Republic of Iraq
Ministry of Higher Education
and Scientific Research
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Department of Mathematics

On Rational Map in The Complex Dynamics

A Thesis
Submitted to the Council of the College of Education for Pure Sciences in University of Babylon in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Education/Mathematics

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2022 A.D 1444 A.H

بسم الله الرحمن الرحيم

فَتَعَالَى اللَّهُ الْمَلِكُ الْحَقُّ ۗ وَلَا
تَعْجَلْ بِالْقُرْآنِ مِنْ قَبْلِ أَنْ يُقْضَى
إِلَيْكَ وَحْيُهُ ۖ وَقُلْ رَبِّ زِدْنِي عِلْمًا

صدق الله العلي العظيم

سورة طه الآية ﴿١١٤﴾

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Acknowledgements

I would like to express my deep gratitude's to my supervisor Professor Dr. Iftichar M. T.AL-Shara'a not only for his help on this research, which has been indispensable, but also for getting me interested in the subject to begin with.

I would like to thank Jianxun Fu, Fei Yang, and Antoni Garijo to support them and help me with resources

I would like to thank my wife for her help on this research through support and encouragement.

Finally, I would like to thank my parents.

Publications

Much of the content of this thesis has previously appeared in the following papers:

[1] Hassanein Q.Al Salami and Iftichar, Al Shara`a , Rational Maps Whose Julia Sets are Quasicircles, International Journal of Nonlinear Analysis and Applications (IJNAA), Online ISSN 2008-6822, Vol. 12 Issue 2/2021.

[2] Hassanein Q.Al Salami and Iftichar, Al Shara`a , Julia Sets Are Cantor Circles and Sierpinski Carpets for Rational Maps , International Journal of Nonlinear Analysis and Applications (IJNAA), , Vol. 13 Issue 1/2022.

[3] Hassanein Q.Al Salami and Iftichar, Al Shara`a , Q_β Has no Doubly Connected Attracting Fatou or Parabolic Fatou components, AIP Publishing Conference Proceedings office, under publication.

Abstract

In this work, we study the family of complex rational maps which is given by

$$Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}} \ ,$$

where $d \geq 2$ and $\beta \in \mathbb{C}\backslash\{0\}$ such that $\beta^{1-d} \neq 1$ and $\beta^{2d-2} \neq 1$. We show that $J(Q_\beta)$ has one from these properties as a quasicircle or a Cantor circle or a Sierpinski carpet or a degenerate Sierpinski carpet whenever the image of one of the free critical points for Q_β is not converged to 0 or ∞ .

As we proved:

- ❖ Since the Julia set of the map Q_β is connected, then the free critical orbits are attracted to 0 or ∞.
- ❖ And Q_β has no Herman ring and no infinitely connected attracting Fatou components or parabolic Fatou components.
- ❖ We prove that the map Q_β is exclusive to the Julia set as a transitive map and because the Julia set has dense periodic points, therefore Q_β will be a chaotic map (according to Devaney's definition).

- We prove the Julia set is a fractal set because it has a Hausdorff dimension whose value is not an integer it is

$$1+\frac{log3}{logd} \leq dim_H J(Q_\beta) < 2,$$

and therefore the Julia set is a strange chaotic attractor.

List of Symbols

X	A metric space
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integer numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
S	Riemann surface
$Q_\beta, f, h_\lambda, \mathcal{G}_\beta, \varphi, \psi, M, \eta, \tau$	maps
$\lvert x - y \rvert$	distance between x and y
$\bar{A}$	closure of the set A
V, U, W	Open set
∂U	boundary of the set U
R	Rational maps
E, K, M, N	Sets
$\mathbb{C}_\infty$	Riemann sphere
$J(R)$	Julia set of R
$\mathbb{F}(R)$	Fatou set of R
γ, Γ, Υ	simple closed curve
$R^n(z) \to z$	The $n-th$ iterate of R of z converge to z
I_0, I_∞	immediate of basin of attraction of 0 or ∞ respect.
$D_r(z)$	open disc of radius r about z
$\overline{D_r(z)}$	closed disc of radius r about z
S^1	unit circle in $\mathbb{C}$
f_n	family of complex analytic maps
$B(0,1)$	Ball with center 0 and radius 1

$\jmath, A_0, W, V$	components
$\varsigma(W)$	the number of connected components
$\mathcal{T}_{\vert\beta\vert}$	The round circle
$\Subset$	compactly contained
γ^{int}	interior of simple closed curve
$M \prec N$	M precedes N
$\mathrm{A}(\beta) \preccurlyeq \mathrm{B}(\beta)$	A precedes or equal to B
$\overline{A^{\circ}}$	closure(interior of A)
$\mathcal{M}$	McMullen domain
$\lambda(\mathfrak{D})$	λ –Ahlfors
E(R)	The set of exceptional points
$f_\lambda(z)$	The McMullen map

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Introduction

Complex dynamics is the iteration of maps on complex numbers for the study of dynamical systems. An important application of complex dynamics is the Julia set, where the Julia set is a type of fractal. We can see the similarity of the topography of a mountain range for example with some parts of this set. Some rendering programs have used these comparisons for color and terrain modeling. In general the rational maps have dynamics more complex than that of polynomials because each polynomial has a totally invariant superattracting focused at ∞. See [31],[33]). However, if we study dynamics behaviors of polynomials when studying the behavior of the rational maps close to the polynomials through the concept of perturbation.

McMullen is the first who used the singular perturbation on z^n (see [25]) he study the family of rational maps

$$f_\lambda(z) = z^m + \frac{\lambda}{z^n},$$

where $m \geq 2, n \geq 1,$ and $\lambda \in \mathbb{C}\backslash\{0\}$, this map called the McMullen maps. The McMullen map has been studied by several authors. The authors in [7] gave (The Escape Trichotomy Theorem) for f_λ by the orbits of the free critical points. After that through several people, they found a generalization of McMullen maps see [7],[14], [15], [41]. Fu and Yang [12], They studied the following maps

$$h_\lambda(z) = \frac{z^d(z^{2d} - \lambda^{d+1})}{z^{2d} - \lambda^{3d-1}} \quad ,$$

where $d \geq 2$ and $\lambda \in \mathbb{C}\backslash\{0\}$, such that $\lambda^{2d-2} \neq 1$. They got several things, including Julia sets is cantor circles or quasicircles or

Sierpinski carpet according to the iterate of the free critical points. See [12] ,[13],[42].

In 1980, McMullen is the first discovery the Julia set is cantor of circles from the maps as

$$f_\lambda(z) = z^2 + \frac{\lambda}{z^3} ,$$

where λ is small and not equal to zero, see [25]. The authors came the generalization of the McMullen maps

$$f_\lambda(z) = z^m + \frac{\lambda}{z^n} ,$$

which was studied by some people through some dynamical phenomena, see [8], [14], [15], [31] and [41].

The appearance of the Julia sets is the Sierpinski carpet or the cantor circle of the McMullen map or a generalization of the McMullen map as in (see [41]). In [24, Appendix F], the Sierpinski carpet Julia set of the rational map have given by Milnor and Tan. In [4], the quasisymmetric geometry on Julia sets for rational maps of post-critically-finite was studied by Lyubich, Bonk and Merenkov. Moreover, the Julia sets are Sierpinski carpets of the rational maps, see [7]. From [2], if the Julia set is connected, which is equivalent to showing every the Fatou components are simply connected for Q_β. However, the infinitely connected attracting or parabolic basin and the Herman ring are only two types of periodic Fatou components, which are not simply connected. By Shishikura, the connectivity of the Julia sets of rational maps was studied as in [34]. By using Newton's method for polynomials, we can apply the results of Shishikura to the rational maps. Also, this concept was studied by Yin, see [45]. family of renormalization transformation maps was studied by Qiao and Li also obtained the Julia sets are connected for each real parameters as in [29] also, the result has been generalized by Yang and Zeng to all complex parameters in [44]. The Julia set for the McMullen maps is a connection was studied in [9], [18] and [40], when constructing Herman rings there are two methods.

This work is divided into four chapters. Many of the results of chapter one is known, hence we state some of them without proof.

In chapter one, section one, we recall the elementary definitions of Julia sets, and we give some fundamental definitions and theorems which we need them through this work.

In section two, we offer the properties of Julia set for rational and polynomial maps to give some theorems and propositions. In section three, we offer the concept of the singular perturbation.

In chapter two, section one we introduce the dynamical and parameter planes of Q_β, we offer "The Escape Quartation Theorem", where we show that the Julia set is a quasicircle or cantor circles or Sierpinski carpet or degenerate Sierpinski carpet whenever the image of one of the free critical points for Q_β is not converging to 0 or ∞. In section two, we study some properties of the parameter plane, as the McMullen domain.

Chapter three is divided into two sections, in the first, we give the Julia set of Q_β is connected. Section two shows the map of Q_β is exclusive to the Julia set where the rest of the properties of chaos are achieved, as the set is a strange chaotic attractors, since it has Hausdorff dimension as the value between greater than one and less than two.

In chapter four, we give the Conclusions and Future works for this work.

At the end of this thesis, we set examples of Julia set and give an appendix showing the practical part of the subject.

CHAPTER ONE

The Julia Sets

1.1 Elementary Definitions and Theorems

Our goal in this section, is the presentation of definitions and theorems that we will use later in this work.

In our work, we will adopt the plane as the extended complex plane which is $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, so that the Riemann sphere is a topological equivalent to extended complex plane. The metric space is the chordal metric.

Definition (1.1.1) [2]

If $U \subset \mathbb{C}$ is an open set of complex numbers. A map $f : U \to \mathbb{C}$ is **holomorphic** (or complex analytic) in U if the derivative f' exists at each point of U. The map $f : U \to \mathbb{C}_\infty$ is **meromorphic** in U if each point of U has a neighborhood on which either f or $\frac{1}{f}$ is holomorphic.

Definition (1.1.2) [2]

If $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a **rational map** of the form

$$R(z) = \frac{f(z)}{g(z)},$$

where f and g are both polynomials, not both being the zero polynomial. If $f = 0$, then $R = 0$ and if $g = 0$, then $R = \infty$. So the

f and g are coprime if we assume that f and g have no common zeros. Thus the **degree** of R is

$$deg(R) = \max\{deg(f), deg(g)\}.$$

In [30] , for any $q \in \{0,1\}$, for each integer $d \geq 2$ and $n_1, .., n_d \in \mathbb{Z}^+$ such that $\sum_{i=1}^{d} \frac{1}{n_i} < 1$. Define

$$f_{q,n_1,..,n_d}(z) = z^{n_1(-1)^{d-q}} \prod_{i=1}^{d-1} \left(z^{n_i+n_{i+1}} - \beta_i^{n_i+n_{i+1}}\right)^{(-1)^{d-i-q}},$$

for $\beta_1, .., \beta_{d-1}$ are $d-1$ small complex numbers holds

$$0 < |\beta_1| < \ldots < |\beta_{d-1}| < 1.$$

Special case , if $d = 2$ and $q = 1$, then $f_{1,n_1,n_2}(z)$ is the McMullen map studied by a number of authors . However, $f_{0,n_1,n_2}(z)$ is conformally conjugate to the McMullen map

$$f_\lambda(z) = z^{n_1} + \frac{\lambda}{z^{n_2}} ,$$

for some $\lambda \neq 0$. If $d \geq 3$, then any $f_{q,n_1,..,n_d}$ is not topologically conjugate to any McMullen maps on their corresponding Julia sets. In particular,

$$h_\lambda(z) = \frac{z^d(z^{2d} - \lambda^{d+1})}{z^{2d} - \lambda^{3d-1}} .$$

We can develop the last map into the map

$$Q_\beta(z) = 2\beta^{1-d} z^d - \frac{z^d(z^{2d} - \beta^{d+1})}{z^{2d} - \beta^{3d-1}} ,$$

and study the dynamics of this map. Q_β have superattracting periodic orbits of 0 and ∞.

Proposition (1.1.3) [21]

If R is a rational map , let $z_0 \in \mathbb{C}_\infty$, the equation $R(z) = z_0$ has exactly d solutions, counting multiplicities, then $deg(R) = d$.

Example (1.1.4)

Suppose that $(z) = \frac{z^2+2z+3}{z-1}$. Then $deg(R) = 2$, $R(1) = \infty$,

Definition (1.1.5) [2]

The Mobius map is a rational map of degree one as follows:

$\varphi(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$.

We say that two rational maps R and ψ are **conjugate** if and only if there is Mobius map φ with $\psi = \varphi R \varphi^{-1}$.

Definition (1.1.6) [3]

Assume that $U \subset \mathbb{C}_\infty$ is open set and $R : U \to \mathbb{C}_\infty$ is a holomorphic map, so $R^p = R \circ R \circ \ldots \circ R$ (p-times).

If $R(z_0) = z_0$, then $z_0 \in \mathbb{C}_\infty$ is a **fixed point** of R and a **periodic point** of period n if it is a fixed point for R^n, for $n \geq 1$.

Definition (1.1.7) [21]

Assume that $z \in W$ and a rational map $R : W \to \mathbb{C}_\infty, W \subset \mathbb{C}_\infty$, we define the **total, forward and backward orbits of** z, respectively, by:

$\mathbb{O}_R(z) = \mathbb{O}_R^+(z) \cup \mathbb{O}_R^-(z)$,

$\mathbb{O}_R^+(z)$ is the set of points $z, r(z), R^2(z), \ldots$

$\mathbb{O}_R^-(z)$ is the set of points $z, R^{-1}(z), R^{-2}(z), \ldots$.

Definition (1.1.8) [23]

Suppose that $z_0 \in \mathbb{C}_\infty$ is a periodic point of period . We denote the $\mathbb{O}_R^+(z_0) = \{z_1, z_2, \dots, z_n = z_0\}$ is a **periodic cycle**. The **multiplier** of a point z of period n is the derivative $(R^n)'(z)$ of
the first return map, where z is any point in the cycle. We say z is:
superattracting if $(R^n)'(z) = 0$;
attracting if $(R^n)'(z) < 1$;
indifferent if $(R^n)'(z) = 1$;
repelling if $(R^n)'(z) > 1$.
An indifferent point is **parabolic** if $(R^n)'(z)$ is a root of unity.

Example (1.1.9)

A map $g: \mathbb{C} \to \mathbb{C}$ such that

(a) $g(z) = z^2$ if $z = 1$, then z is a repelling fixed point. Also

$z = 0$, then z is a superattracting fixed point.

(b) $g(z) = z^2 + z$ if $z = 0,$ then z is indifferent fixed point .

Definition (1.1.10) [3]

If $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a map , Suppose that $z \in \mathbb{C}_\infty$ is an attracting fixed point of R. Then **the basin of attraction of z** defined as

$$A(z) = \{\mathbb{p} \in \mathbb{C}_\infty : R^n(\mathbb{p}) \to z,\ n \to \infty\}.$$

The **immediate basin of attraction** of z is the connected component of the basin of attraction of z . The immediate basin of attraction of 0 and ∞ denoted by I_0 and I_∞ respectively.

Example (1.1.11) [21].

If $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a map, $R(z) = z^2$. The basin of attraction of the fixed point of the origin is $B(0,1)$ (the ball with center 0 and the radius 1), and the basin of attraction of the fixed point of the infinity is $\mathbb{C} \setminus \bar{B}(0,1)$.

Lemma (1.1.12) [21]

If $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a map, let $z \in \mathbb{C}_\infty$ is an attracting fixed point. Then the basin of attraction of z is open.

We recall some results in [2] as:

Proposition (1.1.13) [2]

If $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rational map of degree $d \geq 2$, then

(i) for any periodic cycle, the immediate basin of attraction contains at least one critical point of R.

(ii) there are at most $2d - 2$ attracting periodic cycles.

(iii) the number of non-repelling periodic cycles of R is at most $6d - 6$, so that the number of non-repelling periodic points of R is finite.

Definition (1.1.14) [6]

Let $\{f_n\}$ be a family of holomorphic maps defined on an open set U. The family $\{f_n\}$ is said to be **normal** on U if every sequence of the f_n`s has a subsequence which either

1.converges uniformly on compact subsets of U ,or

2. converges uniformly to ∞ on U.

Example (1.1.15) [6]

Let $U \subset \mathbb{C}$ be an open connected set and let $g : U \to \mathbb{C}$ be a map such that $g(z) = bz$ with $|b| < 1$. We set $\{g_n\} = \{g^n\}$.

$g(z) = bz$

$g^2(z) = b^2 z$.

.

.

$g^n(z) = b^n z \to 0$ as $n \to \infty$ with $|b| < 1$. Hence $\{g_n\}$ converges uniformly to the constant map 0 on compact subsets of U .

Therefore $\{g_n\}$ is a normal family of maps on U .

Definition (1.1.16) [6]

The family $\{f_n\}$ is not normal at z if the family fails to be a normal family in every neighborhood of z.

Definition (1.1.17) [2]

A family $\mathcal{F}$ of maps of (X, d) into (X, d_1) is **equicontinuous** at y if and only if for any $\varepsilon > 0$, there is $\delta > 0$ such that for all x in X, and for each g in $\mathcal{F}$, $d(y, x) < \delta$ implies

$d_1(g(y), g(x)) < \varepsilon$. $\mathcal{F}$ is equicontinuous on $Y \subset X$ if $\mathcal{F}$ is equicontinuous at any point y of X.

The normality is equivalent to equicontinuous from Arzela`-Ascoli Theorem as follows:

Theorem (1.1.18) [2]

Let U be a subdomain of $\mathbb{C}_\infty$ and let $\mathcal{F}$ be a family of continuous maps of U into $\mathbb{C}_\infty$. Then $\mathcal{F}$ is equicontinuous in U if and only if it is a normal family in U.

This latter term is a reference to normal families of holomorphic maps, but we have preferred to base our definition on equicontinuity because of its more immediate geometric appeal.

Definition (1.1.19) [21]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a non-constant rational map and, let $\{R^d\}$ be the family of iterates. The maximal open subset of the Riemann sphere is the **Fatou set** of R such that $\{R^d\}$ is equicontinuous, also the **Julia set** of R is the complement of Fatou set in $\mathbb{C}_\infty$.

We denote $J_0(R)$ is the Julia set .

Remark (1.1.20) [21]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map and $\{R^d\}$ be the family of iterates. If the family $\{R^d\}$ is equicontinuous and uniformly bounded and from Azela`-Ascoli Theorem, then its closure is compact, thus every sequence has a converging subsequence. Hence its normal. Thus we define $J_0(R) = \{z \in \mathbb{C}_\infty : \{R^d\}$ is not normal at $z\}$.

Example (1.1.21) [21]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map, if $R(z) = z^2$, then $J_0(R) = S^1$.

Proposition (1.1.22) [2]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a holomorphic map. Then R is injective at $z \in \mathbb{C}_\infty$ if and only if $R'(z) \neq 0$.

Theorem (1.1.23) [2]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map. The Julia set of R ((R)) is the closure of the repelling periodic points of R.

We see that for example (1.1.21), so $J_0(R) = J(R)$. A natural question to ask if this is true or not for any R. We will prove it is true in case any rational map in the next section.

Definition (1.1.24) [26]

For any holomorphic map $f: \mathbb{C} \to \mathbb{C}$. The **local degree** of z is equal to one or more than the order to which f' takes on the value zero. We denoted by $deg(f, z)$.
For example, $f(z) = z^d$ and $= 0$, thus $deg(z^d, 0) = d$.

It does not affect the generality of the definition (1.1.24), if we define the local degree on $\mathbb{C}_\infty$.

Definition (1.1.25) [23]

If $W \subset \mathbb{C}_\infty$ is a domain. **The Euler characteristic** of W as $\chi(W) = 2 - \varsigma(W)$, where $\varsigma(W)$ is the number of connected components of the complement $\mathbb{C}_\infty \backslash W$.
For example, if a domain with $\varsigma(W) = 1$ is simply connected, so $\chi(W) = 1$.

We give the **Riemann-Hurwitz formula** in the following theorem

Theorem (1.1.26) [36]

Let $W, V \subset \mathbb{C}_\infty$ be a non-empty open sets, and a rational map with d – fold $R : W \to V$. Assume that R has n critical points, counted with multiplicity. Then $(W) = d\, \chi(V) - n$.

Remark (1.1.27) [23]

In the case, $W = V = \mathbb{C}_\infty$, $\chi\ (W) = \chi(V) = 2$, and $R: \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a continuous map with a compact domain of degree d, so the Riemann-Hurwitz formula gives $n = 2d - 2$ for the number of critical points, counted with multiplicity.

Corollary (1.1.28) [2]

If $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rational map of $d \geq 1$ has at most $2d — 2$ critical points in $\mathbb{C}_\infty$. A polynomial of positive degree d has at most $d — 1$ critical points in $\mathbb{C}$.

The degree of Q_β is $3d$. By Corollary (1.1.28), any rational maps have the critical points $(2d-2)$, then the map Q_β has $6d-2$ critical points (counted with multiplicity).

We rewrite some results in [2] as:

Proposition (1.1.29) [2]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map. The backward orbit of z ($\mathbb{O}_R^-(z)$) is finite if and only if z is exceptional point. The set of exceptional points denoted by $\boldsymbol{E}(\boldsymbol{R})$. So $\boldsymbol{E}(\boldsymbol{R})$ lie in $\mathbb{F}(R)$.

Definition (1.1.30) [23]

Let S and $S`$ be a Riemann surface (we mean a connected complex analytic manifold of complex dimension 1). Let $\varphi : S \to S`$ be a holomorphic map. φ is a **covering map** if φ evenly covered a connected neighborhood W contains any point of $S`$.

Definition (1.1.31) [19]

Let $D, D' \subset \mathbb{C}$ and $f : D \to D'$ be an orientation-preserving homeomorphism. If f is continuously differentiable, then it is $\boldsymbol{K}$ **−quasiconformal** if the derivative of f at every point maps circles to ellipses with eccentricity bounded by K.

Definition (1.1.32) [43]

Let Λ be a connected complex manifold. Suppose that $R : \Lambda \times \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a holomorphic family of rational maps parameterized by Λ . Then R has only **one free critical orbit**, if there is a critical point e of R such that for any $e`$ contains in the set of critical points of $R$ without $e$ , either $\mathbb{O}_R^+(e`)$ is finite or $\mathbb{O}_R^+(e`) \cap \mathbb{O}_R^+(e) \neq \emptyset$.

Definition (1.1.33) [12]

A rational map $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is called **hyperbolic** if its critical points are all attracted by attracting periodic orbits.

Definition (1.1.34) [2]

Suppose that $U \subset \mathbb{C}_\infty$, U is said to be a **Cantor set** if it is non-empty, closed, perfect (there are no isolated points), and totally disconnected.

Corollary (1.1.35) [23]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a hyperbolic rational map. Then the Julia set is locally connected if the Julia set of R is connected.

Definition (1.1.36) [27]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map with degree d and let c be a critical point. If $P = \overline{\cup_{n>0,} R^n(c)}$, then R is **nice** if $P \cap J(R)$ is contained in finitely many connected components of $J(R)$.

Proposition (1.1.37) [27]

Every Julia component j is a Jordan curve if R is hyperbolic, or $\mathbb{C}_\infty \backslash j$ has exactly two components if R is nice.

Theorem (1.1.38) [23]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map with degree ≥ 2 . R has a superattracting fixed point at ∞ , there exists a local holomorphic change of coordinate $w = \varphi(z)$, with $\varphi(\infty) = \infty$, which conjugates R to the $d-th$ power map $\varphi(w) = w^d$ throughout some neighborhood of ∞. Furthermore, φ is unique up to multiplication by an $(d - 1)st$ root of unity.

1.2. Properties of Julia Set for Rational Maps

Now, we introduce some properties the Julia set and Fatou set. It can be reformulated thus. Also we give some proposition between the Julia set of polynomial and rational maps.

We recall some theorems in [21] as follows:

Proposition (1.2.1) [21]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map with degree ≥ 2 .

(i) $J_0(R)$ is a perfect set. Hence $J_0(R)$ is uncountable.

(ii) $\emptyset \neq J(R) \subset J_0(R)$. Also $J_0(R)$ has no isolated points, thus $J(R) = J_0(R)$.

(iii) If z_0 attracting fixed point of R, then $\partial A(z_0) = J_0(R)$.

(iv) $R\big(J_0(R)\big) = R^{-1}\big(J_0(R)\big) = J_0(R)$.

(v) The finite total orbits contain at most two points and none belong to Julia set of R.

(vi) For each $z \in \mathbb{C}_\infty$, with at most two exceptions, if $W \subset \mathbb{C}_\infty$ is open, $W \cap J_0(R) \neq \emptyset$, then there is an infinite sequence of $m_j \in \mathbb{N}$ such that $R^{m_j}(z) \cap W \neq \emptyset$. If $z \in J_0(R)$, then $J_0(R) = \overline{\mathbb{O}_R^-(z)}$.

(vii) $J_0(R^n) = J_0(R)$, for any $n \in \mathbb{N}$.

(viii) If the Julia set contains an interior point, then it must be equal to the entire Riemann sphere.

Definition (1.2.2) [2]

If $R:\mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rational map. A Fatou component U is

(i) forward invariant if $R(U) = U$;

(ii) backward invariant if $R^{-1}(U) = U$;

(iii) completely invariant if $R(U) = R^{-1}(U) = U$.

We can define U is a fixed component if $R(U) = U$. Also U is a periodic component if $R^k(U) = U$ for some $k \in \mathbb{N}$.

Theorem (1.2.3) [2]

If f is a polynomial map of $d \geq 2$. Then $\infty \in \mathbb{F}(f)$, $F_\infty \in \mathbb{F}$ containing ∞ is completely invariant component under f.

Proposition (1.2.4) [2]

Let $R:\mathbb{C}_\infty \to \mathbb{C}_\infty$ be a non-constant rational map, let φ be a Mobius map satisfy $\psi = \varphi R \varphi^{-1}$. Then $\mathbb{F}(\psi) = \varphi(\mathbb{F}(R))$ and $J(\psi) = \varphi(J(R))$.

We give The Sullivan classification as follows:

Definition (1.2.5) [2]

A forward invariant component A_0 of $\mathbb{F}(R)$ is:

(1) an **attracting component** if it contains an attracting fixed point ζ of R;

(2) a **super-attracting component** if it contains a super-attracting fixed point ζ of R;

(3) a **parabolic component** if there is a rationally indifferent fixed point ζ of R on the boundary of A_0, and if $R^n \to \zeta$ on A_0;

(4) a **Siegel disc** if $R : A_0 \to A_0$ is analytically conjugate to a Euclidean rotation of the unit disc onto itself;

(5) a **Herman ring** if $R : A_0 \to A_0$ is analytically conjugate to a Euclidean rotation of some annulus onto itself.

Theorem (1.2.6) [2]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map of degree $d \geq 2$, let each open set $U \neq \emptyset$ which meets the Julia set . Then:

(i) $\mathbb{C}_\infty - E(R) \subset \bigcup_{d=0}^{\infty} R^d\,(U)$;and

(ii) $\forall$ sufficiently large $d \in \mathbb{Z}$, $J \subset R^d(U)$.

Theorem (1.2.7) [3]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map of degree $d \geq 2$.

(i) If z is not exceptional, then the Julia set is contained in the closure of $\mathbb{O}_R^-(z)$.

(ii) If $z \in J$, then the Julia set is the closure of $\mathbb{O}_R^-(z)$.

Definition (1.2.8) [3]

Let $R:\mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map. A point z is **eventually periodic** if there is n such that $R^n(z)$ is a periodic point. Also a point z is **pre-periodic** if it is eventually periodic but not periodic.

Theorem (1.2.9) [2]

If each critical point of the rational map R is pre-periodic, then $J(R) = \mathbb{C}_\infty$.

Example (1.2.10)

If $R:\mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rational map. Then $R(z) = 1 - 2/z^2$ has an empty Fatou set . So R has critical point at 0, and as $R(0) = \infty$, $R(\infty) = 1$, $R(1) = -1$, $R(-1) = -1$, thus the critical point 0 is preperiodic point and by Theorem (1.2.9), implies that for this map too, $J(R) = \mathbb{C}_\infty$.

Definition (1.2.11) [25]

A rational map R of degree d is **critically finite** if its critical points are pre-periodic $R^n(c) = R^m(c)$ for some $n \neq m$.

Theorem (1.2.12) [26]

Let R be a critically finite rational map. Then each periodic cycle of R is repelling or superattracting. If R has no super attracting cycles , then $J(R) = \mathbb{C}_\infty$.

Theorem (1.2.13) [2]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map. Then $J(R)$ is connected if and only if any component of $\mathbb{F}(R)$ is simply connected.

Theorem (1.2.14) [2]

If $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rational map of of degree $d \geq 2$, and A_0 is a completely invariant component of $\mathbb{F}$. Then:

(1) $\partial A_0 = J(R)$;

(2) A_0 is either infinitely connected or simply connected;

(3) each other components of $\mathbb{F}(R)$ are simply connected; and

(4) A_0 is simply connected iff $J(R)$ is connected.

But in a polynomial f, where $\{\infty\}$ is completely invariant and so too is the unbounded component of the Fatou set.

Theorem (1.2.15) [2]

Let $\mathbb{F}$ be the Fatou set of a non-linear polynomial. Then

(i) The unbounded component of $\mathbb{F}$ is either infinitely connected or simply connected ; and

(ii) All bounded component of $\mathbb{F}$ is simply connected.

Theorem (1.2.16) [2]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map of degree $d \geq 2$. Then any forward invariant component of $\mathbb{F}(R)$ is simply, doubly, or infinitely, connected.

Corollary (1.2.17) [2]

A forward invariant component of R is doubly connected if and only if it is a Herman ring.

Lemma (1.2.18) [2]

For any compact connected $U \subset \mathbb{C}$. Then $R^{-1}(U)$ has at most d components and each is mapped by R onto U.

Example (1.2.19) [5]

Let $f: \mathbb{C} \to \mathbb{C}$ be a polynomial map such that $f(z) = z^2 + i$. If the Fatou set of f is connected and simply connected, so the Julia set of f is a dendrite, and the Julia set of f is connected, a sufficient condition for this to be so is that each finite critical point of f is preperiodic.

Corollary (1.2.20) [21]

Let f be a polynomial. Then $J_0(f)$ has empty interior. If R is a rational map also $J_0(R)$ has non-empty interior, then $J_0(R) = \mathbb{C}_\infty$.

Theorem (1.2.21) [23]

Let A_0 be the immediate basin of an attracting fixed point (either geometrically attracting or superattracting). Then the complement $\mathbb{C} \setminus A_0$ is either connected or else has uncountably many connected components.

Theorem (1.2.22) [2]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map of degree $d \geq 2$, and let ζ be a superattracting fixed point of R. If all of the critical points of R lie in the immediate attracting basin of ζ. Then:

(a) $J(R)$ is a Cantor set.

(b) for a dense set of ζ in $J(R)$, $\mathbb{O}_R^+(\zeta)$ is dense in $J(R)$; and

(c) the periodic points are dense in $J(R)$.

Definitions of the Julia sets are not always equivalent, but in polynomial maps according to [46], and rational maps according to the following theorem shows that the three definitions of Julia sets are equivalent for rational of degree $d \geq 2$.

Theorem (1.2.23)

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map with degree ≥ 2 . Then the following statements are equivalent.

1. $J(R)$ is the closure of repelling periodic points .

2. $J_0(R)$ is the complement of the Fatou set.

3. $J(R)$ is the boundary of the basin of attraction of a attracting fixed point .

Proof

$1 \Leftrightarrow 2$ By using Proposition (1.2.1) (i).

$2 \Leftrightarrow 3$ By using Proposition (1.2.1) (ii). ■

1.3 Singular Perturbation

In this section, we study the concept of the singular perturbation

Definition (1.3.1) [20]

If the problem in which the parameter ϵ is small, then it causes the **regular perturbation**, but non-zero is qualitatively the same as the problem where ϵ is zero.

If the problem in which the parameter ϵ is small, then it causes the **singular perturbation**, but nonzero is qualitatively different than the problem where ϵ is zero which leads bifurcation.

Remark (1.3.2) [20]

A singular perturbation method may be defined in general as a method which is not regular.

Example (1.3.3) [20]

The regular perturbation for the equation

$$x^2 - x + \epsilon = 0, \tag{1.1}$$

where $x, \epsilon \in \mathbb{R}$

The exact solution is

$$x = \frac{1 \pm \sqrt{1 - 4\epsilon}}{2}.$$

Let $\epsilon = 0$

$x^2 - x = 0$, then $x = 0, 1$ are roots of the equation (1.1).

Also the singular perturbation for the equation

$$\epsilon x^2 + 2x + 1 = 0 \tag{1.2}$$

The exact solution is

$$x = \frac{-2 \pm \sqrt{4-4\epsilon}}{2\epsilon}.$$

So if $\epsilon = 0$, then

$2x + 1 = 0$, thus $x = \frac{-1}{2}$ is a root of the equation (1.2).

Remark (1.3.4) [39]

An equation $I_\epsilon y = 0$ contains a small parameter ϵ. We relate to this equation an unperturbed problem, the equation $I_0 y = 0$. If the difference between the solutions of both equations in a convenient norm does not tend to 0 as ϵ tends to 0, we call the problem a singular perturbation problem.

That is, $I_\epsilon y = 0$ and $I_0 y = 0$, we solve the two equations and ,

$I_\epsilon y - I_0 y = 0$, if the difference tend to 0 whenever ϵ converge to 0, then the case is call singular perturbation.

In [8] Devaney use the same the Newton method since the given map is an example $f_\alpha(z) = z^2 + \alpha$ as singular perturbation such that $N_\alpha(z) = z - \frac{f_\alpha(z)}{f`_\alpha(z)}$. If $\alpha = 0,$ then $N_0(z) = \frac{z}{2}$ and if $\alpha \neq 0,$ then $N_\alpha(z) = \frac{z^2-\alpha}{2z}$. Now we study

$$Q_\beta(z) = 2\beta^{1-d} z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}} \ , \tag{1.3}$$

where $d \geq 2$ and $\beta \in \mathbb{C}\backslash\{0\}$, such that $\beta^{2d-2} \neq 1$ and $\beta^{1-d} \neq 1$. If $d = 1, \beta = 0$ or $\beta^{2d-2} = 1$ and $\beta^{1-d} = 1,$ Q_β degenerates to the polynomial $f_d(z) = -z^d$ or z^d, then the map Q_β perturbed in the polynomial f_d.

CHAPTER TWO

The Dynamical and Parameter Planes of Q_β

The goal of this chapter is studying the properties of the dynamical plane and some properties of parameter plane.

2.1 The Escape Quartation Theorem

In this section, we give the Escape Quartation Theorem. This is done by giving the concepts and theories related to this matter.

Definition (2.1.1) [12]

A simple closed curve is **quasicircle** if is equal to the image of the unit circle for a quasiconformal homeomorphism map from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$.

Definition (2.1.2) [12]

If Γ is a subset of $\mathbb{C}_\infty$ consists of uncountably many simple closed curves which are homeomorphic to (Cantor middle third × unit circle), (in short $\mathrm{C} \times S^1$). Then Γ is called **Cantor circles**.

Definition (2.1.3) [12]

A $S \subset \mathbb{C}_\infty$ is a **Sierpinski carpet** if and only if S has empty interior and $S = \mathbb{C}_\infty \backslash \cup_{m \in \mathbb{N}} V_m$, where $V_m \subseteq \mathbb{C}$ are disjoint Jordan disks for $\partial V_m \cap \partial V_\ell = \emptyset$ for $\ell \neq m$ and diameter $V_m \to 0$ as $m \to \infty$.

If a compact set $S \subset \mathbb{C}$ be the Sierpinski carpet except for the condition $\partial V_m \cap \partial V_\ell \neq \emptyset$, then S is **degenerate Sierpinski carpet.** The standard Sierpinski carpet fractal (A fractal set is a set that has a fractal dimension and it's a figure being of parts analogous to the whole in a certain method) is homeomorphic to a planar set, this set is Sierpinski carpet**.**

We study the Julia set of map of degree $d \geq 2,$ since $d = 1$ is trivial and of little interest, the Julia set is one point or empty.

Lemma (2.1.4)

Let $Q_\beta : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that

$$Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d} - \beta^{d+1})}{z^{2d} - \beta^{3d-1}} \ ,$$

and ω be a complex number satisfying $\omega^{2d} = 1$. Then $Q_\beta^m(\omega z) = \omega^{d^m} Q_\beta^m(z)$ for some $m \geq 1$.

Proof

Suppose that

$$Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d} - \beta^{d+1})}{z^{2d} - \beta^{3d-1}} \ ,$$

and calculation by use $\omega^{2d} = 1$, we have

$$Q_\beta(\omega z) = \omega^d \left(2\beta^{1-d}z^d - \frac{z^d(z^{2d} - \beta^{d+1})}{z^{2d} - \beta^{3d-1}}\right) = \omega^d Q_\beta(z).$$

Assume that $Q_\beta^m(\omega z) = \omega^{d^m} Q_\beta^m(z)$ for some $m \geq 1$. Now we use the Law of Induction,

then $Q_\beta^{m+1}(\omega z) = Q_\beta\left(Q_\beta^m(\omega z)\right) = \omega^{d^{m+1}} Q_\beta^{m+1}(z)$. ■

Lemma (2.1.5)

Let $Q_\beta : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that

$$Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d\left(z^{2d}-\beta^{d+1}\right)}{z^{2d}-\beta^{3d-1}} .$$

For any $\eta(z) = \frac{\beta^2}{z}$, then Q_β satisfies the equation

$$\eta \circ Q_\beta(z) = Q_\beta \circ \eta(z) , \quad \forall z \in \mathbb{C}_\infty.$$

Proof

We notice $\eta^{-1}(z) = \frac{\beta^2}{z} = \eta(z)$. Then

$$Q_\beta \circ \eta(z) = 2\beta^{1-d}\left(\frac{\beta^2}{z}\right)^d - \frac{\left(\frac{\beta^2}{z}\right)^d\left(\left(\frac{\beta^2}{z}\right)^{2d}-\beta^{d+1}\right)}{\left(\frac{\beta^2}{z}\right)^{2d}-\beta^{3d-1}}$$

$$= 2\beta^{1-d}\left(\frac{\beta^2}{z}\right)^d - \frac{\left(\frac{\beta^2}{z}\right)^d\left(\frac{\beta^2}{z}\right)^{2d}-\left(\frac{\beta^2}{z}\right)^d\beta^{d+1}}{\left(\frac{\beta^2}{z}\right)^{2d}-\beta^{3d-1}}$$

$$= \frac{-\beta^{6d}+\beta^{3d+1}z^{2d}+2\beta^{5d+1}-2\beta^{4d}z^{2d}}{z^d(\beta^{4d}-\beta^{3d-1}z^{2d})}$$

$$= \beta^2\frac{z^{2d}-\beta^{3d-1}}{2\beta^{1-n}z^{3d}-2\beta^{2d}z^d-z^d(z^{2d}-\beta^{d+1})}$$

$$= \frac{\beta^2}{Q_\beta(z)} = \eta \circ Q_\beta(z) .\blacksquare$$

Corollary (2.1.6)

Let $Q_\beta : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that

$$Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d\left(z^{2d}-\beta^{d+1}\right)}{z^{2d}-\beta^{3d-1}}.$$

Suppose that W is a Fatou component of Q_β, then $W = \eta(W)$. In particular, $\eta(I_0) = I_\infty$ and $\eta(I_\infty) = I_0$.

Proof

Suppose that W is a Fatou component o f Q_β, then by use Lemma (2.1.5), $\eta(W) = W$, thus W fixed component and $W = \eta(W) = \eta^{-1}(W)$, it follows W is completely invariant .We have only two components I_0 and I_∞, and by Lemma (2.1.5), $\eta(z) = \frac{\beta^2}{z}$, then $\eta(I_0) = I_\infty$ and $\eta(I_\infty) = I_0$. ■

Remark (2.1.7)

Let $Q_\beta : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that

$$Q_\beta(z) = 2\beta^{1-d} z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}$$, we have

$$Q_\beta{}'(z) = dz^{d-1} \frac{(3\beta^{3d-1}-4\beta^d-\beta^{d+1})z^{2d}-z^{4d}(1-2\beta^{1-d})-\beta^{4d}+2\beta^{5d-1}}{(z^{2d}-\beta^{3d-1})^2} \tag{2.1}$$

$$Q_\beta{}'(z) = dz^{d-1} \frac{(3\beta^{3d-1}-4\beta^d-\beta^{d+1})z^{2d}-z^{4d}(1-2\beta^{1-d})-\beta^{4d}+2\beta^{5d-1}}{(z^{2d}-\beta^{3d-1})^2} = 0$$

$dz^{d-1} \frac{(3\beta^{3d-1}-4\beta^d-\beta^{d+1})z^{2d}-z^{4d}(1-2\beta^{1-d})-\beta^{4d}+2\beta^{5d-1}}{(z^{2d}-\beta^{3d-1})^2} = 0$, either $z = 0$ with multiplicity $d-1$ and from Lemma (2.1.5), ∞ is a critical point of Q_β with multiplicity $d-1$, or

$$(3\beta^{3d-1} - 4\beta^d - \beta^{d+1})z^{2d} - z^{4d}(1 - 2\beta^{1-d}) - \beta^{4d} + 2\beta^{5d-1} = 0$$

$$z^{4d}(1 - 2\beta^{1-d}) - (3\beta^{3d-1} - 4\beta^d - \beta^{d+1})z^{2d} + \beta^{4d} - 2\beta^{5d-1} = 0$$

$$e_\beta{}^{2d} = \frac{3\beta^{3d-1}-4\beta^d-\beta^{d+1} \pm \sqrt{(3\beta^{3d-1}-4\beta^d-\beta^{d+1})^2-4(1-2\beta^{1-d})(\beta^{4d}-2\beta^{5d-1})}}{2-4\beta^{1-d}} .$$

There are two roots:

$$e_\beta = \left(\frac{3\beta^{3d-1}-4\beta^d-\beta^{d+1} \pm \sqrt{(3\beta^{3d-1}-4\beta^d-\beta^{d+1})^2-4(1-2\beta^{1-d})(\beta^{4d}-2\beta^{5d-1})}}{2-4\beta^{1-d}}\right)^{\frac{1}{8}},$$

we have $CP(Q_\beta) = \{0, \infty, e_\beta\}$, where e_β is the free critical point.

From Lemma (2.1.4) and (2.1.5), the orbits of points with the form $\omega^m z$, where $m = 0, 1, 2, \cdots, 2d - 1$, or form $(\frac{\beta^2}{z})$ behave symmetry of the iteration of Q_β, for example, if $Q_\beta^k(z)$ tends to 0 (or ∞), then $Q_\beta^k(\omega^m z)$ or $Q_\beta^k(\frac{\beta^2}{z})$ also tends to 0 (or ∞) or ∞ (or 0) respectively , for $1 \le m \le 2d - 1$ as k tends to ∞. of Q_β .

We can write the critical points as form

$CP(Q_\beta)$ =$\{\omega_0^m e_\beta, \omega_0^m \frac{\beta^2}{e_\beta} : 0 \le m \le 2d - 1\}$, where $\omega_0 = e^{\frac{\pi i}{d}}$, therefore Q_β contains only one free critical orbit.

Let $W \subset \mathbb{C}_\infty$ and $c \in \mathbb{C}$. We define $cW = \{cz : z \in W\}$.

Lemma (2.1.8)

If $z \in I_0$ or I_∞ , then $\omega z \in I_0$ or I_∞ respectively, where ω satisfies $\omega^{2d} = 1$. So both I_0 and I_∞ have $2d$ –fold symmetry.

Proof

For any $W \subset I_0$ be define as $\{z \in I_0 : \omega z \in I_0\}$, W is non-empty and open set since I_0 consists of small neighborhood of 0 and $W = I_0 \cap \omega I_0 \cap \ldots \cap \omega^{2d-1} I_0$. Now, if $W \neq I_0$, suppose that $z_0 \in I_0 \cap \partial W$, that is $z_0 \in \partial W$, we have

$z_0 \in I_0$ and $\omega z_0 \notin I_0$. Hence $\omega z_0 \in \partial I_0$ since $\omega z_0 \in \partial W$ and $W \subset I_0$. Then $Q_\beta^k(z) \to 0$ whereas $Q_\beta^k(\omega z_0) \nrightarrow 0$ as $k \to \infty$. But $Q_\beta^k(\omega z_0) = \omega^{d^k} Q_\beta^k(z_0) \to 0$. This is a contradiction. Therefore $\omega I_0 = I_0$ Similarly we can proof that $\omega I_\infty = I_\infty$. ∎

Lemma (2.1.9)

For any Fatou component V of Q_β. Assume that z_0 and $\omega^n z_0$ belong to V, where $\omega^n \neq 1$ with for integer $n \neq 0 \, mod \, 2d$. Then $\omega^m z_0 \in V$ for each m. In special case, V has $2d$ −fold symmetry also surround 0.

Proof

Assume that z_0 , $\omega^n z_0 \in V$ but V no contains $\omega^m z_0$. For any continuous curve Γ_1 connects z_0 to $\omega^n z_0$ in V. We define Γ_2 is a second curve by $\omega^m \Gamma_1$. In a component of the Fatou set contains Γ_2 from symmetry . Because $\omega^n z_0$ lies on Γ_2, thus V contains Γ_2 also $\omega^{2n} z_0$ lies in V. Continuing in this fashion, we have $\omega^{jn} z_0$ lies in V for j and so that V contains curve Γ_j.

Now assume that $\omega^{kn} = 1$. Therefore V contains a closed curve formed from the union of the curves $\Gamma_1, \dots, \Gamma_k$ and around 0. Set this curve $\mathit{\Gamma}$. $\omega^m z_0$ does not lie on $\mathit{\Gamma}$ from the assumption. If we fix $\omega^m \Gamma_l = \Upsilon_l$ for any l. Thus there is a closed curve, say Υ and surrounds 0, also is contained in $\omega^m V$. Because $\omega^m z_0 \in \omega^m V$ but $\omega^m z_0 \notin V$. Hence $\omega^m V \neq V$. Note that V is a Fatou component also $Q_\beta(\omega z) = \omega^d Q_\beta(z)$, so we have $\omega^m V$ is a Fatou component, thus $\omega^m V \cap V = \emptyset$. Since $\Upsilon \subset \omega^m V$ and $\Gamma \subset V$, we have $\Upsilon \cap \mathit{\Gamma} = \emptyset$. Therefore Υ and $\mathit{\Gamma}$ are both curves and around 0 also $\Upsilon = \omega^m \mathit{\Gamma}$, thus Υ and $\mathit{\Gamma}$ must cross. It means that Υ and $\mathit{\Gamma}$ lie in the same Fatou component, this is a contradiction. Therefore V contains $\omega^m z_0$ for each m.

Proposition (2.1.10)

Let $Q_\beta : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that

$$Q_\beta(z) = 2\beta^{1-d} z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}.$$

$J(Q_\beta)$ has the symmetry ($2d - fold$).

Proof

Suppose that $I_\infty = \{ z \in \mathbb{C}_\infty : Q_\beta^m(z)$ tend to ∞ as $m \to \infty \}$, by Lemma (2.1.8), I_∞ has $2d -$ fold symmetry, so that

$Q_\beta^m(\omega z) = \omega^{d^m}\, Q_\beta^m(z)$ for $m \geq 1$, thus $Q_\beta^m(\omega z)$ tend to ∞ if and only if $Q_\beta^m(z)$ tend to ∞ as $m \to \infty$. Since $J(Q_\beta)$ is the boundary of I_∞ also equal to the boundary of $(\cup_{m\geq 0} Q_\beta^{-m}(I_\infty))$. Therefore $J(Q_\beta)$ has $2d -$ fold symmetry.

Proposition (2.1.11)

Assume that $\beta \in \mathbb{R}$. Then $\mathcal{T}_{|\beta|} = \{z \in \mathbb{C}: |z| = |\beta|\}$ be the round circle and $Q_\beta: \mathcal{T}_{|\beta|} \to \mathcal{T}_{|\beta|}$. Moreover, $\mathcal{T}_{|\beta|} \subset J(Q_\beta)$ whenever the free critical orbits are attracted by ∞ and 0 .

Proof

Assume that $= |\beta|e^{i\theta}$, where $\theta \in [0,2\pi)$. Then by using

$|-z| = |z|$

$$|Q_\beta(z)| = |-Q_\beta(z)| = \left| -(2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}) \right|$$

$$= \left| (2\beta^{1-d}z^d + \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}) \right|$$

$$\leq 2|\beta|^{1-d}|\beta|^{d} + |\beta|^{d}\frac{|\beta^{2d}e^{2\pi i\theta} - \beta^{d+1}|}{\left||\beta^{2d}e^{2\pi i\theta} - \beta^{3d-1}|\right|}$$

$$\leq -\left(2|\beta|^{1-d}|\beta|^{d} - |\beta|^{d}\frac{|\beta^{2d}e^{2\pi i\theta} - \beta^{d+1}|}{\left||\beta^{2d}e^{2\pi i\theta} - \beta^{3d-1}|\right|}\right)$$

$$= -(2|\beta| - |\beta|) = -(|\beta|) = |-(|\beta|)| = |\beta|.$$

So $|Q_\beta(z)| \leq |\beta|$. This means that $Q_\beta(\mathcal{T}_{|\beta|}) \subset \mathcal{T}_{|\beta|}$. If β is real, we have $|\beta| \neq 1$ since $\beta^{2d-2} \neq 1, \beta^{1-d} \neq 1$ by use the definition of Q_β. Now, we have two cases:

Case one

If $|\beta| > 1$, we have $|\beta|^{\frac{d+1}{2d}} < |\beta| < |\beta|^{\frac{3d-1}{2d}}$, that is β^{3d-1} is large such that by (1.3), Q_β has $3d$ roots and no poles in

$D_{|\beta|} = \{z \in \mathbb{C} : |z| < |\beta|\}$. From the Argument Theorem, thus $Q_\beta(\mathcal{T}_{|\beta|})$ around the origin $3d$ times anticlockwise. Hence

$Q_\beta(\mathcal{T}_{|\beta|}) = \mathcal{T}_{|\beta|}$.

Case two

If $0 < |\beta| < 1$, we have $|\beta|^{\frac{3d-1}{2d}} < |\beta| < |\beta|^{\frac{d+1}{2d}}$. Note that if β^{d+1} is large and β^{3d-1} is small such that by (1.3), then we have

$Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d} - \beta^{d+1})}{z^{2d} - \beta^{3d-1}}$ since β^{3d-1} is small Q_β has d roots and $2d$ poles in $D_{|\beta|}$, thus $Q_\beta(\mathcal{T}_{|\beta|})$ around the origin d times clockwise. Therefore, $Q_\beta(\mathcal{T}_{|\beta|}) = \mathcal{T}_{|\beta|}$. Therefore $Q_\beta : \mathcal{T}_{|\beta|} \to \mathcal{T}_{|\beta|}$ is a surjection in the two cases. Suppose that $z_0 \in \mathcal{T}_{|\beta|} \subset \mathbb{F}(Q_\beta)$, then $Q_\beta{}^{\ell}(z_0) \to 0$ or ∞ as $\ell \to \infty$. However, on the other side,

$Q_\beta{}^{\ell}(z_0) \in Q_\beta(\mathcal{T}_{|\beta|}) = \mathcal{T}_{|\beta|}$ for each $\ell \geq 0$, that is contradict .

Hence, $\mathcal{T}_{|\beta|} \subset J(Q_\beta)$.■

Example (2.1.12)

Let $Q_\beta : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that

$$Q_\beta(z) = 2\beta^{1-d} z^d - \frac{z^d\left(z^{2d} - \beta^{d+1}\right)}{z^{2d} - \beta^{3d-1}}.$$

If d is odd and $Q_\beta(-\beta) = -\beta$, then $\mathcal{T}_{|\beta|}$ is not contained $J(Q_\beta)$.

If $Q_\beta(-\beta) = -\beta$. By (2.1), we have

$$Q_\beta{}'(z) = dz^{d-1} \frac{-z^{4d} + \left(3\beta^{3d-1} - \beta^{d+1}\right) z^{2d} - \beta^{4d}}{\left(z^{2d} - \beta^{3d-1}\right)^2} + 2d\beta^{1-d} z^{d-1}$$

$$Q_\beta{}'(-\beta) = \quad d(-\beta)^{d-1} \frac{-(-\beta)^{4d} + \left(3\beta^{3d-1} - \beta^{d+1}\right)(-\beta)^{2d} - \beta^{4d}}{\left((-\beta)^{2d} - \beta^{3d-1}\right)^2}$$

$$+ \ 2d\beta^{1-d}(-\beta)^{d-1}$$

$$= -d\beta^{d-1} \frac{\beta^{4d} - 3\beta^{5d-1} + \beta^{3d+1} - \beta^{4d}}{\left((\beta)^{2d} + \beta^{3d-1}\right)^2} - 2d$$

$$Q_\beta{}'(-\beta) = \frac{d\beta^{2d-2} - 4d\beta^{d-1} - 3d}{1 + 2\beta^{d-1} + \beta^{2d-2}}$$

If $Q_\beta{}'(-\beta) = 0$, then $d\beta^{2d-2} - 4d\beta^{d-1} - 3d = 0$

$\beta^{2d-2} - 4\beta^{d-1} - 3 = 0$, thus either $\beta = -(0.6457)^{\frac{1}{d-1}}$ or

$\beta = (4.6457)^{\frac{1}{d-1}}$.

Hence $-\beta$ is superattracting fixed point of Q_β, but this is not attract to 0 or ∞. Then $\mathcal{T}_{|\beta|} \not\subset J(Q_\beta)$.

We study a sufficient and necessary condition for $J(Q_\beta)$ is a quasicircle.

Lemma (2.1.13) [5]

Assume that the rational map is hyperbolic, it has exactly two Fatou components. Then the Julia set is a quasicircle.

Corollary (2.1.14)

Suppose that Q_β is hyperbolic map have exactly two Fatou component I_0 and I_∞ , then $J(Q_\beta)$ is quasicircle .

Proof

By Lemma (2.1.5), I_0 and I_∞ are contain in $\mathbb{F}(Q_\beta)$. From Lemma (2.1.13), then $J(Q_\beta)$ is quasicircle . ■

Proposition (2.1.15)

If one of the free critical points e_β lies in I_0 or I_∞ , then $J(Q_\beta)$ is quasicircle.

Proof

Assume that $e_\beta \in I_0$ .Then by Lemma (2.1.5) $\frac{\beta^2}{e_\beta} \in I_\infty$, also by Lemma (2.1.4) $\omega^j e_\beta \in I_0$ and $\omega^j \frac{\beta^2}{e_\beta} \in I_\infty$ for $0 \leq j \leq 2d-1$, we assume that ${Q_\beta}^{-1}(I_0)$ has the unique component say, I_0 and ${Q_\beta}^{-1}(I_\infty)$ has the unique component say, I_∞. If the degree of the restriction of Q_β is m and $Q_\beta: I_0 \to I_0$ is proper , then $m \geq d$ because d is the local degree of Q_β at 0 . Now in I_0 the preimages of 0 have elements other than 0 by Lemma (2.1.4) , it follows $m \geq 3d$. This means $m = 3d$ because $3d$ is the degree of Q_β. Hence $m = d$ or $m = 3d$. We prove that $m = d$ is impossible. Assume that $e_\beta \in I_0$ is the free critical points of Q_β and $\{\omega^j e_\beta : 0 \leq j \leq 2d-1\}$ lies in I_0 by Lemma (2.1.8). From the definition of Q_β in (1.3), thus there exist preimages $2d$ at least for

$Q_\beta(e_\beta)$ *and* $\{\omega^j e_\beta : 0 \le j \le 2d-1\}$ in I_0 (counted with multiplicity). Hence $m \ge 2d$, which is contradict with $m = d$. Therefore $m = 3d$ and $Q_\beta{}^{-1}(I_0)$ has the unique component I_0, also prove $Q_\beta{}^{-1}(I_\infty)$ has the unique component I_∞. We assume that only there are I_0 and I_∞ contain in $\mathbb{F}(Q_\beta)$. Now, note that there were either superattracting basins or parabolic basins contain one critical point at least, which is contradict because each of the critical points lie in I_0 and I_∞. If there were either Herman rings or Siegel disks, thus $J(Q_\beta)$ contains one critical value at least and is contradicted from Corollary (2.1.14). By Definition(1.1.33), thus Q_β is hyperbolic. According to Lemma (2.1.13), $J(Q_\beta)$ is a quasicircle. See Fig. (1).■

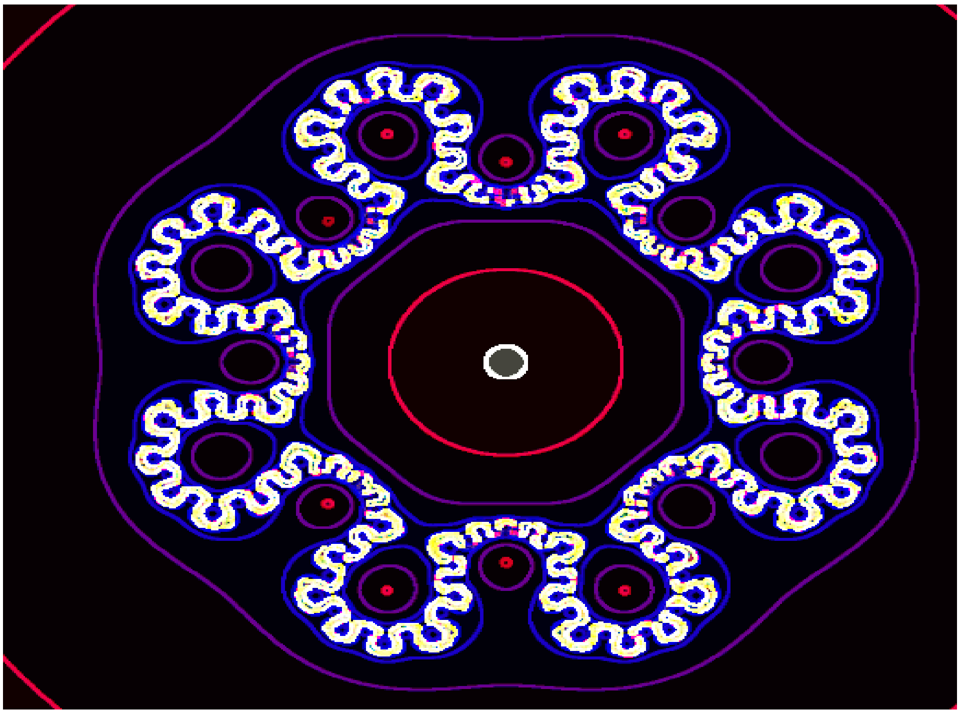


Figure (1) $\boldsymbol{\beta = -1.23i}$ **and** $\boldsymbol{J(Q_\beta)}$ **is a quasicircle**.

Proposition (2.1.16)

Let $Q_\beta : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that

$$Q_\beta(z) = 2\beta^{1-d} z^d - \frac{z^d\left(z^{2d} - \beta^{d+1}\right)}{z^{2d} - \beta^{3d-1}} .$$

Suppose that $|\beta|$ is large enough. Then $J(Q_\beta)$ is a quasicircle.

Proof

From Remark (2.1.7), we have

$$e_\beta{}^{2d} = \frac{3\beta^{3d-1} - 4\beta^d - \beta^{d+1} + \sqrt{(3\beta^{3d-1} - 4\beta^d - \beta^{d+1})^2 - 4(1 - 2\beta^{1-d})(\beta^{4d} - 2\beta^{5d-1})}}{2 - 4\beta^{1-d}} .$$

Thus $|e_\beta| \approx |\beta|^{\frac{3d-1}{2d}}$, if $|\beta|$ is large enough . Since $d \geq 2$,thus $|\beta|^{\frac{3d-1}{2d}} \geq |\beta|^{\frac{5}{4}} > |\beta|^{\frac{6}{5}}$. Define $U = \{z : |z| > |\beta|^{\frac{6}{5}}\}$.

Now , if $|\beta|$ is large enough and $z \in U$, then

$$|Q_\beta(z)| \geq 2|\beta|^{1-d}|z|^d - \frac{|z|^d\left(|z|^{2d} - |\beta|^{d+1}\right)}{|z|^{2d} - |\beta|^{3d-1}}$$

$$> 2|\beta|^{1-d}|\beta|^{\frac{6d}{5}} - \frac{|z|^{\frac{6d}{5}}\left(|z|^{\frac{12d}{5}} - |\beta|^{d+1}\right)}{|z|^{\frac{12d}{5}} - |\beta|^{3d-1}}$$

$$> 2|\beta|^{\frac{d+5}{5}} - \frac{|\beta|^{\frac{6}{5}}}{2} > 2|\beta|^{\frac{6}{5}}.$$

This means that $Q_\beta(U) \subset U$. Therefore U is contained in I_∞.

On the other hand, one of the free critical points e_β holds $|e_\beta| \approx |\beta|^{\frac{3d-1}{2d}} > |\beta|^{\frac{6}{5}}$. It follows that $e_\beta \in U \subset I_\infty$. If $|\beta|$ is large enough . By Proposition (2.1.15), $J(Q_\beta)$ is a quasicircle .■

Proposition (2.1.17)

Let $Q_\beta : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that

$$Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}\ .$$

Assume that $\beta \in \mathbb{R}$. Then $J(Q_\beta)$ is a quasicircle if and only if $J(Q_\beta) = \mathcal{T}_{|\beta|}$, $|\beta| > 1$.

Proof

Suppose that $|\beta| > 1$, we have $Q_\beta(\mathcal{T}_{|\beta|}) = \mathcal{T}_{|\beta|}$ and in the round disk $D_{|\beta|} = \{z \in \mathbb{C} : |z| < |\beta|\}$, Q_β has no poles and $3d$ roots from Proposition (2.1.11). Hence $Q_\beta(D_{|\beta|}) = D_{|\beta|}$. Thus, $D_{|\beta|} \subset \mathbb{F}(Q_\beta)$. From Lemma (2.1.5), $\mathbb{C}_\infty \backslash \overline{D}_{|\beta|} \subset \mathbb{F}(Q_\beta)$. In special case , $D_{|\beta|} \subset I_0$ also $\mathbb{C}_\infty \backslash \overline{D}_{|\beta|} \subset I_\infty$. It follows that $J(Q_\beta) = \mathcal{T}_{|\beta|}$ because $I_0 \cap I_\infty = \emptyset$. We assume that $0 < |\beta| < 1$. And assume that $J(Q_\beta)$ is a quasicircle, thus $J(Q_\beta) = \mathcal{T}_{|\beta|}$ because $Q_\beta(\mathcal{T}_{|\beta|}) = \mathcal{T}_{|\beta|}$. Hence, $D_{|\beta|} \subset I_0$ and $Q_\beta: D_{|\beta|} \to D_{|\beta|}$ has a degree of $3d$ for the covering map . On the other hand, if $0 < |\beta| < 1$, $D_{|\beta|}$ include $2d$ poles and d roots for Q_β, which is a contradict. Hence if $0 < |\beta| < 1$, $J(Q_\beta)$ is not quasicircle. ■

Remark (2.1.18)

Let $\rho_0 = Q_\beta^{-1}(I_0) \backslash I_0$ be the first preimage of I_0 and $\rho_\infty = Q_\beta^{-1}(I_\infty) \backslash I_\infty$ the first preimage of I_∞. If $J(Q_\beta)$ is not a quasicircle, by Proposition (2.1.15).It follows one of the free critical points not lies in I_0 or I_∞, thus one of the free critical points lies in ρ_0 and ρ_∞. So ρ_0 and ρ_∞ are both non-empty.

Remark (2.1.19) [12]

Let $U \subset X$ be an open set of a compact topological space X and $V \Subset U$ an open, **compactly contained** set (i.e., $\bar{V}$ is compact and $\bar{V} \subset U$).

Theorem (2.1.20) [10]

(a) Every polynomial-like map $f: U` \rightarrow U$ of degree d is hybrid equivalent to a polynomial P of degree d.

(b) If K_f is connected, P is unique up to conjugation by an affine map.

Proposition (2.1.21)

Both I_0 , ∂I_∞, and each of the preimages of them is quasicircles around 0 .

Proof

For each closed set $V = \mathbb{C}_\infty \backslash (I_0 \cup I_\infty)$ amidst I_0 and I_∞ divided into closed sets V_0, V_1, V_2 between I_∞ and ρ_0, ρ_0 and ρ_∞, ρ_∞ and I_0 (see Figure (2)). For any smooth simple closed curve $\Gamma \subset \rho_0 \subset V$ around 0. We assume that in V_2 the preimage of Γ is smooth simple closed curve around 0 . Note that, V includes no critical values. Therefore in V_2 the preimage of Γ contains of finitely many smooth simple closed curves. Assume that Γ_2 is not around 0. Therefore in V_2, Γ_2 can disfigure to a point. It follows that in $V,$ $\Gamma = Q_\beta(\Gamma_2)$ can also disfigure to a point. Which is contradicted because $\Gamma \subset V$ is around 0. Hence in V_2 the preimage of Γ are smooth simple closed curves around 0 and ∞. In V_2 there are two components for

${Q_\beta}^{-1}(\Gamma)$, thus between two simple closed curves, the annular region include either poles or roots , this is impossible. Then, ${Q_\beta}^{-1}(\Gamma) \cap V_2$ is a smooth simple closed curve around 0, say $\mathfrak{J}$. Let $\gamma \subset \mathbb{C}$ be a simple closed curve and assume that γ^{int} is the bounded component of $\mathbb{C}\backslash\gamma$. We remark that in Γ^{int} is the Jordan disk include $\mathfrak{J}^{int}$ is compactly contained. By Theorem (2.1.20), Therefore $Q_\beta: \mathfrak{J}^{int} \rightarrow \Gamma^{int}$ is quasiconformally equivalent to $f_d(z) = z^d$. We know that $J(f_d) = S^1$. Then ∂I_0 is a quasicircle. Similarly, also ∂I_∞ is a quasicircle. Because each of the preimages of ∂I_0 and ∂I_∞ are include in $V_0 \cup V_1 \cup V_2$, It follows that each of the preimages of ∂I_0 and ∂I_∞ are quasicircles around 0.■

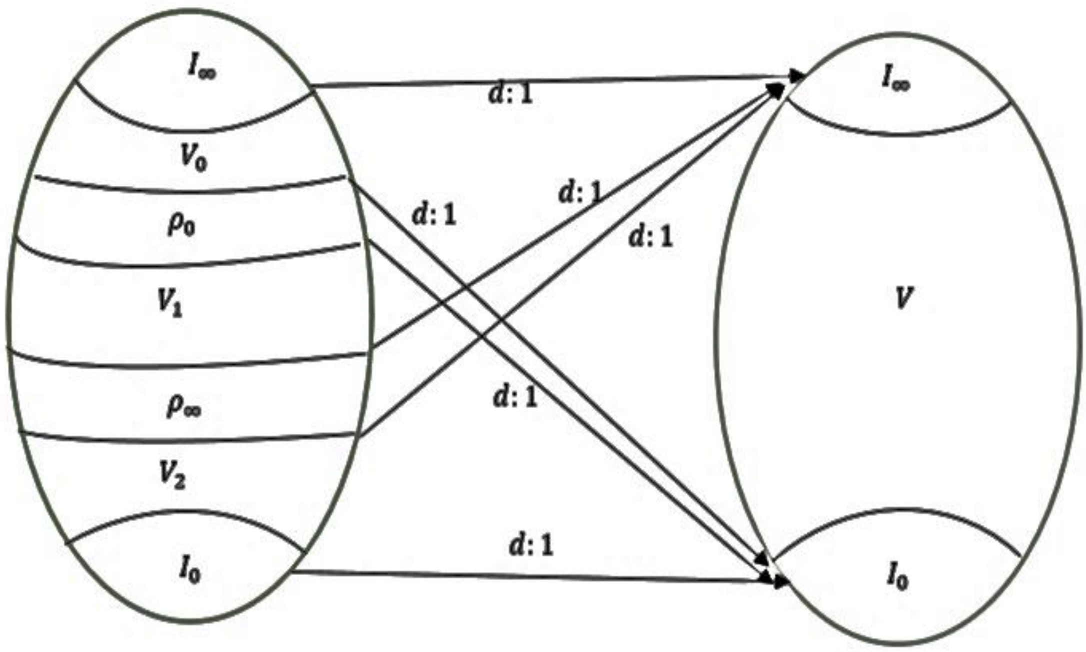


Figure (2) Sketch illustrating of the map relations of Q_β if $Q_\beta(e_\beta) \in I_0$,but $e_\beta \notin I_0$.

Now, we will give the necessary and sufficient condition of Q_β such that $J(Q_\beta)$ is a cantor circles, by studying the location of the critical values and the critical points.

Remark (2.1.22)

If $I_0\ or\ I_\infty$ contains one of the free critical point, we get $J(Q_\beta)$ is quasicircle . Assume that $e_\beta \in \rho_0$ because ρ_0 is the complement of I_0, where $\rho_0 = Q_\beta^{-1}(I_0)$ .Thus $\rho_0 \neq \emptyset$ and by Lemma (2.1.5), it follows that $\frac{\beta^2}{e_\beta} \in \rho_\infty$. Therefore $\rho_\infty \neq \emptyset$. Q_β are both d to one on I_0 and I_∞.

Corollary (2.1.23)

If the immediate basin of attraction of 0 is simple connected, then $\chi(I_0) = 1$.

Proof

Since I_0 is simple connected. So the number of connected components of the complement $\mathbb{C}_\infty \backslash I_0$ is equal one. By Definition (1.1.25), hence $\chi(I_0) = 1$. ■

Proposition (2.1.24)

The I_0 and I_∞ are both simply connected and ρ_0 and ρ_∞ are two annuli surrounding 0 are $2d$ −fold symmetry *.*

Proof

From Theorem (1.2.22), the parabolic basin or the immediate attracting basin is either infinitely connected or simply connected. We assume that V_0 be small open disk around 0 by Remark (2.1.19) $Q_\beta(\overline{V_0}) \subset V_0 \subset I_0$ and the boundary of V_0 is Jordan curve containing no Q_β^m of critical points. Fix $V_m = Q_\beta^{-1}(V_0)$, where $Q_\beta^{-1}(V_0)$ are connected component which contains 0, hence $V_0 \subset V_1 \subset V_2$ ..., with

$\bigcup_m V_m = I_0$. Now the map $Q_\beta : V_m \backslash \{0\} \to V_0 \backslash \{0\}$ are covering by degree d^m, from the Riemann–Hurwitz's formula,

$\chi(V_m) = \chi(V_m^\circ) + \chi(\partial V_m)$. Some finite Jordan curves are restricting for V_m and the complement $(\mathbb{C}_\infty \backslash V_m)$ are disjoint union of Jordan disks, so $\chi(V_m) = 0$. So $V_m \backslash \{0\}$ is an annulus since $V_m \backslash \{0\}$ is no containing Q_β^m of the critical points and $V_0 \backslash \{0\}$ is an annulus, thus V_m is simply connected. Then $\bigcup_m V_m = I_0$ is simply connected. By Lemma (2.1.9), it follows that ρ_0 has either $2d$ component or one. Assume that ρ_0 has $2d$ component, then the origin has $5d$ preimages since has $2d$ component and to add the degree of Q_β is $3d$, thus ρ_0 has $5d$ preimages. Because the map from any component of ρ_0 to I_0 with degree two also there are $2d$ components. Which is contradict because degree of Q_β is $3d$. Thus ρ_0 has only one component and it is connected. By Theorem (1.1.26) for $Q_\beta : \rho_0 \to I_0$,

so $\chi(\rho_0) + 2d = 2d\chi(I_0)$, since I_0 is simply connected. So by Corollary (2.1.23), $\chi(I_0) = 1$. Hence $\chi(\rho_0) = 0$ and ρ_0 is an annulus around 0 with $2d -$ fold symmetry. Similarly, we can to show that the simply connected for I_∞ also ρ_∞ is an annulus around 0 with $2d -$ fold symmetry . ■

Remark (2.1.25) [12]

For each two disjoint sets M and N such that separate the origin and infinity . We write $\boldsymbol{M \prec N}$ if the component $\mathbb{C}_\infty \backslash N$ contains M and the origin .

Proposition (2.1.26)

If ρ_0 , ρ_∞ are two annuli, then $\rho_\infty \prec \rho_0$. However $\overline{\rho_\infty}, \overline{\rho_0}, \overline{I_0}$ and $\overline{I_\infty}$ are disjoint to each other.

Proof

By definition of ρ_0 and ρ_∞, we have $\rho_\infty \cap \rho_0 = \emptyset$ and the intersection of I_∞ and I_0 is an empty set. Now we have two claims either $\rho_\infty \prec \rho_0$ or $\rho_0 \prec \rho_{\infty,}$ since ρ_0 and ρ_∞ are separating the origin and the infinity. Assume that $\rho_0 \prec \rho_\infty$, for each V_0 is bounded component of $\mathbb{C}_\infty \backslash \rho_0$, $Q_\beta(\partial V_0) = Q_\beta(\partial \rho_0)$ because V_0 is compact set.

Thus $Q_\beta(\partial V_0) = Q_\beta(\partial \rho_0) = \partial I_0 = Q_\beta(\partial I_0)$ and

$Q_\beta(V_0) = Q_\beta(I_0) = I_0 \subseteq \overline{I_0}$ since

$\rho_0 = Q_\beta^{-1}(I_0) \backslash I_0$ and $Q_\beta^{-1}(\infty) \subset I_\infty \cup \rho_\infty$,

it follows $Q_\beta(V) \subset \overline{I_0}$. Therefore, the image of $\rho_0 \cup V_0$ is a subset of $\overline{I_0} \subset \rho_0 \cup V_0$, this mean $\rho_0 \cup V_0$ lies in $\mathbb{F}(Q_\beta)$ and $\rho_0 \cup V_0 = I_0$ (in particular) .This is impossible because $\rho_0 \neq \emptyset$,hence $\rho_\infty \prec \rho_0$. By Proposition (2.1.24) ,we have $\overline{I_\infty} \cap \overline{I_0} = \emptyset$.Now, we note that $Q_\beta(\overline{\rho_0}) = Q_\beta(\overline{I_0}) = \overline{I_0}$ and $Q_\beta(\overline{\rho_\infty}) = Q_\beta(\overline{I_\infty}) = \overline{I_\infty}$.

Therefore $\overline{\rho_\infty} \cap \overline{\rho_0} = \emptyset$, $\overline{I_\infty} \cap \overline{\rho_0} = \emptyset$ and $\overline{\rho_\infty} \cap \overline{I_0} = \emptyset$.

Thus $\overline{\rho_0} \cap \overline{I_0} = \emptyset$ and $\overline{I_\infty} \cap \overline{\rho_\infty} = \emptyset$ because $\rho_\infty \prec \rho_0$. ■

Remark (2.1.27)

In our work, we show that all Julia components of simple closed curves (quasicircles). Now we use the technique of symbol dynamics, for each $\Sigma_3 = \{v = (s_0 s_1 s_2, \ldots), v_m \in \{0,1,2\}$

$m \geq 0\}$ be the space of one sided sequences of the symbols $\{0,1,2\}$. For v = ($s_0s_1s_2,\ldots$) ∈ Σ_3 and the shift map

$\sigma: \Sigma_3 \to \Sigma_3$ is denoted by $\sigma(v) = (s_1s_2,\ldots)$.

Let $V_\beta = \{v \in V_0 \cup V_1 \cup V_2 : Q_\beta^d(v) \in V_0 \cup V_1 \cup V_2$ for $d = 1,2,3,\ldots\}$,

all the points in the domain of Q_β either toward 0 or ∞ or stay in V_β. For any $v \in V_\beta$,then each iterate of v either V_0 *or* V_1 *or* V_2 ,so we can associate with v the forward sequence $v = (s_0s_1s_2,\ldots)$, where

$$v_m = \begin{cases} 0 & if\ Q_\beta^d\ is\ in\ V_0 \\ 1 & if\ Q_\beta^d\ is\ in\ V_1 \\ 2 & if\ Q_\beta^d\ is\ in\ V_2. \end{cases}$$

If there is an integer $i > 0$, such that $v_{m+i} = v_m$ for all $m \geq 0$. Suppose that $V_\beta \subset \Lambda_\beta = \{J_{j_0j_1\cdots j_m} : 0 \leq j_m \leq 2\}$. We define the metric in Σ_3 as $d(z,v) = \sum_{i=o}^{\infty} \frac{|s_i - v_i|}{3^i}$.

Proposition (2.1.28)

The map itinerary $s_\beta : \Lambda_\beta \to \Sigma_3$ is homeomorphism, where the set Λ_β is a Cantor set.

Proof

First , to prove s_β is 1-1 map. If $z = (s_0s_1s_2,\ldots)$ and

$v = (v_0v_1v_2,\ldots)$ such that $s_\beta(z) = s_\beta(v)$, it follows

$s_m = v_m\ \forall\, m \geq 0,$ so that z, v lie in the same V_β because the length of V_β is $1/_{3^d}$ and go to 0 when $d \to \infty$. Hence s_β is one to one. Now if $(s_0s_1s_2,\ldots)$ be the sequence of 0`s, 1`s and 2`s, pick V_0 *or* V_1 *or* V_2 satisfying

z in $V_0 \rightarrow s_\beta(z) = s_0$

z in $V_1 \rightarrow s_\beta(z) = s_0 s_1$

z in $V_2 \rightarrow s_\beta(z) = s_0 s_1 s_2$.

So $V_2 \subseteq V_1 \subseteq V_0$ and since each $V_\beta`s$ are closed and bounded, by Heine-Borel Theorem there is $z^* \in V_\beta$ and by definition of s_β. Therefore $s_\beta(z^*) = s_0 s_1 s_2$ and so s_β is onto .To prove s_β is continuous. For any $\varepsilon > 0$ and for any $z \in \Lambda_\beta$, let d be large so $1/_{3^d} < \varepsilon$. Fix $\delta > 0$ is small , let $y \in \Lambda_\beta$ such that $|z - y| < \delta$,then z, y lie in the same V_β. For a y, the sequence $s_\beta(z)$ and $s_\beta(y)$ have the same initial d terms, since definition of s_β. Hence

$\left|s_\beta(z) - s_{\beta(}y)\right| \leq 1/_{3^d} < \varepsilon$, therefore s_β is continuous map.

Since s_β is one to one map, it follows $s_\beta{}^{-1}$ is continuous map . ■

Theorem (2.1.29)

Let $Q_\beta : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ be a map such that

$$Q_\beta(z) = 2\beta^{1-d} z^d - \frac{z^d(z^{2d} - \beta^{d+1})}{z^{2d} - \beta^{3d-1}}.$$

$J(Q_\beta)$ is a Cantor circles if $Q_\beta(e_\beta) \in I_0$ (or I_∞), where $Q_\beta(e_\beta)$ one of the free critical values but $e_\beta \notin I_0$ (or I_∞).

Proof

For each closed set $V \coloneqq \mathbb{C}_\infty \backslash I_\infty \cup I_0$ amidst I_0 and I_∞ divided into closed sets V_0, V_1, V_2 between I_∞ and ρ_0, ρ_0 and ρ_∞, ρ_∞ and I_0 (see Figure (2)). Each the map $Q_\beta : V_m \rightarrow V$ is covering by degree

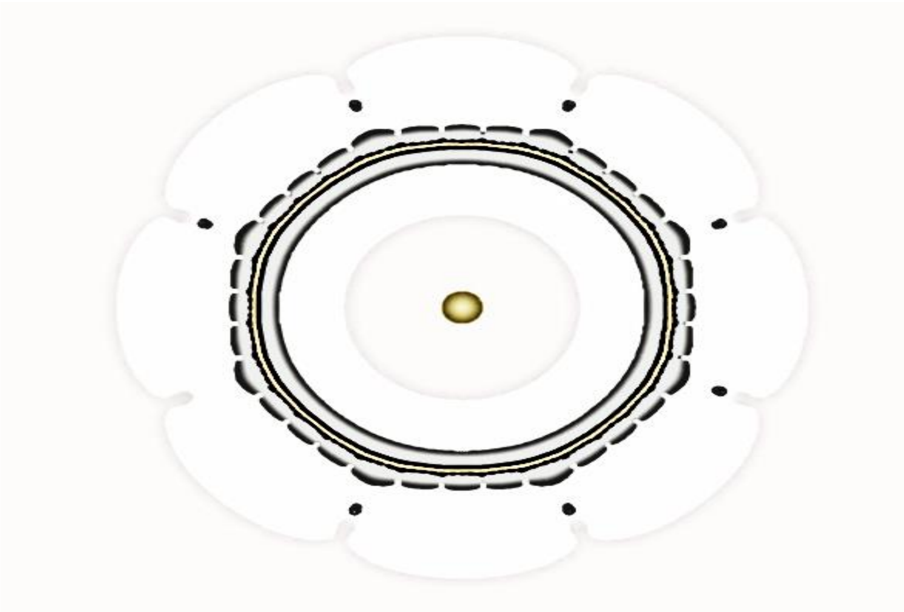
d, for $0 \leq m \leq 2$. So $J(Q_\beta)$ is equal to $\bigcup_{i\geq 0} Q_\beta^{-i}(V)$. For any $h: V \to V_m$ is the inverse branch of Q_β for $0 \leq m \leq 2$. Therefore, any component $j_{m_0,m_1,\ldots,m_i,\ldots} = \bigcap_{i=0}^{\infty} h_{m_i} \circ \ldots \circ h_{m_1} \circ h_{m_0}$, where $(m_0, m_1, \ldots, m_i, \ldots)$ be infinite sequence holding $0 \leq m \leq 2$. For each $j_{m_0,m_1,\ldots,m_i,\ldots}$ is compact set separating the origin and the infinity. By Corollary (1.1.35), thus $j_{m_0,m_1,\ldots,m_i,\ldots}$ is locally connected because Q_β is hyperbolic . Now, for any $\mathrm{E} = \xi \cup \varrho$, $\xi = j_{2,2,\ldots,2,\ldots} = \partial I_0$ and $\varrho = j_{0,0,\ldots,0,\ldots} = \partial I_\infty$. We note $V_m \subset V$, also $g: V_m \hookrightarrow V$ is identity map and not homotopic to a constant map . By Proposition (1.1.37), we get $j_{m_0,m_1,\ldots,m_i,\ldots}$ is a simple closed curve . By Proposition (2.1.21), hence $j_{m_0,m_1,\ldots,m_i,\ldots}$ is a quasicircle since Q_β is hyperbolic. From Remark (2.1.27) and Proposition (2.1.28), it is clear that $s_\beta\left(Q_\beta(z)\right) = \sigma(s_\beta(z))$ for $z \in \Lambda_\beta$. The one-sided shift on the space of 3 symbols
$\Sigma_3 = \{s = (s_0 s_1 s_2, \ldots); s_m \in \{0,1,2\}, m \geq 0\}$ is isomorphic to the dynamics on the Julia components Λ_β. In particular, $J(Q_\beta)$ is homeomorphic to $\Sigma_3 \times S^1$, where this is a Cantor circles. ■

Theorem (2.1.30)

Assume that one of the free critical values lies in I_0 or I_∞ but $e_\beta \notin I_0$ or I_∞ . Then the McMullen map $f_\lambda(z)$ is not topologically conjugate to Q_β corresponding to Julia sets .

Proof

From Theorem (2.1.29), the one-sided shift on the space of 3 symbols $\Sigma_3 = \{s = (s_0 s_1 s_2, \ldots); s_m \in \{0,1,2\}, m \geq 0\}$ is isomorphic to the dynamics on the Julia components Λ_β. Notwithstanding, the dynamics of the one-sided shift on only two symbols $\Sigma_2 = \{s = (s_0 s_1, \dots); s_m \in \{0,1\}, m \geq 0\}$ is isomorphic to dynamics on the set of Julia components of $f_\lambda(z)$. Hence, $f_\lambda(z)$ is not topologically conjugate to Q_β corresponding to Julia sets, see Figure (3). ■



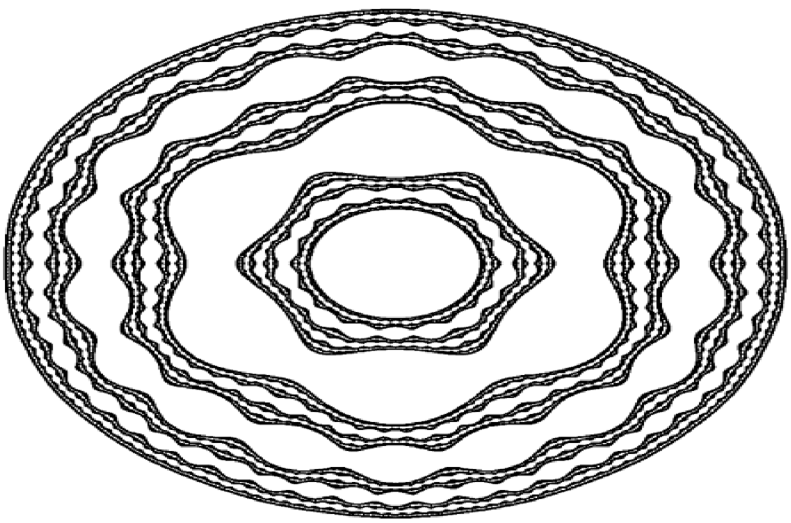


Figure. (3) $J(Q_\beta)$ **if** $d = 4, \beta = 0.7999 + 0.8i$ **and** $f_{0.01}(z) = z^3 + \frac{0.01}{z^3}$**. Are both of them Cantor circles. Hence** f_λ **and** Q_β **are not topologically conjugate corresponding to Julia set**s.

Remark (2.1.31) [12]

Let $\mathrm{A}(\beta)$ and $\mathrm{B}(\beta)$ be two values for the parameter β, where $\mathrm{A}(\beta) \geq 0$ and $\mathrm{B}(\beta) \geq 0$. Then is said to be $\mathbf{A}(\boldsymbol{\beta}) \preccurlyeq \mathbf{B}(\boldsymbol{\beta})$ if there is $\varsigma \geq 0$ such that $\mathrm{A}(\beta) \leq \varsigma.\mathrm{B}(\beta)$ for $0 \neq \beta$ is small.

Theorem (2.1.32)

Let $Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}$ be a map.

Assume that $d \geq 4$. If β is a non-zero and small enough, then $J(Q_\beta)$ is a Cantor circles.

Proof

The $Q_\beta(z)$ has one free point, say e_β such that

$$e_\beta = \left(\frac{3\beta^{3d-1}-4\beta^d-\beta^{d+1}-\sqrt{(3\beta^{3d-1}-4\beta^d-\beta^{d+1})^2-4(1-2\beta^{1-d})(\beta^{4d}-2\beta^{5d-1})}}{2-4\beta^{1-d}}\right)^{\frac{1}{8}}.$$

If $|\beta|$ is small enough, it follows $e_\beta \asymp |\beta|^{\frac{d+1}{2d}}$. We define

$b \coloneqq |\beta|^{\frac{d+1}{2d}}$ $\forall z \in \mathcal{T}_b$, where $\mathcal{T}_b$ is a round circle is defined as $\mathcal{T}_b = \{z: |z| = b\}$. We obtain

$$\left|Q_\beta(z)\right| = \left|(2\beta^{1-d}z^d - \frac{z^d\left(z^{2d}-\beta^{d+1}\right)}{z^{2d}-\beta^{3d-1}})\right|$$

$$\asymp 2|\beta|^{\frac{d+1}{2}}|\beta|^{-d} - \frac{|\beta|^{\frac{d+1}{2}}|(z^{2d}-\beta^{d+1}|)}{|\beta|^{d+1}}$$

$$\preccurlyeq \quad 2|\beta|^{\frac{1-d}{2}}|\beta|^{-d} - |\beta|^{\frac{d+1}{2}} \preccurlyeq 2|\beta|^{\frac{d+1}{2}} \preccurlyeq |\beta|^{\frac{d+1}{2}}.$$

For $d \geq 4$, therefore $5d-3 < d(d+1)$, for any $\alpha > 0$

satisfying $\frac{5d-3}{2d} < \frac{5d-2}{2d} < \frac{5d-1}{2d} < \frac{5d}{2d} = \frac{d+1}{2}$.

Hence $d+1 < 3d-1 < 5d-3 < 2d\alpha$,

define $U = \{z: |z| < |\beta|^\alpha\}$, $\forall z \in U$ and $|\beta|$ is small, it follows $\left|z^{2d}-\beta^{d+1}\right| \asymp |\beta|^{d+1}$ and $\left|z^{2d}-\beta^{3d-1}\right| \asymp |\beta|^{3d-1}$,

we have

$$\left|Q_{\beta}(z)\right| \asymp 2|\beta|^{1-d}|z|^{d}-\frac{|z|^{d}}{|\beta|^{2d-2}}$$

$$\asymp 2|\beta|^{1-d}|z|^{d}-|z|^{d}|\beta|^{-2d+2}$$

$$< 2|\beta|^{1-d}|\beta|^{\alpha d}-|\beta|^{\alpha d}|\beta|^{-2d+2}$$

$$= 2|\beta|^{\alpha d-d}-|\beta|^{d\alpha-2d+2}$$

$$< |\beta|^{d\alpha-2d+2} < |\beta|^{\frac{5d-3}{2}-2d+2} = |\beta|^{d+1}.$$

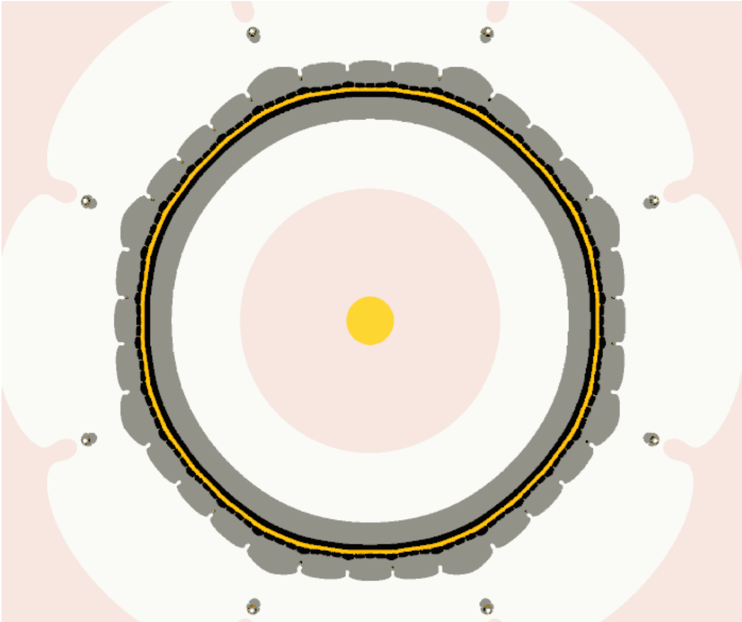
Thus $Q_{\beta}(U) \subset U$ if β is small enough.

Therefore U is lies in I_0 by definition of U. By using that

$\left|Q_{\beta}(z)\right| \preccurlyeq |\beta|^{\frac{d+1}{2}}$ and $\frac{5d-3}{2d} < \alpha < \frac{d+1}{2}$,

we have $Q_{\beta}(\mathcal{T}_b) \subset U \subset I_0$ and $\mathbb{F}(Q_{\beta})$ is contains $\mathcal{T}_b$ if β is small .

Thus $Q_{\beta}(e_{\beta}) \in I_0$, hence $e_{\beta} \notin I_0$, whenever β is small and $Q_{\beta}(\mathcal{T}_b) \subset I_0$ and $\left|e_{\beta}\right| > b$. Now, assume that Q_{β} has critical point $\acute{e}_{\beta}$ such that if $\left|\acute{e}_{\beta}\right| \asymp |\beta|^{\frac{3d-1}{2d}}$ and by Lemma (2.1.4), where $|\beta|$ is small. Then $Q_{\beta}(\acute{e}_{\beta}) \in I_{\infty}$ and $e_{\beta}\acute{} \notin I_{\infty}$ because $\left|\acute{e}_{\beta}\right| < b$ and $\mathcal{T}_b \subset {Q_{\beta}}^{-1}(I_0)$. Hence there is critical point is not contains in I_{∞} or I_0 but the image of this critical point by Q_{β} contains in I_{∞} or I_0. Therefore from Theorem (2.1.29), $J(Q_{\beta})$ is a Cantor circles . See Figure(4) ■



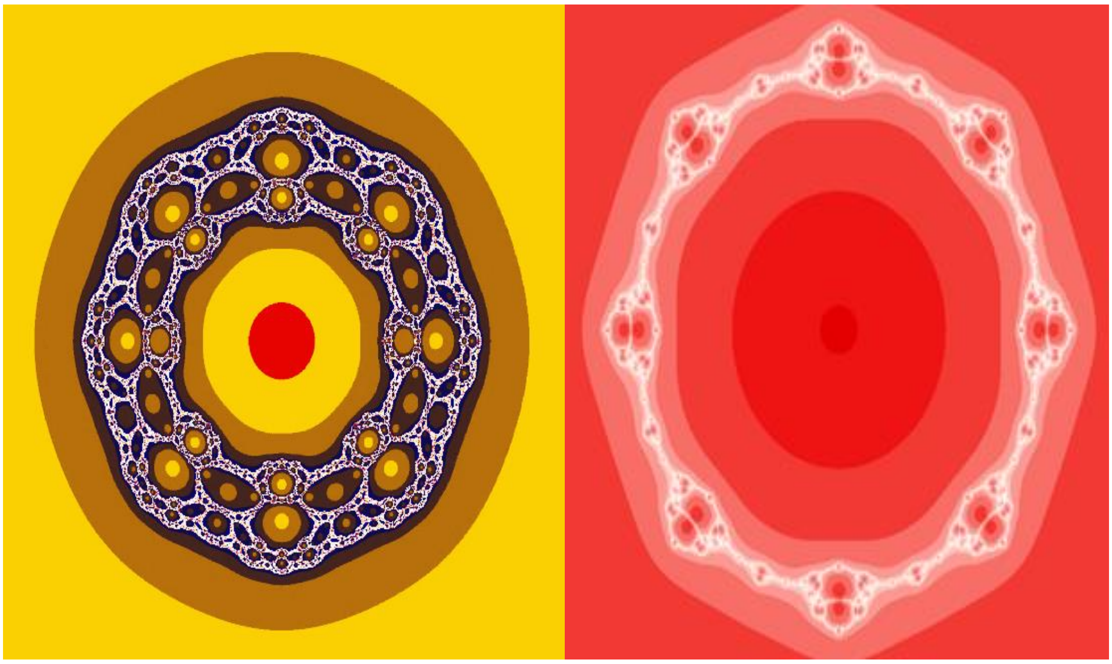


Figure (4) When $d = 4$. Top: $\beta = 0.799 + 0.8i$ and $J(Q_\beta)$ is a Cantor circles; left Bottom: $\beta \approx 1.151442$ and $J(Q_\beta)$ is a Sierpinski carpet; right Bottom: $\beta \approx 1.050$ and $J(Q_\beta)$ is a degenerated carpet.

Proposition (2.1.33)

In any rational map, the Julia set has no critical points if it is a Cantor set of circles.

Proof

Assume that there is $\mathcal{j}$ a Julia component of Q_β and $e_\beta \in \mathcal{j}$ with multiplicity d. Therefore Q_β is not 1-1 map in every small neighborhood of e_β. So $Q_\beta(\mathcal{j})$ containing $Q_\beta(e_\beta)$ from Lemma (1.2.18) . Suppose that V small neighborhood disk of the image of e_β such that $V \cap Q_\beta(\mathcal{j})$ is a simple arc. The preimage of V containing e_β is mapped onto V by $d+1$ to 1 . Now we note that the preimage of $V \cap Q_\beta(\mathcal{j})$ is connected and is in $\mathcal{j}$, thus $\mathcal{j}$ has structure as star, therefore is not simple detour. By assumption that the Julia set is cantor circle and $\mathcal{j}$ is simple closed curve This is impossible. Hence $J(Q_\beta)$ has no critical points. ■

Now, we will study the technique of escaping to the free critical points also to prove $J(Q_\beta)$ is a Sierpinski carpet. Also we give the degenerated Sierpinski carpet if the intersection of the boundaries of complementary domains are non-empty.

Proposition (2.1.34)

Assume that e_β be a free critical point lies in q_0^m for $m \geq 2$, where $Q_\beta^{-1}(I_0)\backslash I_0 = q_0^1$ and $Q_\beta^{-1}(I_\infty)\backslash I_\infty = q_\infty^1$. Therefore each Fatou components of Q_β are simply connected and $J(Q_\beta)$ is compact, connected, nowhere dense and locally connected.

Proof

I_∞ and I_0 are simply connected from Proposition (2.1.24). Suppose that $q_0^1 = Q_\beta^{-1}(I_0)\backslash I_0$ of I_0 consists of Fatou components with $2d$- symmetry. Since Q_β maps each one of them onto I_0 is conformal and $e_\beta \in q_0^m$ for $m \geq 2,$ it follows all component of q_0^i is simply connected $1 \leq i \leq m-1\ \forall i$, the number of components in q_0^m is at least $2d$ and by Proposition (2.1.21) these component $2d$- symmetry surround 0. For any V is simply connected component in the $(m-1\ )$ preimages of I_0. Suppose that the critical orbits does not lie in V, thus all components of $Q_\beta^{-1}(V)$ are simply connected. Now the critical value lies in V also there is U component of $Q_\beta^{-1}(V)$ such that cannot simply connected, therefore U has two critical points at least. Hence there is $2d-1$ different Fatou component from the symmetric Fatou components $\omega_0^i V$ where $\omega_0 = e^{\frac{i\pi}{d}},\ 1 \leq i \leq m-1$. Thus $\omega_0^i V$ has two critical points at least. Therefore Q_β has $4d$ free critical points, this is impossible. Thus each components of $Q_\beta^{-1}(V)$ are simply connected and V has critical value. Hence each components in q_0^m are simply connected. Therefore all components of $Q_\beta^{-1}(I_0)$ are simply connected because q_0^m has no critical values. By Corollary (2.1.6), thus each Fatou components of Q_β are simply connected. Notice that $J(Q_\beta)$ is equal to the complement of $(\bigcup_{i\geq 0} Q_\beta^{-i}(I_0 \cup I_\infty)$, since $I_0 \cup I_\infty$ are simply connected, then $J(Q_\beta)$ is connected and by definition of the Julia set is bounded and closed sets, thus $J(Q_\beta)$ is compact set. Since $J(Q_\beta) \neq \mathbb{C}_\infty$ and by Proposition (1.2.1)(viii), thus $\overline{J(Q_\beta)}^{\circ} = \emptyset$ and

$J(Q_\beta)$ is nowhere dense. By Corollary (1.1.35), it follows $J(Q_\beta)$ is locally connected since Q_β is hyperbolic map. ■

Definition (2.1.35) [27]

Let R be a rational map, if $J(R)$ contains neither parabolic periodic points nor recurrent critical points, then R is **semi-hyperbolic rational map.**

Theorem (2.1.36) [27]

Let R be a semi-hyperbolic rational map such that the boundary of each Fatou component is a Jordan curve. Then the boundaries of all the Fatou components of R are quasicircles.

Theorem (2.1.37)

Suppose that $e_\beta \in q_0^m(or\ q_\infty^m)$ for $m \geq 2$. Therefore each Fatou components of Q_β are Jordan disks. However, if $\partial I_0 \cap \partial I_\infty = \emptyset$, then $J(Q_\beta)$ is a Sierpinski carpet. Otherwise $J(Q_\beta)$ is a degenerate Sierpinski carpet.

Proof

From Corollary (2.1.6) and also Proposition (2.1.34), we must to prove the boundary of I_∞ is a simple closed curve. Because the boundary of I_∞ is locally connected and connected, then the complement of $(\mathbb{C}_\infty \backslash \overline{I_\infty})$ has at most countable Jordan disks. For any Ω_0 component of $\mathbb{C}_\infty \backslash \overline{I_\infty}$ contains 0. Therefore $\partial\Omega_0$ is a simple closed curve. We claim that ${Q_\beta}^{-1}(\Omega_0) \subset \Omega_0$.

Suppose that $0 \in I_0 \subset \Omega_0$, to show that ${Q_\beta}^{-1}(0) \subset \Omega_0$. From Lemmas (2.1.4) and (2.1.9), we have $2d$-roots for ${Q_\beta}^{-1}(0)\backslash\{0\}$ have either in Υ_0 (Fatou component) around 0 or contain in $2d$ different components of Q_β. For the previous case if ${Q_\beta}^{-1}(0)$ is not contain in Ω_0, it follows Υ_0 separate I_∞ from $\overline{\Omega_0}$, which is contradict because $\partial\Omega_0 \subset \partial I_\infty$. Now there is case that $2d$ Fatou component ought contain in $2d$ different component $U_0, \dots, U_{2d-1}$ of $\mathbb{C}_\infty \backslash (\overline{I_\infty} \cup \overline{\Omega_0})$. However ${Q_\beta}^{-1}(\infty)\backslash\{\infty\} \subset \bigcup_{i=0}^{2d-1} \eta(U_i) \subset \Omega_0$, $(\eta(z) = \frac{\beta^2}{z})$.

Hence $Q_\beta(U_i) = \Omega_0 \quad \forall i = 0, \dots, 2d-1$.

Therefore $Q_\beta\left(\bigcup_{i=0}^{2d-1} \partial\, U_i\right) = \partial\Omega_0 \subset \partial I_0$, thus $\partial\Omega_0 \subset \partial I_0$ has $2d$ –preimages on the boundary of I_0, because $Q_\beta\colon \partial I_\infty \to \partial I_\infty$ has degree d. This is impossible.

Hence ${Q_\beta}^{-1}(\Omega_0) \subset \Omega_0$ and $\Omega_0 = \mathbb{C}_\infty \backslash \overline{I_\infty}$. Suppose that $z \in \partial\Omega_0$, since ${Q_\beta}^{-1}(\overline{\Omega_0}) \subset \overline{\Omega_0}$ and we have $\partial\Omega_0 \subset \partial I_\infty$.

It follows $\partial I_\infty \subset J(Q_\beta) = \overline{\bigcup_{m\geq 0} Q_\beta^{-m}(z)} \subset \overline{\Omega_0}$, so $\partial I_\infty \subset \overline{\partial\Omega_0}$ and $\partial I_\infty \subset \partial\Omega_0$.

Thus $\partial I_\infty = \partial\Omega_0$ is simple closed curve and Q_β is hyperbolic.

By Theorem (2.1.36), then ∂I_∞ is quasicircle.

Now we have three cases and we discuss of these cases:

Case one

Let M and N are distinct components of $q_0^i (or\ q_\infty^i)$, $i \geq 1$, such that $\bar{M}$ intersect with $\bar{N}$. Let $z \in \bar{M} \cap \bar{N}$, it follows that ${Q_\beta}^{i-1}(z)$ is a critical point of Q_β because ${Q_\beta}^{i}(M) = {Q_\beta}^{i}(N) = I_0$, also ${Q_\beta}^{i-1}(\bar{M}) \cap {Q_\beta}^{i-1}(\bar{N}) \neq \emptyset$. Which is contradict because that all critical points escape to either the infinity or the origin.

Case two

Let M and N are components of q_0^i and q_0^k $(or\ q_\infty^i$ and $q_\infty^k)$, $0 \leq k < i$ such that $\overline{M}$ intersect $\overline{N}$ is a non-empty. Therefore ${Q_\beta}^{i-1}(\overline{M} \cap \overline{N})$ are critical point of Q_β Which is contradict.

Case three

For any components M and N of q_0^i and q_∞^k, $k \neq 0, i \neq 0$.
Because $\partial I_0 \cap \partial I_\infty = \emptyset$, it follows $\partial M \cap \partial N = \emptyset$.
Therefore $\overline{M} \cap \overline{N} = \emptyset$. By Proposition (2.1.34), $J(Q_\beta)$ is a Sierpinski carpet. Otherwise, if $\partial I_0 \cap \partial I_\infty \neq \emptyset,$ then $J(Q_\beta)$ is a degenerate Sierpinski carpet. ■

Theorem (2.1.38)

Let $Q_\beta : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that

$$Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}\,.$$

For each $d = 4$ *and* $\beta \approx 1.15144239$ such that

$$Q_\beta^2(e_\beta) = 0, \qquad (2.3)$$

where $\mathrm{e}_\beta \approx 1.1592 + 0.4802\mathrm{i}$ is a free critical point of Q_β. Therefore $J(Q_\beta)$ is a Sierpinski carpet.

Proof

From (2.3), it follows that the free critical orbits are escaping to 0 also Q_β is critically-finite. From Proposition (2.1.11),

$\mathcal{T}_\beta = \{z: |z| = \beta\}$ is contained in $J(Q_\beta)$. We have from a direct calculation,

$|e_\beta| \approx 1.254707 > \beta$ and $|Q_\beta(e_\beta)| \approx 3.90962576 > \beta$.

Therefore, $Q_\beta(e_\beta) \in q_0^1$ and $e_\beta \in q_0^2$ because $\mathcal{T}_\beta$ is contained in $J(Q_\beta)$. Now, we prove that $\partial I_0 \cap \partial I_\infty = \emptyset$. Because $\mathcal{T}_\beta$ has no critical values, thus ${Q_\beta}^{-1}(\mathcal{T}_\beta)$ include of finitely many disjoint simple closed curves. From the Argument Principle and since in the

$\mathbb{D}_\beta = \{z: |z| < \beta\}$, there is $d-$ roots and $2d$ poles,

thus $Q_\beta: \mathcal{T}_\beta \to \mathcal{T}_\beta$ has degree d. Therefore ${Q_\beta}^{-1}(\mathcal{T}_\beta) \backslash \mathcal{T}_\beta \neq \emptyset$.

Now, we claim each components of ${Q_\beta}^{-1}(\mathcal{T}_\beta) \backslash \mathcal{T}_\beta$ around 0 .

But if the converse of the claim is satisfy and from by Lemma (2.1.4) and (2.1.5), we have inside of $\mathcal{T}_\beta$ are $2d$ components of ${Q_\beta}^{-1}(\mathcal{T}_\beta)$ and outside of $\mathcal{T}_\beta$ are $2d$ components. Which is contradict with degree of Q_β. Hence the each components of ${Q_\beta}^{-1}(\mathcal{T}_\beta)$ are disjoint and around 0.

Therefore $\partial I_0 \cap \partial I_\infty = \emptyset$ because they are separated by at least there are three disjoint simple closed curves lie in $J(Q_\beta)$. $J(Q_\beta)$ is a Sierpinski carpet from Theorem (2.1.37). See Figure (4). ■

Theorem (2.1.39)

Let $Q_\beta : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that

$$Q_\beta(z) = 2\beta^{1-d} z^d - \frac{z^d(z^{2d} - \beta^{d+1})}{z^{2d} - \beta^{3d-1}} .$$

For any $d = 4$ *and* $\beta \approx 1.050$ such that

$$Q_\beta^2(e_\beta) = \infty, \qquad (2.4)$$

where $\mathrm{e}_\beta \approx -1.8774 - 2.0208\mathrm{i}$ is a free critical point of Q_β. Then $J(Q_\beta)$ is a degenerated Sierpinski carpet.

Proof

From (2.4), thus the critical orbits are escaping also Q_β is critically-finite. By Theorem (2.1.37), to prove $e_\beta \in q_\infty^2$ and the boundary of I_0 intersect with the boundary of I_∞ are a non-empty. Because $Q_\beta^2(e_\beta) = \infty$, it follows that $e_\beta \in q_\infty^2$ if $J(Q_\beta)$ is not cantor circles and not quasicircles from Theorems (2.1.15) and (2.1.29).

To show that $-\beta \in \partial I_0 \cap \partial I_\infty$, for $-\beta$ is a repelling fixed point of Q_β.

Since if d is odd, we have $Q_\beta(-\beta) = -\beta$.

$Q_\beta{}'(-\beta) = \frac{d\beta^{2d-2} - 4d\beta^{d-1} - 3d}{1 + 2\beta^{d-1} + \beta^{2d-2}} \approx -5.696895521$.

Then $|Q_\beta{}'(-\beta)| > 1$ and $-\beta$ is a repelling fixed point.

Our procedure can be analyzed into three steps:

Step one

To find V_0 is a neighborhood of 0 such that $Q_\beta(V_0) \subset V_0$. Therefore $V_0 \subset I_0$.

Step two

To find U_1 and U_2 are two open neighborhoods of $-\beta$ such that

(1) $U_1 \Subset Q_\beta(U_1) \Subset U_2$.

(2) critical values and poles of Q_β not lie in U_2.

(3) the map restriction on U_1 of Q_β is conformal.

Step three

To find $v \in V_0$ and $u_1 \in U_1$ such that I_0 contains the segment $[v, u_1]$.

Now we prove these steps. If $\mathrm{e}_\beta \approx -1.8774 - 2.0208\mathrm{i}$,

then $|\mathrm{e}_\beta| \approx 2.75830$.

For any $V_0 = \{z \in \mathbb{C}_\infty : |z| < 0.4\}$ be the disk centered at zero and radius is 0.4. Suppose that $z \in V_0$,

thus $|z|^{2d} - |\beta|^{3d-1} < -1.709684 < 0$ and therefore $\left|Q_\beta(z)\right| = \left|2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}\right|$

$$< 2|\beta|^{1-d}|z|^d + \frac{|z|^d\left(|z|^{2d}-|\beta|^{d+1}\right)}{|z|^{2d}-|\beta|^{3d-1}}$$

$$< 0.0633 \quad < \; 0.4.$$

Therefore $Q_\beta(\overline{V_0}) \subset V_0$ and $\overline{V_0} \subset I_0$. By using

$CP\ (Q_\beta) = \{\omega_0^m e_\beta, \omega_0^m \frac{\beta^2}{e_\beta} : 0 \le m \le 2d-1\}$, where $\omega_0 = e^{\frac{\pi i}{d}}$.

The set of critical values:

$CV(Q_\beta)$=$\{(\pm(23.6577 + 6.16768i), \pm(-0.02868 - 0.000138i), 0, \infty\}$.

The distance from $-\beta$ to $CV\ (Q_\beta)$ is 1.038267.

By according step two, fix $v_0 = 0.4$, $u_1 = 0.5$, $a = 1.76$ and

$A =$ 2.5. Define $U_1 = \mathbb{D}_a(-\beta) = \{z \in \mathbb{C}: |z+\beta| < a\}$

and $U_2 = \mathbb{D}_A(-\beta) = \{z \in \mathbb{C}: |z+\beta| < A\}$. Thus

$$\max_{z\in[v_0,u_1]} \left|Q_\beta(z)\right| < \max_{y\in[0.4,0.5]} 2|\beta|^{-3}y^4 + \frac{y^d|y^{2d}-\beta^{d+1}|}{|y^{2d}-\beta^{3d-1}|} \approx 0.154578 < 0.4.$$

It follows that $Q_\beta([v_0, u_1]) \subset V_0$ and thus $[v_0, u_1] \subset I_0$.

Since $|u_1 + \beta| \approx 1.55 < a$, therefore $u_1 \in U_1$.

Now to show that $U_1 \Subset Q_\beta(U_1) \Subset U_2$. It means that if $u_1 \in U_1$ and $z \in \overline{\mathbb{D}}_a(-\beta)$, then $\left|Q_\beta(u_1) + \beta\right| < A$.

Also if $z \in \overline{\partial\mathbb{D}}_a(-\beta), so\left|Q_\beta(u_1) + \beta\right| > a$. We take a value of

$u_1 = 0.6$, thus $\max_{z\in\mathbb{D}_a(-\beta)} |Q_\beta(z) + \beta| \approx |0.315938 + 1.05| \approx 1.36 < A$.

Also if $z \in \overline{\partial\mathbb{D}}_a(-\beta)$, we take $u = 0.75 \notin U_1$,

therefore $\min_{z\in\partial\mathbb{D}_a(-\beta)} |Q_\beta(z) + \beta| \approx |0.731 + 1.05| \approx 1.78 > a$.

Hence $U_1 \Subset Q_\beta(U_1) \Subset U_2$, also U_2 has no critical values and poles of Q_β. Because $u_1 \in U_1$ and the segment $[v, u_1]$ lies in I_0,

thus $u_0 = Q_\beta(u_1) \in Q_\beta(U_1) \cap I_0$. $Q_\beta^{-1}: Q_\beta(U_1) \to U_1$ is the inverse of the conformal map $Q_\beta: U_1 \to Q_\beta(U_1)$ is a strict contraction map for the unique fixed point $-\beta$. For each η_0 lies in I_0 is a smooth curve linking u_1 and u_0. Let $m \geq 1$, such that u_m is the $m-th$ preimage of u_0 for Q_β, also η_m is the $m-th$ preimage of η_0 for Q_β linking u_{m+1} and u_m.

Therefore $\eta_m \subset I_0$ $\forall m \geq 0$. Thus $\bigcup_{m\geq 0} \eta_m \subset I_0$. Because

$\lim_{m\to\infty} u_m = -\beta$, then $-\beta$ lies in ∂I_0. By Lemma (2.1.5),

therefore $-\beta$ lies in ∂I_∞. Hence $-\beta \in \partial I_\infty \cap \partial I_0 \neq \emptyset$.

Because $\partial I_\infty \cap \partial I_0 \neq \emptyset$, then $J(Q_\beta)$ is not cantor circles.

Therefore $e_\beta \notin q_\infty^1$, to show that $J(Q_\beta)$ is not quasicircle.

Assume that $J(Q_\beta)$ is quasicircle. From (2.4), thus $e_\beta \in I_0$

and $Q_\beta(e_\beta) \in I_0$ but I_0 is Fatou component of superattracting fixed point $\infty \neq e_\beta$. From Theorem (1.1.38), we have ${Q_\beta}^d(e_\beta)$ in I_0 is infinite. This is impossible with Q_β is critically-finite. It follows $J(Q_\beta)$ is not quasicircle. Since $e_\beta \in q_\infty^2$ and use Theorem (2.1.36), therefore $J(Q_\beta)$ is degenerate Sierpinski carpet . ■

Remark (2.1.40)

From the comparison between our family and McMullen maps, if the free critical orbits are attracted by the cycle to the origin or to infinity, then the Julia set is neither to be a degenerate Sierpinski carpet nor a quasicircle, see Figure (1) and (4).

Now, we give the state of The Escape Quartation Theorem as follows:

Theorem (2.1.41)

Assume that the orbit of one free critical point e_β of Q_β is attracted by 0 (resp. ∞). Then

(1) If $e_\beta \in I_0$ (resp. I_∞), then $J(Q_\beta)$ is a quasicircle.

(2) If $Q_\beta(e_\beta) \in I_0$ (resp. I_∞) but $e_\beta \notin I_0$ (resp. I_∞), then $J(Q_\beta)$ is a Cantor set of circles.

(3) If $Q_\beta{}^m(e_\beta) \in I_0$ (resp. I_∞), for $m \geq 2$ and $Q_\beta{}^i(e_\beta) \notin I_0$(resp. I_∞) for $0 \leq i < m$ and further,

$(3')$ If $\partial I_\infty \cap \partial I_0 = \emptyset$, then $J(Q_\beta)$ is a Sierpiński carpet.

$(3'')$ If $\partial I_\infty \cap \partial I_0 \neq \emptyset$, then $J(Q_\beta)$ is a degenerated Sierpiński carpet .

2.2 Dynamics in Parameter Plane of Q_β

In this section, we give some the properties of the parameter plane of Q_β.

Definition (2.2.1)

We define The non-escape locus sets by:

$M = \{\beta \in \mathbb{C}\backslash\{0\}: Q_\beta^{m+1}(e_\beta)\ not\ tend\ to\ \infty$ as $m \to \infty\}$.

Now, we give the Proposition is to study the symmetry of the parameter plane.

Proposition (2.2.2)

The non-escape locus sets M satisfies:

1. M is symmetric about the real axis.
2. $\upsilon M = M$ with $\upsilon^{2d-1} = 1$.

Proof

1. If $z = x + iy$, then $\bar{z} = x - iy$. Suppose that $Q_\beta(z) = 2\beta^{1-d}(x+iy)^d - \frac{(x+iy)^d((x+iy)^{2d}-\beta^{d+1})}{(x+iy)^{2d}-\beta^{3d-1}}$, thus

$Q_\beta(\bar{z}) = 2\beta^{1-d}(x-iy)^d - \frac{(x-iy)^d((x-iy)^{2d}-\beta^{d+1})}{(x-iy)^{2d}-\beta^{3d-1}}$, so

$\overline{Q_\beta(\bar{z})} = 2\bar{\beta}^{1-d}(x+iy)^d - \frac{(x+iy)^d((x+iy)^{2d}-\bar{\beta}^{d+1})}{(x+iy)^{2d}-\bar{\beta}^{3d-1}}$.

Therefore $\overline{Q_\beta(\bar{z})} = Q_{\bar{\beta}}(z)$, so that Q_β and $Q_{\bar{\beta}}$ are conjugate via map $Q_\beta(z) = Q_\beta(\bar{z})$ by symmetry with the real axis. Hence the parameter plane is symmetry under the map $H(\beta) = \bar{\beta}$, they either both remain bounded or both tend to ∞. Hence M is symmetric about the real axis.

2. We have in the parameter plane $(2d-1)$ –fold symmetry for Q_β, to show this. For any $= e^{\frac{2\pi i}{2d-1}}$, so v is a primitive $(2d-1)^{st}$ root of unity. Therefore $Q_{\beta v}(v^d z) = v^d Q_\beta(z)$, for $v^{2d} = 1$, so that $v = 1$, let $h(z) = ze^{\frac{\pi i}{2d-1}}$. For $m \geq 1$,

$$h^{-1} \circ Q_{\omega\beta}^m \circ h(z) = \begin{cases} (-1)\, {Q_\beta}^m(z), & m\ odd, \\ {Q_\beta}^m(z)\,, & m\ even. \end{cases}$$

For any $\beta \in \mathbb{C}$, thus the maps $Q_{\beta v}$ and Q_β are conjugate under the linear map $h_1(z) = v^d z$. Therefore the parameter plane is symmetry under the map $h_2(\beta) = \beta v$, they either both remain bounded or both tend to ∞. Evenly, $\beta \in M$ if and only if $\beta v \in M$. See Figure (5).

Definition (2.2.3)

For each $d \geq 4$, we define $\mathcal{M} = \{\beta \in \mathbb{C}\backslash\{0\} : J(Q_\beta)$ is a Cantor circle.

If $\mathcal{M}$ is an open domain and contains a punctured neighborhood of 0. This domain is called the **McMullen domain**.

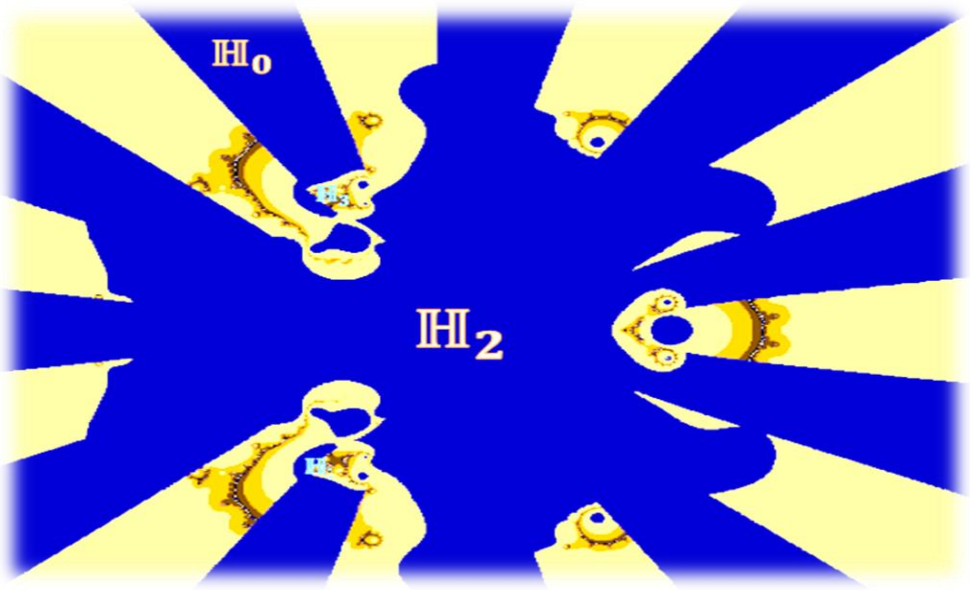
Remark (2.2.4)

Suppose that the set of the critical points such that $CP\ (Q_\beta) = C_\beta \subset I_\infty$. Let $m \geq 0$, $\mathbb{H}_m$ be a parameter set that can be defined as:

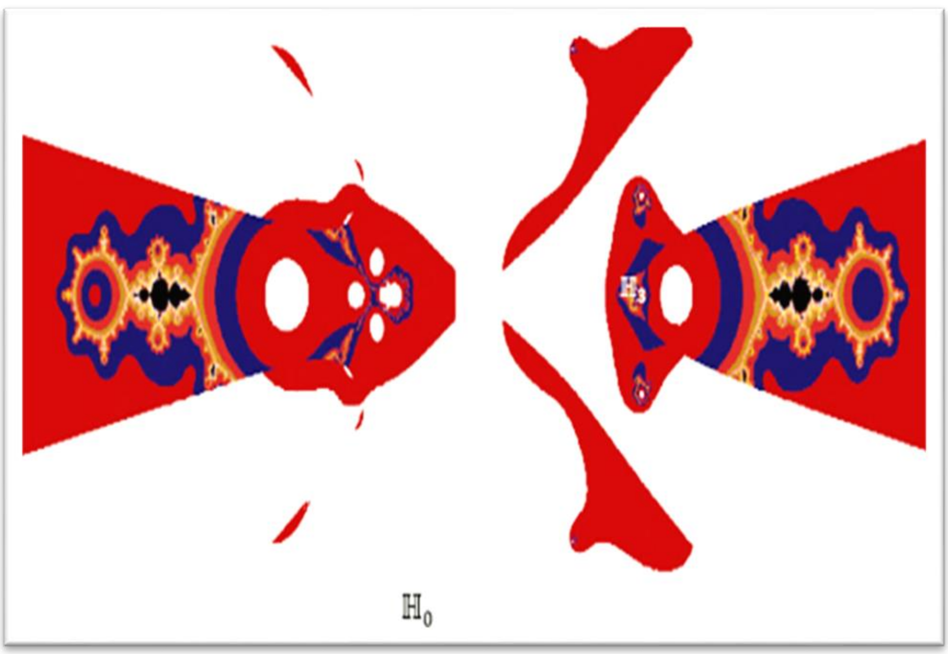
$\mathbb{H}_m = \{\beta \in \mathbb{C}\backslash\{0\}\,; m$ is the first integer such that $Q_\beta^m(C_\beta) \subset I_\infty\}$. A escape domain of level m is a component of $\mathbb{H}_m$.

$\mathbb{H}_0 = \{\beta \in \mathbb{C}\backslash\{0\}\,;\ Q_\beta(e_\beta) \in\ I_\infty\}$

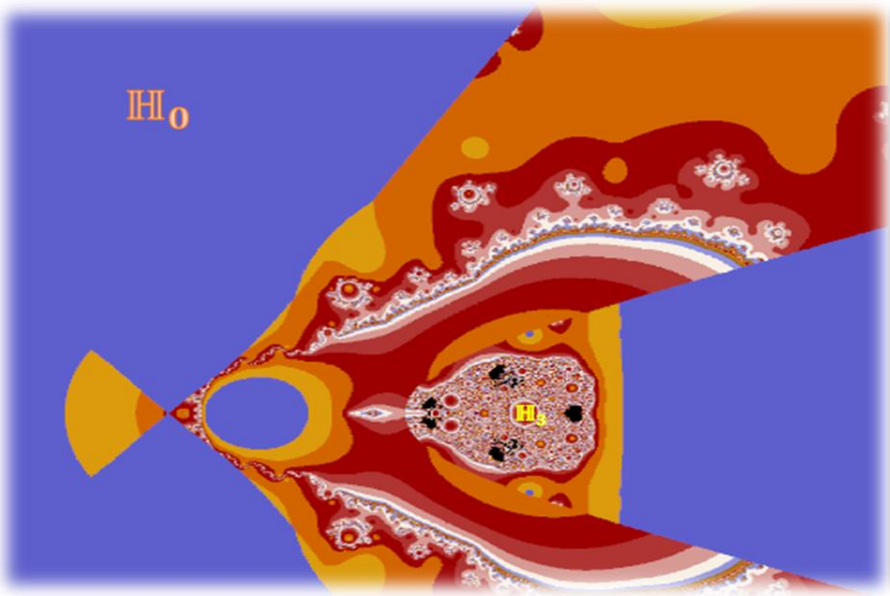
$\mathbb{H}_1 = \emptyset$ and $\mathbb{H}_m = \{\beta \in \mathbb{C}\backslash\{0\};\ Q_\beta^{m-2}(Q_\beta(e_\beta)) \in I_0\}$. See Figure (5). The complement of the escape domains is called the non-escape locus M.



(a)



(b)



(c)

Figure (5) Parameter Plane for the map Q_β when degree (a) $d = 4$, (b) $d = 3$ and (c) $d = 2$. Where shown here are the areas $\mathbb{H}_0$, $\mathbb{H}_2$ and $\mathbb{H}_3$, where $\mathbb{H}_2$ represents the MacMullen domain.

By Figure (5), we define $\mathbb{H}_0$ is the Cantor set locus, $\mathbb{H}_2$ is the McMullen domain, $\mathbb{H}_m$ for $m \geq 3$ this is Sierpinski locus also any of its component is called a Sierpinski hole.

Theorem (2.2.5)

Let $Q_\beta : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that

$$Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}} .$$

The McMullen domain exists in the map Q_β if and only if $d \geq 4$.

Proof

Assume that $J(Q_\beta)$ is a Cantor circles. Therefore I_0 and I_∞ are simply connected and for any Fatou components but except I_∞ and I_0 are annuli which separate ∞ from 0. By Proposition (2.1.9), the

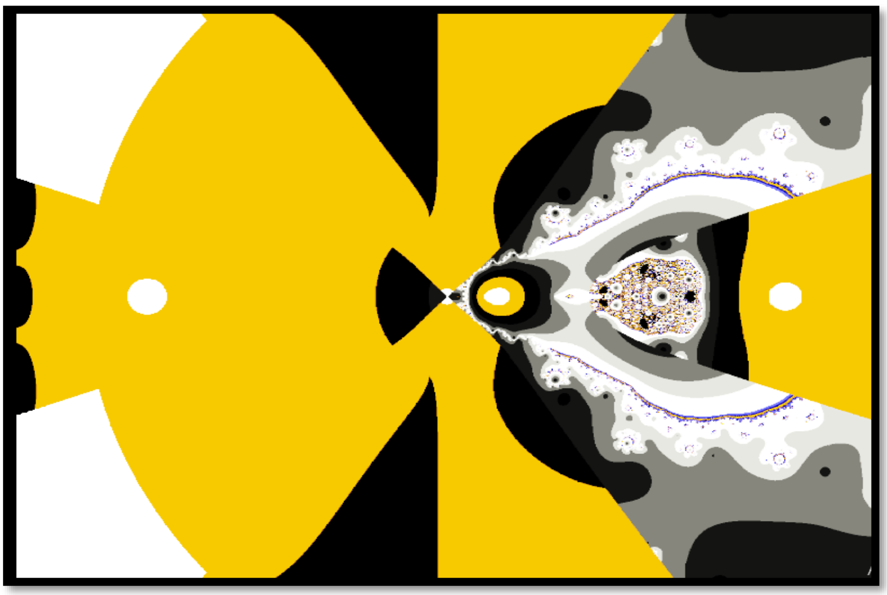
Fatou components consists of two annular such that contain $2d$(critical points). From Riemann–Hurwitz's formula, the first preimage of I_∞ and I_0 contain all free critical points . However, each the free critical points does not lie in I_∞ and I_0 because $J(Q_\beta)$ is Cantor circles. By using Proposition (2.1.21) and from Figure (2), it follows that the conformal moduli of annuli holds

$$mod(V_0) = mod(V_1) = mod(V_2) = mod(V)/d$$

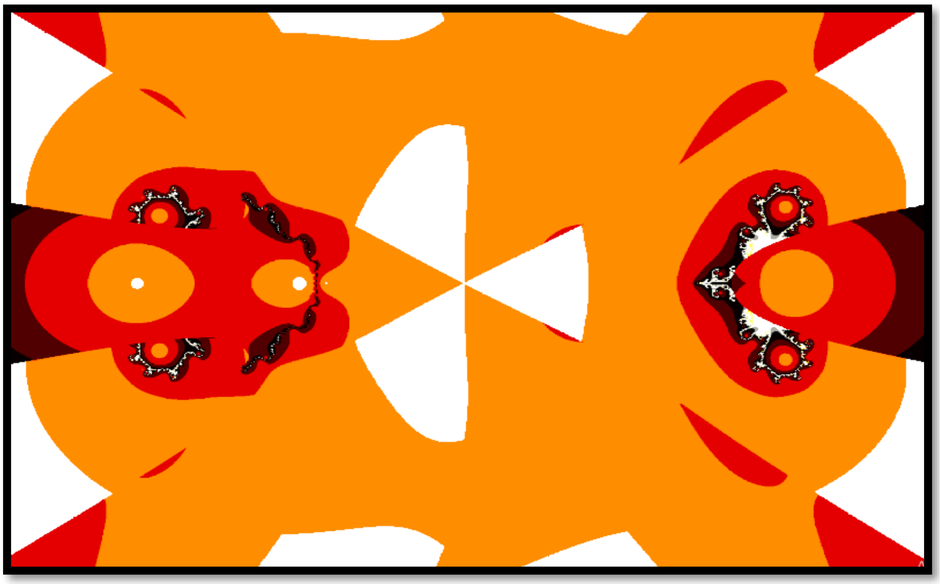
because $Q_\beta: V_m \to V$ for $m = 0,1,2$ is a covering map $d\ to\ 1$. Moreover V essentially contains on V_0, V_1, V_2 also $V \setminus (V_0 \cup V_1 \cup V_2) \neq \emptyset$. By the Grötzsch's modulus inequality, we get

$$mod(V_0) + mod(V_1) + mod(V_2) = \frac{3}{d} mod(V) < mod(V),$$

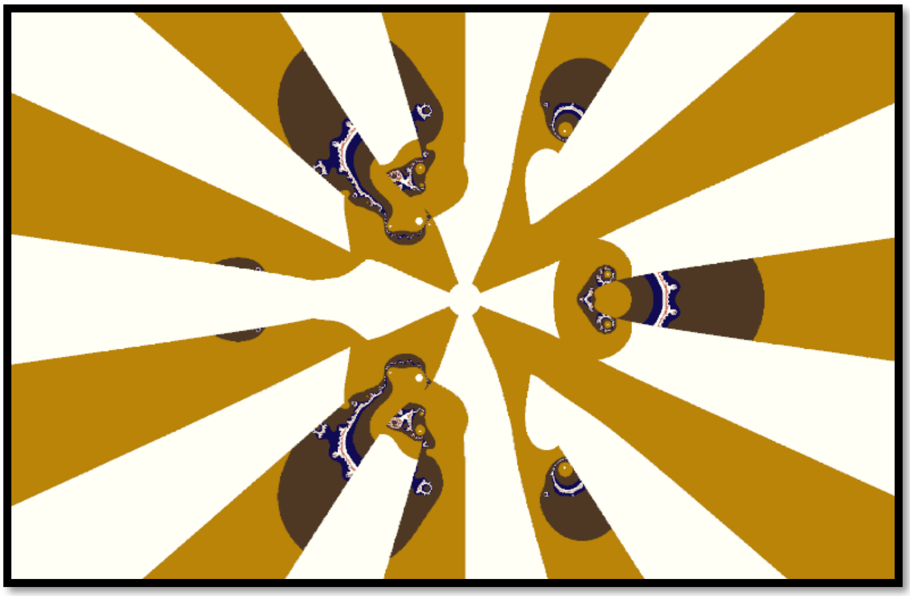
that is $\frac{3}{d} < 1$. We need a $\frac{3}{d}$ of cycles to cover the circle which is equivalent if and only if $d \geq 4$. See Figure (6). ■



(a)



(b)



(c)

Figure (6) : The non-escaping loci of Q_β, where $d = 2, 3$ and 4. (a) and (b) If $d \leq 3$, then Q_β has no McMullen domain .(c) If $d \geq 4$, then there is a McMullen domain centered at 0 this is a white disk.

CHAPTER THREE

The Connectivity of Julia sets of Q_β

In this chapter, We talk about the connection of the Julia set and we study the Julia set as a stranger chaotic attractor to give the Julia set of Q_β has Hausdorff dimension.

3.1 Connectivity of the Julia sets of Q_β

In this section, we prove that the orbits of all free critical points are not attracted by the superattracting cycle 0 or ∞. For any $d \geq 2$ and $\beta \in \Lambda \coloneqq \mathbb{C}\backslash\{0\} - \{\beta : \beta^{1-d} = 1 \; or \; \beta^{2d-2} = 1\}$, , then $J(Q_\beta)$ is connected.

Proposition (3.1.1)

Suppose that the orbit of one of the free critical points e_β does not tend to the origin or infinity, then Q_β has no doubly connected attracting Fatou components or parabolic components.

Proof

By the previous section, we have I_∞ and I_0 are simply connected and all $Q_\beta^{-\ell}(I_\infty)$ and $Q_\beta^{-\ell}(I_0)$ are simply connected for any natural number ℓ. Suppose that there is a parabolic basin or periodic attracting basin $\mathbb{V}$ for period q linking a periodic point z_0, where is different from I_∞ and I_0. Thus q is the period of z_0 but $z_0(\neq \infty, 0)$. Suppose that $\mathbb{V}$ has at least a critical point e_β. Notice that each the components of parabolic basin or q- periodic attracting basin have the same connectivity. Therefore the connectivity of $\mathbb{V}$ have the same of ${Q_\beta}^i(\mathbb{V})$ $(1 \leq i \leq q-1)$.

For any a maximal number of the critical points $\mathcal{K} \geq 1$ lie in each component of the orbit of the parabolic basin or periodic attracting basin. We have three cases for this proof:

Case one

Assume that $\mathcal{K} = 1$, $\mathbb{V}$ is simply connected from using the Riemann–Hurwitz formula . There is $U_0 \subset \mathbb{V}$ is a simply connected neighborhood of periodic point z_0 such that ${Q_\beta}^q(U_0) \subset U_0$ from using the local dynamics of the parabolic and attracting periodic points (but with the case of parabolic $U_0 \subset \mathbb{V}$ such that $\partial U_0 \cap \partial \mathbb{V} = \{z_0\}$) . Assume that U_1 of U_0 be the preimage under Q_β. The parabolic basin or q –periodic attracting orbit has this the preimage. Because $\mathcal{K} = 1$, U_1 has at most one critical point of Q_β. If $e_\beta \notin U_1$ of Q_β, then the branched covering from U_1 to U_0 has a degree is one because the Riemann–Hurwitz formula, thus U_1 is simply connected. If U_1 has one critical point, from using the Theorem(1.1.26), the branched covering has degree two from U_1 to U_0. So U_1 is simply connected. Continuing this process, we obtain that $\mathbb{V} = \bigcup_{\mathcal{K}>0} Q_\beta^{-q\mathcal{K}}(U_0)$ and thus $\mathbb{V}$ is simply connected.

Case two

We set $\mathcal{K}=2$, It follows one component $\mathbb{V}$ of the q- periodic attracting basin or parabolic basin has 2 critical points e_0 and e_1 of Q_β. We claim that $e_1=\omega_0^{j_0}\eta(e_0)$ for some $j_0\in\mathbb{Z}$.

On the other hand, by CP (Q_β) = $\{\omega_0^m e_\beta, \omega_0^m \frac{\beta^2}{e_\beta} : 0\leq m\leq 2d-1\}$, we have $e_1=\omega_0^{m_0}(e_0)$ for some $m_0\in\mathbb{Z}$. By Lemma (2.1.4), (2.1.5) and (2.1.9), thus we have $2d$ critical points

$\{\omega_0^m e_\beta\colon\ 0\leq\ m\ \leq 2d-\ 1\}$ are contained in $\mathbb{V}$, this is impossible. Then we claim that $\deg(Q_\beta|\mathbb{V})=3$. Otherwise, if $\deg(Q_\beta|\mathbb{V})$ is greater than 3, for each $z\in Q_\beta(\mathbb{V})$ from using Lemma (2.1.4), (2.1.5) and (2.1.9) warranty that the preimage of $Q_\beta(z)$ has greater than $3d$ (counted with multiplicity). This is impossible since degree of Q_β. Otherwise, for each z_0 be the periodic point in $\mathbb{V}$, there are at least $three$ preimages of $Q_\beta(z_0)$ which contain in $\mathbb{V}$ (with case of the parabolic, we can find 3 preimages of $Q_\beta(z_0)\in Q_\beta(\partial\mathbb{V})$ in $\partial\mathbb{V}$ as a similar method). Each the $4d-$free critical points contain in $\bigcup_{m=0}^{2d-1}\omega_0^m\mathbb{V}$. From Lemma (2.1.4), (2.1.5) and (2.1.9), so the Fatou component $\omega_0^m\eta(\mathbb{V})$ has two free critical points is $\omega_0^m\eta(e_0)$ and $\omega_0^m\eta(e_1)=\omega_0^{m-j_0}\ e_0$, it follows $\omega_0^{j_0}\eta(\mathbb{V})=\mathbb{V}$. Because the points in the parabolic or attracting basin can be only attracted by one parabolic or attracting point, we have $\omega_0^{j_0}\eta(z_0)=z_0$.

For any $z_1\in\mathbb{V}$ be a preimage of $Q_\beta(z_0)$ and $\omega_0^{j_0}\eta(z_1)=z_2$. We claim that $Q_\beta(z_2)=Q_\beta(z_0)$. Then

$$Q_\beta(z_2)=\omega_0^{dj_0}\eta\left(Q_\beta(z_1)\right)=\omega_0^{dj_0}\eta\left(Q_\beta(z_0)\right)$$

$$=\omega_0^{dj_0}\eta\left(Q_\beta\left(\omega_0^{j_0}\eta(z_0)\right)\right)=\omega_0^{dj_0}\eta(\omega_0^{dj_0}\eta(\ Q_\beta(z_0)))$$

$$= \omega_0^{dj_0} \frac{\beta^2}{\omega_0^{dj_0}\eta(\,Q_\beta(z_0))} = \eta^2\left(Q_\beta(z_0)\right) = Q_\beta(z_0).$$

Assume that z_0 not equal to z_1. But claim that z_1 not equal to z_2. With on the other hand, if $z_1 = z_2$, we obtain $z_1 = \mp z_0$. From Lemma (2.1.4), (2.1.5) and (2.1.9) and combining, we preclude $z_1 = -z_0$, because $\mathbb{V}$ around 0 and satisfies $\omega_0^j\mathbb{V} = \mathbb{V}$ for each $j \in \mathbb{Z}$ *.* This is impossible. Therefore z_1 not equal to z_2 and so z_1, z_0 *and* z_2 are each distinct. Thus $\omega_0^{j_0}\eta(\mathbb{V}) = \mathbb{V}$ and Q_β has degree three on $\mathbb{V}$. We assume that $z_1 = z_0$, so $z_0 = z_2$.

Hence z_0 equals to the free critical point with at least three for local degree, so equals to three and Q_β has degree three on $\mathbb{V}$ in every cases. From the Riemann–Hurwitz formula, we can to show the parabolic basin or the periodic attracting and is simply connected. By the same process, we can take $\omega_0^{j_0}\eta(U_0) = U_0$. This warranty that U_1 has 2 critical points e_0 and e_1 of Q_β .

Case three

Assume that $2 < \mathcal{K} \leq 4d$. By using Lemma (2.1.4), (2.1.5) and (2.1.9), then there is $\mathbb{V}$ a component of the q −periodic attracting or parabolic basin such that satisfies $\omega_0^j(\mathbb{V}) = \mathbb{V}$*,* for each $j \in \mathbb{Z}$ also has exactly $2d$ or $4d -$ critical points with the form $\{\omega_0^m e_\beta : 0 \leq m \leq 2d-1\}$ for some e_β.

For any $\Omega \subset \mathbb{V}$ a simple closed curve around 0 such that satisfies $\omega_0^j\Omega = \Omega$*,* for each $\in \mathbb{Z}$. From Lemma (2.1.4), (2.1.5) and (2.1.9) warranty that ${Q_\beta}^{dq}(\Omega)$ always around 0 for each positive integer d. But $\mathbb{V}$ is disjoint with I_∞ and I_0, which is contradict with

${Q_\beta}^q(\Omega) \to z_0$ is converge uniformly. Hence each the components of the q −periodic attracting or parabolic basin have the same connectivity. ■

Lemma (3.1.2) [13]

If V is a Fatou (or Julia) component of h_λ. Then either V does not around 0 and there are $2d$ such Fatou (or Julia) components $\omega_0^m V$, where $1 \leq m \leq 2d$ such that $\omega_0^m V \cap \omega_0^i V = \emptyset$,
for each $m \neq i \ (mod\ 2d)$, or V component of Q_β around 0 and satisfies $\omega_0^m V = V$ for any $m \in \mathbb{Z}$.

Remark (3.1.3) [13]

Let $A \subset \mathbb{C}$ be an annulus. The core curve of A is defined as $\psi^{-1}(\sqrt{r})$, where
$\psi : A \to A_r := \{z \in \mathbb{C} : 0 < r < |z| < 1\}$ is a conformal isomorphism. Suppose that A_+ is the outer boundary of A and A_- is the inner boundary of A. Let $z \in \mathbb{C}_\infty$, $\mathbb{O}_R^+ (z_1)$ and $\mathbb{O}_R^+ (z_2)$ are disjoint if $\mathbb{O}_R^+ (z_1) \cap \mathbb{O}_R^+ (z_2) = \emptyset$.

Proposition (3.1.4) [13]

Assume that R be a rational map with $d \geq 1$ has fixed Herman rings $A_0, \dots, A_{d-1}$. Call the $\Upsilon_m \subset A_m$ is the core curve whose union divides $\mathbb{C}_\infty$ into $d+1$ connected components $U_0, U_1, \dots, U_d$, where $0 \leq m \leq d-1$. Therefore R has at least $(d+1)$ disjoint infinite critical orbits $\mathbb{O}_R^+ (e_{\beta_m})$ in $J(R)$ such that $\mathbb{O}_R^+ (e_{\beta_m}) \subseteq U_m \cap J(R)$, where e_{β_m}, $0 \leq m \leq d$ is the critical point of R.

We prove that $J(Q_\beta)$ is connected, we need to exclude the case in which Q_β is Hermann rings.

Definition (3.1.5) [43]

A rational map R has exactly one infinite critical orbit in $J(R)$ is **eventually** if there is a critical point e_β in $J(R)$ such that the forward orbit of e_β is infinite and for each other critical point $e_\beta' \in J(R)$ either $\mathbb{O}_R^+(e_\beta')$ is finite or e_β' has the same grand orbit as e_β, $R^m(e_\beta) = R^i(e_\beta')$ for some $m, i \in \mathbb{N}$.

Proposition (3.1.6)

Let $Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}$ be a map. The map Q_β has no Herman rings.

Proof

By contradiction we suppose that Q_β has a cycle of Herman rings. Assume that Q_β has a q −periodic Herman ring, where $q \geq 1$. For any component V_0 of the cycle of these Herman rings. By semiconjugacy, we assume a new rational map because Q_β has the symmetric properties in Lemma (3.1.3), For any $h(\mathbb{z}) = \mathbb{z}^{2d}$ also define $\mathcal{G}_\beta(\mathbb{z}) = \mathbb{z}^d\left[2\beta^{1-d} - \frac{\mathbb{z}-\beta^{d+1}}{\mathbb{z}-\beta^{3d-1}}\right]^{2d}$.

Therefore h is continuous and onto, thus h is a semiconjugacy between Q_β and $\mathcal{G}_\beta$ such that $h \circ Q_\beta = \mathcal{G}_\beta \circ h$. Hence $\mathcal{G}_\beta$ has also a cycle of Herman rings if Q_β has a cycle of Herman rings. From Lemma (3.1.3) and because Q_β is one to one in component V_0, this component can not around 0. Therefore there are $2d$ –Fatou components $\omega_0^m V_0$ satisfying

$\omega_0^m V \cap \omega_0^i V = \emptyset$, for $1 \le m \le 2d$ also for each $m \ne i \ (mod\ 2d)$.

So $h|V_0$ is one to one also $h(V_0)$ is a periodic Herman ring of $\mathcal{G}_\beta$.

Suppose that q is the period of $h(V_0)$, we remark that q is maybe or not equal to $\acute{q}$ but q is a divisor of $\acute{q}$. Hope to get a discrepancy using Proposition (3.1.4). For any $U_0 = h(V_0)$ and $U_i = {\mathcal{G}_\beta}^i(U_0)$, where $0 \le i \le q-1$. In particular, $\mathcal{G}_\beta(U_{q-1}) = U_0$ also $\{U_0, U_1, \dots, U_q = U_0\}$ is a q –periodic Herman rings of , $\mathcal{G}_\beta$. Set $\eta_0(\mathbb{z}) = \frac{\beta^{4d}}{\mathbb{z}}$ and from Lemma (2.1.5), we get

$$\eta_0 \circ \mathcal{G}_\beta(\mathbb{z}) = \mathcal{G}_\beta \circ \eta_0(\mathbb{z}) \ . \qquad (3.1)$$

Because $\deg(\mathcal{G}_\beta)$ is equal to $3d$ also $\mathcal{G}_\beta$ contains $6d-2$ critical points (counted with multiplicity). The local degrees of the origin and the infinity are both d. The local degree of the zero at β^{d+1} is $2d$ also the local degree of the pole at β^{3d-1} of $\mathcal{G}_\beta$ is $2d$. Therefore, this leaves only two free critical points we can denoted by $\acute{e}_\beta$ and $\eta_0(\acute{e}_\beta) = \frac{\beta^{4d}}{\acute{e}_\beta}$ from (3.1), also the local degrees of $\acute{e}_\beta$ is 2 and the local degrees of $\eta_0(\acute{e}_\beta)$ is two.

Because for $0 \le i \le q-1$, U_i is bounded and does not around 0.

We now consider ${\mathcal{G}_\beta}^q$ and the critical points of ${\mathcal{G}_\beta}^q$ are $\left(\bigcup_{i=0}^{q-1} {\mathcal{G}_\beta}^{-i}(\acute{e}_\beta)\right) \cup \left(\bigcup_{i=0}^{q-1} {\mathcal{G}_\beta}^{-i}\left(\frac{\beta^{4d}}{\acute{e}_\beta}\right)\right)$.

Therefore at most there exist $2q$ disjoint critical orbits of ${\mathcal{G}_\beta}^q$, where we have the following form

$\left\{\mathbb{O}^+_{{\mathcal{G}_\beta}^q}(e_{\beta_0}), \mathbb{O}^+_{{\mathcal{G}_\beta}^q}(\eta_0(e_{\beta_0})), \dots, \mathbb{O}^+_{{\mathcal{G}_\beta}^q}\left(e_{\beta_{q-1}}\right), \mathbb{O}^+_{{\mathcal{G}_\beta}^q}\left(\eta_0(e_{\beta_{q-1}})\right)\right\}$.

Assume that they q –periodic Herman rings have the collection of core curves $\{\Gamma_0, \Gamma_1, \dots, \Gamma_{q-1}\}$ for $\mathcal{G}_\beta$ such that there are $q+1$ connected components $\varpi_0, \varpi_1, \dots, \varpi_q$ separate $\mathbb{C}_\infty$.We have two cases:

Case one

Assume that $\eta_0(U_0) = U_0$. Thus from (3.1), $\eta_0(U_i) = U_i$, for each $0 \leq i \leq q-1$. We have the interior and exterior boundary components of U_i are ι_i and E_i respectively, for $0 \leq i \leq q-1$. First claim that $\eta_0(\mathrm{E}_i) = \mathrm{E}_i$ for $i \in \mathbb{N}$, because U_i is bounded also does not around 0. Hence E_i must have the points in the closure of U_i for the smallest and largest modulus. Note that the map η_0 the point of the closure of U_i for largest modulus to the smallest one. Hence $\eta_0(\mathrm{E}_i) \neq \iota_i$, we conclude $\eta_0(\mathrm{E}_i) = \mathrm{E}_i$ with $i \in \mathbb{N}$, this the claim is true. Therefore component ϖ_i has $\mathbb{O}^+_{\mathcal{G}_\beta{}^q}(e_{\beta_i})$ also $\mathbb{O}^+_{\mathcal{G}_\beta{}^q}(\eta_0(e_{\beta_i}))$. From Proposition (3.1.4), $J(\mathcal{G}_\beta{}^q)$ contain at least $q+1$ disjoint infinite critical orbits $\mathbb{O}^+_{\mathcal{G}_\beta{}^q}(e_{\beta_i})$ such that

$$\mathbb{O}^+_{\mathcal{G}_\beta{}^q}(e_{\beta_i}) \subseteq \varpi_i \cap J(\mathcal{G}_\beta{}^q).$$

Nevertheless for $\mathcal{G}_\beta{}^q$, the critical orbits ($2q$) only belong to q of $q+1$ component of ϖ_i for $i = 0, \dots, q$. This is contradicting.

Case two

Assume that $\eta_0(U_0) \neq U_0$. From (3.1), we conclude $\eta_0(U_i) = U_i$ for each $0 \leq i \leq q-1$. Hence the $2q$ disjoint fixed Herman rings belong in $\mathcal{G}_\beta{}^q$. Otherwise, at most only there are $2q$ disjoint critical orbits. From Proposition (3.1.4), this is impossible. So $\mathcal{G}_\beta$ and Q_β has no Herman rings. ■

Theorem (3.1.7)

Let $d \geq 2$ and $\beta \in \Lambda$, $J(Q_\beta)$ is connected if and only if it is not a Cantor circles.

Proof

Assume that the free critical orbits of Q_β are not attracted by 0 and ∞. From Propositions (3.1.1), there are no infinitely connected Fatou components for Q_β and by Propositions (3.1.6), Q_β has no Herman rings. Hence Q_β have any of the periodic Fatou components are simply connected. Because each of the preimages of the periodic Fatou components has no critical points. Therefore, the preimages are also simply connected. By Theorem (1.2.13), we conclude that $J(Q_\beta)$ is connected. If the free critical orbits of Q_β are attracted by the origin and infinity, then $J(Q_\beta$ is connected if it is not a Cantor set of circles from The Escape Quartation Theorem . ■

3.2 Julia Set is chaotic stranger attractor

In this section, we show that the Julia set has a stranger attractor and is chaotic and is therefore considered a fractal set because the Julia set is an irregular set, it has a Hausdorff dimension. we will study the concept "Hausdorff dimension" and to find the dimension of the Julia set of Q_β . Finally, we get chaotic Julia set.

Definition (3.2.1) [11]

If x, y are points of $\mathbb{R}^d$, the distance between them is $d(x,y) = \left(\sum_{i=1}^{d}|x_i - y_i|^2\right)^{1/2}$. We define the diameter $|U|$ of a non-empty subset U of $\mathbb{R}^d$ as the greatest distance apart of pairs of points in U. Thus $|U| = sup\,\{d(x,y) : x, y \in U\}$.

Definition (3.2.2) [11]

If $\{W_m\}$ is a countable collection of sets of diameter at most ε that cover K, where $K \subset \mathbb{R}^d$, that is $K \subset \bigcup_{m=1}^{\infty} W_m$ for $0 \le |W_m| \le \varepsilon$, for any $t > 0$, for each $\varepsilon > 0$.

$\mathcal{H}_\varepsilon^t(K) = \inf\{\sum_{m=1}^{\infty}|W_m|^t : \{W_m\}$ is a ε –cover of $\}$. If $\mathcal{H}^t(K) = \lim_{\varepsilon \to 0}\mathcal{H}_\varepsilon^t(K)$, then is called t –Hausdorff dimension measure of K.

Remark (3.2.3) [2]

K has a dimension $\mathbb{d}(K)$, if $\mathbb{d}(K) > t$, then $\mathcal{H}^t(K) = +\infty$ but if $\mathbb{d}(K) < t$, then $\mathcal{H}^t(K) = 0$.

Definition (3.2.4) [11]

The Hausdorff dimension is defined as $dim_H(K) = \inf\{t \geq 0 : \mathcal{H}^t(K) = 0\} = \sup\{t : \mathcal{H}^t(K) = \infty\}$.

$$\mathcal{H}^t(K) = \begin{cases} \infty & if \;\; 0 \leq t < dim_H(K) \\ 0 & if \;\; t > dim_H(K) . \end{cases}$$

If $t = dim_H(K)$, then $\mathcal{H}^t(K)$ may be zero or infinite, or may satisfy $0 < \mathcal{H}^t(K) < \infty$.

Example (3.2.5) [11]

For any K be the Cantor dust constructed from the unit square such that any stage of the construction the squares are divided into 16 squares with a quarter of the side length, of which the same pattern of four squares is retained. Then $1 \leq \mathcal{H}^1(K) \leq \sqrt{2}$, therefore $dim_H(K) = 1$.

To find $dim_H(K) = 1$. By the construction for the nth stage such that is M_n consists of 4^n squares of length 4^{-n} and has diameter $4^{-n}\sqrt{2}$. We have the squares of M_n as a ε –cover of K where $\varepsilon = 4^{-n}\sqrt{2}$, thus an estimate $\mathcal{H}^1_\varepsilon(K) \leq 4^n 4^{-n}\sqrt{2}$ for the infimum in definition of $\mathcal{H}^t_\varepsilon(K)$. As $n \to \infty$ so $\varepsilon \to 0$, hence giving $\mathcal{H}^1(K) \leq \sqrt{2}$.

For any $proj$ denote orthogonal projection onto the $x - axis$. So $|proj\, x - proj\, y| \leq |x - y|$ if $x, y \in \mathbb{R}^2$. For the lower estimate, so $proj$ is a Lipschitz map. By the construction of K, $proj\, K$ is the unit interval [0, 1]. By using $\mathcal{H}^t(proj\, K) \leq c^t\, \mathcal{H}^t(K)$, for constant $c > 0$. Therefore

$1 = length\,[0,1] = \mathcal{H}^1([0,1]) = \mathcal{H}^1(proj\, K) \leq \mathcal{H}^1(K)$. ■

Definition (3.2.6) [16]

For any $K \subset \mathbb{R}^d$, where $d = 1,2, or\ 3$. The box dimension of K define by

$$dim_B K = \lim_{\epsilon \to 0} \frac{lnM(\epsilon)}{\ln(1/\epsilon)},$$

where $M(\epsilon)$ the smallest number of d-dimensional boxes of side length ϵ required in order to completely cover K

Example (3.2.7) [11]

For any K be the middle third Cantor set . Then $dim_B K = log\ 2/\ log 3$.

To find $dim_B K = log\ 2/\ log 3$. It's clear cover by the 2^n level$-n$ intervals of M_n of length 3^{-n}

gives that $M(\epsilon) \leq 2^n$, if $3^{-n} \leq \epsilon \leq 3^{-n+1}$. From Definition (3.2.6) $dim_B K = \lim_{\epsilon \to 0} \frac{lnM(\epsilon)}{\ln(1/\epsilon)} \leq \lim_{n \to \infty} \frac{ln 2^n}{ln 3^{n-1}} = log\ 2/\ log 3$. On the other hand, each interval of length ϵ with $3^{-n-1} \leq \epsilon \leq 3^{-n}$, at most intersects the one of the level $-n$ intervals of length 3^{-n} used in the construction of K. There are 2^n such intervals, so at least 2^n intervals of length ϵ are required to cover K. Therefore $M(\epsilon) \geq 2^n$ go to $dim_B K \geq log\ 2/\ log 3$. Hence, at least for the Cantor set, $dim_H(K) = dim_B K$. ■

Theorem (3.2.8) [2]

For any rational map R with degree d, where $d \geq 2$. If $\infty \in \mathbb{F}(R)$ and $\mathcal{K}_0 = \max\{ |R'(z)| : z \in J(R)\} > 1$, then $dim_H(J_R) \geq \frac{\log d}{\log \mathcal{K}_0}$, also the lower bound is betterest.

Remark (3.2.9) [2]

The lower bound is attained if $R(z) = z^d$. Therefore $dim_H(J_R) \leq 2$, so $\mathcal{K}_0 \geq \sqrt{d} \geq \sqrt{2}$.

Example (3.2.10) [23]

$B_0(z) = z$, $B_\infty(z) = \frac{1}{z}$, whenever $a \neq \infty$, ingeneral

$$B_a(z) = \frac{1-\bar{a}}{1-a} . \frac{z-a}{1-\bar{a}z}.$$

If $|a| < 1$, then B_a preserves orientation on the circle and maps the unit disk into itself. But if $|a| > 1$, then B_a reverses orientation on $\partial\mathbb{D} = S^1$ and maps $\mathbb{D}$ to its complement.

Proposition (3.2.11) [23]

Suppose that $F : S^1 \to S^1$ is a rational map with degree d if and only if can be written as a Blaschke product as

$$F(\text{z}) = e^{2\pi it} B_{a_1}(z) \ldots B_{a_d}(z)$$

for some constants $e^{2\pi it} \in \partial\mathbb{D}$ and $a_1, \ldots, a_d \in \mathbb{C}_\infty \backslash \partial\mathbb{D}$.

Remark (3.2.12) [28]

The hyperbolic Hausdorff dimension of R –invariant U is the supremum of the Hausdorff dimensions of $R-$ invariant subsets V of U such that $|(R^n|V)'| > 1,$ for an integer n.

Theorem (3.2.13) [28]

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map with degree $d \geq 2$ and Ω be the simply connected immediate basin of attraction to a periodic attracting orbit. Then, provided R is not a finite Blaschke product in some holomorphic coordinates, or a quotient of a Blaschke product by a rational map of degree two, the hyperbolic Hausdorff dimension of the boundary of Ω is larger than 1.

Theorem (3.2.14) [37]

The Hausdorff dimensions of $J(R)$ is strictly greater than one and strictly less than two.

Remark (3.2.15)

In Theorem (3.2.14) Sullivan proved that hyperbolic Julia sets have zero area. Thus, from the proof of Theorem (3.2.14), can obtain that the Hausdorff dimension is strictly less than two directly.

Remark (3.2.16) [17]

If X is a λ –Ahlfors regular metric space and $ARconfdim(X)$ is the infimum of the Hausdorff dimensions of all metric spaces quasisymmetrically equivalent to X, then

$ARconfdim\,(X) = 1 + \lambda(\mathfrak{D})$, where $\lambda = \lambda(\mathfrak{D})$ is the unique real number , for any $\mathcal{D} := (d_0, \ldots, d_{m-1})$ be a sequence of positive integers satisfying $\sum_{m=0}^{i-1} \frac{1}{d_m^{\lambda}} < 1$, an even integer $i \geq 2$.

Corollary (3.2.17) [17]

Assume that $R: \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rational map and N be a closed annulus for R also there is a sub annulus pairwise disjoint $N_0, N_1, \ldots, N_{i-1}$, i even, contained in the interior of N such that (with respect to a linear ordering induced by N) $N_0 < N_1 < \cdots < N_{i-1}$. Let A_0, respectively A_1, be the disk bounded by the least, respectively greatest, boundary component of A. Further, Assume that for each

$m = 0, \dots, i-1$, $R|N_m \to N$ is a proper covering map of degree d_m, with R map the greatest component of N_m and the least component of N_{m+1} to the boundary of A_1 if m is even, and to the boundary of A_0, if m is odd. Set $\mathfrak{D} = (d_0, d_1, \dots, d_{i-1})$. For any $\tilde{g} = R\big|\bigcup_{m=0}^{i-1} N_m$ and put $\mathbb{Y} = \bigcap_{n\geq 0} \tilde{g}^{-n}(N)$. Then $\mathbb{Y} \subset J_R, \tilde{g}(\mathbb{Y}) = \mathbb{Y} = \tilde{g}^{-1}(\mathbb{Y})$ and there is a quasisymmetric homeomorphism $\hbar : \mathbb{Y} \to X$ conjugating $\tilde{g}|\mathbb{Y} : \mathbb{Y} \to \mathbb{Y}$ to $\tilde{f}|X : X \to X$ for $\tilde{f}$ is the family of annulus maps defined by the data $\mathfrak{D}$. So $ARconfdim(J_R) \geq 1 + \lambda(\mathfrak{D})$, with equality if $\mathbb{Y} = J_R$.

Theorem (3.2.18) [38]

Assume that $R: \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a hyperbolic rational map, then $dim_B(J(R) = diam_H J(R) < 2$.

Definition (3.2.19) [1]

The **$\boldsymbol{\omega}$-limit set** of x lies in a metric space X is define as the set of cluster points of the forward orbit $\{f^n(x)\}_{n\in\mathbb{N}}$ of the iterated map. That is $y \in \omega(x)$ if and only if there is a strictly increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that $f^{n_k}(x) \longrightarrow y \; as \; k \longrightarrow \infty$.

For example, for each fixed point x of a map, $lim_\omega \, x = x$.

Definition (3.2.20) [38]

A point x **recurrent** if $x \in \omega(x)$. Otherwise x is called **non-recurrent**. We call any rational map from $\mathbb{C}_\infty$ to itself is a non-recurrent **($\boldsymbol{NCP}$)** map if all critical points of this map contained in the Julia set of this map are non-recurrent.

Theorem (3.2.21) [38]

If $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is an NCP rational map, then either $J(R) = \mathbb{C}_\infty$ or $diam_H J(R) < 2$.

Theorem (3.2.22)

Assume that the orbit of $Q_\beta(e_\beta)$ is attracted by the infinity or the origin. Then the Hausdorff dimension of $J(Q_\beta)$ satisfies $1 < dim_H J\left(Q_\beta\right) < 2$.

Proof

Before prove the theorem, we have three cases of $dim_H J\left(Q_\beta\right) < 2$.

Case one

If the free critical points are attracted by the cycle 0 or ∞ and by Corollary (2.1.14), then Q_β is hyperbolic. From Theorem (3.2.14), the Julia set of Q_β has Hausdorff dimension strictly less than two.

Case two

Since the free critical points are attracted by the cycle 0 or ∞ and by Corollary (2.1.14), then Q_β is hyperbolic. From Theorem (3.2.18) , we have $dim_H J\left(Q_\beta\right) < 2$.

Case three

We assume that the critical point is not belongs to limit set and by Definition (3.2.20), thus Q_β is NCP. From Theorem (3.2.21), we have $dim_H J\left(Q_\beta\right) < 2$. Because Q_β contains $2-$ Fatou components

I_∞ and I_0 which are Jordan domains from Proposition (2.1.15) , it follows $dim_H J(Q_\beta) \geq 1$. Because Q_β^2 has two fixed superattracting basins I_∞, I_0, also Q_β^2 is neither conjugate to a Blaschke product nor a quotient of a Blaschke product and by Theorem (3.2.14), therefore $dim_H \partial I_0 > 1$.

Notice that $\partial I_0 \subset J(Q_\beta^2) = J(Q_\beta)$ by properties of Julia sets . Hence $1 < dim_H J(Q_\beta) < 2$. ■

Remark (3.2.23)

We compute the value $\lambda = \lambda(\mathcal{D})$ is $\frac{log3}{logd} = t$, we use

$No. of\ processors$	$M(X,r)$	$1/r$
1	3	$\frac{1}{d}$
2	3^2	$\frac{1}{d^2}$
.	.	.
.	.	.
.	.	.
k	3^k	$\frac{1}{d^k}$.

We compute $t = dim_B(J(Q_\beta))$ by Definition (3.2.6), we get

$$dim_B(J(Q_\beta)) = \lim_{r\to 0} \frac{logM(X,r)}{\log(1/r)} = \lim_{k\to\infty} \frac{log3^k}{logd^k} = \lim_{k\to\infty} \frac{klog3}{klogd} = \frac{log3}{logd} \text{ . ■}$$

After this Remark (3.2.23), in the following theorem, we will give the relationship between the box dimension and the Hausdorff dimension.

Theorem (3.2.24)

Assume that $J(Q_\beta)$ is a Cantor circles. Then the Hausdorff dimension of $J(Q_\beta)$ satisfies $1+\frac{log3}{logd} \leq dim_H J(Q_\beta) < 2$.

Proof

Suppose that the Julia set of Q_β is a Cantor circles, by Proposition (2.1.21). For each closed set $V \coloneqq \mathbb{C}_\infty \backslash (I_\infty \cup I_0)$ amidst I_0 and I_∞ divided into closed sets V_0, V_1, V_2 between I_∞ and ρ_0, ρ_0 and ρ_∞, ρ_∞ and I_0 (see Figure (2)). Each $Q_\beta : V_m \to V$ is a covering map with degree d, where $0 \leq m \leq 2$, so that the boundaries of I_0 and I_∞ are simple closed curves . Note that from Figure (2), the Julia set of Q_β belongs to $V_0 \cup V_1 \cup V_2 \subset V$. By Corollary (3.2.17) and Remark (3.2.23), Therefore

$$dim_H J(Q_\beta) = dim_H \bigcap_{i=0}^{\infty} Q_\beta^{-i}(V) \geq 1 + \frac{log3}{logd} .$$

So from Theorem (3.2.18), thus $dim_H J(Q_\beta) < 2$ because Q_β is hyperbolic. Hence $1+\frac{log3}{logd} \leq dim_H J(Q_\beta) < 2$.■

Definition (3.2.25) [6]

Let X be a metric space. Suppose that $f : X \to X$ is a map. Then f has sensitive dependence on initial conditions if there exists $\delta > 0$ such that for any $x \in X$ and for any neighborhood U of x, there exists $y \in U$ and $n \geq 1$ such that $d(f^n(x), f^n(y)) > \delta$.

Example (3.2.26) [6]

Let $f : S^1 \to S^1$ be a map given by $(\vartheta) = 2\vartheta$. Hence f is sensitive to initial conditions .

Definition (3.2.27) [6]

Let $f : X \to X$ be a map , f is transitive if for any M and N be two open nonempty subsets of X, there exists an integer $n \geq 1$ such that $f^n(M) \cap N \neq \emptyset$. For example $f(\vartheta) = 2\vartheta$ is transitive.

Now, we give the Devaney`s definition of chaos.

Definition (3.2.28) [6]

Let $f : X \to X$ be a map . f chaotic on X if

(a) the periodic points for f dense in X .

(b) f is transitive .

(c) f has sensitive dependence on initial conditions .

Theorem (3.2.29)

For any rational map $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ with degree $d \geq 2,$ then the map R is chaotic on its Julia set $J(R)$.

Proof

It suffices only to prove that the set of periodic points of the map R is dense as well as that the map R is transitive and thus the map is chaotic. By Theorem (1.1.23) and by Theorem (1.2.24),

the periodic points are dense in $J(R)$. Now, to prove transitivity of R on $J(R)$.

Let M and N be two nonempty subsets of $\mathbb{C}_\infty$ that intersect $J(R)$.

Let $z_0 \in N \cap J(R)$. By Theorem (1.2.7) (ii), then

$J(R) = \overline{\bigcup_{d=1}^{\infty} R^{-d}(z_0)}$. Therefore, $R^{-n}(z_0) \cap M \neq \emptyset$, for some

$n \geq 1$. Hence, if $z \in R^{-n}(z_0) \cap M$, then $R^{-n}(z) = z_0$ and therefore,

$R^n(M) \cap N = \emptyset$. Therefore R is chaotic on $J(R)$. ■

From this Theorem, we can generalize for Q_β and by using The Escape Quartation Theorem (2.1.41) case (2) and (3), where the free critical points are periodic points and thus these points are dense in $J(Q_\beta)$, then Q_β is a chaotic map on the Julia set.

Chapter Four

Conclusions and Future Works

Conclusions

Suppose that $Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}$, $d \geq 2$ and $\beta \in \Lambda = \{\, \beta \in \mathbb{C}\backslash\{0\} - (\beta^{2d-2}\; and\; \beta^{1-d} = 1)\}$.

We get the results the following:

1. $J(Q_\beta)$ is a quasicircle
2. $J(Q_\beta)$ is Cantor circle
3. $J(Q_\beta)$ is a Sierpinski carpet
4. $J(Q_\beta)$ is a degenerate Sierpinski carpet
5. The connectivity of the Julia set of Q_β whenever the free critical orbits of Q_β are attracted by 0 and ∞
6. Q_β has no Herman ring and no infinitely connected attracting Fatou components or parabolic Fatou components.
7. We prove that the map Q_β is exclusive to the Julia set as a transitive map and because the Julia set has dense

periodic points, therefore Q_β will be a chaotic map (according to Devaney's definition).

8. We prove the Julia set is a fractal set because it has a Hausdorff dimension whose value is not an integer it is

$$1 + \frac{log3}{logd} \leq dim_H J(Q_\beta) < 2,$$

and therefore the Julia set is a strange chaotic attractor.

Future Works

Suppose that $Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}$, $d \geq 2$ and $\beta \in \Lambda = \{\, \beta \in \mathbb{C}\backslash\{0\} - (\beta^{2d-2}\ and\ \beta^{1-d} = 1)\}$.

We will present the outlines of the future study as follows:

- In the study of the parameter plane, what are the properties of Julia sets with a different parameter so that as follows:

a1. $J(Q_\beta)$ is a Cantor set.

a2. $J(Q_\beta)$ is a Cantor circles.

a3. $J(Q_\beta)$ is a Sierpinski curve.

a4. $J(Q_\beta)$ is a connected set.

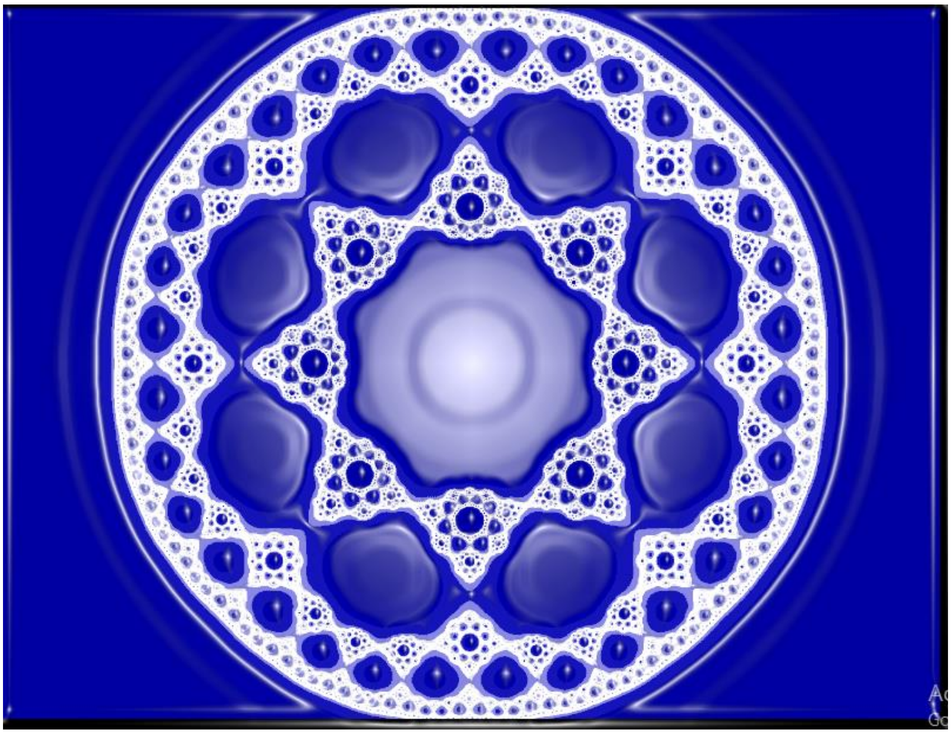
a5. Q_β has no Siegel disk.

a6. β is in escape locus.

- What are the conditions on the parameter plane for any Sierpinski hole in which is an open simply connected set and the parameter is unique.
- Does it depend on the behavior of the critical points of the map Q_β.
- Is there another method to calculate the dimension of the Julia set?

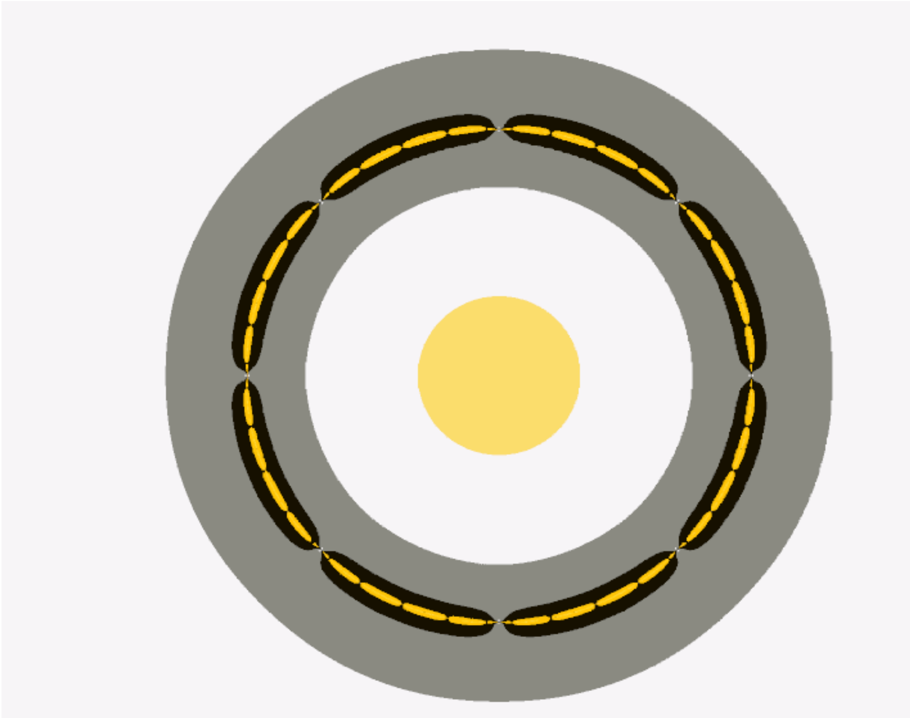
Examples of Julia sets:

By using the program is Ultra fractals 6, to find all the Figures of the Julia sets and the Figures in the parameter plane.



1. $The\ Julia\ set\ is\ homeomorphic\ to\ Sierpinski\ triangle$

$$If\ \ d = 4\ \ , initial\ condition\ (-4{,}3)$$
$$Julia\ seed\ (Re)\ =\ 0\ \ , Center\ (0{,}0)$$
$$Julia\ seed\ (Im)\ =\ -1.2\ \ \ , where\ \ \beta = -1.2\, i\ \ or\ \beta = 1.2$$
$$Bailout\ value\ =\ 1000\ \ .$$



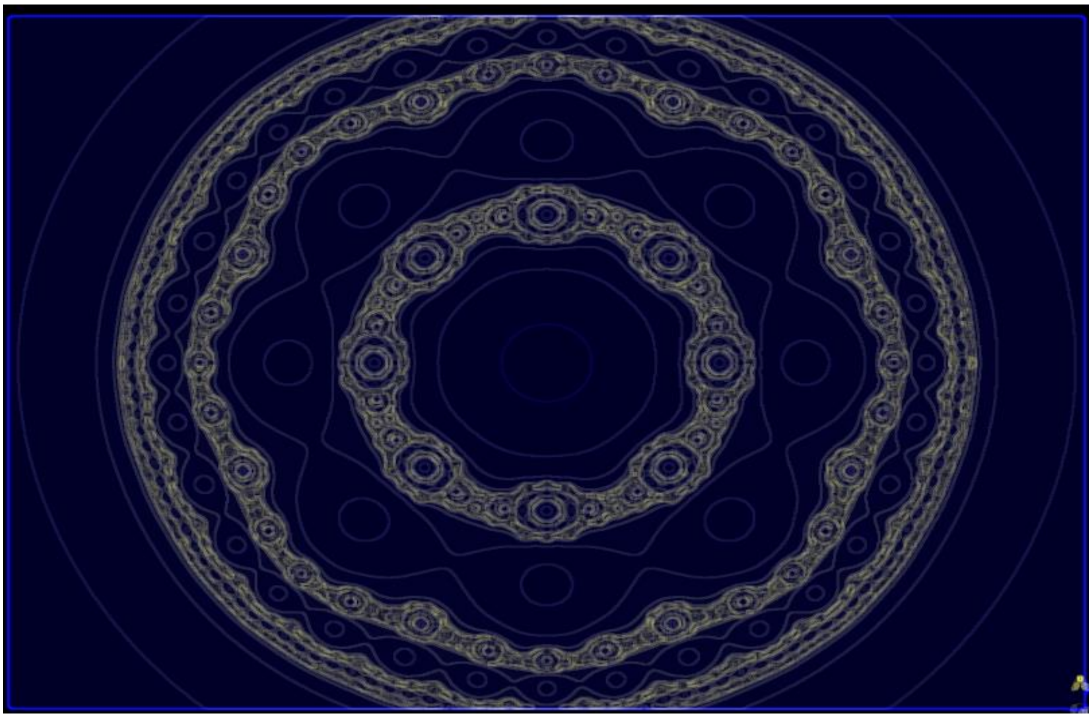
2. $The\ Julia\ set\ is\ homeomorphic\ to\ cantor\ set$

$$If\ \ d = 4\ \ , initial\ condition\ (-4,3)$$

$$Julia\ seed\ (Re)\ =\ 0\ \ , Center\ (0,0)$$

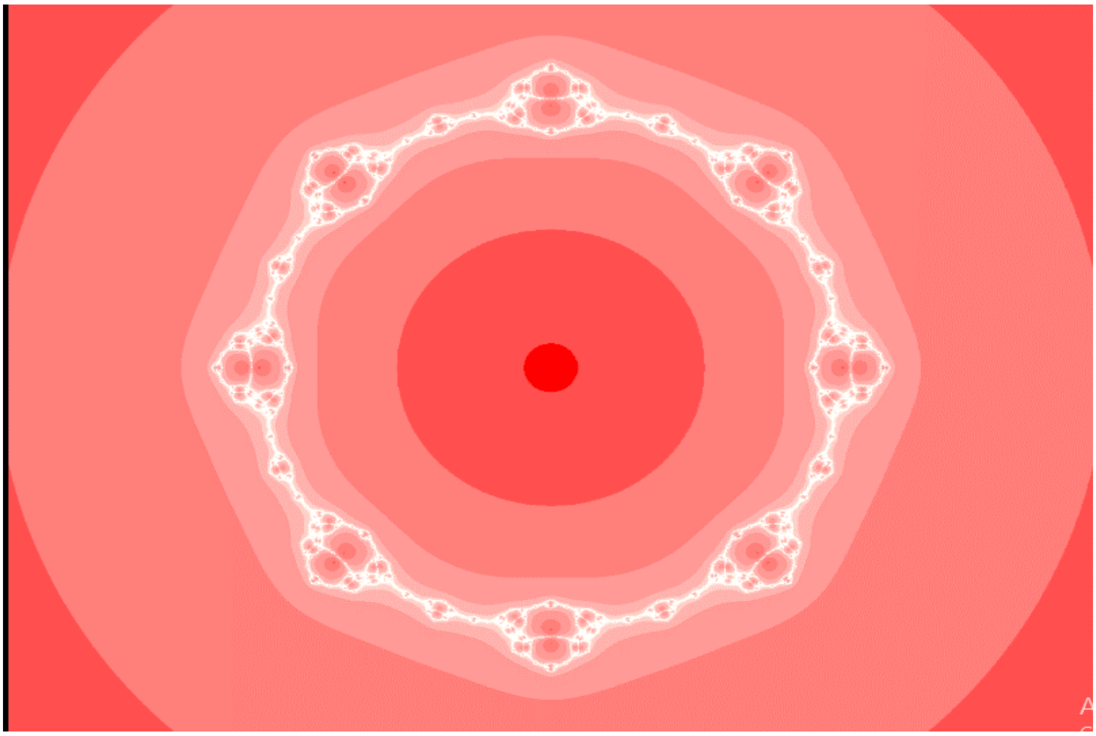
$$Julia\ seed\ (Im)\ =\ -0.99\ \ \ , where\ \ \beta = -0.99\ i$$

$$Bailout\ value\ =\ 10000\ \ .$$

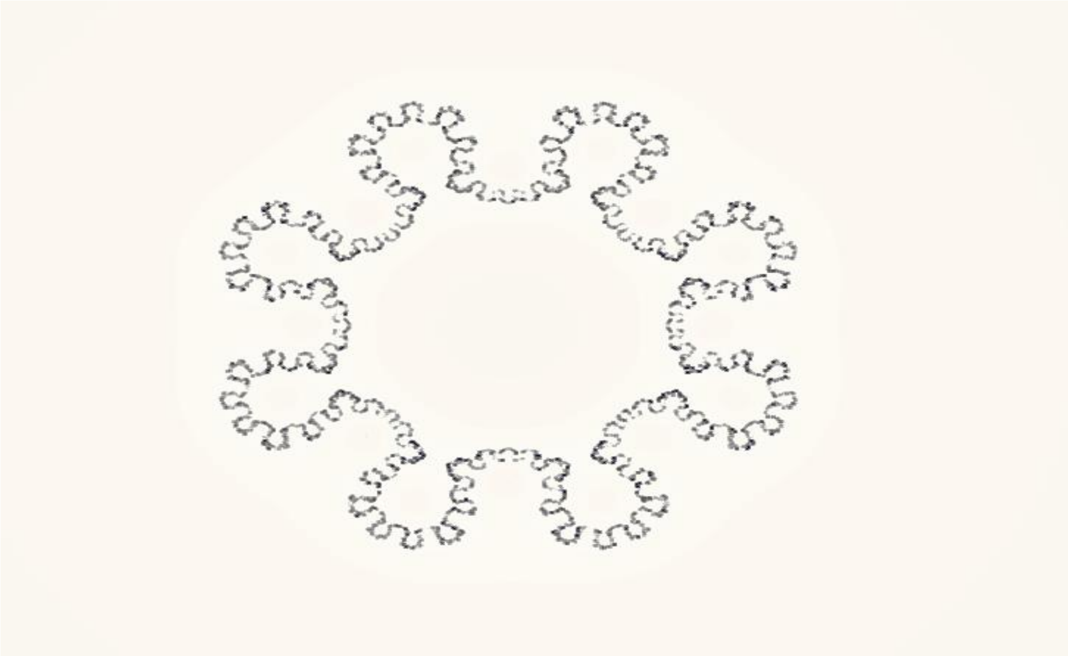


3. $The\ Julia\ set\ is\ homeomorphic\ to\ degenerate\ Sierpinski\ carpet$

$$If\ \ d = 4\ \ , initial\ condition\ (-4{,}3)$$
$$Julia\ seed\ (Re)\ =\ 1.18\ \ , Center\ (0{,}0)$$
$$Julia\ seed\ (Im)\ =\ 0\ \ \ , where\ \ \ \beta = 1.18$$
$$Bailout\ value\ =\ 10000\ .$$



$$4.\ The\ Julia\ set\ is\ generating\ to\ degenerate\ Sierpinski\ carpet$$
$$If\ \ d = 4\ \ , intial\ condition\ (-2{,}1.5)$$
$$Julia\ seed\ (Re)\ =\ 1.0500\ , Center\ (0{,}0)$$
$$Julia\ seed\ (Im)\ =\ 0\ \ , where\ \ \beta = 1.0500$$
$$Bailout\ value\ =\ 10000\ \ .$$



5. $The\ Julia\ set\ is\ generating\ to\ quasicircle$

$$If\ \ d = 4\ \ , initial\ condition\ (-4, 3)$$
$$Julia\ seed\ (Re)\ =\ 0\ \ , Center\ (0,0)$$
$$Julia\ seed\ (Im)\ =\ -1.233871346431\ \ , where$$
$$\beta = -1.233871346431\ i$$
$$Bailout\ value\ =\ 10000\ .$$

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جمهورية العراق
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جامعة بابل
كلية التربية للعلوم الصرفة
قسم الرياضيات

حول الدوال الكسرية للدينمية المعقدة

أطروحة

مقدمة الى مجلس كلية التربية للعلوم الصرفة في جامعة بابل كجزء
من متطلبات نيل درجة الدكتوراه فلسفة في التربية / الرياضيات

من قبل
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١٤٤٤هـ ٢٠٢٢م

الخلاصة

في هذا العمل، ندرس عائلة الدوال النسبية المعقدة ذات الصيغة:

$$Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}} \ ,$$

حيث $d \geq 2$ **, وان** $\beta \in \mathbb{C}\backslash\{0\}$ **، بحيث ان** $\beta^{1-d} \neq 1$ **و** $\beta^{2d-2} \neq 1$.

بيّنا أن $J(Q_\beta)$ **له خاصية من هذه الخصائص مثل شبه دائرة و دائرة كنتورية و مجموعة سربنسكية أو مجموعة سربنسكية المنحلة كلما كانت صورة إحدى النقاط الحرجة الطليقة لـ** Q_β **غير متقاربة إلى 0 أو ∞ .**

كما برهنا:

- **اذا كانت المجموعة الجوليا للدالة** Q_β **متصلة فأن المسارات الحرجة الطليقة تنجذب الى 0 أو ∞ .**
- **اذا كانت** Q_β **لا تمتلك حلقة هيرمانية ولا تمتلك مركبات فاتو جاذبة متصلة لانهائية ولا مركبات فاتو مكافئة .**
- **اثبتنا ان دالة** Q_β **مقصورة على مجموعة جوليا بأنها دالة متعدية ولكونها مجموعة جوليا تمتلك نقاط دورية كثيفة وبالتالي ستكون** Q_β **دالة فوضوية (حسب تعريف دفني).**
- **اثبتنا ان مجموعة جوليا مجموعة كسورية لأنها تمتلك بعد هاوسدورفي غير صحيح قيمته** $2 > dim_H J(Q_\beta) \geq 1 + \frac{log3}{logd}$ **ولذلك تكون مجموعة جوليا جاذبا غريب فوضويا.**