cap \mathcal{W}_2 \neq 0$, then $(\mathcal{V}_1 \oplus \mathcal{V}_2) \cap \mathcal{B} \neq 0$.

Assume that, $\mathcal{B} \cap \mathcal{W}_1 = \mathcal{B} \cap \mathcal{W}_2 = 0$. Thus 0 is a semi-prime subsemimodule in $\mathcal{W}_i$, $(i = 1, 2)$. Then any subsemimodule of $\mathcal{W}_i$ is semi-prime subsemimodule by (Remark 3.1.2 (4)) $(i = 1, 2)$. Then $\lambda_i(\mathcal{B})$ is a semi-prime subsemimodule in $\mathcal{W}_i, i = 1,2$, since $\mathcal{V}_i \leq_{J.es} \mathcal{W}_i$ then, $\mathcal{V}_i \cap \lambda_i(\mathcal{B}) \neq 0$ for $(i = 1, 2)$. Let $0 \neq b = \lambda_1(b) + \lambda_2(b) \in \mathcal{B}$, if $\lambda_2(b) = 0$ then $0 \neq b = \lambda_1(b) \in \lambda_1(\mathcal{B}) \cap \mathcal{W}_1 \subseteq \mathcal{B} \cap \mathcal{W}_1 = 0$, a contradiction. Then $\lambda_2(b) \neq 0$, also $\lambda_1(b) \neq 0$, Hence $0 \neq b = \lambda_1(b) + \lambda_2(b) \in (\lambda_1(\mathcal{B}) \cap \mathcal{V}_1) \oplus (\lambda_2(\mathcal{B}) \cap \mathcal{V}_2) \subseteq \mathcal{V}_1 \oplus \mathcal{V}_2$. That is, $0 \neq b \in (\mathcal{V}_1 \oplus \mathcal{V}_2) \cap \mathcal{B}$. Therefore $(\mathcal{V}_1 \oplus \mathcal{V}_2) \leq_{J.es} \mathcal{W}$. □

In the following, we show that the converse of (Proposition 3.2.12) is not true, in general.

Example 3.2.13: Consider the $\mathbb{Z}$-semimodule $\mathcal{W} = \mathbb{N}_6 \oplus \mathbb{N}_4$. And $\mathcal{V}_1 =< \overline{3} > \leq \mathbb{N}_6, \mathcal{V}_2 =< \overline{2} > \leq \mathbb{N}_4$, then, $(\mathcal{V}_1 \oplus \mathcal{V}_2) = \left(< \overline{3} > \oplus < \overline{2} >\right) \leq_{J.es} (\mathbb{N}_6 \oplus \mathbb{Z}_4) = (\mathcal{W}_1 \oplus \mathcal{W}_2)$ (since $\left(\left(< \overline{3} > \oplus < \overline{2} >\right) \cap \mathcal{A} \neq 0\right)$) for any $\mathcal{A}$ is a semi-prime subsemimodule in $\mathcal{W} = \mathbb{N}_6 \oplus \mathbb{N}_4$. However, $< \overline{3} >$ is not J-essential subsemimodule in $\mathbb{Z}_6$ (since $< \overline{3} > \cap < \overline{2} > = < \overline{0} >$). □

Proposition 3.2.14: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a $\mathcal{T}$-semimodule, where $\mathcal{W}_i \leq \mathcal{W}, i = 1,2$, $0 \neq \mathcal{V}_1 \leq \mathcal{W}_1$ and $0 \neq \mathcal{V}_2 \leq \mathcal{W}_2$, if $(\mathcal{V}_1 \oplus \mathcal{V}_2) \leq_{J.es} \mathcal{W}$ then $\mathcal{V}_1 \leq_{J.es} \mathcal{W}_1$, provided that every semi-prime subsemimodule in $\mathcal{W}_1$ is a semi-prime subsemimodule of $\mathcal{W}$.

Proof:

Let $\mathcal{A}_1$ be a semi-prime subsemimodule of $\mathcal{W}_1$, by condition, then $\mathcal{A}_1$ is a semi-prime subsemimodule in $\mathcal{W}$, since $\mathcal{V}_1 \oplus \mathcal{V}_2 \leq_{J.es} \mathcal{W}$, then

$(\mathcal{V}_1 \oplus \mathcal{V}_2) \cap \mathcal{A} \neq 0$. However, $\mathcal{V}_2 \cap \mathcal{A} = 0$, then $\mathcal{V}_1 \cap \mathcal{A} \neq 0$. Therefore, $\mathcal{V}_1 \leq_{J.es} \mathcal{W}_1$. □

Proposition 3.2.15: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a distributive fully essential $\mathcal{T}$-semimodule, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are subsemimodules of $\mathcal{W}$, $< 0 > \neq \mathcal{N}_i \leq \mathcal{W}_i$, $i = 1, 2$, then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{J.es} \mathcal{W}$ if and only if $\mathcal{N}_i \leq_{J.es} \mathcal{W}_i, i = 1, 2$.

Proof:

($\Longrightarrow$) Assume that, $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{J.es} \mathcal{W}$. Since $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{J.es} \mathcal{W}$, then by (Remark 3.2.2 (4)) $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$, but $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ is a fully essential $\mathcal{T}$-semimodule, then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_e \mathcal{W}$. Let $0 \neq \mathcal{H}_1 \leq \mathcal{W}_1 \leq \mathcal{W}$, then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \cap \mathcal{H}_1 \neq 0$, implies $(\mathcal{N}_1 \cap \mathcal{H}_1) + (\mathcal{N}_2 \cap \mathcal{H}_1) \neq 0$. However, $(\mathcal{N}_2 \cap \mathcal{H}_1) = 0$, conesquntly $(\mathcal{N}_1 \cap \mathcal{H}_1) \neq 0$, then $\mathcal{N}_1 \leq_e \mathcal{W}_1$. Therefore $\mathcal{N}_1 \leq_{J.es} \mathcal{W}_1$. Similarly, $\mathcal{N}_2 \leq_{J.es} \mathcal{W}_2$.

($\Longleftarrow$) By (Proposition. 3.2.12), $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{J.es} \mathcal{W}$. □

Proposition 3.2.16: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a distributive and fully prime $\mathcal{T}$-semimodule, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are subsemimodules of $\mathcal{W}$, $< 0 > \neq \mathcal{N}_i \leq \mathcal{W}_i$ $(i = 1, 2)$, then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{J.es} \mathcal{W}$ if and only if $\mathcal{N}_i \leq_{J.es} \mathcal{W}_i, (i = 1, 2)$.

Proof:

($\Longrightarrow$) Assume that, $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{J.es} \mathcal{W}$. Let $\mathcal{P}_1$ be a semi-prime subsemimodule in $\mathcal{W}_1$ such that $\mathcal{N}_1 \cap \mathcal{P}_1 = 0$, since $\mathcal{W}$ is fully prime and $\mathcal{P}_1 \leq \mathcal{W}_1 \leq \mathcal{W}$ then $\mathcal{P}_1$ is a prime subsemimodule in $\mathcal{W}$, by (Remark 3.1.2 (1)), then $\mathcal{P}_1$ is a semi-prime subsemimodule in $\mathcal{W}$. Since $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ thus $\mathcal{W}_1 \cap \mathcal{W}_2 = 0$, implies $\mathcal{P}_1 \cap \mathcal{N}_2 = 0$ for each $\mathcal{N}_2 \leq \mathcal{W}_2$. Now, $(\mathcal{N}_1 \cap \mathcal{P}_1) \oplus (\mathcal{N}_2 \cap \mathcal{P}_1) = 0$, since $\mathcal{W}$ is distributive $\mathcal{T}$-semimodule thus

$(\mathcal{N}_1 \oplus \mathcal{N}_2) \cap \mathcal{P}_1 = 0$, but $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{J.es} \mathcal{W}$, therefore $\mathcal{P}_1 = 0$, consequently $\mathcal{N}_1 \leq_{J.es} \mathcal{W}_1$. Similarly, $\mathcal{N}_2 \leq_{J.es} \mathcal{W}_2$.

($\Leftarrow$) By (Proposition 3.2.12), $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{J.es} \mathcal{W}$. □

3.3. J-extending Semimodules

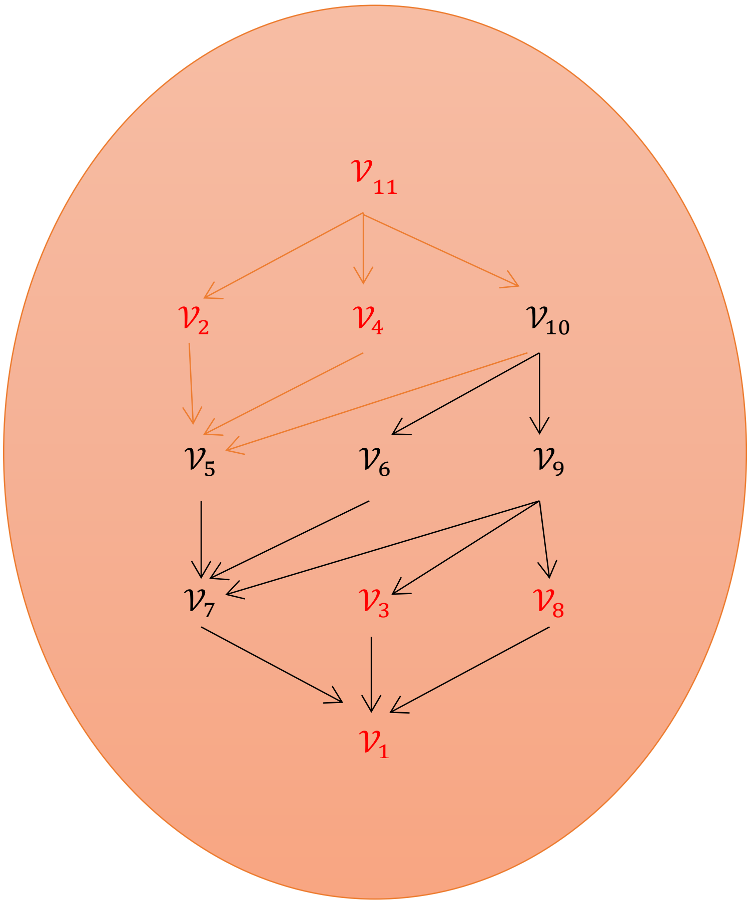
In this section, the J-extending semimodule will be introduced and investigated throughout this section. This concept lies between extending semimodule and semi-extending semimodule. Also, some results will be obtained.

Definition 3.3.1: A $\mathcal{T}$-semimodule $\mathcal{W}$ is called J-extending if each subsemimodule in $\mathcal{W}$ is J-essential in direct summand of $\mathcal{W}$. A semiring $\mathcal{T}$ is said to be J-extending, if $\mathcal{T}$ is J-extending semimodule.

Remarks and Examples 3.3.2:

1. The $\mathbb{N}$-semimodule $\mathbb{N}_{20}$ is J-extending semimodule, (since, there are six subsemimodules which are: $\mathcal{N}_1 =< \overline{0} >, \mathcal{N}_2 =< \overline{2} >, \mathcal{N}_3 =< \overline{5} >, \mathcal{N}_4 =< \overline{4} >, \mathcal{N}_5 =< \overline{10} >$ and $\mathcal{N}_6 = \mathbb{N}_{20}$. There are four subsemimodules direct summands $\mathcal{N}_1, \mathcal{N}_3, \mathcal{N}_4$ and $\mathcal{N}_6$. While there are three nonzero semi-prime subsemimodules $\mathcal{N}_2, \mathcal{N}_3$ and $\mathcal{N}_5$. Now, any direct summand is J-essential in itself, $\mathcal{N}_2 \leq_{J.es} \mathcal{N}_6 \leq^{\oplus} \mathbb{N}_{20}$ and $\mathcal{N}_5 \leq_{J.es} \mathcal{N}_6 \leq^{\oplus} \mathbb{N}_{20}$).
2. Every semisimple semimodule is a J-extending.
   e.g. The $\mathbb{N}$-semimodule $\mathbb{N}_6$ is a J-Extending, (since for each subsemimodule $\mathcal{N}$ of $\mathbb{N}_6$, $\mathcal{N} \leq_{J.es} \mathcal{N}$, with $\mathcal{N} \leq^{\oplus} \mathbb{N}_6$).
3. Every CS-semimodule is J-Extending. (since, if $\mathcal{V} \leq_e \mathcal{W}$, then $\mathcal{V} \leq_{J.es} \mathcal{W}, \forall\, \mathcal{V} \leq \mathcal{W}$).

4. If $\mathcal{W}$ is J-extending, then it is not necessary $\mathcal{W}$ is a CS-semimodule. Consider the $\mathbb{N}$-semimdule $\mathcal{W} = \mathbb{Z}_8 \oplus \mathbb{Z}_2$, by (Example 1.2.9). Which, it is has four semi-prime subsemimodules of $\mathcal{W}$ which are $\mathcal{V}_2, \mathcal{V}_4, \mathcal{V}_5$ and $\mathcal{V}_{10}$, and six subsemimodules direct summands of $\mathcal{W}$ which are $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_8$ and $\mathcal{V}_{11}$. As in the diagram below.



Now, since every $\mathcal{V}_i \leq_{J.es} \mathcal{V}_i \leq^{\oplus} \mathcal{W}, i = 1, 2, 3, 4, 8, 11,$ and $\mathcal{V}_k \leq_{J.es} \mathcal{W} \leq^{\oplus} \mathcal{W}, k = 5, 6, 7, 9, 10$ then $\mathcal{W}$ is J-Extending semimodule. However, this $\mathcal{W}$ is not CS-semimodule, (since, $\mathcal{V}_6 \cap \mathcal{V}_8 = 0$, thus $\mathcal{V}_6$ is not essential in a direct summand of $\mathcal{W}$). □

5. Any uniform semimodule is a J-extending semimodule. But the converse is not true.

   For example: Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{36}$, by (Example 1.3.2.1). Now, there are four direct summands in $\mathbb{N}_{36}$ which are

$\mathcal{V}_1, \mathcal{V}_4, \mathcal{V}_6$ and $\mathcal{V}_9$. While $\mathcal{V}_2, \mathcal{V}_3$ and $\mathcal{V}_5$ are a semi-prime subsemimodules in $\mathcal{V}_9$. $\mathcal{V}_7$ is a semi-prime subsemimodule in $\mathcal{V}_4$. While $\mathcal{V}_8$ is a semi-prime subsemimodule in $\mathcal{V}_6$. Since $\mathcal{V}_4$ and $\mathcal{V}_6$ are direct summand, $\mathcal{V}_2, \mathcal{V}_3$ and $\mathcal{V}_5$ are J-essential in $\mathbb{N}_{36}$. It is enough to examin $\mathcal{V}_7$ and $\mathcal{V}_8$. Since $\mathcal{V}_7 \cap \mathcal{V}_7 = \mathcal{V}_7 \neq 0 \Longrightarrow \mathcal{V}_7 \leq_{J.es} \mathcal{V}_4 \leq^{\oplus} \mathbb{N}_{36}$, also, since $\mathcal{V}_8 \cap \mathcal{V}_8 = \mathcal{V}_8 \neq 0 \Longrightarrow \mathcal{V}_8 \leq_{J.es} \mathcal{V}_6 \leq^{\oplus} \mathbb{N}_{36}$. Therefore, the $\mathbb{N}$-semimodule $\mathbb{N}_{36}$ is J-extending semimodule. But $\mathcal{V}_8 \cap \mathcal{V}_7 =< \overline{18} > \cap < \overline{12} >= 0$, then $\mathcal{V}_8 =< \overline{18} >$ is not essential subsemimodule in $\mathbb{N}_{36}$, therefore the $\mathbb{N}$-semimodule $\mathbb{N}_{36}$ is not uniform.

6. If $\mathcal{W}$ is J-extending then it is not necessary $\mathcal{W}$ is semi-uniform semimodule.

   For example: Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{24}$. by (Example 1.3.2.2). There are four direct summands in $\mathbb{N}_{24}$ and they $\mathcal{V}_1, \mathcal{V}_3, \mathcal{V}_6$ and $\mathcal{V}_8$. While $\mathcal{V}_2, \mathcal{V}_3$ and $\mathcal{V}_5$ are a semi-prime subsemimodules in $\mathcal{V}_8$. So $\mathcal{V}_5$ is a semi-prime subsemimodule in $\mathcal{V}_3$. Now, since $\mathcal{V}_3$ and $\mathcal{V}_8$ are direct summand, $\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5$ and $\mathcal{V}_7$ are J-essential in $\mathbb{N}_{24}$. Therefore, the $\mathbb{N}$-semimodule $\mathbb{N}_{24}$ is J-Extending semimodule. But $\mathcal{V}_6 \cap \mathcal{V}_3 =< \overline{8} > \cap < \overline{3} >=< \overline{0} >$, thus $\mathcal{V}_6 =< \overline{8} >$ is not semi-essential subsemimodule in $\mathbb{N}_{24}$, therefore the $\mathbb{N}$-semimodule $\mathbb{N}_{24}$ is not semi-uniform.

Proposition 3.3.3: Every J-extending semimodule is an SCS-semimodule.

Proof: Let $\mathcal{N} \leq \mathcal{W}$, with $\mathcal{W}$ is a J-Extending semimodule, then there is $\mathcal{H} \leq^{\oplus} \mathcal{W}$ such that $\mathcal{N} \leq_{J.es} \mathcal{H}$. Since every J-essential subsemimodule is semi-essential, then $\mathcal{N} \leq_{sem} \mathcal{H} \leq^{\oplus} \mathcal{W}$. Therefore, $\mathcal{W}$ is a SCS-semimodule. □

Proposition 3.3.4: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, if $\mathcal{W}$ is a J-extending semimodule and indecomposable, then $\mathcal{W}$ is a semi-uniform semimodule.

Proof:

Let $\mathcal{V}$ be a nonzero subsemimodule of $\mathcal{W}$, since $\mathcal{W}$ is a J-extending semimodule, by (Proposition 3.3.3), then $\mathcal{W}$ is an SCS-semimodule and by (Proposition 2.1.3). Thus $\mathcal{W}$ is a semi-uniform semimodule. □

Theorem 3.3.5: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule. Then $\mathcal{W}$ is a J-extending semimodule if and only if for each St-closed subsemimodule in $\mathcal{W}$ is a direct summand of $\mathcal{W}$, provided every semi-prime subsemimodule in $\mathcal{W}$ is prime.

Proof:

(⟹) Suppose that $\mathcal{W}$ is a J-extending semimodule. Let $\mathcal{V} \leq_{Stc} \mathcal{W}$, since $\mathcal{W}$ is a J-Extending semimodule then by (Proposition 3.3.3) $\mathcal{W}$ is an SCS-semimodule, thus there is a subsemimodule $\mathcal{H}$ of $\mathcal{W}$ such that $\mathcal{V} \leq_{sem} \mathcal{H} \leq^{\oplus} \mathcal{W}$. But $\mathcal{V} \leq_{Stc} \mathcal{W}$, thus $\mathcal{V} = \mathcal{H}$. Therefore, $\mathcal{V} \leq^{\oplus} \mathcal{W}$.

(⟸) Suppose that any St-closed subsemimodule in $\mathcal{W}$ is a direct summand of $\mathcal{W}$. Let $\mathcal{V} \leq \mathcal{W}$, if $\mathcal{V} = 0$ thus $\mathcal{V} \leq_{J.es} < \overline{0} >$ and $< \overline{0} > \leq^{\oplus} \mathcal{W}$. If $\mathcal{V} \neq 0$ then by (Lemma 2.1.5 (b)) there exists $\mathcal{A} \leq_{Stc} \mathcal{W}$ such that $\mathcal{V} \leq_{sem} \mathcal{A}$. Since every semi-prime subsemimodule in $\mathcal{W}$ is prime and by (Proposition 3.2.7) then $\mathcal{V} \leq_{J.es} \mathcal{A}$. But every St-closed subsemimodule in $\mathcal{W}$ is a direct summand of $\mathcal{W}$, thus $\mathcal{A} \leq^{\oplus} \mathcal{W}$. Therefore, $\mathcal{W}$ is an J-Extending semimodule. □

Proposition 3.3.6: Let $\mathcal{V}, \mathcal{H} \leq \mathcal{W}$, with $\mathcal{W}$ is a J-extending $\mathcal{T}$-semimodule. If $(\mathcal{V} \cap \mathcal{H})$ is an St-closed in $\mathcal{W}$, then $(\mathcal{V} \cap \mathcal{H}) \leq^{\oplus} \mathcal{V}$ and $(\mathcal{V} \cap \mathcal{H}) \leq^{\oplus} \mathcal{H}$. Provided $\mathcal{V}$ and $\mathcal{H}$ are subtractive in $\mathcal{W}$.

Proof: Follows from (Theorem 3.3.5 and Lemma 2.1.7). □

Proposition 3.3.7: Let $\mathcal{W}$ be a fully prime $\mathcal{T}$-semimodule. Then $\mathcal{W}$ is a J-extending semimodule if and only if $\mathcal{W}$ is an SCS-semimodule.

Proof:

(⟹) Follows from (Proposition 3.3.3).

(⟸) Assume that $\mathcal{W}$ is an SCS-semimodule. To prove that $\mathcal{W}$ is a J-extending semimodule. Let $0 \neq \mathcal{V} \leq \mathcal{W}$, since $\mathcal{W}$ is an SCS-semimodule then there is a $\mathcal{B} \leq^{\oplus} \mathcal{W}$ such that $\mathcal{V} \leq_{sem} \mathcal{B}$, but $\mathcal{W}$ is a fully prime semimodule, thus $\mathcal{V} \cap \mathcal{H} \neq 0$ for any $0 \neq \mathcal{H} \leq \mathcal{W}$, thus $\mathcal{V} \leq_{J.es} \mathcal{B} \leq^{\oplus} \mathcal{W}$. Therefore, $\mathcal{W}$ is a J-extending semimodule. □

Proposition 3.3.8: Let $\mathcal{W}$ be a fully essential $\mathcal{T}$-semimodule. Then $\mathcal{W}$ is a J-extending semimodule if and only if $\mathcal{W}$ is a CS-semimodule.

Proof:

(⟹) Assume that $\mathcal{W}$ is a J-extending semimodule. To prove that $\mathcal{W}$ is a CS-semimodule. Let $0 \neq \mathcal{N} \leq \mathcal{W}$, Since $\mathcal{W}$ is a J-extending semimodule then there is $\mathcal{H} \leq^{\oplus} \mathcal{W}$ such that $\mathcal{N} \leq_{J.es} \mathcal{H}$, thus $\mathcal{N} \leq_{sem} \mathcal{H}$. But $\mathcal{W}$ is fully essential semimodule then $\mathcal{N} \leq_{e} \mathcal{H}$. Therefore, $\mathcal{W}$ is a CS-semimodule.

(⟸) Follows from (Remark 3.3.2 (3)). □

3.4. J-closed subsemimodules

Definition 3.4.1: Let $0 \neq \mathcal{N}$ be a subsemimodule of $\mathcal{T}$-semimodule $\mathcal{W}$. Then, $\mathcal{N}$ is called J-closed in $\mathcal{W}$ (briefly $\mathcal{N} \leq_{Jc} \mathcal{W}$) if $\mathcal{N}$ has no proper J-essential extension in $\mathcal{W}$, i.e., if there is $\mathcal{H} \leq \mathcal{W}$ such that $\mathcal{N} \leq_{J.es} \mathcal{H}$ then

$\mathcal{N} = \mathcal{H}$. An ideal E of $\mathcal{T}$ is said to be J-closed, if it is a J-closed $\mathcal{T}$-subsemimodule.

Remarks and Examples 3.4.2:

1. Every $\mathcal{T}$-semimodule $\mathcal{W}$ is a J-closed subsemimodule in itself.
2. Consider the $\mathbb{N}$-semimodule $\mathcal{W} = \mathbb{N}_{12}$, the number subsemimodules in $\mathbb{N}_{12}$ is six subsemimodules which are $\mathcal{V}_1 =< \overline{0} >, \mathcal{V}_2 =< \overline{2} >, \mathcal{V}_3 =< \overline{3} >, \mathcal{V}_4 =< \overline{4} >, \mathcal{V}_5 =< \overline{6} >$ and $\mathcal{V}_6 = \mathbb{N}_{12}$. Now, the subsemimodules $\mathcal{V}_4$ and $\mathcal{V}_6$ are J-closed in $\mathcal{W}$ (since, $\mathcal{V}_4$ and $\mathcal{V}_6$ have no proper J-essential extensions in $\mathcal{W}$).
3. Any St-closed subsemimodule is J-closed.

   Assume that, $\mathcal{V} \leq_{Stc} \mathcal{W}$, to prove $\mathcal{V} \leq_{Jc} \mathcal{W}$. Let $\mathcal{V} \leq_{J.es} \mathcal{H}$, with $\mathcal{H} \leq \mathcal{W}$, thus $\mathcal{V} \leq_{sem} \mathcal{H}$. But, $\mathcal{V} \leq_{Stc} \mathcal{W}$, then $\mathcal{V} = \mathcal{H}$. Therefore, $\mathcal{V} \leq_{Jc} \mathcal{W}$. □
4. If $\mathcal{V} \leq_{Jc} \mathcal{W}$ then it is not necessary $\mathcal{V} \leq_{Stc} \mathcal{W}$. For example: Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{30}$, note that: there are three non-zero prime subsemimodules in $\mathbb{N}_{30}$ $\mathcal{V}_1 =< \overline{2} >, \mathcal{V}_2 =< \overline{3} >$ and $\mathcal{V}_3 =< \overline{5} >$. There are six non-zero semi-prime subsemimodules $\mathcal{V}_1 =< \overline{2} >, \mathcal{V}_2 =< \overline{3} >, \mathcal{V}_3 =< \overline{5} >, \mathcal{V}_4 =< \overline{6} >, \mathcal{V}_5 =< \overline{10} >$ and $\mathcal{V}_6 =< \overline{15} >$. Now, since $\mathcal{V}_1 \cap \mathcal{V}_6 = 0$ and $\mathcal{V}_6 \neq 0$ then $\mathcal{V}_1 =< \overline{2} >$ is not J-essential subsemimodule in $\mathbb{N}_{30}$, thus $\mathcal{V}_1 \leq_{Jc} \mathbb{N}_{30}$. But $\begin{cases} \mathcal{V}_1 \cap \mathcal{V}_1 \neq 0 \\ \mathcal{V}_1 \cap \mathcal{V}_2 \neq 0 \\ \mathcal{V}_1 \cap \mathcal{V}_3 \neq 0 \end{cases}$, thus $\mathcal{V}_1 \leq_{sem} \mathbb{N}_{30}$. Therefore, $\mathcal{V}_1 =< \overline{2} >$ is not St-closed subsemimodule in $\mathbb{N}_{30}$.
5. If $\mathcal{H}$ is a direct summand of $\mathcal{T}$-semimodule $\mathcal{W}$ then it is not necessary $\mathcal{H} \leq_{Jc} \mathcal{W}$. For example: Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{20}$, note that $< \overline{4} > \oplus < \overline{5} > = \mathbb{N}_{20}$ and $< \overline{4} >$ is not J-essential

subsemimodule in $\mathbb{N}_{20}$, therefore $<\overline{4}>\leq_{Jc}\mathbb{N}_{20}$. But $<\overline{5}>\leq_{J.es}\mathbb{N}_{20}$, therefore $<\overline{5}>$ is not J-closed in $\mathbb{N}_{20}$.

<u>Proposition 3.4.3</u>: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule. If $\mathcal{V}_1\leq_{Jc}\mathcal{V}_2$ and $\mathcal{V}_2\leq_{Jc}\mathcal{V}_3$ for any $\mathcal{V}_i\leq\mathcal{W}, i=1,2,3$. Then $\mathcal{V}_1\leq_{Jc}\mathcal{V}_3$, provided that $\mathcal{V}_2$ contained in (or containing) any J-essential extension of $\mathcal{V}_1$.

<u>*Proof*</u>: Let $\mathcal{N}\leq\mathcal{V}_3$ such that $\mathcal{V}_1\leq_{J.es}\mathcal{N}\leq\mathcal{V}_3$. there are two cases:

- If $\mathcal{N}\leq\mathcal{V}_2$ then $\mathcal{V}_1=\mathcal{N}$, (since, $\mathcal{V}_1\leq_{Jc}\mathcal{V}_2$). Hence $\mathcal{V}_1\leq_{Jc}\mathcal{V}_3$.
- If $\mathcal{V}_2\leq\mathcal{N}$ then $\mathcal{V}_2\leq_{J.es}\mathcal{N}$ (since, $\mathcal{V}_1\leq_{J.es}\mathcal{N}$), but $\mathcal{V}_2\leq_{Jc}\mathcal{V}_3$ then $\mathcal{V}_2=\mathcal{N}$. That is $\mathcal{V}_1\leq_{J.es}\mathcal{V}_2$. Since $\mathcal{V}_1\leq_{Jc}\mathcal{V}_2$, thus $\mathcal{V}_1=\mathcal{V}_2$. Therefore, $\mathcal{V}_1\leq_{Jc}\mathcal{V}_3$. □

<u>Remark 3.4.4</u>: Let $0\neq\mathcal{N}\leq\mathcal{H}\leq\mathcal{W}$. If $\mathcal{N}\leq_{Jc}\mathcal{W}$ then it is not necessary $\mathcal{H}\leq_{Jc}\mathcal{W}$. For example: Consider the $\mathbb{N}$-semimodule $\mathbb{N}_{24}$. Let $\mathcal{N}=<\overline{8}>$, $\mathcal{H}=<\overline{4}>$, then $\mathcal{N}\leq\mathcal{H}\leq\mathbb{N}_{24}$. Now, since $\begin{cases}\mathcal{N}\cap<\overline{12}>=0\\ \mathcal{N}\cap<\overline{6}>=0\\ \mathcal{N}\cap<\overline{3}>=0\end{cases}$, where $\begin{cases}<\overline{12}>\text{ is semiprime in }<\overline{4}>\\ <\overline{6}>\text{ is semiprime in }<\overline{2}>\\ <\overline{3}>\text{ is semiprime in }\mathbb{N}_{24}\end{cases}$, thus $\mathcal{N}$ is J-essential. Therefore, $\mathcal{N}\leq_{Jc}\mathcal{W}$. But, $\begin{cases}\mathcal{H}\cap<\overline{12}>\neq 0\\ \mathcal{H}\cap<\overline{6}>\neq 0\end{cases}$, thus $\mathcal{H}\leq_{J.es}\mathbb{N}_{24}$. Therefore, $\mathcal{H}$ is not J-closed in $\mathbb{N}_{24}$. □

<u>Lemma 3.4.5</u>: Any subsemimodule $\mathcal{N}$ in $\mathcal{W}$ is a J-essential in J-closed subsemimodule of $\mathcal{W}$.

<u>*Proof*</u>: Let $\mathcal{N}\leq\mathcal{W}$. If $\mathcal{N}\leq_{J.es}\mathcal{W}$ then $\mathcal{W}\leq_{Jc}\mathcal{W}$. Now, if $\mathcal{N}\leq_{J.es}\mathcal{H}\leq\mathcal{W}$, for some $\mathcal{H}$ then, either $\mathcal{H}\leq_{Jc}\mathcal{W}$, thus $\mathcal{N}\leq_{J.es}\mathcal{H}\leq_{Jc}\mathcal{W}$ or $\mathcal{H}\leq_{J.es}\mathcal{W}$, by (Proposition 3.4.3) then $\mathcal{N}\leq_{J.es}\mathcal{W}$ contradiction. If $\mathcal{N}$ is

not J-essential in any subsemimodule containing it properly, then $\mathcal{N} \leq_{Jc} \mathcal{W}$. Therefore, $\mathcal{N} \leq_{J.es} \mathcal{N} \leq_{Jc} \mathcal{W}$. □

Theorem 3.4.6: A $\mathcal{T}$-semimodule $\mathcal{W}$ is a J-extending semimodule if and only if any J-closed subsemimodule in $\mathcal{W}$ is a direct summand of $\mathcal{W}$.

Proof:

(⟹) Assume that $\mathcal{W}$ is a J-extending semimodule . To prove every J-closed subsemimodule in $\mathcal{W}$ is a direct summand in $\mathcal{W}$.

Let $\mathcal{V} \leq_{Jc} \mathcal{W}$. Since $\mathcal{W}$ is a J-extending semimodule, then $\mathcal{V} \leq_{J.es} \mathcal{B}$ with $\mathcal{B} \leq^{\oplus} \mathcal{W}$, since $\mathcal{V} \leq_{Jc} \mathcal{W}$, then $\mathcal{V} = \mathcal{B}$. Therefore, $\mathcal{V} \leq^{\oplus} \mathcal{W}$.

(⟸) Suppose that every J-closed subsemimodule in $\mathcal{W}$ is a direct summand in $\mathcal{W}$. To prove $\mathcal{W}$ is a J-extending semimodule.

Let $\mathcal{V} \leq \mathcal{W}$, if $\mathcal{V} = < \overline{0} >$ then $\mathcal{V} \leq_{J.es} < \overline{0} >, < \overline{0} > \leq^{\oplus} \mathcal{W}$. If $\mathcal{V} \neq < \overline{0} >$ then by (Lemma 3.4.5) there is $\mathcal{A} \leq_{Jc} \mathcal{W}$ that is $\mathcal{V} \leq_{J.es} \mathcal{A}$. But every J-closed subsemimodule in $\mathcal{W}$ is a direct summand in $\mathcal{W}$, thus $\mathcal{A} \leq^{\oplus} \mathcal{W}$. Therefore, $\mathcal{W}$ is a J-extending semimodule. □

Conclusions

Semi-extending has been studied on modules and got some results. Extending the semimodule was also studied and they obtained some properties.

In this work, the researcher has a semi-extending study on semimodules, obtained some properties and results. Some other concepts were also studied: such as Prime-extending semimodule, J-essential subsemimodule, J-extending semimodule and J-closed subsemimodule. Among the most important results obtained are:

- Every extending semimodule is semi-extending semimodule. But, the converse is not true.
- Every prime-extending semimodule is semi-extending semimodule. But, the converse is not true.
- Extending semimodule $\longrightarrow$ J-extending semimodule $\longrightarrow$ semi-extending semimodule.
- Any proper subsemimodule of a semisimple $\mathcal{T}$-semimodule is semi-prime.
- A $\mathcal{T}$-semimodule $\mathcal{W}$ is a J-extending semimodule if and only if any J-closed subsemimodule in $\mathcal{W}$ is a direct summand of $\mathcal{W}$.

Future works

In the future, some generalizations of J-closed subsemimodule will be studied, Direct sum and direct summand of J-extending on modules (or semimodules). Examples, properties for these concepts and some relationships of them will be discussed.

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الخلاصة

في هذا العمل ، تم تقديم بعض التعميمات حول شبه المقاسات شبه الأساسية. حيث قام الباحث بدراسة هذه النتائج من خلال وضع الشروط المناسبة ، وتحديد خصائص جديدة ، اطلقنا عليها حسب مفهومنا (شبه المقاس الاولي التام و شبه المقاس التام وشبه مقاسات جزئية شبه مكملة) حيث: إذا كان كل شبه مقاس جزئي شبه أساسي في شبه المقاس هو أساسي فنسمى شبه المقاس بـ أساسي تام. وإذا كان كل شبه مقاس جزئي فعلي من شبه المقاس هي شبه مقاس اولي فنسمى شبه المقاس بـ شبه المقاس الأولي التام، شبه المقاسات الجزئية الأولية من شبه المقاس تسمى بـ شبه مقاسات جزئية شبه مكملة لشبه مقاس جزئي اخر إذا كان الأول هو اعظمي في صفة ان تقاطعه مع الاخر هو صفر.

أيضًا ، درس الباحث مفهوم شبه التوسع على شبه المقاسات حيث: يُطلق على شبه المقاس $\mathcal{W}$ بـ شبه مقاسات شبه التوسع (إذا كان كل شبه مقاس جزئي هو شبه مقاس جزئي اساسي في مركبة جمع مباشر). كما، تمت دراسة مفهوم شبه مقاسات شبه التوسع الاولي حيث: (يُقال إن شبه المقاس $\mathcal{W}$ هو شبه مقاس شبه التوسع الاولي إذا كان أي شبه مقاس جزئي من $\mathcal{W}$ هو شبه مقاس جزئي اساسي في مركبة جمع مباشر اولي.

أخيرًا ، إذا كانت $\mathcal{A}$ عبارة عن شبه مقاس جزئي من شبه المقاس $\mathcal{W}$ ، فإن $\mathcal{A}$ يسمى شبه أولي إذا كان ($t^2x \in \mathcal{A}$ حيث ان $x \in \mathcal{W}$ و $t \in \mathcal{T}$ يؤدي الى $tx \in \mathcal{A}$). ويطلق على شبه المقاس الجزئي الغير الصفري $\mathcal{V}$من شبه المقاس $\mathcal{W}$ ب اساسي من النمط J في $\mathcal{W}$ إذا كان تقاطعه مع اي شبه مقاس جزئي غير صفري و شبه اولي لا يساوي صفر.

يسمى شبه المقاس $\mathcal{W}$ بـ شبه مقاس التوسع من النمط J إذا كان أي شبه مقاس جزئي من $\mathcal{W}$ شبه مقاس جزئي اساسي من النمط J في مركبة جمع مباشر. ويدعى شبه مقاس جزئي غير صفري $\mathcal{N}$ من شبه المقاس $\mathcal{W}$ مغلق من النمط J إذا لم تكن $\mathcal{N}$ محتواة فعليا في شبه مقاس جزئي وتكون اساسي من النمط J فيه. وقد تم الحصول على بعض النتائج والخصائص المثيرة للاهتمام للمفاهيم الجديدة.

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قسم الرياضيات

شبه مقاسات شبه التوسع

رسالة

مقدمة الى مجلس كلية التربية للعلوم الصرفة / جامعة بابل كجزء من متطلبات نيل درجة الماجستير في التربية / الرياضيات

من قبل
زيدون وهاب عبد محسن

بإشراف
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1444 هـ 2022 م