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Analytic Study on Domination Graphs Via Fuzzy Graphs

A Dissertation
Submitted to the Council of the College of Education for Pure Sciences in University of Babylon in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Education / Mathematics

By

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بسم الله الرحمن الرحيم

يرفع الله الذين آمنوا منكم والذين أوتوا العلم درجات

صدق الله العظيم

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Thanks for everything…

Haneen H. Aljanaby
2022

Dedication

To the Prophet of Mercy “Mohammad”

To the moon, my father

To the sun, my mother

To the stars, my brother and sisters

Haneen H. Aljanaby

2022

Supervisor's Certificate

I certify that this dissertation "**Analytic Study on Domination Graphs Via Fuzzy Graphs**" by student "**Haneen Hamed Oda kreemsh**" was prepared under my supervision at the University of Babylon, Faculty of Education for Pure Sciences, in a partial fulfillment of the requirements for the degree of Doctor of Philosophy in Education / Mathematics.

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Contents

List of Symbols

Symbol	Description
$G(n, m)$ or $G(V, E)$	Graph of order n and size m
$V(G)$	Vertex set in a graph G
E(G)	Edge set in a graph G
$\|V\|$	Number of vertices in a graph G or G
| E |	Number of edges in a graph G or G
|A|	Cardinality of a set A
⌈A⌉	Cardinality integer ≥A (ceiling)
⌊A⌋	Cardinality integer ≤A (floor)
$deg_G(v)$ or $deg(v)$	Degree of vertex v in a graph G
$\delta(G)$	Minimum degree of vertices in G
$\Delta(G)$	Maximum degree of vertices in G
$d(v,u)$	Shortest path joining v and u
$N(v)$	Open neighborhood of v in a graph G
$N[v]$	Closed neighborhood of v in a graph G
$G[H]$ or $<H>$	Induced subgraph of a graph G
$\bar{G}$	Complement of graph G
$G_1 \cong G_2$	G_1 is isomorphic to G_2
$G_1 + G_2$	Join of G_1 and G_2
$G_1 \cup G_2$	Union of G_1 and G_2
$G_1 \odot G_2$	Corona of G_1 and G_2
P_n and $\overline{P_n}$	Path and complement of path graph
C_n and $\overline{C_n}$	Cycle and complement of cycle graphs
N_n	Null graph
K_n and $\overline{K_n}$	Complete and complement of complete graphs
$K_{n,m}$ and $\overline{K_{m,n}}$	Complete bipartite complement of bipartite graph
S_n and $\overline{S_n}$	Star and complement star graph
W_n and $\overline{W_n}$	Wheel and complement of wheel graphs

H_n and $\overline{H_n}$	Helm and complement of helm graphs
$T_{m,n}$	Tadpole graph
$L_{m,n}$	Lollipop graph
$D(n_1, n_2, \ldots, n_k)$	Daisies graph
D_n^m	Dutch windmill graph
D	Dominating set
D^c	Complement of dominating set
$\gamma(G)$	Domination number of G
$\gamma_t(G)$	Total domination number of G
$\gamma_{hn}(G)$	Hn-domination number of G
X	Finite and non-empty set
σ	Fuzzy subset of X
ρ	Fuzzy relation on X
G_f	Fuzzy graph
G_f^*	Crisp graph of G_f
P	Fuzzy order
q	Fuzzy size of G_f
P_{n_f}	Fuzzy path graph
C_{n_f}	Fuzzy cycle graph
$\overline{G_f}$	Complement fuzzy graph
I_f	Independent fuzzy set
K_{n_f}	Fuzzy complete graph
K_{m,n_f}	Fuzzy complete bipartite graph
W_{n_f}	Fuzzy wheel graph
H_{n_f}	Fuzzy helm graph
T_{m,n_f}	Fuzzy tadpole graph
L_{m,n_f}	Fuzzy lollipop graph
S_{n_f}	Fuzzy star graph

List of Abbreviations

Abbreviation	Description
DS	Dominating Set
OD^cS	Odd Neighbor in D^c Dominating set
MOD^cS	Minimal Odd Neighbor in D^c Dominating set
TOD^cS	Total Odd Neighbor in D^c Dominating set
$MTOD^cS$	Minimal Total Odd Neighbor in D^c Dominating set
FDS	Fuzzy Dominating Set
FOD^cS	Fuzzy Odd Neighbor in D^c Dominating set
$MFOD^cS$	Minimum Fuzzy Odd Neighbor in D^c Dominating set
$FTOD^cS$	Fuzzy Total Odd Neighbor in D^c Dominating set
$MFTOD^cS$	Minimum Fuzzy Total Odd Neighbor in D^c Dominating set

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ABSTRACT

Several different research papers were presented in graph and fuzzy graphs on the topic of domination. As it has been studied by researchers once for the graph and another time for the fuzzy graph. The domination in the fuzzy graph was found by the effective edge once and the strong edge again.

In this thesis, a set of new concepts in the graph and the fuzzy graph by focusing on the domination by the effective edge. Two concepts are studied in the graph. The first concept, the set D is called an odd neighbor in the D^Cdominating set as it focuses on the adjacent of each vertex in the dominating set on an odd number of vertices in the D^C of dominating set D. The other concept, the set D is called the total odd neighbor in the D^C dominating set, where this concept carries the properties of the first concept in addition to the characteristic of the total dominating set, which was defined by the researchers. As for the fuzzy graph, three concepts were studied as follows first, the set D is the fuzzy odd neighbor in the D^C dominating set, if every vertex in D is adjacent to the odd number of vertices in the fuzzy complement set of fuzzy dominating set D. As for the second, it is called the total odd neighbor in the D^C dominating set as in the second concept and then finding the least weight of the weights of the minimum fuzzy odd neighbor in the D^C dominating sets. In addition to the third and final type called fuzzy hn-dominating set it includes finding the fuzzy dominating set that fulfills the following condition that any two vertices in the complementary set of the fuzzy dominating set dominate two adjacent vertices in the complement of the fuzzy dominating set are also adjacent.

This modern method of finding fuzzy domination is complementary to finding domination and has many applications in life, such as commercial projects. These concepts have been studied on many types of graphs and fuzzy graphs in addition to some characteristics of each.

Publications

[1] Fuzzy hn-domination of strong fuzzy graphs. In AIP Conference Proceedings (Vol. 2386, No. 1, p. 060001). AIP Publishing LLC. January,2022.

[2] Some Properties of Odd Neighbor in D^c Dominating Sets. Applied Mathematics and Information Sciences. Vol. 14, p. 575-579. 2022

[3] Odd Neighbor in D^c Domination in Graphs. Accepted in AIP Conference Proceedings. June 2021.

Introduction

Graph theory is one of the best mathematical methods for solving various complex problems and herein lies its importance. Where problems and their relationships are represented by vertices and edges. In 1736, the foundation stone for graph theory was laid when scientist Euler published his paper on the well-known "Seven Bridges of Konigsberg" problem. It is used in all fields of science and engineering, for example, electrical networks, cryptography, wireless communications, image processing and computer science, as well as scientific fields including chemistry, physics, and life sciences (Balakrishnan and Ranganathan, 2012) [1] and (Berge,1962) [2].

In some cases, there are uncertainties in the description of the objects, in their relationships, or in both that must be modeled as a fuzzy graph. The fuzzy concept is a recent concept that emerged as recently as (Zadeh ,1965) [3] defined the fuzzy subgroup of a fragile group, as a generalization of the classical subgroup. It was entered on the graph by giving each vertex a specific value ranging from 0 to 1 in addition to giving values for the edges provided that their values do not exceed the values of the vertices connected to them.

In (Kaufmann, 1975) [4], the first definition of the fuzzy graph by was based on Zadeh's ambiguous relationships. Then in (Rosenfeld, 1975) [5], it developed. Fuzzy graph theory and its fuzzy analogues have been introduced to many graphs theoretical concepts such as branch graphs, paths, interconnections, bridges, vertices, forests and trees, etc. During the same time, (Yeh and Bang, 1975) [6] also introduced independent blur charts.

In (Bhutany, 1989) [7] discussed the instrumental forms of fuzzy graphs and introduced the concept of eccentricity and centerless in fuzzy graphs. In (Mordison, 1993) [8] introduced a fuzzy line graph. In (Mordison and Chang, 1994) [9] dealt with the processes and properties of fuzzy graphs. In (Mordeson and Nair, 1996 and 2012) [10-11] studied cycles and joint cycles and provided a definition of the fuzzy graph diagram.

In (Sunitha and Vijayakumar, 1999 and 2002) [12-13-14] introduced some properties of fuzzy bridges, fuzzy severed head and misty tree and got many results on metric spaces as they defined the complement of fuzzy graph in another way that gives a better understanding of this concept. In (Bhutani and Rosenfeld, 2003) [15] discussed strong arcs in fuzzy graphs.

Because of the great importance of both the graph and the fuzzy graph, this framework was researched in more detail and many topics were found, each of which has a specific application and benefit in our lives. What is the minimum number of pieces (queens specifically) to control the chessboard?" (Berge,1962) [2] identified the "Control number" in the graph in his book. In (Ore, 1962) [16] first used the terms “dominanting set” and “domination number” in his book. In (Cockayne and Hedetnemi, 1977) [17] published some findings about the dominating set. In (Haynes, Hedetniemi and Slater, 1998) [18] published a book in which the basics of excellent domination are presented in graphs. The definition of domination of vertices is the set of vertices that dominates all the vertices of the graph, and it is worth noting that each vertex dominates itself as well as all the vertices adjacent to it. Researchers have continued to this day to present and develop new concepts related to domination (Omran and Aljanaby, 2018) [19], (Al-harere and

Breesam, 2017) [20], (Jabor and Omran, 2017) [21] and (Al-Harere, Omran and Breesam, 2020) [22].

The concept of domination in fuzzy graphs was investigated by (A. Somasundaram and Somasundaram, 1998) [23] by using effective edges. While in (NagoorGani and Chandrasekaran,2006) [24] presented domination in fuzzy graph using strong arc. In (NagoorGani and Vadivel, 2009) [25] and (NagoorGani, and Devi, 2014) [26] were discussed some concepts of domination in fuzzy graph. In (Omran, and Ibrahim, 2021) [27] introduced Fuzzy co-even domination of strong fuzzy graphs.

The aim of this thesis is to find new parameters related to the subject of domination and study them in graphs and fuzzy graphs and study the relationship between them and selected parameters. In addition, to finding properties of each of them in the fuzzy graph and graph. We will discuss some of the contents of each chapter of the thesis, which consists of four chapters as follows:

Chapter One: In this chapter, deals with a basic definitions and concepts in graph and fuzzy graph that have been used in this work, which are prerequisites in constructing the properties relating to the study of this dissertation.

Chapter Two: In this chapter, new concepts such as odd neighbor in D^c dominating set and total odd neighbor in D^c dominating set, the relationships between them, and some boundaries that pertain to each concept are introduced.

Chapter Three: In this chapter, new domination fuzzy concept is introduced called fuzzy odd neighbor in D^c dominating set calculating their boundaries for different types of strong fuzzy graphs exclusively using effective edge and their relationships.

Chapter Four: Finally, the new fuzzy concepts to domination are introduced such as fuzzy Total odd neighbor in D^c dominating set and fuzzy Hn-dominating set to strong fuzzy graphs using the effective edge and finding some limits for the same fuzzy graphs to check for differences between the studied definitions.

Chapter One

Basic Definitions and Concepts

Chapter One

Basic Definitions and Concepts

1.1 Introduction

The first chapter includes the background, which is represented by some basic definitions that are specific to this thesis in addition to some theories and examples. It consists of three main sections as following the first section explains something related to the graph. The second section presents the concepts of the fuzzy graph. The last section deals with the concept of domination and some of the different definitions of domination with some properties.

1.2 Basic Definitions of graph theory

In this section, we introduce some of the graph concepts we rely on in the thesis, including the types of graphs that have been worked on.

Definition 1.2.1. [28]

A ***graph*** is an ordered triple $G(V(G), E(G))$ where $V(G)$ is a nonempty set, $E(G)$ is a set disjoint from $V(G)$; each element of $E(G)$ an unordered pair of elements (same or distinct) of $V(G)$. Elements of $V(G)$ are called the *vertices* (or *nodes* or *points*) of G; and elements of $E(G)$ are called the *edges* (or *lines*) of G. $V(G)$ and $E(G)$ are the *vertex set* and *edge set* of G, respectively.

Definition 1.2.2. [28]

The **order** of G is $n = |V(G)|$ and the **size** of G is $m = |E(G)|$.

Example 1.2.3.

In Fig.1.1, a graph G has $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ such that $|V(G)| = 5$ and $|E(G)| = 7$.

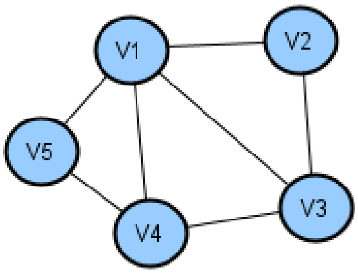


Figure 1.1. A graph of order five

Definition 1.2.4 [1]

A graph G is **trivial** if G has only one vertex and has no edges.

Definition 1.2.5 [28]

undirected graph *G* is a graph has all edges (or line) of G have no direction. Otherwise G is called **directed graph or digraph.**

Definition 1.2.6 [29]

An edge e has the same pair of end-vertices is called **loop.**

Definition 1.2.7 [29]

The two or more edges that have the same pair of end-vertices is called **Multiple edges.**

Definition 1.2.8 [30]

A Simple graph is graph that has no loops and multiple edges.

Definition 1.2.9 [30]

Any graph has $V(G)$ and $E(G)$ are finite, this graph be **finite graph**. Otherwise, it is called **infinite graph**.

Note that a graph in Fig 1.1 is simple, finite and undirected graph.

Definition 1.2.10 [31]

For any graph, **Adjacent vertices** are vertices linked by an edge.

Definition 1.2.11 [32]

For any graph G, If k of edges;$k \geq 2$ are called **adjacent edges** if have a common vertex. In addition, any vertex u joined to edge e, we said e is **incident** to u.

Definition 1.2.12 [31]

A **bridge** is an edge in a graph G whose removal increase the number of components.

Definition 1.2.13 [32]

The number of all edges that an incident to a vertex x is called the **degree** of x in a graph G with symbol $deg_G(x)$ or $deg\ (x)$. For example, in Fig. 1.1 $deg(v_1) = 4$.

Definition 1.2.14 [16]

Any vertex has maximum value we called its degree by **maximum degree** of G with symbol $\Delta(\boldsymbol{G})$. Similarly, any vertex has minimum value we called its degree by **minimum degree** of G with symbol $\boldsymbol{\delta}(\boldsymbol{G})$.

Definition 1.2.15 [16]

An **isolated vertex $\boldsymbol{u}$** of G is a vertex that not adjacent to all other vertices in graph (that is u has degree 0), A **pendant vertex** (or end-vertex or leaf) of G is a vertex adjacent to only one vertex in G(that means has degree 1). Its neighbor is called support vertex.

Definition 1.2.16. [18]

The set of all vertices adjacent to a vertex x is called **open neighborhood and** denoted by $N(x)$ of the vertex x is, $N(x) = \{y \in V: xy \in E(G)\}$. **The closed neighborhood** of u is equal to $N(u) \cup \{u\}$ with symbol $N[u]$.

Definition 1.2.17 [32]

A graph H is called **subgraph** of a graph $G = \big(V(G), E(G)\big)$ if the vertex set of H is subset of $V(G)$ and the edge set of H is subset of $E(G)$.

Definition 1.2.18 [1]

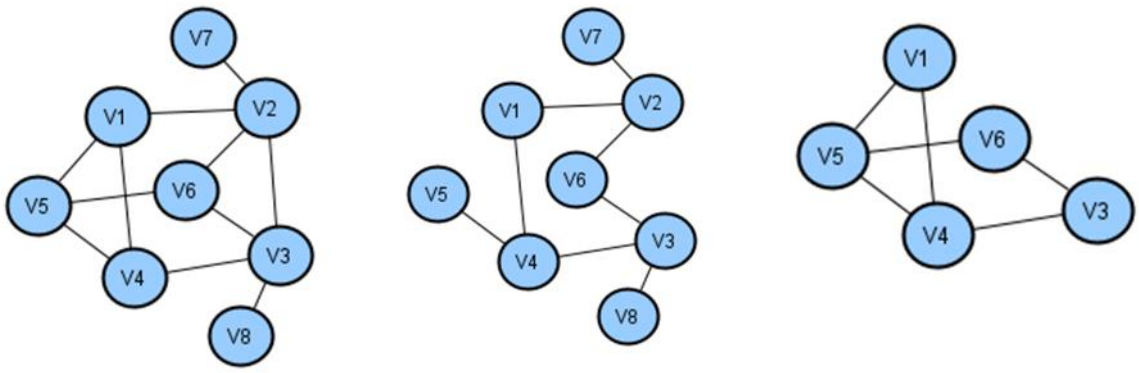
A subgraph H of G is called an **induced subgraph** of G if every line of G which has its ends in $V(H)$ is also line of H and is denoted by $\boldsymbol{G}[\boldsymbol{H}]$ also denoted by $\langle \boldsymbol{H} \rangle$.

Definition 1.2.19 [32]

A **spanning subgraph** is a subgraph containing all the vertices of G.

Example 1.2.20

In Fig.1.2 shown an induced subgraph and spanning subgraph.



a-A graph b- spanning graph c- An induced subgraph of G

Figure 1.2. A graph G with its subgraphs.

Definition 1.2.21 [33]

A graph G has n vertices is called **k-regular graph** if all vertices in G are adjacent to the same number of vertices, that is $deg(v) = k$ for every vertex v in G, such that $k \leq n - 1$.

Definition 1.2.22 [34]

For a graph G; $u, v \in V(G)$. A $u - v$ **walk** W in G is a sequence of vertices in G, such that beginning with u and ending with v such that consecutive vertices in the sequence are adjacent.

Definition 1.2.23 [34]

For a graph G; $u, v \in V(G)$. A $u - v$ walk in a graph which no vertices are repeated is called a **path** and denoted by $\boldsymbol{P_n}$.

Definition 1.2.24 [31]

A graph is obtained by joining the two pendent vertices of a path is called **a cycle**, such that the degree of all vertices in a cycle equal to two. A cycle is denoted by C_n.

Definition 1.2.25 [29]

A graph G is **connected** if $\forall\, u, v$ there is a path from u to v. Otherwise, G is **disconnected**.

Definition 1.2.26 [29]

The disjoint graphs G_1 and G_2, **the join** $G = G_1 + G_2$ is a graph which has a vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv: u \in V(G_1), v \in V(G_2)\}$. (see Fig.1.3).

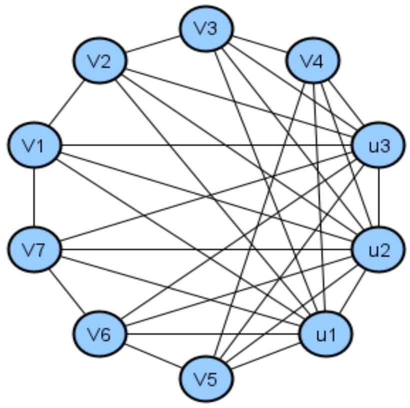


Figure 1.3. The join graph $G = C_7 + P_3$.

Definition 1.2.27 [34]

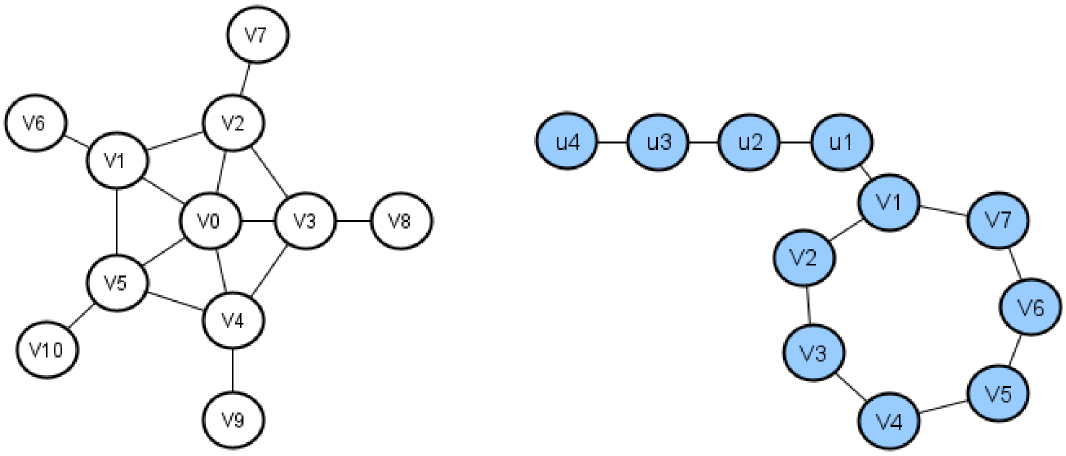
A **wheel** W_n with n vertices is the graph $C_{n-1} + K_1$.

Definition 1.2.28 [35]

A **helm** graph with $n \geq 3$ vertices, is the graph obtained from wheel W_m graph by creating a new vertex for each vertex of cycle C_{m-1} and denoted by $\boldsymbol{H_n}$ such that $n = m - 1$.

Definition 1.2.29 [36]

A **tadpole graph** is also called **dragon graph** $\boldsymbol{T_{m,n}}$ that obtained by adjacent C_m to P_n by single edge or we called bridge.



a- Helm graph H_5 b- Tadpole graph $T_{7,4}$

Figure 1.4. Helm and Tadpole graphs.

Definition 1.2.30 [37]

A **lollipop graph** $\boldsymbol{L_{m,n}}$ is a graph obtained by adjacent K_m to P_n by single edge or we called bridge.

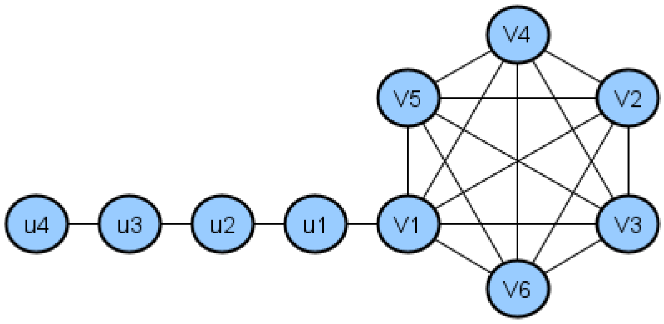


Figure 1.5. lollipop graph $L_{6,4}$.

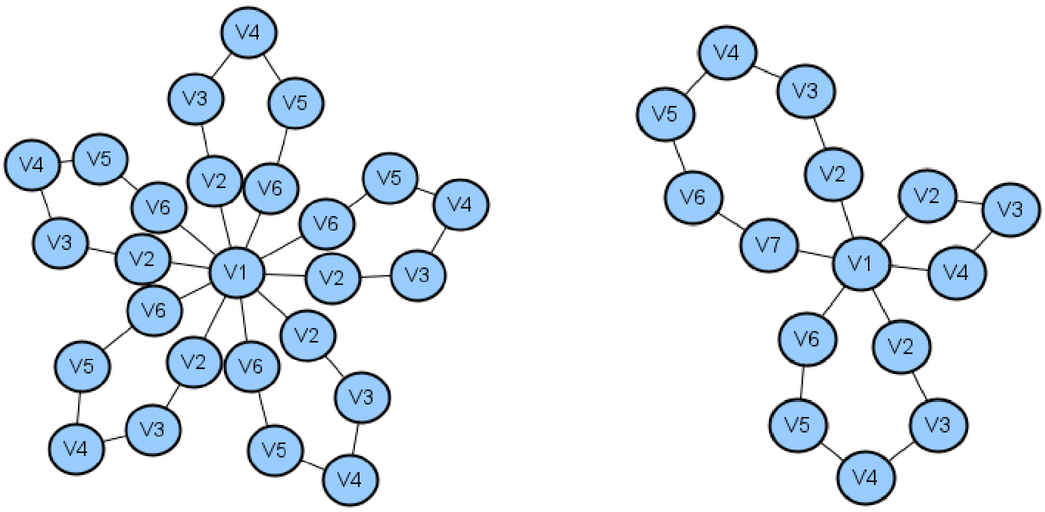
Definition 1.2.31 [38]

A **Dutch windmill graph** $\boldsymbol{D_n^m}$ is a graph obtained by taking m copies of the cycle C_n with a vertex in common. Dutch windmill graph D_n^m contains $(n-1)m+1$ vertices and mn edges.

Definition 1.2.32 [39]

A **daisies graph** $\boldsymbol{D(n_1, n_2, \dots, n_k)}$ is k cycles have lengths $n_1, n_2, \dots, n_k$ we denote the daisy graph.

Further, if $n_1 = n_2 = \cdots = n_k$ then, $D(n_1, n_2, \dots, n_k)$ is simply as $D_k(n_1)$ and it is also known as, Dutch windmill graph.



a- Dutch windmill graph D_6^5 b- Daisies graph $D(4,6,7)$

Figure 1.6. Dutch windmill and Daisies graphs.

Definition 1.2.33 [18]

For any graph G with vertex set V. The **complement** $\bar{G}$ of G the same vertex set of G, where every two vertices are adjacent in $\bar{G}$ if and only if these vertices are not adjacent in G.

Definition 1.2.34 [31]

If the edge set is empty in $G = (V, E)$ then a graph G called a **null graph** and denoted by $\boldsymbol{N_n}$ with n vertices.

Definition 1.2.35 [1]

A graph that no contains any cycle and connected, it is called a **tree** and denoted by T.

Definition 1.2.36 [32]

A **bipartite** graph G is a graph with set V that split into only two subsets V_1 and V_2 were every edge of G adjacent V_1 to V_2.

Definition 1.2.37 [31]

A bipartite graph G called **complete bipartite** if $\forall\, v \in V_1$ is adjacent to all vertices $u \in V_2$ and it is denoted by $\boldsymbol{K_{m,n}}$.

Definition 1.2.38 [16]

A complete bipartite $K_{m,n}$ is called **star** if $K_{1,n-1}$, $n \geq 2$ and denoted by S_n.

Definition 1.2.39 [40]

A set $N \subseteq V(G)$ for a graph G is **independent** if all vertices in N are not adjacent.

Definition 1.2.40 [34]

A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is a **component** of G.

Definition 1.2.41 [31]

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. **the union** of G_1 and G_2, denoted by $G_1 \cup G_2$, is another graph $G = (V, E)$ whose vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

Definition 1.2.42 [29]

A graph G is **complete** if every two distinct vertices of G are adjacent and denoted by $\boldsymbol{K_n}$.

Definition 1.2.43 [1]

A graph $G_1 = (V_1, E_1)$ is said to be **isomorphic** to the graph $G_2 = (V_2, E_2)$ if there is a one-to-one correspondence between the vertex sets V_1 and V_2 and a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1 is an edge with end vertices $u_1$and u_2 in G_1 then the corresponding edge $e_2$in G_2 has its end vertices v_1 and v_2 in G_2 which corresponds to u_1 and u_2, respectively. Such a pair of correspondence is called a graph isomorphism and denoted by $(G_1 \cong G_2)$.

Definition 1.2.44 [29]

The **corona product** of two graphs G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2. And denoted by $G_1 \odot G_2$.

Proposition 1.2.45 [35]

Let $G = (V, E)$ be a graph and let R be super hereditary property. Then a set D is a 1-minimal R-set iff D is a minimal R-set.

1.3 Fuzzy Graph

In this section we present some of them that were worked on in the thesis.

Definition 1.3.1 [3]

For set $X \neq \emptyset$, let $\sigma : X \longrightarrow [0,1]$ be mapping. Then σ is called **fuzzy subset of** a set X.

Definition 1.3.2 [3]

For any fuzzy subset σ of X. A relation ρ on X define by $\rho: X \times X \rightarrow [0,1]$ is called **fuzzy relation** on fuzzy subset σ if $\rho\,(x,y) \leq min\{\sigma(x), \sigma(y)\}, \forall x, y \in X$. A fuzzy relation ρ is a symmetric**.**

Definition 1.3.3 [15]

Let $V \neq \emptyset$. **A fuzzy graph** is represented by pair of the following functions $\sigma: V \rightarrow [0,1]$ and $\rho: V \times V \rightarrow [0,1]$ where $\rho(x,y) \leq min\,\{\sigma(x), \sigma(y)\}\ \forall x, y \in V$ and denoted by $G_f = (\sigma, \rho)$.

Definition 1.3.4 [15]

For any fuzzy graph $G_f = (\sigma, \rho)$. Let $\sigma^* = \{x \in V: \sigma(x) > 0\}$ and $\rho^* = \{(x,y) \in E(G_f)$ are functions and $\rho^*(x,y) = \sigma^*(x) = \sigma^*(y) = 1$ $\forall x, y \in V$ (that means all weights of vertices and all weights of edges in G^* are equal to 1), then ${G_f}^* = (\sigma^*, \rho^*)$ is called **the crisp (or underlying) graph** that is a special case of a fuzzy graph $G = (\sigma, \rho)$. (see Fig, 1.7).

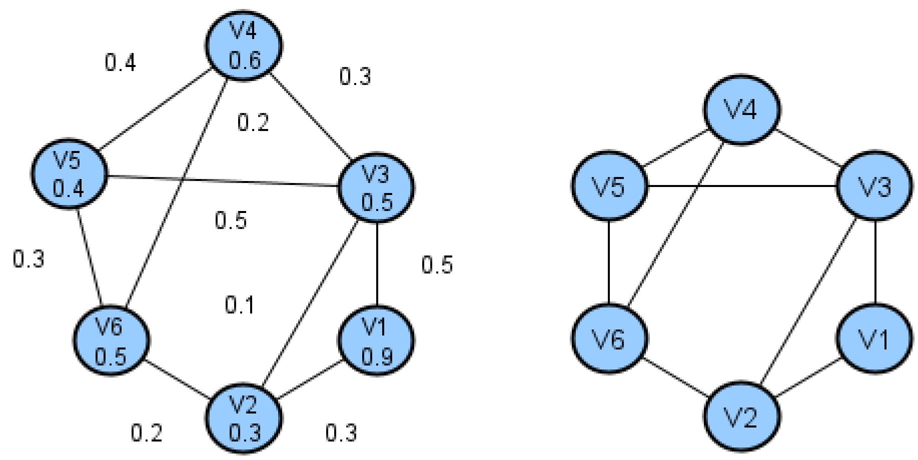


Figure 1.7. Fuzzy graph and crisp graph

Definition 1.3.5 [41]

For any fuzzy graph $H = (\vartheta, \tau)$ is called **a fuzzy subgraph** of $G_f = (\sigma, \rho)$ if $\vartheta \subseteq \sigma$ and $\tau \subseteq \rho$, that is $\vartheta(x) \le \sigma(x)$ $\forall$ $x \in V$ and $\tau(x, y) \le \rho(x, y)$ for all $e = (x, y) \in E$.

Definition 1.3.6 [41]

For any fuzzy graph $H = (\vartheta, \tau)$ is called **a fuzzy spanning subgraph** of $G_f = (\sigma, \rho)$ if $\vartheta(x) = \sigma(x)$ $\forall$ $x \in V$, and $\tau(x, y) \le \rho(x, y)$ for all $e = (x, y) \in E$.

Definition 1.3.7 [42]

The fuzzy cardinality of subset A of $V(G_f)$ is $\sum_{x \in A}$ (x), and denote the cardinality and fuzzy cardinality of $A \subseteq V(G_f)$ by $|A|, ||A||$ respectively.

Definition 1.3.8 [43]

For any fuzzy graph $G_f = (\sigma, \rho)$. The **order (we will call fuzzy order)** of G_f is represented by $P = \sum_{v \in V(G_f)} \sigma(v)$.

Definition 1.3.9 [43]

For any fuzzy graph $G_f = (\sigma, \rho)$. The **size ((we will call fuzzy size)** of G_f is represented by $q = \sum_{v \neq u} \rho(u, v)$.

In Fig. 1.7 $P = 3.2$ and $q = 2.8$.

Definition 1.3.10 [42]

A line $e = (x, y)$ of a fuzzy graph G_f is called an **effective edge** if $\rho(x, y) = min\,\{\sigma(x), \sigma(y)\}$. In Fig.1.7 the edges v_1v_2, v_1v_3 *and* v_4v_5 are effective edges

Definition 1.3.11 [46]

For any fuzzy graph $G_f = (\sigma, \rho)$ is called to be **fuzzy null graph** if $\rho(x, y) = 0$ $\forall x, y \in V$

Definition 1.3.12 [44]

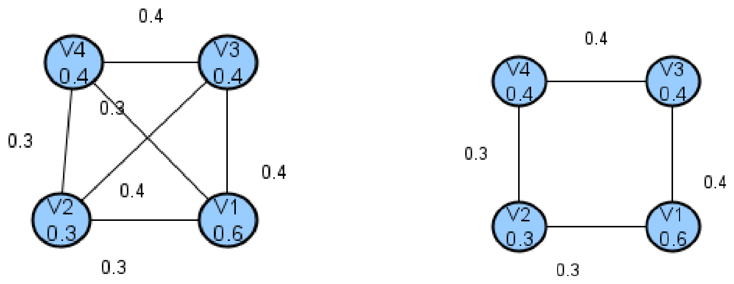
For any fuzzy graph $G_f = (\sigma, \rho)$ is called to be **strong fuzzy graph** if $\rho(x, y) = min\,\{\sigma(x), \sigma(y)\}$ $\forall (x, y) \in E(G_f)$.

Definition 1.3.13 [45]

The set **open neighborhood** of a node x in $G_f = (\sigma, \rho)$ is define as $N(x) = \{y \in V: (x, y)$ an effective edge$\}$. The **closed neighborhood** represents by $N[x] = N(x) \cup \{x\}$. For example, in Fig.1.7 $N(v_5) = \{v_3, v_4\}$.

Definition 1.3.14 [46]

A fuzzy graph $G_f = (\sigma, \rho)$ is called a **fuzzy complete graph** if $\rho(x, y) = min\,\{\sigma(x), \sigma(y)\}\ \forall x, y \in V(G_f)$ and denoted by K_n.



a- Complete fuzzy graph b- Strong fuzzy graph

Figure 1.8. The different of complete and strong fuzzy graph

Definition 1.3.15 [46]

The **complement** of a fuzzy graph $G_f = (\sigma, \rho)$ is $\overline{G_f} = (\bar{\sigma}, \bar{\rho})$**,** where $\bar{\sigma} = \sigma$ and $\overline{\rho\ (x, y)} = min\{\sigma(x), \sigma(y)\} - \rho(x, y)\ \forall\ x, y \in V$.

a- Fuzzy graph G_f b- complement fuzzy graph $\overline{G_f}$

Figure 1.9. Fuzzy graph G_f and its complement $\overline{G_f}$

Definition 1.3.16 [47]

A **fuzzy path** in a fuzzy graph $G_f = (\sigma, \rho)$ is a sequence of distinct nodes $x_0, x_1, \dots, x_n$ $(n \geq 2)$, $x_1 \neq x_n$ such that $\rho(x_{i-1}, x_i) > 0,\ 1 \leq i \leq n$ and denoted by P_{n_f}.

Definition 1.3.17 [47]

A fuzzy graph $G_f = (\sigma, \rho)$ is said to be **connected** if any two vertices are joined by a fuzzy path. Otherwise, it is **disconnected**.

Definition 1.3.18 [48]

A fuzzy graph $G_f = (\sigma, \rho)$ is called a **fuzzy cycle** if $G_f{}^* = (\sigma^*, \rho^*)$ is a cycle and denoted by C_{n_f}.

Definition 1.3.19 [48]

A fuzzy graph $G_f = (\sigma, \rho)$ is called **tree** if $G_f{}^* = (\sigma^*, \rho^*)$ is a tree and denoted by T_f.

Definition 1.3.20 [48]

A vertex x is said to be **end or pendant vertex** in $G_f = (\sigma, \rho)$ if it has exactly one effective neighbor in G_f.

Definition 1.3.21 [49]

A vertex x of a fuzzy graph $G_f = (\sigma, \rho)$ is called an **isolated vertex** if $\rho(x, y) < min\,\{\sigma(x), \sigma(y)\}$ for all $y \in V - \{x\}$, that is $N(x) = \emptyset$.

Definition 1.3.22 [49]

A set I_f of vertices of a fuzzy graph is called an **fuzzy independent set** if $\rho(x, y) < min\,\{\sigma(x), \sigma(y)\}$, for all $x, y \in I_f$.

Definition 1.3.23 [50]

An independent fuzzy set I_f of G_f is said to be **maximal fuzzy independent set** if for every vertex $x \in V - I_f$, the set $I_f \cup \{x\}$ is not independent fuzzy set.

Definition 1.3.24 [51]

A fuzzy graph $G_f = (\sigma, \rho)$ is called **bipartite** if the node set V can be split into two nonempty sets V_1 and V_2 such that if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$, then $\rho(v_1, v_2) = 0$. If $\rho(v_1, v_2) = min\{\sigma(v_1), \sigma(v_2)\}$ for all $v_1 \in V_1$ and $v_2 \in V_2$, then $G_f = (\sigma, \rho)$ is called **fuzzy complete bipartite graph** with symbol K_{m,n_f}.

Definition 1.3.25 [52]

A fuzzy graph $G_f = (\sigma, \rho)$ is called a f**uzzy star** if consist of two vertex sets X and Y with $|X| = 1$ and $|Y| > 1$ such that $\rho(x, y_i) > 0$ and $\rho(y_i, y_{i+1}) = 0$, $1 \leq i \leq n$ and denoted by K_{1,n_f}.

Definition 1.3.26 [53]

A **fuzzy wheel graph** W_{n_f} is a fuzzy graph that is formed from C_{n-1_f} by adding one node and adjacent by single edge to every vertex of C_{n-1_f}.

Definition 1.3.27 [54]

A Helm graph is a **fuzzy Helm graph** $\boldsymbol{H_{n_f}}$ consists of two vertex sets U and V with $U = \{u_1, \dots, u_n, u_{n+1}, u_{n+2}, \dots, u_{2n}\}$ and $|V| = 1$ such that $\sigma: V \rightarrow [0,1]$ and $\sigma: U \rightarrow [0,1]$, $\rho(v, u_i) > 0$, $\rho(u_i, u_{n+1}) > 0$ and $\rho(u_i, u_{i+1}) = 0$ for $1 \leq i \leq n$.

Definition 1.3.28 [54]

A **fuzzy lollipop graph** $\boldsymbol{L_{m,n_f}}$ is a fuzzy graph obtained by adjacent K_{m_f} to P_{n_f} by edge such that $\rho(u_i, v_1) \leq min\{\sigma(u_i), \sigma(v_1)\}$; $u_i \in V\left(K_{m_f}\right)$ and $v_1 \in V(P_{n_f})$.

1.4 Domination in graph and fuzzy graph

In fuzzy graph, Somasundaram and Somasundaram [23] discussed about domination in fuzzy graph. In this section, we introduce some of the domination fuzzy graph with graph concepts as following.

Definition 1.4.1 [7]

A set $D \subseteq V$ in a graph $G = (V, E)$ is called a **dominating set** if every vertex $v \in V$ is either an element of D or is adjacent to an element of D.

Definition 1.4.2 [18]

A dominating set D is called a **minimal dominating set** if there is no subset $S \subset D$ is dominating set.

Definition 1.4.3 [19]

The **domination number** $\gamma(G)$ of a graph G is the minimum cardinality of a set of minimal dominating set of G. Such a set is called the γ −set of G.

Definition 1.4.4 [23]

Let $G_f = (\sigma, \rho)$ be fuzzy graph, $D \subseteq V(G_f)$ is called a **fuzzy dominating *set*** (FDS) of G if $\forall\ v \notin D\ \exists\ u \in D$ such that $\rho(u, v) = min\,\{\sigma(u), \sigma(v)\}$ and we say u is dominates v or v dominated by u.

Definition 1.4.5 [56]

A fuzzy dominating set D of $G_f = (\sigma, \rho)$ is called a **minimal fuzzy dominating set** if there is no subset of D is a fuzzy dominating set of G_f.

Definition 1.4.6 [56]

A fuzzy dominating set of a fuzzy graph $G_f = (\sigma, \rho)$ with minimum number of vertices is called a **minimum fuzzy dominating set** (MFDS) of G_f.

Definition 1.4.7 [56]

A **fuzzy domination number** of a fuzzy graph $G_f = (\sigma, \rho)$ is the minimum sum of membership values of the vertices for all minimum fuzzy dominating sets and denoted by $\gamma_f(G_f)$ or simply γ_f.

Definition 1.4.8 [57]

$D \subseteq V(G)$ is called a **total dominating set (TDS)** of G if $\langle D \rangle$ has no isolated vertices.

Definition 1.4.9 [57]

The **total domination number** of G equals the minimum cardinality of a TDS of G with symbol $\gamma_t(G)$.

Definition 1.4.10 [58]

For any graph G. The dominating set is called **hn-dominating set** if for all adjacent for all two vertices u, v in D^c there are two adjacent $w, z \in D$ such that u is adjacent to$w$ and v is adjacent to z (may be $w = z$).

Definition 1.4.11 [58]

For any graph G has hn-dominating set D. The set D is called **minimal hn-dominating set** if it has no proper hn-dominating set.

Definition 1.4.12 [58]

The **hn-domination number** $\gamma_{hn}(G)$ of G is a minimum cardinality of a minimal hn-dominating set.

Definition 1.4.13 [58]

A set D is called $\boldsymbol{\gamma_{hn}}$ **−set** in case it is hn-dominating set with cardinality $\gamma_{hn}(G)$.

Proposition 1.4.14

Let G be a graph has k support vertices, then $\gamma(G) \geq k$.

Proof.

Let D be minimum dominating set of G. Suppose that $|D| < k$, then there are at least one of support vertices is adjacent to vertex from D. that's mean it's isolated vertex not adjacent to any vertex from D and this vertex is not dominated by any vertex of dominating set, then D is not dominating set. But that contradiction. So, we get $\gamma(G) \geq k$.

Chapter Two

Odd Neighbor in D^c Domination Number

Chapter Two

Odd Neighbor in D^c Domination in Graphs

2.1 Introduction

Two new definitions of domination are presented as odd neighbor in D^c domination (OD^cS) and total odd neighbor in D^c domination (TOD^cS) and the relationships between them were found, in addition to studying some of the parameters of each.

2.2. $OD^cS -$ Dominating Set of Graph

This part of the chapter studies the new concept an odd neighbor in D^c domination and focus on some of its characteristics while studying a number of selected diagrams.

Definition 2.2.1

Let G be a graph and $D \subseteq V$ is a dominating set (briefly DS), if $|N(v) \cap (V - D)| = 0$ or odd $\forall v \in D$, then D is called **odd neighbor in D^c dominating set** (briefly OD^cS).

Definition 2.2.2

For a graph G, D is OD^cS. The set D is called **minimal odd neighbor in D^c dominating set** (MOD^cS)if it has no proper MOD^cS. (See Fig. 2.1).

Definition 2.2.3

The minimum cardinality of all D is called the **odd neighbor in D^c domination** and denoted by $\gamma_{odc}(G)$.

Definition 2.2.4

A set D is called $\boldsymbol{\gamma_{odc}}$ **−set** in case it is MOD^cS with cardinality $\gamma_{odc}(G)$.

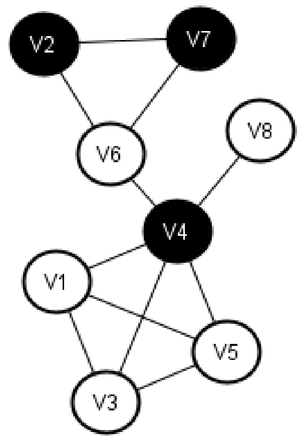


Figure 2.1. MOD^cS of a graph of order 8.

Observation 2.2.5

let G be a graph and D be OD^cS, then

1) If G that has an isolated vertex, then this vertex belonging to each OD^cS.

2) The induced subgraph $\langle D \rangle$ may have an isolated vertex.

Proposition 2.2.6

If every vertex of a graph G has even degree and D is an OD^cS, then there is no isolated vertex in an induced subgraph $\langle D \rangle$.

Proof.

Suppose that there is an isolated vertex say v in the induced subgraph $\langle D \rangle$, then there are two cases one of them that $deg(v) = 0$, and this a contradiction. The other case that all neighborhoods of this vertex are in the set $V - D$, again this is a contradiction with the fact that the set D is OD^cS. Thus, the result is obtained. (See Fig. 2.2).

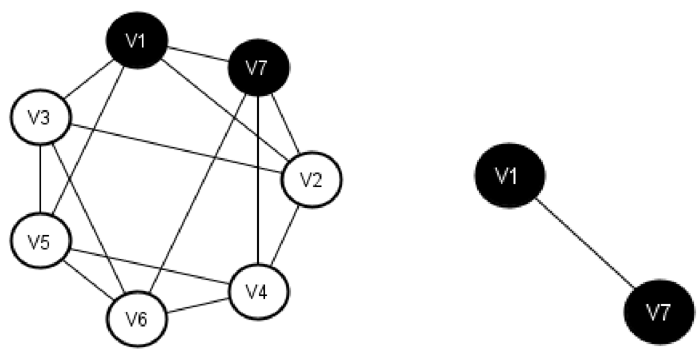


Figure 2.2. $4-regular$ graph and an induced subgraph of OD^cS.

Proposition 2.2.7

Let G be a graph and D is an OD^cS such that the induced subgraph $\langle D\rangle$ is independent, then the degree of all vertices of D is odd.

Proof.

Let the induced subgraph $\langle D\rangle$ be independent and $v \in D$, then all neighbors of the vertex v are in the set $V - D$, and $deg(v) = |N(v)|$. Now, if $|N(v)|$ is even, then we get a contradiction with Definition 2.2.1, thus $|N(v)|$ is odd and gets the result.

Remark 2.2.8

The converse of Proposition 2.2.7., not necessarily true. The counterexample as in the following figure, which give the graph G has odd dominating set D and the degree of all vertices is odd, but the induced subgraph $\langle D\rangle$ is not independent, where $D = \{v, w, z\}$. (See Fig. 2.3)

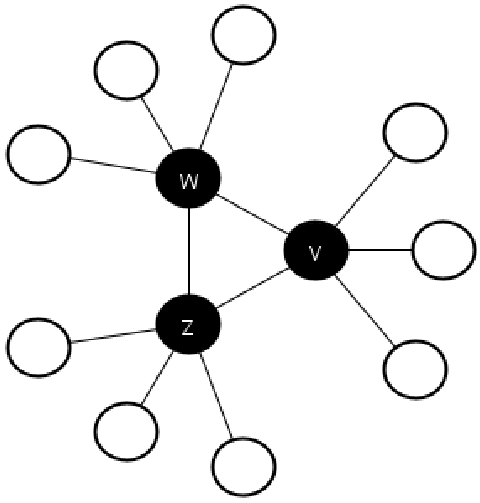


Figure 2.3. The graph with OD^cS.

Proposition 2.2.9

Let G be a graph of order n has an OD^cS and $k = 0,1, \dots, n-2$, then

1) $\forall v \in D,$ if $deg(v) = m$ such that m is odd, then v either adjacent to m or $m - 2k$ vertices in $V - D$.

2) $\forall v \in D,$ if $deg(v) = h,$ such that h is even, then either adjacent to $h - (2k + 1)$ vertices or h vertices in D.

Proof.

Case 1. Let $\forall v \in D, deg(v) = m,$ such that m is odd, then D is OD^cS by two subcases.

Subcase1. If a vertex v is adjacent to m vertices in $V - D$, the case above holds.

Subcase 2. If a vertex v is adjacent to number of vertices less than m in $V - D$, then if v is adjacent to $m - 2k;\ k = 0,1, \dots, n-2$ vertices in $V - D$ and since $m - 2k$ is odd. Then we get the result.

Case 2. Let $\forall v \in D, deg(v) = h$, such that h is even, then if a vertex v is adjacent to odd vertices in the induced subgraph $\langle D \rangle$, then $|N(v) \cap (V - D)|$ is odd, then the set D is OD^cS, otherwise the set D is not OD^cS.

By previous cases, the result is obtained.

Proposition 2.2.10 Let P_n be a path graph with n vertices, so $\gamma_{odc}(P_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proof.

Let $\{v_1, v_2, \dots, v_n\}$ be vertices of P_n. So, three cases are discussed below.

Case 1. If $n = 1,2$, so the MOD^cS is $D = \{v_1\}$. Thus, $\gamma_{odc}(P_n) = 1$.

Case 2. If $n = 3$, then the MOD^cS is $D = \{v_1, v_3\}$. Thus, $\gamma_{odc}(P_n) = 2$.

Case 3. If $n > 3$, then there are two subcases as follows.

Subcases 1. Suppose that $D_1 = \{v_{2+4i}, v_{3+4i}; i = 0, \dots, \left\lceil \frac{n}{4} \right\rceil - 1\}$. There are two cases:

1) If $n \equiv 0 \ mod\ 4$, so let $D = D_1$.

2) If $n \equiv 1,2 \ mod\ 4$, so let $D = D_1 \cup \{v_n\}$.

Subcases 2. If $n \equiv 3 \ mod\ 4$, so $D = \{v_{2+4i}, v_{3+4i}; i = 0, \dots, \left\lceil \frac{n}{4} \right\rceil - 1\}$.

In each above three subcases mentioned earlier, it is obvious that the set D is a DS and all vertices in the set D are adjacent to at most one vertex of the set $V - D$. Therefore, D is an OD^cS. Now, to prove that the set D is MOD^cS, it is enough to prove that the set $D^* = D - \{v\}$ is not OD^cS, since an OD^cS is super hereditary. To do these three cases that depend on the deletion of a vertex are discussed below.

I) If $v = v_n$, then D^* is not OD^cS, since it not dominates the vertex v_n.

II) If $v = v_{2+4i}, i = 0, \dots, \left\lceil \frac{n}{2} \right\rceil - 1$, then the vertex v_{3+4i} in D^* is join with two vertices. Thus, D^* is not OD^cS.

III) If $v = v_{3+4i}, i = 0, \dots, \left\lceil \frac{n}{2} \right\rceil - 1$, then the vertex v_{2+4i} in D^* is join with two vertices. Thus, D^* is not OD^cS.

Therefore, depending on I, II, and III are mentioned earlier and by proposition 1.2.44, D^* is not OD^cS. Thus, the set D is MOD^cS and $\gamma_{odc}(P_n) = \left\lceil \frac{n}{2} \right\rceil$. □

Proposition 2.2.11 If C_n is cycle graph, then $\gamma_{odc}(C_n) = \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil, & if\ n \equiv 1,2,3\ mod\ 4 \\ \frac{n}{2}, & if\ n \equiv 0\ mod\ 4 \end{cases}$

Proof.

Two cases are discussed below.

Case1. If $n \equiv 1,2,3\ mod\ 4$, let $D = \{v_{2+4i}, v_{3+4i}; i = 0, \dots, \left\lceil \frac{n+1}{4} \right\rceil - 1\}$. It is clear that the set D is DS and each vertex in the set D is adjacent to at most one vertex of the set $V - D$. Thus, the set D is an OD^cS. To prove that the set D is MOD^cS, it is enough to prove that the set $D^* = D - \{v\}$ is not OD^cS, since an OD^cS is super hereditary. Suppose that $D^* = D - \{u\}$, then there is at least one vertex adjacent to two vertices from $V - D$, then D^* is not OD^cS. So, D is the MOD^cS with $\gamma_{odc}(C_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Case2. If $n \equiv 0\ mod\ 4$, let $D = \{v_{2+4i}, v_{3+4i}; i = 0, \dots, \frac{n}{4} - 1\}$, one can easily conclude that D is OD^cS. Since, every vertex in D is adjacent to at most one vertex in $V-D$, then D OD^cS. By the same manner in case1, then D is MOD^cS with $\gamma_{odc}(C_n)=\frac{n}{2}$.

From the cases mentioned earlier, the required is getting.

Proposition 2.2.12 If W_n is a wheel graph with n vertices, then $\gamma_{odc}(W_n)=$

$$\begin{cases} 1 & , \ if\ n\ is\ even \\ 2, & if\ n=5 \\ \left\lfloor \frac{n}{2} \right\rfloor -1 & , \ if\ n\ \ is\ odd\ and\ n=7,9 \\ 4 & , \ if\ \ n\ is\ odd\ and\ n \geq 11 \end{cases}$$

Proof.

There are two cases as follows.

Case1. If n is even, let v_1 be a vertex of K_1. Since $D=\{v_1\}$ is DS and v_1 is adjacent to $n-1$ vertices in $V-D$ and since $n-1$ is an odd number, then D is OD^cS and it is clear that D is MOD^cS, so, $\gamma_{odc}(W_n)=1$.

Case2. If n is odd, there are three subcases as follows.

Subcase1. If $n=5$, then $D=\{v_2,v_5\}$ is MOD^cS. $\gamma_{odc}(W_5)=2$.

Subcase2. If $n=7$ *or* 9, let $D=\{v_{2+2i}; i=0,\dots,\left\lfloor \frac{n}{2} \right\rfloor -2\}$, it is obvious that $D \subseteq V(C_{n-1})$ and it is DS of the graph W_n. Now, $\forall i$ v_i in D dominates to two vertices in C_{n-1} plus v_1, then every vertex in D dominates to an odd number of vertices in $V-D$, thus, D is OD^cS. To prove that the set D is MOD^cS, let $D^*=D-\{v\}$, then there is at least one vertex not dominated by any vertex in D^*, then D^* is not DS. And so, it not OD^cS. Thus, by using Proposition 1.1, D is MOD^cS. Thus, $\gamma_{odc}(W_n)=\left\lfloor \frac{n}{2} \right\rfloor -1$.

Subcase3. If $n \geq 11$, let $D=\{v_1,v_2,v_3,v_n\}$ be DS. Since v_1 is adjacent to $n-4$ in D^c and v_3 and v_n are adjacent to only one vertex in $V-D$ and

$N[v_2] = D$. So D is OD^cS. For any subset $D^* = D - \{v\}$ is not OD^cS because any vertex in D^* adjacent to even number. Then, $\gamma_{odc}(W_n) = 4$.

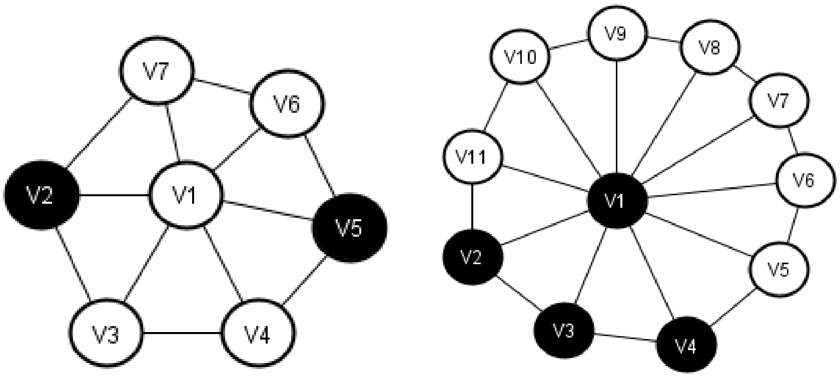


Figure 2.4. MOD^cS of wheel graphs w_7 and w_{11}.

Proposition 2.2.13 Consider K_n be a complete graph, so $\gamma_{odc}(K_n) = \begin{cases} 1 & , if\ n\ is\ even \\ 2 & , if\ n\ is\ odd \end{cases}$

Proof.

Three cases are discussed below.

Case1. let n be even, so every vertex has $n - 1$ degree (odd), then $\gamma_{odc}(K_n) = 1$.

Case2. If n is odd, then let $D = \{v, u\}; u\ and\ v \in K_n$, it is clear that D is the DS of the graph K_n. Every vertex in D is adjacent to $n - 2$ vertices and $n - 2$ is odd, then D is OD^cS. Now, if there is a dominating set contains only one vertex, then the cardinal of the neighborhood of it is even which mean this DS is not OD^cS. Thus, $\gamma_{odc}(K_n) = 2$.

Proposition 2.2.14 Consider $K_{m,n}$ be a complete bipartite graph, then

$$\gamma_{odc}(K_{m,n}) = \begin{cases} 1, & if\ n = 1\ and\ m\ is\ odd \\ 2, & if\ n = 1\ and\ m\ is\ even\ or \\ & n = 2\ and\ m\ is\ odd\ or \\ & n\ and\ m\ are\ even \\ 3, & if eithor\ n\ or\ m\ is\ odd\ ; n, m \neq 1 \\ 4, & if\ m\ and\ n\ are\ odd; n, m \neq 1 \end{cases}$$

Proof.

By definition of $K_{m,n}; n \leq m$, $V(K_{m,n}) = V_m \cup V_n$ such that $V_m = \{v_1, v_2, \dots, v_m\}$ has m vertices and $V_n = \{u_1, u_2, \dots, u_n\}$ has n vertices ; $V_m \cap V_n = \emptyset$, $\langle V_m \rangle$ and $\langle V_n \rangle$ isomorphic to null graph.

Four cases are discussed below.

Case 1. Let $n = 1$ and m be odd, then $D = \{u_1\}$ is the DS of the graph $K_{m,1}$, moreover $|N(u_1) \cap (V - D)|$ is odd, then D is OD^cS and it is obvious that D is MOD^cS. Thus, $\gamma_{odc}(K_{m,1}) = 1$.

Case 2. Two subcases are discussed below.

Subcase 1. let $n = 1$ and m be even or $n\ and\ m$ are even, then let $D = \{u_1, v_1\}$, it is obvious that the set D is the DS, moreover $|N(z) \cap (V - D); z = u_1\ or\ v_1|$ is odd, then the set D is OD^cS and it is obvious that D is MOD^cS.

Subcase 2. Let $n = 2$ and m be odd, then let $D = \{u_1, u_2\}$, it is obvious that D is the DS, moreover $|N(u_i) \cap (V - D); i = 1,2|$ is odd, then the set D is OD^cS and it is obvious that D is MOD^cS.

From two subcases mentioned earlier, $\gamma_{odc}(K_{m,n}) = 2$.

Case 3. If either $m\ or\ n$ is odd, then assume that n is odd and let $D = \{u_1, u_2, v_1\}$, by the same technique in previous case, one can easily concluded that $\gamma_{odc}(K_{m,n}) = 3$.

Case 4. If $m\ and\ n$ are odd, then let $D = \{u_1, u_2, v_1, v_2\}$, again, by the same technique in case 2, one can easily concluded that $\gamma_{odc}(K_{m,1}) = 4$.

Therefore, depend on the results in above cases the required is obtained.

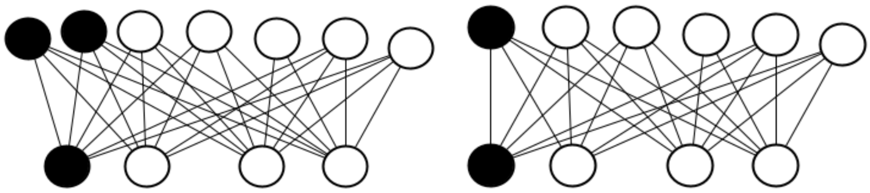


Figure 2.5. MOD^cS of bipartite graphs $K_{7,4}\ and\ K_{6,4}$.

Proposition 2.2.15 If P_n is path graph, then $\gamma_{odc}(\overline{P_n}) = \begin{cases} 1, & if\ n = 1 \\ 4, & if\ n = 5 \\ 2, & otherwise \end{cases}$

Proof.

To get the result, four cases are discussed as the following.

Case1. If $n = 1$, then it is trivial.

Case2. If $n = 2,3$, then the graph $\overline{P_n} \equiv N_2\ or\ N_1 \cup K_2$, so, $\gamma_{odc}(\overline{P_n}) = 2$.

Case3. If $n = 4$, since $\overline{P_n} \cong P_n$, then by proposition 2.2.7 $\gamma_{odc}(P_4) = \gamma_{odc}(\overline{P_4}) = 2$.

Case2. If $n = 5$, then in $\overline{P_5}$ there is no vertex has $(n-1)$ degree, so $\gamma_{odc}(P_4) > 1$. Again, if D contains two or three vertices, so there is a vertex say v such that $|N(v) \cap (V - D)|$ is even. Thus, there is no OD^cS contains less than four vertices. Now, let D contains four vertices from $V(\overline{P_5})$, then $|N(v_i) \cap (V - D); v_i \in D\ | \leq 1$. Therefore, $\gamma_{odc}(\overline{P_n}) = 4$.

Case4. If $n \geq 6$, then there are two subcases.

Subcase1. If n is even, then let $D = \{v_1, v_n\}$, in the graph $\overline{P_n}$, the vertex $v_1($ or $v_n)$ is adjacent to all other vertices except the vertex $v_2(or\ v_{n-1})$, so D is a DS. Moreover, the degree of the two vertices v_1 and v_n is $n-2$ and they adjacent, so $|N(v_i) \cap (V-D); i = 1,2|$ is odd, then the set D is OD^cS.

Subcase2. If n is odd, let $D = \{v_2, v_5\}$, Since $d(v_2, v_5) = 3$ in P_n, then v_2 is dominated to all vertices that not dominated by v_5 and v_5 is dominated to all vertices that not dominated by v_2, then D is DS. Since v_2 and v_5 have degree $n-3$ in $\overline{P_n}$ and they adjacent, then every vertex in D is dominated to $n-4$ vertices in $V-D$, so D is OD^cS.

In two subcases above, there is no vertex adjacent to all other vertices in the graph $\overline{P_n}$, then D is MOD^cS and $\gamma_{odc}(\overline{P_n}) = 2$.

From the cases mentioned earlier, the required is getting.

Proposition 2.2.16 In the graph $\overline{P_n}$ if $n \geq 6$ and odd and D is OD^cS, then each vertex in D has $(n-3)$ degree.

Proof.

1) If n is odd, let at least one vertex in D has degree $n-2$ say v_1. Since v_1 is join with each vertex in $V(\overline{P_n})$ except v_2, so two cases are discussed below.

Case1. Let $D = \{v_1, v_2\}$ is OD^cS, then v_2 is adjacent to $n-3$ vertices in the set $V-D$ and $n-3$ is even, that contradiction so D is not OD^cS.

Case2. Let $D = \{v_1, v_3\}$ be OD^cS such that $i = 3, \ldots, n-1$, since every vertex v_i is not adjacent to vertices v_{i-1} and v_{i+1} then v_2 is not dominated by any vertex in D, so D is not DS.

Case3. Let $D = \{v_1, v_i\}$ be OD^cS such that $i = 4, \dots, n-1$, then v_1 is adjacent to $n-2$ vertices in V, so v_1 is dominated to $n-3$ vertices in $V - D$ and since $n-3$ is even, then it is a contradiction. So D is not OD^cS.

Thus, each vertex in D has $n-3$.

Observation 2.2.17 If K_n is complete graph, then $\gamma_{odc}(\overline{K_n}) = n$.

Theorem 2.2.18 If C_n is cycle graph, then $\gamma_{odc}(\overline{C_n}) = \begin{cases} 3, & if\ n = 3,5 \\ 2, & otherwise \end{cases}$

Proof: Let $\{v_1, v_2, \dots, v_n\}$ be in C_n, four cases are discussed below.

Case1. Let $n = 3$, so $\overline{C_n} \equiv N_n$, then the required is obtained.

Case2. If $n = 4$, then $\overline{C_n} \equiv K_2 \cup K_2$, again, the required is obtained.

Case3. Let $n = 5$, then each vertex in $\overline{C_n}$ is join with only two vertices, thus, $\gamma_{odc}(\overline{C_n}) > 2$. Now, let $D = \{v_1, v_3, v_5\}$, it is obvious that the set D is DS, moreover, $|N(v_i) \cap (V - D); i = 1,5| = 1$ and $|N(v_3) \cap (V - D)| = 0$. Thus, the set D is OD^cS and $\gamma_{odc}(\overline{C_5}) = 3$.

Case. If $n \geq 6$, two subcases are discussed below.

Subcase1. Let n be even, and $D = \{v_i, v_{i+1}\}$, $i = 1, \dots, n-1$, since v_i and v_{i+1} have $n-3$ degree in $\overline{C_n}$ and since v_i and v_{i+1} are not adjacent, then v_i and v_{i+1} are dominated to $n-3$ vertices in $V - D$. Since $n-3$ is odd. Then D is OD^cS, and there is no vertex in $\overline{C_n}$ has $(n-1)$ degree, so D is MOD^cS and $\gamma_{odc}(\overline{C_n}) = |D| = 2$.

Subcase2. If n is odd, let $D = \{v_i, v_j\}$ such that v_i is adjacent to v_j in the graph $\overline{C_n}$. Moreover, the vertex v_j is dominated to all vertices that do not dominate by the vertex v_i in $\overline{C_n}$ and the vertex v_i is dominated to all vertices that do not dominate by the vertex v_j in $\overline{C_n}$. Then v_i and v_j are dominated to $n-4$ vertices in $V - D$, since $n-4$ is odd. Thus, D is OD^cS and by the same technique in previous subcase, we get that D is MOD^cS and $\gamma_{odc}(\overline{C_n}) = |D| = 2$.

From the cases mentioned earlier, the required is getting.

Proposition 2.2.19 Assume that G be a wheel graph of order n, then $\gamma_{odc}(\overline{W_n}) = \begin{cases} 4, & if\ n = 3,5 \\ 3, & otherwise \end{cases}$

Proof.

Since $W_n = \overline{C_{n-1}} \cup K_1$, then $\gamma_{odc}(\overline{W_n}) = \gamma_{odc}(\overline{C_{n-1}}) + \gamma_{odc}(K_1)$. Thus, by using Theorem 2.2.18 and observation 2.2.5, the result is obtained.

Proposition 2.2.20 If $K_{m,n}$ is a complete bipartite graph, then $\gamma_{odc}(\overline{K_{m,n}}) = \begin{cases} 2, & if\ n\ and\ m\ are\ even \\ 3, & if\ either\ n\ or\ m\ is\ even \\ 4, & if\ n\ and\ m\ are\ odd \end{cases}$

Proof.

The graph $\overline{K_{m,n}} \equiv K_n \cup K_m$, so $\overline{K_{m,n}}$ has two components such that $\gamma_{odc}(\overline{K_{m,n}}) = \gamma_{odc}(K_m) + \gamma_{odc}(K_n)$ by using the Proposition 2.2.13 to find $\gamma_{odc}(K_m)$ and $\gamma_{odc}(K_n)$, the required is obtained.

Theorem 2.2.21 For any graph G with n vertices and maximum degree Δ :

I. If Δ is odd, then $\left\lceil \frac{n}{\Delta+1} \right\rceil \leq \gamma_{odc} \leq n - \Delta$.

II. If Δ is even, then $\left\lceil \frac{n}{\Delta} \right\rceil \leq \gamma_{odc} \leq n - \Delta + 1$.

Proof.

Let D be a $\gamma_{odc} - set$.

Case1. If Δ is odd, then to proof the lower bound since every vertex can dominate at most itself and $\Delta(G)$ other vertices such that every vertex in D is adjacent to odd number of vertices in $V - D$. Hence, $\gamma_{odc} \geq \left\lceil \frac{n}{\Delta+1} \right\rceil$.

Now, to proof the upper bound. Let u be vertex that has maximum degree $\Delta(G)$.Then the vertex u dominates $N[u]$ vertices. Suppose the other vertices in $V - N[u]$ are dominate themselves only such that achieve the definition of OD^cS. Then the set $V - N(u)$ is odd neighbor in D^c dominating set since Δ is odd. Since $|V| = n$ and $|N(u)| = \Delta$, then $|D| = |V - N(u)| = n - \Delta$. So $\gamma_{odc} \leq n - \Delta$.

Case2. If Δ is even by the same way in case1 every vertex can dominate at most itself and $\Delta(G) - 1$, then $\gamma_{odc} \geq \left\lceil \frac{n}{\Delta} \right\rceil$.

To proof upper bound by the same hypothesis in case1u dominates $N[u] - \{w\}$ vertices for any vertex $w \in N[u]$. So $|D| = |V - (N(u) - \{w\}| = n - \Delta + 1$. Hence $\gamma_{odc} \leq n - \Delta + 1$.

Theorem 2.2.22 Let G be a graph of order n and $\forall v \in V(G)$, $deg(v) = even \geq 2$. Then $\gamma_{odc}(G) \geq \frac{n}{\Delta}$.

Proof.

Let D be $\gamma_{odc} - set$ of G and $\forall v \in V(G)$, $deg(v) = even$. Then by definition every vertex of D must adjacent to at least some odd number (say m) such that $m \leq \Delta$. Since all vertices in V is even, then every vertex in D must adjacent to $2k + 1$ vertices in $V - D$ such that $k = 1,2, \dots, \frac{n}{2} - 1$. That is, $N(D) = V(G)$. Since every vertex $v \in D$ has at most Δ neighbors. Then $\Delta\gamma_{odc} \geq |V| = n$. So, by dividing this inequality by Δ we get $\gamma_{odc}(G) \geq \frac{n}{\Delta}$.

Corollary 2.2.23 Let G be a graph of order n and $\forall v \in V(G)$, $deg(v) = even \geq 2$. If $\Delta \leq \frac{n}{k}$ for some positive integer k, then $\gamma_{odc}(G) \geq k$.

Proof.

By Theorem, $\gamma_{odc}(G) \geq \frac{n}{\Delta}$. If $\Delta \leq \frac{n}{k}$, then substitution yields $\gamma_{odc}(G) \geq k$.

Theorem 2.2.24 For any connected graph of order n has at least three vertices of degree $n-2$, then $\gamma_{odc}(G) \leq 4$.

Proof.

Suppose that G has three vertices say $V(H) = \{v_i, i = 1,2,3\}$ have degree $n-2$ such that H is subgraph of G. If one of those vertices independent to others, then this vertex has $n-3$ degree but that contradiction, so every vertex in H is adjacent to at least one vertex of them. So, an induced subgraph of H is connected subgraph and it is path of order three. In case the vertex that not adjacent to $v_i \in V(H)\ \forall i$ lie in $V(G) - V(H)$ then an induced subgraph of H is complete of order three, so $H \equiv K_3\ or\ P_3$. Then there are two cases depending on whether the order is even or odd.

Case1. If n is odd, then there are two subcases.

Subcase1. If $\langle\{v_1, v_2, v_3\}\rangle \equiv K_3$, then we distinguish three cases as follows.

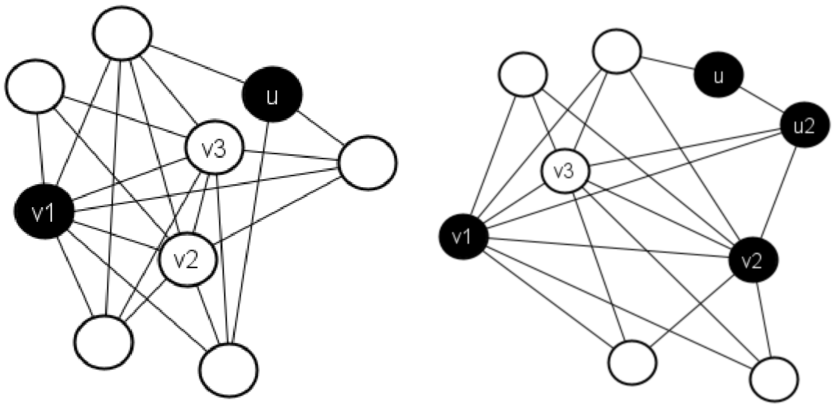
1) If these three vertices are not adjacent to the same vertex (say u) then there are two cases as follows.

I) If u has odd or zero degree, then $D = \{u, v_1\}$ is MOD^cS, so $\gamma_{odc}(G) = 2$. (See Fig 2.6 (a)).

II) If u has even degree, then there are two cases as follows.

A. If $N(u)$ has a vertex of odd degree say u_1, then $D = \{u_1, v_1, v_2\}$ is MOD^cS, so $\gamma_{odc}(G) = 3$.

B. If all vertices belong to the set $N(u)$ have even degree, then $D = \{u, u_2, v_1, v_2\}$, where $u_2 \in N(u)$. One can be concluded that the set D is MOD^cS, so $\gamma_{odc}(G) = 4$. (See Fig 2.6 (b))



(a) (b)

Figure 2.6. MOD^cS of order 2 and 4 of a graph of even order.

2) If two vertices say $v_1, and\ v_2$ are not adjacent to the same vertex say u_1 and the vertex v_3 is not adjacent to the vertex u_2, where $u_1\ and\ u_2$ are different, then there are four cases as follows.

A) If the vertex u_1 has odd degree and the vertex u_2 has even degree, then the set $D = \{u_1, v_2\}$ is MOD^cS, so $\gamma_{odc}(G) = 2$.

B) If the vertex u_1 has even degree and the vertex u_2 has odd degree, then the set $D = \{u_2, v_3\}$ is MOD^cS, so $\gamma_{odc}(G) = 2$.

C) If two vertices $u_1\ and\ u_2$ have odd degree, then the set D in case A or B is obtained the result. (See Fig. 2.7).

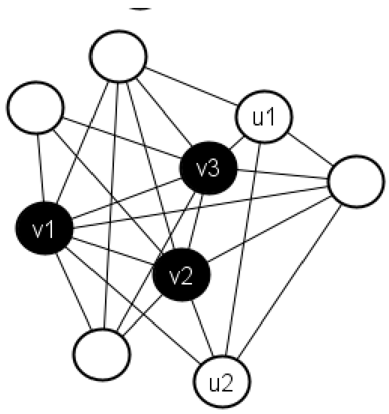


Figure 2.7. MOD^cS of order 3 of a graph G of odd order

***D*)** If two vertices u_1 and u_2 have even degree, $D = \{v_1, v_2, v_3\}$ is MOD^cS. So, $\gamma_{odc}(G) = 3$.

3) If The vertex v_1 is not adjacent to the vertex say u_1 and the vertex v_2 is not adjacent to the vertex say u_2 and the vertex v_3 is not adjacent to the vertex say u_3 where the vertices$u_1, u_2, and\ u_3$ are different, then there are cases as follows.

A. If at least one vertex from the set $S = \{u_1, u_2, u_3\}$ has odd degree say u_1 , then the set $D = \{u_1, v_1\}$ is MOD^cS, so $\gamma_{odc}(G) = 2$.

B. If all vertices in the set S have even degree, then there are two cases as follows.

$\boldsymbol{B_1}$**.** If there are two vertices of the set S are adjacent say $\{u_1, u_2\}$, the if $N(u_1) \cup N(u_2) = V(G)$, then the set $D = \{u_1, u_2\}$ is MOD^cS, so $\gamma_{odc}(G) = 2$.

$\boldsymbol{B_1}$**.** If there are no two vertices of the set S are adjacent, then the set $D = \{v_1, v_2, v_3\}$ is MOD^cS, so $\gamma_{odc}(G) = 3$.

Subcase2. If $\langle\{v_1, v_2, v_3\}\rangle \equiv P_3$, then the vertices v_1 and v_3 are not adjacent. Then the set $D = \{v_1, v_3\}$ is MOD^cS. So, $\gamma_{odc}(G) = 2$.

Case 2. If n is even, then if G has at least one vertex v of degree $n-1$, then $D=\{v\}$ and $\gamma_{odc}(G)=1$. Otherwise, there are two subcases:

Subcase1. If $\langle\{v_1,v_2,v_3\}\rangle \equiv K_3$, then we distinguish three cases as follows.

1) If these three vertices are not adjacent to the same vertex (say u) then there are two cases as follows.

I) If u has odd, then there are two cases as follows.

A) If $\exists v \in N(u)$ such that v has even degree, then $D=\{v,v_1\}$ is MOD^cS.

B) If all vertices in $N(u)$ has odd degree, then $D=\{u,v_1,v_2\}$ is the MOD^cS, so $\gamma_{odc}(G)=3$.

II) If u has even degree, then there are three cases as follows.

A. If $N(u)$ has a vertex of even degree say u_1, then $D=\{u_1,v_1\}$ is MOD^cS, so $\gamma_{odc}(G)=2$. (See Fig. 2.8 (a)).

B. If all vertices belong to the set $N(u)$ have odd degree, then let $D=\{u,u_2,v_1\}$, where $u_2 \in N(u)$. One can be concluded that the set D is MOD^cS, so $\gamma_{odc}(G)=3$.

2) If two vertices say v_1 *and* v_2 are not adjacent to the same vertex say u_1 and the vertex v_3 is not adjacent to the vertex u_2, where u_1 *and* u_2 are different, then there are two cases as follows.

A. If the vertex u_1 has even degree, then the set $D=\{u_1,v_3\}$ is a MOD^cS, so $\gamma_{odc}(G)=2$.

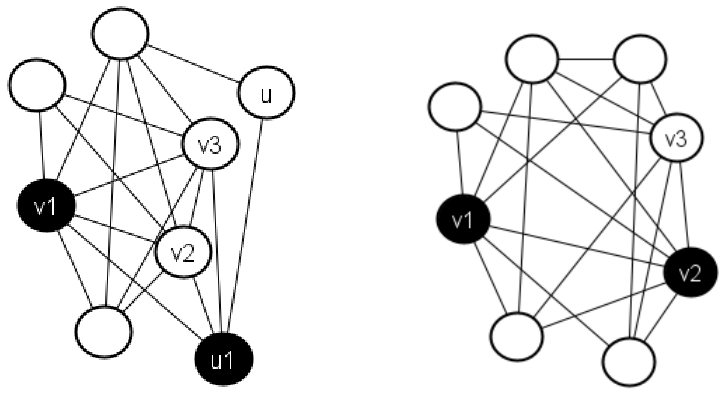
B. If the vertex u_2 has even degree, then the set $D=\{u_2,v_1\}$ is a MOD^cS, so $\gamma_{odc}(G)=2$.

C. If the vertex u_1 *and* u_2 have odd degree, then the set $D=\{u_1,v_1,v_2\}$ is a MOD^cS, so $\gamma_{odc}(G)=3$.

3) If The vertex v_1 is not adjacent to the vertex say u_1 and the vertex v_2 is not adjacent to the vertex say u_2 and the vertex v_3 is not adjacent to the vertex say u_3 where the vertices$u_1, u_2, and\ u_3$ are different, then the set $D = \{v_1, v_2\}$ is MOD^cS, so $\gamma_{odc}(G) = 2$.

Subcase2. If $\langle\{v_1, v_2, v_3\}\rangle \equiv P_3$, by hypothesis v_1 are v_3 pendent vertices of P_3. Then $D = \{v_1, v_2\}$ is MOD^cS. So, $\gamma_{odc}(G) = 2$. (See Fig. 2.8 (b)).

From all cases above, the result is obtained.



(a) (b)

Figure 2.8. MOD^cS of order 2 of a graph G of even order.

Corollary 2.2.25. For any graph of order n has at least three vertices of degree $n-k-2$ and k isolated vertices, then $\gamma_{odc}(G) \leq 4 + k$.

Proof.

Suppose that G has at least three vertices of order $n-k-2$ and k isolated vertices, then G has $k+1$ component such that k isolated vertices and subgraph H of order $m = n-k$. Then $\gamma_{odc}(G-H) = k$. It is clear H has at least three vertices of order $m-2$ and by theorem 2.2.24 $\gamma_{odc}(H) \leq 4$. So, $\gamma_{odc}(G) \leq 4 + k$.

Theorem 2.2.26 Let H_n be a helm graph, $n \geq 3$, then $\gamma_{odc}(H_n) = n$.

Proof.

Let the vertices of H_n are represented in the set $V(H_n) = \{v_0, v_1, v_2, \dots, v_{2n}\}$ such that v_0 is center vertex. By proposition 1.4.14 the MOD^cS can not become less than n, we take sets of order n and proof it is minimum. Then there are three cases as follows.

Case1. If $n \equiv 0\ (mod\ 3)$, then $D = \left\{v_{1+3i}, v_{2+3i}, i = 0, \dots, \left\lceil\frac{n}{4}\right\rceil\right\} \cup \left\{v_{3i+n}; i = 1, \dots, \left\lceil\frac{n}{4}\right\rceil\right\}$ is dominating set. Since every vertex adjacent to one or three vertices in $V - D$, then D is OD^cS. Any vertex removing from D, then it becomes no OD^cS. So, D is MOD^cS with order n and $\gamma_{odc}(H_n) = n$.

Case2. If $n \equiv 1\ (mod\ 3)$, then $D = \left\{v_{1+3i}, v_{2+3i}, i = 0, \dots, \left\lceil\frac{n}{4}\right\rceil - 1\right\} \cup \left\{v_{3i+n}; i = 1, \dots, \left\lceil\frac{n}{4}\right\rceil\right\} \cup \{v_{2n}\}$ is minimum dominating set such that $|D| = n$. since every vertex is adjacent to one or three vertices from $V - D$, so D is MOD^cS. $\gamma_{odc}(H_n) = n$.

Case3. If $n \equiv 2\ (mod\ 3)$, then $D = \left\{v_{1+3i}, v_{2+3i}, i = 0, \dots, \left\lceil\frac{n}{4}\right\rceil - 1\right\} \cup \left\{v_{3i+n}; i = 1, \dots, \left\lceil\frac{n}{4}\right\rceil\right\} \cup \{v_{2n-1}, v_{2n}\}$ is dominating set. By the same way in above cases, it clear D is MOD^cS . $\gamma_{odc}(H_n) = n$

Thus, we get the result. □

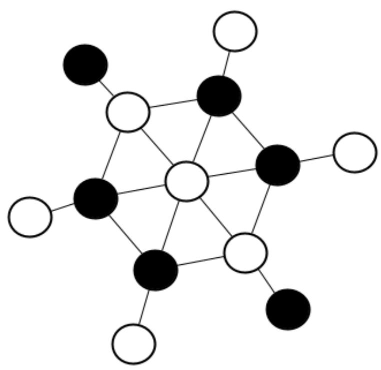


Figure 2.9. MOD^cS of helm H_6

Theorem 2.2.27 Let $T_{m,n}$ be tadpole graph, $m \geq 3, n \geq 1$, then

$$\gamma_{odc}(T_{m,n}) = \begin{cases} \left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor, & if \begin{cases} m = 0(mod\ 4)\ or \\ m = 1,2\ (mod\ 4) and\ n = 0,2\ (mod\ 4) \end{cases} \\ \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil - 1, & if\ m = 1,2\ (mod\ 4)\ and\ n = 1,3\ (mod\ 4) \\ \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil, & if\ m \equiv 3\ (mod4)\ and\ n = 0,2\ (mod4) \\ \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil - 1, & if\ m \equiv 3\ (mod4)\ and\ n = 1,3\ (mod4) \end{cases}$$

Proof.

The tadpole graph consists of cycle C_m and path P_n common by bridge. Such that the vertex there exist one vertex in C_m has degree three and only vertex in P_n has degree one. Let the set of vertices in the path P_n be $\{v_1, v_2, \dots, v_n\}$ and the set of vertices of the cycle graph C_m is $\{u_1, u_2, \dots, u_m\}$. There is one vertex of degree one (say v_n) and one vertex of degree three (say u_1). Let $D_1 = \{u_1\} \cup$

$$\begin{cases} \left\{u_{4+4i}, u_{5+4i}; i = 0, \dots, \frac{m}{4} - 2\right\} \cup \{u_{m-2}\}, & if\ m \equiv 0\ (mod\ 4); m \neq 4 \\ \left\{u_{4+4i}, u_{5+4i}; i = 0, \dots, \left\lfloor \frac{m}{4} \right\rfloor - 2\right\} \cup \{u_{m-2}, u_{m-1}\}, & if\ m \equiv 1\ (mod\ 4); m \neq 5 \\ \left\{u_{4+4i}, u_{5+4i}; i = 0, \dots, \left\lfloor \frac{m}{4} \right\rfloor - 1\right\}, & if\ m \equiv 2,3\ (mod\ 4) \end{cases}$$

$$D_2 = \begin{cases} \{v_{3+4i}, v_{4+4i}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\}, & if\ n \equiv 0,1\ (mod\ 4); n \neq 1 \\ \{v_{3+4i}, v_{4+4i}, i = 0, \dots, \left\lceil \frac{n}{4} \right\rceil - 1\} \cup \{v_n\}, & if\ n \equiv 3\ (mod\ 4) \\ \{v_{3+4i}, v_{4+4i}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\} \cup \{v_n\}, & if\ n \equiv 2\ (mod\ 4); n \neq 2 \end{cases}$$

We obtained cases as follows.

Case1. If $n = 1$, there are four subcases as follows.

Subcase1. If $m = 3$, it is clear $D = \{u_1\}$ is MOD^cS. Then $\gamma_{odc}(T_{m,n}) = 1$.

Subcase2. If $m = 4$, then $D = \{u_1, u_2, u_4\}$ is MOD^cS. Then $\gamma_{odc}(T_{m,n}) = 3$.

Subcase3. If $m = 5$, then $D = \{u_1, u_3, u_4\}$ is MOD^cS. Then $\gamma_{odc}(T_{m,n}) = 3$.

Subcase4. If $m \geq 6$, it is obvious $D = D_1$ is MOD^cS.

Case2. If $n = 2$, there are three subcases as follows.

Subcase1. If $m = 3$, it is clear $D = \{u_1, v_2\}$ is MOD^cS. Then $\gamma_{odc}(T_{m,n}) = 2$.

Subcase2. If $m = 4$, then $D = \{u_1, u_2, v_2\}$ is MOD^cS. Then $\gamma_{odc}(T_{m,n}) = 3$.

Subcase3. If $m = 5$, then $D = \{u_1, u_2, u_3, v_1\}$ is MOD^cS. Then $\gamma_{odc}(T_{m,n}) = 4$.

Subcase4. If $m \geq 6$, let $D = \{v_2\} \cup D_1$ be dominating set and it is clear D is OD^cS.

Case3. Otherwise, if $m \geq 3$ and $m \neq 5$, let $D = D_1 \cup D_2$ be dominating set. By the definition of OD^cS, D is OD^cS.

In each above cases if we removing any vertex, the sets become do not OD^cS because at least one of vertices is adjacent to two vertices in $V - D$. So, D is MOD^cS.Thus,we get the result. □

Proposition 2.2.28. Let $L_{m,n}$ is lollipop graph, $m \geq 3, n \geq 2$ then $\gamma_{odc}(L_{m,n}) = \begin{cases} 2 + \left\lceil \frac{n-1}{2} \right\rceil, & if\ m\ is\ even\ and\ n\ is\ odd \\ 1 + \left\lceil \frac{n-1}{2} \right\rceil, & otherwise \end{cases}$.

Proof.

The lollipop graph consists of the complete graph K_m and the path graph P_n , these graphs are common by a bridge. Such that one vertex in K_m has degree m and one vertex in P_n has degree one. The set of vertices in P_n is $\{v_1, v_2, \dots, v_n\}$ and the set of vertices in K_m is $\{u_1, u_2, \dots, u_m\}$ such that u_1 is vertex has degree m and v_n has degree one.

Case1. If m is even, then by the proposition 2.2.10 we get D as

$$D = \{u_1, v_1\} \cup \begin{cases} \{v_{4+4i}, v_{5+4i}; i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\} & , \quad if\ n \equiv 0\ (mod\ 4) \\ \{v_{4+4i}, v_{5+4i}; i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\} & , \quad if\ n \equiv 1{,}2\ (mod\ 4)\ and\ n \neq 1{,}2 \\ \left\{v_{4+4i}, v_{5+4i}; i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_{n-1}\} & , \quad if\ n \equiv 3\ (mod\ 4) \end{cases}$$

If $n = 1\ or\ 2$, then it is clear $D = \{u_1, v_1\}$ is MOD^cS. Otherwise D is dominating set, since every vertex is dominate to one vertex in $V - D$ and u_1 dominate $m - 2$ vertices from K_m and one vertex from P_n then it dominates $m - 1$ vertices in $V - D$. Since D is OD^cS of P_n and $D - \{u_1\}$ is not OD^cS. Hence, D is MOD^cS.

Case2. If m is odd, then

$$D_1 = \{u_1\} \cup \begin{cases} \{v_{3+4i}, v_{4+4i}; i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\} & , \quad if\ n \equiv 1\ (mod\ 4) \\ \{v_{3+4i}, v_{4+4i}; i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\} \cup \{v_n\} & , \quad if\ n \equiv 2{,}3\ (mod\ 4) \\ \left\{v_{3+4i}, v_{4+4i}; i = 0, \dots, \left\lceil\frac{n}{4}\right\rceil - 1\right\} & , \quad if\ n \equiv 0\ (mod\ 4) \end{cases}$$

In case $n = 1 \ or \ 2$, then it is obvious $D_1 = \{u_1\}$ and $D_2 = \{u_1, v_2\}$ are MOD^cS respectively. By the same way in case1 we obtained that D is MOD^cS. So, we get the result.

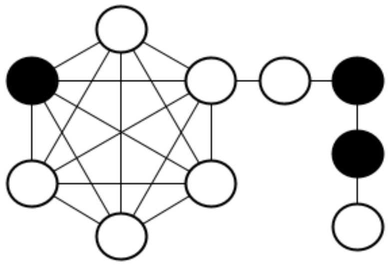


Figure 2.10. OD^cS of lollipop graph $L_{5,4}$.

Proposition 2.2.29. Let $\overline{H_n}$ be a complement of helm graph, $n \geq 3$, then $\gamma_{odc}(\overline{H_n}) = 3$.

Proof.

Since H_n has at least three pendent vertices, then $\overline{H_n}$ has at least three vertices of degree $n-2$ (say $\{v_1, v_2, v_3\}$). So, by theorem 2.2.24 $\gamma_{odc}(\overline{H_n}) \leq 4$ and since $\langle\{v_1, v_2, v_3\}\rangle \equiv K_3$ and $\forall v \in V(\overline{H_n})$, then $v \in N(v_1) \cup N(v_2) \cup N(v_3)$. So, $D = \{v_1, v_2, v_3\}$ is MOD^cS. Hence, $\gamma_{odc}(\overline{H_n}) = 3$.

Theorem 2.2.30 Let D_{n_1,n_2} be daisies graph has order $n_1 + n_2 - 1$ $and \ n_1, n_2 \geq 3 \ and \ n_1, n_2 \neq 4$ then $\gamma_{odc}(D_{n_1,n_2}) =$

$$\begin{cases} \left\lceil \frac{n_1+1}{2} \right\rceil + \left\lceil \frac{n_2+1}{2} \right\rceil - 2 \ , \ if \ n_1 \equiv 2,3 \ (mod\ 4) \ and \ n_2 \equiv 1,2,3 \ (mod\ 4) \\ \left\lceil \frac{n_1+1}{2} \right\rceil + \left\lceil \frac{n_2+1}{2} \right\rceil - 1 \ , \ if \ n_1 \ and \ n_2 \equiv 1 \ (mod\ 4) \\ \frac{n_1}{2} + \frac{n_2}{2} - 1 \ , \ if \ n_1 \ and \ n_2 \equiv 0 \ (mod\ 4) \\ \frac{n_1}{2} + \left\lceil \frac{n_2+1}{2} \right\rceil - 2 \ , \ if \ n_1 \equiv 0 \ (mod\ 4) \ and \ n_2 \equiv 2,3 \ (mod\ 4) \\ \frac{n_1}{2} + \left\lceil \frac{n_2+1}{2} \right\rceil - 1 \ , \ if \ n_1 \equiv 0 \ (mod\ 4) \ and \ n_2 \equiv 1 \ (mod\ 4) \end{cases}$$

Proof.

Let the set of vertices in the cycle C_{n_1} be $\{v_1, v_2, \dots, v_{n_1}\}$ and the set of vertices in the cycle C_{n_2} is $\{u_1, u_2, \dots, u_{n_2}\}$ such that $v_1 = u_1$.Thus, we can use the set D_1 to dominate all vertices in the cycle of order n_1 and D_2 to dominate all vertices in the cycle of order n_2 as follows. Let $D \subset V(D_{n_1,n_2})$ and $D = D_1 \cup D_2$. Therefore, there are six cases as follows.

Case1. If n_1 *and* $n_2 \equiv 0 \ (mod\ 4)$, let $D_1 = \{ v_{1+4i}, v_{2+4i}, i = 0, \dots, \frac{n_1}{4} - 1\}$ and $D_2 = \left\{ u_{4+4i}, u_{5+4i}, i = 0, \dots, \frac{n_2}{4} - 2\right\} \cup \{ u_{n_2-2}\}$

Case2. If n_1 *and* $n_2 \equiv 1 \ (mod\ 4)$; there are two cases as follows.

Subcase1. If $n_1 = n_2 = 5$,then $D = \left\{ v_{1+4i}, v_{2+4i}, i = 0, \dots, \left\lceil\frac{n_1+1}{4}\right\rceil - 2\right\} \cup \left\{ v_{n_1-2}\right\} \cup \{u_3, u_4\}$ is OD^cS.

Subcase2. Otherwise, let $D_1 = \left\{ v_{1+4i}, v_{2+4i}, i = 0, \dots, \left\lceil\frac{n_1+1}{4}\right\rceil - 2\right\} \cup \{ v_{n_1-2}\}$ and $D_2 = \left\{ u_{4+4i}, u_{5+4i}, i = 0, \dots, \left\lceil\frac{n_2+1}{4}\right\rceil - 2\right\} \cup \{ u_{n_2-3}, u_{n_2-2}\}$.

Case3. If n_1 *and* $n_2 \equiv 2{,}3 \ (mod\ 4)$, let $D_1 = \left\{ v_{1+4i}, v_{2+4i}, i = 0, \dots, \left\lceil\frac{n_1+1}{4}\right\rceil - 2\right\} \cup \{ v_{n_1-2}, v_{n_1-3}\}$ and $D_2 = \left\{ u_{4+4i}, u_{5+4i}, i = 0, \dots, \left\lceil\frac{n_2+1}{4}\right\rceil - 1\right\}$.

Case4. If $n_1 \equiv 0 \ (mod\ 4)$ *and* $n_2 \equiv 2{,}3 \ (mod\ 4)$, let $D_1 = \left\{ v_{1+4i}, v_{2+4i}, i = 0, \dots, \frac{n_1}{4} - 1\right\}$ and $D_2 = \left\{ u_{4+4i}, u_{5+4i}, i = 0, \dots, \left\lceil\frac{n_2+1}{4}\right\rceil - 2\right\}$.

Case 5. If $n_1 \equiv 1 \ (mod\ 4)$ *and* $n_2 \equiv 2{,}3 \ (mod\ 4)$,then D_1 is equal to D_1 in case 2 and D_2 is equal to D_2 in case 4.

Case 6. If $n_1 \equiv 1 \ (mod\ 4)$ *and* $n_2 \equiv 0 \ (mod\ 4)$,then D_1 is equal to D_1 in case 2 and D_2 is equal to D_2 in case 1.

Since $D = D_1 \cup D_2$ is dominating set for chosen in all cases above. Since these vertices in all sets are adjacent to zero or one or three vertices in $V - D$, then D is OD^cS. To proof minimum, if we removing any vertex from D, we lose the property of domination or OD^cS. Hence, D is MOD^cS and we get the proof.

Corollary 2.2.31 For daisies graph D_{n_1,n_2}, the following is hold.

1) If $n_1 = n_2 = 1,2 \ (mod\ 4)$ $\gamma_{odc}(D_{n_1,n_2}) = n_1$.

2) If $n_1 = n_2 = 0,3 \ (mod\ 4)$ $\gamma_{odc}(D_{n_1,n_2}) = n_1 - 1$.

Proof.

By the proof of theorem 2.2.30, we get the result.

Theorem 2.2.32 The Dutch windmill graph, $n \geq 3$, then $\gamma_{odc}(D_n^m) =$

$$\begin{cases} 2m, & if\ n = 4 \\ m\binom{n}{2} - m + 1, & if\ n \equiv 0\ (mod\ 4)\ and\ n \neq 4 \\ m\left\lceil\frac{n+1}{2}\right\rceil - m + 1, & if\ n \equiv 1\ (mod\ 4) \\ m\left\lceil\frac{n+1}{2}\right\rceil - 2m + 2, & if\ n \equiv 2,3\ (mod\ 4) \end{cases}$$

Proof.

The set of vertices D_n^m represent the set $V(D_n^m) = \{v, u_i^j, i = 1,2, \dots, n - 1\}$ $\forall\, j = 1, \dots, m$ such that v is common vertex. Let $D \subset V(D_n^m)$ and $D = D_1 \cup D_2$. Therefore, there are three cases as follows.

Case1. If $n \equiv 0\ (mod\ 4)$, in case $n = 4$, it obvious $D = \{u_1^j, u_2^j\}$ is OD^cS; $\forall j = 1, \dots, m$, so $|D| = 2m = \gamma_{odc}(D_4^m)$. In case $n \neq 4$, the OD^cS as follow $D = D_1 \cup D_2$ such that D_1 is OD^cS of subgraph H of vertex set $V(H) = \{v, u_1^1, u_2^1, \dots, u_{n-1}^1, u_1^j, u_{n-1}^j, j = 2, \dots, m\}$ such that $D_1 = \{v, u_1^1\} \cup \left\{u_{4+4i}^1, u_{5+4i}^1, i = 0, \dots, \left\lfloor\frac{n-1}{4}\right\rfloor - 1\right\}$. The other vertices in D_n^m are dominated by $D_2 = \left\{u^j_{3+4i}, u^j_{4+4i}, i = 0, \dots, \left\lfloor\frac{n-1}{4}\right\rfloor - 1\right\} \cup \{u^j_{n-3}\}\ j = 2, \dots, m.$ By proposition 2.2.11 D is MOD^cS. So, $\gamma_{odc}(D^m_n) = \frac{n}{2} + (m-1)(\frac{n}{2} - 1)$.

Case2. If $n \equiv 1\ (mod\ 4)$, by the same proposition that used in case1 we have the OD^cS as follow $D = D_1 \cup D_2$ as follows

$$D_1 = \{v, u^1_1\} \cup \left\{u^1_{4+4i}, u^1_{5+4i}, i = 0, \dots, \left\lfloor\frac{n-1}{4}\right\rfloor - 1\right\} \cup \{u^j_{n-3}\}$$

$$D_2 = \left\{u^j_{3+4i}, u^j_{4+4i}, i = 0, \dots, \left\lfloor\frac{n-1}{4}\right\rfloor - 1\right\}, j = 2, \dots, m$$

It is clear $|D_1| = \left\lceil\frac{n+1}{2}\right\rceil$ and $|D_2| = (m-1)(\left\lceil\frac{n+1}{2}\right\rceil - 1)$. So, $\gamma_{odc}(D^m_n) = \left\lceil\frac{n+1}{2}\right\rceil + (m-1)(\left\lceil\frac{n+1}{2}\right\rceil - 1)$.

Case3. If $n \equiv 2,3\ (mod\ 4)$, the two subsets that obtained OD^cS as follow

$D_1 = \{v, u^1_1\} \cup \left\{u^1_{4+4i}, u^1_{5+4i}, i = 0, \dots, \left\lfloor\frac{n-1}{4}\right\rfloor - 1\right\}$ and $D_2 = \left\{u^j_{3+4i}, u^j_{4+4i}, i = 0, \dots, \left\lfloor\frac{n-1}{4}\right\rfloor - 1\right\}, j = 2, \dots, m$

The OD^cS D is minimum by the same away in two cases above. Then $\gamma_{odc}(D^m_n) = \left\lceil\frac{n+1}{2}\right\rceil + (m-1)(\left\lceil\frac{n+1}{2}\right\rceil - 2)$. Hence, we get the result.

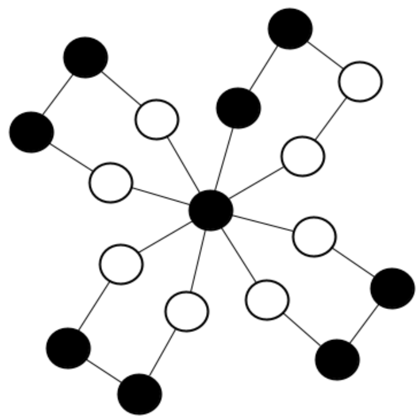


Figure 2.11. OD^cS of Dutch windmill graph D^4_5 .

Corollary 2.2.33 For any tree graph T of order n and $l \geq 3$ such that l is the number of pendent and s is the number of support vertices in T,

$$\gamma_{odc}(\bar{T}) = \begin{cases} 3, & \text{if } n \text{ is even and } s = 1 \\ & \text{or } n \text{ is odd and } s > 1 \text{ and all support vertices in } T \\ & \text{have even degree in } \bar{T} \\ 2, & \text{if } n \text{ is even and } s > 1 \text{ or } n \text{ is odd and } s = 1 \\ & \text{or } n \text{ is odd and } s > 1 \text{ and there is a support vertex in } T \\ & \text{has odd degree in } \bar{T} \end{cases}$$

Proof.

Suppose that $\{v_1, v_2, \dots, v_l\}$ the set of pendent vertices and $\{u_1, u_2, \dots, u_s\}$ is set of support vertices. It is clear that $deg(v_i) = n - 2$. If $l < 3$, then T is path and proof this case by proposition 2.2.10. If $l \geq 3$, there are two cases as follows.

Case1. If n is even, since $deg(v_i)$ is even for $i = 1, \dots, l$, then there are two cases:

I) If all pendant adjacent to one support vertex say u that mean G is isomorphic to star graph, then $D = \{v_1, v_2, u\}$ is MOD^cS. Thus, $\gamma_{odc}(\bar{T}) = 3$.

II) If the pendant vertices are adjacent to more than one support vertex, then the set $D = \{v_1, v_2\}$ where the vertices v_1, v_2 are adjacent to different two support vertices is MOD^cS. Thus, $\gamma_{odc}(\bar{T}) = 2$.

Case2. If n is odd, then there are two cases:

I) If all pendant adjacent to one support vertex say u that mean G is isomorphic to star graph, then $D = \{v_1, u\}$ is MOD^cS. Thus, $\gamma_{odc}(\bar{T}) = 2$.

II) If the pendant vertices are adjacent to more than one support vertex, then there are two cases:

A) If there is a support vertex say w_1 has odd degree, then $D = \{v_i, w_1\}$, where v_i is a pendant vertex that adjacent to w_1 in the graph T is MOD^cS in the graph $\bar{T}$. Thus, $\gamma_{odc}(\bar{T}) = 2$.

B) If all support vertices have even degree, then the set $D = \{v_1, v_2, v_3\}$ is MOD^cS in the graph $\bar{T}$. Thus, $\gamma_{odc}(\bar{T}) = 3$.

Then, we get the result.

Observation 2.2.34 Let G be any graph of order n, then $\gamma_{odc}(G) = 1$ if and only if n is even and $\Delta = n - 1$.

Proof.

If $\gamma_{odc}(G) = 1$, then G has OD^cS say D contains one vertex say v such that it dominates all other vertices in $V(G)$, then degree of this vertex is $n - 1$, and since $\Delta(G) \leq n - 1$, so $\Delta = n - 1$. Since D is OD^cS, then $deg(v) = n - 1$ is odd, then n is even.

Conversely, let n be even and $\Delta = n - 1$, then the vertex has degree Δ dominates all other vertices in $V(G)$ and since has degree odd since n is even. So $\gamma_{odc}(G) = 1$.

Proposition 2.2.35 Let G be any graph of order n, then $\gamma_{odc}(\bar{G}) = 1$ if and only if n is even and G has at least one isolated vertex.

2.3 TOD^cS − Dominating Set of Graph

In this part, we represent to new definition of domination is called Total odd neighbor in D^c dominating set such that denoted by TOD^cS and studies some properties.

Definition 2.3.1

Let G be a graph and $D \subseteq V$ is OD^cS , then if D is total dominating set, then D is called total odd neighbor in D^c dominating set (TOD^cS).

Definition 2.3.2

If D is OD^cS, Then D is called **minimal total odd neighbor in D^c dominating set** ($MTOD^cS$)if it has no proper $MTOD^cS$.

Definition 2.3.3

The minimum cardinality of all D is called the **total neighbor in D^c domination** and denoted by $\gamma_{todc}(G)$.

Definition 2.3.4

A set D is called **γ_{todc} −set** in case it is $MTOD^cS$ with cardinality $\gamma_{todc}(G)$.

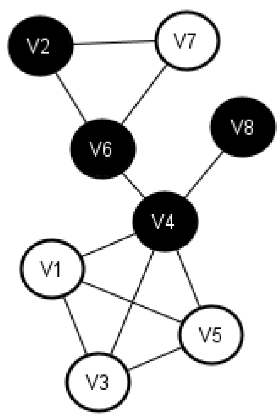


Figure 2.12. TOD^cS of a graph G of order 8.

Proposition 2.3.5 For any graph G:

1) If all vertices in $V(G)$ have even degree, then $\gamma_{odc}(G) = \gamma_{todc}(G)$.

2) If G has isolated vertices, then G has no TOD^cS.

3) $\gamma_{todc}(G) \geq 2$.

Proof.

1) Suppose that every vertex has even degree and D is OD^cS .Since $\forall v \in D$ is adjacent to odd number of vertices of $V - D$, then $\forall v \in D$ is adjacent to at least one vertex of D. So, the induced subgraph of D has no isolated vertices. Thus, D is TOD^cS and $\gamma_{odc}(G) = \gamma_{todc}(G)$.

2) It is obvious.

3) It is obvious. □

Proposition 2.3.6 Consider P_n be a path graph with $n; n \geq 2$ vertices, so

$$\gamma_{todc}(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & , \ if \ \ n \equiv 0,1,3 \ (mod\ 4) \\ \left\lceil \frac{n+1}{2} \right\rceil & , \ if \ \ n \equiv 2 \ (mod\ 4) \end{cases}$$

Proof.

Let $\{v_1, v_2, \dots, v_n\}$ be vertices of P_n. So, three cases are discussed below.

Case 1. If $n = 1$, so G has no TOD^cS.

Case2. If $n = 2,3$, then by proposition 2.1 (3) $D = \{v_1, v_2\}$ is TOD^cS. Thus, $\gamma_{todc}(P_n) = 2$.

Case 3. If $n > 3$, then there are four subcases as follows.

Subcases 1. If $n \equiv 0\ (mod\ 4)$, Let $D_1 = \{v_{2+4i}, v_{3+4i}; i = 0, \dots, \frac{n}{4} - 1\}$ be dominating set. since every vertex of D_1 is adjacent to only one vertex of

$V - D_1$ and an induced subgraph of D has no isolated vertex, then D_1 is TOD^cS. To prove the minimum, suppose that F is TOD^cS such that $|F| < |D_1|$. If $|F| = |D| - 1$ or $|D| - 2$, then F is not total and not OD^cS since F has isolated vertex and at least one of vertices of D is adjacent to two vertices of $V - D$. So, D is MTOD^cS.

Subcases 2. If $n \equiv 1 \, mod \, 4$, let $D_2 = \left\{v_{2+4i}, v_{3+4i}; i = 0, \ldots, \left\lceil \frac{n}{4} \right\rceil - 1\right\} \cup \{v_{n-1}\}$ is dominating set. Since every vertex of D is adjacent to at least one vertex and an induced subgraph of D has no isolated vertex, So, D is TOD^cS. By the same technique of subcase1 suppose that F is TOD^cS such that $|F| < |D_2|$. If $|F| = |D_2| - 1$ then there are two cases if $F = D_2 - \{v_{n-1}\}$, then F is total dominating set but not OD^cS $or |F| = |D_2| - 2$, then F is not TOD^cS, we get D is MTOD^cS.

Subcases 3. If $n \equiv 2 \, mod \, 4$, let $D_3 = \left\{v_{2+4i}, v_{3+4i}; i = 0, \ldots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\} \cup \{v_{n-1}, v_n\}$ is dominating set. Then by the same technique of subcase1 D_3 is TOD^cS.

Subcases 4. If $n \equiv 3 \, mod \, 4$, et $D_4 = \{v_{2+4i}, v_{3+4i}; i = 0, \ldots, \left\lceil \frac{n}{4} \right\rceil - 1\}$ is dominating set, then by the same technique of above cases D_4 is $MTOD^cS$.

Thus, we get the result. □

Corollary 2.3.7 If D is total dominating set of path graph, then D is OD^cS of path graph.

Proof.

Let D be total dominating set, since every vertex of D adjacent to only one vertex of $V - D$. Then D is OD^cS.

Observation 2.3.8 If C_n is cycle graph, then $\gamma_{todc}(C_n) = \gamma_{odc}(C_n)$.

Proof.

Since every vertex in cycle has even degree, then by proposition 2.2.11 we get the result.

Proposition 2.3.9 If G is a wheel graph $W_n = C_{n-1} + K_1$, then $\gamma_{todc}(W_n) = \begin{cases} 3 & , \ if\ n\ is\ even \\ 4 & , \ if\ n\ \ is\ odd \end{cases}$

Proof.

Let v_1 be a center vertex and $\{v_2, v_3, \dots, v_n\}$ is set of other vertices. Then there are two cases.

Case1. If n is even, let $D = \{v_1, v_2, v_3\}$ be dominating set. Since v_1 has $n-1$ degree, then it dominates $n-3$ vertices in $V - D$. Since the vertices v_2 and v_3 are adjacent, it is clear that these vertices are dominating only one vertex in the set $V - D$. So, D is total MOD^cS and $\gamma_{todc}(W_n) = 3$

Case2. If n is odd, the set $D = \{v_1, v_2, v_3, v_4\}$ is dominating set. Since v_1 dominates $n-4$ vertices in $V - D$. The vertex v_2 (or v_4) dominates only one vertex in $V - D$, the vertex v_3 is adjacent to only vertices of D, then D is TOD^cS. To prove D is $MTOD^cS$, suppose that $D_i = D - \{v_i\},\ i = 2,3,4$ are total OD^cS, but v_1 in these sets dominates $n-2$ vertices in $V - D$ that mean it dominates even vertices, then its contradiction, so $D_i\ , i = 2,3,4$ are not total OD^cS. So, D is minimum total OD^cS and $\gamma_{todc}(W_n) = 4$. □

Proposition 2.3.10 Consider $K_n; n \geq 2$ be a complete graph, so $\gamma_{todc}(K_n) = \begin{cases} 3 & , if\ n\ is\ even\ ; n \neq 2 \\ 2 & , \ if\ n\ is\ odd\ or\ n = 2 \end{cases}$

Proof.

Two cases are discussed below.

Case1. If n is even, there are two subcases as follows.

Subcase1. If $n = 2$, it obvious $\gamma_{todc}(K_2) = 2$.

Subcase2. If $n > 2$, then $D = \{v, u, w\}$ is $MTOD^cS$ because if we dominate by only two vertices, then every vertex from it dominates even number of vertices of $V - D$ and this set total but not D^cS. So, $\gamma_{todc}(K_n) = 3$.

Case2. If n is odd, then by the same way in proposition 2.2.13 $\gamma_{todc}(K_n) = 2$.

Thus, we get the result. □

Proposition 2.3.11 Consider $K_{m,n}$ be a bipartite graph, then $\gamma_{todc}\left(K_{m,n}\right) = \begin{cases} 2 & , \; if\ n = 1\ and\ m = 1 \\ 3 & , \; if\ n = 1\ and\ m\ is\ odd; m \neq 1 \\ \gamma_{odc} & , \quad otherwise \end{cases}$

Proof.

By definition of $K_{m,n}; n \leq m$, $V(K_{m,n}) = V_m \cup V_n$ such that $V_m = \{v_1, v_2, \dots, v_m\}$ has m vertices and $V_n = \{u_1, u_2, \dots, u_n\}$ has n vertices ; $V_m \cap V_n = \emptyset$, $\langle V_m \rangle$ and $\langle V_n \rangle$ isomorphic to null graph. Let $n = 1\ and\ m$ is odd, then two cases are discussed below.

Subcase1. If $m = 1$, then it obvious by proposition 2.3.5 (3) $\gamma_{todc}\left(k_{1,1}\right) = 2$.

Subcase2. If $m > 1$, let $D = \{u_1, v_1\}$ be dominating set, it obvious D is total dominating set but not OD^cS because u_1 is adjacent to even number of vertices of V_m. Then $D_1 = \{v_1, v_2, u_1\}$ is $MTOD^cS$. So, $\gamma_{todc}\left(K_{m,n}\right) = 3$. Otherwise, by proposition 2.2.13 all sets are hold $MTOD^cS$. So, $\gamma_{todc}(K_{m,n}) = \gamma_{odc}(K_{m,n})$. □

Proposition 2.3.12 Let H_n be a helm graph with $n \geq 3$, then $\gamma_{todc}(H_n) = n+1$.

Proof.

Let the vertices of H_n are as follows.

$V(H_n) = \{v_0, v_1, v_2, \dots, v_{2n}\}$ such that v_0 is center vertex. Let $D = \{v_1, v_2, \dots, v_n\} \cup \{v_0\}$ be dominating set. since all vertices of D is adjacent to at most one vertex of $V - D$, then D is TOD^cS. If we remove v_0, the set obtained by removing is not OD^cS and if remove any other vertex, then the set obtained is not dominating set. so, $\gamma_{todc}(H_n) = n+1$. □

Proposition 2.3.13 Let $T_{m,n}$ is tadpole graph; $m > 3, n \geq 1$, then

$$\gamma_{todc}(T_{m,n}) = \begin{cases} \begin{cases} \left\lceil \frac{m+3}{2} \right\rceil, & if\ m \equiv 1,2,3\ (mod\ 4) \\ \left\lceil \frac{m+1}{2} \right\rceil, & if\ m \equiv 0\ (mod\ 4) \end{cases} & if\ n = 1, m \geq 4 \\ \left\lceil \frac{n+6}{2} \right\rceil, & if\ m = 3, n \geq 1 \\ \frac{m}{2} + \left\lceil \frac{n+1}{2} \right\rceil, & if\ n = 0,1,3\ (mod\ 4)\ and\ m = 0\ (mod\ 4) \\ \frac{m}{2} + \left\lceil \frac{n}{2} \right\rceil, & if\ n = 2\ (mod\ 4)\ and\ m = 0\ (mod\ 4) \\ \left\lceil \frac{m+1}{2} \right\rceil + \left\lceil \frac{n+1}{2} \right\rceil, & if\ n = 0,1,3\ (mod\ 4)\ and\ = 2,3\ (mod\ 4) \\ \left\lceil \frac{m+1}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil, & if\ n = 2\ (mod\ 4)\ and\ = 2,3\ (mod\ 4) \\ \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{n+1}{2} \right\rceil, & if\ n = 0,1,3\ (mod\ 4)\ and\ = 1\ (mod\ 4) \\ \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil, & if\ n = 2\ (mod\ 4)\ and\ = 1\ (mod\ 4) \end{cases}$$

Proof.

The tadpole graph there consist of cycle C_m and path P_n common by bridge. Such that the vertex there exist one vertex in C_m has degree three and only vertex in P_n has degree one. Let the set of vertices in the path P_n be $\{v_1, v_2, \dots, v_n\}$ and the set of vertices of the cycle graph C_m is $\{u_1, u_2, \dots, u_m\}$. There is one vertex of degree one (say v_n) and one vertex of degree three (say u_1). We obtained three cases as follows.

Case1. If $n = 1$, four cases are discussed below.

Subcase1. If $m = 0\ (mod\ 4)$, let $D = \left\{u_{5+4i}, u_{6+4i}; i = 0, \dots, \left\lceil \frac{m}{8} \right\rceil - 1\right\} \cup \{u_1, u_2, v_1\}$ is dominating set, it obvious every vertex of D is adjacent to at most one vertex and an induced subgraph of D has no isolated vertices. So, D is TOD^cS. If we want to build a TOD^cS of lower order, we cannot take a single vertex because we will lose the total property, so we exclude any of the vertices of set $\{u_1, u_2, v_1\}$, but when any vertex is excluded from the set $\{u_1, u_2, v_1\}$,, we lose the OD^cS property. So, D is $MTOD^cS$.

Subcase2. If $m = 1\ (mod\ 4)$,in case $m = 5$, then it is clear $D = \{u_1, u_2, u_5\}$ is $MTOD^cS$. If $m > 5$, let $D = \left\{u_{5+4i}, u_{6+4i}; i = 0, \dots, \left\lfloor \frac{m}{8} \right\rfloor - 1\right\} \cup \{u_1, u_2, u_n\}$ is dominating set, then by the same way in case1 D is $MTOD^cS$.

Subcase3. If $m = 2\ (mod\ 4)$, suppose that $D = \left\{u_{5+4i}, u_{6+4i}; i = 0, \dots, \left\lceil \frac{m}{8} \right\rceil - 1\right\} \cup \{u_1, u_2, v_1\}$ is dominating set.

Case2. If $m = 3$, two cases are discussed below.

Subcase1. If $n = 1,2$, it is clear $D = \{u_1, u_2, v_1\}$ is $MTOD^cS$. So, $\gamma_{todc}(T_{m,n}) = 3$.

Subcase2. If $n \geq 3$, let $D = \{u_2, u_3\} \cup$

$$\begin{cases} \{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\}, & if\ n \equiv 0,1\ (mod\ 4), n \neq 1 \\ \{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lceil \frac{n}{4} \right\rceil - 1\}, & if\ n \equiv 3\ (mod\ 4) \\ \{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\} \cup \{v_{n-1}, v_n\}, & if\ n \equiv 2\ (mod\ 4) \end{cases}$$

is dominating set, if we remove any vertex from D, then D is not TOD^cS because one of vertices of D is adjacent to two vertices or D is not dominating set. So, D is $MTOD^cS$, $\gamma_{todc}(T_{m,n}) = \left\lceil \frac{n}{2} \right\rceil + 2$.

Case3. Otherwise, let $D = D_1 \cup D_2$ be dominating set such that the set D_1 to dominate C_m and D_2 to dominate P_n as follows.

$$D_1 = \begin{cases} \left\{u_{2+4i}, u_{3+4i}; i = 0, \dots, \frac{m}{4} - 1\right\}, & if\ m \equiv 0\ (mod\ 4) \\ \left\{u_{2+4i}, u_{3+4i}; i = 0, \dots, \left\lfloor \frac{m}{4} \right\rfloor - 1\right\} \cup \{u_{m-1}\}, & if\ m \equiv 1\ (mod\ 4) \\ \left\{u_{2+4i}, u_{3+4i}; i = 0, \dots, \left\lfloor \frac{m}{4} \right\rfloor - 1\right\} \cup \{u_{m-1}, u_m\}, & if\ m \equiv 2\ (mod\ 4) \\ \left\{u_{2+4i}, u_{3+4i}; i = 0, \dots, \left\lfloor \frac{m}{4} \right\rfloor\right\}, & if\ m \equiv 3\ (mod\ 4); m \neq 3 \end{cases}$$

And

$$D_2 = \begin{cases} \{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\}, & if\ n \equiv 0,1\ (mod\ 4), n \neq 1 \\ \{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lceil \frac{n}{4} \right\rceil - 1\}, & if\ n \equiv 3\ (mod\ 4) \\ \{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\} \cup \{v_{n-1}, v_n\}, & if\ n \equiv 2\ (mod\ 4) \end{cases}$$

The separation dominating is minimum because if u_1 belongs to dominating set say D^*, then we must take at least two neighbors of vertex u_1, then $|D^*| \geq |D|$. By the same away in above cases, the set D is $MTOD^cS$. We get the result. □

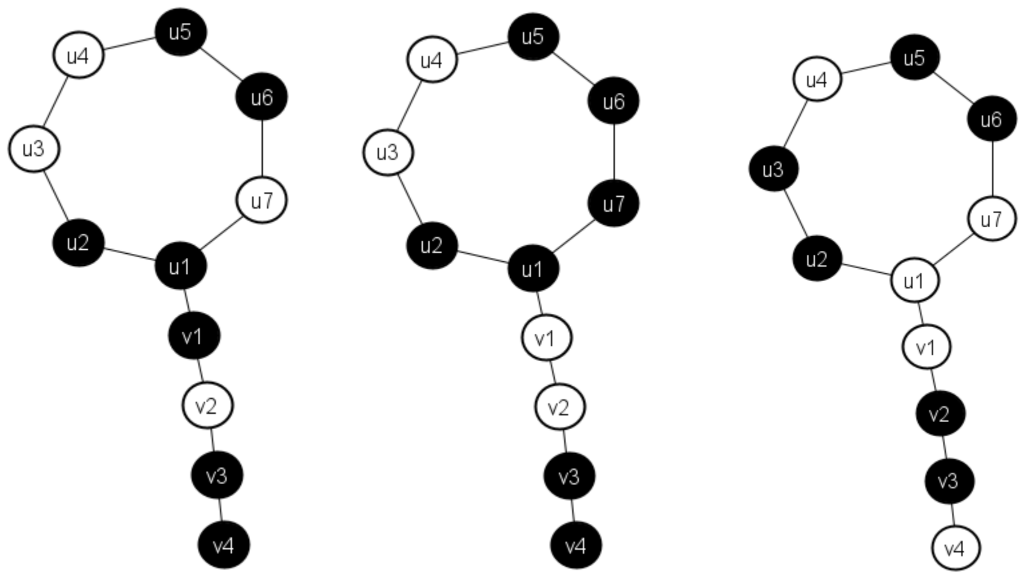


Figure 2.13. TOD^cS of tadpole graph $T_{7,4}$.

In Fig.2.12 shown TOD^cS with take u_1 and without take u_1.

Proposition 2.3.14 Let $L_{m,n}$ be lollipop graph; $m \geq 3, n \geq 2$, then $\gamma_{todc}(L_{m,n}) = 1 + \left\lceil \frac{n-1}{2} \right\rceil$.

$$= \begin{cases} \left\lceil \dfrac{n+1}{2} \right\rceil, & if\ n \equiv 0\ (mod\ 4) \\ \left\lceil \dfrac{n}{2} \right\rceil, & if\ n \equiv 1,2,3\ (mod\ 4) \end{cases}$$

Proof.

By the same hypothesis of proposition 2. The set of vertices in P_n is $\{v_1, v_2, \dots, v_n\}$ and the set of vertices in K_m is $\{u_1, u_2, \dots, u_m\}$ such that u_1 is vertex has degree m and v_n has degree one. There are the following two cases.

Case1. If m is even, let

$$D = \begin{cases} \{v_{4+4i}, v_{5+4i}; i = 0, \dots, \frac{n}{4} - 2\} \cup \{v_1, v_{n-1}, v_n, u_1\}, & if\ n \equiv 0\ (mod\ 4) \\ \{v_{4+4i}, v_{5+4i}; i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\} \cup \{v_1, u_1\}, & if\ n \equiv 1,2\ (mod\ 4) \\ \left\{v_{4+4i}, v_{5+4i}; i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\} \cup \{v_1, v_{n-1}, u_1\}, & if\ n \equiv 3\ (mod\ 4) \end{cases}$$

Since the vertex u_1 is adjacent to $m-1$ vertices and the other vertices of D is adjacent to one vertex from $V - D$, then D is TOD^cS.

Case2. If m is odd, then there are the following two subcases.

Subcase1. If $m = 3$, then it the same in proposition 2.3.13 case 2 in proof.

Subcase2. If $m \geq 5$, let $D = D_1 \cup D_2$ be dominating set such that

$$D_1 = \begin{cases} \{v_{2+4i}, v_{3+4i}; i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\}, & if\ n \equiv 0,3\ (mod\ 4)\ and\ n \neq 3 \\ \{v_{2+4i}, v_{3+4i}; i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\} \cup \{v_{n-1}\}, & if\ n \equiv 1 (mod\ 4) \\ \left\{v_{2+4i}, v_{3+4i}; i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\} \cup \{v_{n-1}, v_n\}, & if\ n \equiv 2 (mod\ 4) \end{cases}$$

And $D_2 = \{u_2, u_3\}$. By the same way in case1 we obtained that D is TOD^cS. In addition, by the same way of proposition 2.3.13 D in case1 and case2 are $MTOD^cS$. □

Theorem 2.3.15. For any connected graph of order n has at least three vertices of degree $n-2$, then $\gamma_{todc}(G) \leq 5$

Proof.

Let G has three vertices say $V(H) = \{v_1, v_2, v_3\}$ have degree $n-2$ such that H is subgraph of G. If one of them vertices independent to others, then this vertex has $n-3$ degree but that contradiction, so every vertex in H is adjacent to at least one vertex of them. So, an induced subgraph of H is connected subgraph and it is path of order three. In case the vertex that not adjacent to $v_i \in V(H)\ \forall i$ lie in $V(G)-V(H)$ then an induced subgraph of H is complete of order three, so $H \equiv K_3\ or\ P_3$. If there exist two adjacent vertices w_1, w_2 have even degree such $N(w_1) \cup N(w_2) = V(G)$, then $\gamma_{todc}(G) = 2$. Otherwise, then we distinguish two cases depending on whether order of graph is even or odd.

Case1. If n is odd, then there are two subcases.

Subcase1. If $\langle \{v_1, v_2, v_3\} \rangle \equiv K_3$, then we distinguish three cases as follows.

1) If these three vertices are not adjacent to the same vertex (say u), then there are two cases as follows.

I. If $N(u)$ has a vertex of odd degree say u_1 and u has even or odd degree, then $D = \{u_1, v_1, v_2\}$ is $MTOD^cS$ since $\langle D \rangle \equiv K_3$, so $\gamma_{todc}(G) = 3$.

II. If all vertices belong to the set $N(u)$ have even degree, then there are two cases.

A) If u has even degree, then $D = \{u, u_2, v_1, v_2\}$, where $u_2 \in N(u)$. One can be concluded that the set D is $MTOD^cS$, so $\gamma_{todc}(G) = 4$.

B) if u has odd degree, then if $\exists u_1, u_2 \in N(u)$ and adjacent, then $D_1 = \{u_1, u_2, v_1, v_2\}$ and $D_2 = \{u_1, u_2, v_1, u\}$ are $MTOD^cS$, so $\gamma_{todc}(G) = 4$. In case $\forall u_1, u_2$ are not adjacent, then $D_1 = \{u, u_1, u_2, v_1, v_2\}$ and $D_2 = \{u_1, u_2, v_1, v_2, v_3\}$ are $MTOD^cS$. So, $\gamma_{todc}(G) = 5$.

2) If two vertices say $v_1, and\ v_2$ are not adjacent to the same vertex say u_1 and the vertex v_3 is not adjacent to the vertex u_2, where $u_1\ and\ u_2$ are different, then $D = \{v_1, v_2, v_3\}$ is $MTOD^cS$. So, $\gamma_{todc}(G) = 3$.

3) If The vertex v_1 is not adjacent to the vertex say u_1 and the vertex v_2 is not adjacent to the vertex say u_2 and the vertex v_3 is not adjacent to the vertex say u_3 where the vertices $u_1, u_2\ and\ u_3$ are different, then $D = \{v_1, v_2, v_3\}$ is $MTOD^cS$, so $\gamma_{todc}(G) = 3$.

Subcase2. If $\langle\{v_1, v_2, v_3\}\rangle \equiv P_3$, then the vertices v_1 and v_3 are not adjacent. Then the set $D = \{v_1, v_2, v_3\}$ is $MTOD^cS$. So, $\gamma_{todc}(G) = 2$.

Case 2. If n is even, then there are two subcases as follows.

Subcase1. If $\langle\{v_1, v_2, v_3\}\rangle \equiv K_3$, then we distinguish three cases as follows.

1) If these three vertices are not adjacent to the same vertex (say u) then there are two cases as follows.

I) If u has odd, then two cases are discussed below.

A) If $\exists v \in N(u)$ such that v has even degree, then $D = \{v, v_1\}$ is MTOD^cS.

B) If all vertices in $N(u)$ has odd degree, then two cases are discussed below.

$\boldsymbol{B_1}$. If there are two vertices u_1 and u_2 are not adjacent, then $D = \{u_1, u_2, v_1, v_2\}$ is the $MTOD^cS$, so $\gamma_{todc}(G) = 4$.

$\boldsymbol{B_2}$. If all vertices in $N(u)$ are adjacent, then $D = \{u, u_1, u_2, v_1, v_2\}$ is the $MTOD^cS$, so $\gamma_{todc}(G) = 5$.

II) If u has even degree, then three cases are discussed below.

A. If $N(u)$ has a vertex of even degree say u_1, then $D = \{u_1, v_1\}$ is $MTOD^cS$, so $\gamma_{todc}(G) = 2$.

B. If all vertices belong to the set $N(u)$ have odd degree, then let $D = \{u, u_2, v_1\}$, where $u_2 \in N(u)$. One can be concluded that the set D is $MTOD^cS$, so $\gamma_{todc}(G) = 3$.

2) If two vertices say $v_1, and\ v_2$ are not adjacent to the same vertex say u_1 and the vertex v_3 is not adjacent to the vertex u_2, where $u_1\ and\ u_2$ are different, then there are two cases as follows.

A. If the vertex u_1 has even degree, then the set $D = \{u_1, v_3\}$ is a $MTOD^cS$, so $\gamma_{todc}(G) = 2$.

B. If the vertex u_2 has even degree, then the set $D = \{u_2, v_1\}$ is a $MTOD^cS$, so $\gamma_{todc}(G) = 2$.

C. If the vertex $u_1\ and\ u_2$ have odd degree, then the set $D = \{u_2, v_1, v_2, v_3\}$ is a $MTOD^cS$, so $\gamma_{todc}(G) = 4$.

3) If v_1 is not adjacent to a vertex say u_1 and a vertex v_2 is not adjacent to a vertex say u_2 and the vertex v_3 is not adjacent to the vertex say u_3 where the vertices$u_1, u_2, and\ u_3$ are different, then the set $D = \{v_1, v_2\}$ is $MTOD^cS$, so $\gamma_{todc}(G) = 2$.

Subcase2. If $\langle\{v_1, v_2, v_3\}\rangle \equiv P_3$, by hypothesis v_1 are v_3 pendent vertices of P_3. Then $D = \{v_1, v_2\}$ is $MTOD^cS$. So, $\gamma_{todc}(G) = 2$.

From all cases above, the result is obtained. □

Chapter Three

Fuzzy Odd Neighbor in D^c Domination Number in Fuzzy Graphs

Chapter Three

Fuzzy Odd Neighbor in D^c Domination in Graphs

3.1 Introduction

In this chapter, another concept of fuzzy domination is studied in the so-called fuzzy odd neighbor of D^c dominating set in the fuzzy graph, and many of its properties and examples are found for different types of fuzzy graphs, where many details of the fuzzy domination and domination are compared.

3.2 $FOD^cS -$ In Fuzzy Graphs

In this section, we review the results of the concept fuzzy odd neighbor of D^c dominating set.

For the fuzzy graphs, we used in this work only the strong fuzzy graphs, so the weight of the edges will not be mentioned because they are known by the weights of the vertices.

Definition 3.2.1

Let $G_f = (\sigma, \rho)$ be a fuzzy graph. A subset D (or D_f) of $V(G_f)$ is called a **fuzzy odd neighbor of D^c dominating set** (FOD^cS) of G_f if for every vertex $v \in D$, v is adjacent to odd number of vertices in D^c (or $V - D$)**.**

Definition 3.2.2

A fuzzy odd neighbor of D^c dominating set D of $G_f = (\sigma, \rho)$ is called a **minimal fuzzy odd neighbor of D^c dominating set** if there is no subset of D is a FOD^cS of G_f.

Definition 3.2.3

A fuzzy odd neighbor of D^c dominating set of a fuzzy graph $G_f = (\sigma, \rho)$ with minimum number of vertices is called a **minimum fuzzy odd neighbor of D^cdominating set** (MFOD^cS) of G_f.

Definition 3.2.4

A **fuzzy odd neighbor of D^c domination number** of a fuzzy graph $G_f = (\sigma, \rho)$ is the minimum sum of membership values of the vertices for all minimum fuzzy odd neighbor of D^c dominating sets and denoted by $\gamma_{fodc}(G_f)$ or simply γ_{fodc}.

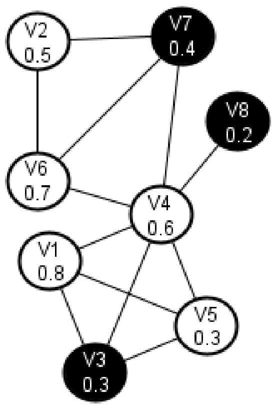


Figure 3.1. MFOD^cS of fuzzy graph G_f.

In Fig 3.1, $D_1 = \{v_5, v_7, v_8\}$ and $D_2 = \{v_3, v_7, v_8\}$ are MFOD^cS. It is clear $|D_1| = |D_2|$ so, $\gamma_{fodc}(G_f) = 0.9$.

Observation 3.2.5 Let G_f be strong fuzzy graph with n vertices, then the following are hold:

1) $0.1 \le \gamma_{fodc}(G_f) \le p$.

2) If G_f has set of isolated vertices $\{v_1, \dots, v_k\}$, then $\gamma_{fodc}(G_f) = \gamma_{fodc}(G_f - \{v_1, \dots, v_k\}) + \sum_{i=1}^{k} \sigma(v_i)$.

3) $\forall v \in D$; $\sigma(v) = 0.5$, then $\gamma_{fodc}(G_f) = \frac{\gamma_{odc}(G_f)}{2}$.

Proposition 3.2.6 If G_f is a strong fuzzy Path has n vertices, then $\gamma_{fodc}\left(P_{n_f}\right) =$

$$
\begin{cases}
\sigma(v_1), \quad if\ n = 1 \\
min\,\{\sigma(v_1), \sigma(v_2)\}, \quad if\ n = 2 \\
min\left\{\begin{matrix} \sigma(v_1) + \sigma(v_n) + \sum_{i=0}^{\frac{n}{4}-2}(\sigma(v_{4+4i}) + \sigma(v_{5+4i})), \\ \sum_{i=0}^{\frac{n}{4}-1}\left(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\right) \end{matrix}\right\}, if\ \ n \equiv 0\ (mod\ 4) \\
min\left\{\begin{matrix} \sigma(v_1) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{3+4i}) + \sigma(v_{4+4i})\right), \\ \sigma(v_1) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{4+4i}) + \sigma(v_{5+4i})\right), \\ \sigma(v_2) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\{(\sigma(v_{3+4i}) + \sigma(v_{4+4i}))\}, \\ \sigma(v_{n-1}) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\right), \\ \sigma(v_n) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\right), \\ \sigma(v_n) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{1+4i}) + \sigma(v_{2+4i})\right) \end{matrix}\right\}, if\ n \equiv 1\ (mod\ 4); n \neq 1 \\
min\left\{\begin{matrix} \sigma(v_n) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\right), \\ \sigma(v_1) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{4+4i}) + \sigma(v_{5+4i})) \end{matrix}\right\}, if\ n \equiv 2\ (mod\ 4); n \neq 2 \\
min\left\{\begin{matrix} \sum_{i=0}^{\left\lceil\frac{n}{4}\right\rceil-1}\left(\sigma(v_{1+4i}) + \sigma(v_{2+4i})\right), \\ \sum_{i=0}^{\left\lceil\frac{n}{4}\right\rceil-1}\left(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\right), \\ \sigma(v_2) + \sigma(v_n) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{3+4i}) + \sigma(v_{4+4i})\right), \\ \sigma(v_1) + \sigma(v_n) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{3+4i}) + \sigma(v_{4+4i})\right), \\ \sigma(v_1) + \sigma(v_{n-1}) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{4+4i}) + \sigma(v_{5+4i})\right), \\ \sigma(v_1) + \sigma(v_n) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{4+4i}) + \sigma(v_{5+4i})\right), \\ \sigma(v_{n-2}) + \sigma(v_{n-1}) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\right), \\ \sigma(v_2) + \sigma(v_3) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{5+4i}) + \sigma(v_{6+4i})\right), \end{matrix}\right\}, \\
if\ n \equiv 3\ (mod\ 4)
\end{cases}
$$

Proof.

There are four cases as follows.

Case1. If $n \equiv 0\ (mod\ 4)$, then there are only two $MFOD^cS$ of as follows.

$D_1 = \left\{v_{4+4i}, v_{5+4i}, i = 0, \ldots, \frac{n}{4} - 2\right\} \cup \{v_1, v_n\}$ and $D_2 = \left\{v_{2+4i}, v_{3+4i}, i = 0, \ldots, \frac{n}{4} - 1\right\}$, because if we take any set by leaving one vertex or more from these sets, then the set obtained is not dominating set or not FOD^cS. Since $|D_1| = |D_2| = \frac{n}{2}$, then $\gamma_{fodc}\left(P_{n_f}\right) = min\ \{\sum_{i=0}^{\frac{n}{4}-2}(\sigma(v_{4+4i}) + \sigma(v_{5+4i}) + \sigma(v_1) + \sigma(v_n)), \sum_{i=0}^{\frac{n}{4}-1}(\sigma(v_{2+4i}) + \sigma(v_{3+4i}))\}$.

Note that it is clear that ${D_1}^c = D_2$

Case2. If $n \equiv 1\ (mod\ 4)$, in case $n = 1$, $D = \{v_1\}$ is $MFOD^cS$ and $\gamma_{fodc}\left(P_{1_f}\right) = \sigma(v_1)$. Otherwise, there are six $MFOD^cS$ cases as follows.

$$D_1 = \left\{v_{1+4i}, v_{2+4i}, i = 0, \ldots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_n\}$$

$$D_2 = \left\{v_{2+4i}, v_{3+4i}, i = 0, \ldots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_{n-1}\}$$

$$D_3 = \left\{v_{2+4i}, v_{3+4i}, i = 0, \ldots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_n\}$$

$$D_4 = \left\{v_{3+4i}, v_{4+4i}, i = 0, \ldots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_1\}$$

$$D_5 = \left\{v_{3+4i}, v_{4+4i}, i = 0, \ldots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_2\}$$

$$D_6 = \left\{v_{4+4i}, v_{5+4i}, i = 0, \ldots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_1\}$$

All these sets are minimum and has the same order equal to $\left\lceil\frac{n}{2}\right\rceil$ because all vertices are adjacent to only one vertex in $V - D$ and any set has number of vertices less than $\left\lceil\frac{n}{2}\right\rceil$, then it is not FOD^cS. So, all above sets are $MFOD^cS$

and $\gamma_{fodc}\left(P_{n_f}\right) = min\,\{\sigma(v_n) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\big(\sigma(v_{1+4i}) + \sigma(v_{2+4i})\big), min\{\sigma(v_{n-1}), \sigma(v_n)\} + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\big((v_{2+4i}) + \sigma(v_{3+4i})\big), min\{\sigma(v_1), \sigma(v_2)\} + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{3+4i}) + \sigma(v_{4+4i}) +), \sigma(v_1) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{4+4i}) + \sigma(v_{5+4i}))\}$.

Case3. If $n \equiv 2\ (mod\ 4)$, in case $n = 2$, then $D_1 = \{v_1\}$ and $D_2 = \{v_2\}$ are $MFOD^cS$, then $\gamma_{fodc}\left(P_{2_f}\right) = min\{\sigma(v_1), \sigma(v_2)\}$. Otherwise, the sets D_3 and D_6 in case2 are unique $MFOD^cS$ in this case.

Case4. If $n \equiv 3\ (mod\ 4)$, then are eight FOD^cS as follows.

$$D_1 = \left\{v_{4+4i}, v_{5+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_1, v_n\}$$

$$D_2 = \left\{v_{4+4i}, v_{5+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_1, v_{n-1}\}$$

$$D_3 = \left\{v_{3+4i}, v_{4+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_1, v_n\}$$

$$D_4 = \left\{v_{3+4i}, v_{4+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_2, v_n\}$$

$$D_5 = \left\{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lceil\frac{n}{4}\right\rceil - 1\right\}$$

$$D_6 = \left\{v_{1+4i}, v_{2+4i}, i = 0, \dots, \left\lceil\frac{n}{4}\right\rceil - 1\right\}$$

$$D_7 = \left\{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_{n-2}, v_{n-1}\}$$

$$D_8 = \left\{v_{5+4i}, v_{6+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_2, v_3\}$$

By the same procedures above these sets are $MFOD^cS$. So, $\gamma_{fodc}\left(P_{n_f}\right) = min\,\{\sum_{i=0}^{\lceil\frac{n}{4}\rceil-1}\left(\sigma(v_{1+4i}) + \sigma(v_{2+4i})\right), \sum_{i=0}^{\lceil\frac{n}{4}\rceil-1}\left(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\right), \sigma(v_n) + min\{\sigma(v_1), \sigma(v_2)\} + \sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}\left(\sigma(v_{3+4i}) + \sigma(v_{4+4i})\right), \sigma(v_1) + min\{\sigma(v_{n-1}), \sigma(v_n)\} + \sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}\left(\sigma(v_{4+4i}) + \sigma(v_{5+4i})\right), \sigma(v_{n-2}) + \sigma(v_{n-1}) + \sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}\left(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\right), \sigma(v_2) + \sigma(v_3) + \sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}\left(\sigma(v_{5+4i}) + \sigma(v_{6+4i})\right)\}$. □

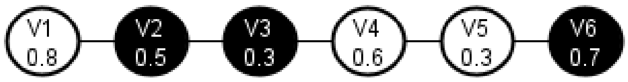


Figure 3.2. $MFOD^cS$ of path graph P_6.

Proposition 3.2.7 If G_f is a strong fuzzy Cycle has n vertices, then $\gamma_{fodc}\left(C_{n_f}\right) =$

$$\begin{cases} \left\{min\,\{\sum_{i=0}^{\frac{n}{4}-1}(\sigma(v_{j+4i}) + \sigma(v_{j+4i+1})); j = 1, \dots, n-1\}, if\ n \equiv 0\ (mod\ 4)\right\} \\ \left\{\begin{matrix} \min\left\{\left\{\begin{matrix}\sigma(v_{j+n-k}) + \sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}(\sigma(v_{j+4i}) + \sigma(v_{j+4i+1}); \\ j = 1, \dots n-1\end{matrix}\right\}; k = 1,3\right\} \\ j+n-k\ (mod\ n), if\ n \equiv 1\ (mod\ 4)\end{matrix}\right\} \\ \left\{\begin{matrix} min\left\{\begin{matrix}\sum_{i=0}^{3}\sigma(v_{j+i}) + \sum_{i=0}^{\lceil\frac{n}{4}\rceil-2}\left(\sigma(v_{j+6i}) + \sigma(v_{j+6i+1})\right); \\ j = 1, \dots, n-1\end{matrix}\right\}; \\ j+i, j+6i, j+6i+1\ (mod\ n) \\ if\ n \equiv 2\ (mod\ 4)\end{matrix}\right\} \\ \left\{\begin{matrix} min\left\{\sum_{i=0}^{\lfloor\frac{n}{4}\rfloor}\left(\sigma(v_{j+4i}) + \sigma(v_{j+4i+1})\right)\right\}, j = 1, \dots n-1, \\ j+4i, j+4i+1\ (mod\ n);\ if\ n \equiv 3\ (mod\ 4)\end{matrix}\right\} \end{cases}$$

Proof.

There are four cases as follows.

Case1. If $n \equiv 0\ (mod\ 4)$, then there are only four dominating set

$D_j = \left\{v_{j+6i}, v_{j+6i+1}, i = 0, \dots, \left\lceil \frac{n}{4} \right\rceil - 2\right\} \cup \{v_{j+i}, i = 0, \dots, 3\}; j = 1, \dots, n - 1$ such that D_j are unique dominating sets, the other sets are equal to one of these sets. By proposition 2.2.11, it is obvious that all sets are MOD^cS and so it is $MFOD^cS$. Thus, $\gamma_{fodc}\left(C_{n_f}\right) = min\,\{\sum_{i=0}^{\left\lceil \frac{n}{4} \right\rceil - 2}\left(\sigma\left(v_{j+6i}\right) + \sigma\left(v_{j+6i+1}\right)\right) + \sum_{i=0}^{3}\left(\sigma\left(v_{i+j}\right)\right)\}$,

Case2. If $n \equiv 1\ (mod\ 4)$, then let $D_j = \left\{v_{j+4i}, v_{j+4i+1}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\} \cup \left\{v_{j+n-k}\right\}$ $such\ that\ j = 1, \dots, n-1$ ans $k = 1,3$ and $j + n - k\ (mod\ n)$ are fuzzy dominating sets such that every vertex of them is adjacent to one vertex in $V - D$, then D_j are FOD^cS and have the same order equal to $\left\lceil \frac{n}{2} \right\rceil$ and it is minimum because any other sets have order less than $\left\lceil \frac{n}{2} \right\rceil$ are not FOD^cS. So, $\gamma_{fodc}\left(C_{n_f}\right) = min\,\{\sigma\left(v_{j+n-k}\right) + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma\left(v_{j+4i}\right) + \sigma\left(v_{j+4i+1}\right)\right)\}$.

Case3. If $n \equiv 2\ (mod\ 4)$, then there are only the following sets are FOD^cS

$D_1 = \left\{v_{j+4i}, v_{j+4i+1}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor\right\}, D_2 = \left\{v_{j+4i}, v_{j+4i+1}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\} \cup \left\{v_{j+4\left\lfloor \frac{n}{4} \right\rfloor - 1}, v_{j+4\left\lfloor \frac{n}{4} \right\rfloor}\right\}$, $D_3 = \left\{v_{j+4i}, v_{j+4i+1}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\} \cup \left\{v_{j+4\left\lfloor \frac{n}{4} \right\rfloor - 1}, v_{j+4\left\lfloor \frac{n}{4} \right\rfloor - 2}\right\}$ and $D_4 = \left\{v_{j+4i}, v_{j+4i+1}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\} \cup \left\{v_{j+4\left\lfloor \frac{n}{4} \right\rfloor - 2}, v_{j+4\left\lfloor \frac{n}{4} \right\rfloor + 1}\right\}$ are all dominating sets such that $j = 1, \dots, n$ for all sets above. All sets above are $MFOD^cS$ and has order $\left\lceil \frac{n+1}{2} \right\rceil$ such that any other sets have order less than $\left\lceil \frac{n+1}{2} \right\rceil$, it be not FOD^cS. Thus,

$$\gamma_{fodc}\left(C_{n_f}\right) = min\,\{\sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor}\left(\sigma(v_{j+4i}) + \sigma(v_{j+4i+1})\right), \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{j+4i}) + \sigma(v_{j+4i+1}) + \sigma\left(v_{j+4\left\lfloor\frac{n}{4}\right\rfloor-1}\right) + \sigma\left(v_{j+4\left\lfloor\frac{n}{4}\right\rfloor}\right), \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{j+4i}) + \sigma(v_{j+4i+1}) + \sigma\left(v_{j+4\left\lfloor\frac{n}{4}\right\rfloor-1}\right) + \sigma\left(v_{j+4\left\lfloor\frac{n}{4}\right\rfloor} - 2\right), \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{j+4i}) + \sigma(v_{j+4i+1}) + \sigma\left(v_{j+4\left\lfloor\frac{n}{4}\right\rfloor-2}\right) + \sigma\left(v_{j+4\left\lfloor\frac{n}{4}\right\rfloor+1}\right)\}.$$

Case4. If $n \equiv 3\ (mod\ 4)$, then

$D_1 = \left\{v_{j+4i}, v_{j+4i+1}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor\right\}$ and $D_2 = \left\{v_{j+4i}, v_{j+4i+1}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \left\{v_{j+4\left\lfloor\frac{n}{4}\right\rfloor-1}, v_{j+4\left\lfloor\frac{n}{4}\right\rfloor}\right\}$ are dominating sets such that $j = 1, \dots, n-1$. By the same procedures we get the result. □

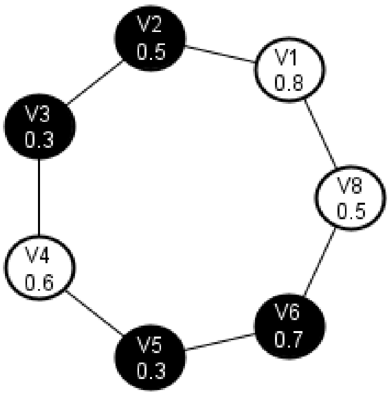


Figure 3.3. $MFOD^cS$ of cycle graph C_7.

Proposition 3.2.8 If G_f is a strong fuzzy complete has n vertices, then

$$\gamma_{fodc}\left(K_{n_f}\right) = \begin{cases} min\,\{\sigma(v_i), i = 1, \dots, n\} & , if\ n\ is\ even \\ min\{\sigma(v_i) + \sigma(v_j), i\ and\ j = 1, \dots, n\}\ , i \neq j, & if\ n\ is\ odd \end{cases}$$

Proof.

For complete graph there are two cases as follows.

Case1. If n is even, then $D_i = \{v_i\}$ such that $i = 1, \dots, n$. In this case we have n of FOD^cS and $\gamma_{fodc}(K_n) = min\{\sigma(v_i)\}$ $such\ that\ \ i = 1, \dots, n.$

Case2. If n is odd, then $D_j = \{v_i, v_j, i = 1, \dots, n\}$ are FOD^cS such that $j = 1, \dots, n$ and $i \neq j$. It is clear all these sets are $MFOD^cS$. So, we get the result. □

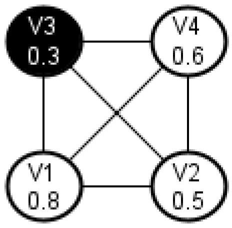


Figure 3.4. $MFOD^cS$ of complete graph K_4.

In Fig. 3.4, it is clear $D = \{v_3\}$ is $MFOD^cS$.

Proposition 3.2.9 If G is a strong fuzzy complete bipartite has $n + m$ vertices, then $\gamma_{fodc}\left(K_{m,n_f}\right) =$

$$\begin{cases} \sigma(u_1), \quad if\ n = 1\ and\ m\ is\ odd \\ \left\{min\left\{\sigma(u_1) + \sigma\left(v_j\right), j = 1, \dots, m\right\}, if\ n = 1\ and\ m\ is\ even\right\} \\ \{\sum_{i=1}^{2}(u_i) \quad , if\ n = 2\ and\ m\ is\ odd\ ;\mathrm{m} \neq 1\} \\ \left\{min\left\{\begin{matrix} \sigma(u_i) + \sigma\left(v_j\right), i = 1, \dots, n \\ and\ j = 1, \dots, m \end{matrix}\right\}, if\ n\ and\ m\ are\ even, n, n > 2\right\} \\ \left\{\begin{matrix} min\left\{\sigma(u_i) + \sigma\left(v_j\right) + \sigma(v_l), i = 1, \dots n\ and\ j, l = 1, \dots m\right\}; j \neq l\ , \\ if\ n\ is\ even\ and\ m\ is\ odd; n \neq 2\ and\ m \neq 1 \end{matrix}\right\} \\ \left\{\begin{matrix} min\left\{\sigma\left(v_j\right) + \sigma(u_i) + \sigma(u_k), i, k = 1, \dots n\ and\ j = 1, \dots m\right\}; i \neq k, \\ if\ m\ is\ even\ and\ n\ is\ odd; m \neq 2\ and\ n \neq 1 \end{matrix}\right\} \\ \left\{\begin{matrix} min\left\{\sigma(u_i) + \sigma(u_k) + \sigma\left(v_j\right) + \sigma(v_l), i, k = 1, \dots, n\ and\ j, l = 1, \dots, m\right\}; \\ i \neq k\ and\ j \neq l, if\ n\ and\ m\ are\ odd; n, m \neq 1 \end{matrix}\right\} \end{cases}$$

Such that $u_i \in V_n$ and $v_j \in V_m$; $V\left(K_{m,n_f}\right) = V_n \cup V_m$.

Proof.

Let $V_m = \{v_1, v_2, \dots, v_m\}$ has m vertices and $V_n = \{u_1, u_2, \dots, u_n\}$ has n vertices such that $n \leq m$. The cases we obtained as follows.

Case1. If $n = 1$ (if $m = 1$ is same), there are two subcases as follows.

Subcase1. If m is odd, in case $m = 1$ then it is clear that there is only one $MFOD^cS$ as $D_1 = \{u_1\}$ and $D_2 = \{v_1\}$, then $\gamma_{fodc}\left(K_{m,n_f}\right) = \min\{\sigma(u_1), \sigma(v_1)\}$. Otherwise, if $m \neq 1$ so $D_1 = \{u_1\}$ and $\gamma_{fodc}\left(K_{m,n_f}\right) = \sigma(u_1)$.

Subcase2. If m is even, then $D = \{u_1, v_j\}$ are $MFOD^cS$ such that $j = 1, \dots, m$. So, $\gamma_{fodc}\left(K_{m,n_f}\right) = \sigma(u_1) + min\{\sigma(v_j)\}$.

Case2. If $n = 2$ and m is odd; $m \neq 1$, there is only one FOD^cS $D = \{u_1, u_2\}$ and $\gamma_{fodc}\left(K_{m,n_f}\right) = \sigma(u_1) + \sigma(u_2)$.

Case3. If $n\ and\ m$ are even, then there are FOD^cS as $D = \{u_i, v_j\}$. It is obvious these are minimum and has the same order. Thus, $\gamma_{fodc}\left(K_{m,n_f}\right) = min\{\sigma(u_i) + \sigma(v_j), i = 1, \dots n\ and\ j = 1, \dots, m.\}$.

Case4. If $m\ is\ odd$; $m \neq 1$ and n is even; $n \neq 2$, then $D_j = \{u_i, v_j, v_l\}$ are $MFOD^cS$ such that $i = 1, \dots, n$ and $j, l = 1, \dots, m$; $j \neq l$. The converse (n is odd and n is even) is the same. Thus, we get the result.

Case5. If $n\ and\ m$ are odd, then $D = \{u_i, u_k, v_j, v_l\}; i, k = 1, \dots, n\ and\ j, l = 1, \dots, m.$ Then $\gamma_{fodc}\left(K_{m,n_f}\right) = min\{\sigma(u_i) + \sigma(u_k) + \sigma(v_j) + (v_l)\}; i \neq k\ and\ j \neq l.$

From all cases above, the result is obtained. □

Proposition 3.2.10 If G is a strong fuzzy wheel has n vertices, then $\gamma_{fodc}\left(W_{n_f}\right) =$

$$\begin{cases} \sigma(v_1)\,, & if\ n\ is\ even \\ min\left\{\sum_{i=0}^{\left\lfloor\frac{n-1}{4}\right\rfloor}\left(\sigma\left(v_{j+2i}\right)\right), j = 2,\dots,n\right\}, & if\ n = 5 \\ min\left\{\sum_{i=0}^{\left\lfloor\frac{n-1}{4}\right\rfloor}\left(\sigma\left(v_{j+3i}\right)\right), j = 2,\dots,n\right\}, & if\ n = 7,9 \\ \min\left\{\begin{matrix} \left\{\sum_{i=0}^{\left\lfloor\frac{n-1}{4}\right\rfloor}\left(\sigma\left(v_{j+3i}\right)\right), j = 2,\dots,n\right\} \cup \left\{v_{j+\left\lfloor\frac{n-1}{4}\right\rfloor+2}\right\} \\ ;\ j + \left\lfloor\frac{n-1}{4}\right\rfloor + 2\ (mod\ n-1), \\ \sigma(v_1) + \sigma(v_{i-1}) + \sigma(v_i) + \sigma(v_{i+1}), i = 3,\dots,n \\ ;\ \text{if } i-1 = 1, \text{then } v_{i-1} = v_n \text{ and} \\ \text{if } i+1 = 1\ (mod\ n), \text{then } v_{i+1} = v_2 \end{matrix}\right\}, & if\ n = 11 \\ \min\left\{\begin{matrix} \left\{\sum_{i=0}^{\left\lfloor\frac{n-1}{4}\right\rfloor}\left(\sigma\left(v_{j+3i}\right)\right), j = 2,\dots,n\right\} \\ \sigma(v_1) + \sigma(v_{i-1}) + \sigma(v_i) + \sigma(v_{i+1}), i = 3,\dots,n \\ ;\ \text{if } i-1 = 1, \text{then } v_{i-1} = v_n \text{ and} \\ \text{if } i+1 = 1\ (mod\ n), \text{then } v_{i+1} = v_2 \end{matrix}\right\}, & if\ n = 13 \\ min\left\{\begin{matrix} \sigma(v_1) + \sigma(v_{i-1}) + \sigma(v_i) + \sigma(v_{i+1}), i = 3,\dots,n \\ ;\ \text{if } i-1 = 1, \text{then } v_{i-1} = v_n \text{ and} \\ \text{if } i+1 = 1\ (mod\ n), \text{then } v_{i+1} = v_2 \end{matrix}\right\}, & if\ n \geq 15\ and\ odd \end{cases}$$

Proof.

Let $V = \{v_1, v_2, \dots, v_n\}$ such that v_1 is center vertex of strong fuzzy wheel graph. There are two cases as follows.

Case1. If n is even, then there are only one FOD^cS $D = \{v_1\}$. So, $\gamma_{fodc}\left(w_{n_f}\right) =$

Case2. If n is odd, then there are five subcases as follows.

Subcase1. If $n = 5$, $D_{j1} = \left\{v_{j+2i}, i = 0,\dots,\left\lfloor\frac{n-1}{4}\right\rfloor\ and\ j+2i\ (mod\ n)\right\}$; $j = 2,\dots,n$ is are FOD^cS. By proposition 2.2.12 of wheel in OD^cS, these sets are $MFOD^cS$ since it is MOD^cS. So, $\gamma_{fodc}\left(w_{n_f}\right) = min\left\{\sum_{i=0}^{\left\lfloor\frac{n-1}{4}\right\rfloor}(\sigma(v_{j+2i}))\right\}$.

Subcase2. If $n = 7,9$, then $D_{j2} = \left\{v_{j+3i}, i = 0, \dots, \left\lfloor\frac{n-1}{4}\right\rfloor \; and \; j + 2i \; (mod \; n)\right\}; j = 2, \dots, n$ are FOD^cS. By the same proposition above we get D_{j2} is $MFOD^cS$ and $\gamma_{fodc}\left(w_{n_f}\right) = min\left\{\sum_{i=0}^{\left\lfloor\frac{n-1}{4}\right\rfloor}(\sigma(v_{j+3i}))\right\}$.

Subcase3. If $n = 11$, let $D_{j3} = D_{j2} \cup \left\{v_{j+\left\lfloor\frac{n-1}{4}\right\rfloor+2}, j + \left\lfloor\frac{n-1}{4}\right\rfloor 2 \; (mod \; n)\right\}; j = 2, \dots n$ and $D_{j4} = \{v_1, v_{i-1}, v_i, v_{i+1}\}$ are fuzzy dominating sets; $i = 2, \dots, n$ and if $i - 1 = 1$, then $v_{i-1} = v_n$ and if $i + 1 = 1 \; (mod \; n)$, then $v_{i+1} = v_2$. Since all vertices in these sets are adjacent to zero or one or $n - 4$ and, then these sets are $MFOD^cS$ because in case leave any vertex it become do not FOD^cS. Thus, $\gamma_{fodc}\left(w_{n_f}\right) = min\left\{\sum_{i=0}^{\left\lfloor\frac{n-1}{4}\right\rfloor}(\sigma(v_{j+3i})) + v_{j+\left\lfloor\frac{n-1}{4}\right\rfloor+2}; j = 2, \dots, n, \sigma(v_1) + \sigma(v_{i-1}) + \sigma(v_i) + \sigma(v_{i+1}); i = 2, \dots, n\right\}$.

Subcase4. If $n = 13$, then it is clear D_{j2} and D_{j4} are $MFOD^cS$. $\gamma_{fodc}\left(w_{n_f}\right) = min\left\{\sum_{i=0}^{\left\lfloor\frac{n-1}{4}\right\rfloor}(\sigma(v_{j+3i})) \; ; j = 2, \dots, n, \sigma(v_1) + \sigma(v_{i-1}) + \sigma(v_i) + \sigma(v_{i+1}); i = 2, \dots, n\right\}$

Subcase5. If $n \geq 15$, then it is clear D_{j4} represent $MFOD^cS$ in this case. $\gamma_{fodc}\left(w_{n_f}\right) = min\{\sigma(v_1) + \sigma(v_{i-1}) + \sigma(v_i) + \sigma(v_{i+1}); i = 2, \dots, n\}$ □

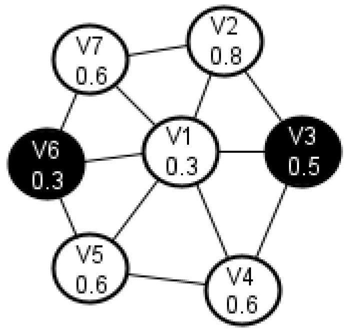


Figure 3.5. $MFOD^cS$ of wheel graph W_7.

Theorem 3.2.11 Let H_{n_f} be a strong fuzzy helm graph that has n vertices; $n \geq 3$, then $\gamma_{fodc}\left(H_{n_f}\right) =$

$$\begin{cases}
\left\{ \min\left\{ \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor - 1} \left(\sigma(v_{j+1+3i}) + \sigma(v_{j+2+3i}) + \sigma(v_{j+3i+m})\right) ; j = 1, \dots, n \right\} \quad if\ n \equiv 0\ (mod\ 3) \right\} \\
\left\{ min\left\{ \sigma(v_{j+n}) + \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor - 1} \left(\sigma(v_{j+1+3i}) + \sigma(v_{j+2+3i}) + (v_{j+m+3i})\right) ; j = 1, \dots, n \right\} \quad if\ n \equiv 1\ (mod\ 3) \right\} \\
\left\{ \min\left\{ \sigma(v_{j+n}) + \sigma(j+n+1) + \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor - 1} \left(\sigma(v_{j+2+3i}) + \sigma(v_{j+3+3i}) + \sigma(v_{j+m+3i})\right) ; j = 1, \dots, n \right\} \quad if\ n \equiv 2\ (mod\ 3) \right\} \\
\left\{ \min\left\{ \sum_{i=0}^{\frac{n}{4} - 1} \left(\sigma(v_{j+2+3i}) + \sigma(v_{j+3+3i}) + \sigma(v_{j+m+3i}) + \sigma(v_{j+1+m+3i})\right) ; j = 1, \dots, n \right\} \quad if\ n \equiv 0\ (mod\ 4) \right\} \\
\left\{ \sigma(v_{j+n}) + \min\left\{ \sum_{i=0}^{\lfloor \frac{n}{4} \rfloor - 1} \left(\sigma(v_{j+2+3i}) + \sigma(v_{j+3+3i}) + \sigma(v_{j+m+3i}) + \sigma(v_{j+1+m+3i})\right) \right\} \quad if\ n \equiv 1\ (mod\ 4) \right\} \\
\left\{ \min\left\{ \sum_{i=0}^{\lfloor \frac{n}{4} \rfloor} \left(\sigma(v_{j+2+3i}) + \sigma(v_{j+3+3i})\right) + \sum_{i=0}^{\lfloor \frac{n}{4} \rfloor - 1} \left(\sigma(v_{j+m+3i}) + \sigma(v_{j+1+m+3i})\right) \right\} \quad if\ n \equiv 2\ (mod\ 4) \right\}
\end{cases}$$

$$\begin{cases} \sigma(v_{j+n}) + \min\left\{ \begin{array}{l} \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor} \left(\sigma(v_{j+2+3i}) + \sigma(v_{j+3+3i})\right) + \\ \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1} (\sigma(v_{j+m+3i}) + \sigma(v_{j+1+m+3i})) \end{array} \right\} \\ if\ n \equiv 3\ (mod\ 4) \end{cases}$$

Such that $m = n\ or\ 0;\ if\ j + 3i \leq n, then\ m = n\ and\ if\ j + 3i > n, then\ m = 0$

Proof.

Let $V\left(H_{n_f}\right) = \{v_0, v_1, v_2, \dots, v_{2n}\}$ such that v_0 is center vertex; v_0 is adjacent to $\{v_1, v_2, \dots, v_n\}$. The set $\{v_{1+n}, v_{2+n}, \dots, v_{2n}\}$ represent all pendent vertices such that v_j is adjacent to v_{j+n}; $j = 1, \dots, n$. We obtained two cases as follows.

Case1. If take n is modulo three, then there three subcases as follows.

Subcase1. If $n \equiv 0\ (mod\ 3)$, let $D_{j1} = \left\{v_{j+1+3i}, v_{j+2+3i}, v_{j+m+3i}, i = 0, \dots, \left\lfloor \frac{n}{3} \right\rfloor - 1\right\}$ $m = 0\ or\ n\ such\ that\ if\ j + 3i \leq n, then\ m = n\ and\ if\ j + 3i > n\ , then\ m = 0$.

Subcase2. If $n \equiv 1\ (mod\ 3)$, let $D_{j2} = \left\{v_{j+1+3i}, v_{j+2+3i}, v_{j+m+3i}, i = 0, \dots, \left\lfloor \frac{n}{3} \right\rfloor - 1\right\} \cup \left\{v_{j+n}\right\}$ such that $m = 0\ or\ n\ ;\ if\ j + 3i \leq n, then\ m = n\ and\ if\ j + 3i > n\ , then\ m = 0$.

Subcase3. If $n \equiv 2\ (mod\ 3)$, let $D_{j3} = \left\{v_{j+2+3i}, v_{j+3+3i}, v_{j+m+3i}, i = 0, \dots, \left\lfloor \frac{n}{3} \right\rfloor - 1\right\} \cup \left\{v_{j+n}, v_{j+1+n}\right\}$.

It is clear that D_{j1}, D_{j2} and D_{j3} are dominating sets; $j = 1, \dots, n$. Since the vertices v_{j+1+3i}, v_{j+2+3i} and v_{j+3+3i} in every one of sets above are adjacent to only three vertices in $V - D_j$, and the vertices v_{j+n} and v_{j+1+n} are adjacent to only one vertex in $V - D_j$ and $|D_{j1}| = |D_{j2}| = |D_{j3}| = n$, so by theorem 2.2.26 D_{j1}, D_{j2} and D_{j3} are $MFOD^cS$. Then we obtained the result.

Case2. If take n is modulo four, then there three subcases as follows.

Subcase1. If $n \equiv 0 \ (mod\ 4)$, let $D_{j1} = \left\{v_{j+2+4i}, v_{j+3+4i}, v_{j+m+4i}, v_{j+1+m+4i}, i = 0, \dots, \frac{n}{4} - 1\right\}$ such that $m = 0\ or\ n;\ if\ j + 3i\ , j + 1 + 3i \leq n, then\ m = n\ and\ if\ j + 3i, j + 1 + 3i > n\ , then\ m = 0$.

Subcase2. If $n \equiv 1 \ (mod\ 4)$, let $D_{j2} = \left\{v_{j+2+4i}, v_{j+3+4i}, v_{j+m+4i}, v_{j+1+m+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \left\{v_{j+n}\right\}$ such that $m = 0\ or\ n;\ if\ j + 3i\ , j + 1 + 3i \leq n, then\ m = n\ and\ if\ j + 3i, j + 1 + 3i > n\ , then\ m = 0$.

Subcase3. If $n \equiv 2 \ (mod\ 4)$, let $D_{j3} = \left\{v_{j+2+4i}, v_{j+3+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor\right\} \cup \{v_{j+m+4i}, v_{j+1+m+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\}$ such that $m = 0\ or\ n;\ if\ j + 3i\ , j + 1 + 3i \leq n, then\ m = n\ and\ if\ j + 3i, j + 1 + 3i > n\ , then\ m = 0$.

Subcase4. If $n \equiv 3 \ (mod\ 4)$, $D_{j4} = \left\{v_{j+2+4i}, v_{j+3+4i+1}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor\right\} \cup \{v_{j+m+4i}, v_{j+1+m+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \left\{v_{j+n}\right\}$ such that $m = 0\ or\ n;\ if\ j + 3i\ , j + 1 + 3i \leq n, then\ m = n\ and\ if\ j + 3i, j + 1 + 3i > n\ , then\ m = 0$.

All cases above are dominating sets and since every vertex in D_{jk}; $k = 1, \dots, 4$ is adjacent to one or three vertices in $V - D_{jk}$ and $|D_{j1}| = |D_{j2}| =$

$|D_{j3}| = n$. So, by theorem 2.2.26 D_{j1}, D_{j2}, D_{j3} and D_{j4} are $MFOD^cS$.Thus, we get the result. □

Proposition 3.2.12 Let L_{m,n_f} be strong fuzzy lollipop graph, $m \geq 3, n \geq 2$ then if m is even, $\gamma_{fodc}(L_{m,n_f}) =$

$$\left\{\begin{array}{c}
\left\{\begin{array}{c} min\left\{\begin{array}{c}\sigma(u_1)+\sigma(v_1), \sigma(v_1)+min\{\sigma(u_j), j=2,\dots,m\} \\ ,\sigma(u_1)+min\{\sigma(u_j), j=2,\dots,m\}\end{array}\right\} \\ if\ n=1 \end{array}\right\} \\
\left\{\begin{array}{c} min\{\sigma(u_1)+\sigma(v_1), \sigma(v_2)+min\{\sigma(u_j), j=2,\dots,m\}\} \\ if\ n=2 \end{array}\right\} \\
\left\{\begin{array}{c} min\left\{\begin{array}{c}
\sum_{i=0}^{\lfloor\frac{n}{4}\rfloor}(\sigma(v_{1+4i})+\sigma(v_{2+4i}))+min\{\sigma(u_j), j=2,\dots,m\} \\
\sum_{i=0}^{\lfloor\frac{n}{4}\rfloor}(\sigma(v_{2+4i})+\sigma(v_{3+4i}))+min\{\sigma(u_j), j=2,\dots,m\} \\
\sigma(v_1)+\sigma(v_n)+\sigma(u_1)+\sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}(\sigma(v_{3+4i})+\sigma(v_{4+4i})) \\
\left\{\begin{array}{c}\sigma(v_2)+\sigma(v_n)+min\{\sigma(u_j), j=2,\dots,m\}+ \\ \sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}(\sigma(v_{3+4i})+\sigma(v_{4+4i}))\end{array}\right\} \\
\sigma(u_1)+\sigma(v_1)+\sigma(v_n)+\sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}(\sigma(v_{4+4i})+\sigma(v_{5+4i})) \\
\sigma(u_1)+\sigma(v_1)+\sigma(v_{n-1})+\sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}(\sigma(v_{4+4i})+\sigma(v_{5+4i}))
\end{array}\right\} \\ if\ n\equiv 3\ (mod\ 4) \end{array}\right\} \\
\left\{\begin{array}{c} min\left\{\begin{array}{c}
\sigma(u_1)+\sigma(v_1)+\sigma(v_n)+\sum_{i=0}^{\lfloor\frac{n-1}{4}\rfloor-1}(\sigma(v_{4+4i}))+\sigma(v_{5+4i}) \\
min\{\sigma(u_j); j=2,\dots,m\}+\sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}(\sigma(v_{1+4i}))+\sigma(v_{2+4i}) \\
min\{\sigma(u_j); j=2,\dots,m\}+\sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}(\sigma(v_{2+4i}))+\sigma(v_{3+4i})
\end{array}\right\} \\ if\ n\equiv 0\ (mod\ 4) \end{array}\right\}
\end{array}\right\}$$

$$\left\{\left\{\left\{ min \left\{ \begin{array}{c} \sigma(u_1)+\sigma(v_1)+\sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{4+4i})\right)+\sigma(v_{5+4i}) \\ \left\{ \begin{array}{c} \sigma(v_n)+min\{\sigma(u_j); j=2,\dots,m\}+ \\ \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{1+4i})\right)+\sigma(v_{2+4i}) \end{array} \right\} \\ \left\{ \begin{array}{c} \sigma\left(v_{4\left\lfloor\frac{n}{4}\right\rfloor}\right)+min\{\sigma(u_j); j=2,\dots,m\}+ \\ \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{2+4i})\right)+\sigma(v_{3+4i}) \end{array} \right\} \\ \left\{ \begin{array}{c} \sigma(v_n)+min\{\sigma(u_j); j=2,\dots,m\}+ \\ \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{2+4i})\right)+\sigma(v_{3+4i}) \end{array} \right\} \\ \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{3+4i})+\sigma(v_{4+4i})\right)+\sigma(v_2)+min\{\sigma(u_j), j=2,\dots,m\} \end{array} \right\} \right. \\ if\ n \equiv 1\ (mod\ 4) \right\} \\ \left\{ min \left\{ \begin{array}{c} \sigma(u_1)+\sigma(v_1)+\sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{4+4i})\right)+\sigma(v_{5+4i}) \\ \left\{ \begin{array}{c} \sigma(v_n)+min\{\sigma(u_j); j=2,\dots,m\}+ \\ \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\left(\sigma(v_{1+4i})\right)+\sigma(v_{2+4i}) \end{array} \right\} \end{array} \right\} \right. \\ if\ n \equiv 2\ (mod\ 4) \right\} \right\}$$

If m is odd, then $\gamma_{fodc}(L_{m,n_f}) =$

$$
\left\{
\begin{array}{c}
\left\{\begin{array}{c} \sigma(u_1) \\ if\ n = 1 \end{array}\right\} \\
\{\sigma(u_1) + \sigma(v_2)\ , if\ n = 2\} \\
\left\{\begin{array}{c} \sigma(u_1) + min\left\{\begin{array}{c} \sum_{i=0}^{\frac{n}{4}-1}\big(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\big) \\ \sum_{i=0}^{\frac{n}{4}-1}\big(\sigma(v_{3+4i}) + \sigma(v_{4+4i})\big) \end{array}\right\} \\ if\ n \equiv 0\ (mod\ 4) \end{array}\right\} \\
\left\{\begin{array}{c} \sigma(u_1) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\big(\sigma(v_{3+4i}) + \sigma(v_{4+4i})\big) \\ if\ n \equiv 1\ (mod\ 4); n \neq 1 \end{array}\right\} \\
\left\{\begin{array}{c} min\left\{\begin{array}{c} \sigma(u_1) + \sigma(v_n) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\big(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\big) \\ \sigma(u_1) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\big(\sigma(v_{3+4i}) + \sigma(v_{4+4i})\big) + min\{\sigma(v_{n-1}), \sigma(v_n)\} \\ \sigma(u_1) + \sigma(v_3) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\big(\sigma(v_{4+4i}) + \sigma(v_{5+4i})\big) \end{array}\right\} \\ n \equiv 2\ (mod\ 4); n \neq 2 \end{array}\right\} \\
\left\{\begin{array}{c} \sigma(u_1) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}\big(\sigma(v_{3+4i}) + \sigma(v_{4+4i})\big) + \sigma(v_n) \\ if\ n \equiv 3\ (mod\ 4) \end{array}\right\}
\end{array}
\right\}
$$

Proof.

Let the set of vertices in strong fuzzy path P_n be $\{v_1, v_2, \dots, v_n\}$ and the set of vertices in strong fuzzy complete K_m is $\{u_1, u_2, \dots, u_m\}$ such that u_1 is adjacent to m vertices represented by set $\{v_1, u_2, \dots, u_m\}$. We obtained two cases as follows.

Case1. If m is even, then we get six subcases as follows.

Subcase1. If $n = 1$, then $D_1 = \{u_j, v_1\}; j = 2, \dots, m$, $D_2 = \{u_j, u_1\}; j = 2, \dots, m$ and $D_3 = \{u_1, v_1\}$ are the unique $MFOD^cS$. So, $\gamma_{fodc}(L_{m,1_f}) = min\{\sigma(u_1) + \sigma(v_1), \sigma(v_1) + min\{\sigma(u_j), j = 2, \dots, m\}, \sigma(u_1) + min\{\sigma(u_j), j = 2, \dots, m\}\}$.

Subcase2. If $n = 2$, then $D_1 = \{u_j, v_2\}; j = 2, \dots, m$ and $D_2 = \{u_1, v_1\}$ are the unique $MFOD^cS$. So, $\gamma_{fodc}(L_{m,2_f}) = min\{\sigma(u_1) + \sigma(v_1), \sigma(v_2) + min\{\sigma(u_j), j = 2, \dots, m\}\}$.

Subcase3. If $n \equiv 3\ (mod\ 4)$, then there are six sets as follows.

$D_1 = \{u_1, v_1, v_n\} \cup \left\{v_{4+4i}, v_{5+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\}$, $D_2 = \{u_1, v_1, v_{n-1}\} \cup \left\{v_{4+4i}, v_{5+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\}$, $D_3 = \{u_j\} \cup \left\{v_{1+4i}, v_{2+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor\right\}; j = 2, \dots, m$, $D_4 = \{u_1, v_1, v_n\} \cup \left\{v_{3+4i}, v_{4+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\}$, $D_5 = \{u_j, v_2, v_n\} \cup \left\{v_{3+4i}, v_{4+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\}$ and $D_6 = \{u_j\} \cup \left\{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor\right\}; j = 2, \dots, m$ are the FOD^cS . Because the order of every set above is equal to $2 + \left\lceil\frac{n-1}{2}\right\rceil$, then by proposition 2.2.28 all sets above are FOD^cS. So, $\gamma_{fodc}(L_{m,n_f}) = min\Big\{\sigma(u_1) + \sum_{i=0}^{\lfloor\frac{n}{4}\rfloor}(\sigma(v_{1+4i}) + \sigma(v_{2+4i})), \sum_{i=0}^{\lfloor\frac{n}{4}\rfloor}(\sigma(v_{1+4i}) + \sigma(v_{2+4i})) + min\{\sigma(u_j), j = 2, \dots, m\}), \sum_{i=0}^{\lfloor\frac{n}{4}\rfloor}(\sigma(v_{2+4i}) + \sigma(v_{3+4i})) + min\{\sigma(u_j), j = 2, \dots, m\}), \sigma(u_1) + \sigma(v_1) + \sigma(v_n) + \sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}(\sigma(v_{4+4i}) + \sigma(v_{5+4i})), \sigma(u_1) + \sigma(v_1) + \sigma(v_{n-1}) + \sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}(\sigma(v_{4+4i}) + \sigma(v_{5+4i})), \sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}\big(\sigma(v_{3+4i}) + \sigma(v_{4+4i})\big) + \sigma(v_n) + \sigma(u_1) +$

$min\{\sigma(u_j), j = 2, \dots, m\}, \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma(v_{3+4i}) + \sigma(v_{4+4i})\right) + \sigma(v_n) + \sigma(u_1) + min\{\sigma(u_j), j = 2, \dots, m\}\Big\}$.

Subcase4. If $n \equiv 1\ (mod\ 4)$, then $D_1 = \{u_1, v_1\} \cup \left\{v_{4+4i}, v_{5+4i}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\}$, $D_2 = \{u_j, v_n\} \cup \left\{v_{1+4i}, v_{2+4i}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\}$, $D_3 = \left\{u_j, u_{4\left\lfloor \frac{n}{4} \right\rfloor}\right\} \cup \left\{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\}$, $D_4 = \{v_2, u_j\} \cup \left\{v_{3+4i}, v_{4+4i}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\}$ and $D_5 = \{u_j, v_n\} \cup \left\{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\}$ such that $j = 2, \dots, m$ are dominating sets and $MFOD^cS$ since u_j is adjacent to $m-1$ vertices in complement set of dominating set and $v_{1+4i}, v_{2+4i}, v_{3+4i}, v_{4+4i}, v_{5+4i}$ and v_n are adjacent to only one vertex in complement set of dominating set. So, $\gamma_{fodc}(L_{m,n_f}) = min\Big\{\sigma(u_1) + \sigma(v_1) + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma(v_{4+4i})\right) + \sigma(v_{5+4i}), \sigma(v_n) + min\{\sigma(u_j); j = 2, \dots, m\} + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma(v_{1+4i})\right) + \sigma(v_{2+4i}), \sigma(v_n) + min\{\sigma(u_j); j = 2, \dots, m\} + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma(v_{2+4i})\right) + \sigma(v_{3+4i}), \sigma(v_n) + min\{\sigma(u_j); j = 2, \dots, m\} + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma(v_{2+4i})\right) + \sigma(v_{3+4i}), \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma(v_{3+4i})\right) + \sigma(v_{4+4i}) + \sigma(v_2) + min\{\sigma(u_j); j = 2, \dots, m\}\Big\}$.

Subcase5. If $n \equiv 2\ (mod\ 4)$, then $D_1 = \{u_1, v_1\} \cup \left\{v_{4+4i}, v_{5+4i}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\}$ and $D_2 = \{u_j, v_n\} \cup \left\{v_{1+4i}, v_{2+4i}, i = 0, \dots, \left\lfloor \frac{n}{4} \right\rfloor - 1\right\}; j = 2, \dots, m$ are dominating set and by the same way above, we get $MFOD^cS$. So, $\gamma_{fodc}(L_{m,n_f}) = min\Big\{\sigma(u_1) + \sigma(v_1) + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma(v_{4+4i})\right) +$

$\sigma(v_{5+4i}),\sigma(v_n)+min\{\sigma(u_j);j=2,\dots,m\}+\sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}(\sigma(v_{1+4i}))+\sigma(v_{2+4i})\Big\}$.

Subcase6. If $n\equiv 0\ (mod\ 4)$, then there are three sets as follows.

$D_1=\{u_1,v_1,v_n\}\cup\left\{v_{4+4i},v_{5+4i},i=0,\dots,\left\lfloor\frac{n-1}{4}\right\rfloor-1\right\}$, $D_2=\{u_j\}\cup\left\{v_{1+4i},v_{2+4i},i=0,\dots,\frac{n}{4}-1\right\}$ and $D_3=\{u_j\}\cup\left\{v_{2+4i},v_{3+4i},i=0,\dots,\frac{n}{4}-1\right\}$ $j=2,\dots,m$ are dominating sets and has order $1+\left\lceil\frac{n-1}{2}\right\rceil$, so it is $MFOD^cS$. Thus, $\gamma_{fodc}(L_{m,n_f})=min\Big\{\sigma(u_1)+\sigma(v_1)+\sigma(v_n)+\sum_{i=0}^{\lfloor\frac{n-1}{4}\rfloor-1}(\sigma(v_{4+4i}))+\sigma(v_{5+4i}),min\{\sigma(u_j);j=2,\dots,m\}+\sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}(\sigma(v_{1+4i}))+\sigma(v_{2+4i}),min\{\sigma(u_j);j=2,\dots,m\}+\sum_{i=0}^{\lfloor\frac{n}{4}\rfloor-1}(\sigma(v_{2+4i}))+\sigma(v_{3+4i})\Big\}$.

Case2. If m is odd, then we obtained seven subcases as follows.

Subcase1. If $n=1$, then $D=\{u_1\}$ is $MFOD^cS$. $\gamma_{fodc}(L_{m,1_f})=\sigma(u_1)$.

Subcase2. If $n=2$, then $D=\{u_1,v_2\}$ is $MFOD^cS$. $\gamma_{fodc}(L_{m,2_f})=\sigma(u_1)+\sigma(v_2)$.

Subcase3. If $n=3$, then $D=\{u_1,v_3\}$ is $MFOD^cS$. $\gamma_{fodc}(L_{m,3_f})=\sigma(u_1)+\sigma(v_3)$.

Subcase4. If $n\equiv 0\ (mod\ 4)$, then $D_1=\{u_1\}\cup\left\{v_{2+4i},v_{3+4i},i=0,\dots,\frac{n}{4}-1\right\}$ and $D_2=\{u_1\}\cup\left\{v_{3+4i},v_{4+4i},i=0,\dots,\frac{n}{4}-1\right\}$ are the unique

$MFOD^cS$. $\gamma_{fodc}(L_{m,n_f}) = \sigma(u_1) + min\left\{\sum_{i=0}^{\frac{n}{4}-1}(\sigma(v_{2+4i}) + \sigma(v_{3+4i})), \sum_{i=0}^{\frac{n}{4}-1}(\sigma(v_{3+4i}) + \sigma(v_{4+4i}))\right\}$.

Subcase5. If $n \equiv 1\ (mod\ 4); n \neq 1$, then $D = \{u_1\} \cup \left\{v_{3+4i}, v_{4+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\}$ is the unique $MFOD^cS$. $\gamma_{fodc}(L_{m,n_f}) = \sigma(u_1) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{3+4i}) + \sigma(v_{4+4i}))$.

Subcase6. If $n \equiv 2\ (mod\ 4); n \neq 2$, then $D_1 = \{u_1, v_n\} \cup \left\{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\}$, $D_2 = \{u_1, v_{n-1}\} \cup \left\{v_{3+4i}, v_{4+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\}$ and $D_3 = \{u_1, v_n\} \cup \left\{v_{3+4i}, v_{4+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\}$ and $D_4 = \{u_1, v_3\} \cup \left\{v_{4+4i}, v_{5+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\}$ are dominating sets and FOD^cS. Since order of all sets above is equal to $\left\lceil\frac{n-1}{2}\right\rceil$. So, it is $MFOD^cS$ and $\gamma_{fodc}(L_{m,n_f}) = \sigma(u_1) + min\left\{\sigma(v_n) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{2+4i}) + \sigma(v_{3+4i})), \sigma(v_n) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{3+4i}) + \sigma(v_{4+4i})), \sigma(v_{n-1}) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{3+4i}) + \sigma(v_{4+4i}))\right\}$.

Subcase7. If $n \equiv 3\ (mod\ 4)$, then $D = \{u_1, v_n\} \cup \left\{v_{3+4i}, v_{4+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\}$ is the unique $MFOD^cS$. $\gamma_{fodc}(L_{m,n_f}) = \sigma(u_1) + \sigma(v_n) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{3+4i}) + \sigma(v_{4+4i}))$. □

Theorem 3.2.13 Let G_f be strong fuzzy graph has only three vertices are adjacent to $n-2$ vertices; n is number of vertices of G. If n is odd

Then $\gamma_{fodc}(G_f) =$

$$
\left\{\begin{array}{l}
\left\{\begin{array}{c}
\text{if } \langle\{v_1,v_2,v_3\}\rangle \equiv K_3, \text{then}\\
a)\left\{\begin{array}{c}
\text{if } u \text{ is not adjacent to all vertices in } \{v_1,v_2,v_3\}, \text{then}\\
a_1)\left\{\begin{array}{c} \sigma(u)+\min\{\sigma(v_i);\ i=1,2,3\}, \text{if } u \text{ is isolated}\\ \text{vertex or adjacent to odd number of vertices}\end{array}\right\}\\
a_2)\left\{\begin{array}{c} \min\{\sigma(u_k)+\sigma(v_i)+\sigma(v_j)\};\ k=1,\dots,n-4 \text{ and}\\ i,j=1,2,3; i\neq j, \text{if } u \text{ is adjacent to even number of}\\ \text{vertices and adjacent to vertex } u_k \text{ and}\\ u_k \text{ is adjacent to odd number of vertices}\end{array}\right\}\\
a_3)\left\{\begin{array}{c} \min\{\sigma(u_k)+\sigma(v_i)+\sigma(v_j)+\sigma(u)\};\\ k=1,\dots,n-4 \text{ and}\\ i,j=1,2,3, \text{if } u \text{ is adjacent to even number of}\\ \text{vertices and adjacent to } u_k \text{ and}\\ u_k \text{ is adjacent to even number of vertices}\end{array}\right\}
\end{array}\right\}\\
b)\left\{\begin{array}{c}
\text{if } u_1 \notin N(v_1)=N(v_2) \text{ and } u_2 \notin N(v_3), \text{then}\\
b_1)\left\{\begin{array}{c} \sigma(u_1)+\min\{\sigma(v_1),\sigma(v_2)\}, \text{if } u_1 \text{ is adjacent to}\\ \text{odd number of vertices}\end{array}\right\}\\
b_2)\left\{\begin{array}{c} \sigma(u_2)+(v_3), \text{if } u_2 \text{ is adjacent to odd}\\ \text{number of vertices}\end{array}\right\}\\
b_3)\left\{\begin{array}{c} \min\{\sigma(u_2)+\sigma(v_3),\sigma(u_1)+\sigma(v_1),\sigma(u_1)+\sigma(v_2)\},\\ \text{if } u_1 \text{ and } u_2 \text{ is adjacent to odd number of vertices}\end{array}\right\}\\
b_4)\left\{\begin{array}{c} \sum_{i=1}^{3}\sigma(v_i), \text{if } u_1 \text{ and } u_2 \text{ is adjacent to even number}\\ \text{of vertices}\end{array}\right\}
\end{array}\right\}
\end{array}\right\}\\
c)\left\{\begin{array}{c}
\text{if } u_1 \notin N(v_1), u_2 \notin N(v_2) \text{ and } u_3 \notin N(v_3), \text{then}\\
\text{if } \{u_i; i=1,2,3\} \text{ are adjacent to even number of vertices, then}\\
c_1)\left\{\begin{array}{c} \sigma(u_i)+\sigma(u_j); i,j=1,2,3 \text{ andi } \neq j,\\ \text{if } N(u_i)\cup N(u_j)=V(G) \text{ and } u_i \text{ and } u_j \text{ are}\\ \text{adjacent}\end{array}\right\}\\
c_2)\left\{\sum_{i=1}^{3}\sigma(v_i), \text{otherwise}\right\}
\end{array}\right\}\\
\{\text{if } \langle\{v_1,v_2,v_3\}\rangle \equiv P_3, \text{then } \sigma(v_1)+\sigma(v_3);\ v_1 \text{ and } v_3 \text{ are not adjacent}\}
\end{array}\right\}
$$

If n is even, then

$$
\gamma_{fodc}(G_f) = \left\{ \begin{array}{c}
\{\sigma(v); if\ v\ is\ adjacent\ to\ n-1\ vertices\} \\
\left\{ \begin{array}{c}
\left\{ \begin{array}{c}
\left\{ \begin{array}{c}
\left\{ \begin{array}{c}
\left\{ \begin{array}{c} min\,\{\sigma(u_k)+\sigma(v_i); i=1,2,3 \\ and\ k=1,\dots,n-5\}, if\ u_k \in N(u) \\ and\ adjacent\ to\ even\ number\ of\ vertices \end{array} \right\} \\
\left\{ \begin{array}{c} \sigma(u)+min\{\sigma(v_i)+(v_j); i,j=1,2,3\ and\ i \neq j\} \\ if\ \forall u_k \in N(u) is\ adjacent\ to \\ odd\ number\ of\ vertices \end{array} \right\} \\
if\ u\ is\ adjacent\ to\ odd\ number\ of\ vertices
\end{array} \right\} \\
\left\{ \begin{array}{c}
\left\{ \begin{array}{c} min\{\sigma(u_k)+\sigma(v_i); k=1,\dots,n-5\ and\ i=1,2,3\}, \\ if\ u_k \in N(u)\ and\ adjacent\ to \\ even\ number\ of\ vertices \end{array} \right\} \\
\left\{ \begin{array}{c} \sigma(u)+min\left\{ \begin{array}{c} \sigma(v_i)+\sigma(u_k); i=1,2,3\ and\ k=1,\dots \\ ,n-5 \end{array} \right\} \\ if\ \forall u_k \in N(u) is\ adjacent\ to \\ odd\ number\ of\ vertices \end{array} \right\} \\
if\ u\ is\ adjacent\ to\ even\ number\ of\ vertices
\end{array} \right\}
\end{array} \right\} \\
if\ u \notin N(v_1) \cup N(v_2) \cup N(v_3) \\
\left\{ \begin{array}{c}
\left\{ \begin{array}{c}
\left\{ \begin{array}{c} \sigma(u_1)+(v_3), if\ u_1\ is\ adjacent\ to \\ even\ number\ of\ vertices \end{array} \right\} \\
\left\{ \begin{array}{c} \sigma(u_2)+min\{\sigma(v_1),\sigma(v_2)\}, if\ u_2\ is\ adjacent\ to \\ even\ number\ of\ vertices \end{array} \right\} \\
\left\{ \begin{array}{c} \sigma(u_1)+\sum_{i=1}^{2}\sigma(v_i), if\ u_1\ and\ u_2\ are\ adjacent\ to \\ even\ number\ of\ vertices \end{array} \right\}
\end{array} \right\} \\
if\ u_1 \notin N(v_1)=N(v_2)\ and\ u_2 \notin N(v_3)
\end{array} \right\} \\
\left\{ \begin{array}{c} min\,\{\sigma(v_i)+\sigma(v_j); i,j=1,2,3\ and\ i \neq j\} \\ if\ u_1 \notin N(v_1), u_2 \notin N(v_2)\ and\ u_3 \notin N(v_3) \end{array} \right\}
\end{array} \right\} \\
if\ \langle\{v_1,v_2,v_3\}\rangle \equiv K_3
\end{array} \right\} \\
\left\{ \begin{array}{c} min\{\sigma(v_1)+\sigma(v_2),\sigma(v_2)+\sigma(v_3)\} \\ if\ \langle\{v_1,v_2,v_3\}\rangle \equiv P_3;\ v_1\ and\ v_3 \\ are\ not\ adjacent \end{array} \right\}
\end{array} \right\}
$$

Such that v_1, v_2, v_3 are vertices has property above and u is independent vertex to v_1, v_2, v_3.

Proof.

Let G has three vertices say $\{v_1, v_2, v_3\}$ are adjacent to $n-2$ vertices; n is number of vertices of fuzzy graph. If one of them vertices independent to others, then this vertex adjacent to only $n-3$ vertices but that contradiction, so every vertex of $\{v_1, v_2, v_3\}$ is adjacent to at least one vertex of them. By the same hypothesis in theorem 2.2.24 so $\langle\{v_1, v_2, v_3\}\rangle \equiv K_3\ or\ P_3$. Then there are two cases depending on whether the number of vertices n of Gas follows.

Case1. If n is odd, then there are two subcases.

Subcase1. If $\langle\{v_1, v_2, v_3\}\ \rangle \equiv K_3$, then we distinguish three cases as follows.

1) If these three vertices are not adjacent to the same vertex (say u) note that this vertex is unique because hypothesis above. Then we obtained two subcases as follows.

I. If u is isolated vertex or adjacent to odd number of vertices, then $D = \{u, v_i\}; i = 1,2,3$ are $MFOD^cS$, so $\gamma_{fodc}(G) = min\{\sigma(u) + \sigma(v_i)\}; i = 1,2,3$.

II. If u is adjacent to even number of vertices, then there are two cases as follows.

A. If the set $M = \{u_k; k = 1, \dots, n-4\}$ is a set of vertices that adjacent to u and adjacent to odd number of vertices. Then $D = \{u_k, v_i, v_j\}; j \neq i\ and\ i,j = 1,2,3$ are $MFOD^cS$, so $\gamma_{fodc}(G) = min\{\sigma(u_k) + \sigma(v_i) + \sigma(v_j)\}; k = 1, \dots, n-5\ and\ i,j = 1,2,3; i \neq j.$

B. If all vertices that adjacent to u are adjacent to even number of vertices say $H = \{u_k; k = 1, \dots, n-5\}$, then $D = \{u, u_k, v_i, v_j\}$. Then the set D is

$MFOD^cS$, so $\gamma_{fodc}(G) = min\{\sigma(u_k) + \sigma(v_i) + \sigma(v_j) + \sigma(u)\}; k = 1, \dots, n-5\ and\ i, j = 1,2,3; i \neq j$.

2) If two vertices say $v_1, and\ v_2$ are not adjacent to the same vertex say u_1 (that means $N(v_1) = N(v_2)$) and the vertex v_3 is not adjacent to the vertex u_2, where $u_1\ and\ u_2$ are different, then there are four cases as follows.

A) If the vertex u_1 is adjacent to odd number of vertices and the vertex u_2 is adjacent to even number of vertices, then the set $D_1 = \{u_1, v_1\}$ and $D_2 = \{u_1, v_2\}$ are $MFOD^cS$, so $\gamma_{fodc}(G) = min\{\sigma(u_1) + \sigma(v_1), \sigma(u_1) + \sigma(v_2)\}$.

B) If the vertex u_1 is adjacent to even number of vertices and the vertex u_2 is adjacent to odd number of vertices, then the set $D = \{u_2, v_3\}$ is MOD^cS, so $\gamma_{fodc}(G) = \sigma(u_2) + \sigma(v_3)$.

C) If the vertices $u_1\ and\ u_2$ are adjacent to odd number of vertices, then $D_1 = \{u_1, v_1\}$, $D_2 = \{u_1, v_2\}$ and $D_3 = \{u_2, v_3\}$ are $MFOD^cS$, so $\gamma_{fodc}(G) = min\{\sigma(u_2) + \sigma(v_3), \sigma(u_1) + \sigma(v_1), \sigma(u_1) + \sigma(v_2)\}$.

***D*)** If the vertices $u_1\ and\ u_2$ are adjacent to even number of vertices, then $D = \{v_1, v_2, v_3\}$ is $MFOD^cS$. So, $\gamma_{fodc}(G) = \sum_{i=1}^{3} \sigma(v_i)$.

3) If The vertex v_1 is not adjacent to the vertex say u_1 and the vertex v_2 is not adjacent to the vertex say u_2 and the vertex v_3 is not adjacent to the vertex say u_3 where the vertices$u_1, u_2, and\ u_3$ are different, then there are cases as follows.

A. If at least one vertex from the set $H = \{u_1, u_2, u_3\}$ is adjacent to odd number of vertices or all vertices in H is adjacent to odd number of vertices, then $D = \{u_i, v_i\}$ is $MFOD^cS$ such that $i = 1,2,3$, so $\gamma_{fodc}(G) = min\{\sigma(u_i) + \sigma(v_i); i = 1,2,3\}$.

B. If all vertices in the set H have even degree, then there are two cases as follows.

$\boldsymbol{B_1}$**.** If there are two vertices of the set H are adjacent say $\{u_i, u_j\}; i, j = 1,2,3$ and $i \neq j$. If $N(u_i) \cup N(u_j) = V(G)$, then the set $D = \{u_i, u_j\}$ is $MFOD^cS$, so $\gamma_{fodc}(G) = min\{\sigma(u_i) + \sigma(u_j); i, j = 1,2,3\ andi \neq j\}$

$\boldsymbol{B_1}$**.** If there are no two vertices of the set H are adjacent, then the set $D = \{v_1, v_2, v_3\}$ is $MFOD^cS$, so $\gamma_{fodc}(G) = \sum_{i=1}^{3} \sigma(v_i)$.

Subcase2. If $\langle\{v_1, v_2, v_3\}\rangle \equiv P_3$, then the vertices v_1 and v_3 are not adjacent. Then the set $D = \{v_1, v_3\}$ is $MFOD^cS$. So, $\gamma_{fodc}(G) = \sigma(v_1) + \sigma(v_3)$.

Case 2. If n is even, then if G has at least one vertex v is adjacent to $n-1$, then $D = \{v\}$ and $\gamma_{fodc}(G) = \sigma(v)$. Otherwise, there are two subcases:

Subcase1. If $\langle\{v_1, v_2, v_3\}\rangle \equiv K_3$, then we distinguish three cases as follows.

1) If these three vertices are not adjacent to the same vertex (say u) then there are two cases as follows.

I) If u has odd, then there are two cases as follows.

A. If $u_k \in N(u); k = 1, \dots, n-5$ are adjacent to even degree, then $D = \{u_k, v_i\}; k = 1, \dots, n-5\ and\ i = 1,2,3$ is $MFOD^cS$. Then $\gamma_{fodc}(G) = min\,\{\sigma(u_k) + \sigma(v_i), i = 1,2,3\}$.

B. If all vertices in $N(u)$ are adjacent to odd number of vertices, then $D = \{u, v_i, v_j\}; i, j = 1,2,3\ and\ i \neq j$ is the $MFOD^cS$, so $\gamma_{fodc}(G) = \sigma(u) + min\{\sigma(v_i) + (v_j); i, j = 1,2,3\ and\ i \neq j\}$.

II) If u has even degree, then there are three cases as follows.

A. If $u_k \in N(u); k = 1, \dots, n-5$ such that u_k is adjacent to even number of vertices, then $D = \{u_k, v_i\}; k = 1, \dots, n-5$ and $i = 1,2,3$ are $MFOD^cS$, so $\gamma_{fodc}(G) = min\,\{\sigma(u_k) + \sigma(v_i); k = 1, \dots, n-5\ and\ i = 1,2,3\}$.

B. If all vertices belong to the set $N(u)$ are adjacent to odd number of vertices, then let $D = \{u, u_k, v_i\}$, where $u_k \in N(u); k = 1, \dots, n-5\ and\ i = 1,2,3$. It is obvious the set D is $MFOD^cS$, so $\gamma_{fodc}(G) = \sigma(u) + min\{\sigma(v_i) + \sigma(u_k); i = 1,2,3\ and\ k = 1, \dots, n-5\}$.

2) If two vertices say $v_1, and\ v_2$ are not adjacent to the same vertex say u_1 and the vertex v_3 is not adjacent to the vertex u_2, where $u_1\ and\ u_2$ are different, then there are two cases as follows.

A. If the vertex u_1 is adjacent to even number of vertices, then the set $D = \{u_1, v_3\}$ is a $MFOD^cS$, so $\gamma_{fodc}(G) = \sigma(u_1) + \sigma(v_3)$.

B. If the vertex u_2 has even degree, then the set $D = \{u_2, v_i\}; i = 1,2$ are a $MFOD^cS$, so $\gamma_{fodc}(G) = \sigma(u_2) + min\{\sigma(v_1), \sigma(v_2)\}$.

C. If the vertex $u_1\ and\ u_2$ are adjacent to odd number of vertices, then the set $D = \{u_1, v_1, v_2\}$ is unique $MFOD^cS$, so $\gamma_{fodc}(G) = \sigma(u_1) + \sigma(v_1) + \sigma(v_2)$.

3) If The vertex v_1 is not adjacent to the vertex say u_1 and the vertex v_2 is not adjacent to the vertex say u_2 and the vertex v_3 is not adjacent to the vertex say u_3 where the vertices $u_1, u_2, and\ u_3$ are different, then the set $D = \{v_i, v_j\}; i,j = 1,2,3\ and\ i \neq j$ are $MFOD^cS$, so $\gamma_{fodc}(G) = min\,\{\sigma(v_i) + \sigma(v_j); i,j = 1,2,3\ and\ i \neq j\}$.

Subcase2. If $\langle \{v_1, v_2, v_3\} \rangle \equiv P_3$, by hypothesis v_1 are v_3 pendent vertices of P_3. Then $D_1 = \{v_1, v_2\}$ and $D_1 = \{v_2, v_3\}$ are $MFOD^cS$. So, $\gamma_{fodc}(G) = min\{\sigma(v_1) + \sigma(v_2), \sigma(v_2) + \sigma(v_3)\}$.

From all cases above, the result is obtained. □

Corollary 3.2.14 If G_f is strong fuzzy graph has three vertices are adjacent to $n-2$ vertices; n is number of vertices of G, then $\forall v \in V(G), 0.4 \leq \gamma_{fodc}(G) \leq 3.6$.

Proof.

Let G_f be strong fuzzy graph has n number of vertices and three vertices adjacent to $n-2$ vertices. Let D be $MFOD^cS$, then by the same hypothesis in theorem 2.2.24, there are four cases as follows.

Case1. If $|D| = 1$, then $0.1 \leq \gamma_{fodc}(G_f) \leq 0.9$.

Case2. If $|D| = 2$, then $0.2 \leq \gamma_{fodc}(G_f) \leq 0.18$.

Case2. If $|D| = 3$, then $0.3 \leq \gamma_{fodc}(G_f) \leq 2.7$.

Case2. If $|D| = 4$, then $0.4 \leq \gamma_{fodc}(G_f) \leq 3.6$.

It is possible to be all vertices of D have weight 0.1, then this is minimum of domination. And if all vertices of D have weight 0.9, then that is maximum. Then all cases above is achieved. □

Corollary 3.2.14 If G_f is strong fuzzy graph and G_f^* is crisp strong fuzzy graph has n vertices such that D is OD^cS of G_f^* and D_f is $MFOD^cS$ of G_f; $\forall \sigma(v_i) \in D_f$ $i = 1, \dots, k$ and $k \leq n$; $\sigma(v_1) = \sigma(v_2) = \cdots = \sigma(v_k)$. Then $\gamma_{fodc}\left(G_f\right) = \sigma(v)\gamma_{odc}(G)$.

Proof.

Let G_f be strong fuzzy graph and G_f^* is crisp strong fuzzy graph has n vertices and $|D| = \gamma_{odc}(G_f^*)$. Since every vertex in D has the same weight, then $\gamma_{fodc}\left(G_f\right) = \sigma(v)|D| = \sigma(v)\gamma_{odc}(G)$. □

Corollary 3.2.15 For any strong fuzzy tree graph T_f that has n vertices and $k \geq 3$ such that k is the number of pendent and s is the number of support vertices in T, then the FOD^c domination number of complement of strong fuzzy tree graph $\gamma_{fodc}(\overline{T_f})$

$$= \begin{cases} \begin{cases} \begin{Bmatrix} \sigma(u) + min\{\sigma(v_i) + \sigma(v_j); i,j = 1, \dots, k \; and i \neq j\} \\ if\; u \; is \; unique \; support \; vertex \; and \; v_i, v_j \; areadjacent \; to \; u \; in \; T_f \end{Bmatrix} \\ \begin{Bmatrix} min\{\sigma(v_i) + \sigma(v_j); i,j = 1, \dots, k \; and \; i \neq j\} \\ if \; v_i, v_j \; are \; adjacent \; to \; different \; support \; vertices \\ in \; T_f \end{Bmatrix} \end{cases} \\ if \; n \; is \; even \\ \begin{cases} \begin{Bmatrix} \sigma(u) + min\{\sigma(v_i); i = 1, \dots, k\} \\ if \; u \; is \; unique \; support \; vertex \end{Bmatrix} \\ \begin{Bmatrix} min\{\sigma(v_i) + \sigma(w_j); i = 1, \dots, k \; and \; j = 1, \dots l; l \leq k\} \\ v_i \; is \; a \; pendant \; vertex \; that \; adjacent \; to \; w_j \; in \; the \; graph \; T_f \\ and \; w_j \; is \; support \; vertex \end{Bmatrix} \\ \{min\{\sigma(v_i) + \sigma(v_j) + \sigma(v_r); i,j,r = 1, \dots, k \; and \; i \neq j \neq r\}\} \end{cases} \\ if \; n \; is \; odd \end{cases}$$

such that $H = \{v_1, v_2, \dots, v_k\}$ is the set of pendent vertices.

Proof.

Suppose that $H = \{v_1, v_2, \dots, v_k\}$ is the set of pendent vertices and $M = \{u_1, u_2, \dots, u_s\}$ is set of support vertices. It is clear that $\forall v_i \in H$ is adjacent to $n-2$ vertices. If $k < 3$, then T_f is strong fuzzy path and proof this case by proposition 3.2.6. If $k \geq 3$, we obtained two cases as follows.

Case1. If n is even, since $\forall v_i \in H$ is adjacent to even number of vertices for $i = 1, \dots, k$, then there are two cases:

I) If all pendant adjacent to one support vertex say u that mean T_f is isomorphic to strong fuzzy star graph, then $D = \{v_i, v_j, u\}$ such that $i,j =$

$1, \dots k$ and $i \neq j$ are $MFOD^cS$. So, $\gamma_{fodc}(\overline{T_f}) = \sigma(u) + min\{\sigma(v_i) + \sigma(v_j); i, j = 1, \dots, k \ and \ i \neq j\}$.

II) If the pendant vertices are adjacent to more than one support vertex, then the set $D = \{v_i, v_j\}$ where $i, j = 1, \dots, k$ and the vertices v_i, v_j are adjacent to different two support vertices are $MFOD^cS$. Thus, $\gamma_{fodc}(\overline{T_f}) = min\{\sigma(v_i) + \sigma(v_j); i, j = 1, \dots, k \ and \ i \neq j\}$.

Case2. If n is odd, then there are two cases:

I) If all pendant adjacent to one support vertex say u that mean T_f is isomorphic to strong fuzzy star graph, then $D = \{v_i, u\}$ such that $i = 1, \dots, k$ are $MFOD^cS$. So, $\gamma_{fodc}(\overline{T_f}) = \sigma(u) + min\{\sigma(v_i); i = 1, \dots, k\}$.

II) If the pendant vertices are adjacent to more than one support vertex, then there are two cases:

A) If there is a set of support vertex $\{w_j; j = 1, \dots, l \ and \ l \leq k\}$ such that w_j is adjacent to odd number of vertices in $\overline{T_f}$, then $D = \{v_i, w_j\}$, where v_i is a pendant vertex that adjacent to w_j in the graph T_f are $MFOD^cS$ of $\overline{T_f}$. Thus, $\gamma_{fodc}(\overline{T_f}) = min\{\sigma(v_i) + \sigma(w_j); i = 1, \dots, k \ and \ j = 1, \dots l; l \leq k\}$.

B) If all support vertices are adjacent to even number of vertices, then the set $D = \{v_i, v_j, v_r\}$ where $i, j, r = 1, \dots, k$ and $i \neq j \neq r$ are $MFOD^cS$ in the strong fuzzy graph $\overline{T_f}$. Thus, $\gamma_{fodc}(\overline{T_f}) = min\{\sigma(v_i) + \sigma(v_j) + \sigma(v_r); i, j, r = 1, \dots, k \ and \ i \neq j \neq r\}$.

Then, we get the result. □

Proposition 3.2.15 Let G be any strong fuzzy graph has n vertices, then $\gamma_{fodc}(G_f) = \sigma(v); v \in V(G_f)$ if and only if n is even and $\exists v$ is adjacent to $n - 1$ vertices.

Proof.

If $\gamma_{fodc}(G_f) = \sigma(v); v \in V(G_f)$, then $D = \{v\}$ is $MFOD^cS$ such that v dominates all other vertices in $V(G_f)$, then by definition of $MFOD^cS$ v is dominate to odd number of vertices in $V - D$, that means is adjacent to $n - 1$ vertices, so n is even.

Conversely, let n be even and there exit a vertex say v is adjacent to $n - 1$ vertices, then $D = \{v\}$ is $MFOD^cS$. So, $\gamma_{fodc}(G_f) = \sigma(v)$. □

Proposition 3.2.16 Let G_f be any strong fuzzy graph has n vertices, then $\gamma_{fodc}(\overline{G_f}) = \sigma(v); v \in V(G_f)$ if and only if n is even and v is isolated vertex in G_f.

Proof.

Let $\gamma_{fodc}(\overline{G_f}) = \sigma(v); v \in V(\overline{G_f})$, then v is adjacent to $n - 1$ vertices in G_f , that means v is isolated vertex in G_f and since $D = \{v\}$ is $MFOD^cS$, then n is even.

Conversely, if n is even and G_f has at least one isolated vertex, then this isolated vertex is adjacent to $n - 1$ vertices in $\overline{G_f}$, this means v is adjacent to odd number of vertices in $\overline{G_f}$. Hence, $D = \{v\}$ and $\gamma_{fodc}(\overline{G_f}) = \sigma(v)$.

Then we get the result. □

Chapter Four

Fuzzy Total Odd Neighbor in D^c Domination Number and Fuzzy Hn-Domination Number

Chapter Four

Fuzzy Total Odd Neighbor in D^c and Hn-Domination in Graphs

4.1 Introduction

In this chapter, the concepts of fuzzy domination is studied in the so-called fuzzy total odd neighbor of D^c dominating set and fuzzy hn-dominating set in the fuzzy graph, and many of its properties and examples.

4.2 $FTOD^cS -$ In Fuzzy Graphs

In this section, we review the results of the concept fuzzy total odd neighbor of D^c dominating set.

Definition 4.2.1

Let $G_f = (\sigma, \rho)$ be a fuzzy graph. A subset D_f FOD^cS of G_f is **fuzzy total odd neighbor of D^c dominating set** if for every vertex $x, y \in D_f$, x is dominate to odd number of vertices in D^c by effective edge.

Definition 4.2.2

A fuzzy total odd neighbor of D^c dominating set D of $G_f = (\sigma, \rho)$ is called a **minimal fuzzy total odd neighbor of D^c dominating set** if there is no subset of D is a $FTOD^cS$ of G_f.

Definition 4.2.3

A fuzzy total odd neighbor of D^c dominating set of a fuzzy graph $G_f = (\sigma, \rho)$ with minimum number of vertices is called a **minimum fuzzy total odd neighbor of D^cdominating set** ($MFOD^cS$) of G_f.

Definition 4.2.4

A **fuzzy total odd neighbor of D^c domination number** of a fuzzy graph $G_f = (\sigma, \rho)$ is the minimum sum of membership values of the vertices for all minimum fuzzy total odd neighbor of D^c dominating sets and denoted by $\gamma_{ftodc}(G_f)$ or simply γ_{ftodc}.

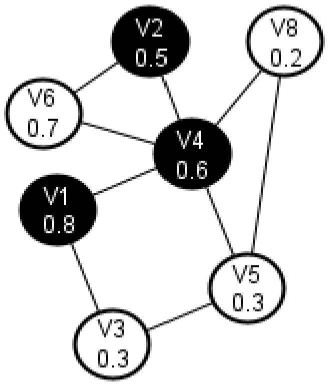


Figure 4.1. $MFTOD^cS$ of a fuzzy graph G_f

Proposition 4.2.5 Consider P_{n_f} be a strong fuzzy path graph with n vertices; $n \geq 2$, so $\gamma_{ftodc}\left(P_{n_f}\right) =$

$$\begin{cases}
\{\sigma(v_1) + \sigma(v_2), if\ n = 2\} \\
\left\{\sum_{i=0}^{\frac{n}{4}-1}\left(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\right), if\ n \equiv 0\ (mod\ 4)\right\} \\
\left\{ min \left\{ \begin{matrix} \sigma(v_2) + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\{(\sigma(v_{3+4i}) + \sigma(v_{4+4i}))\}, \\ \sigma(v_{n-1}) + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\right) \end{matrix} \right\}, if\ n \equiv 1\ (mod\ 4) \atop and\ n \neq 1 \right\} \\
\left\{ min \left\{ \begin{matrix} \sigma(v_{n-1}) + min\left\{ \begin{matrix} \sigma(v_{n-2}) \\ \sigma(v_n) \end{matrix} \right\} + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma(v_{1+4i}) + \sigma(v_{2+4i})\right) \\ \sigma(v_{n-1}) + min\left\{ \begin{matrix} \sigma(v_{n-2}) \\ \sigma(v_n) \end{matrix} \right\} + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\right) \\ \sigma(v_2) + min\left\{ \begin{matrix} \sigma(v_1) \\ \sigma(v_3) \end{matrix} \right\} + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma(v_{4+4i}) + \sigma(v_{5+4i})\right) \end{matrix} \right\} \atop , if\ n \equiv 2\ (mod\ 4)\ and\ n \neq 2 \right\} \\
\left\{ min \left\{ \begin{matrix} \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor}\left(\sigma(v_{1+4i}) + \sigma(v_{2+4i})\right), \\ \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor}\left(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\right), \\ \sigma(v_{n-2}) + \sigma(v_{n-1}) + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma(v_{2+4i}) + \sigma(v_{3+4i})\right), \\ \sigma(v_2) + \sigma(v_3) + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1}\left(\sigma(v_{5+4i}) + \sigma(v_{6+4i})\right), \end{matrix} \right\} \atop , if\ n \equiv 3\ (mod\ 4) \right\}
\end{cases}$$

Proof.

Let the set $\{v_1, v_2, \dots, v_n\}$ be set of vertices of strong fuzzy path graph such then v_1 and v_n are pendent vertices, then we obtained six cases as follows.

Case1. If $n = 1$, then P_{1_f} has no $FTOD^cS$.

Case2. If $n = 2$, then there is unique $FTOD^cS$ as $D = \{v_1, v_2\}$, so $\gamma_{ftodc}\left(P_{2_f}\right) = \sigma(v_1) + \sigma(v_2)$.

Case3. If $n \equiv 0\ (mod\ 4)$, then $D = \left\{v_{2+4i}, v_{3+4i}, i = 0, \dots, \frac{n}{4} - 1\right\}$ is the unique $MFTOD^cS$ because all other sets have order more than $\frac{n}{2}$. So, $\gamma_{ftodc}\left(P_{n_f}\right) = \sum_{i=0}^{\frac{n}{4}-1}(\sigma(v_{2+4i}) + \sigma(v_{3+4i}))$.

Case4. If $n \equiv 1\ (mod\ 4)$, then $D = \left\{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_{n-1}\}$ is the unique $MFTOD^cS$ because all other sets have order more than $\frac{n}{2}$. So, $\gamma_{ftodc}\left(P_{n_f}\right) = \sigma(v_{n-1}) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{2+4i}) + \sigma(v_{3+4i}))$.

Case5. If $n \equiv 2\ (mod\ 4)$, then $D_1 = \left\{v_{1+4i}, v_{2+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor\right\}$, $D_2 = \left\{v_{1+4i}, v_{2+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_{n-2}, v_{n-1}\}$, $D_3 = \left\{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_{n-1}, v_n\}$ and $D_4 = \left\{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_{n-1}, v_{n-2}\}$ are $MFTOD^cS$ because any vertex remove from any sets above, this set become not $FTOD^cS$. So, $\gamma_{ftodc}\left(P_{n_f}\right) = min\left\{\sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor}(\sigma(v_{1+4i}) + \sigma(v_{2+4i})), \sigma(v_{n-1}) + \sigma(v_{n-2}) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{1+4i}) + \sigma(v_{2+4i})), \sigma(v_{n-1}) + \sigma(v_n) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{2+4i}) + \sigma(v_{3+4i})), \sigma(v_{n-1}) + \sigma(v_{n-2}) + \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor-1}(\sigma(v_{2+4i}) + \sigma(v_{3+4i}))\right\}$.

Case6. If $n \equiv 3\ (mod\ 4)$, then $D_1 = \left\{v_{1+4i}, v_{2+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor\right\}$, $D_2 = \left\{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor\right\}$ $D_3 = \left\{v_{2+4i}, v_{3+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_{n-2}, v_{n-1}\}$ and $D_4 = \left\{v_{5+4i}, v_{6+4i}, i = 0, \dots, \left\lfloor\frac{n}{4}\right\rfloor - 1\right\} \cup \{v_2, v_3\}$ are $MFTOD^cS$. So, $\gamma_{ftodc}\left(P_{n_f}\right) = min\left\{\sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor}(\sigma(v_{1+4i}) + \sigma(v_{2+4i})), \sum_{i=0}^{\left\lfloor\frac{n}{4}\right\rfloor}(\sigma(v_{2+4i}) + \sigma(v_{3+4i})), \sigma(v_{n-1}) + \sigma(v_{n-2}) +\right.$

$\sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor - 1} (\sigma(v_{2+4i}) + \sigma(v_{3+4i}))\,, (v_2) + \sigma(v_3) + \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor} (\sigma(v_{5+4i}) +$

$\sigma(v_{6+4i}))\Big\}.$

Thus, we get the result. □

Observation 4.3.6 Consider C_{n_f} be a strong fuzzy cycle graph with n vertices; $n \geq 2$, so $\gamma_{ftodc}\left(C_{n_f}\right) = \gamma_{fodc}\left(C_{n_f}\right)$.

Proposition 4.2.7 Consider K_{n_f} be a strong fuzzy complete graph with n vertices; $n \geq 2$, so $\gamma_{ftodc}\left(K_{n_f}\right) =$

$$\begin{Bmatrix} min\{\sigma(v_i) + \sigma(v_j) + \sigma(v_k)\}, if\ n\ is\ even; i,j,k = 1, \dots, n\ and\ i \neq j \neq k \\ min\{\sigma(v_i) + \sigma(v_j)\}, if\ n\ is\ odd;\ ; i,j = 1, \dots, n\ and\ i \neq j \end{Bmatrix}$$

Proof.

Let $\{v_1, \dots, v_n\}$ be set of vertices of strong fuzzy complete graph, then we obtained the following two cases as follows.

Case1. If n is even, then $D = \{v_i, v_j, v_k\}$ such that $v_i, v_j, v_k \in \{v_1, \dots, v_n\}$ and $i,j,k = 1, \dots, n\ and\ i \neq j \neq k$ are $MFTOD^cS$ because if $D_1 = \{v_i, v_j\}$, then every vertex from D_1 is adjacent to $n-2$ vertices and $n-2$ is even, but that contradiction and so D_1 is not $FTOD^cS$. Then $\gamma_{ftodc}\left(K_{n_f}\right) = min\{\sigma(v_i) + \sigma(v_j) + \sigma(v_k)\}$.

Case2. If n is odd, then by the same away in case above we get $D = \{v_i, v_j\}$ such that $v_i, v_j \in \{v_1, \dots, v_n\}$ and $i,j = 1, \dots, n\ and\ i \neq j$ are $MFTOD^cS$. Then $\gamma_{ftodc}\left(K_{n_f}\right) = min\{\sigma(v_i) + \sigma(v_j)\}$.

Proposition 4.2.8 Consider W_{n_f} be a strong fuzzy wheel graph with n vertices; $n \geq 4$, so $\gamma_{ftodc}\left(W_{n_f}\right) = \sigma(v_1) + min\{\sigma(v_i), \sigma(v_j); i, j = 2, \dots, n \; and \; i \neq j\}$ such that v_1 is center vertex of W_{n_f} and v_i, v_j are adjacent.

Proof.

Let $\{v_1, v_2, \dots, v_n\}$ be set of vertices of W_{n_f} such that v_1 is center vertex. Then $D = \{v_1\} \cup \{v_i, v_j\}$, $i, j = 1, \dots, n; i \neq j$ are $FTOD^cS$ such that v_i, v_j are adjacent because if not adjacent, then v_i, v_j are adjacent to even number of vertices of $V - D$. To proof D is $MFTOD^cS$, let D has only two vertices. There are three cases, one of them case, say $D = \{v_1, v_i\}$ it is clear v_i is adjacent to two vertices from $V - D$, then D is not $FTOD^cS$. In case $D = \{v_i, v_j\}$ such that v_i, v_j are adjacent, then D is not $FTOD^cS$ because every vertex in D is adjacent to even number of vertices and D is not DS. In addition, if $D = \{v_i, v_j\}$ such that v_i, v_j are not adjacent, then D is not total dominating set. Thus, we get the result. □

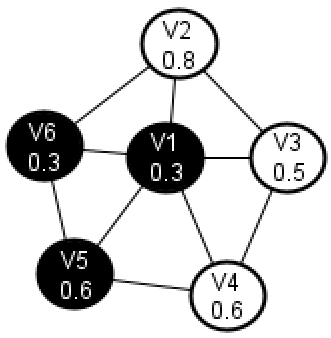


Figure 4.2. $MFTOD^cS$ of wheel graph W_6.

Proposition 4.2.9 Let K_{m,n_f} be a strong fuzzy complete bipartite graph with $n+m$ vertices, so $\gamma_{ftodc}\left(K_{m,n_f}\right) =$

$$\begin{cases} \sigma(u_1)+\sigma(v_1), \quad if\ n=1\ and\ m=1 \\ \sigma(u_1)+min\{\sigma(v_j),\sigma(v_l); j,l=1,\dots,m\ and\ m\neq l\}, \quad if\ n=1\ and\ m\ is \\ ;odd\ m\neq 1 \\ \gamma_{fodc}\left(K_{m,n_f}\right), otherwise \end{cases}$$

Proof.

By definition of $K_{m,n}$, $n \leq m$,suppose that $V(K_{m,n}) = V_m \cup V_n$ such that $V_m = \{v_1, v_2, \dots, v_m\}$ has m vertices and $V_n = \{u_1, u_2, \dots, u_n\}$ has n vertices. The cases we obtained as follows.

Case1. If $n = 1$ and $m = 1$, it is obvious $D = \{u_1, v_1\}$ is $MFTOD^cS$. So, $\gamma_{ftodc}\left(K_{m,n_f}\right) = \sigma(u_1) + \sigma(v_1)$.

Case2. If If $n = 1$ and m is odd; $m \neq 1$, then $D = \{u_1, v_j\}; j = 1, \dots, m$ are $MFTOD^cS$, $\gamma_{ftodc}\left(K_{m,n_f}\right) = \sigma(u_1) + min\{\sigma(v_j)\}; j = 1, \dots, m$.

Case3. Otherwise, by proposition 3.2.9 since every $MFOD^cS$ is no contains isolated vertex in an induced subgraph, then it is $MFTOD^cS$. So, $\gamma_{ftodc}\left(K_{m,n_f}\right) = \gamma_{ftodc}\left(K_{m,n_f}\right)$.

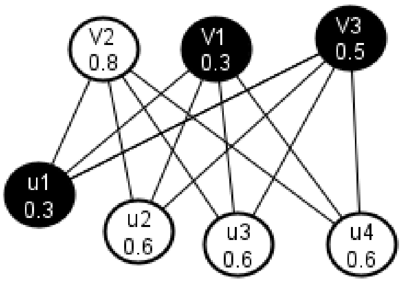


Figure 4.3. $MFTOD^cS$ of complete bipartite graph $K_{4,3_f}$.

Theorem 4.2.10 For any strong fuzzy graph G_f with n vertices. If it has three vertices v_1, v_2, v_3 are adjacent to $n-2$ vertices, then if n is odd, then $\gamma_{ftodc}(G_f) =$

$$
\left\{\begin{array}{c}
\left\{\begin{array}{c}
\left\{\begin{array}{c}
min\left\{\begin{array}{c} min\{v_i, v_j; i,j = 1,2,3 \text{ and } i \neq j\} + min\{u_k; k = 1,\dots,m\}, \\ min\{v_i; i = 1,2,3\} + min\{u_k, u_l; k,l = 1,\dots,m \text{ and } k \neq l\}, \end{array}\right\} \\
\text{if } u \notin N(v_i); i = 1,2,3 \text{ and } u_k, u_l \in N(u) \text{ are adjacent to} \\
\text{odd number of vertices}
\end{array}\right\} \\
\left\{\begin{array}{c}
\left\{\begin{array}{c}
\left\{\begin{array}{c} \sigma(u) + min\{\sigma(u_k); u_k \in N(u)\} + \\ min\{\sigma(v_i), \sigma(v_j); i,j = 1,2,3 \text{ and } i \neq j\} \\ \text{if } u \text{ is adjacent to even number of vertices} \end{array}\right\} \\
\left\{\begin{array}{c}
\left\{\begin{array}{c}
\left\{\begin{array}{c}
min\left\{\begin{array}{c} \left\{\begin{array}{c} min\{\sigma(v_i) + \sigma(v_j); i,j = 1,2,3\} + \\ min\{\sigma(u_k) + \sigma(u_l); k,l = 1,\dots,n-5\} \end{array}\right\} \\ \left\{\begin{array}{c} min\{\sigma(v_i); i = 1,2,3\} + \\ min\{\sigma(u_k) + \sigma(u_l); k,l = 1,\dots,n-5\} + \sigma(u) \end{array}\right\} \end{array}\right\} \\
\text{if } \forall u_k, u_l \in N(u) \text{ are adjacent}
\end{array}\right\} \\
\left\{\begin{array}{c}
\left\{\begin{array}{c}
min\left\{\begin{array}{c} min\{\sigma(v_i) + \sigma(v_j); i,j = 1,2,3\} + \\ min\{\sigma(u_k) + \sigma(u_l); k,l = 1,\dots,n-5\} + \sigma(u), \\ \left\{\begin{array}{c} \sigma(v_1) + \sigma(v_2) + \\ \sigma(v_3) + min\left\{\begin{array}{c} \sigma(u_k) + \sigma(u_l); \\ k,l = 1,\dots,n-5 \end{array}\right\} \end{array}\right\} \end{array}\right\}
\end{array}\right\} \\
\forall u_k, u_l \in N(u) \text{ are not adjacent}
\end{array}\right\}
\end{array}\right\} \\
\text{if } u \text{ is adjacent to odd number of vertices}
\end{array}\right\}
\end{array}\right\} \\
\text{if } u \notin N(v_i) \text{ and } \forall u_k, u_l \in N(u) \text{ are adjacent to even} \\
\text{number of vertices}
\end{array}\right\} \\
\left\{\begin{array}{c} \sigma(v_1) + \sigma(v_2) + \sigma(v_3)\text{ , if } u_1 \notin N(v_1) \cup N(v_2) \text{ and} \\ u_2 \notin N(v_3) \text{ or if } u_1 \notin N(v_1),\ u_2 \notin N(v_2) \text{ and } u_2 \notin N(v_3) \end{array}\right\} \\
\text{if } \langle\{v_1, v_2, v_3\}\rangle \equiv K_{3_f}
\end{array}\right\} \\
\left\{\begin{array}{c}
min\left\{\begin{array}{c} \sigma(v_1) + \sigma(v_3) + min\{\sigma(u_k); k = 1,\dots,m\}, \\ \sigma(v_2) + min\{\sigma(u_k); k = 1,\dots,m\} + min\left\{\begin{array}{c} \sigma(w_k); \\ k = 1,\dots,m \end{array}\right\} \\ ,\sigma(v_2) + min\left\{\begin{array}{c} \sigma(w_k) + \sigma(w_l); k,l = 1,\dots,m \text{ and} \\ w_k, w_l \text{ are adjacent}; k \neq l \end{array}\right\} \end{array}\right\} \\
\text{such that } u_k \in N(v_i); i = 1,2 \text{ and } w_k, w_l \in N(v_3) \text{ and} \\
u_k, w_k, w_l \text{ are adjacent to odd number of vertices} \\
\text{if } \langle\{v_1, v_2, v_3\}\rangle \equiv P_{3_f}
\end{array}\right\}
\end{array}\right\}
$$

If n is odd, then $\gamma_{ftodc}\left(G_f\right) =$

$$
\left\{\begin{array}{c}
\left\{\begin{array}{c}
\left\{\begin{array}{c}
\left\{\begin{array}{c}
min\{\sigma(u_k); k = 1, \dots, m \; and \; m \le n-5\} + min\{\sigma(v_i); i = 1,2,3\} \\
if \; u_k \in N(u) \; are \; adjacent \; to \; odd \; number \; of \; vertices
\end{array}\right\} \\
\left\{\begin{array}{c}
\left\{\begin{array}{c}
min\left\{\sigma(v_i) + \sigma\left(v_j\right); i \neq j\right\} + min\left\{\begin{array}{c}\sigma(u_k) + \sigma(u_l); \\ k, l = 1, \dots, m \; a\end{array}\right\} \\
if \; u_k, u_l \; are \; not \; adjacent
\end{array}\right\} \\
\left\{\begin{array}{c}
\sigma(u) + min\left\{\sigma(v_i) + \sigma\left(v_j\right); i, j = 1,2,3, i \neq j\right\} + \\
min\{\sigma(u_k) + \sigma(u_l); k, l = 1, \dots, m\} \\
if \; u_k, u_l \; are \; adjacent
\end{array}\right\} \\
if \; u_k, u_l \in N(u); k, l = 1, \dots, m \; and \; m \le n-5
\end{array}\right\}
\end{array}\right\} \\
if \; u \; is \; adjacent \; to \; odd \; number \; of \; vertices \\
\left\{\begin{array}{c}
\left\{\begin{array}{c}
min\{\sigma(v_i); i = 1,2,3,\} + min\{\sigma(u_k); k = 1, \dots, m\} \\
u_k \in N(u) \; is \; adjacent \; to \; even \; number \; of \; vertices
\end{array}\right\} \\
\left\{\begin{array}{c}
\sigma(u) \, min\{\sigma(v_i); i = 1,2,3,\} + min\{\sigma(u_k); k = 1, \dots, m\} \\
\forall u_k \in N(u) \; are \; adjacent \; to \; odd \; number \; of \; vertices
\end{array}\right\} \\
if \; u \; is \; adjacent \; to \; even \; number \; of \; vertices
\end{array}\right\} \\
if \; u \notin N(v_i); i = 1,2,3
\end{array}\right\} \\
\left\{\begin{array}{c}
\{\sigma(v_3) + \sigma(u_1), if \; u_1 \; is \; adjacent \; to \; even \; number \; of \; vertices\} \\
\left\{\begin{array}{c}
min\{\sigma(v_1), \sigma(v_2)\} + \sigma(u_2) \\
if \; u_2 \; is \; adjacent \; to \; even \; number \; of \; vertices
\end{array}\right\} \\
\left\{\begin{array}{c}
min\{\sigma(u_1), \sigma(u_2)\} + \sigma(v_1) + \sigma(v_2) + \sigma(v_3). \\
u_1 \; and \; u_2 \; are \; adjacent \; to \; odd \; number \; of \; vertices
\end{array}\right\} \\
\sigma(v_1) + \sigma(v_2) + \sigma(v_3), if \; u_1 \notin N(v_1) \cup N(v_2) \; and \; u_2 \notin N(v_3)
\end{array}\right\} \\
\left\{\begin{array}{c}
min\{\sigma(v_1) + \sigma(v_2), \sigma(v_1) + \sigma(v_3), \sigma(v_2) + \sigma(v_3)\} \\
u_1 \notin N(v_1), u_2 \notin N(v_2) \; and \; u_3 \notin N(v_3)
\end{array}\right\} \\
if \; \langle\{v_1, v_2, v_3\}\rangle \equiv K_{3_f}
\end{array}\right\}
$$

$$
\left\{\begin{array}{c}
min\{\sigma(v_1) + \sigma(v_2), \sigma(v_2) + \sigma(v_3)\} \\
if \; \langle\{v_1, v_2, v_3\}\rangle \equiv P_{3_f}
\end{array}\right\}
$$

Proof.

Let G_f has three vertices say $\{v_1, v_2, v_3\}$ are adjacent to $n-2$ vertices. By the same away in proof of theorem 3.2.12 then $\langle\{v_1, v_2, v_3\}\rangle \equiv K_3 \; or \; P_3$. Then we distinguish two cases as follows.

Case1. If n is odd, then there are two subcases.

Subcase1. If $\langle\{v_1, v_2, v_3\}\rangle \equiv K_3$, then we distinguish three cases as follows.

1) If these three vertices are not adjacent to the same vertex (say u) and u has even or odd degree then there are two cases as follows.

I. If $H = \{u_k; k = 1, \dots, m\}; m \leq n - 5$ is set of vertices that adjacent to odd number of vertices. Then $D_1 = \{u_k, v_i, v_j\}$ such that $i, j = 1,2,3; i \neq j$ and $D_2 = \{u_k, u_l, v_i\}$ such that u_k, u_l are adjacent and $k, l = 1, \dots, m; k \neq l$ are $MFTOD^cS$, so $\gamma_{ftodc}(G_f) = min\{min\{v_i, v_j; i, j = 1,2,3 \text{ } and \text{ } i \neq j\} + min\{u_k; k = 1, \dots, m\}, min\{v_i; i = 1,2,3\} + min\{u_k, u_l; k, l = 1, \dots, m \text{ } and \text{ } k \neq l\},\}$.

II. If all vertices belong to the set $N(u)$ are adjacent to even number of vertices, then there are two cases as follows.

A) If u is adjacent to even number of vertices, then $D = \{u, u_k, v_1, v_2\}$ and $u_k \in N(u)$ are $MFTOD^cS$, so $\gamma_{ftodc}(G_f) = min\{\sigma(u_k); u_k \in N(u)\} + min\{\sigma(v_i), \sigma(v_j); i, j = 1,2,3 \text{ } and \text{ } i \neq j\} + \sigma(u)$.

B) If u is adjacent to odd number of vertices, then if $\forall u_k, u_l \in N(u)$ are adjacent; $k, l = 1, \dots, n - 5; k \neq l$, then $D_1 = \{u_k, u_l, v_i, v_j\}; i, j = 1,2,3 \text{ } and \text{ } i \neq j$ and $D_2 = \{u_k, u_l, v_i, u\}; i = 1,2,3$ are $MFTOD^cS$, so $\gamma_{ftodc}(G_f) = min\{min\{\sigma(v_i) + \sigma(v_j); i, j = 1,2,3\} + min\{\sigma(u_k) + \sigma(u_l); k, l = 1, \dots, n - 5\}, min\{\sigma(v_i); i = 1,2,3\} + min\{\sigma(u_k) + \sigma(u_l); k, l = 1, \dots, n - 5\} + \sigma(u)\}$.

In case $\forall u_k, u_l$ are not adjacent, then $D_1 = \{u, u_k, u_l, v_i, v_j\}$ and $D_2 = \{u_k, u_l, v_1, v_2, v_3\}$ are $MFTOD^cS$. So, $\gamma_{ftodc}(G_f) = min\{min\{\sigma(v_i) + \sigma(v_j); i, j = 1,2,3\} + min\{\sigma(u_k) + \sigma(u_l); k, l = 1, \dots, n - 5\} + \sigma(u), \sigma(v_1) + \sigma(v_2) + \sigma(v_3) + min\{\sigma(u_k) + \sigma(u_l); k, l = 1, \dots, n - 5\}\}$.

2) If two vertices say $v_1, and\ v_2$ are not adjacent to the same vertex say u_1 and the vertex v_3 is not adjacent to the vertex u_2, where $u_1\ and\ u_2$ are different, then $D = \{v_1, v_2, v_3\}$ is $MFTOD^cS$. So, $\gamma_{ftodc}(G_f) = \sigma(v_1) + \sigma(v_2) + \sigma(v_3)$.

3) If The vertex v_1 is not adjacent to the vertex say u_1 and the vertex v_2 is not adjacent to the vertex say u_2 and the vertex v_3 is not adjacent to the vertex say u_3 where the vertices $u_1, u_2, and\ u_3$ are different, then $D = \{v_1, v_2, v_3\}$ is $MFTOD^cS$. So, $\gamma_{ftodc}(G_f) = \sigma(v_1) + \sigma(v_2) + \sigma(v_3)$.

Subcase2. If $\langle\{v_1, v_2, v_3\}\rangle \equiv P_{3_f}$, then the vertices v_1 and v_3 are not adjacent and u is a vertex that not adjacent to v_2.If $\{u_k, k = 1, \dots, m; m \leq n - 5\} \subseteq N(v_i); i = 1,2$ is set of vertices that adjacent to odd number of vertices and $\{w_k, k = 1, \dots, m; m \leq n - 5\} \subseteq N(v_3)$ is the set of vertices that adjacent to odd number of vertices, then $D_1 = \{v_1, u_k, v_3\}$, $D_2 = \{v_2, u_k, w_k\}$ and $D_3 = \{v_2, w_k, w_l\}$ are $MFTOD^cS$. So, $\gamma_{ftodc}(G_f) = min\{\sigma(v_1) + \sigma(v_3) + min\{\sigma(u_k); k = 1, \dots, m\}, \sigma(v_2) + min\{\sigma(u_k); k = 1, \dots, m\} + min\{\sigma(w_k); k = 1, \dots, m\}, \sigma(v_2) + min\{\sigma(w_k) + \sigma(w_l); k, l = 1, \dots, m\ and\ w_k, w_l\ are\ adjacent; k \neq l\}\}$.

Case 2. If n is even, then there are two subcases as follows.

Subcase1. If $\langle\{v_1, v_2, v_3\}\rangle \equiv K_{3_f}$, then we distinguish three cases as follows.

1) If these three vertices are not adjacent to the same vertex (say u) then there are two cases as follows.

I) If u is adjacent to odd number of vertices, then two cases are discussed below.

A) If $\{u_k; k = 1, \dots, m \; and \; m \leq n-5\} \subseteq N(u)$ is set of vertices that adjacent to even number of vertices, then $D = \{u_k, v_i\}$ are $MFTOD^cS$. So, $\gamma_{ftodc}(G_f) = min\{\sigma(u_k); k = 1, \dots, m \; and \; m \leq n-5\} + min\{\sigma(v_i); i = 1,2,3\}$.

B) If all vertices in $N(u)$ are adjacent to odd number of vertices, then two cases are discussed below.

$\boldsymbol{B_1}$. If $u_k, u_l \in N(u); k, l = 1, \dots, m \; and \; m \leq n-5$ are not adjacent, then $D = \{u_k, u_l, v_i, v_j\}; k \neq l \; and \; ij$ are the $MFTOD^cS$. Thus, $\gamma_{ftodc}(G_f) = min\{\sigma(v_i) + \sigma(v_j); i \neq j\} + min\{\sigma(u_k) + \sigma(u_l); k, l = 1, \dots, m \; and \; m \leq n-5\}$.

$\boldsymbol{B_2}$**.** If all vertices in $N(u)$ are adjacent, then $D = \{u, u_k, u_l, v_i, v_j\}; i \neq j \; and \; k \neq l$ are $MFTOD^cS$, Thus, $\gamma_{ftodc}(G_f) = \sigma(u) + min\{\sigma(v_i) + \sigma(v_j); i, j = 1,2,3, i \neq j\} + min\{\sigma(u_k) + \sigma(u_l); k, l = 1, \dots, m \; and \; m \leq n-5\}$.

II) If u is adjacent to even number of vertices, then three cases are discussed below.

A. If $\{u_k; k = 1, \dots, m \; and \; m \leq n-5\} \subseteq N(u)$ is the set of vertices that adjacent to even number of vertices, then $D = \{u_k, v_i\}; k = 1, \dots, m \; and \; i = 1,2,3$ are $MFTOD^cS$. Thus, $\gamma_{ftodc}(G_f) = min\{\sigma(v_i); i = 1,2,3,\} + min\{\sigma(u_k); k = 1, \dots, m\}$.

B. If all vertices belong to the set $N(u)$ are adjacent to odd number of vertices, then $D = \{u, u_k, v_i\}$, where $u_k \in N(u)$ are $MFTOD^cS$. Thus, $\gamma_{ftodc}(G_f) = \sigma(u) + min\{\sigma(v_i); i = 1,2,3,\} + min\{\sigma(u_k); k = 1, \dots, m\}$.

2) If two vertices say $v_1, and\ v_2$ are not adjacent to the same vertex say u_1 and the vertex v_3 is not adjacent to the vertex u_2, where $u_1\ and\ u_2$ are different, then there are two cases as follows.

A. If the vertex u_1 is adjacent to even number of vertices, then the set $D = \{u_1, v_3\}$ are $MFTOD^cS$. Thus, $\gamma_{ftodc}(G_f) = \sigma(v_3) + \sigma(u_1)$.

B. If the vertex u_2 is adjacent to even number of vertices, then the set $D_1 = \{u_2, v_1\}$ and $D_2 = \{u_2, v_2\}$ are $MFTOD^cS$. Thus, $\gamma_{ftodc}(G_f) = min\{\sigma(v_1), \sigma(v_2)\} + \sigma(u_2)$.

C. If the vertex $u_1\ and\ u_2$ are adjacent to odd number of vertices, then the set $D_1 = \{u_1, v_1, v_2, v_3\}$ and $D_2 = \{u_2, v_1, v_2, v_3\}$ are $MFTOD^cS$. Thus, $\gamma_{ftodc}(G_f) = min\{\sigma(u_1), \sigma(u_2)\} + \sigma(v_1) + \sigma(v_2) + \sigma(v_3)$.

3) If v_1 is not adjacent to a vertex say u_1 and a vertex v_2 is not adjacent to a vertex say u_2 and the vertex v_3 is not adjacent to the vertex say u_3 where the vertices$u_1, u_2, and\ u_3$ are different, then the set $D_1 = \{v_1, v_2\}, D_2 = \{v_1, v_3\}$ and $D_3 = \{v_2, v_3\}$ are $MFTOD^cS$. Thus, $\gamma_{ftodc}(G_f) = min\{\sigma(v_1) + \sigma(v_2), \sigma(v_1) + \sigma(v_3), \sigma(v_2) + \sigma(v_3)\}$.

Subcase2. If $\langle\{v_1, v_2, v_3\}\rangle \equiv P_3$, by hypothesis v_1 are v_3 pendent vertices of P_3. Then $D_1 = \{v_1, v_2\}$ and $D_2 = \{v_2, v_3\}$ are $MFTOD^cS$. Thus, $\gamma_{ftodc}(G_f) = min\{\sigma(v_1) + \sigma(v_2), \sigma(v_2) + \sigma(v_3)\}$.

From all cases above, the result is obtained. □

Remark 4.2.11 Let G be strong fuzzy graph G_f and n number of vertices of G_f, $0.2 \leq \gamma_{ftodc}(G) \leq n(0.9)$.

Proposition 4.2.12 Let G be strong fuzzy graph has three vertices are adjacent to $n - 2$ vertices; n is number of vertices of G, then $\forall v \in V(G), 0.5 \leq \gamma_{ftodc}(G) \leq 4.5$.

Proof.

Let G be strong fuzzy graph has n number of vertices and three vertices adjacent to $n-2$ vertices. Let D be $MFTOD^cS$, then by the same hypothesis in theorem 2.2.24, there are four cases as follows.

Case1. If $|D| = 2$, then $0.2 \leq \gamma_{ftodc}(G_f) \leq 1.8$.

Case2. If $|D| = 3$, then $0.3 \leq \gamma_{ftodc}(G) \leq 2.7$.

Case3. If $|D| = 2$, then $0.4 \leq \gamma_{ftodc}(G) \leq 3.6$.

Case4. If $|D| = 1$, then $0.1 \leq \gamma_{ftodc}(G) \leq 0.9$.

Then we get the result. □

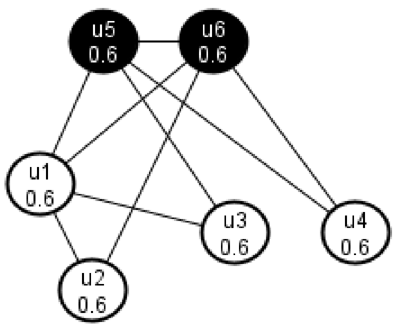


Figure 4.4. Application of proposition 4.2.12.

In Fig. 4.4, $|D| = 2$, then $\gamma_{ftodc}(G_f) = 2 \times (0.6) = 1.2$. Then the above proposition is confirmed.

4.3 Fuzzy Hn-Domination in Fuzzy Graph

Definition 4.3.1

Let $G_f = (\sigma, \rho)$ be a fuzzy graph of a graph (V, E), if there is a set $D \subseteq V$ and $\forall\, v \in V - D$ there is a vertex u such that u is adjacent to v by effective edge (u dominates v) and for all $v_1, v_2 \epsilon V -$

D *there are two adjacent* $u_1, u_2 \epsilon D$ such that v_1, v_2 are adjacent to u_1, u_2 respectively. Then D is called a fuzzy hn- dominating set on G.

Definition 4.3.2

A fuzzy dominating set D in a fuzzy graph $G_f = (\sigma, \rho)$ is called minimum fuzzy hn-dominating if the number of vertices of all fuzzy hn-dominating set greater than or equal the number of vertices in D.

Definition 4.3.3

Consider $W(D_i) = \{\sum \sigma(v);\ \forall v \in D_i; D_i\ is\ a\ minimum\ fuzzy\ hn-dominating\ set\}$, then the fuzzy hn-domination number of a fuzzy graph is $\gamma_{fhn}(G) = \min\{W(D_i);\ D_i\ is\ a\ minimum\ hn-dominating\ set\}$.

Remark 4.3.4

Every isolated vertex belongs to every fuzzy hn-dominating set.

Proposition 4.3.5 If G is a strong fuzzy Path graph has n vertices, then

$$\gamma_{fhn}\left(P_{n_f}\right) = \left\{\begin{matrix} \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \sigma(v_{2i+2}) \ , & if \quad n\ is\ odd \\ min\left\{\sum_{i=0}^{\frac{n}{2}-1} \sigma\left(v_{2i+j}\right)\right\} \ , j = 1,2 \ if & n\ is\ even \end{matrix}\right\}.$$

Proof.

There are two cases.

Case1. If n is odd, then there are exactly two minimal fuzzy hn-dominating sets which are $D_1 = \left\{v_{2i+1}, i = 0,1, \dots, \left\lceil \frac{n}{2} \right\rceil - 1\right\}$ and $D_2 = \left\{v_{2i+2}, i = 0,1, \dots, \left\lceil \frac{n}{2} \right\rceil - 1\right\}$, since if the other dominating set is taken say D_3, this dominating must conclude at least two adjacent vertices and these vertices dominated by other non-adjacent two vertices, so this dominating set is not hn-dominating. It is clear that $|D_1| = \left\lceil \frac{n}{2} \right\rceil$ and $|D_2| = \left\lceil \frac{n}{2} \right\rceil$. Then the minimum fuzzy hn-dominating set is D_2 and so $\gamma_{fhn}\left(P_{n_f}\right) = min\left\{\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1} \sigma(v_{2i+2})\right\}$.

Case2. If n is even, then there are only two minimal fuzzy dominating sets, $D_1 = \left\{v_{2i+1}, i = 0,1, \dots, \frac{n}{2} - 1\right\}$ and $D_2 = \left\{v_{2i+2}, i = 0,1, \dots, \frac{n}{2} - 1\right\}$. It is clear that it have the same order such that $|D_1| = |D_2| = \frac{n}{2}$, then the minimum fuzzy hn-dominating sets two sets, so $\gamma_{fhn}\left(P_{n_f}\right) = min\left\{\sum_{i=0}^{\frac{n}{2}-1} \sigma\left(v_{2i+j}\right) \ , j = 1,2\right\}$. □

Proposition 4.3.6 If G be strong fuzzy cycle graph has $n \geq 4$ vertices, then

$$\gamma_{fhn}\left(C_{n_f}\right) = \begin{cases} min\left\{\sum_{i=0}^{\frac{n}{2}-1} \sigma\left(v_{2i+j}\right) \ , j = 1,2\right\}, & if\ n\ is\ even \\ min\left\{\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1} \left(\sigma\left(v_j\right) + \sigma\left(v_{j+2i+1}\right)\right), j = 1,2,\dots,n; (j+2i+1) taken\ mod\ n\right\}, & if\ n\ is\ odd \end{cases}$$

Proof.

There are two cases as follows.

Case1. If n is even, then there are only two minimal fuzzy hn-dominating set as in case 2 of proposition 4.3.5 and so, $\gamma_{fhn}\left(C_{n_f}\right) = min\left\{\sum_{i=0}^{\frac{n}{2}-1} \sigma\left(v_{2i+j}\right)\} \ , j = 1,2\right\}$

Case2. If n is odd, then to build the hn-dominating set D the following steps are followed. If we take a vertex and leave a vertex then there will be two adjacent vertices left, and then the set D will be dominating, but it is not hn-dominating. To remedy this problem, we will take two adjacent vertices in the set D and obtain the remained vertices in the set D, from leaving the vertex that adjacent to the two adjacent vertices and take the next vertex and continue at the same procedure leave a vertex and take the next vertex. Thus, $D_j = \left\{v_j, v_{j+2i+1},\ i = 0,1,\dots,\left\lfloor\frac{n}{2}\right\rfloor - 1\right\}; j = 1,2,\dots,n,$ such that the number $j + 2i + 1$ is taken modulo n. It is clear that all sets D_j are hn-dominating. One can be concluded that each of these sets has the minimum number of vertices since if we leave a vertex from any one of these sets, the conditions of hn-dominating set do not satisfy in this set. Thus, $\gamma_{fhn}\left(C_{n_f}\right) = min\left\{\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1}(\sigma(v_j) + \sigma(v_{j+2i+1})), j = 1,2,\dots,n; (j + 2i + 1)\ taken\ mod\ n\right\}$ □

Proposition 4.3.7 If G is a strong fuzzy complete graph has n vertices, then $\gamma_{fhn}\left(K_{n_f}\right) = \begin{cases} \sigma(v), & if\ n = 1 \\ min\{\sigma(v_i), i = 1,\dots,n\}, & if\ \ n \geq 2 \end{cases}$.

Proof.

If $n = 1$, then the result is obvious. If $n \geq 2$,then it is clear that K_{n_f} has n minimal fuzzy hn-dominating set which are $D_i = \{v_i\}; i = 1,\dots,n$ and it have order equal to one, so $\gamma_{fhn}\left(K_{n_f}\right) = min\{\sigma(v_i)\};\ i = 1,\dots,n\}$. □

Proposition 4.3.8 If G is a strong fuzzy wheel graph has n vertices and let v_1 be vertex of K_1, then $\gamma_{fhn}\left(W_{n_f}\right) = \begin{cases} min\{\sigma(v_i), i = 1,\dots,n\}, & if\ \ n = 4 \\ \sigma(v_1)\ , & if\ \ n \geq 5 \end{cases}$, *where* v_1 is the vertex of K_1.

Proof.

There are two cases as follows.

Case 1. If $n = 4$, then $W_{4_f} \equiv K_{4_f}$, so according to proposition 4.3.7 $\gamma_{fhn}\left(W_{4_f}\right) = min\{\sigma(v_i), i = 1,\dots,4\}$.

Case 2. If $n > 4$, then let $D = \{v_1\}$, it is clear that the set D is hn-dominating set and has the minimum number of vertices. Thus, $\gamma_{fhn}\left(W_{n_f}\right) = \sigma(v_1)$.

From two cases above, the proof is done. □

Proposition 4.3.9 If G is a fuzzy complete bipartite graph $K_{n,m_f}; n \geq m$ contains two partite sets V_1 and V_2 suth that V_1 has n vertices $\{u_1, u_2, \dots, u_n\}$ and V_2 has m vertices $\{v_1, v_2, \dots, v_m\}$, then $\gamma_{fhn}\left(K_{m,n_f}\right) =$

$$\left\{\begin{array}{c} min\{\sigma(u_1), \sigma(v_1)\}\, if m = n = 1 \\ \sigma(u_1), if\ n = 1\ and\ m \geq 2 \\ \sigma(v_1), if\ m = 1\ and\ n \geq 2 \\ min\{\sigma(u_1) + \sigma(u_2), \sigma(v_1) + \sigma(v_2), min\{\sigma(u_i) + \sigma(v_j); i, j = 1,2\}\}, \\ if m = n = 2 \\ min\{min\{\sigma(u_i) + \sigma(v_j); i = 1,2; j = 1, \dots, n\}, \sigma(u_1) + \sigma(u_2)\}, \\ if\ n = 2\ and\ m \geq 3 \\ min\{\sigma(u_i)\} + min\{\sigma(v_j)\},\ if\ n, m \geq 3;\ i = 1, \dots, n\ and\ j = 1, \dots, m \end{array}\right\}$$

Proof.

There are five cases as follows.

Case1. If $m = n = 1$, then there two hn-dominating set that are $D_1 = \{u_1\}$ and $D_2 = \{v_1\}$, so $\gamma_{fhn}\left(K_{1,1_f}\right) = min\{\sigma(u_1), \sigma(v_1)\}$.

Case2. If $n = 1\ and\ m \geq 2$, then one can be concluded that the set $D = \{u_1\}$ is hn-dominating set and has minimum number of vertices. Thus, $\gamma_{fhn}\left(K_{m,1_f}\right) = \sigma(u_1)$, moreover in same manner $\gamma_{fhn}\left(K_{1,n_f}\right) = \sigma(v_1)$ when, $m = 1\ and\ n \geq 2$.

Case3. If $m = n = 2$, then it is clear that the set contains only one vertex cannot be hn-dominating set. The hn-dominating set contains two vertices are $D_1 = \{u_1, u_2\}$,, $D_2 = \{v_1, v_2\}$, $D_3 = \{u_1, v_1\}$, $D_4 = \{u_1, v_2\}$, $D_5 =$

$\{u_2, v_1\}$ and $D_6 = \{u_2, v_2\}$. Thus, $\gamma_{fhn}\left(K_{2,2_f}\right) = min\{\sigma(u_1) + \sigma(u_2), \sigma(v_1) + \sigma(v_2), min\{\sigma(u_i) + \sigma(v_j); i, j = 1,2\}\}$.

Case4. In the same manner in case 2 and case 3, it is clear that $\gamma_{fhn}\left(K_{2,m_f}\right) = min\ \{min\{\sigma(u_i) + \sigma(v_j); i = 1,2; j = 1, \dots, n\}, \sigma(u_1) + \sigma(u_2)\}$, where $m \geq 3$ and $\gamma_{fhn}\left(K_{n,2_f}\right) = min\ \{min\{\sigma(u_i) + \sigma(v_j); i = 1, \dots, n; j = 1,2\}$, where ≥ 3 .

Case5. If $n, m \geq 3$, then again, the hn-dominating set cannot contain only one vertex. Let $D_{ij} = \{u_i, v_j\}$; $i = 1, \dots, n\ and\ j = 1, \dots, m$, all the sets D_{ij} are hn-dominating sets and have the minimum number of vertices. Thus, $\gamma_{fhn}\left(K_{n,2_f}\right) = min\{\sigma(u_i)\} + min\{\sigma(v_j)\}$.

From all cases above, the result is obtained. □

Corollary 4.3.10 If S_{n_f} is a strong fuzzy star has n vertices such that $S_{n_f} = K_{1,n-1_f}$, then $\gamma_{fhn}\left(S_{n_f}\right) = \begin{Bmatrix} min\{\sigma(u_1), \sigma(v_1)\}\ if m = n = 1 \\ \sigma(u_1), if\ n = 1\ and\ m \geq 2 \\ \sigma(v_1), if\ m = 1\ and\ n \geq 2 \end{Bmatrix}$.

Proof.

By the same way in proof above, $min\{\sigma(u_1), \sigma(v_1)\}\ if m = n = 1$.

If $n = 1$, then u_1 is adjacent to all other vertices $D = \{u_1\}$, then $\gamma_{fhn}\left(S_{n_f}\right) = \sigma(u_1)$. The same way if $m = 1$, so we get the result. □

Remark 4.3.11 If G_1 and G_2 are disconnected fuzzy graphs which have fuzzy hn-dominating set $\gamma_{fhn}(G_{1_f})$ and $\gamma_{fhn}(G_{2_f})$, respectively. Then $\gamma_{fhn}\left(G_{1_f} \cup G_{2_f}\right) = \gamma_{fhn}\left(G_{1_f}\right) + \gamma_{fhn}(G_{2_f})$.

Proof.

Since the graph $G_{1_f} \cup G_{2_f}$ is disconnected fuzzy graph and has two components, then every component has fuzzy hn-dominating set. So, ta fuzzy hn-dominating set of $G_{1_f} \cup G_{2_f}$ is $\gamma_{fhn}\left(G_{1_f}\right) + \gamma_{fhn}(G_{2_f})$. □

Proposition 4.3.12 If G be a strong fuzzy path graph has n vertices, then $\gamma_{fhn}\left(\overline{P_{n_f}}\right) =$

$$\begin{cases} \sum_{i=1}^{n} \sigma(v_i), & if \quad n = 1,2 \\ \sigma(v_2) + min\{\sigma(v_i), i = 1,3\}, & if \quad n = 3\ , v_2\ is\ isolated \\ min\left\{\sum_{i=0}^{\frac{n}{2}-1} \sigma\left(v_{2i+j}\right)\}\ \ , j = 1,2\right\}, & if \quad n = 4 \\ min\ \{\left(\sigma(v_i) + \sigma\left(v_j\right); i = 1, \dots, n; j \geq 3 + i\right\}, & if\ n \geq 5 \end{cases}$$

Proof.

There are three cases as follows.

Case1. If $n = 1,2$, then $\overline{P_{n_f}}$ is null, so by Remark 4.3.4 $\gamma_{fhn}\left(\overline{P_{n_f}}\right) = \sum_{i=1}^{n} \sigma(v_i)$.

Case2. If $n = 3$, then $\overline{P_{n_f}}$ has two component one of them is an isolated vertex v_2 and the other is P_2, so by Remark 4.3.4 and Proposition 4.3.5, $\gamma_{fhn}\left(\overline{P_{n_f}}\right) = \sigma(v_2) + min\{\sigma(v_i)\}\,; i = 1,3$.

Case3. If $n = 4$, since $\overline{P_{n_f}} \cong P_{n_f}$, then $\gamma_{fhn}\left(\overline{P}_{n_f}\right) = \gamma_{fhn}\left(P_{n_f}\right) = min\left\{\sum_{i=0}^{\frac{n}{2}-1} \sigma\left(v_{2i+j}\right)\}\ \ , j = 1,2\right\}$.

Case4. If $n \geq 5$, suppose that $D = \{v\}$, then v is adjacent to all other vertices in $\overline{P}_n$ and $deg(v) = n - 1$.But $\Delta\left(\overline{P}_{n_f}\right) = n - 2$, then D is not fuzzy hn-dominating set. then $|D| \geq 2$

If $|D| = 2$, then $D = \{v_i, v_j\}$, so there are to cases as follows.

(i) If $j < 3 + i$, then either $j = 1 + i$ or $j = 2 + i$. If $j = 1 + i$, then $\langle D \rangle$ is null, but there are at least two vertices is adjacent in $V - D$ and dominating by different vertices in D, so D is not fuzzy hn-dominating set. Now, if $j = 2 + i$, then the vertex v_{i+1} is not dominating by any vertex in D, so D is not fuzzy dominating set.

(ii) If $j \geq 3 + i$, then $d(v_i, v_j) \geq 3$, so v_i is adjacent to v_j in $\overline{P_n}_f$ and all independent vertices with one of vertices in D is dominating by the other vertex in D. Then D is fuzzy hn-dominating set. so $\gamma_{fhn}(\overline{P_n}_f) = min\,\{(\sigma(v_i) + \sigma(v_j); i = 1, \dots, n; j \geq 3 + i\}$. □

Proposition 4.3.13 If G be a strong fuzzy cycle graph has n vertices, then $\gamma_{fhn}\left(\overline{C_n}_f\right) =$

$$\begin{cases} \sum_{i=1}^{n} \sigma(v_i)\,, & if \quad n = 3 \\ min\{\sigma(v_i) + \sigma(v_j)\};\ v_i and\ v_j\ are\ independent, & if \quad n = 4 \\ mi\,n\{\sigma(v_i) + \sigma(v_j)\}, i = 1, \dots, n, ; j \geq 3 + i, & if \quad n \geq 5 \end{cases}$$

Proof.

Let $\{v_i, i = 1, \dots, n\}$ be set of the vertices of which are incident clockwise in C_n. There are to cases as follows.

Case1. If $n = 3$, then $\overline{C_3}_f$ is null and by remark 4.3.4, $\gamma_{fhn}\left(\overline{C_3}_f\right) = \sum_{i=1}^{3} \sigma(v_i)$.

Case2. If $n = 4$, then there are only two hn-dominating set which are $D_1 = \{v_1, v_2\}$ and $D_2 = \{v_3, v_4\}$. It is clear that $\langle D_1 \rangle$ and $\langle D_2 \rangle$ are null and $|D_1| = |D_2|$. So $\gamma_{fhn}\left(\overline{C_n}_f\right) = min\,\{\sigma(v_i) + \sigma(v_j)\}$ such that v_i and v_j are independent.

Case3. If $n \geq 5$, By the same procedures in proof of Proposition 4.3.12 case 4, we get the result. □

Observation 4.3.14 If G is a fuzzy complete or null graph has n vertices, then $\gamma_{fhn}\left(\overline{K_n}_f\right) = \gamma_{fhn}(N_n) = \sum_{i=1}^{n} \sigma(v_i)$.

Proof.

It is obvious by remark 4.3.4 □

Proposition 4.3.15 If G be strong fuzzy wheel graph such that $W_{n_f} = C_n + K_1$, then $\gamma_{fhn}\left(\overline{W_{n_f}}\right) = \sigma(v_1) +$

$$\begin{cases} \sum_{i=1}^{n} \sigma(v_i)\,, & if \quad n = 4 \\ min\{\sigma(v_i) + \sigma(v_j)\};\ v_i and\ v_j\ are\ idependent\ in\ \overline{W_4}_f, & if \quad n = 5 \\ mi\,n\{\sigma(v_i) + \sigma(v_j)\}, i = 1, \dots, n; j = 3 + i, & if \quad n \geq 6 \end{cases}$$

, such that v_1 is a vertex of K_1.

Proof.

The graph $\overline{W_{n_f}}$ has two components which are K_{1_f} and $\overline{C_{n-1}}$ such that $\overline{W_{n_f}} = K_1 \cup \overline{C_{n-1}}$. Then by Remark 4.3.11 $\gamma_{fhn}(\overline{W_n}) = \gamma_{fhn}(K_1) + \gamma_{fhn}(\overline{C_{n-1}})$ and by Remark 4.3.4 $\gamma_{fhn}(K_1) = \sigma(v_1)$ since v_1 is a vertex of K_1 and by Proposition 4.3.13 we get the result. □

Proposition 4.3.16 If G is a strong fuzzy complete bipartite K_{n,m_f} contains two partite sets V_1 and V_2 such that V_1 has n vertices $\{u_1, u_2, \dots, u_n\}$ and V_2 has m vertices $\{v_1, v_2, \dots, v_m\}$, then $\gamma_{fhn}\left(\overline{K_{m,n_f}}\right) = min\{\sigma(v_i), i = 1, \dots, m\} + min\,\{\sigma(v_i), i = 1, \dots, n\}$.

Proof.

There are two components in $\overline{K_{m,n_f}}$ such that every component is complete. Then by Proposition 4.3.7 $\gamma_{fhn}\left(\overline{K_{m,n_f}}\right) = min\{\sigma(v_i), i = 1, \dots, m\} + min\,\{\sigma(v_i), i = 1, \dots, n\}$. □

Proposition 4.3.16 If S_n be strong fuzzy star graph, then $\gamma_{fhn}\left(\overline{S_{n_f}}\right) = \sigma(v_1) + min\,\{\sigma(v_i)\}$ such that v_1 is a center of S_{n_f} and v_i is a vertex that independent in S_n; $i = 2, \dots, n$.

Proof.

Since $\overline{S_{n_f}}$ has two component which are K_1 and K_{n-1}, then by Proposition 4.3.7 and Remark 4.3.11, thus $\gamma_{fhn}\left(\overline{S_{n_f}}\right) = \sigma(v_1) + min\,\{\sigma(v_i)\}$. □

Proposition 4.3.17 If C_{n_f} is a strong fuzzy cycle graph and K_1 is a strong fuzzy complete graph and $C_{n_f} \odot K_{1_f}$ is a corona of these graphs, then

$$\gamma_{fhn}\left(C_{n_f} \odot K_{1_f}\right) = min \left\{ \begin{array}{c} min\left\{\left(\sigma(u_i) + \sigma(u_t) + \sigma(v_j); i,t,j = 1,2,3 \; and \; i \neq j \neq t\right), \sum_{i=1}^{3} \sigma(u_i)\right\}, \\ if\; n = 3 \\ min\left\{ \begin{array}{c} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} \sigma(v_{k+1+2j}) + \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \sigma(u_{k+2i}), \sum_{i=1}^{n} \sigma(u_i) \\ k = 1, \ldots, n; (k + 2i \; mod \; n) and (k + 1 + 2j \; mod \; n) \end{array} \right\}, if\; n \geq 4 \end{array} \right\}$$

,such that $u_i \epsilon V(C_n)$ and v_j is vertex of copies of K_1.

Proof.

To obtain the required two cases depend on the order of cycle as follows.

Case1. If $n = 3$, then four ways to choose the vertices of hn-dominating set.

Subcase1. If take one vertex from the cycle and two vertices from the copy of K_1 which are not adjacent to this vertex to make a set D, then the other two vertices in the cycle are adjacent and the vertices in the set D that dominate these vertices are not adjacent. Thus, the set D is not hn-dominating.

Subcase2. If the set D contains all vertices of copies of K_1, then D is dominating set but not hn-dominating set, since all vertices in $V - D$ are adjacent and the vertices in the set are independent.

Subcase3. If the set D contains two vertices from the cycle and one vertex that not adjacent for them from vertices of copies of K_1. In this case the set

D is hn-dominating set and it is clear that it has minimum cardinality, since all vertices in $\langle V - D \rangle$ are independent.

Subcase4. If the set D contains all vertices of the cycle, then it is clear that D is hn-dominating set.

Case2. If $n = 4$, then two ways to choose the vertices of hn-dominating set.

Subcase1. let $D = \left\{u_{k+2i},\ i = 0,1, \dots, \left\lceil \frac{n}{2} \right\rceil - 1 \right\} \cup \left\{v_{k+1+2j}, j = 0,1, \dots, \left\lceil \frac{n}{2} \right\rceil - 1 \right\},\ k = 1, \dots, n; (k + 2i\ mod\ n) and (k + 1 + 2j\ mod\ n)$. It is clear that the set D is a dominating and the set $\langle V - D \rangle$ is independent and it has minimum cardinality. Thus, the set D is a hn-dominating and $\gamma_{fhn}(C_n \odot K_1) = \sum_{j=0}^{\left\lceil \frac{n}{2} \right\rceil - 1} \sigma(v_{k+1+2j}) + \sum_{i=0}^{\left\lceil \frac{n}{2} \right\rceil - 1} \sigma(u_{k+2i})$.

Subcase2. let $D = V\left(C_{n_f}\right)$, again by using the same technique in the case2(subcase 1), one can be obtained that $\gamma_{fhn}\left(C_{n_f} \odot K_1\right) = \sum_{i=1}^{n} \sigma(u_i)$.

From all cases that mentioned above, the required is obtained. □

Proposition 4.3.18 If C_n is a strong fuzzy cycle graph and $\overline{K_{m_f}}$ is the complement of the strong fuzzy complete graph and $C_{n_f} \odot \overline{K_m}$ is a corona of these graphs, then $\gamma_{fhn}\left(C_{n_f} \odot \overline{K_{m_f}}\right) = \sum_{i=1}^{n} \sigma(v_i), v_i \in V(C_{n_f})$.

Proof.

Let $V\left(C_{n_f}\right) = \{u_1, u_2, \dots, u_n\}$, *it is clear that* the set $D = \{u_1, u_2, \dots, u_n\}$ is a minimal fuzzy hn-dominating set. Now, we must prove that D is minimal fuzzy hn-dominating set that has less order among the other minimal sets. To build fuzzy hn-dominating set if we take all vertices of copies of $\overline{K_{m_f}}$, then this set is fuzzy dominating set but not fuzzy hn-dominating set. Then every fuzzy hn-dominating set must contain vertices from C_{n_f}. Suppose that D_1 obtained by taking vertices from copies of $\overline{K_{m_f}}$

and vertices from C_{n_f} by the same way in proof of Proposition 4.3.17, but if taking vertices from copies of $\overline{K_m}_f$ (say v_1), then we must take a vertex from C_{n_f} (say u_1) that adjacent to v_1 to dominate on the other vertices from copies of $\overline{K_m}_f$ which are adjacent to u_1. Then $|D_1| = n + \left\lfloor \frac{n}{2} \right\rfloor$, so $|D| < |D_1|$. Then D is the minimum fuzzy hn-dominating set, so $\gamma_{fhn}\left(C_n \odot \overline{K_m}_f\right) = \sum_{i=1}^{n} \sigma(v_i)\,, v_i \in V(C_{n_f})$. □

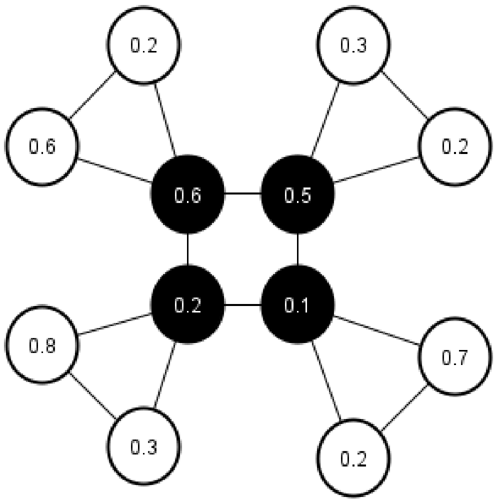


Figure 4.5. Fuzzy hn-domination of corona graph $C_4 \odot K_2$

Proposition 4.3.19 If P_n is a strong fuzzy path graph and K_m is a strong fuzzy complete graph and $P_n \odot K_1$ is a corona of these graphs, then $\gamma_{fhn}\left(P_n \odot \overline{K_m}_f\right) = \sum_{i=1}^{n} u_i;\ u_i \epsilon V(P_n); m, n \geq 2$.

Proof.

Let $V(P_n) = \{u_1, u_2, \dots, u_n\}$, since $m \geq 2$ it is obvious that $D = V(P_n)$ is dominating set and has minimum cardinality of the graph $P_{n_f} \odot \overline{K_m}_f$. Also, the set D is hn-dominating set, since $\langle D \rangle$ is connected and $\langle V - D \rangle$ is independent. Thus, $\gamma_{fhn}\left(P_{n_f} \odot \overline{K_m}_f\right) = \sum_{i=1}^{n} u_i$

Proposition 4.3.20 If K_n is a strong fuzzy path graph and $\overline{K_{m}}_{f}$ is a strong fuzzy complete graph and $K_{n_f} \odot \overline{K_{m_f}}$ is a corona of these graphs, then, $\gamma_{fhn}\left(K_{n_f} \odot \overline{K_{m_f}}\right) = \sum_{i=1}^{n} \sigma(u_i)$, $u_i \in V(K_{n_f})$.

Proof.

Since every vertex in $V(K_{n_f})$ is adjacent to all other vertices, then if D is the set of all vertices from copies of $\overline{K_{m_f}}$, so this set is fuzzy dominating set but not fuzzy hn-dominating set. Also, if D is set of vertices from $\overline{K_{m_f}}$ and K_{n_f} is not fuzzy hn-dominating set. It is clear $D = \{u_1, u_2, \dots, u_n\}$ is minimal fuzzy hn-dominating set. So, $\gamma_{fhn}\left(K_{n_f} \odot \overline{K_{m_f}}\right) = \sum_{i=1}^{n} \sigma(u_i)$.

Proposition 4.3.21 If N_n is a strong fuzzy null graph and $\overline{K_m}$ is a strong fuzzy complete graph and $N_{n_f} \odot \overline{K_{m_f}}$ is a corona of these graphs, then, $\gamma_{fhn}\left(N_{n_f} \odot \overline{K_{m_f}}\right) = \sum_{i=1}^{n} \sigma(u_i)$ $u_i \in V(N_{n_f})$.

Proof.

Since every vertex in N_{n_f} is adjacent to all vertices of one of copies of $\overline{K_{m_f}}$ and N_{n_f} has no any edge, then $N_{n_f} \odot \overline{K_{m_f}}$ has m components which are stars such that the center of every star in $N_{n_f} \odot \overline{K_{m_f}}$ is a vertex from N_{n_f}. Thus, by Corollary 4.3.10 and Remark 4.3.11 we get the result.

Conclusions

and

Future Work

CONCLUSIONS

In this dissertation, the focus is on the domination and fuzzy domination of vertices in graphs and fuzzy graphs. The study comes out with the following conclusions:

1. We can still generate other concepts of domination, project them on the blurry charts and compare them.
2. Odd neighbor D^c domination and total odd neighbor D^c domination are available for all types of graphs.
3. For any graph G with order n and has at least three vertices of degree $n-2$, then $\gamma_{odc}(G) \leq 4$ and $\gamma_{todc}(G) \leq 5$.
4. A graph has even degree for all its vertices, then $\gamma_{odc}(G) = \gamma_{todc}(G)$.
5. For any fuzzy graph has fuzzy odd neighbor D^c domination and fuzzy total odd neighbor D^c domination and fuzzy hn-domination.
6. In graph G and fuzzy graph G_f, $\gamma_{odc}(G) \leq \gamma_{todc}(G)$ and $\gamma_{fodc}(G_f) \leq \gamma_{ftodc}(G_f)$.

   All parameters mentioned have been studied in detail so that all results can be compared.

Future work

We recommend having the following points for future work:

1- Studying fuzzy odd neighbor in D^c domination, fuzzy total odd neighbor in D^c domination and fuzzy hn-domination in fuzzy graph by strong arc.

2- Studying some new graphs with odd neighbor in D^c domination and total odd neighbor in D^c domination.

3- Studying inverse odd neighbor in D^c domination and inverse total odd neighbor in D^c domination.

4- Studying odd neighbor in D^c domination, total odd neighbor in D^c Domination and hn-dominatin in digraph.

5- Studying odd neighbor in D^c domination, total odd neighbor in D^c Domination and hn-dominatin on two operations in some graphs.

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الملخص

في موضوع الهيمنة، تم تقديم العديد من الأوراق البحثية المختلفة في شكل بيان وبيان ضبابي. كما تمت دراستها من قبل الباحثين مرة للرسم البياني ومرة أخرى للرسم البياني الضبابي. تم العثور على الهيمنة في الرسم البياني الضبابي من خلال الحافة الفعالة مرة واحدة والحافة القوية مرة أخرى. في هذه الأطروحة، مجموعة من المفاهيم الجديدة في الرسم البياني والرسم البياني الضبابي من خلال التركيز على هيمنة الحافة الفعالة. يتم دراسة مفهومين في الرسم البياني المفهوم الأول المجموعة D تسمى الجار الفردي في المجموعة المسيطرة D^C لأنها تركز على المجاور لكل رأس في المجموعة المسيطرة على عدد فردي من الرؤوس في D^C للمجموعة المسيطرة D . المفهوم الآخر، المجموعة D تسمى الجار الفردي الكلي في المجموعة المسيطرة D^C ، حيث يحمل هذا المفهوم خصائص المفهوم الأول بالإضافة إلى خاصية المجموعة المهيمنة الكلية، والتي حددها الباحثون .أما الرسم البياني الضبابي فقد تمت دراسة ثلاثة مفاهيم كالتالي: أولاً، المجموعة D هي الجار الغامض في المجموعة المسيطرة D^C ، إذا كانت كل قمة في D مجاورة للعدد الفردي للرؤوس في المجموعة التكميلية الضبابية للمجموعة المسيطرة الضبابية D .

أما بالنسبة للثاني، فيسمى الجار الفردي الإجمالي في المجموعة المسيطرة D^C كما في المفهوم الثاني ثم إيجاد الوزن الأقل لأوزان الجار الفردي الغامض الأدنى في مجموعات D^C المسيطرة. بالإضافة إلى النوع الثالث والأخير الذي يسمى مجموعة مهيمنة ضبابية – hn يتضمن إيجاد المجموعة المسيطرة الغامضة التي تحقق الشرط التالي بأن أي رأسين في المجموعة التكميلية للمجموعة المسيطرة الغامضة يسيطران على رأسين متجاورين في تكملة المجموعة المسيطرة الضبابية متجاورة أيضًا .

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

دراسة تحليلية على هيمنة البيان عبر البيانات الضبابية

اطروحة

مقدمة إلى مجلس كلية التربية للعلوم الصرفة في جامعة بابل

كجزء من متطلبات نيل درجة الدكتوراه فلسفة في التربية / الرياضيات

من قبل

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بإشراف

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