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Study the Chaos in g −Non Autonomous Discrete Systems

A Thesis

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By

Baraa Ali Hussein AL-Aamery

Supervised by

Prof. Dr. Iftichar Mudhar Talb AL-Shara'a

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بسم الله الرحمن الرحيم

﴿ يرفع الله الذين امنوا منكم والذين اتوا العلم درجات والله بما تعملون خبير ﴾

صدق الله العلي العظيم

سوره المجادلة (اية ١١)

Dedicated To

My father , mother and brothers,

Grandpa to my mother,

My teachers,

And my best friends

Acknowledgments

Many thanks and gratitude to God Almighty Who responded to my repeated and constant supplications to accept my postgraduate studies and rise to a valuable scientific rank with his kindness and satisfaction .

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I would like to think members of the department of Mathematics in the college of Education for pure Sciences in the University of Babylon .

All thinks and love to my family , friends and all the people who supported me in my scientific career .

List of Contents

List of Symbols

Symbol	*Explanation*
$\mathbb{N}$	Natural number
$\mathbb{Z}$	Integer number
$\mathbb{R}$	Real number
$\emptyset$	Empty set
$\boldsymbol{d}$	Metric of matric space
$\boldsymbol{Cl}\ (\boldsymbol{B})$	Closure for the set B
$\boldsymbol{\mathcal{B}_r(q)}$	Ball with reduced r and center q
$\boldsymbol{Y}$	Topological non autonomous space
$\boldsymbol{S^1}$	Unit circle
$\boldsymbol{g}$	Generalize for the non autonomous discrete systems
$\boldsymbol{h_n}: \boldsymbol{Y} \longrightarrow \boldsymbol{Y}$	itself maps on Y
$\boldsymbol{\pi}$	Factor map
$\boldsymbol{\beta}(\boldsymbol{y}, \boldsymbol{h}_{\boldsymbol{n},\infty})$	Orbit for the point y
$\boldsymbol{y}^+$	Forward orbit
$\boldsymbol{y}^-$	Backward orbit
$\boldsymbol{Min}\ (\boldsymbol{h}_{\boldsymbol{n},\infty})$	Minimal set of generalize for non autonomous discrete systems
$\boldsymbol{R}(\boldsymbol{h}_{\boldsymbol{n},\infty})$	Recurrent set of generalize for non autonomous discrete systems
$\boldsymbol{E}(\boldsymbol{h}_{\boldsymbol{n},\infty})$	Eventually periodic point set for generalize of non autonomous discrete systems
$\boldsymbol{\Omega}(\boldsymbol{h}_{\boldsymbol{n},\infty})$	Non wandering set of generalize for non autonomous discrete systems
$\boldsymbol{P}(\boldsymbol{h}_{\boldsymbol{n},\infty})$	Periodic point set of generalize for non autonomous discrete systems
Tran $(\boldsymbol{h}_{\boldsymbol{n},\infty})$	Transitive set of generalize for non autonomous discrete systems

Publications

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Abstract

The aim of this work to generalized some chaotic properties for g-Non Autonomous discrete systems such that properties has been studied is transitivity : When $\{Y_i\}_{i=1,\dots,n}$ be family of transitive sets Then $\cup_{i=1}^{n} Y_i$ is transitive set . Also , we study other chaotic properties : Touhey property , strongly blending , dense periodic point , locally eventually onto $((\mathcal{L}.e.o)$, equicontinuous and sensitive dependent on initial, we get : If h_n is strongly blending and has dense periodic point points , then h_n is satisfy Touhey property . If $h_n: Y \longrightarrow Y$ is equicontinuous piecewise monotone map and $h_{n,\infty}$ have dense homoclinic points with $h_{n,\infty}$ are strongly blending then $h_{n,\infty}$ is locally eventually onto. If $h_n: Y \longrightarrow Y$ is locally eventually onto then $h_{n,\infty}$ is sensitive dependent on initial condition , also we studied weakly mixing set by factor map such that if B is weakly mixing subset of Y we obtained $\pi(B)$ is weakly mixing subset of Z when $\pi : (Y.h) \rightarrow (Z, g)$ is factor map and Y , Z are two g- non autonomous discrete systems .

In addition generalized above properties by product multiplication , so we get : If Let $(Y, h_{n,\infty})$ and $(X, g_{n,\infty})$ be two g-non autonomous discrete systems , then the set of periodic point of $g_{n,\infty} \times h_{n,\infty}$ is dense of $X \times Y$ if and only if the sets of $g_{n,\infty}$ and $h_{n,\infty}$ have dense periodic points in X and Y respectively . When $g_n$and $h_n$are maps , then

h_n , g_n are strongly blending if and only if $g_n \times h_n$ is strongly blending . $g_n$and h_n are satisfy Touhey property if and only if $g_n \times h_n$ is satisfy Touhey property . $g_{n,\infty} \times h_{n,\infty}$ is topological transitive then $g_{n,\infty}$ and $h_{n,\infty}$ are both topological transitive and the convers is not true .If g_n hasTouhey property and h_n is chaotic with topological mixing then $g_{n,\infty} \times h_{n,\infty}$ is chaotic (Devaney) .

Moreover generalized minimality and topological mixing by : g_n and h_n are minimal if and only if $g_{n,\infty} \times h_{n,\infty}$ is minimal . $g_{n,\infty} \times h_{n,\infty}$ is sensitive dependent on initial condition If and only if at least $h_{n,\infty}$ or $g_{n,\infty}$ is sensitive dependent on initial condition . If $g_n$and $h_n$are topological mixing ,then also $g_{n,\infty} \times h_{n,\infty}$ is topological mixing .

Introduction

Word "chaos" means " a state of disorder " . And in the dynamical system , Li and york are first to the interdicted of the word "chaos" in 1975 [27] . In 1989 Devaney give definition of chaos stronger than Li-york chaos , chaos of Devaney implies three conditions : transitivity , dense of periodic point and sensitive on initial condition [11] . In 1992 , Banks .J and anther authors [7] showed that the sensitivity on initial condition is not necessary condition to make maps are chaos with implies another two condition (transitivity and density periodic point) . In 1994 the author prove that the transitivity maps of compact intervals implies Devaney chaos [28] . . Devaney chaos implies Li-York chaos a according to Hung and Ye [13] , who established this in 2002 by demonstrating that anon-periodic transitive system with aperiodic point is Li-York chaotic .

In the dynamical system (autonomous discrete systems) , all the definition of the concept , chaotic properties , metrical properties and definition of chaotic were dealing with a single map which implies certain condition , In 1996 Kolyada , S, Snha [16] introduced system called non autonomous discrete systems , and they are denoted by $(Y, h_{1,\infty})$ when Y be compact topological space and $h_n: Y \longrightarrow Y$, $\forall\, n \in \mathbb{N}$, such that $h_{1,\infty}$ is the sequences of the continuous maps with $h_{1,\infty} = (h_n)_{n=1}^{\infty}$ (i.e $h_{1,\infty} = (h_1, h_2, \dots, h_n, \dots))$, so when $h_{1,\infty}$ convergent uniformly to continuous map h (i.e $h_m = h, \forall m \in \mathbb{N}$) , then we will obtained a dynamical system [30] ; Non autonomous discrete systems are suitable for modeling some phenomena in applied sciences ,including biology [10] , [29] , physics[15] and economy [31], they are also useful for solving mathematical puzzles (see[16]) .

In [18] Kolyada.S and Snoh.L discussed minimality of non autonomous discrete systems. Three interpretations of chaos according to (Devaney) , (Wiggins) and a strong sense of Li-York [25], were presented by Shi and Chen in 2009 .

In 2014 [20] , Lei Liu and Yuejuan introduced weakly mixing set and transitive set in case sequences of maps and they give definition of this concept and some result by dealing with sequences of different maps in non autonomous discrete system instead of a single maps in autonomous discrete system .

At our work , we study non autonomous discrete systems in general called general-non autonomous discrete systems or g-non autonomous discrete system , we can write it by $(Y, h_{n,\infty})$

The thesis consists of two chapters : ***in chapter one*** , we take two sections . ***In section one*** , we take some definition of concept and chaotic property in dynamical system (autonomous discrete system) ***. In section two*** , we give some result and examples in this system

In chapter two , we take g-non autonomous discrete systems by three sections : ***in section one*** , we define some definition of concept in space more general than (autonomous and non autonomous discrete systems respectively) .

In section two , we discussed and generalized some result by composite this maps in for any $h_n\colon Y \longrightarrow Y$, $n \in \mathbb{N}$.

In section three , we generalize some chaotic properties in maps and sets by product multiplication property in g-non autonomous discrete systems, and we concluded that some chaotic properties don't generalize by product multiplication property like : transitivity , totally transitive in the direct direction .

Chapter One

The Chaos in Autonomous Discrete system

The Chaos in Autonomous Discrete System

In this chapter , we will introduced definitions of some chaotic properties (transitivity , topological mixing , weakly mixing , (strongly and weakly) blending and locally eventually onto) of the maps , also introduced definitions for (transitive set , weakly mixing set , non wandering set , recurrent set , eventually periodic set and periodic point set) in space called autonomous discrete systems (Discrete Dynamical system) , this properties studied under effect single map $h : Y \longrightarrow Y$, where Y topological space instead of sequences an infinite maps in non autonomous discrete systems and we find relationship between these properties . So for the orbit of any point in the domain is obtained by iterating such map . So in special case , when the non autonomous discrete systems $h_{1,\infty}$ is constant sequences $(h , h , h , \dots .)$, the pair (Y , h) called autonomous discrete systems .

1.1. Preliminaries of the Autonomous Discrete System

In this section , we introduce the concept which we need them in our work

Definition 1.1.1 [12]

Let (Y,h) be topological dynamical system . Then the set of

$$Orb(y) = \left\{y\ , h(y), h\big(h(y)\big), h\Big(h\big(h(y)\big)\Big), \dots\right\}$$

$$= \{y\ , h(y), h^2(y), h^3(y), \dots\}$$

is called ***orbit*** of the point $y \in Y$, and denoted by $\beta\ (y,h)$.

Definition 1.1.2 [2]

Let (Y,h) be topological dynamical system and y $\in Y$. Then the set

$$y^- = \{x\ \in Y\ ;\ h^n(x) = y, for\ some\ n\ \in \mathbb{N}\}$$

is called ***backward orbit*** and denoted by y^-.

Definition 1.1.3 **[2]**

Let(Y,h) be topological dynamical system and y $\in Y$. Then the set

$$y^+ = \{x \in Y\ ;\ h^n(y) = x\ ;\ \ for\ some\ n\ \in \mathbb{N}\ \}$$

is called ***forward orbit*** and denoted by y^+.

Definition 1.1.4 [12]

Let (Y, h) be topological dynamical system . the point $y \in Y$ is *called* ***fixed point*** where

$$\boldsymbol{h(y) = y.}$$

0 , 1 are two fixed points of $h_1(y) = y^2$,where , $h_1: \mathbb{R} \longrightarrow \mathbb{R}$.

On the other hand ,there are maps has no fixed point for example $h_2(y) = y + 1$, where $h_2 : \mathbb{R} \longrightarrow \mathbb{R}$.

Definition 1.1.5 **[2]**

Let (Y, h) be topological dynamical system . Apoint $y \in Y$ is called ***periodic Point*** with period n where there is $n \in \mathbb{N}$:

$$\boldsymbol{h^n(y) = y \text{, and } h^k(y) \neq y}$$

$\forall k < n$, and n called period of point y . Denoted to periodic points set by $P(h)$.

Example 1.1.6 *[12]*

Let T : [0,1] ⟶ [0,1] be the tent map , with

$$T(y) = \begin{cases} 2y & if\ 0 \leq y \leq 0.5 \\ 2 - 2y & if\ 0.5 \leq y \leq 1 \end{cases}$$

Then the point $\frac{2}{7}$ is periodic point of period 3 .

So , the fixed points are periodic point of period 1[2] .

Definition 1.1.7 **[22]**

Let (Y, h) be topological dynamical system and y_1 be periodic point of period k . The point y ∈ Y is called ***eventually periodic point*** if

$$\boldsymbol{h^{k_1}(y) = y_1 \text{ , for some } k_1 \in \mathbb{N} .}$$

and denoted by $E(h)$.

Example 1.1.8

Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ such that $h(y) = y^2 - 1$, so when $y = 1$,we get :

$h(1) = 0$

$h^2(1) = -1$

$h^3(1) = h\Big(h\big(h(1)\big)\Big) = 0$

So , $h(1) = h^3(1)$

Then $y = 1$ is eventually periodic point .

Remark 1.1.9 **[22]**

Let (Y, h) be topological dynamical system , then

$$P(h) \subseteq E(h) \qquad (1.1)$$

Definition 1.1.10 **[12]**

Let (Y, h) be topological dynamical system **.** Apoint $y \in Y$ called repelling fixed point of h if there is an interval $(y - \epsilon, y + \epsilon)$ containing y and $x \in Y$ such that

$$|\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{y}| > |\boldsymbol{x} - \boldsymbol{y}|$$

Definition 1.1.11 **[6]**

Let (Y, h) be topological dynamical system . Apoint $y \in Y$ is said to be ***recurrent point*** if there is $\{n_i\}$ increasing sequences of positive Integer n such that

$$\boldsymbol{lim_{n\to\infty}\, h^{nk}\,(y) = y \;\; where\; n_i} \to \infty\ .$$

So the set of all recurrent point called ***recurrent set*** and denoted by $R(h)$.

Definition 1.1.12 **[6]**

Let (Y, h) be topological dynamical system . Then $y \in Y$ is called ***non wandering point*** if $\exists\, m \in \mathbb{N} \ni$

$$\boldsymbol{h^m(W) \cap W \neq \emptyset}$$

where W is an open set that containing y . So the set of all non-wandering points is called ***non-wandering set*** and denoted by $\Omega\ (h)$.

Note : In [22], Moothathu gave the following relation :

$$\boldsymbol{P}\ (\boldsymbol{h}) \subseteq \boldsymbol{R}\ (\boldsymbol{h})\ \subseteq \boldsymbol{\Omega}\ (\boldsymbol{h}) \qquad (1.2)$$

and by [8], Block.L and Conven.E with another authors showed when the map is continuous of the circle and $P\ (h)$ non-empty closed set , then

$$P\ (h) = \Omega\ (h)$$

In [24] , Oproch and Zhang defined $\mathcal{G}(\ A, B)$,where A and B are subsets Y and $h: Y \longrightarrow Y$ is a map by :

$$\mathcal{G}(\boldsymbol{A}, \boldsymbol{B})\ =\ \{\boldsymbol{n}\ \in \mathbb{N}\ ;\ \boldsymbol{h}^{\boldsymbol{n}}(\boldsymbol{A}) \cap \boldsymbol{B}\ \neq\ \emptyset\ \} \qquad (1.3)$$

Definition 1.1.13 **[24]**

Let (Y, h) be topological dynamical system and the non-empty $B\ \subseteq\ Y$, Then B is said to be ***transitive set*** if for any W and M are open subsets of Y with $B \cap M\ \neq \emptyset$ and $B \cap W\ \neq \emptyset$

$\mathcal{G}(\boldsymbol{B} \cap \boldsymbol{M}\ , \boldsymbol{W})$ ***is an infinite***

and denoted by $Tran(h)$.

Definition 1.1.14 **[24]**

Let (Y, h) be Topological dynamical system , a non-empty closed subset $B\ \subseteq\ Y$ is called ***weakly mixing set*** with at least two element if for any open subsets W_1 ,…, W_n of B and M_1 ,…, M_n non-empty open subset of Y, There is $k > 0$ such that

$$\boldsymbol{h^k(W_i \cap B) \cap M_i \neq \emptyset}$$

for $i = 1, \dots, n$.

Definition 1.1.15 **[4]**

Let (Y, h) be topological dynamical system. Then $h : Y \longrightarrow Y$ is

Said to be ***Topologically transitive*** if for any $\emptyset \neq W, M$ open subset

of Y , $\exists\, n > 0$, such that

$$\boldsymbol{h^n(W) \cap M \neq \emptyset} \textbf{ .}$$

We can take the tent map example for the Topologically transitive [2] .

Definition 1.1.16 **[2]**

Let (Y, h) be topological dynamical system . Then $h : Y \longrightarrow Y$

Is said to be ***totally transitive*** if h^n is transitive , $\forall n \in \mathbb{N}$.

Remark 1.1.17 .

Transitivity property not transmitted by composition , so in case h is

transitive , it is not necessary that h^n is transitive [17] .

And by definition . The totally transitive map is transitive . The opposite, however , is not generally true .

***Example 1.1.18* [5]**

Let h : [0,1] → [0,1] be continuous map defined by :

$$h(y) = \begin{cases} 3y + \frac{2}{5} & 0 \leq y \leq 1/5 \\ -3y + \frac{8}{5} & 1/5 \leq y \leq 2/5 \\ \frac{2}{3}y - \frac{2}{3} & 2/5 \leq y \leq 1 \end{cases}$$

So that h^2[0,2/5] ⊂ [0,2/5] and h^2[2/5,1] ⊂ [2/5,1] . Then h is transitive but not totally transitive .

***Definition 1.1.19* [14]**

Let (Y, h) be topological dynamical system . $h : Y \longrightarrow Y$ is said to be ***Topological mixing*** when there are W and M non-empty open subsets of Y :

$$\boldsymbol{h^n (M) \cap W \neq \emptyset}$$

$\forall\, \boldsymbol{n \geq m}$ are positive integer

In general , the product of two transitive maps need not be transitive map but if one of them mixing then the rustle hold [2] .

Definition 1.1.20 **[2]**

Let (Y, h) be topological dynamical system . $h : Y \longrightarrow Y$ is said to be ***weakly mixing*** when there are M_i and W_i non-empty open set :

$$\boldsymbol{h^n(M_i) \cap W_i \neq \emptyset ,}$$

$i = 1,2$, $n \in \mathbb{N}$

Remarks 1.1.21

1. On $\mathbb{R}$, mixing , weakly mixing and totally transitive are equivalent [2] . But on the other spaces : weakly mixing maps are totally transitive .

2. (Y , h) is weakly mixing if (Y^2, h^2) = ($Y \times Y , h \times h$) is transitive , In general by Induction Law (Y^n, h^n) is transitive $\forall\, n \geq 2$ [23] .

The Figure 1.1 show that the relation between mixing , weakly mixing , transitive and totally transitive that Moothathu introduced in [22]

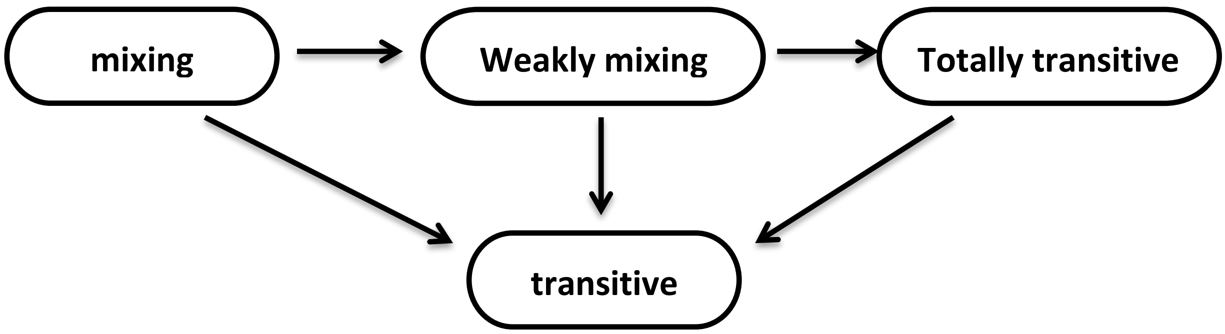

Figure 1.1 : The relations between mixing , weakly mixing , transitive and totally transitive

Definition 1.1.22 **[2]**

Let (Y, h) be discrete dynamical system . $h : Y \to Y$ is called ***minimal*** when every orbit is dense.

In [21] Li.J and Ye. X shows that $h : Y \to Y$ is minimal if it contain no proper subsystem

Example 1.1.23

Let $h : S^1 \longrightarrow S^1$ such that $h(\vartheta) = n\theta + k$, $k \in Q^C$ and $\vartheta \in S^1$. Then

$$\beta(\vartheta, h) = \{\vartheta , h(\vartheta), h^2(\vartheta) , \dots \}$$

$$\beta(\vartheta, h) = \{\vartheta , n\theta + k , n^2(\vartheta) + 2k , \dots , n^r(\vartheta) + rk , \dots \}$$

for all $r \geq 1$. Hence $\beta(\vartheta, h) = S^1$, then h is minimal map .

Definition 1.1.24 **[21]**

Let (Y, h) be discrete dynamical system and $B \subseteq Y$. B is called ***minimal set*** if the orbit of every point in B is dense , and denoted by $Min(h)$.

In [1], Akin .E and Kolyada showed that is if Trans$(h) = Y$, then h is minimal

Definition 1.1.25 **[14]**

Let (Y,h) be topological dynamical system . $h\colon Y \to Y$ is called ***weakly blending*** where for any open $\emptyset \neq W, M \subseteq Y \ \exists\ \boldsymbol{n} > \boldsymbol{0} \ni$

$$\boldsymbol{h^n(W)} \cap \boldsymbol{h^n(M)} \neq \emptyset \ .$$

Definition 1.1.26 **[3]**

Let (Y,h) be topological dynamical system . $h\colon Y \to Y$ is said to be ***strongly blending*** when any open $\emptyset \neq W, M \subseteq Y \ \exists\ n \in \mathbb{N} \ni$

$$\boldsymbol{h^n(W)} \cap \boldsymbol{h^n(M)} \ \textbf{\textit{Contain an open set}} \ .$$

by above definition we get ,the Strongly blending implies weakly blending directly .

Definition 1.1.27 **[19]**

Let (Y,h) be topological dynamical system . $h\colon Y \to Y$ is said to be ***locally eventually onto*** $(\mathcal{L}.e.o)$ when any open $\emptyset \neq W \subseteq Y \ \exists\ n \in \mathbb{N} \ni$

$$\boldsymbol{h^n(W)} = \boldsymbol{Y}$$

1.2. Some Theorems in the Autonomous Discrete System

In this section , we recall some results which need them in our work without proof :

Proposition 1.2.1 **[24]**

Let (Y, h) be topological dynamical system ,when a non-empty set $B \subseteq Y_0 \subseteq Y$, and $y \in Y$. Then

1. $y \in R(h)$ if and only if $\{y\}$ is transitive set .

2. h is transitive if and only if Y is transitive set .

3. If Y_0 is subsystem of Y , then B is transitive set with respect to Y if and only if B is transitive set with respect to Y_0 .

4. If B is transitive set , then $B \subseteq \Omega(h)$, when $\Omega(h)$ is non-wandering set .

5. B is transitive if and only if $cl(B)$ is transitive .

so by above proposition , we get the following result :

Proposition 1.2.2

Let (Y, h) be topological dynamical system . If $R(h)$ is non-empty subset of $\Omega(h)$. Then $\{y\}$ is transitive .

Proof :

By hypothesis $\emptyset \neq R(h) \subseteq \Omega(h)$. That mean there is $y \in R(h) \cap \Omega(h)$

, so $y \in R(h)$, then by proposition 1.2.1 , we get $\{y\}$ is transitive set . ■

Proposition 1.2.3

Let (Y, h) be topological dynamical system and B is non-empty open subset of Y . If $\cap_{i=1}^{n} R_i(h)$ is non-empty recurrent set , then $cl(B)$ is transitive set .

Proof :

By assumption $\cap_{i=1}^{n} R_i(h) \neq \emptyset$, then $\exists y \in \cap_{i=1}^{n} R_i(h)$ so $y \in R(h)\ \forall i = 1, \dots, n$. By proposition 1.2.1(1) , $\{y\}$ is transitive set . If $\{y\} = B \subseteq Y$, then by proposition 1.2.1(5) , then $cl(B)$ is transitive set .■

In the following proposition , we show that generalize transitivity property in the set from family to the union of this family :

Proposition 1.2.4

Let $(Y, h\,)$ be topological dynamical system and $\{Y_i\}_{i=1,\dots,n}$ be family of transitive sets . Then $\cup_{i=1}^{n} \{Y_i\}$ is transitive set .

Proof :

Let $\{Y_i\}$ be transitive sets for $i = 1, \dots, n$. Then by Definition 1.1.13, there is two non-empty open subsets W and M such that

$$\mathcal{G}(\cup_{i=1}^{n} Y_i \cap M, W) = \{n \in \mathbb{N}, h^n(\cup_{i=1}^{n} Y_i \cap M) \cap W \neq \emptyset\}$$

$\mathcal{G}(\cup_{i=1}^{n} Y_i \cap M, W) = \{ n \in \mathbb{N}, h^n[(Y_1 \cap M) \cup (Y_2 \cap M) \cup \ldots \cup (Y_n \cap M)] \cap W \neq \emptyset\}$

$$= \{n \in \mathbb{N}, [h^n(Y_1 \cap M) \cup h^n(Y_2 \cap M) \cup \ldots h^n(Y_n \cap M)] \cap W \neq \emptyset\}$$

$$= \{ n \in \mathbb{N}, h^n(Y_1 \cap M) \cap W \cup h^n(Y_2 \cap M) \cap W \cup \ldots \cup h^n(Y_n \cap M) \cap W \neq \emptyset \}$$

Since $\{Y_i\}$ is transitive set for all $i = 1, \ldots, n$, then$\mathcal{G}(Y_i \cap M, W)$ is an Infinite, so that $\mathcal{G}(\cup_{i=1}^{n} Y_i \cap M, W)$ is infinite . Then $\cup_{i=1}^{n} \{Y_i\}$ is transitive set . ■

The following result can be proved , because the union of transitive set is transitive .

Proposition 1.2.5

Let the union of transitive sets is transitive . Then the non-empty set $R(h)$ is transitive .

Proof :

Let $R(h) \neq \emptyset$, hence there is $y \in R(h)$ then by Proposition 1.2.1(1) we get $\{y\}$ is transitive , so $R(h) = \cup_{y_{\in R(h)}} \{y\}$, by hypothesis we obtained $R(h)$ is transitive . ■

Now , we prove the following proposition :

Proposition 1.2.6

Let (Y, h) be topological dynamical system . If h has periodic point Then $R(h) \cap E(h) \neq \emptyset$.

Proof :

Let y $\in Y$ be periodic point with period n. then by (1.1) we get y $\in$ E $(h)$and by (1.2) we get y $\in$ R (h) that mean $y \in R(h) \cap E(h) \neq \emptyset$. Then $R(h) \cap E(h) \neq \emptyset$. ■

Theorem 1.2.7 **[2]**

Let (Y, h) be topological dynamical system which $h : Y \longrightarrow Y$ surjective maps and y^{-} is dense for all y $\in Y$. Then h is topological transitive .

Proposition 1.2.8 **[23]**

Let (Y, h) be topological dynamical system . If B_1 and B_2 are non-empty subsets of Y . And both B_1 and B are transitive sets with $cl(B_1) - cl(B_2) \neq \emptyset$ and $cl(B_2) - cl(B_1) \neq \emptyset$. Then $B_1 \cup B_2$ is transitive set if and only if cl($\beta^{+}(B_1, h)$) =cl($\beta^{+}(B_2, h)$) $= cl$ $(\beta^{+}(B_1 \cup B , h))$.

In the maps , the transitive property not generalize by product multiplication but in the sets the result hold , the below proposition explain it :

Proposition 1.2.9.

Let (Y, h) be topological dynamical system and $\emptyset \neq A , B \subseteq Y$. If A , B are transitive set . Then $A \times B$ also transitive .

Proof :

Let $\emptyset \neq W , M$ be open subsets of B and $\emptyset \neq C , D$ be open subset of A .

Then $\emptyset \neq C \times W$ and $D \times M$ are open subset of $A \times B$,

so by definition , we get :

$$\mathcal{G}\big((A \times B) \cap (C \times W), M \times D\big) = \{n \in \mathbb{N}: h^n(A \times B) \cap C \times W) \cap M \times D \neq \emptyset\}$$

$$= \{n \in \mathbb{N} : h^n[(A \cap C) \times (B \cap W)] \cap M \times D \neq \emptyset\}$$

$$= \{n \in \mathbb{N} : h^n(A \cap C) \times h^n(B \cap W) \cap M \times D \neq \emptyset\}$$

$$= \{n \in \mathbb{N} : h^n(A \cap C) \cap M \times h^n(B \cap W) \cap D \neq \emptyset\}$$

Since A is transitive set , then $\mathcal{G}(A \cap C, M)$ is an infinite and B is transitive, then $\mathcal{G}(B \cap W, D)$ is an infinite . Hence $\mathcal{G}(A \times B \cap C \times W , M \times D)$ is an infinite . Then $A \times B$ is transitive . ■

In particular case , the product for two transitive set $A \times A$ is transitive , so by Induction Law , we get the product for more than two transitive set also transitive set .

Theorem 1.2.10 **[9]**

Let $h : I \rightarrow I$ be a continuous transitive map with a repelling fixed point y_0 , and I is compact subset of $\mathbb{R}$. Then h is weakly blending .

In [3] ,we get that $(\mathcal{L}. e. o)$ is stronger than totally transitive , mixing and (weakly and strongly) blending . So the following theorem to show that the condition to make the transitive map is $(\mathcal{L}. e. o)$.

Theorem 1.2.11. **[26]**

Let $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an equicontinuous piecewise monotone map . If h is transitive map , then h is locally eventually onto .

Chapter Two

The Chaos in g-Non Autonomous Discrete Systems

The Chaos ing-Non Autonomous Discrete Systems

Kolyada, S,Snoha in 1996 [16] discussed space called non autonomous discrete systems defined by $h_n: Y \longrightarrow Y$ where Y compact topological space , $n \in \mathbb{N}$. So $h_{1,\infty}$ denoted to sequences from 1 to an infinite such that $(h_1, h_2, \dots ,\ h_n)$ with $h_1^n(y) = h_n \circ h_{n-1} \circ \dots \circ h_1(y)$, and h_1^0 is the identity map on Y , the orbits are produced using iteration method by composition different maps in every time , so that the set $\{ y, h_1(y) , h_1^2(y), \dots , h_1^n(y), \dots \}$ called orbit of the point $y \in Y$ and denoted by $\beta(y, h_{1,\infty})$.

In this chapter , we speaks on non autonomous discrete systems in General , that is these sequences will be begin from any $n \in \mathbb{N}$ to be an Infinite instead to begin from 1 , so $(Y, h_{n,\infty})$ denoted to g-non autonomous discrete systems , where Y is compact space and $h_{n,\infty} = (h_n, h_{n+1}, \dots , h_m, \dots)$ with $h_n^m(y) = h_m \circ h_{m-1} \circ \dots \circ h_n(y)$ $\forall n < m \in \mathbb{N}$ and h_n^0 is the identity map , so that the sequences for maps affecting a single point or open set to study chaotic and topological properties instead single map in the dynamical system .

2.1. ***Fundamental Definitions for g-Non Autonomous Discrete Systems*** .

In this section , we will define some concept of dynamical system and chaotic properties in g- non autonomous discrete system . That mean , this concept and properties will be generalized from single map in dynamical system to sequences an infinite maps in g- non autonomous discrete system .

Definition 2.1.1

Let $(Y, h_{n,\infty})$ be g- non autonomous discrete system and y $\in Y$.Then

h_n(y) : the first iterate of y for h_n .

$h_n^2(y) : h_2\big(h_n(y)\big) :$ the second for h_n .

In general , if $n < m \in \mathbb{N}$, then h_n^m(y) is the m ***iterate*** of y for h_n.

Definition 2.1.2

Let $(Y, h_{n,\infty})$ be g- non autonomous discrete systems . The set of all

these iteration of y :

$$\{ y , h_n(y) , h_2(h_n(y) , \dots , h_m(h_{m-1}(h_n(y), \dots . \}$$

$$\{ y , h_n(y) , \ h_n^2(y) , \dots , h_n^m(y) , \dots \}$$

is called ***orbit*** for $y \in Y$ and denoted by $\beta(y, h_{n,\infty})$.

Example 2.1.3

Let $h_n\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a maps , for all $n \in \mathbb{R}$ such that :

$h_1(y) = 2y$

$h_2(y) = y^2$

$h_3(y) = 2y^2 + 1$, then the orbit for $y = \frac{1}{2}$ when $n = 1$ and $m = 3$

we get : $h_1\left(\frac{1}{2}\right) = 1$

$h_1^2(y) = h_2 \circ h_1\left(\frac{1}{2}\right) = h_2(1) = 1$

$h_1^3\left(\frac{1}{2}\right) = h_3 \circ h_2 \circ h_1\left(\frac{1}{2}\right)$

$= h_3 \circ h_2(1)$

$= h_3(1)$

$= 3$

Then $\beta\left(\frac{1}{2} , Y, h_{n,\infty}\right) = \left\{ \frac{1}{2} , 1 , 3 , \dots \right\}$

Definition 2.1.4.

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete system and y $\in Y$. the set

$$\boldsymbol{y^{+}} = \{\boldsymbol{x} \in \boldsymbol{Y} \,;\, \boldsymbol{h_n^m}(\boldsymbol{y}) = \boldsymbol{x} \,. \boldsymbol{For\ some\ n} < \boldsymbol{m} \in \mathbb{N}\}$$

is called ***forward orbit*** .

Definition 2.1.5 .

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete system and $y \in Y$. the set

$$\boldsymbol{y^{-}} = \{\boldsymbol{x} \in \boldsymbol{Y} \,;\, \boldsymbol{h_n^m}(\boldsymbol{x}) = \boldsymbol{y} \,. \textbf{\textit{For some}}\ \boldsymbol{n} < \boldsymbol{m} \in \mathbb{N}\}$$

is called ***backward orbit*** .

Definition 2.1.6.

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete system . Apoint y $\in Y$

is called ***fixed point*** when :

$$\boldsymbol{h_n^m}(\boldsymbol{y}) = \boldsymbol{y}\,, \forall\, \boldsymbol{n} < \boldsymbol{m} \in \mathbb{N}\,.$$

Example 2.1.7

Let $h_n : \mathbb{R} \longrightarrow \mathbb{R}$ be a maps , $\forall n \in \mathbb{N}$, where

$h_2(y) = y^2$

$h_3(y) = 3y^2 - 1$

$h_4(y) = y - 1$

if $y = 1$, Hence

$h_2^4(y) = h_4 \circ h_3 \circ h_2(1)$

$h_2^4(1) = h_4 \circ h_3 \circ (1)$

$= h_4(2)$

$= 1$

Then 1 is fixed point for $h_{2,\infty}$.

Definition 2.1.8

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete system . Apoint y ∈ Y

y is called ***eventually fixed point*** on g-non autonomous discrete system if $\exists \boldsymbol{n} < \boldsymbol{m} \in \mathbb{N} \ni$

$$\boldsymbol{h_n^m(y) = p}$$

Where p is fixed point .

So by using Example 2.1.7 , we can show that $y = -1$ is eventually fixed point by :

$$h_2^4(-1) = h_4 \circ h_3 \circ h_2(-1)$$

$$= h_4 \circ h_3(1)$$

$$= h_4(2)$$

$$= 1$$

So $h_2^4(-1) = 1$, that is fixed point . Then -1 is eventually fixed point .

Definition 2.1.9.

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete system . Apoint y $\in Y$ is called ***periodic point*** of h_n if $\exists\, m \in \mathbb{N} \ni$

$$\boldsymbol{h_n^m(y) = y} \quad \textit{and}\ \boldsymbol{h_n^k(y) \neq y}\ , \forall\ \boldsymbol{k < m} \qquad (2.1)$$

Definition 2.1.10

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete system and $y_1 \in Y$ be periodic point of period k . Apoint $y \in Y$ is called an ***eventually periodic point*** for h_n where

$$\exists\ \boldsymbol{k_1} \in \mathbb{N} : \boldsymbol{h_n^{k_1}(y) = y_1}\ , \forall \boldsymbol{n >\ k_1}$$

.***Remark 2.1.11 .***

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete system with $P(h_{n,\infty})$ and $E\ (h_{n,\infty})$ be the periodic point set and eventually periodic point set respectively , then we get :

$$P\ (h_{n,\infty})\ \subseteq E\ (h_{n,\infty}) \qquad (2.2)$$

Definition 2.1.12.

Let $(Y, h_{n,\infty})$ be g -non autonomous discrete system and $y\ \in Y$ **.** Then y is called ***non-wandering point*** if for any $\emptyset \neq W$ open set containing y :

$$\boldsymbol{h_n^m(W) \cap W \neq \emptyset\ , \forall n < m} \in\ \mathbb{N}\ .$$

The set of all the non-wandering points is known as the ***non-wandering set*** and denoted by $\Omega\ (h_{n,\infty})$.

Definition 2.1.13.

Let $(Y, h_{n,\infty})$ be g -non autonomous discrete systems. Apoint y $\in Y$ is called ***recurrent point*** where there is sequences $\{k_i\}_{i=0}^{\infty}$ there is $n < m \in \mathbb{N}$. such that

$$\lim_{k_i \to \infty} h_n^{mk_i}(y) = y \ .$$

The set of all recurrent point called ***recurrent set*** and denoted by $R(h_{n,\infty})$

Remark 2.1.14

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete system . Then

$$P\ (h_{n,\infty}) \subseteq R\ (h_{n,\infty}) \subseteq \Omega\ (h_{n,\infty}) \qquad (2.3)$$

Definition 2.1.15

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete systems and $\emptyset \neq W\ , M$ be open subset of Y. A map $h_n : Y \longrightarrow Y$ are called ***topological transitive*** where

$$h_n^m(W) \cap M \neq \emptyset \ , \forall n < m \in \mathbb{N}$$

In [24] , Oprocha and Zhang are defined $\mathcal{G}(A, B)$ in discrete dynamical system , so we will generalize (1.3) for in g -non autonomous discrete system by :

$$\mathcal{G}(A, B) = \{n < m \in \mathbb{N}\ ;\ h_n^m(A) \cap B \neq \emptyset\ \} \ . \qquad (2.4)$$

Definition 2.1.16.

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete systems . Then a non-empty subset B of Y is said to be ***transitive set*** if for any $\emptyset \neq W, M$ are open subset of Y with $B \cap W \neq \emptyset$ and $B \cap M \neq \emptyset$:

$$\mathcal{G}(\boldsymbol{B} \cap \boldsymbol{M}, \boldsymbol{W}) = \{\boldsymbol{n} < \boldsymbol{m} \in \mathbb{N} : \boldsymbol{h}_{\boldsymbol{n}}^{\boldsymbol{m}}(\boldsymbol{B} \cap \boldsymbol{M}) \cap \boldsymbol{W} \neq \emptyset\} \tag{2.5}$$

is an infinite ,

This is an intermediate situation between the ***topological mixing*** and ***topological transitive*** . If the map is topological mixing , then there is transitive set while if the map has transitive set , then it is transitive map but the converse is not true . That mean , there is transitive map but has no transitive set because (2.5) an infinite is not hold .

Definition 2.1.17

Let $(Y, h_{n,\infty})$ be g -non autonomous discrete systems and B subset of Y. B is said to be ***weakly mixing set*** on g -non autonomous discrete system of Y with contain at least two elements if for any non-empty open subset $W_1, \dots, W_n$ of B and non-empty open subsets $M_1, \dots, M_n$ of Y with

$B \cap M_i \neq \emptyset$,there is $n < m \in \mathbb{N}$, such that

$$\boldsymbol{h_n^m(W_i \cap B) \cap M_i \neq \emptyset},$$

For $i = 1 , \dots , n$.

Definition 2.1.18

Let $(Y, h_{n,\infty})$ be g -non autonomous discrete systems . The map $h_n: Y \to Y$ is called ***weakly blending*** on g -non autonomous discrete system when any open subsets $\emptyset \neq \boldsymbol{W, M} \subseteq \boldsymbol{Y} \exists \boldsymbol{n} < \boldsymbol{m} \in \mathbb{N} \ni$

$$\boldsymbol{h_n^m(W) \cap h_n^m(M) \neq \emptyset} .$$

Definition 2.1.19

Let $(Y, h_{n,\infty})$ be g -non autonomous discrete systems . A map $h_n: Y \to Y$ is called ***strongly blending*** on g -non autonomous discrete system where for any open subsets $\emptyset \neq W, M \exists n < m \in \mathbb{N} \ni$

$\boldsymbol{h_n^m(W) \cap h_n^m(M)}$ ***contain an open set on Y .***

Note : By above definition , if $h_n: Y \longrightarrow Y$ is strongly blending , then it is weakly blending .

Definition 2.1.20

Let $(Y, h_{n,\infty})$ be g -non autonomous discrete systems . A map $h_n: Y \to Y$ is called ***locally eventually onto*** $(\mathcal{L}.e.o)$ on g-non autonomous discrete system when any non-empty open subset W of Y $\exists$ $\boldsymbol{n} < \boldsymbol{m} \in \mathbb{N}$ $\ni$

$$\boldsymbol{h_n^m(W) = Y} \ .$$

The Figure 2.1 show the relation between $\mathcal{L}.e.o$property and some other chaotic properties for g -non autonomous discrete systems .

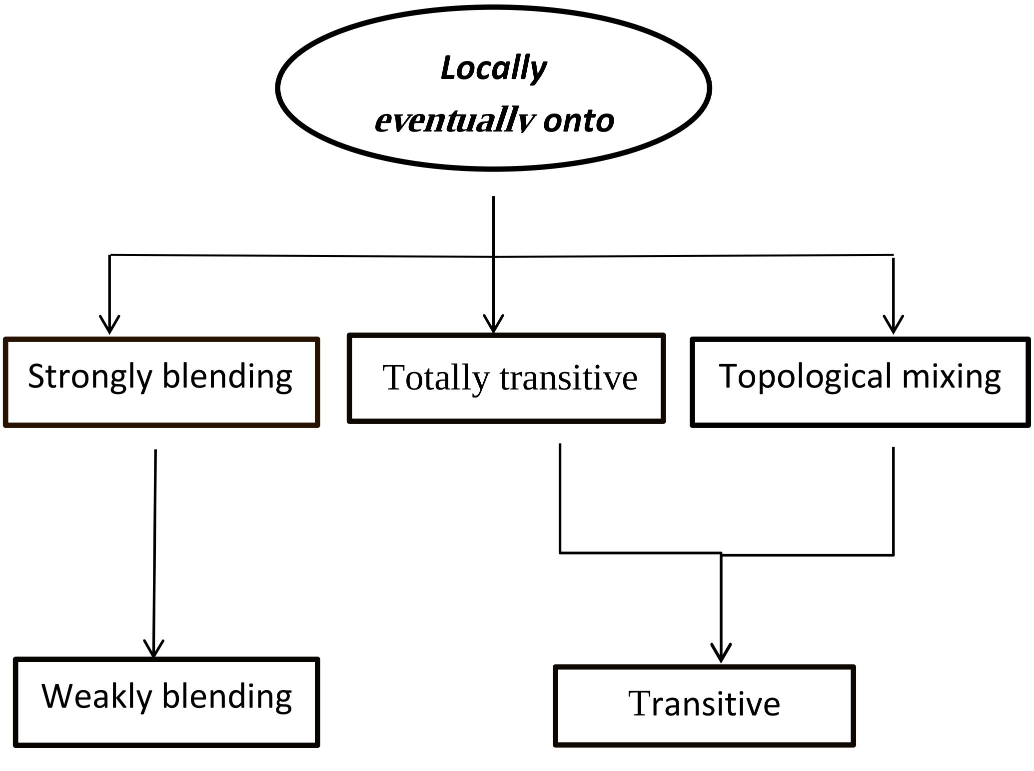

Figure 2.1 **:** relation between $(\mathcal{L}.e.o)$property and some other chaotic properties for g -non autonomous discrete systems

Definition 2.1.21

Let $(Y, h_{n,\infty})$ be g -non autonomous discrete systems with $h_n: Y \longrightarrow Y$ are continuous maps . h_n satisfy ***Touhey property*** on$g$ -non autonomous discrete systems where for any two open subset W, M of Y , there is periodic point $y \in W$ and $n < m \in \mathbb{N} \ni$

$$\boldsymbol{h_n^m(y) \in M}$$

Definition 2.1.22

Let $(Y, h_{n,\infty})$ be g -non autonomous discrete systems with Y is metric space with metric d .A map $h_n: Y \longrightarrow Y$ are called ***equicontinuous maps*** on g -non autonomous discrete systems where for any $\epsilon > 0$ $\exists \delta > 0 \ni$

$$\boldsymbol{d(h_n^m(x) , h_n^m(y)) < \epsilon}$$

whenever $d(x, y) < \delta$, $\forall$ x , $y \in Y$ and $n < m \in \mathbb{N}$.

Definition 2.1.23

Let $(Y, h_{n,\infty})$ be g -non autonomous discrete systems and Y be metrical space with metric d, $h_n: Y \longrightarrow Y$ are called ***sensitive dependent on***

initial conditions on g -non autonomous discrete systems if there exists $\delta > 0$ that for any $y \in Y$ and W any neighborhood of y there is $x \in W$ and $n < m \in \mathbb{N} \ni$

$$\boldsymbol{d}\left(\boldsymbol{h}_{\boldsymbol{n}}^{\boldsymbol{m}}(\boldsymbol{y}), \boldsymbol{h}_{\boldsymbol{n}}^{\boldsymbol{m}}(\boldsymbol{x})\right) > \boldsymbol{\delta}$$

Definition 2.1.24

Let $(Y, h_{n,\infty})$ be g -non autonomous discrete systems and $x, y \in Y$.

Apoint y is called ***homoclinic*** to x on g -non autonomous discrete systems where y local unstable of x and there is $n < m \in \mathbb{N} \ni$

$$\boldsymbol{h}_{\boldsymbol{n}}^{\boldsymbol{m}}(\boldsymbol{y}) = \boldsymbol{x} .$$

Example 2.1.25

Let $h_n: \mathbb{R} \longrightarrow \mathbb{R}$ be a maps , for all $n \in \mathbb{N}$. We take ,

$h_1(y) = 2y$

$h_2(y) = y^2$

$h_3(y) = y - 2$

Where $n = 1$, $m = 3$, then when $y = 2$,we obtained :

$h_1^3(2) = h_3 \circ h_2 \circ h_1(2) = h_3 \circ h_2(4) = h_3(16) = 14$

Then 2 is homoclinic to 14 .

Definition 2.1.26

Let $(Y , h_{n,\infty})$ be g -non autonomous discrete system . Then

$h_n\colon Y \longrightarrow Y$ are called ***minimal*** on g-non autonomous discrete system if

for all $y \in Y$ and $n < m \in \mathbb{N}$.

$$\boldsymbol{h_n^m} \textbf{ (y) is dense}$$

Example 2.1.27

Let $h_n\colon S^1 \longrightarrow S^1$ be a maps such that :

$h_1(\vartheta) = \vartheta + k$

$h_2(\vartheta) = 2\vartheta + k$

$h_3(\vartheta) = 3\vartheta + k$

For all $\vartheta \in S^1$ and $k \in Q^{`}$. we take $\vartheta = \frac{\pi}{2} \in S^1$ and $n = 1 , m = 3$, then :

$$h_1^3\left(\frac{\pi}{2}\right) = h_3 \circ h_2 \circ h_1\left(\frac{\pi}{2}\right)$$

$$h_1^3\left(\frac{\pi}{2}\right) = h_3 \circ h_2\left(\frac{\pi}{2} + k\right)$$

$$h_1^3\left(\frac{\pi}{2}\right) = h_3(\pi + 2k)$$

$$h_1^3\left(\frac{\pi}{2}\right) = 3\pi + 2k$$

Then $h_1^3\left(\frac{\pi}{2}\right)$ is dense in S^1 , then h_n are minimal maps .

Remark 2.1.28

$\{h_n^m(y)\}_{n=1}^{\infty}$ is dense if and only if it is an infinite sets , so y is not periodic point , Hence if $h_{n,\infty}$ are minimal . Then they are have no periodic points .

Example 2.1.28

Let $h_n: S^1 \longrightarrow S^1$ be a maps such that $h_n(\vartheta) = n\vartheta$, where $\vartheta \in S^1$.

Then $\beta(\vartheta) = \{\vartheta ,\ h_n(\vartheta) , h_n^2(\vartheta) , h_n^m(\vartheta) , \ldots\}$,

so $h_n^m(\vartheta) \neq S^1$, so $\beta\left(h_{n,\infty}, \vartheta\right) \neq S^1$.

Then h_n is not minimal .

Note : $Min\left(h_{n,\infty}\right) = Y$. So Trans $\left(h_{n,\infty}\right) = \{y \in Y :\ h_n^m(y)$ is dense \} . That mean :

$$\boldsymbol{Trans}\ \left(\boldsymbol{h}_{\boldsymbol{n},\infty}\right) = \boldsymbol{Min}\left(\boldsymbol{h}_{\boldsymbol{n},\infty}\right) = \boldsymbol{Y} .$$

2.2 . *Some Result of Chaotic Properties in g-Non Autonomous Discrete System*

In this section , we will prove some result of chaotic properties in the sets and the maps for space more general called g-non autonomous discrete systems .

Theorem 2.2.1

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete system with $h_n: Y \longrightarrow Y$ surjective maps and y^- is dense for every $y \in Y$. Then ($h_{n,\infty}$) are topological transitive .

Proof :

Let $\emptyset \neq W, M$ be open subsets of Y and let $y \in M$. By hypothesis , backward orbit is dense , then $\forall\, W \in Y\ \exists n < m \in \mathbb{N} \ni h_n^{-m}(y) \in W$, so $y^- \cap W \neq \emptyset$, then $x \in y^-$and . Since $x \in y^-$ and $x \in W$, then we get , $h_n^m(x) =$ y so $h_n^m(x) \in M$, so $h_n^m(M) \cap W \neq \emptyset$. Then $h_{n,\infty}$ is topological transitive . ■

Proposition 2.2.2

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete system and $\{Y_i\}_{i=1,\dots,n}$ be family of transitive sets where $Y_i \subseteq Y_j$, if $i \leq j$, Then $\cup_{i=1}^{n} \{Y_i\}$ is transitive set .

proof :

Let $B \subseteq Y$. Then by (2.4) , for any W and M are non-empty open subsets of Y such that

$$\mathcal{G}(\cup_{i=1}^{n} Y_i \cap M, W) = \{n < m \in \mathbb{N} : h_n^m(\cup_{i=1}^{n} Y_i \cap M) \cap W \neq \emptyset \}$$

$$=\{n < m \in \mathbb{N} : h_n^m\big((Y_1 \cap M) \cup (Y_2 \cap M) \cup \dots \cup (Y_n \cap M)\big) \cap W \neq \emptyset\}$$

$$= \{n < m \in \mathbb{N}: h_n^m(Y_1 \cap M) \cup h_n^m(Y_2 \cap M) \cup \dots \cup h_n^m(Y_n \cap M)) \cap W \neq \emptyset\}$$

$$=\{n < m \in \mathbb{N}, h_n^m(Y_1 \cap M) \cap W \cup h_n^m(Y_2 \cap M) \cap W \cup \dots \cup h_n^m(Y_n \cap M) \cap W \neq \emptyset\}$$

Since Y_i are transitive set, for all $i = 1,\dots, n$, then $\mathcal{G}(Y_i \cap M, W)$ is an infinite . Then $\mathcal{G}(\cup_{i=1}^{n} Y_i \cap M, W)$ is an infinite so $\cup_{i=1}^{n} \{Y_i\}$ is transitive set . ■

proposition 2.2.3

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete systems and B is transitive set of Y . Then $A \subseteq B$ also transitive set .

proof :

Let $\emptyset \neq W, M$ be open subset of B , then U and V are open subset of W and M respectively . So by (2.5)

$$\mathcal{G}(\boldsymbol{A} \cap \boldsymbol{U}, \boldsymbol{V}) = \{ \boldsymbol{n} < \boldsymbol{m} \in \mathbb{N} : \ \boldsymbol{h}_{\boldsymbol{n}}^{\boldsymbol{m}}(\boldsymbol{A} \cap \boldsymbol{U}) \cap \boldsymbol{M} \neq \emptyset \}$$

Since $A \subseteq B$, then

$$\mathcal{G}(\boldsymbol{A} \cap \boldsymbol{U}, \boldsymbol{V}) \subseteq \mathcal{G}(\boldsymbol{B} \cap \boldsymbol{W}, \boldsymbol{M})$$

Since B is transitive set , then $\mathcal{G}(B \cap W, M)$ is infinite , so $\mathcal{G}(A \cap U, V)$ is an infinite . Then A is transitive set .■

In particular case , the intersection of finite family of transitive sets is transitive .

Proposition 2.2.4.

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete system , if h has periodic point . Then $R(h_{n,\infty}) \cap E(h_{n,\infty}) \neq \emptyset$.

Proof :

Let $y \in Y$ be periodic point with period m , Then y $\in P(h_{n,\infty})$ that mean by (2.2) , y $\in E(h_{n,\infty})$, and by (2.3) we obtained y $\in R(h_{n,\infty})$. Then y $\in R(h_{n,\infty}) \cap E(h_{n,\infty})$. Then $R(h_{n,\infty}) \cap E(h_{n,\infty}) \neq \emptyset$.■

Proposition 2.2.5.

Let $(Y, h_{n,\infty})$ and $(Z, g_{n,\infty})$ be two g-non autonomous discrete systems with $\pi : (Y, h_{n,\infty}) \longrightarrow (Z, g_{n,\infty})$ be factor map and B be weakly mixing subset of Y . Then $\pi(B)$ is weakly mixing subset of Z .

Proof :

In this proof , we have two cases :

Case I : if $\pi\ (B)$ is singleton set . then the result hold and trivial .

Case II : Let $\pi\ (B)$ has contain at least two element and $B\ \subseteq\ Y$ is weakly mixing of Y . Then $\exists\ i,k \in\ \mathbb{N}\ \ni\ M_1^i , M_2^i , \dots , M_K^i$ are open subsets of $\pi\ (B)$ and $W_1^i , W_2^i , \dots , W_K^i$ are open subsets of Z with $W_j^i \cap\ \pi\ (B) \neq \emptyset$, $j\ =\ 1 ,2 , \dots ,$ k .Then $B\ \cap\ \pi^{-1}(M_j)$ are open subset of B and $\pi^{-1}\left(W_j^i\right)$ is subset of Y with $\pi^{-1}\left(W_j^i\right) \cap B \neq \emptyset$, where $j\ =\ 1 , \dots , k$. Since B is weakly mixing of Y then $\exists\ n < m \in \mathbb{N}$

$$\ni\ h_n^m(B \cap \pi_i^{-1}(M_j^i) \cap \pi_i^{-1}(W_j^i) \neq \emptyset ,$$

$$\pi \circ h_n^m \left[\ B \cap \pi_i^{-1}\left(M_j^i\right)\right]\ \cap \pi\left[\pi_i^{-1}\left(W_j^i\right)\right] \neq \pi(\emptyset) = \emptyset$$

$$g_n^m \circ \pi \left[B \cap \pi_i^{-1}\left(M_j^i\right) \cap \left[\ \left(W_j^i\right)\right]\right] \neq \emptyset$$

$$g_n^m(\pi(B) \cap M_j^i) \cap W_j^i \neq \emptyset$$

Then $\pi(B)$ is weakly mixing set of Y .■

Proposition 2.2.6

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete system . If $cl(B)$ is transitive set , then B is transitive set .

Proof :

Assume that $cl(B)$ is transitive set , then by Definition 2.1.15 , for any non-empty open subsets W, M of $Y$with $W \cap B \neq \emptyset$ and $M \cap B \neq \emptyset$ such that

$$\mathcal{G}(M \cap cl(B), W) = \{n < m \in \mathbb{N}, h_n^m(M \cap cl(B)) \cap W \neq \emptyset\}$$

is an infinite . Since $B \subseteq cl(B)$,which impels that

$$\mathcal{G}(M \cap B, W) = \{n < m \in \mathbb{N}, h_n^m(M \cap B) \cap W \neq \emptyset\}$$

is an infinite so $\mathcal{G}(M \cap B, W) \neq \emptyset$. Then B is transitive set .■

Before starting with the proof of the following theorem , we will introduced a pricewise monotone map : let be the $Y = \mathbb{R}^n$. The continuous maps $h_n: Y \longrightarrow Y$ are called piecewise monotone if there are points $u = a_0 \propto a_1 \propto \cdots \propto a_j \propto a_{j+1} = v$, such that h_n are strictly monotone on neighborhood of a_{j-1}, a_j .

Theorem 2.2.7

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete systems ,where $Y = \mathbb{R}^k$ and $h_n: \mathbb{R}^k \longrightarrow \mathbb{R}^k$ is an equicontinuous piecewise monotone maps where $k \in \mathbb{N}$, if h_n is transitive ,then h_n is locally eventually onto.

Proof : The method used to prove this theorem is the same as that used to prove Theorem 3.3 (see[26])

Theorem 2.2.8

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete systems with ,where $Y = \mathbb{R}^n$,$n \in \mathbb{N}$ and $h_n : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be equicontinuous piecewise monotone maps and $h_{n,\infty}$ have dense homoclinic points . If $h_{n,\infty}$ isstrongly blending . Then $h_{n,\infty}$ are locally eventually on to .

Proof :

We take non-empty open subsets W , M of Y , since $h_{n,\infty}$ is strongly blending , then $\exists\, n < m \in \mathbb{N} \ni h_n^m(W) \cap h_n^m(M)$ contain an open set say A . Let $X = h_n^{-r}(A) \cap M$ and q homoclinic point and dense , then $h_n^r(q) \in A$ and $\exists y \in W \ni h_n^b(y) = h_n^b(q)\ \forall b > r$, by Definition (2.1.23) $\exists \gamma \ni h_n^{\gamma}(q) = p$, so take three cases :

Case I : $\boldsymbol{\gamma = r}$

$h_n^b(y) = h_n^{b-r}\left(h_n^r(q)\right) = h_n^{b-r}(p)$ so we obtained $p \in h_n^b(W) \cap M$, then $h_n^b(W) \cap M \neq \emptyset$.

Case II : $\boldsymbol{\gamma < r}$

$h_n^b(y) = h_n^{b-r}(h_n^{r-\gamma}\left(h_n^{\gamma}(q)\right) = h_n^{b-r}\left(h_n^{r-\gamma}(p)\right)$, then $h_n^{r-\gamma}(p) \in h_n^b(W)$ so $h_n^b(W) \cap M \neq \emptyset$.

Case III : $\boldsymbol{\gamma > r}$

$h_n^b(y) = h_n^{b-\gamma}\left(h_n^{\gamma}(q)\right) = h_n^{b-\gamma}(p)$, so $h_n^{b-\gamma}(p) \in h_n^b(W)$. Then $h_n^b(W) \cap M \neq \emptyset$, so $h_{n,\infty}$ is transitive . Then by Theorem (2.2.7) we get $h_{n,\infty}$ is locally eventually onto .

Theorem 2.2.9

Let $(Y, h_{n,\infty})$ be$g$-non autonomous discrete systems with $h_n: Y \longrightarrow Y$ be continuous maps . If h_n is Locally eventually onto , then it is topologically mixing .

Proof

Let W, M be non-empty open subset of Y. Since h_n is ($\mathcal{L}$.e.o) , then for any $W \in Y \exists n < m \in \mathbb{N} \ni h_n^m(W) = Y$, then $h_n^m(W) \cap M \neq \emptyset$, so we choose N $> 0 \ni h_n^m(W) \cap M \neq \emptyset \ \forall m > N$. Then h_n is topological mixing . ■

So by definition of topological mixing , we get h_n is topological transitive .

Theorem 2.2.10

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete systems with $h_n: Y \longrightarrow Y$ be continuous maps . If h_n is locally eventually onto , then it is totally transitive .

Proof

Let W, M be non-empty open subsets of Y . Since h_n is ($\mathcal{L}$.e.o) , then $W \in Y \exists n < m \in \mathbb{N} \ni h_n^m(W) = Y$, let $r > 0,$ then

$$(h_n^r)^m(W) = h_n^r\big(h_n^m(W)\big) = h_n^r(Y) = Y$$

So $(h_n^r)^m(W) \cap M \neq \emptyset$. Then h_n is totally transitive . ■

So by Definition of totally transitive , we get h_n is topological transitive .

Theorem 2.2.11

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete systems . with $h_n: Y \longrightarrow Y$ continuous maps , if h_n is ($\mathcal{L}$.e.o) , then it is strongly blending .

Proof

Let W, M be any non-empty open subsets of Y . Since h_n is ($\mathcal{L}$.e.o) , then $W \in Y \exists n < m \in \mathbb{N} \ni h_n^m(W) = Y$, so we can find $c > 0 \ni h_n^c(W) = h_n^c(M) = Y$, then $h_n^c(W) \cap h_n^c(M) = Y$, which is open set , so h_n is strongly blending . ■

So by definition of strongly blending , we can get h_n is weakly blending .

Theorem 2.2.12

Let Y be a matric space with metric d and let $(Y, h_{n,\infty})$ be g-non autonomous discrete systems . If $h_n: Y \longrightarrow Y$ is locally eventually onto . Then h_n is sensitive dependent on initial conditions .

Proof :

Let $\mathcal{P}$, $q \in Y$ such that $d(\mathcal{P} , q) > 3r$, where r positive integer number and let $\mathcal{B}_r(\mathcal{P})$ be ball with radius r and center $\mathcal{P}$ and $\mathcal{B}_r(q)$ ball with radius r and center q . Let $y \in Y$ and $B_\epsilon(y)$ open neighborhood of y for some $\epsilon > 0$. Since $h_{n,\infty}$ is locally eventually onto , there is $c_1 \in \mathbb{N}$ such that $h_n^{c_1}\big(\mathcal{B}_r(\mathcal{P})\big) = Y$, so there exists $b_1 < c_1 \in \mathbb{N} \ni h_n^{b_1}(B_r(p) \cap B_\epsilon(Y) \neq \emptyset$

and $c_2 \in \mathbb{N} \ni\ h_n^{c_2}\big(B_r(q)\big) = Y$, so there is $b_2 < c_2 \in \mathbb{N}$ such that

$$(h_n^{b_2}(\mathcal{B}_r(p) \cap B_\epsilon(y)) \neq \emptyset$$

We take $b = max\{b_1, b_2\}$, then there exists $y_1, y_2 \in B_\epsilon(y)$ such that

$(h_n^b(y_1) \in \mathcal{B}_r(\mathcal{P})$ and $(h_n^b(y_2) \in \mathcal{B}_r(q)$. Then we have

$$d(h_n^b(y_1)\,, h_n^b(y_2)) \geq \delta$$

By triangular inequality , we get

$$d(\ h_n^b(y_1)\,, h_n^b(y) \geq \delta \quad \text{or } d(h_n^b(y_2), h_n^b(y)\ \geq \delta$$

Then $h_{n,\infty}$ is sensitive dependent on initial conditions . ■

Theorem 2.2.13

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete systems with $h_n\colon Y \longrightarrow Y$ Be continuous map . if h_n is strongly blending and has dense periodic points , then h_n is satisfy Touhey property .

proof :

Let $\emptyset \neq W, M$ be open subsets of Y ,there is $n < m \in \mathbb{N}$:

$h_n^m(W) \cap h_n^m(M)$ contain open set say U ,$U \subset Y\colon U \subset h_n^m(W) \cap h_n^m(M)$.

Now. Let $M_1 = h_n^{-m}(U) \cap M$, since U open set and h_n continuous maps , then $h_n^{-m}(U)$ is open set , so M_1 is open set . Since the periodic points are dense then there is periodic point y of period m such that $h_n^m(y) \in U$ and $y \in M$, so there is $p \in W : h_n^k(y) = h_n^k(p)$, therefor

$$h_n^m(p) = h_n^{m-k}\left(h_n^k(p)\right) = h_n^{m-k}\left(h_n^k(y)\right) = h_n^m(y) = y$$

So $y \in h_n^m(W) \cap M$,then $y \in h_n^m(W)$. Then we obtained $h_{n,\infty}$ satisfy Touhey property . ■

Theorem 2.2.14

Let $(Y, h_{n,\infty})$ be g-non autonomous discrete systems . If there is $y \in Y$ has dense orbit , then $h_{n,\infty}$ is topological transitive .

Proof :

Let $y \in Y$ has dense orbit and W , M be two non-empty subset of Y . Then

$$\forall\, W \in Y\ \exists\, n < m \in \mathbb{N} :\ h_n^m(y) \in W$$

Since y has dense orbit , then $\forall\, h_n^m(y) \in \beta\left(y, h_{n,\infty}\right)$, also $h_n^m(y)$ has dense orbit , so by using density of $h_n^m(y)$, then

$$\forall M \in Y\ \exists n < m \in \mathbb{N} \ni\ h_n^m\left(h_n^m(y)\right) \in M$$

That mean $h_n^m(y) \in W$ and $y \in M$, so $h_n^m(M) \cap W \neq \emptyset$. Then $h_{n,\infty}$ is transitive . ■

2.3 .The Product Multiplication in g-Non Autonomous Discrete Systems .

In this section , we will generalized of some chaotic properties for maps through product multiplication in more general space than dynamical system called g-non autonomous discrete systems and we will see that there are properties not generalized by product expect with some conditions on these maps .

Proposition 2.3.1

Let $(Y, h_{n,\infty})$ and $(X, g_{n,\infty})$ be two g-non autonomous discrete systems .the set periodic point of $g_{n,\infty} \times h_{n,\infty}$ is dense of $X \times Y$ if and only if the set periodic points of $g_{n,\infty}$ and $h_{n,\infty}$ are dense in X and Y respectively .

Proof :

$\Longrightarrow$) Let $\emptyset \neq W, M$ be open subsets of X and Y respectively

Then $W \times M$ is open subset of $X \times Y$. Since the set of periodic point in $g_{n,\infty} \times h_{n,\infty}$ is dense in $X \times Y$. Then there exists $p = (x, y) \in W \times M$,

so$(g_n \times h_n)^m(x, y) = (g_n^m(x), h_n^m(y)) = (x, y)$, for any $n < m \in \mathbb{N}$.

Then we obtained $g_n^m(x) = x,$ for $x \in W$ and $h_n^m(y) = y$, for $y \in Y$,so $g_{n,\infty}$ and $h_{n,\infty}$ have dense periodic points .

($\Longleftarrow$ Assume that $g_{n,\infty}$ and $h_{n,\infty}$ have dense periodic point and let $U \subset X \times Y$ be non-empty open set . Then there is non-empty open sets

$W \subset X$ and $M \subset Y$ such that $M \times W \subset U$. by hypothesis there is $y \in W$ such that $h_n^{km}(y) = y, n < m \in \mathbb{N}$ and there is $x \in M$ such that $h_n^{dm}(x) = x$, $n < m \in \mathbb{N}$. For $s = (x, y) \in U$, we let $m = kd$, we obtained :

$(g_n \times h_n)^m(s) = (g_n \times h_n)^m(x, y) = \left(g_n^m(x), h_n^m(y)\right) = (x, y)$. Then U contain periodic point and the set of periodic point of $g_n \times h_n$ is dense in $X \times Y$. ■

Theorem 2.3.2

Let $(Y, h_{n,\infty})$ and $(X, g_{n,\infty})$ be two g-non autonomous discrete systems with $g_n: X \to X$ and $h_n: Y \to Y$ be a maps . Then h_n and g_n are strongly blending if and only if $g_n \times h_n$ is strongly blending .

proof :

$\Longrightarrow$) we take g_n and h_n are strongly blending ,and we went to show that $g_n \times h_n$ is strongly blending .

Now; let $\emptyset \neq U_1, U_2$ be open subsets of $X \times Y$. Hence There are $\emptyset \neq W_1, W_2$ open subsets in X and $\emptyset \neq M_1, M_2$ open subsets of Y such that $U_1 = W_1 \times M_1$ and $U_2 = W_2 \times M_2$. Since g_n are strongly blending , then there is $m_1 > 0$ such that $g_n^{m_1}(W_1) \cap g_n^{m_1}(W_2)$ contain open set , then there is open set say $B_1 \subset X$ such that $B_1 \subset g_n^m(W_1) \cap g_n^m(W_2)$,Since h_n is strongly blending , then there is $m_2 > 0$ such that

$h_n^{m_2}(M_1) \cap h_n^{m_2}(M_2)$ contain open set say $B_2 \subset Y$. Then

$B_2 \subset h_n^{m_2}(M_1) \cap h_n^{m_2}(M_2)$. We let $m = m_1 + m_2$, there we get :

$(g_n \times h_n)^m(U_1) \cap (g_n \times h_n)^m(U_2) = g_n^m(W_1) \times h_n^m(M_1) \cap g_n^m(W_2) \times h_n^m(M_2)$, that mean :

$$(g_n^m(W_1) \cap g_n^m(W_2)) \times h_n^m(M_1) \cap h_n^m(M_2) \supset B_1 \times B_2$$

Then $B_2 \times B_1$ open set , then $g_n \times h_n$ are strongly blending

($\Leftarrow$ Let $g_n \times h_n$ be strongly blending , we went to prove that both of them are strongly blending . We will prove that $g_{n,\infty}$ are strongly blending and on the other is same .

Now, let W_1 and M_1 is non-empty open subset in X , there is $W = W_1 \times Y$ And $M = M_1 \times Y$, there is $m > 0$, such that

$(g_n \times h_n)^m(W_1 \times Y) \cap (g_n \times h_n)^m(M_1 \times Y) = (g_n^m(W_1) \cap g_n^m(M_1)) \times (h_n^m(Y) \cap h_n^m(Y))$ contain open set .

Since $g_n^m(W_1) \cap g_n^m(M_1)$ an open set and $(h_n^m(Y) \cap h_n^m(Y))$ is non-empty set . Then $g_{n,\infty}$ is strongly blending . ■

In the same a way , can be show $h_{n,\infty}$ is strongly blending .

Note : Since the condition for strongly blending is hold weakly blending , so by above Theorem , we can get that the product for two weakly blending maps also weakly blending directly .

Theorem 2.3.3.

Let $(X, g_{n,\infty})$ and $(Y, h_{n,\infty})$ be two$g$-non autonomous discrete systems with $g_n: X \longrightarrow X$ and $h_n: Y \longrightarrow Y$ be continuous maps . Then g_n and h_n is satisfy Touhey property if and only if $g_n \times h_n$ is satisfy Touhey property .

proof :

$\Longrightarrow$) let $\emptyset \neq W, M$ be open subsets of $X \times Y$, then there exists $\emptyset \neq W_1$, M_1 are open subsets in X and $\emptyset \neq W_2$,M_2 are open subsets in Y such that $W = W_1 \times W_2$ and $M = M_1 \times M_2$. Since g_n satisfy Touhey property , then there is $y_1 \in W_1$ periodic point of g_n with period m_1 , that is $h_n^{m_1}(y_1) = y_1$ and there is $k_1 > 0 :$ $g_n^{k_1}(y_1) \in M_1$. Also there is $y_2 \in W_2$ is periodic point of h_n for period m_2 that is $h_n^{m_2}(y_2) = y_2$ and $\exists\ k_2 > 0 : h_n^{k_2}(y_2) \in M_2$. We note $g_n^{m}(y_1) = y_1$,

for all m multiple of m_1 . Then $(y_1, y_2) \in W_1 \times W_2$ is a periodic point of $g_n \times h_n$ with period m ,

that is $(g_n \times h_n)^{\ m}(y_1, y_2) = (g_n^{m}(y_1)\ , h_n^{m}(y_2))$, that mean $\left(g_n^{m}(y_1)\ , h_n^{m}(y_2)\right) = (y_1, y_2)$. Then $W = W_1 \times W_2$ contain a periodic point of period m . Let $k = k_1 \cdot k_2$,we get

$(g_n \times h_n)^{mk}(y_1, y_2) = g_n^{mk}(y_1)\ , h_n^{mk}(y_2)) = (g_n^{mk_1}(y_1)\ , h_n^{mk_2}(y_2))$

Belong to $M_1 \times M_2$. Then $g_{n,\infty} \times h_{n,\infty}$ satisfy Touhey property .

($\Leftarrow$ Let $\emptyset \neq W_1$,M_1 be open subsets in X . Then there is $W = W_1 \times Y$ and $M = M_1 \times Y$ are open subsets of $X \times Y$. By assumption $g_{n,\infty} \times h_{n,\infty}$ is satisfy Touhey property , then there is $y = (y_1, y_2) \in W$ a periodic point of period m such that $(g_n \times h_n)^{mk}(y) \in M$, then

$$(g_n \times h_n)^m(y_1, y_2) = \left(g_n^m(y_1) \times h_n^m(y_2)\right) = (y_1, y_2)$$

Then $g_n^m(y_1) = y_1 \in W_1$ and $h_n^m(y_2) = y_2 \in M_1$. Then $y_1$is periodic point of g_n with period m and y_2 is periodic point of h_n with period m , So $(g_n \times h_n)^k(y) = (g_n \times h_n)^k(y_1, y_2) \in M$, then $g_n^k(y_1) \in M_1$ and $h_n^k(y_2) \in Y$. Then $g_{n,\infty}$ is satisfy Touhey property . ■

Note : Transitivity is a chaotic property that is not generalized by product of maps. In the following proposition we will show that generalized this property between the sets chaotic by the product

Proposition 2.3.4.

Let $(Y, h_{n,\infty})$ be g -non autonomous discrete system and A, B be two non-empty transitive sets . Then $A \times B$ also transitive set .

Proof :

Let $\emptyset \neq W, M$ be open subsets of B and $\emptyset \neq C, D$ be open subsets of A . Then $\emptyset \neq C \times W$, $D \times M$ are open subsets of $A \times B$.Then by definition , we get :

$$\mathcal{G}(A \times B \cap C \times W, M \times D) = \{n < m \in \mathbb{N} : h_n^m(A \times B \cap C \times W) \cap M \times D \neq \emptyset\}$$

$$= \{n < m \in \mathbb{N} : h_n^m(A \cap C \times B \cap W) \cap M \times D \neq \emptyset\}$$

$$= \{n < m \in \mathbb{N} : h_n^m(A \cap C) \times h_n^m(B \cap W) \cap M \times D \neq \emptyset\}$$

$$\mathcal{G}(A \times B \cap C \times W, M \times D) = \{n < m \in \mathbb{N} : h_n^m(A \cap C) \cap M \times h_n^m(B \cap W) \cap D \neq \emptyset\} .$$

Since A is transitive set , then $\mathcal{G}(A \cap C, M)$ is an infinite and since B is transitive set , then $\mathcal{G}(B \cap W, D)$ is an infinite . So $\mathcal{G}(A \times B \cap C \times W, M \times D)$ is an infinite .Then $A \times B$ is transitive set .■

In special case , if B is transitive set then $B \times B$ is transitive set . Then we can generalize this case by Induction Law that $B \times B \times \ldots \times B$ (n-time) also transitive set.

Proposition 2.3.5.

Let $(X, g_{n,\infty})$ and $(Y, h_{n,\infty})$ be two g-non autonomous discrete systems with $g_n: X \longrightarrow X$ and $h_n: Y \longrightarrow Y$ are maps . If $g_{n,\infty} \times h_{n,\infty}$is topological transitive . Then $g_{n,\infty}$ and $h_{n,\infty}$ are both topological transitive on X and Y respectively .

Proof:

Let W_1 , M_1 be non-empty open subsets in X , so $W = W_1 \times Y$ and $M = M_1 \times Y$ be open subsets in $X \times Y$. Since $g_{n,\infty} \times h_{n,\infty}$ is transitive ,then by Definition (2.1.15) , there is $n < m \in \mathbb{N}$ and $k > 0 \ni$

$$(g_{n,\infty} \times h_{n,\infty})^{mk}(W) \cap M \neq \emptyset$$

$(g_{n,\infty} \times h_{n,\infty})^{mk}(W) \cap M = g_n^{mk}(W_1) \times h_n^{mk}(Y) \cap (M_1 \times Y)$, that mean

$$= [\, g_n^{mk}(W_1) \cap M_1] \times [h_n^{mk}(Y) \cap Y].$$Then

We obtained : $[\, g_n^{mk}(W_1) \cap M_1] \times [h_n^{mk}(Y) \cap Y] \neq \emptyset$. Then $g_n^{mk}(W_1) \cap M_1 \neq \emptyset$, then $g_{n,\infty}$ is topological transitive . We can show that $h_{n,\infty}$ is also topological transitive in the same away . ■

Theorem 2.3.6

Let $(X, g_{n,\infty})$ and $(Y, h_{n,\infty})$ be two g-non autonomous discrete systems with $g_n: X \longrightarrow X$ and $h_n: Y \longrightarrow Y$ be topological mixing maps . Then also $g_{n,\infty} \times h_{n,\infty}$ is topological mixing .

Proof :

Let $U_1, U_2 \subset X \times Y$, so there exists open subsets $W_1, W_2 \subset X$ and $M_1, M_2 \subset Y$,such that $W_1 \times M_2 \subset U_1$ and $W_2 \times M_2 \subset U_2$. By assumption $g_{n,\infty}$ is topological mixing , there are m_1 , m_2:

$$g_n^m(W_1) \cap W_2 \neq \emptyset \text{ , for } m \geq m_1$$

and $h_{n,\infty}$ is topological mixing , then

$$h_n^m(M_1) \cap M_2 \neq \emptyset \text{ , for } m \geq m_2$$

We take $m \geq m_0 = max\,\{m_1 , m_2\}$, then we get

$$[(g_n \times h_n)^m(W_1 \times M_1)] \cap (W_2 \times M_2) = [g_n^m(W_1) \times h_n^m(M_1)] \cap (W_2 \times M_2)$$

, which implies that

$$[g_n^m(W_1) \cap W_2] \times [\, h_n^m(M_1) \ \cap M_2] \ \neq \emptyset$$

Then $g_{n,\infty} \times h_{n,\infty}$ is topological mixing . ■

Theorem 2.3.7.

Let $(X, g_{n,\infty})$ and $(Y, h_{n,\infty})$ be g-non autonomous discrete systems with $g_n: X \longrightarrow X$ and $h_n: Y \longrightarrow Y$ are chaotic (Devaney) and topological mixing maps . Then $g_n \times h_n: X \times Y \longrightarrow X \times Y$ is chaotic .

Proof :

Since g_n and h_n are two topological mixing maps , then by Theorem (2.3.6) we get $g_n \times h_n$ is topological mixing so $g_n \times h_n$ is topological transitive . By assumption g_n and h_n have dense periodic points , then by Theorem (2.3.1) we get $g_n \times h_n$ also have dense periodic points , by chaotic of (Devaney) , the density of periodic points with transitivity are satisfy sensitive dependent on initial conditions, so $g_n \times h_n$ is hold conditions chaotic . Then $g_n \times h_n$ is chaotic . ■

Theorem 2.3.8

Let $(X, g_{n,\infty})$ and $(Y, h_{n,\infty})$ be two g-non autonomous discrete systems . Then $g_{n,\infty} \times h_{n,\infty}$ is sensitive dependent on initial condition.

If and only if at least $h_{n,\infty}$ or $g_{n,\infty}$ is sensitive dependent on initial condition

Proof :

($\Leftarrow$ Let us assume that $g_{n,\infty}$ be sensitive dependent on initial condition , we show that $g_{n,\infty} \times h_{n,\infty}$ is so . let $\mathcal{P} = (x, y) \in X \times Y$ and U be neighborhood of $\mathcal{P}$, then there exists open neighborhood W of x and M neighborhood of y in Y such that $W \times M \subset U$. Since g_n is sensitive dependent on initial condition , there is $\epsilon > 0$ such that $x^{`} \in W$ and

$$n < m \in \mathbb{N} , d_1(g_n^m(x) , g_n^m(x^{`})) > \epsilon ,$$

any $y^{`} \in M$ , so $\mathcal{P}^{`}(x^{`}, y^{`}) \in U$ and

$$d((g_n \times h_n)^m(\mathcal{P}) , \left(g_n \times h_n)^m(\mathcal{P}^{`})\right) = d(g_n^m(x) \times h_n^m(y) , g_n^m(x^{`}) \times h_n^m(y^{`}) ,$$

which implies that

$$d((g_n \times h_n)^m(\mathcal{P}) , \left(g_n \times h_n)^m(\mathcal{P}^{`})\right) = \max\{d_1(g_n^m(x) , g_n^m(x^{`}), d_2(h_n^m(y) , h_n^m(y^{`}))$$

, that mean $d((g_n \times h_n)^m(\mathcal{P}) , \left(g_n \times h_n)^m(\mathcal{P}^{`})\right) \geq \epsilon$.

Then $g_{n,\infty} \times h_{n,\infty}$ is sensitive dependent on initial condition .

$\Longrightarrow$) Let neither $h_{n,\infty}$ nor $g_{n,\infty}$ be sensitive dependent on initial condition .

Then for any

$$\epsilon > 0 , \exists x \in X \ni d_1\left(g_n^m(x) , g_n^m(x^{`})\right) < \frac{\epsilon}{2}$$

when $x^{`} \in W$ and W open subsets containing x . Similarly for any

$$\epsilon > 0 , \exists\, y \in Y \ni d_2(h_n^m(y) , h_n^m(y^{`})) < \frac{\epsilon}{2}$$

when $y^{`} \in M$ and M open subsets containing y . Then

$$d\left((g_n \times h_n)^m(\mathcal{P}) , (g_n \times h_n)^m(\mathcal{P}^{`})\right) = d_1(g_n^m(x) , g_n^m(x^{`}) +$$

$$d_2\left(h_n^m(y) , h_n^m(y^{`})\right) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon$$

for $(x^{`}, y^{`}) \in W \times M$. Then $g_{n,\infty} \times h_{n,\infty}$ is not sensitive dependent on initial condition , which is contradiction . Then $g_{n,\infty}$ or $h_{n,\infty}$ is sensitive dependent on initial condition . ■

Theorem 2.3.9

Let $(X, g_{n,\infty})$and $(Y, h_{n,\infty})$ be g-non autonomous discrete systems with $g_n: X \longrightarrow X$ have the Touhey property and let $h_n: Y \longrightarrow Y$ be chaotic (Devaney) and topological mixing map . Then $g_{n,\infty} \times h_{n,\infty}$ is chaotic .

Proof :

Since $h_{n,\infty}$ is sensitive dependent on initial condition , then $g_{n,\infty} \times h_{n,\infty}$ also sensitive dependent on initial condition , on the other hand the Touhey property implies density of periodic points for $g_{n,\infty}$ by assumption $h_{n,\infty}$

also implies that the periodic points are dense , that mean $g_{n,\infty} \times h_{n,\infty}$ have dense periodic points . Therefore ,the topological transitive is hold by the product of two topological mixing are topological mixing ,then $g_{n,\infty} \times h_{n,\infty}$ topological transitive . Then $g_{n,\infty} \times h_{n,\infty}$ is chaotic maps (Devaney) . ■

Theorem 2.3.10.

Let $(X, g_{n,\infty})$ and $(Y, h_{n,\infty})$ be g-non autonomous discrete systems . Then $g_{n,\infty}$ and $h_{n,\infty}$ are minimal if and only if $g_{n,\infty} \times h_{n,\infty}$ is minimal .

Proof :

$\Longrightarrow$) Let $x \in X$ and $y \in Y$, then $\mathcal{P} = (x, y) \in X \times Y$, there is W an open subset in X and M an open subset in Y such that $U = W \times M$ an open subset in $X \times Y$. Since $g_{n,\infty}$ is minimal , then $\forall x \in X$, $\exists m_1 > n \in \mathbb{N}$ such that $g_n^{m_1}(x)$ is dense (i.e $g_n^{m_1}(x) \in W$) , on the other hand $h_{n,\infty}$ is minimal , then $\forall\ y \in Y$, $\exists m_2 > n$ such that $h_n^{m_2}(y)$ is dense in Y (i.e $h_n^{m_2}(y) \in M$) , Then we get $(g_n^{m_1}(x)\ , h_n^{m_2}(y)) \in U$, we choose $m = m_1 . m_2$, then

$$(g_n \times h_n)^m(\mathcal{P}) = (g_n^m(x)\ , h_n^m(y)) \in W \times \mathrm{M}$$

. Then $(g_n \times h_n)^m(\mathcal{P}) \in U = W \times M$ is dense . Then $g_{n,\infty} \times h_{n,\infty}$ is minimal .

($\Leftarrow$ Let $\emptyset \neq W$ be open subset of X and $\emptyset \neq M$ open subset of Y then $U = W \times M$ open subset of $X \times Y$. Since $g_{n,\infty} \times h_{n,\infty}$ is minimal , then

$$\forall \mathcal{P} = (x\,, y) \in X \times Y \,\exists\, m > n \in \mathbb{N} : (\, g_{n,\infty} \times h_{n,\infty})^m(\mathcal{P})$$

is dense (i.e $(g_{n,\infty} \times h_{n,\infty})^m(\mathcal{P}) \in U)$, so $g_n^m(x) \times h_n^m(y)$ are dense (i.e $g_n^m(x) \times h_n^m(y) \in U)$ so we get $g_n^m(x) \in W$, then $\{g_n^m(x)\}_{n=1}^{\infty}$ are dense in X , it followed we obtained $g_{n,\infty}$ are minimal . Also $h_n^m(y) \in M$, We get $\{h_n^m(y)\}_{n=1}^{\infty}$ is dense in Y , it followed we obtained $h_{n,\infty}$ is minimal . ■

Note : Since if h_n and g_n are transitive maps , then not necessary $h_n \times g_n$ is transitive , then so the totally transitive . The following proposition to show this fact :

Proposition 2.3.11

Let $(X\,, g_{n,\infty})$ and $(Y\,, h_{n,\infty})$ be two g-non autonomous discrete systems . If $g_{n,\infty} \times h_{n,\infty}$ is totally transitive ,then both of $g_{n,\infty}$ and $h_{n,\infty}$ are totally transitive .

Proof :

Let $\emptyset \neq W_1, M_1$ be open subsets of X , then $W = W_1 \times Y$ and $M = M_1 \times Y$ is open subsets of $X \times Y$. Since $g_{n,\infty} \times h_{n,\infty}$ is totally transitive , then $(g_{n,\infty} \times h_{n,\infty})^m$ is transitive , so

$(g_{n,\infty} \times h_{n,\infty})^m(W) \cap M \neq \emptyset$, that mean

$$g_n^m(W_1) \times h_n^m(Y) \cap M_1 \times Y \neq \emptyset$$

Which implies $\quad g_n^m(W_1) \cap M_1 \times h_n^m(Y) \cap Y \neq \emptyset$

Since $h_n^m(Y) \cap Y$ is non-empty . Then $g_n^m(W_1) \cap M_1 \neq \emptyset$. Then g_n^m is transitive , it followed g_n is totally transitive , By the same away we can show that $h_{n,\infty}$ is totally transitive . ■

Theorem 2.3.12

Let $(X, g_{n,\infty})$ and $(Y, h_{n,\infty})$ be two g-non autonomous discrete systems . Then $g_{n,\infty}$ and $h_{n,\infty}$ are locally eventually onto if and only if $g_{n,\infty} \times h_{n,\infty}$ is locally eventually onto .

Proof :

$\Longrightarrow$) Let $\emptyset \neq W$ be subset of X and $\emptyset \neq M$ be subset of Y , then $\emptyset \neq V = W \times M$ be subset of $X \times Y$. Since $g_{n,\infty}$ is locally eventually onto , then

$$\exists n < m_1 \in \mathbb{N} : \ g_n^{m_1}(W) = X$$

Since $h_{n,\infty}$ is locally eventually onto , then

$$\exists n < m_2 \in \mathbb{N} : \ h_n^{m_2}(M) = Y$$

We take and $m = \max\ \{m_1 , m_2\}$ That hold

$$(g_{n,\infty} \times h_{n,\infty})^m(V)(g_{n,\infty} \times h_{n,\infty})^m(W \times M)$$

$$= \ g_n^m(W) \times g_n^m(M)$$

$$(g_{n,\infty} \times h_{n,\infty})^m(V) \ = X \times Y$$

Then $g_{n,\infty} \times h_{n,\infty}$ is locally eventually onto .

($\Leftarrow$ Let $g_{n,\infty} \times h_{n,\infty}$ be a locally eventually onto , then for every

$V = W \times M$ non-empty subset of $X \times Y$, then

$$(g_{n,\infty} \times h_{n,\infty})^m(V) = X \times Y$$

$$(g_{n,\infty} \times h_{n,\infty})^m(W \times M) = X \times Y$$

Which hold

$$g_n^m(W) \times h_n^m(M) = X \times Y$$

So we obtained $g_n^m(W) = X$. Then $g_{n,\infty}$ is locally eventually onto , and

$h_n^m(M) = Y$. Then $h_{n,\infty}$ is locally eventually onto . ■

Chapter Three

Conclusion and Future work

Conclusions and Future work

3.1. Conclusions

In this work , we prove the following results :

1. Let $(Y, h_{n,\infty})$ be g -non autonomous discrete system and $\{Y_i\}_{i=1,\dots,n}$ be family of transitive sets where $Y_i \subseteq Y_j$ if i $\leq$ j , Then $\cup_{i=1}^{n} Y_i$ is transitive set .

2 . h_n is topological transitive when y^- is dense .

3. if h has periodic point . Then R$(h_{n,\infty}) \cap E(h_{n,\infty}) \neq \emptyset$.

4. Let $(Y, h_{n,\infty})$ and (Z,$g_{n,\infty}$) be two g -non autonomous discrete systems with $\pi : (Y.h) \rightarrow (Z, g)$ be factor map and B be weakly mixing subset of Y . Then $\pi(B)$ is weakly mixing subset of Z .

5 . when $h_n: Y \longrightarrow Y$ be equicontinuous piecewise monotone map and $h_{n,\infty}$ has dense homoclinic points . If $h_{n,\infty}$ are strongly blending . Then $h_{n,\infty}$ is locally eventually onto .

6 . If $h_n: Y \longrightarrow Y$ is locally eventually onto . Then $h_{n,\infty}$ is sensitive dependent on initial conditions .

7. If $h_n^m(y)$ is dense for all $y \in Y$, then $h_{n,\infty}$ is transitive .

8. Let $(Y, h_{n,\infty})$ and $(X, g_{n,\infty})$ be two g -non autonomous discrete systems and the set of periodic point of $g_{n,\infty} \times h_{n,\infty}$ is dense of $X \times Y$ if and only if both $g_{n,\infty}$ and $h_{n,\infty}$ the set of periodic points in X and Y are

dense .

9 . When $g_n: X \to X$ and $h_n: Y \to Y$ be a maps . Then h_n and g_n are strongly blending if and only if $g_n \times h_n$ is strongly blending .

10. $g_n: X \longrightarrow X$ and $h_n: Y \longrightarrow Y$ are continuous maps . Then g_n and h_n is satisfy Touhey property if and only if $g_n \times h_n$ is satisfy Touhey property .

11 . A, B are two non-empty transitive set . Then $A \times B$ also transitive .

12. if h_n is strongly blending and has dense periodic points , then h_n is satisfy Touhey property

13 . $g_n: X \longrightarrow X$ and $h_n: Y \longrightarrow Y$ are maps . Let $g_{n,\infty} \times h_{n,\infty}$ is topological transitive . Then the maps $g_{n,\infty}$ and $h_{n,\infty}$ are both topological transitive .

14 . $g_n: X \longrightarrow X$ and $h_n: Y \longrightarrow Y$ are topological mixing maps . Then also $g_{n,\infty} \times h_{n,\infty}$ is topological mixing .

15. $g_n: X \longrightarrow X$ and $h_n: Y \longrightarrow Y$ are chaotic(Devaney) and topological mixing maps . Then $g_n \times h_n: X \times Y \longrightarrow X \times Y$ is chaotic (Devaney) .

16 . $g_n: X \longrightarrow X$ have the Touhey property and let $h_n: Y \longrightarrow Y$ is chaotic (Devaney) and topological mixing maps . Then $g_{n,\infty} \times h_{n,\infty}$ is chaotic

17. $g_{n,\infty} \times h_{n,\infty}$ is sensitive dependent on initial condition.

If and only if at least $h_{n,\infty}$ or $g_{n,\infty}$ is sensitive dependent on initial

Condition .

18 . $g_{n,\infty}$ and $h_{n,\infty}$ are minimal if and only if $g_{n,\infty} \times h_{n,\infty}$ is minimal .

19 . If $g_{n,\infty} \times \ h_{n,\infty}$ is totally transitive ,then both of $g_{n,\infty}$ and $h_{n,\infty}$ are totally transitive .

3.2. Future work

In the future work , we will study anther chaotic properties and metric properties : expensive , Laypanouv exponent ,ect … of maps and sets . And We will generalized of them in g-non autonomous discrete systems , also

1. We find relationship between there maps .We try answer this question : Is the result in the maps are hold in the sets ?

2. We will generalize this chaotic properties by composition between the maps and the set in g −non autonomous discrete systems .

3. We will study the entropy in $g$non autonomous discrete systems .

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الملخص

الهدف من هذا العمل هو تعميم بعض الخصائص الفوضوية في الانظمة غير مستقلة g ومن هذه الخصائص هي خاصية التعدي التبولوجي حيث تم برهان : اتحاد عائلة منتهية من المجاميع المتعدية تكون متعدية ايضا . كما درسنا الخصائص اخرى : خاصية توهاي و الخلط القوي وكثافة النقاط الدورية وخاصية $(\mathcal{L}.e.o)$والحساسية المعتمدة على الشروط الابتدائية وخاصية ***equicontinuous*** عن طريق برهان ما يأتي :اذ كانت الدالة تمتلك خاصية الخلط القوي والنقاط الدورية كثيفة فأن الدالة تحقق خاصية توهاي . اذا كانت الدالة تحقق خاصية الخلط القوي ومجموعة نقاط ***homoclinic*** كثيفة فأن الدالة تحقق خاصية $(\mathcal{L}.e.o)$. اذا كانت الدالة تحقق خاصية $(\mathcal{L}.e.o)$ فأنها تكون حساسة عند الشروط الابتدائية . كما درسنا انتقال مجموعة المزج الضعيف وبرهنا اذا كانت مجموعة المزج الضعيف في المجال فأن صورتها كذلك تكون مجموعة مزج ضعيف في المجال المقابل عن طريق factor map .

بالإضافة الى تعميم جميع الخصائص اعلاه بواسطة الضرب الجدائي حيث تم برهان ما يأتي: دالتان تمتلكان نقاط دورية كثيفة اذا وفقط اذا كان الضرب الجدائي لهما يمتلك نقاط دورية كثيفة ايضا . اذا كانت دالتان تمتلكان خاصية الخلط القوي فأن الضرب الجدائي لهما يمتلك خاصية الخلط القوي والعكس صحيح ايضا . دالتان تحققان خاصية توهاي اذا وفقط اذا كانت الضرب الجدائي لهما يحقق خاصية توهاي ايضا . اذا كان الضرب الجدائي لدالتين يحقق التعدي التبولوجي فأن كلتا الدالتان تحققان خاصية التعدي التبولوجي والعكس غير صحيح . اذا كانت احدى الدوال تحقق خاصية توهاي والاخرى تحقق فوضوية ديفني مع المزج التبولوجي فأن الضرب الجدائي لهما يحقق فوضوية ديفني ايضا . اذا كانت هناك دالتان تحققان خاصية minimality فأن

الضرب الجدائي يحقق خاصية minimality والعكس صحيح ايضا . اذا كان الضرب الجدائي لدالتان يحقق خاصية الحساسية المعتمدة على الشروط الابتدائية فأن على الاقل الدالة الاولى او الثانية تمتلك الحساسية المعتمدة على الشروط الابتدائية والعكس صحيح . اذا كانت هناك دالتان يحققان خاصية المزج التبولوجي فأن الضرب الجدائي لهما يحقق المزج التبولوجي ايضا .

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من قبل

براء علي حسين العامري

بإشراف

أ.د.افتخار مضرطالب الشرع

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