Transmuted Survival Models with Application

A Thesis

Submitted to the Council of the College of Education for Pure Sciences in University of Babylon in a Partial Fulfillment of the Requirements for the Degree of Master in Education / Mathematics

By

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2022 A.D.   **1444 A.H**

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Dedication

To my family... husband and children.

To my big family ... parents, sister and brothers.

To all who teach me a letter or a lesson in my life.

Acknowledgments

First , I wish to thank Allah, my Lord and Creator for His help and inspiration that make this work more than a dream after all what happened. I would like to express my deep thanks and gratitude to my supervisor **prof. Dr . Kareema Abed AL-Kadim** for her valuable guidance and encouragement during my work.

I 'm very grateful to the faculty members of Mathematics Department in the college of Education for Pure Sciences.

Moreover , I am very grateful to my father , mother , sister ,and brothers for their engorgement.

Finally , I 'd like to thank my husband 'Amer Mohammed ' for his support in all days of finishing this work. I would also like to thank my daughter and son for the great help.

Abstract

In this work, we discuss constricting a mathematical formula based on survival function, this formula has a basic parameter called transmuted parameter. In addition, experimenting this formula within Rayleigh and Pareto distributions to produce two different lifetime distributions. The process of this production contains applying this formula on two of lifetime distributions. The need for modeling some data that do not fit the available distributions, so that the objective of this thesis is to formulate a distribution that is more flexible and accurate such these data. In order to do this, the new transmuted distribution based on the survival function is derived. A new distributions gives more accurate results in the application of data. The probability density function, the probability cumulative function and hazard function were extracted also, explaining the shapes of these functions for two models. Furthermore, derived the mathematical and statistical characteristics of two models as the rth moment about the origin, moment generation function, median, mean, the coefficient of Skewness and other properties of models with the dependability performance. The parameters of these models are also estimated by using a classical method(Maximum Likelihood estimation method).Using the Akaike information criterion, and Bayesian information criterion to compare the work of the models. Finally, we proved the superiority of our new distributions compared to the original distributions. We also compared the new distributions with each other for the same data, and we noticed that the performance of the transmuted survival Rayleigh distribution is higher than the performance of the transmuted survival Pareto distribution. In this study the shapes of the functions of our two new distributions and the comparisons in the given tables are introduced by MATLAB(R2012b) software.

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List of Symbols

T	Random variable
t	Value of random variable
U	Quantile function
λ	Transmuted parameter
S(t)	Survival Function
$\mathbb{R}$	Real numbers
S_{TSR}	Survival Function of TSR distribution
F_{TSR}	Cumulative Function of TSR distribution
f_{TSR}	Density Function of TSR distribution
S_{TSP}	Survival Function of TSP distribution
F_{TSP}	Cumulative Function of TSP distribution
f_{TSP}	Density Function of TSP distribution
h(t)	Hazard Function
E(T)	Mathematical expectation of a Random Variable T
θ	Parameter Vector
M_o	Mode
$f_p(t, b)$	Density function of Pareto distribution
$f_r(t, b)$	Density function of Rayleigh distribution
μ	The mean
$F_P(t, b)$	Cumulative Function of Pareto Distribution
$S_r(t, b)$	Survival Function of Rayleigh Distribution
$S_P(t, b)$	Survival Function of Pareto Distribution
M_e	Median
Γ	Gamma function
b	parameter
h_{TSR}	Hazard Function of TSR distribution
h_{TSP}	Hazard Function of TSPD

List of Abbreviations

AIC	Akaike information criterion
BIC	Bayesian information criterion
cdf	Cumulative Density Function
CK	Coefficient of kurtosis
CS	Coefficient of Skewness
CV	Coefficient of variation
MGF	The Moment Generation Function
MLE	Maximum Likelihood Estimator
MSE	Mean Square Error
pdf	Probability Density Function
TSRD	Transmuted Survival Rayleigh Distribution
TSPD	Transmuted Survival Pareto Distribution
Var(T)	Variance of a Random Variable T

Introduction

It is known that there is growing by researchers and acadimics interested in the topic of survival function because of its great importance in the study of the rate of time and the probability of survival of the organism after a period defined time t. The survival function is one of the functions are important in the science of statistics , which has a key role in most of the analytic phenomena depending on the data and statistical information available on this phenomenon : Many researchers have taken great interest in studying the survival functions, until these studies became curricula for different levels of study, thus it became a science concerned with the study of estimation, prediction and optimization.

Statistical modeling is an important concept, in the study of statistical analysis, it helps us in the understanding of many natural phenomena. In many areas, statisticians try for a long time to work hard to produce new distributions and expand those distributions to get rid of the problems of modern life, especially in the areas of age phenomena, such as science medicine, devices the construction is taken in considered of an equation from which to create a new distribution using prevailing formula (traditional) in order to take advantage of the new distribution . In addition, it is very important that we compare the health and performance of the new distributions through the use of information standards of the statistical such as standard (Akaike, Bayesian) information criterion , the right quantity and quality, the new formula is defined and tested with two popular distributions Rayleigh and Pareto distribution to extract different two distributions lifetime.

This thesis consists of four chapters as follows each includes respectively:

The first chapter includes some definitions and principal concepts.

The second chapter deals with the methodology of the new formula that gives new distribution (TSRD) transmuted survival of Rayleigh distribution and studies the mathematical and statistical characteristics of the new distribution and estimation method of (TSRD) transmuted

survival of Rayleigh distribution. The third chapter deals with the methodology of the new formula that introduce the new distribution (TSP) transmuted survival of Pareto distribution and studies the mathematical and statistical characteristics of the new distribution (TSP) estimation method of (TSPD) transmuted survival of Pareto distribution.

The fourth chapter Consists of the Application, a comparison of the new distributions and the origin distributions is made ,by using the (MLE) Maximum Likelihood Estimation method, based on using some comprise criterion.

Literature Review

One new formula depends on two concepts: transmuted formula, survival function. There are many researchers studied these concepts to produce new distributions especially lifetime distributions.(Warren Gilchrist,2000)[23] gave attention to transmutation map especially the quadratic form:

$$F(x) = (1 + \lambda)F_*(x) - \lambda F_*^2(x) \qquad ; |\lambda| \leq 1$$

Where $F_*(x)$ is the cdf of base distribution. Such map has attracted the attention of statistician (Gilchris 2000, Marshall and Olkin 2007 and William T. Show and Ian R.C. Bucklay 2009)[27][5][21]. Lai, C.D. (2013) [9], there are many applications that he presented, and many recent literatures that use the Rayleigh distribution with a two-parameter scale. Dey,S.Dey,T. & Kundu, D. (2014) [4] provid the transformed distributions as extended models.And in 2015 Abdus Saboor, Mustafa Kamal and Munir Ahmad [10] apply the transmuted technique to Exponential-Weibull (TEW) distribution to make Weibull distribution more flexible, and make it wider application in lifetime of system. Daniele Cristina Tita Granzotto 2017 [19] propose the transmuted log-logistic model using the quadratic rank transmutation map and in his book he presents transmuted family of model which has the property that the extra parameter λ can take any real value hence without restricted parameter space λ transmuted technique be a convenient way of constricting new distributions. Kareema A. Al-Kadim and Ashraf A. Mahdi (2018) combine Transmuted map and Exponentiated formula [1].

A new survival model for the transformed Rayleigh distribution with application is used, AL-Kadim, K.(2018) [1] she derived the general formula for the transmuted distribution and presented properties the formula helps us build models for a new distribution. Mohamaad.S.F & AL-Kadim ,K.(2021) [13] are presented new model of transmuted survival of exponential distribution to help us build model for a new distribution .

Publications

Marwa Abbas Madloom and Kareema Abed AL-Kadim "Transmuted Survival of Rayleigh ". Journal of Positive School Psychology:2022,vol.6,No.4,4827-4836.

Chapter One

Some Definitions and Principal Concepts

1.1 Some Reliability Functions

A random variable T is a function, from the sample space S into the set of real numbers $\mathbb{R}$ such that for each interval I in $\mathbb{R}$

$$T: S \rightarrow \mathbb{R} \tag{1.1}$$

, the set $\{s \in S / T(s) \in I\}$ is an event in S ,[23].

Let T be a continuous random variable of distribution that has a parameter vector $\theta = \{\theta_1, \theta_2, \theta_3, \dots, \theta_n\}$, and let $f(T; \theta)$ be the pdf of t, then the cdf of t is given by:

$$F(t; \theta) = p(T \leq t) = \int_{-\infty}^{t} f(t; \theta) dt \tag{1.2}$$

The reliability function (survival function) is defined ,[18].as the probability that a system will operate satisfactorily for a given period of time and denoted by $S(t; \theta) = \bar{F}(t)$ where

$$S(t; \theta) = P (T > t) = 1 - F(t; \theta) \tag{1.3}$$

The hazard   function $h (t; \theta)$[18] for any continuous distribution with pdf $f(t; \theta)$ and cdf $F(t; \theta)$ is defined as follows:

$$h(t; \theta) = \frac{f(t; \theta)}{S(t; \theta)} \tag{1.4}$$

1.2 Statistical Properties

1.2.1 Coefficient of Skewness[15][20]

The coefficient of Skewness is the degree of symmetry of the distribution's curve, it is denoted by

$$CS = \frac{\mu_3}{\sigma^3} = \frac{E(T - \mu)^3}{[E(T - \mu)^2]^{\frac{3}{2}}} \tag{1.5}$$

Where μ_3 is the third moment about the mean and σ is the Standard deviation of the distribution.

The skewness is zero for the symmetric distribution when mean $(\mu) =$ median$(M_e) =$ mode (M_o),positive for the skewed to the right

distribution When $(\mu > M_e > M_o)$ and negative if the distribution is skewed to the left when ($\mu < M_e < M_o$).

1.2.2 Coefficient of Kurtosis [20]

It measures the flatness of the top of the curve and it is defined by:

$$CK = \frac{\mu_4}{\sigma^4} = \frac{E(T-\mu)^4}{[E(T-\mu)^2]^2} \tag{1.6}$$

where μ_4 is the forth moment about the mean and σ is standard

deviation of the distribution. The kurtosis equals 3 for the normal distribution , positive for the more tall and slim curves than the normal one in the neighborhood of the mode when (CK > 3) (in this case it is said to be leptokurtic) and it is negative for platykurtic distribution when (CK < 3)(it means that the distribution is flatter than the normal distribution).

1.2.3 Coefficient of Variation [2]

The coefficient of variation is a dimensionless number which quantifies the degree of variability relative to the mean. The population coefficient of variation is defined as:

$$CV = \frac{\sigma}{\mu} = \frac{\sqrt{E(T-\mu)^2}}{\mu} \tag{1.7}$$

Where μ is the mean of the distribution and σ is the standard division.

1.2.4 Mode[10]

The mode is most useful as a measure of central tendency and the mode is a approach of wording distinguish value. This value is no need to be unique. In addition, if the pdf of the distribution has more than one maximum point then we can tell this distribution has more than one mode. The mode of probability distribution is given as:

Mode $= \frac{\partial f(t)}{\partial t} = 0$, if $\frac{\partial^2 f(t)}{\partial t^2} < 0$

1.2.5 Median[14]

The common definition of the median is the value of random variable t where half the probability comes before it and half comes after it

and $P\ (T\ \le t\ ) \ge \frac{1}{2}$

$P\ (T\ \le t\ ) \le \frac{1}{2}$

Hence we can find it by solve the equation$F\ (t) = \frac{1}{2}$,for $t \in \mathbb{R}$.

1.2.6 Moment Generation Function[14]

Moment generation function has major functional connexion not only because one can use them to soft drive moments but also because a probability distribution is uniquely specified by its mgf. The moments of a random variable or of distribution are the expectations of the powers of the random variable which has the given distribution.

$m(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$

1.2.7 Quantile Function[14]

The quintile function or inverse cumulative distribution function returns the value t such that $t = Q(P) = F^{-1}(P)$ where $0 < p < 1$ so this function is every important tool , in statistical modeling precisely in simulate data.

1.3 Useful Functions[22]

1) The gamma function [6] is given by :

$$\Gamma(t) = \int_0^{\infty} t^{x-1} e^{-t} \, dt \ , x > 0 \tag{1.8}$$

2) The binomial theorem is defined as follows:

$$(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \tag{1.9}$$

where n is positive integer number.

1.4 Rayleigh Distribution [11]

Rayleigh distribution is commonly used to model waiting times between occurrence of rare events and lifetime of electrical or mechanical devices.

A continuous random variable t is said to have a Rayleigh distribution (RD) with parameter (b > 0) if it has the following probability density function , cdf and survival functions are respectively as :

$$f(t,b) = \frac{t}{b^2} \exp\left(\frac{-t^2}{2b^2}\right) \ , t > 0 \ , b > 0 \tag{1.10}$$

$$F(t,b) = 1 - \exp\left(\frac{-t^2}{2b^2}\right) \ , t > 0 \ , b > 0 \tag{1.11}$$

$$S(t) = \exp\left(\frac{-t^2}{2b^2}\right) , t > 0 \ , b > 0 \tag{1.12}$$

Where b is a scale parameter.

1.5 Pareto Distribution[27]

Pareto distribution is named after an Italian-born Swiss professor of economics, Vilfredo Pareto (1848-1923), some times this distribution used to model that distribution incomes and used in description of social, quality control, scientific, geophysical, actuarial, and many other types of observable phenomena. The Pareto distribution is used in many applied fields , including the distribution of oil reserves, sand particles and their size measurement , the burring of part of the forests , the distribution of income and precipitation rain and others. Pareto is continuous distribution and it is a lifetime with $t_\circ$ (necessarily positive) minimum possible value of T.If t is a random variable with a Pareto distribution then the Pdf

,Cdf , Survival function of Pareto distribution is the respectively as :

$$f(t,a,t_\circ) = \frac{at_\circ{}^a}{t^{a+1}} \tag{1.13}$$

For all , $t \geq t_\circ , t_\circ > 0$,a is a positive shape parameter.

$$F(t, a, t_\circ) = 1 - (\frac{t_\circ}{t})^a \qquad (1.14)$$

$$S(t, a, t_\circ) = (\frac{t_\circ}{t})^a \qquad (1.15)$$

1.6 Survival Data

Numerous assays outcome in data on event studied and collected over time, and on the study of elements that associated and effect on the appearance for those events the survival data observes and measure the time until appearance of precise or fixed events. Our example for such events the lifetime of machine components in industrial applications, disease progression, or time to death of Cancer patients.

In addition , the statistical methods that deal and analysis such data usually known as "Survival Analysis ". More over there are three major functions that related to this branch of statistics survival function , probability density function and hazard function The statistical model is a mathematical exemplification and formulation of spotted data set,[24]. The statistical modeling is a set of procedures of applying statistical analysis to a data set moreover when the survival data observes and measures the time until the appearance of precise or fixed event, then the statistical model called "Survival model" or "lifetime model".

1.6.1 Distribution Using Survival Time

Assume T be a non-negative continuous random variable. The most commonly ways to characterize the distribution of survival data are the probability density function (pdf) and cumulative distribution function (cdf), [14]. we shall denote these by f(.) and F(.),

1.6.2 The Transmuted Survival Formula [1]

The proposed quadric transmuted family of distribution with cdf

$$F(t) = (1 + \lambda)G(t) - \lambda G^2(t) \quad |\lambda| \le 1 \qquad (1.16)$$

We find the modern distribution relies on survival function (S(t)) which is determined by :

$$S(t) = (T > t), t > 0$$

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$$S(t) = 1 - F(t) \qquad (1.17)$$

The new formula[14] relies initially, on survival function S(t) which is determined by:

$$S(t) = (1 + \lambda)S_+^2(t) - \lambda S_+(t) \qquad (1.18)$$

as $f: \mathbb{R} \to [\,0, \infty\,]$ ((is a probability density function))

If satisfies the following condition :

1) $f(t) \geq 0$ for all $t > 0$

2) $\int_{-\infty}^{\infty} f(t)dt = 1$

1.7 Information Criterion [17]

Information standards are statistical tools to deal with choosing a suitable   model among other models. The goal of   calculating information standards is to know how a good model works in explaining the relationship between variables and determining where is the best model among the selected models.

In this work we use several information standards which are the Akaike Information Criterion (AIC), Bayesian Information Criterion

(BIC). The Akaike's information criterion (AIC)- calculated using

$$AIC = -2 \ln L(\hat{\theta}) + 2m \qquad (1.19)$$

where $L(\hat{\theta}) = L\,(\,t_1, t_2, \ldots, t_n, \hat{\theta})$, $\hat{\theta}$ is the maximum likelihood estimation of the parameter θ which maximized the likelihood function , m is number of parameters to be estimated, The Bayesian information criterion (BIC) – calculated using.

$$BIC = -2 \ln(\hat{\theta}) + m \ln(n) \qquad (1.20)$$

The Mean Square error (MSE) – calculated using

$$MSE = \frac{1}{n} \sum_{i=1}^{n} (f_n(t_i) - \hat{f}(t_i))^2 \qquad (1.21)$$

where n is the number of data points , $f_n(t_i)$ the value returned by The model , and $\hat{f}(t_i)$ the actual value for data point (i).

Kolmogorow   Smirnow (KS) is the absolute maximum difference between two samples taken from the data set, and calculated by

$$KS = \max(abs(F_f - F_r)) \qquad (1.22)$$

Chapter Two

Transmuted Survival Rayleigh Distribution

2.1 Introduction

This chapter includes the derivation of the density function (Pdf),the probability cumulative distribution function (CdF) and the survival function (S) of the transmuted Rayleigh distribution and they are presented graphically. And constructing a new distribution transmuted survival Rayleigh (TSR) by use survival function of major distribution. We proved that this formula is well defined statistically. Therefore, the statistical characteristics, r^{th} moments, the moment about the origin were derived, mean, variance, median , hazard function and reliability with the presentation of graphs.

2.2 The Transmuted Survival Rayleigh Distribution(TSRD)

The cdf ,pdf and survival function Rayleigh distribution indicated in the respectively equations (1.10) ,(1.11),(1.12) and the shape of pdf of Rayleigh distribution is:
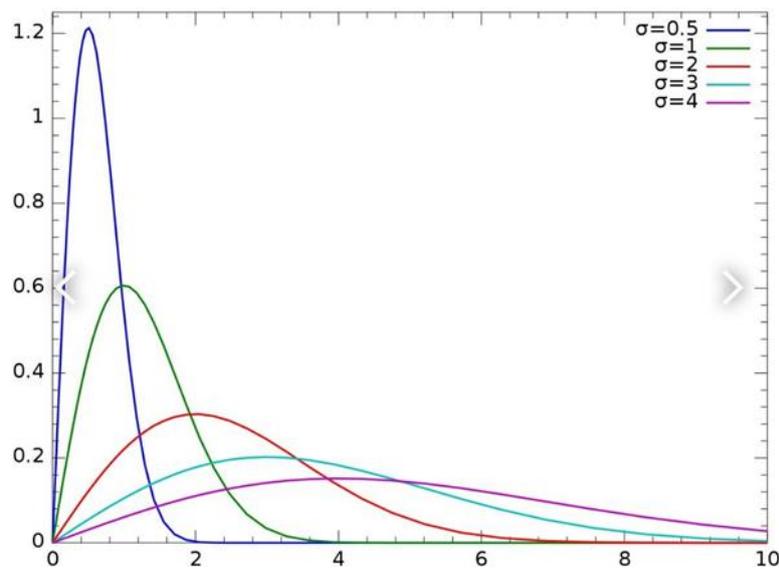
Figure(2-1): The pdf of Rayleigh distribution.

The survival function of Rayleigh distribution and by applying (1.18) then the transmuted-survival-Rayleigh distribution is gotten

$$S_{TSR}(t; b, \lambda) = (1 + \lambda)[exp2\left(\frac{-t^2}{2b^2}\right)] - \lambda[\exp\left(\frac{-t^2}{2b^2}\right)]$$

So we have

$$S_{TSR}(t; b, \lambda) = \exp\left(\frac{-t^2}{2b^2}\right)[(1 + \lambda)exp\left(\frac{-t^2}{2b^2}\right) - \lambda] \qquad (2.1)$$

The Cdf of this distribution is

$$F_{TSR}(t; b, \lambda) = 1 - exp\left(\frac{-t^2}{2b^2}\right)[(1 + \lambda)exp\left(\frac{-t^2}{2b^2}\right) - \lambda] \qquad (2.2)$$

Formerly, the Pdf of the new distribution from the derivative of the Cdf, where b > 0 is the scale parameter.

$$f_{TSR}(t; b, \lambda) = \frac{2t}{b^2}(1 + \lambda)exp\left(\frac{-t^2}{b^2}\right) - \frac{t}{b^2}\lambda exp(\frac{-t^2}{2b^2}) \qquad (2.3)$$

Now , we must prove that the $f_{TSR}(t; b, \lambda)$ is a Pdf

1) $f_{TSR}(t; b, \lambda) \geq 0$

2) $\int_0^\infty f_{TSR}(t; b, \lambda)dt = 1$

1)$f_{TSR}(t; b, \lambda) \geq 0$

$$\frac{2t}{b^2}(1 + \lambda)exp\left(\frac{-t^2}{b^2}\right) - \frac{t}{b^2}\lambda \exp\left(\frac{-t^2}{2b^2}\right) \geq 0$$

$$\frac{t}{b^2}\exp\left(\frac{-t^2}{2b^2}\right)\left[2(1 + \lambda)\exp\left(\frac{-t^2}{2b^2}\right) - \lambda\right] \geq 0$$

$$2(1 + \lambda)exp\left(\frac{-t^2}{2b^2}\right) \geq \lambda$$

$$exp\left(\frac{-t^2}{2b^2}\right) \geq \frac{\lambda}{2(1 + \lambda)}$$

$$S(t) \geq \frac{\lambda}{2(1 + \lambda)}$$

2)$\int_0^\infty f_{TSR}(t; b, \lambda)dt = 1$

$$= \int_0^\infty \frac{2t}{b^2}(1 + \lambda)\exp\left(\frac{-t^2}{b^2}\right)dt + \int_0^\infty \frac{-t}{b^2}\lambda\exp\left(\frac{-t^2}{2b^2}\right)dt$$

$$= -(1 + \lambda) \exp\left(\frac{-t^2}{b^2}\right)\Big|_0^\infty + \lambda \exp\left(\frac{-t^2}{2b^2}\right)\Big|_0^\infty$$

$$= -(1 + \lambda)[0 - 1] + \lambda[0 - 1]$$

$$= 1 + \lambda - \lambda$$

$$= 1$$

The following Figure shows us the shape of the pdf of TSRD for varous values of the parameter b also fixed value $\lambda = 0.2$
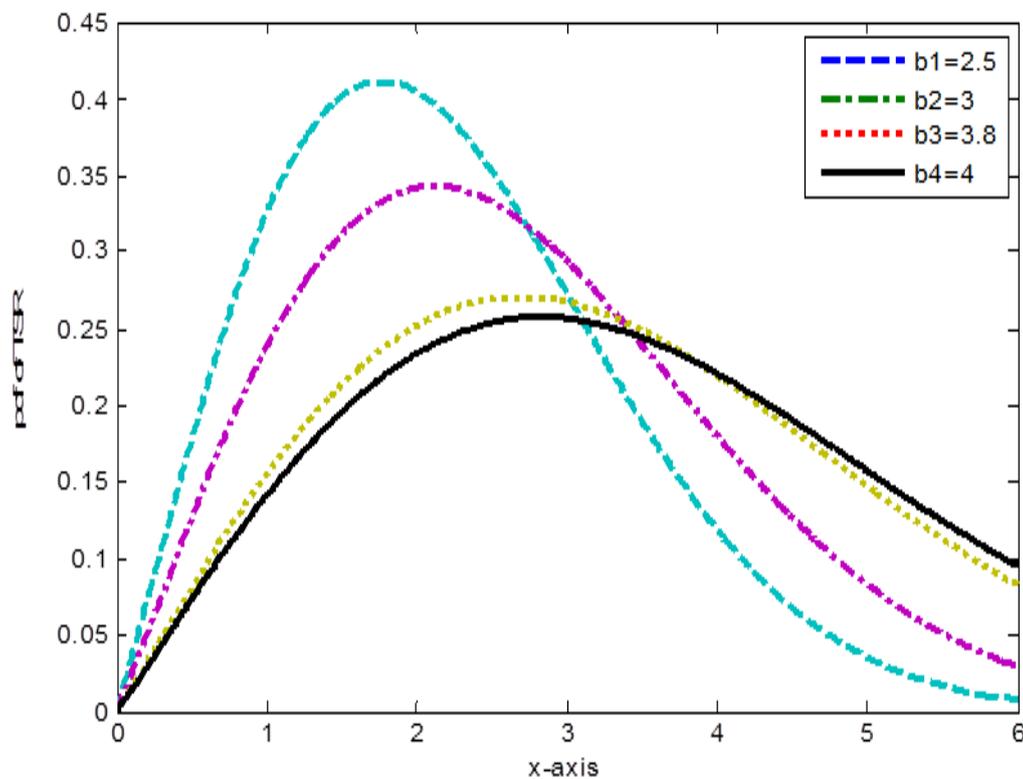
Figure (2-2) Pdf of TSRD with λ= 0.2

Figure (2-2) show us the curve of the Pdf of TSRD is increasing until its maximum at t = 0.4 also it starts to be decreased after that , with increasing value of α random value T , also at $\lambda = 0.2$
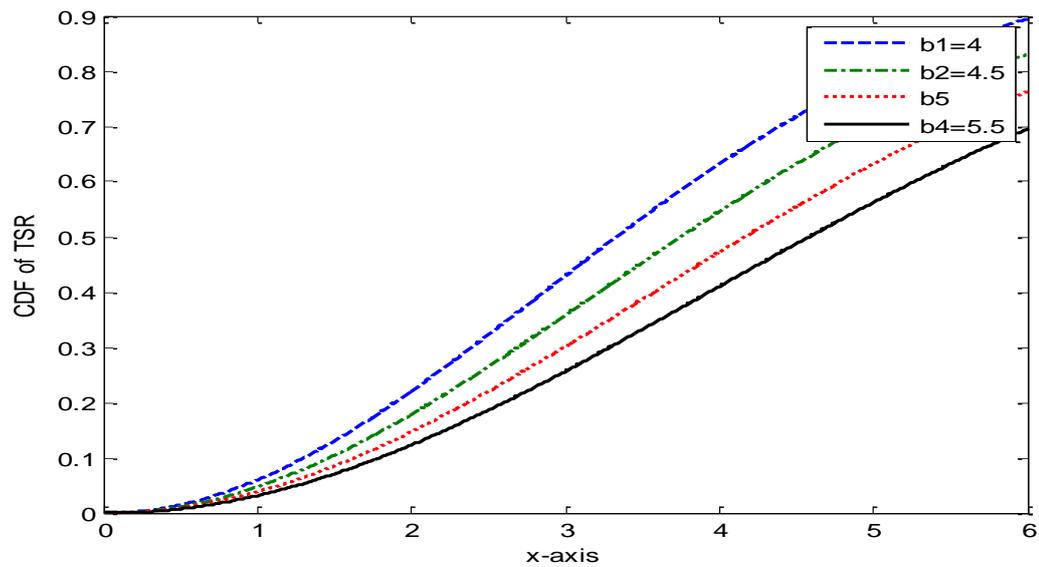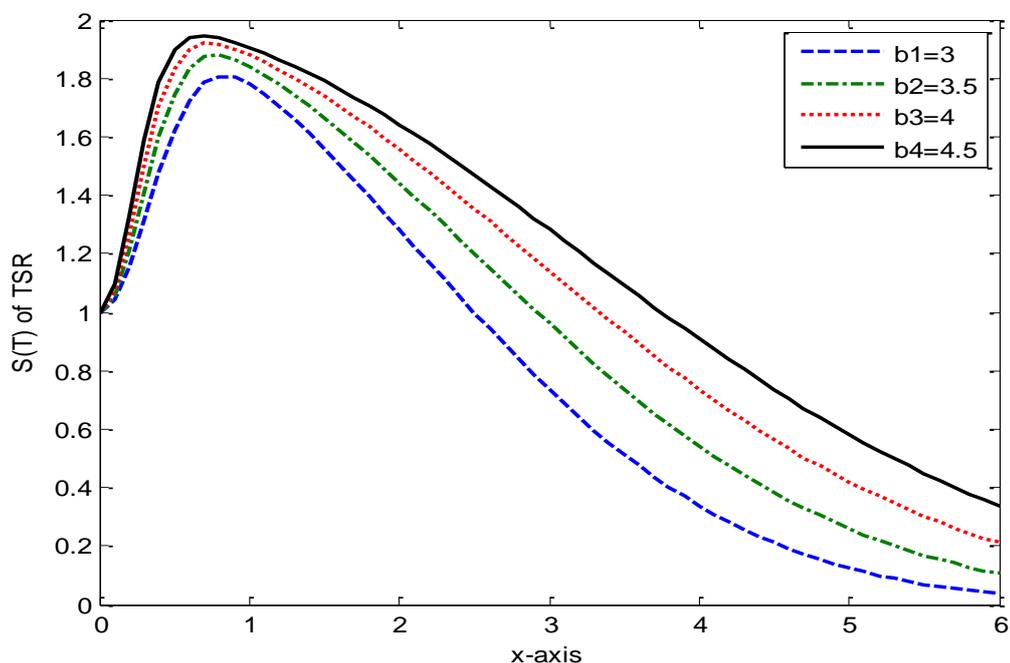
Figure (2-3) Cdf of TSRD with λ=0.001

Figure (2-3) explain that the curve of the Cdf of TSRD is increasing with increasing T and λ = 0.001

Figure (2-4) The survival function of TSRD with λ=1

In Figure (2-4) ,it is noted that the curve of S(t) is right skewed , it is increasing until maximum value and then it is decreasing , until increasing T , λ = 1 at varous value of b.

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2.3 Shape of TSRD

2.3.1 The Limit of Pdf of TSRD

$$\lim_{t \to 0} f_{TSR}(t; b, \lambda) = \lim_{t \to 0} [\frac{2t}{b^2}(1 + \lambda) \, exp\left(\frac{-t^2}{b^2}\right) - \left(\frac{t}{b^2}\right)\lambda \, exp\left(\frac{-t^2}{2b^2}\right)]$$

$$= \lim_{t \to 0} \left(\frac{2t}{b^2}\right)(1 + \lambda)\lim_{t \to 0} exp\left(\frac{-t^2}{b^2}\right) - \lim_{t \to 0} \left(\frac{t}{b^2}\right)\lambda\lim_{t \to 0} exp\left(\frac{-t^2}{2b^2}\right)]$$

$$\lim_{t \to 0} f_{TSR}(t; b, \lambda) = 0$$

Now,

$$\lim_{t \to \infty} f_{TSR}(t; b, \lambda) = \lim_{t \to \infty} [\frac{2t}{b^2}(1 + \lambda) \, exp\left(\frac{-t^2}{b^2}\right) - \left(\frac{t}{b^2}\right)\lambda \, exp\left(\frac{-t^2}{2b^2}\right)]$$

$$= \lim_{t \to \infty}(\frac{2t}{b^2})(1 + \lambda) \lim_{t \to \infty} exp\left(\frac{-t^2}{b^2}\right) - \lim_{t \to \infty}(\frac{t}{b^2})\lambda \lim_{t \to \infty} exp(\frac{-t^2}{2b^2})$$

$$\lim_{t \to \infty} f_{TSR}(t; b, \lambda) = 0$$

Show that in Figure (2-1)

2.3.2 The Limit of Cdf of TSRD

$$\lim_{t \to 0} F_{TSR}(t; b, \lambda) = \lim_{t \to 0} [1 - (1 + \lambda) \, exp\left(\frac{-t^2}{b^2}\right) + \lambda \, exp\left(\frac{-t^2}{2b^2}\right)]$$

$$= 1 - (1 + \lambda)\lim_{t \to 0} exp\left(\frac{-t^2}{b^2}\right) + \lambda\lim_{t \to 0} exp\left(\frac{-t^2}{2b^2}\right)]$$

$$\lim_{t \to 0} F_{TSR}(t; b, \lambda) = 1 - 1 - \lambda + \lambda$$

$$\lim_{t \to 0} F_{TSR}(t; b, \lambda) = 0$$

In addition , it is clear that

$$\lim_{t \to \infty} F_{TSR}(t; b, \lambda) = \lim_{t \to \infty} [1 - (1 + \lambda) \, exp\left(\frac{-t^2}{b^2}\right) + \lambda \, exp\left(\frac{-t^2}{2b^2}\right)]$$

$$= 1 - (1 + \lambda) \lim_{t \to \infty} exp\left(\frac{-t^2}{b^2}\right) + \lambda \lim_{t \to \infty} exp\left(\frac{-t^2}{2b^2}\right)]$$

$$\lim_{t \to \infty} F_{TSR}(t; b, \lambda) = 1$$

To have more clarify about distribution's curve ,we need to find first and second derivative of $f_{TSR}(t)$ in order to decide the maximum point, so

$$f_{TSR}(t; b, \lambda) = \left(\frac{2t}{b^2}\right)(1 + \lambda) \, exp\left(\frac{-t^2}{b^2}\right) - \left(\frac{t}{b}\right)\lambda exp\left(\frac{-t^2}{2b^2}\right)$$

$$\frac{\partial f_{TSR}}{\partial t} = \left(\frac{2t}{b^2}\right)(1 + \lambda)\left(\frac{-2t}{b^2}\right)\exp\left(\frac{-t^2}{b^2}\right) + \exp\left(\frac{-t^2}{b^2}\right)(1 + \lambda)\left(\frac{2}{b^2}\right)$$

$$- \left(\frac{t}{b^2}\right)\lambda\left(\frac{-t}{b^2}\right)\exp\left(\frac{-t^2}{2b^2}\right) + \lambda \exp\left(\frac{-t^2}{2b^2}\right)\left(\frac{1}{b^2}\right)$$

$$\frac{\partial f_{TSR}}{\partial t} = \left(\frac{-4t^2}{b^4}\right)(1 + \lambda)\exp\left(\frac{-t^2}{b^2}\right) + \left(\frac{2}{b^2}\right)(1 + \lambda)\exp\left(\frac{-t^2}{b^2}\right)$$

$$+ \left(\frac{t^2}{b^4}\right)\lambda \exp\left(\frac{-t^2}{2b^2}\right) + \left(\frac{\lambda}{b^2}\right)\exp\left(\frac{-t^2}{2b^2}\right) \qquad (2.4)$$

There is more than one root of this equation which can be founded numerically so, suppose t=t, then ,it depending to local maximum ,minimum or appoint of inflection which depending on the

$$\frac{\partial^2 f(t_,)}{\partial t^2} > 0 \qquad or \frac{\partial^2 f(t_,)}{\partial t^2} < 0 \qquad or \frac{\partial^2 f(t_,)}{\partial t^2} = 0$$

Where ,if the second derivative of the function is less than zero $\frac{\partial^2 f(t_,)}{\partial t^2} < 0$,then this means that the function is in its maximum limit

$$\frac{\partial^2 f_{TSR}(t)}{\partial t^2} = \left(\frac{8t^3}{b^6}\right)(1 + \lambda) \, exp\left(\frac{-t^2}{b^2}\right) - \left(\frac{8t}{b^4}\right)(1 + \lambda) \, exp\left(\frac{-t^2}{b^2}\right) -$$
$$\left(\frac{4t}{b^4}\right)(1 + \lambda) \, exp\left(\frac{-t^2}{b^2}\right) - \left(\frac{t^3}{b^6}\right)\lambda \, exp\left(\frac{-t^2}{2b^2}\right) + \left(\frac{2t}{b^4}\right)\lambda \, exp\left(\frac{-t^2}{2b^2}\right) +$$
$$\left(\frac{t}{b^4}\right)\lambda \, exp\left(\frac{-t^2}{2b^2}\right) \qquad (2.5)$$

2.4 Hazard Rate Function

The hazard rate function also known as the "Force of mortality," is defined by

$$h_{TSR}(t; b, \lambda) = \frac{f_{TSR}(t)}{S_{TSR}(t)}$$

$$= \frac{\left(\frac{2t}{b^2}\right)(1+\lambda)\exp\left(\frac{-t^2}{b^2}\right) - \left(\frac{t}{b^2}\right)\lambda\exp\left(\frac{-t^2}{2b^2}\right)}{(1+\lambda)\exp\left(\frac{-t^2}{b^2}\right) - \lambda\exp\left(\frac{-t^2}{2b^2}\right)} \qquad (2.6)$$

Now we can find the limit:

$$\lim_{t\to 0} h_{TSR}(t;b,\lambda) = \lim_{t\to 0} \frac{\left(\frac{2t}{b^2}\right)(1+\lambda)\,exp\left(\frac{-t^2}{b^2}\right) - \left(\frac{t}{b^2}\right)\lambda\,exp\left(\frac{-t^2}{2b^2}\right)}{(1+\lambda)\,exp\left(\frac{-t^2}{b^2}\right) - \lambda\,exp\left(\frac{-t^2}{2b^2}\right)}$$

$$\lim_{t\to 0} h_{TSR}(t;b,\lambda) = 0$$

And $\lim_{t\to\infty} h_{TSR}(t;b,\lambda) = \infty$
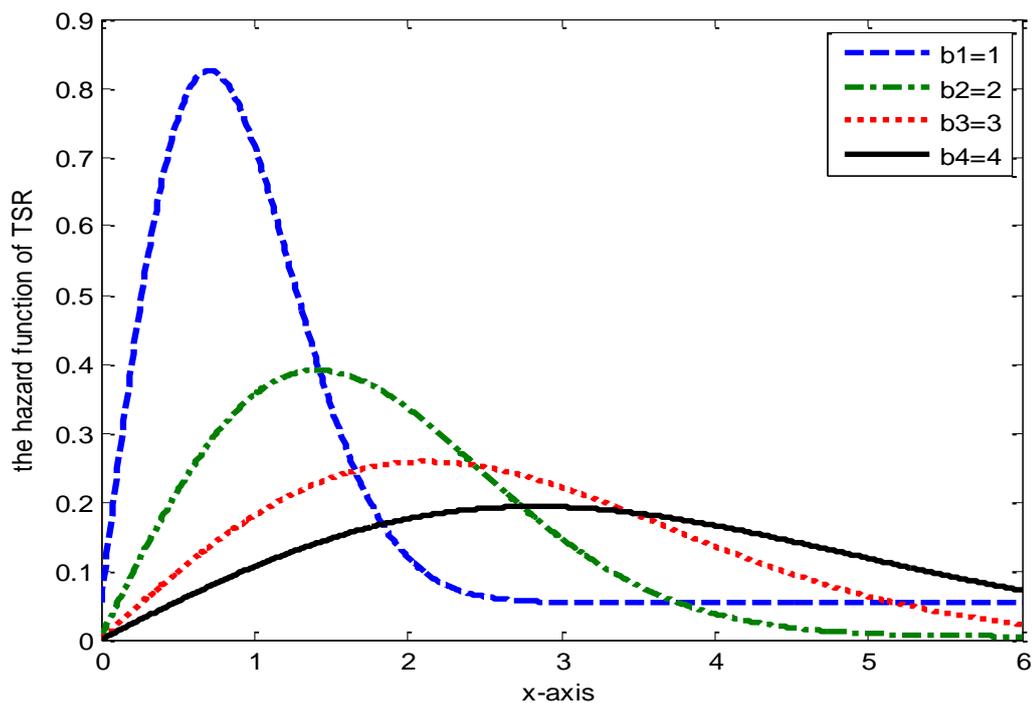
Figure (2-5) The hazard function of TSRD

Figure (2-5): the Hazard function's shape of TSR with altered values of (b) also statistic of $\lambda = -0.1$
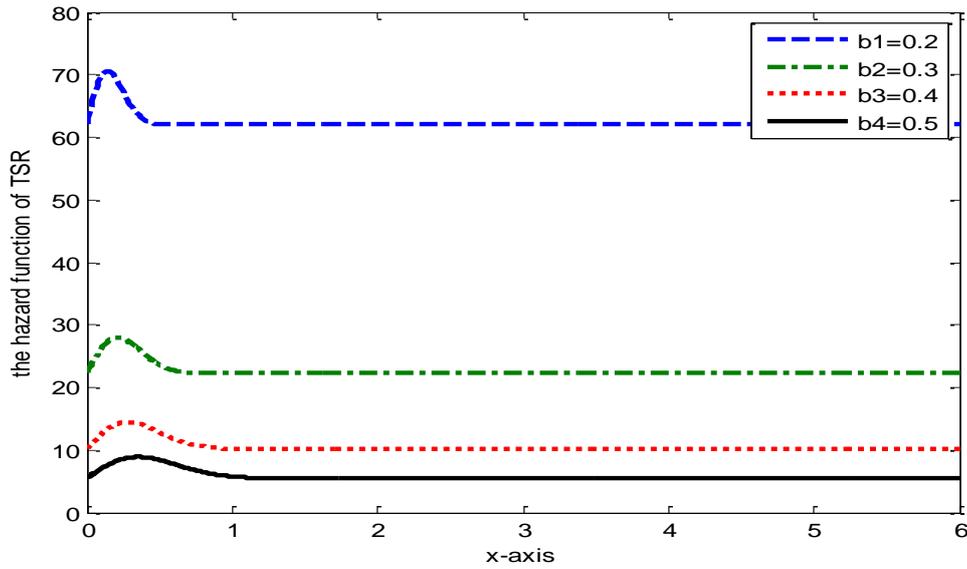
Figure (2-6) The hazard function of TSRD

In this Figure(2-6) the curve of the hazard function of TSRD explains that it increasing until the maximum point of T which gives the high beak , the curve decreases to be stabilize , with increasing T , $\lambda = 1$, and altered value of b.

2.5 Statistical Properties of TSRD

2.5.1 Mode of TSRD

Consider the pdf of TSRD (2.4) and the first derivative (2.5) with respectively:

$$\frac{\partial f_{TSR}(t)}{\partial t} = 0 \text{ ,hence}$$

$$= \frac{-4t^2}{b^4}(1+\lambda)\exp\left(\frac{-t^2}{b^2}\right) + \frac{2}{b^2}(1+\lambda)\exp\left(\frac{-t^2}{b^2}\right) + \frac{t^2}{b^4}\lambda\exp\left(\frac{-t^2}{2b^2}\right) +$$
$$\frac{\lambda}{b^2}\exp\left(\frac{-t^2}{2b^2}\right) = 0$$

And

$$\exp\left(\frac{-t^2}{b^2}\right)\left[\frac{-4t^2}{b^4}(1+\lambda) + \frac{2}{b^2}(1+\lambda)\right] + \exp\left(\frac{-t^2}{2b^2}\right)\left[\frac{t^2}{b^4}\lambda + \frac{\lambda}{b^2}\right] = 0$$

$$exp\left(\frac{-t^2}{b^2}\right)\left[\frac{-4t^2}{b^4}(1+\lambda) + \frac{2}{b^2}(1+\lambda)\right] = - exp\left(\frac{-t^2}{2b^2}\right)\left[\frac{t^2}{b^4}\lambda + \frac{\lambda}{b^2}\right]$$

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$$exp\left(\frac{-t^2}{2b^2}\right)\left[\frac{-4t^2}{b^4}(1+\lambda)+\frac{2}{b^2}(1+\lambda)\right]=-\left[\frac{t^2}{b^4}\lambda+\frac{\lambda}{b^2}\right]$$

$$exp\left(\frac{-t^2}{2b^2}\right)(1+\lambda)\left[\frac{-4t^2}{b^4}+\frac{2}{b^2}\right]=-\lambda\left[\frac{t^2}{b^4}+\frac{1}{b^2}\right]$$

$$exp\left(\frac{-t^2}{2b^2}\right)(1+\lambda)\left(\frac{2}{b^2}\right)\left[\frac{-2t^2}{b^2}+1\right]=-\lambda\left(\frac{1}{b^2}\right)\left[\frac{t^2}{b^2}+1\right]$$

$$exp\left(\frac{-t^2}{2b^2}\right)\frac{2(1+\lambda)}{\lambda}\left[\frac{-2t^2}{b^2}+1\right]=-\left[\frac{t^2}{b^2}+1\right]$$

$$exp\left(\frac{-t^2}{2b^2}\right)\left[-\frac{\left[\frac{-2t^2}{b^2}+1\right]}{\left[\frac{t^2}{b^2}+1\right]}\right]=\frac{\lambda}{2(1+\lambda)}$$

$$ln\,exp\left(\frac{-t^2}{2b^2}\right)\left[-\frac{\left[\frac{-2t^2}{b^2}+1\right]}{\left[\frac{t^2}{b^2}+1\right]}\right]=ln\frac{\lambda}{2(1+\lambda)}$$

$$\left[-\frac{\left[\frac{-2t^2}{b^2}+1\right]}{\left[\frac{t^2}{b^2}+1\right]}\right]=2b^2\ln\frac{\lambda}{2(1+\lambda)}\ln t^2$$

Which is nonlinear equation that can be solved for t numerically.

2.5.2 Moments of TSRD

Theorem1:

1- The rth moment of TSRD about the origin is

$$E(T^r)=b^r\Gamma\left(\frac{r}{2}+1\right)\left[1+\frac{1}{2}\lambda\right]\,,r\in Z^+$$

The mean of TSRD is

$$E(T)=\mu_1=b\Gamma\frac{3}{2}\left[1+\frac{1}{2}\lambda\right]\,,r\in Z^+$$

2- The rth moment $E(T-\mu)^r$ of TSRD about the mean is

$$E(T - \mu)^r = \sum_{i=0}^{r}(-1)^{r-i} \binom{r}{i} (\mu'_1)^{r-i}(\mu'_i)$$

$$E(T - \mu)^r = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}[b\Gamma\frac{3}{2}[1+\frac{1}{2}\lambda]]^{r-i}[b^i\Gamma\left(\frac{i}{2}+1\right)\left[1+\frac{1}{2}\lambda\right]]$$

Proof:

$$E(T^r) = \int_0^\infty t^r f_{TSR}(t)dt$$

$$= \int_0^\infty t^r \left[\frac{2t}{b^2}(1+\lambda)\exp\left(\frac{-t^2}{b^2}\right) - \frac{t}{b^2}\lambda\exp\left(\frac{-t^2}{2b^2}\right)\right]dt$$

$$= \int_0^\infty t^{r+1}\frac{2}{b^2}(1+\lambda)\,exp\left(\frac{-t^2}{b^2}\right)dt - \int_0^\infty t^{r+1}\frac{\lambda}{b^2}exp\left(\frac{-t^2}{2b^2}\right)dt$$

$$= \frac{2(1+\lambda)}{b^2}\int_0^\infty t^{r+1}\,exp\left(\frac{-t^2}{b^2}\right)dt - \frac{\lambda}{b^2}\int_0^\infty t^{r+1}\,exp\left(\frac{-t^2}{2b^2}\right)dt$$

Let

$$y = \frac{t^2}{b^2} \quad , \quad y^{\frac{1}{2}} = \frac{t}{b} \quad , \quad t = by^{\frac{1}{2}} \quad , dt = \frac{1}{2}\,by^{\frac{-1}{2}}dy$$

Now ,

$$E(T^r) = \frac{2(1+\lambda)}{b^2}\int_0^\infty (by^{\frac{1}{2}})^{r+1}\exp(-y)\left[\frac{1}{2}\,by^{\frac{-1}{2}}dy\right]$$
$$- \frac{\lambda}{b^2}\int_0^\infty (by^{\frac{1}{2}})^{r+1}\exp\left(\frac{-1}{2}y\right)\left[\frac{1}{2}\,by^{\frac{-1}{2}}dy\right]$$

$$E(T^r) = (1+\lambda)b^r\int_0^\infty y^{\frac{r}{2}}\exp(-y)\,dy - \frac{1}{2}\lambda b^r\int_0^\infty y^{\frac{r}{2}}\exp\left(\frac{-1}{2}y\right)dy$$

$$E(T^r) = b^r\Gamma\left(\frac{r}{2}+1\right)\left[1+\frac{1}{2}\lambda\right] \quad ,r \in Z^+ \tag{2.7}$$

Now,

Where μ_1' is the mean of TSRD calculated from (2-10), the mean of the TSRD is (where r = 1)

$$E(T) = b\Gamma\frac{3}{2}\left[1+\frac{1}{2}\lambda\right] \quad ,r \in Z^+ \tag{2.8}$$

Now ,

The rth moments about the mean of TSRD is

E(T- μ)^r= $\sum_{i=0}^{r}$ (-1)$^{r-i}$ $\binom{r}{i}$ μ$_i^{'}$ (μ$_1^{'}$)$^{r-i}$

E(T- μ)^r= $\sum_{i=0}^{r}$ (-1)$^{r-i}$ $\binom{r}{i}$ [b Γ$\frac{3}{2}$ (1+λ)]$^{r-i}$[b^{i}Γ $\left(\frac{i}{2}+1\right)$ $\left[1+\frac{λ}{2}\right]$]   (2.9)

  ,$r \in Z^+$

And the second moment of the TSRD is(when r=2):

$$E(T^2) = b^2\Gamma\left(\frac{2}{2}+1\right)\left[1+\frac{1}{2}\lambda\right]$$

$$E(T^2) = b^2\Gamma 2\left[1+\frac{1}{2}\lambda\right]$$ (2.10)

Then ,the variance of TSRD is

$$var(T) = E(T-\mu)^2$$

$$var(T) = \sum_{i=0}^{2}(-1)^{2-i}\binom{2}{i}[b\Gamma\frac{3}{2}\left[1+\frac{1}{2}\lambda\right]]^{2-i}\left[b^i\Gamma\left(\frac{i}{2}+1\right)\left[1+\frac{1}{2}\lambda\right]\right]$$

(2.11)

The coefficient of variation, CV, is given by

$$CV = \frac{\sqrt{var(T)}}{E(T)}$$

CV

$$= \frac{\sqrt{\sum_{i=0}^{2}(-1)^{2-i}\binom{2}{i}[b\Gamma\frac{3}{2}\left[1+\frac{1}{2}\lambda\right]]^{2-i}\left[b^i\Gamma\left(\frac{i}{2}+1\right)\left[1+\frac{1}{2}\lambda\right]\right]}}{b\Gamma\frac{3}{2}\left[1+\frac{1}{2}\lambda\right]}$$ (2.12)

Theorem 2:

The (mgf) Moments Generation Function of the TSRD is

$$M_t(z) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \left[b^j \Gamma\left(\frac{j}{2} + 1\right) \left[1 + \frac{1}{2}\lambda\right] \right]$$

Proof: to proof the (mgf) of (T) for TSRD by

$$M_t(z) = E(e^{tz}) = \int_0^{\infty} \exp(tz) f_{TSR}(z) dz$$

When $\exp(tz) = \sum_{j=0}^{\infty} \frac{(tz)^j}{j!}$,

then $m_t(z) = \sum_{j=0}^{\infty} \frac{t^j}{j!} E(z^j)$

$$m_t(z) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \left[b^j \Gamma\left(\frac{j}{2} + 1\right) \left[1 + \frac{1}{2}\lambda\right] \right] \tag{2.13}$$

2.5.3 Quintile Function of TSRD

Consider the cdf in equations (2.2), the quintile function of the TSRD may be expressed with in implicit form as:

$$U = 1 - (1 + \lambda) \exp\left(\frac{-t^2}{b^2}\right) + \lambda \exp\left(\frac{-t^2}{2b^2}\right)$$

$$1 - U = (1 + \lambda) \exp\left(\frac{-t^2}{b^2}\right) - \lambda \exp\left(\frac{-t^2}{2b^2}\right)$$

$$ln(1 - U) = ln \exp\left(\frac{-t^2}{2b^2}\right) \left[(1 + \lambda) \exp\left(\frac{-t^2}{2b^2}\right) - \lambda\right]$$

$$ln(1 - U) = \frac{-t^2}{2b^2} + \ln\left[(1 + \lambda) exp\left(\frac{-t^2}{2b^2}\right) - \lambda\right] \tag{2.14}$$

Which is non-linear equation can be solved for t numerically.

2.5.4 Median of TSRD

Consider (2- 2) and let $F_{TSR}(t) = \frac{1}{2}$

$$1 - (1 + \lambda) \exp\left(\frac{-t^2}{b^2}\right) + \lambda \exp\left(\frac{-t^2}{2b^2}\right) = \frac{1}{2}$$

$$(1 + \lambda) exp\left(\frac{-t^2}{b^2}\right) - \lambda exp\left(\frac{-t^2}{2b^2}\right) = \frac{1}{2}$$

$$(1 + \lambda) exp\left(\frac{-t^2}{b^2}\right) - \lambda exp\left(\frac{-t^2}{2b^2}\right) - \frac{1}{2} = 0$$

$$exp\left(\frac{-t^2}{b^2}\right) - \frac{\lambda}{(1 + \lambda)} exp\left(\frac{-t^2}{2b^2}\right) - \frac{1}{2(1 + \lambda)} = 0$$

Let $x = exp\left(\frac{-t^2}{2b^2}\right)$, $x^2 = exp\left(\frac{-t^2}{b^2}\right)$

$$x^2 - \frac{\lambda}{(1 + \lambda)} x - \frac{1}{2(1 + \lambda)} = 0$$

and

$$c = \frac{1}{2(1 + \lambda)}, \qquad let \ k = \frac{\lambda}{(1 + \lambda)}$$

The quadratic equation can be solved algebraically for x as:

$$X = \frac{-k \pm \sqrt{k^2 - 4ac}}{2a}$$

If $k^2 - 4ac > 0$ then $x < 0$ or x> 0, x is found

If $k^2 - 4ac < 0$ then $x > 0$, x is neglect

Then we can find t_{med} after solving the equation $x = exp(\frac{-t^2}{2b^2})$ such that

$$f(t, b^2) = \frac{t}{b^2} \exp\left(\frac{-t^2}{2b^2}\right) I_{(0,\infty)}(t) \quad , b > 0 , \ b^2 > 0$$

where b² is the scale parameter

Now,

$$lnx = lnexp(\frac{-t^2}{2b^2}), \text{hence}$$

$$lnx = ln\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 - 4\left(\frac{1}{2(1+\lambda)}\right)}}{2*1}$$

$$\ln\left(exp\left(\frac{-t^2}{2b^2}\right)\right) = \ln\left(\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 - \left(\frac{2}{(1+\lambda)}\right)}}{2}\right)$$

$$\frac{-t^2}{2b^2} = ln\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 - \left(\frac{2}{(1+\lambda)}\right)}}{2}$$

$$t^2 = -2b^2 \ln\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 - \left(\frac{2}{(1+\lambda)}\right)}}{2}$$

$$t^2 = \ln[\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 - \left(\frac{2}{(1+\lambda)}\right)}}{2}]^{-2b^2}$$

$$t = \sqrt{\ln[\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 - \left(\frac{2}{(1+\lambda)}\right)}}{2}]^{-2b^2}} \qquad (2.15)$$

when t > 0

$$\text{that is } \sqrt{\ln[\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 - \left(\frac{2}{(1+\lambda)}\right)}}{2}]^{-2b^2}} \in \mathbb{R}$$

2.5.5 The Coefficient of Skewness of TSRD

The coefficient of Skewness of a random variable (t) CS is given by

$$CS = \frac{E(T-\mu)^3}{P^3}$$

Where $P^3 = (var(T))^{\frac{3}{2}}$

μ is the mean (E(T)) , so

$$CS$$

$$= \frac{\sum_{i=0}^{3}(-1)^{3-i}\binom{3}{i}[b\Gamma\frac{3}{2}(1+\frac{\lambda}{2})]^{3-i}\left[b^i\Gamma\left(\frac{i}{2}+1\right)\left[1+\frac{1}{2}\lambda\right]\right]}{[\sum_{i=0}^{2}(-1)^{2-i}\binom{2}{i}[b\Gamma\frac{3}{2}(1+\frac{\lambda}{2})]^{2-i}[b^i\Gamma\left(\frac{i}{2}+1\right)\left[1+\frac{1}{2}\lambda\right]]]^{\frac{3}{2}}} \quad (2.16)$$

2.5.6 Coefficient of Kurtosis of TSRD

The coefficient of kurtosis is a tool that identity if the tails of given distribution have extreme values or not.The coefficient of kurtosis CK is given by

$$CK = \frac{E(T-\mu)^4}{P^4}$$

where

$$E(T-\mu)^4 = \sum_{i=0}^{4}(-1)^{4-i}\binom{4}{i}[b\Gamma\frac{3}{2}(1+\frac{\lambda}{2})]^{4-i}[b^i\Gamma\left(\frac{i}{2}+1\right)\left[1+\frac{1}{2}\lambda\right]]$$

And $P^4 = [var(t)^2]^2$

According to theorem 1 the kurtosis of TSRD is

$$CK$$

$$= \frac{\sum_{i=0}^{4}(-1)^{4-i}\binom{4}{i}[b\Gamma\frac{3}{2}(1+\frac{\lambda}{2})]^{4-i}[b^i\Gamma\left(\frac{i}{2}+1\right)\left[1+\frac{1}{2}\lambda\right]]}{[\sum_{i=0}^{2}(-1)^{2-i}\binom{2}{i}[b\Gamma\frac{3}{2}(1+\frac{\lambda}{2})]^{2-i}[b^i\Gamma\left(\frac{i}{2}+1\right)\left[1+\frac{1}{2}\lambda\right]]]^2} \quad (2.17)$$

2.5.7 Order Statistics

For sample of size n , we know that if $T_{(1)} \leq \cdots \leq T_{(n)}$ denotes the order statistic of random sample $T_1, T_2, \ldots, T_n$ from a continuous population with cdf $F_x(t)$ and pdf $f_x(t)$ then the pdf of T_j is given by

$$f_{Tj}(t) = \frac{n!}{(j-1)!\,(n-j)!} f_t(t)(F_t(t))^{j-1}(1 - F_t(t))^{n-j}$$

For j=1,…,n . The pdf of the jth order statistic for a TSRD is given by

$$f_{Tj}(t) = \frac{n!}{(j-1)!\,(n-j)!} f_{TSR}(t)(F_{TSR}(t))^{j-1}(1 - F_{TSR}(t))^{n-j}$$

$$= \frac{n!}{(j-1)!(n-j)!} \left[\frac{2t}{b^2}(1+\lambda)\exp\left(\frac{-t^2}{b^2}\right) - \frac{t}{b^2}\lambda\exp\left(\frac{-t^2}{2b^2}\right)\right][1 - (1 +$$

$$\lambda)exp\left(\frac{-t^2}{b^2}\right) + \lambda\,exp\left(\frac{-t^2}{2b^2}\right)]^{j-1}[1 - [1 - (1 + \lambda)\,exp\left(\frac{-t^2}{b^2}\right) +$$

$$\lambda\,exp\left(\frac{-t^2}{2b^2}\right)]]^{n-j}$$

$$= \frac{n!}{(j-1)!(n-j)!} \left[\frac{2t}{b^2}(1+\lambda)\,exp\left(\frac{-t^2}{b^2}\right) - \frac{t}{b^2}\lambda\,exp\left(\frac{-t^2}{2b^2}\right)\right][1 - (1 +$$

$$\lambda)exp\left(\frac{-t^2}{b^2}\right) + \lambda\,exp\left(\frac{-t^2}{2b^2}\right)]^{j-1}[(1 + \lambda)exp\left(\frac{-t^2}{b^2}\right) - \lambda\,exp\left(\frac{-t^2}{2b^2}\right)]^{n-j}$$

Therefore ,the pdf of the largest order statistic T_n is

$$f_{Tn}(t) = [\frac{2nt}{b^2}(1+\lambda)\,exp\left(\frac{-t^2}{b^2}\right) - \frac{nt}{b^2}\lambda\,exp\left(\frac{-t^2}{2b^2}\right)][1 - (1 +$$

$$\lambda)exp\left(\frac{-t^2}{b^2}\right) + \lambda\,exp\left(\frac{-t^2}{2b^2}\right)]^{n-1} \tag{2.18}$$

And the pdf of the smallest order statistic T_1 is

$$f_{T1}(t) = [\frac{2nt}{b^2}(1+\lambda)\,exp\left(\frac{-t^2}{b^2}\right) - \frac{nt}{b^2}\lambda\,exp\left(\frac{-t^2}{2b^2}\right)][(1 + \lambda)exp\left(\frac{-t^2}{b^2}\right) -$$

$$\lambda\,exp\left(\frac{-t^2}{2b^2}\right)]^{n-1} \tag{2.19}$$

We can find median $= T_{\frac{n+1}{2}}$, range r.s $= T_n - T_1$

2.6 Estimation Method of TSRD[7]

We estimate the two parameters (λ) and (b) of TSRD by the following estimations method:

2.6.1 Maximum Likelihood Estimation of TSRD

We will apply this method in order to estimate the parameters of the TSR distribution, Let $\gamma = (\lambda, b)^{\mathrm{T}}$ be the factor vector of that we want to guess and let $(T_1, ..., T_n)$ be a random variables of size (n) with a recognized probability density function $f_{TSR}(t; b, \lambda)$, the likelihood function is defined as a famous joint probability density function of the data.it can be written as follows:

$$L(t_1, t_2, \ldots, t_n, b; \lambda) = \prod_{i=1}^{n} f_{TSR}(t_i, b; \lambda)$$

$$= \prod_{i=1}^{n} \left[\frac{2t}{b^2} (1 + \lambda) \exp\left(\frac{-t^2}{b^2}\right) - \frac{t}{b^2} \lambda \exp\left(\frac{-t^2}{2b^2}\right) \right] \qquad (2.20)$$

It can be presented the log-likelihood function for the vector of parameters

$$l = \ln L(t_i, b; \lambda)$$

$$l = \ln \prod_{i=1}^{n} \left[\frac{2t_i}{b^2} (1 + \lambda) \exp\left(\frac{-t_i^2}{b^2}\right) - \frac{t_i}{b^2} \lambda \exp\left(\frac{-t_i^2}{2b^2}\right) \right]$$

$$l = \sum_{i=1}^{n} \ln\left[\frac{2t_i}{b^2} (1 + \lambda) \exp\left(\frac{-t_i^2}{b^2}\right) - \frac{t_i}{b^2} \lambda \exp\left(\frac{-t_i^2}{2b^2}\right) \right] \qquad (2.21)$$

Then, by taking the fractional derivatives of l with respect to unidentified parameters (λ, b) as :

$$G(\lambda) = \frac{\partial l}{\partial \lambda} = \sum_{i=1}^{n} \left[\frac{\frac{2t_i}{b^2} exp\left(\frac{-t_i{}^2}{b^2}\right) - \frac{t_i}{b^2} exp\left(\frac{-t_i{}^2}{2b^2}\right)}{\frac{2t_i}{b^2}(1+\lambda) exp\left(\frac{-t_i{}^2}{b^2}\right) - \frac{t_i}{b^2}\lambda\, exp\left(\frac{-t_i{}^2}{2b^2}\right)} \right] \quad (2.22)$$

$w(b)$

$$= \sum_{i=1}^{n} \frac{\frac{2t_i}{b^2}(1+\lambda)\left(\frac{2t_i{}^2}{b^3}\right)\exp\left(-\frac{t_i{}^2}{b^2}\right) + \exp\left(\frac{-t_i{}^2}{b^2}\right)(1+\lambda)\left(\frac{-4t_i}{b^3}\right)}{\left(\frac{2t_i}{b^2}\right)(1+\lambda)\exp\left(\frac{-t_i{}^2}{b^2}\right) - \left(\frac{t_i}{b^2}\right)\lambda\exp\left(\frac{-t_i{}^2}{2b^2}\right)}$$

$$- \sum_{i=1}^{n} \frac{\left(\frac{t_i}{b^2}\right)\lambda\left(\frac{t_i{}^2}{b^3}\right)\exp\left(\frac{-t_i{}^2}{b^2}\right) + \exp\left(\frac{-t_i{}^2}{2b^2}\right)\left(\frac{2t_i}{b^3}\right)\lambda}{\left(\frac{2t_i}{b^2}\right)(1+\lambda)\,exp\left(\frac{-t_i{}^2}{b^2}\right) - \left(\frac{t_i}{b^2}\right)\lambda\, exp\left(\frac{-t_i{}^2}{2b^2}\right)}$$

$$(2.23)$$

Now,let

$$\frac{\partial l}{\partial \lambda} = 0 \quad \text{and} \quad \frac{\partial l}{\partial b} = 0$$

There is no closed solution of (2-22) and (2-23) therefore , numerical technique (Newton- Raphso - method) should be apply to solve these equation as follows:

$$\begin{pmatrix} \hat{\lambda} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} \lambda \\ b \end{pmatrix} - J^{-1}\begin{pmatrix} G(\lambda) \\ W(b) \end{pmatrix}$$

Where

$$J = \begin{pmatrix} \dfrac{\partial G(\lambda)}{\partial \lambda} & \dfrac{\partial G(\lambda)}{\partial b} \\ \dfrac{\partial W(b)}{\partial \lambda} & \dfrac{\partial W(b)}{\partial b} \end{pmatrix}$$

It is the Jacobean matrix.

Chapter Three

Transmuted Survival of Pareto Distribution

3.1 Introduction

In this chapter ,we have contributed to finding life distribution which is the transmuted survival of Pareto distribution depending on the survival function and prove it as a probability function as well as finding its mathematical and statistical properties, , rth moments, the moment about the origin were derived, mean, variance, median , hazard function and survival with the presentation of graphs.

3.2 The Transmuted Survival Pareto Distribution TSPD:

The Cdf and Pdf functions are indicated in the respectively equations(1.13),(1.14),(1.15) and the shape of the pdf of Pareto distribution is :[12]
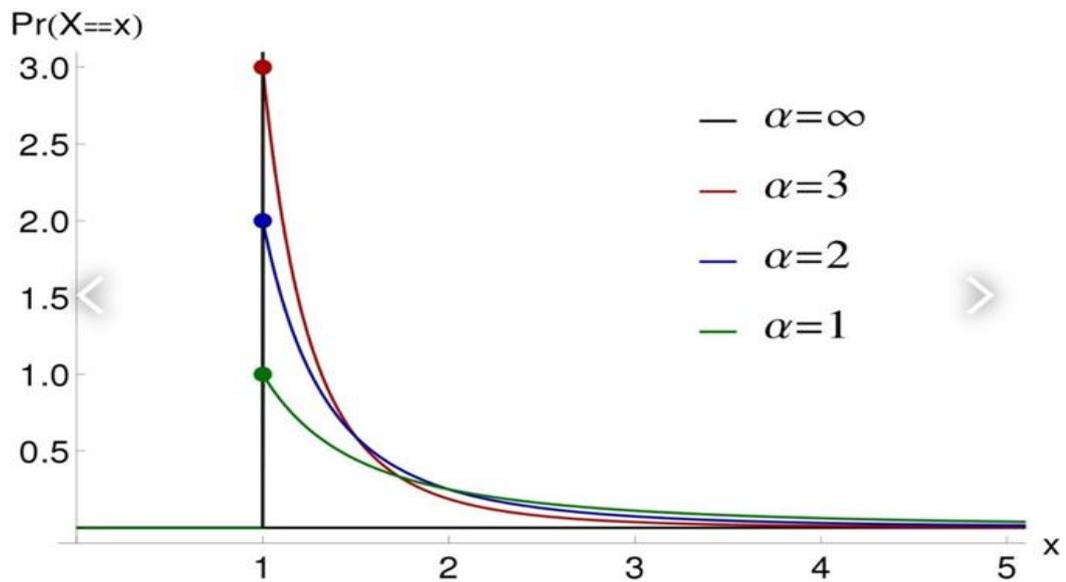
Figure(3-1) the pdf of Pareto distribution

In addition, Survival function that clearly gives the same information as cdf is the right tail distribution function [26]

$$S_P(t; a, t_\circ) = 1 - F_P(t; a, t_\circ)$$

$$S_P(t; a, t_\circ) = (\frac{t_\circ}{t})^a \tag{3.1}$$

Apply formula (3.1) we get:

$$S_{TSP}(t) = (1 + \lambda)S_P(t)^2 - \lambda S_P(t) \tag{3.2}$$
$$= (1 + \lambda)(\frac{t_\circ}{t})^{2a} - \lambda(\frac{t_\circ}{t})^a$$

$$S_{TSP}(t) = (\frac{t_\circ}{t})^a[(1 + \lambda)(\frac{t_\circ}{t})^a - \lambda]$$

(3.3)

Where $-1 \leq \lambda \leq 1$ also λ is called the transmuted parameter.

$$F_{TSP}(t) = 1 - S_{TSP}(t)$$

$$F_{TSP}(t) = 1 - (1 + \lambda)(\frac{t_\circ}{t})^{2a} + \lambda(\frac{t_\circ}{t})^a$$

(3.4)

Then the pdf of the new distribution, from the derivative of the Cdf distribution where $a > 0$

$$f_{TSP}(t) = \frac{(1 + \lambda)t_\circ^{2a}(2at^{2a-1})}{t^{4a}} - \frac{\lambda t_\circ^a(at^{a-1})}{t^{2a}}$$

$$f_{TSP}(t) = \frac{2a(1 + \lambda)t_\circ^{2a}}{t^{2a+1}} - \frac{a\lambda t_\circ^a}{t^{a+1}} I_{(t_\circ, \infty)}$$

(3.5)

Now ,we want to prove that f_{TSP} is a probability density function therefore ,we must fulfill the Conditions of pdf ,it is easy to show that the pdf of TSPD is positive

$$f_{TSP}(t) \geq 0$$

$$\frac{2a(1 + \lambda)t_\circ^{2a}}{t^{2a+1}} - \frac{a\lambda t_\circ^a}{t^{a+1}} \geq 0$$

$$\frac{2a(1 + \lambda)t_\circ^{2a}}{t^{2a+1}} \geq \frac{a\lambda t_\circ^a}{t^{a+1}}$$

$$\frac{(t^{a+1})2a(1 + \lambda)t_\circ^{2a}}{t^{2a+1}} \geq a\lambda t_\circ^a$$

$$\frac{2a(1 + \lambda)t_\circ^{2a}}{t^a} \geq a\lambda t_\circ^a$$

$$\frac{2t_\circ^a}{t^a} \geq \frac{\lambda}{(1 + \lambda)}$$

If $\lambda > 0$ then $f_{TSP} > 0$

If $\lambda = 0$ then $f_{TSP} = 0$

Now to prove that $\int_{t_\circ}^\infty f_{TSP}(t)dt = 1$

$$\int_{t_\circ}^{\infty} \left[\frac{2a(1+\lambda)t_\circ^{2a}}{t^{2a+1}} - \frac{a\lambda t_\circ^{a}}{t^{a+1}} \right] dt$$

$$= \int_{t_\circ}^{\infty} \frac{2a(1+\lambda)t_\circ^{2a}}{t^{2a+1}} dt - \int_{t_\circ}^{\infty} \frac{a\lambda t_\circ^{a}}{t^{a+1}} dt$$

$$2a(1+\lambda)t_\circ^{2a} \int_{t_\circ}^{\infty} \frac{1}{t^{2a+1}} dt - a\lambda t_\circ^{a} \int_{t_\circ}^{\infty} \frac{1}{t^{a+1}} dt$$

$$2a(1+\lambda)t_\circ^{2a} \int_{t_\circ}^{\infty} t^{-2a-1} dt - a\lambda t_\circ^{a} \int_{t_\circ}^{\infty} t^{-a-1} dt$$

$$2a(1+\lambda)t_\circ^{2a} \left[\frac{t^{-2a}}{-2a} \right]_{t_\circ}^{\infty} + a\lambda t_\circ^{a} [\frac{t^{-a}}{a}]_{t_\circ}^{\infty}$$

$$0+1+\lambda-\lambda = 1$$

$$\int_{t_\circ}^{\infty} f_{TSP}(t)dt = 1$$

Figure (3-2): the pdf of TSPD
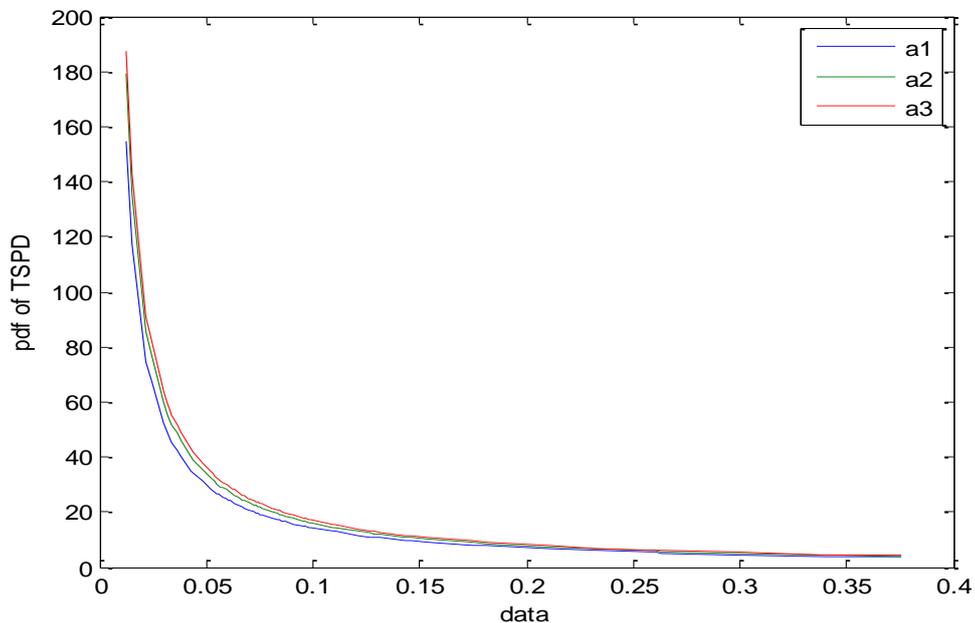
In this figure starts of maximum value and decreases to zero at various value of a.It is noted that the curve of pdf is right skewed.
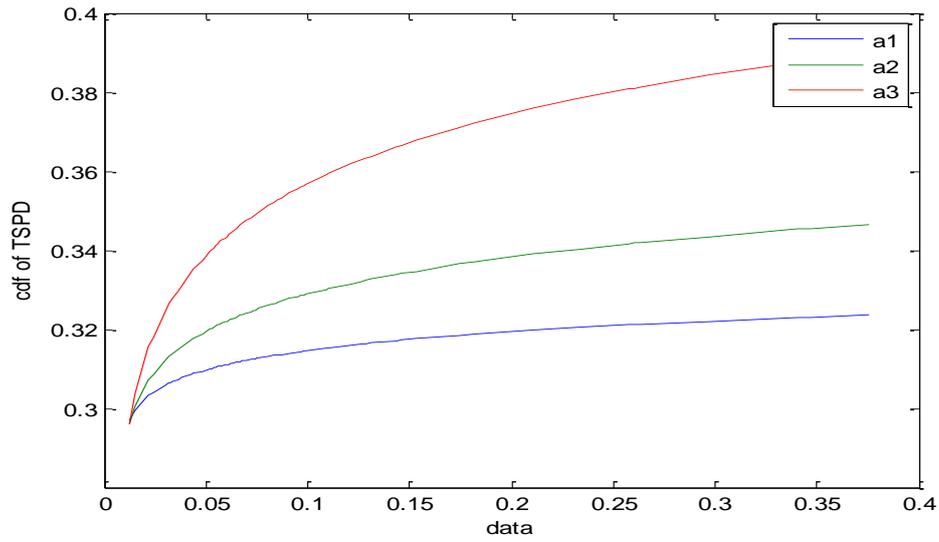
Figure (3-3):the Cdf of TSPD

in this figure the function is starts from minimum value and increases, at various value of a .
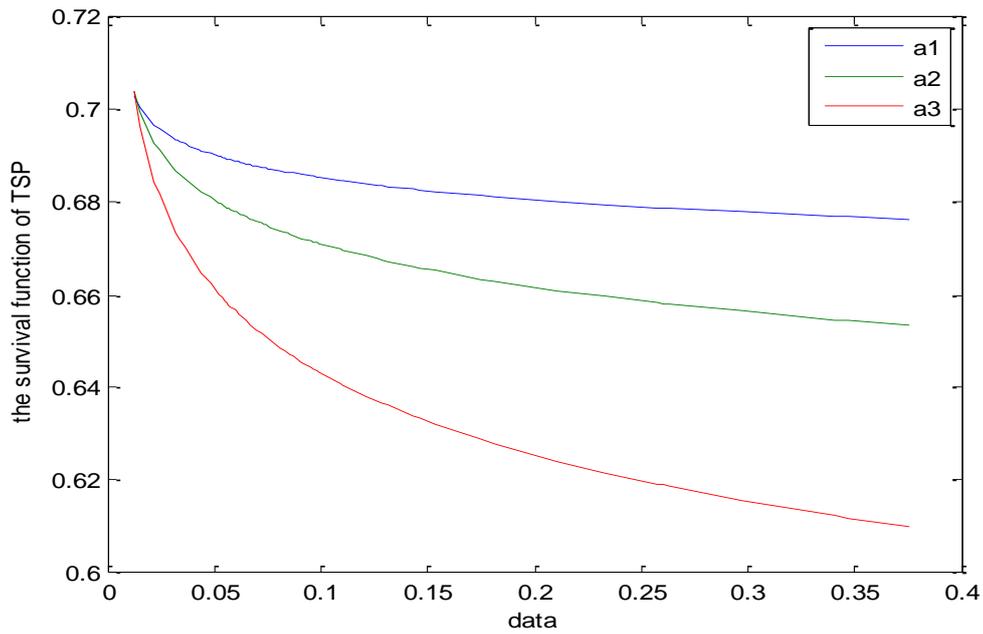
Figure (3-4): The survival function of TSPD

In this figure the function starts in maximum value and decreases at various value of a

3.3 Shape of TSPD

3.3.1 The Limit of Pdf of TSPD

$$\lim_{t \to t_\circ} f_{TSP}(t, a, t_\circ; \lambda) = \lim_{t \to t_\circ}\left[\frac{2a(1 + \lambda)t_\circ^{2a}}{t^{2a+1}} - \frac{a\lambda t_\circ^a}{t^{a+1}}\right]$$

$$2a(1 + \lambda)t_\circ^{2a} \lim_{t \to t_\circ}\frac{1}{t^{2a+1}} - a\lambda t_\circ^a \lim_{t \to t_\circ}\frac{1}{t^{a+1}}$$

$$\lim_{t \to t_\circ} f_{TSP}(t, a, t_\circ; \lambda) = \frac{2a}{t_\circ}$$

Now,

$$\lim_{t \to \infty} f_{TSP}(t, a, t_\circ; \lambda) = \lim_{t \to \infty}\left[\frac{2a(1 + \lambda)t_\circ^{2a}}{t^{2a+1}} - \frac{a\lambda t_\circ^a}{t^{a+1}}\right]$$

$$2a(1 + \lambda)t_\circ^{2a} \lim_{t \to \infty}\frac{1}{t^{2a+1}} - a\lambda t_\circ^a \lim_{t \to \infty}\frac{1}{t^{a+1}}$$

$$\lim_{t \to \infty} f_{TSP}(t, a, t_\circ; \lambda) = 0$$

3.3.2 The Limit of CDF of TSPD

$$\lim_{t \to t_\circ} F_{TSP}(t, a, t_\circ; \lambda) = \lim_{t \to t_\circ}\left[1 - (1 + \lambda)(\frac{t_\circ}{t})^{2a} + \lambda(\frac{t_\circ}{t})^a\right]$$

$$\lim_{t \to t_\circ}\left[1 - (1 + \lambda)(\frac{t_\circ}{t})^{2a}\right] + \lim_{t \to t_\circ}\left[\lambda(\frac{t_\circ}{t})^a\right]$$

$$\lim_{t \to t_\circ} F_{TSP}(t, a, t_\circ; \lambda) = 0$$

Now, $\lim_{t \to \infty} F_{TSP}(t, a, t_\circ; \lambda) = \lim_{t \to \infty}\left[1 - (1 + \lambda)(\frac{t_\circ}{t})^{2a} + \lambda(\frac{t_\circ}{t})^a\right]$

$$= \lim_{t \to \infty}\left[1 - (1 + \lambda)(\frac{t_\circ}{t})^{2a}\right] + \lim_{t \to \infty}\left[\lambda(\frac{t_\circ}{t})^a\right]$$

$$\lim_{t \to \infty} F_{TSP}(t, a, t_\circ; \lambda) = 1$$

To have more clarify about distribution's curve ,we need to find first and second derivative of $f_{TSP}(t)$ in order to decide the stationary point, so

$$f_{TSP}(\text{t}) = \frac{2a(1+\lambda)t_{\circ}^{2a}}{t^{2a+1}} - \frac{a\lambda t_{\circ}^{a}}{t^{a+1}}$$

$$\frac{\partial f_{TSP}(t)}{\partial t} = \frac{-2a(1+\lambda)t_{\circ}^{2a}(2a+1)}{t^{2a+2}} + \frac{a\lambda t_{\circ}^{a}(a+1)}{t^{a+2}} \qquad (3.6)$$

$$\frac{\partial^2 f_{TSP}(t)}{\partial t^2} = \frac{2a(1+\lambda)t_{\circ}^{2a}(2a+1)(2a+2)}{t^{2a+3}} + \frac{-a\lambda t_{\circ}^{a}(a+1)(a+2)}{t^{a+3}}$$

$$(3.7)$$

There is more than one root of this equation which can be founded numerically.

3.4 Hazard Rate Function of TSPD

The hazard rate function also known as the "Force of mortality" is defined by(1.4)

$$h_{TSP}(t) = \frac{\dfrac{2a(1+\lambda)t_{\circ}^{2a}}{t^{2a+1}} - \dfrac{a\lambda t_{\circ}^{a}}{t^{a+1}}}{(\frac{t_{\circ}}{t})^{a}[(1+\lambda)(\frac{t_{\circ}}{t})^{a} - \lambda]} \qquad (3.8)$$

Now ,we can find the limit;

$$\lim_{t\to t_{\circ}} h_{TSP}(t, a, t_{\circ}; \lambda) = \lim_{t\to t_{\circ}} \frac{\dfrac{2a(1+\lambda)t_{\circ}^{2a}}{t^{2a+1}} - \dfrac{a\lambda t_{\circ}^{a}}{t^{a+1}}}{(\frac{t_{\circ}}{t})^{a}[(1+\lambda)(\frac{t_{\circ}}{t})^{a} - \lambda]}$$

$$\lim_{t\to t_{\circ}} h_{TSP}(t, a, t_{\circ}; \lambda) = \infty$$

Now,

$$\lim_{t\to\infty} h_{TSP}(t, a, t_{\circ}; \lambda) = \lim_{t\to\infty} \frac{\dfrac{2a(1+\lambda)t_{\circ}^{2a}}{t^{2a+1}} - \dfrac{a\lambda t_{\circ}^{a}}{t^{a+1}}}{(\frac{t_{\circ}}{t})^{a}[(1+\lambda)(\frac{t_{\circ}}{t})^{a} - \lambda]}$$

$$\lim_{t\to\infty} h_{TSP}(t, a, t_{\circ}; \lambda) = 0$$

Figure (3-5): The hazard function of TSPD

3.5 Statistical Properties of The Pareto Distribution TSPD

3.5.1 Mode of TSPD

Consider the pdf of TSPD(3.5) and the first derivative (3.6) with

$$\frac{\partial f_{TSP}(t)}{\partial t} = 0 \ , hence$$

$$\frac{-2a(1+\lambda)t_{\circ}^{2a}(2a+1)}{t^{2a+2}} + \frac{a\lambda t_{\circ}^{a}(a+1)}{t^{a+2}} = 0$$

also

$$\frac{a\lambda t_{\circ}^{a}(a+1)}{t^{a+2}} = \frac{2a(1+\lambda)t_{\circ}^{2a}(2a+1)}{t^{2a+2}}$$

$$t^{2a+2}[a\lambda t_{\circ}^{a}(a+1)] = t^{a+2}[2a(1+\lambda)t_{\circ}^{2a}(2a+1)]$$
$$\frac{t^{2a+2}}{t^{a+2}}[a\lambda t_{\circ}^{a}(a+1)] = 2a(1+\lambda)t_{\circ}^{2a}(2a+1)$$

$$\frac{t^{2a+2}}{t^{a+2}} = \frac{2a(1+\lambda)t_{\circ}^{2a}(2a+1)}{a\lambda t_{\circ}^{a}(a+1)}$$

$$t^a = \frac{2(1+\lambda)(2a+1)t_o^a}{\lambda(a+1)} \tag{3.9}$$

3.5.2 The Moments of TSPD

Theorem1:

1-the rth moment $E(T^r)$ of TSPD about the origin is

$$E(T^r) = \frac{a}{r-a}\lambda t_o^r - \frac{2a}{r-2a}(1+\lambda)t_o^r$$

therefore the mean of TSPD about the means is

$$E(T) = \frac{a}{1-a}\lambda t_o - \frac{2a}{1-2a}(1+\lambda)t_o$$

2-the rth moment μ_r of TSPD about the origin

$$E(T-\mu)^r = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}(\mu_1{}')^{r-i}\mu_i{}'$$

$$E(T-\mu)^r = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}[\frac{a}{1-a}\lambda t_o - \frac{2a}{1-2a}(1+\lambda)t_o]^{r-i}\left[\frac{a}{i-a}\lambda t_o^i - \frac{2a}{i-2a}(1+\lambda)t_o^i\right]$$

Proof 1:

$$E(T^r) = \int_{t_o}^{\infty} t^r f_{TSP}(t)dt \quad \text{where (} t_o \text{ is positive)}$$

$$= \int_{t_o}^{\infty} t^r \left[\frac{2a(1+\lambda)t_o^{2a}}{t^{2a+1}} - \frac{a\lambda t_o^a}{t^{a+1}}\right] dt$$

$$= \int_{t_o}^{\infty} \left[\frac{2a(1+\lambda)t_o^{2a}t^r}{t^{2a+1}} - \frac{a\lambda t_o^a t^r}{t^{a+1}}\right] dt$$

$$= 2a(1+\lambda)t_o^{2a} \int_{t_o}^{\infty} t^{r-2a-1}\,dt - a\lambda t_o^a \int_{t_o}^{\infty} t^{r-a-1}\,dt$$

$$2a(1+\lambda)t_o^{2a}\left[\frac{t^{r-2a}}{r-2a}\right]_{t_o}^{\infty} - a\lambda t_o^a\left[\frac{t^{r-a}}{r-a}\right]_{t_o}^{\infty} \quad , r=1,2,3,\dots, a>0$$

$$= -\left[2a(1+\lambda)t_\circ^{2a}\left[\frac{t_\circ^{r-2a}}{r-2a}\right]\right] + a\lambda t_\circ^a\left[\frac{t_\circ^{r-a}}{r-a}\right]$$

$$E(T^r) = \frac{a}{r-a}\,\lambda t_\circ^r - \frac{2a}{r-2a}(1+\lambda)t_\circ^r \tag{3.10}$$

If $0 < a$, $a > r$, $r \in Z^+$

Then $a - r > 0$

Where r = 1,2,3,...

E(T) is the mean of TSPD, when r =1,

$$E(T) = \frac{a}{1-a}\,\lambda\,t_\circ - \frac{2a}{1-2a}(1+\lambda)t_\circ \tag{3.11}$$

Proof 2

Therefore since that is $E(T-\mu)^r = \sum_{i=0}^r (-1)^{r-i}\binom{r}{i}\mu_i{'}(\mu_1)^{r-i}$

$$E(T-\mu)^r = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}\left[\frac{a}{1-a}\,\lambda t\right.$$
$$\left.- \frac{2a}{1-2a}{}_\circ(1+\lambda)t_\circ\right]^{r-i}\left[\frac{a}{i-a}\,\lambda t_\circ^i - \frac{2a}{i-2a}(1+\lambda)t_\circ^i\right] \tag{3.12}$$

Where the second moment of TSPD is

$$E(T^2) = \frac{a}{2-a}\,\lambda\,t_\circ^2 - \frac{2a}{2-2a}(1+\lambda)t_\circ^2 \tag{3.13}$$

And the variance of TSPD is

$$var(T) = E(T-\mu)^2$$

$$= \sum_{i=0}^2 (-1)^{2-i}\binom{2}{i}\left[\frac{a}{1-a}\,\lambda\,t_\circ - \frac{2a}{1-2a}(1+\lambda)t_\circ\right]^{2-i}\left[\frac{a}{i-a}\,\lambda\,t_\circ^i - \right.$$
$$\left.\frac{2a}{i-2a}(1+\lambda)t_\circ^i\right]\quad,\text{i=1,2,3,...} \tag{3.14}$$

The coefficient of Skewness measures can be obtained by

$$CS(T) = \frac{E(T-\mu)^3}{P^3}$$

Where $P^3 = [var(T)]^{\frac{3}{2}}$

$$CS = \frac{\sum_{i=0}^{3}(-1)^{3-i}\binom{3}{i}\left[\frac{a}{1-a}\lambda\,t_\circ - \frac{2a}{1-2a}(1+\lambda)t_\circ\right]^{3-i}\left[\frac{a}{i-a}\lambda\,t_\circ{}^i - \frac{2a}{i-2a}(1+\lambda)t_\circ{}^i\right]}{\left[\sum_{i=0}^{2}(-1)^{2-i}\binom{2}{i}\left[\frac{a}{1-a}\lambda\,t_\circ - \frac{2a}{1-2a}(1+\lambda)t_\circ\right]^{2-i}\left[\frac{a}{i-a}\lambda\,t_\circ{}^i - \frac{2a}{i-2a}(1+\lambda)t_\circ{}^i\right]\right]^{\frac{3}{2}}} \qquad (3.15)$$

The coefficient of kurtosis(T) measures can be obtained by

$$CK(T) = \frac{E(T-\mu)^4}{P^4}$$

Where $P^4 = (var(T))^2$

$$CK =$$

$$\frac{\sum_{i=0}^{4}(-1)^{4-i}\binom{4}{i}\left[\frac{a}{1-a}\lambda\,t_\circ - \frac{2a}{1-2a}(1+\lambda)t_\circ\right]^{4-i}\left[\frac{a}{i-a}\lambda\,t_\circ{}^i - \frac{2a}{i-2a}(1+\lambda)t_\circ{}^i\right]}{\left[\sum_{i=0}^{2}(-1)^{2-i}\binom{2}{i}\left[\frac{a}{1-a}\lambda\,t_\circ - \frac{2a}{1-2a}(1+\lambda)t_\circ\right]^{2-i}\left[\frac{a}{i-a}\lambda\,t_\circ{}^i - \frac{2a}{i-2a}(1+\lambda)t_\circ{}^i\right]\right]^{2}} \qquad (3.16)$$

Theorm2:

The(mgf) Moment Generating Function of T for the TSPD is

$$\sum_{j=0}^{\infty}\frac{t^j}{j!}\left[\frac{a}{j-a}\lambda t_\circ{}^j - \frac{2a}{j-2a}(1+\lambda)t_\circ{}^j\right] \qquad (3.17)$$

Proof:

$$\mu_t(Z) = E(\exp(tz))$$

$$= \int_{t_\circ}^{\infty}\exp(tz)\,f_{TSP}(z)dz$$

Where $\exp(tz) = \displaystyle\sum_{j=0}^{\infty}\frac{(tz)^j}{j!}$

$$= \sum_{j=0}^{\infty}\frac{t^j}{j!}\,E(Z^j)$$

$$= \sum_{j=0}^{\infty}\frac{t^j}{j!}\left[\frac{a}{j-a}\lambda t_\circ{}^j - \frac{2a}{j-2a}(1+\lambda)t_\circ{}^j\right] \qquad (3.17)$$

3.5.3 Quantile Function of TSPD

Consider the cdf in (3.2) the quintile function of the TSP distribution may be expressed with in implicit from as

$$F_{TSP}(t) = 1 - (1 + \lambda)(\frac{t_\circ}{t})^{2a} + \lambda(\frac{t_\circ}{t})^a$$

$$U = 1 - (1 + \lambda)(\frac{t_\circ}{t})^{2a} + \lambda(\frac{t_\circ}{t})^a$$

$$1 - U = (1 + \lambda)(\frac{t_\circ}{t})^{2a} - \lambda(\frac{t_\circ}{t})^a$$

$$1 - U = (1 + \lambda)t_\circ^{2a}t^{-2a} - \lambda t_\circ^a t^{-a} \tag{3.18}$$

Which nonlinear equation can be solved numerically .

3.5.4 Median of TSPD

Consider (3.2), let $F_{TSP}(t) = \frac{1}{2}$

$$1 - (1 + \lambda)(\frac{t_\circ}{t})^{2a} + \lambda(\frac{t_\circ}{t})^a = \frac{1}{2}$$

$$(\frac{t_\circ}{t})^{2a} + \frac{\lambda}{(1 + \lambda)}(\frac{t_\circ}{t})^a - \frac{1}{2(1 + \lambda)} = 0$$

Let $x = (\frac{t_\circ}{t})^a$, $x^2 = (\frac{t_\circ}{t})^{2a}$

Then by last equation become

$$x^2 + \frac{\lambda}{(1 + \lambda)}x - \frac{1}{2(1 + \lambda)} = 0$$

$$let \ k = \frac{\lambda}{(1 + \lambda)} \qquad , let \ c = \frac{1}{2(1 + \lambda)}$$

The quadratic equation can be solved algebraically for x as

$$X = \frac{-k \pm \sqrt{k^2 - 4ac}}{2a}$$

Then we can find after solving the $x = (\frac{t_\circ}{t})^a$ such that $\ln x = ln(\frac{t_\circ}{t})^a$

hence

$$\ln x = \ln\left[\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 + 4(\frac{1}{2(1+\lambda)})}}{2}\right]$$

$$ln(\frac{t_\circ}{t})^a = ln\left[\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 + (\frac{2}{(1+\lambda)})}}{2}\right]$$

$$aln\frac{t_\circ}{t} = ln\left[\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 + (\frac{2}{(1+\lambda)})}}{2}\right]$$

$$a[\ln(t_\circ) - \ln(t)] = ln\left[\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 + (\frac{2}{(1+\lambda)})}}{2}\right]$$

$$aln(\,t_\circ) - aln(t) = ln\left[\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 + (\frac{2}{(1+\lambda)})}}{2}\right]$$

$$\ln(t) = \ln(t_\circ) - \frac{1}{a}ln\left[\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 + (\frac{2}{(1+\lambda)})}}{2}\right]$$

$$t = t_\circ - \frac{1}{a}\left[\frac{\frac{-\lambda}{(1+\lambda)} \pm \sqrt{(\frac{\lambda}{(1+\lambda)})^2 + (\frac{2}{(1+\lambda)})}}{2}\right] \qquad (3.19)$$

3.5.5 Order Statistics of TSPD

It is known that if $T_{(1)} \leq \ldots \leq T_{(n)}$ denotes the order statistics of the random sample from a continuous population with cdf and pdf , then the pdf of distribution is given by:[12]

$$f_{Tj}(t) = \frac{n!}{(j-1)!\,(n-j)!}\, f_t(t)(F_t(t))^{j-1}(1 - F_t(t))^{n-j}$$

for j=1,…a Pdf of the jth order statistic for a TSPD may be found by using the formula:

$$f_{Tj}(t) = \frac{n!}{(j-1)!\,(n-j)!}\, f_{TSP}(t)(F_{TSP}(t))^{j-1}(1 - F_{TSP}(t))^{n-j}$$

$$= \frac{n!}{(j-1)!\,(n-j)!}\left[\frac{2a(1+\lambda)t_\circ^{2a}}{t^{2a+1}} - \frac{a\lambda t_\circ^{a}}{t^{a+1}}\right]\left[1 - (1+\lambda)(\frac{t_\circ}{t})^{2a}\right.$$

$$\left. + \lambda(\frac{t_\circ}{t})^{a}\right]^{j-1}\left[1 - [1 - (1+\lambda)(\frac{t_\circ}{t})^{2a} + \lambda(\frac{t_\circ}{t})^{a}]\right]^{n-j}$$

As a result, the maximum order statistic's Pdf as a result, the maximum order statistic's pdf is t_n is

$$f_{Tn}(t) = \left[\frac{2a(1+\lambda)t_\circ^{2a}}{t^{2a+1}} - \frac{a\lambda t_\circ^{a}}{t^{a+1}}\right]\left[1 - (1+\lambda)(\frac{t_\circ}{t})^{2a} + \lambda(\frac{t_\circ}{t})^{a}\right]^{n-1}$$

$$(3.20)$$

also the pdf of smallest order statistic t_1 is

$$f_{T1}(t) = \left[\frac{2a(1+\lambda)t_\circ^{2a}}{t^{2a+1}} - \frac{a\lambda t_\circ^{a}}{t^{a+1}}\right]\left[(1+\lambda)(\frac{t_\circ}{t})^{2a} + \lambda(\frac{t_\circ}{t})^{a}\right]^{n-1}$$

$$(3.21)$$

3.6 Estimation Method of TSPD

In this sub section, the two parameters λ also a of the TSPD using the MLE method are estimated. [7]

3.6.1 Maximum Likelihood Estimation of TSPD

Let L (λ , a) denotes likelihood function of the TSPD , then the maximum likelihood estimator $\hat{\lambda}$ also $\hat{a}$

can be formulation as follows:

$$L(t_1, t_2, \ldots, t_n, t_\circ; \lambda) = \prod_{i=1}^{n} f_{TSP}(t_i, a, t_\circ; \lambda)$$

$$= \prod_{i=1}^{n} \left[\frac{2a(1+\lambda)t_\circ^{2a}}{t_i^{2a+1}} - \frac{a\lambda t_\circ^{a}}{t_i^{a+1}} \right]$$

$$= \prod_{i=1}^{n} \left[\frac{2a(1+\lambda)t_\circ^{2a}t_i^{a+1}}{t_i^{2a+1}} - \frac{a\lambda t_\circ^{a}t_i^{2a+1}}{t_i^{a+1}} \right] \tag{3.22}$$

$$= \frac{\prod_{i=1}^{n}[2a(1+\lambda)t_\circ^{2a}t_i^{a+1} - a\lambda t_\circ^{a}t_i^{2a+1}]}{\prod_{i=1}^{n}(t_i^{2a+1}) \prod_{i=1}^{n}(t_i^{a+1})}$$

Taking the natural logarithm

$$\ln L = \sum_{i=1}^{n} \ln[2a(1+\lambda)t_\circ^{2a}t_i^{a+1} - a\lambda t_\circ^{a}t_i^{2a+1}] - [(2a+1)\sum_{i=1}^{n}\ln(t_i) + (a+1)\sum_{i=1}^{n}\ln(t_i)] \tag{3.23}$$

By taking partial derivative for equation with respect to (a) and equality to zero.

$$\frac{\partial lnL}{\partial a} = \frac{[2(1+\lambda)t_\circ^{2a}t_i^{a+1} - \lambda t_\circ^{a}t_i^{2a+1}]}{2a(1+\lambda)t_\circ^{2a}t_i^{a+1} - a\lambda t_\circ^{a}t_i^{2a+1}}$$

$$+ \frac{[2a(1+\lambda)(2\ln(t_\circ)\,t_\circ^{2a}(t_i^{a+1}) - \lambda t_\circ^{a}\ln(t_\circ)\,t_i^{2a+1}]}{2a(1+\lambda)t_\circ^{2a}t_i^{a+1} - a\lambda t_\circ^{a}t_i^{2a+1}}$$

$$+ \frac{[2a(1+\lambda)t_\circ^{2a}(\ln(t_i)\,t_i^{a+1}) - 2a\lambda t_\circ^{a}t_i^{2a+1}\ln(t_i)]}{2a(1+\lambda)t_\circ^{2a}t_i^{a+1} - a\lambda t_\circ^{a}t_i^{2a+1}}$$

$$- 3\sum_{i=1}^{n}\ln(t_i) \tag{3.24}$$

Now ,

$$\frac{\partial \ln L}{\partial \lambda} = \frac{2at_{\circ}{}^{2a}t_i{}^{a+1} - at_{\circ}{}^{a}t_i{}^{2a+1}}{\sum_{i=1}^{n}[2a(1+\lambda)t_{\circ}{}^{2a}t_i{}^{a+1} - a\lambda t_{\circ}{}^{a}t_i{}^{2a+1}}$$

(3.25)

We can solve these nonlinear equations by (Newton – Raphson method) as follows

$$\binom{\hat{\lambda}}{\hat{a}} = \binom{\lambda}{a} - J^{-1}\binom{G(a)}{W(\lambda)}$$

Where

$$J = \begin{pmatrix} \dfrac{\partial G(a)}{\partial a} & \dfrac{\partial G(\lambda)}{\partial \lambda} \\ \dfrac{\partial W(a)}{\partial a} & \dfrac{\partial W(\lambda)}{\partial \lambda} \end{pmatrix}$$

It is the Jacobean matrix.

Chapter Four

Applied Part

4.1 Introduction

In this chapter, the TSRD and TSPD are applied to data set by use several information criterions,

Information criterians (AIC, BIC) are used to compare distributions with each other and with the original distributions. Through a real data set to enhance the results.

4.2 Numerical Results

This section , the two models TSRD and TSPD are tested with the following real data sets and compare the results using information criterions .

Data set: Guinea Pig Data[3]

All this data system includes survival periods through days for guinea pigs inserted in dissimilar makes for tubercle bacilli , this data are presented after breaking the ties.

0.012 0.015 0.022 0.024 0.032 0.033 0.034 0.038 0.043 0.044
0.048 0.052 0.053 0.054 0.055 0.056 0.057 0.058 0.059 0.060
0.061 0.062 0.063 0.065 0.067 0.068 0.070 0.072 0.073 0.075
0.076 0.081 0.083 0.084 0.085 0.087 0.091 0.095 0.096 0.098
0.099 0.109 0.110 0.121 0.127 0.129 0.131 0.143 0.146 0.175
0.211 0.233 0.258 0.263 0.297 0.341 0.376 0.030 0.036 0.043
0.061 0.060 0.063 0.063 0.63 0.072 0.081 0.153 0.181 0.260 0.347
0.074

Now , we will study the estimation of the densty function for the same data

Summary of Guinea pig data: Original distribution Rayleigh vs Transmuted survival Rayleigh (TSR)

Table (4- 1) The estimation of parameters of TSRD

Distribution	Estimated parameters	Log lik	AIC	BIC	MSE	KS
Rayleigh	$\hat{b}$= 0.1035	91.48	-178.96	-174.41	2.5806	0.3553
TSR	$\hat{b}$ =0.1035 **λ=-0.3524**	95.8517	-186.9867	-182.4334	0.1482	0.2205

In this work we use several information standards which is define in chapter one (AIC), (BIC) , (MSE) ,(KS) . We notice that the log lik values were lower in the Pareto distribution and in the TSRD the value became larger, for Akaike criterion, the original distribution was large and became less in the TSRD , in BIC the percentage was greater in the original distribution and became less in the TSRD and the error rate also became less in the TSRD.

Where the KS test was used for the goodness of matching with the real data ,and show that the data is distributed TSRD.
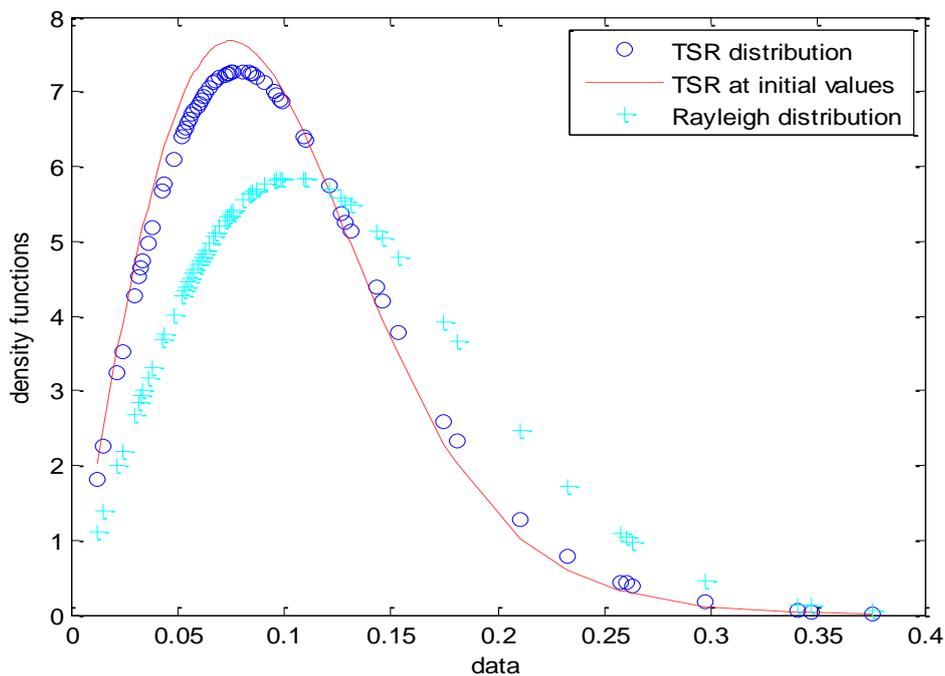
Figure(4-1) the pdf of Rayleigh distribution and TSRD after estimation of parameters

Figure (4-1) show us the of the curve of the Rayleigh distribution and transmuted survival Rayleigh distribution (TSRD), there were drawn using the estimated parameters. The shapes of these functions correspond to the numerical results in the previous table, so that the

curves of the estimated functions calculated by MLE are close to the curve of the initial parameters.

This figure and by information criterians shows the optimization of the new distribution compared to the original distribution.

In the post the estimation of the survival function is examined for transmuted survival of Rayleigh distribution compared to time.

Table (4-2)the estimation of the survival function of TSRD

t	S(t)	t	S(t)	t	S(t)	t	S(t)	t	S(t)
0.012	0.988	0.053	0.807	0.063	0.740	0.085	0.581	0.153	0.191
0.015	0.982	0.054	0.800	0.065	0.725	0.087	0.567	0.175	0.121
0.022	0.963	0.055	0.794	0.067	0.711	0.091	0.538	0.181	0.106
0.024	0.956	0.056	0.787	0.068	0.704	0.095	0.510	0.211	0.054
0.03	0.933	0.057	0.781	0.07	0.690	0.096	0.503	0.233	0.032
0.032	0.924	0.058	0.774	0.072	0.675	0.098	0.489	0.258	0.017
0.033	0.919	0.059	0.767	0.072	0.675	0.099	0.482	0.26	0.016
0.034	0.915	0.06	0.760	0.073	0.668	0.109	0.416	0.263	0.015
0.036	0.905	0.06	0.760	0.074	0.661	0.11	0.409	0.297	0.005
0.038	0.895	0.061	0.753	0.075	0.654	0.121	0.343	0.341	0.0015
0.043	0.868	0.061	0.753	0.076	0.646	0.127	0.309	0.347	0.0012
0.043	0.868	0.062	0.746	0.081	0.610	0.129	0.299	0.376	0.0004
0.044	0.862	0.063	0.740	0.081	0.610	0.131	0.288	——	——
0.048	0.838	0.063	0.740	0.083	0.596	0.143	0.231	——	——
0.052	0.813	0.063	0.740	0.084	0.588	0.146	0.218	——	——

In the table (4-2), it is noted that the values of the (t) column represent the time to failure (time) , while the values of the other column s(t) represent the estimated survival function values , which are an arithmetic mean for the values of the survival function and are decreasing . there is an inverse proportion between the survival function and time.
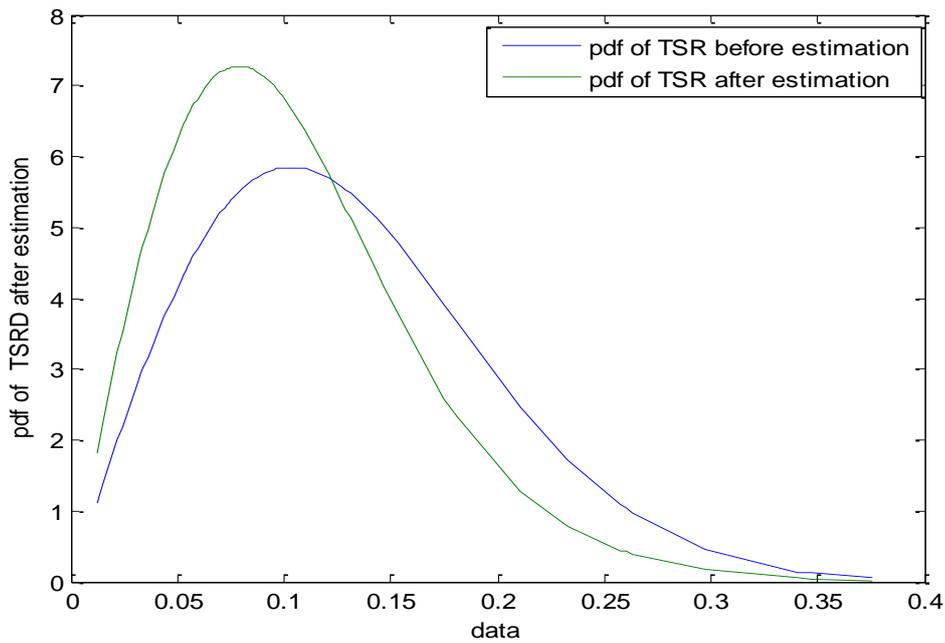
Figure (4- 2) The pdf of TSRD after and before estimation of parameters

In this figure, the pdf function increases reaches it's highest value at 0.1 , and then decreases to be almost vanished with zero.
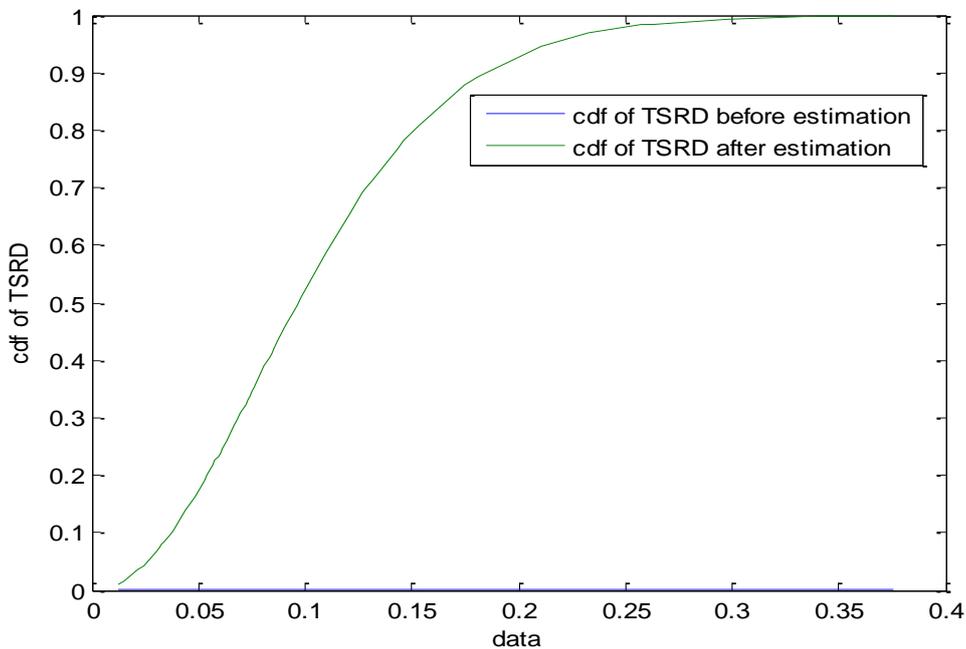

Figure (4-3) The CDF of TSRD after and before estimation of parameters.

In this figure ,The function is insert tamically increasing to be starting after that with increasing T.
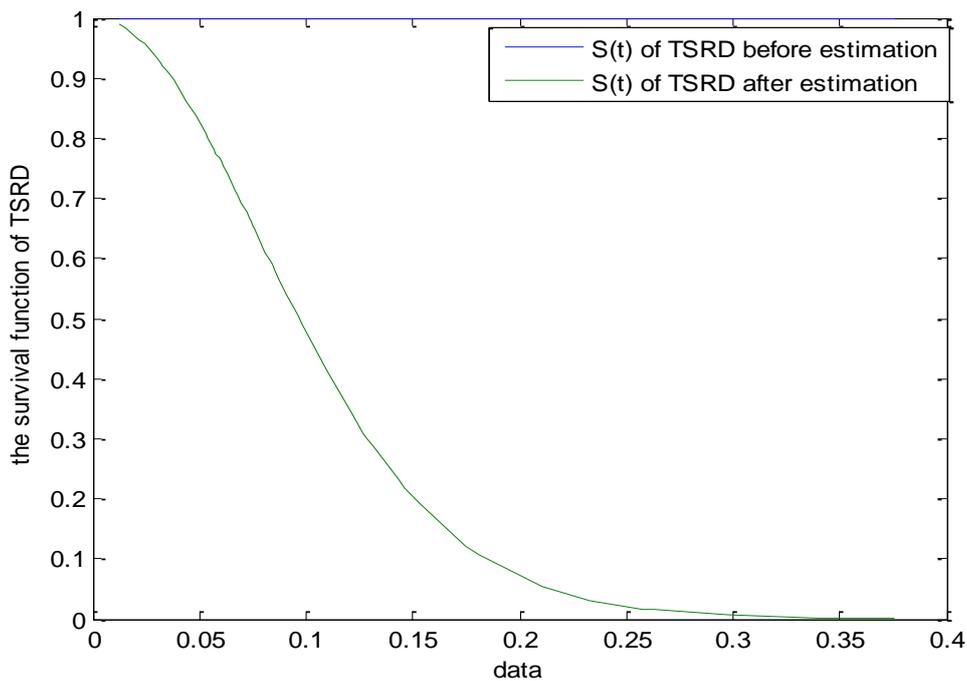
Figure (4-4) The survival function of (TSRD) after and before estimation

In Figure (4-4)   the function the curve of survival function decreases until it vanish to zero

Summary of Guinea pig data: Original distribution Pareto vs Transmuted survival Pareto (TSPD)

Table (4-3)the estimation of the parameters of TSPD

Distribution	Estimated parametrs	Log lik	AIC	BIC	MSE	KS
Pareto	a 0.3713 $t_{\circ}$0.0120	62.0223	-120.0445	-115.4912	0.2713	0.278
TSP	a 0.3713 λ **-0.5393**	62.5311	-121.0622	-116.5089	0.2713 **0.2393**	0.5194

In this work we use several information standards(AIC),(BIC), (KS),(MSE) are used they defined in chapter one.

We notice that the log lik values were lower in the Pareto distribution and in the TSPD the value became larger, for Akaike criterion, the original distribution was large and became less in the TSPD , in BIC the percentage was greater in the original distribution and became less in the TSPD and the error rate also became less in the TSPD.
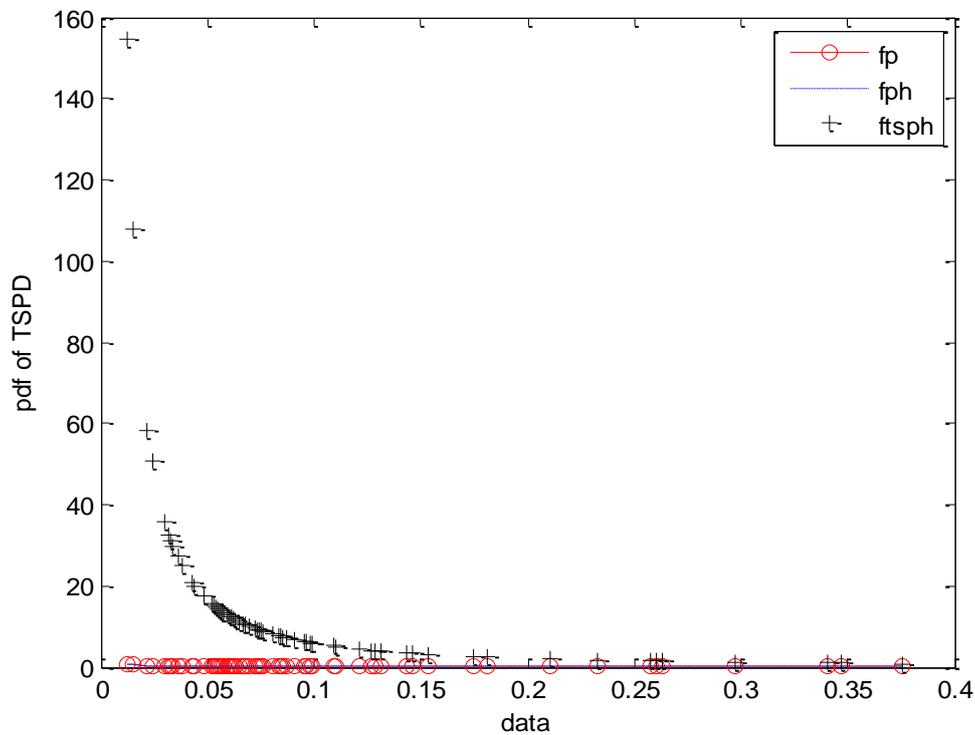
Figure(4-5)The pdf of pareto distribution and TSPD after estimation of parameters

In Figure (4-5) , it is noted that the curve of the TSPD with estimated parameters is better fitted than the Pareto distribution and the TSPD with initial values of parameters. there were drawn using the estimated parameters. The shapes of these functions correspond to the numerical results in the previous table, so that the curves of the estimated functions calculated by MLE are close to the curve of the initial parameters.

In the post the estimation of the survival function is estimated for transmuted survival Pareto distribution compared to time us in the following table:

Table (4-4)the estimation of the survival function of TSPD

t	S(t)	t	S(t)	t	S(t)	t	S(t)	t	S(t)
0.012	0.710	0.053	0.319	0.063	0.297	0.085	0.264	0.153	0.210
0.015	0.605	0.054	0.317	0.065	0.293	0.087	0.261	0.175	0.200
0.022	0.481	0.055	0.314	0.067	0.290	0.091	0.257	0.181	0.197
0.024	0.459	0.056	0.312	0.068	0.288	0.095	0.252	0.211	0.186
0.030	0.411	0.057	0.310	0.07	0.285	0.096	0.251	0.233	0.179
0.032	0.398	0.058	0.307	0.072	0.282	0.098	0.249	0.258	0.173
0.033	0.393	0.059	0.305	0.072	0.282	0.099	0.248	0.26	0.172
0.034	0.387	0.06	0.303	0.073	0.280	0.109	0.239	0.263	0.171
0.036	0.377	0.06	0.303	0.074	0.278	0.11	0.238	0.297	0.164
0.038	0.368	0.061	0.301	0.075	0.277	0.121	0.230	0.341	0.155
0.043	0.349	0.061	0.301	0.076	0.276	0.127	0.226	0.347	0.154
0.043	0.349	0.062	0.299	0.081	0.269	0.129	0.224	0.376	0.150
0.044	0.345	0.063	0.297	0.081	0.269	0.131	0.223	——	——
0.048	0.333	0.063	0.297	0.083	0.266	0.143	0.216	——	——
0.052	0.322	0.063	0.297	0.084	0.265	0.146	0.214	——	——

In the table (4-4) we note that the values of the (t) column represent the time to failure time , while the values of the other column S(t) represent the estimated survival function values , which are an arithmetic mean for the values of the survival function and are decreasing . there is an inverse proportion between the survival function and time .
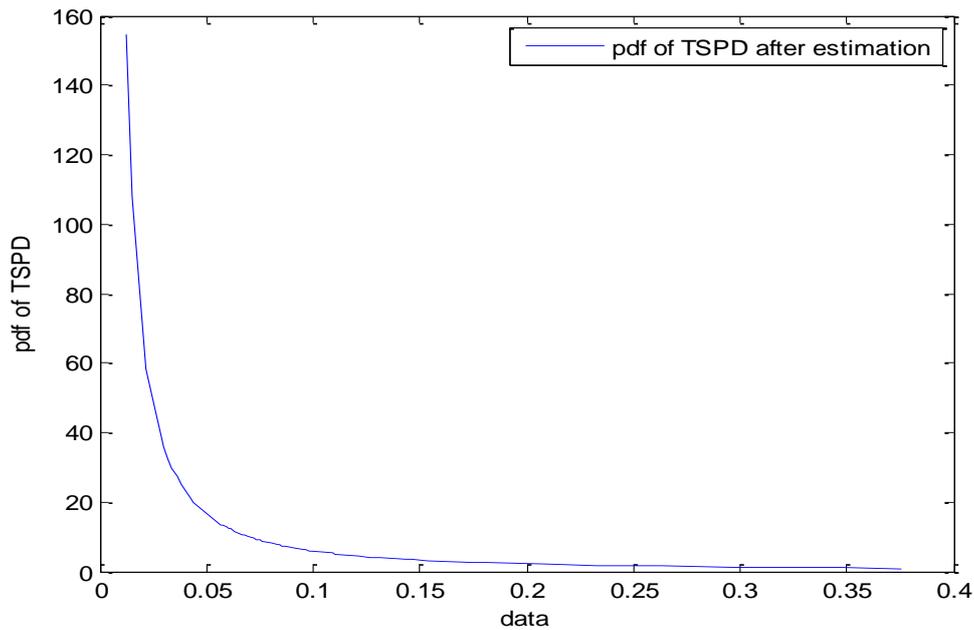
Figure (4-6):The Pdf of TSPD after estimation of parameters

In Figure(4-6) it is shown that the density function (pdf) of the transmuted survival Pareto distribution (TSPD) as its shape after the

Estimation based on the real data.the pdf is closer in shape to the normal distribution.
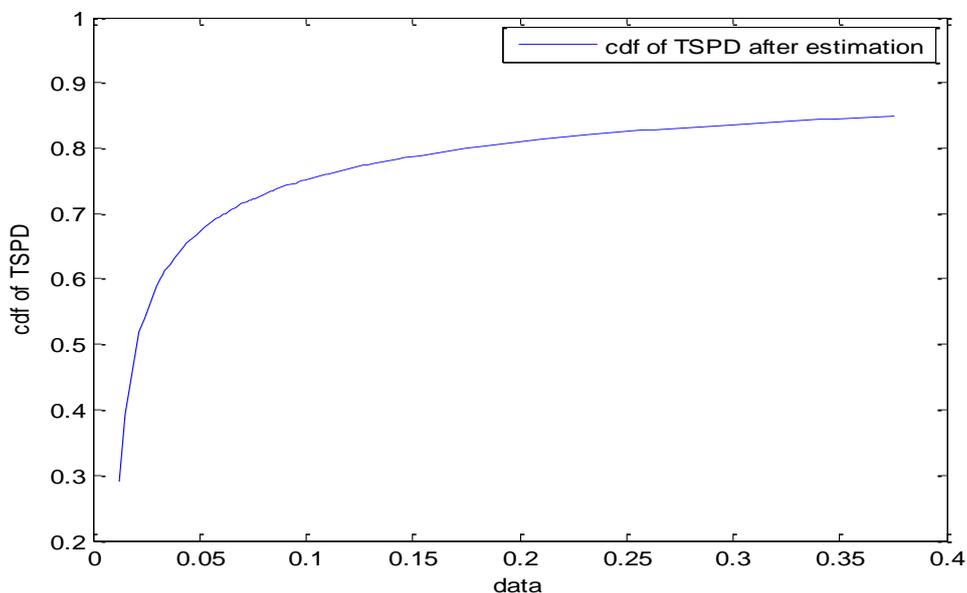

Figure (4-7) The CDF of TSPD after estimation of parameters

This Figure(4-7) shows that the CDF function of (TSPD) after estimation as its shape after the estimated based data.the CDF is closer in shape to the normal distribution.
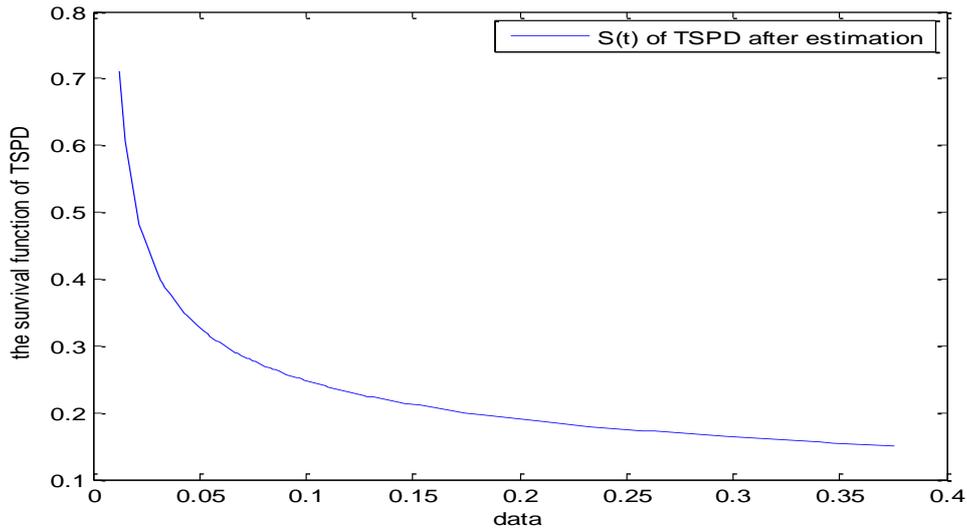
Figure (4-8) the survival function of TSPD after estimation of parameters

This Figure of the survival function is as its shape after the estimation based on the real data.the S(t) is closer in shape to the normal distribution.

Summary of Guinea pig data:

Original distribution Rayleigh vs Transmuted survival Rayleigh (TSRD) and Original distribution Pareto vs Transmuted survival Pareto (TSPD):

Table (4-5)the estimation of parameters between the original distribution of Rayleigh and Pareto distribution, and between TSRD and TSPD

Distribution	Estimated parameters	Log lik	AIC	BIC	Mse	KS
Rayleigh	$\hat{b}$= 0.1035	91.48	-178.96	-174.41	2.5806	0.3553
TSR	$\hat{b}$=0.1035 **λ=-0.3524**	95.8517	-186.9867	-182.4334	0.1482	0.2205
Pareto	a 0.3713 **$t_\circ$ 0.0120**	62.0223	-120.0445	-115.4912	0.2713	0.2783
TSP	a 0.3713 **λ -0.5393**	62.5311	-121.0622	-116.5089	0.2393	0.5194

In a Table (4-5) we compare the new distributions that we generated using the survival function after estimation , ,we note by comparing with their original distributions, that the transmuted survival of Rayleigh distribution is the best because it is more flexible and accurate in applying the data Rayleigh distribution as well as better than the original Pareto distribution and the Pareto- transmuted survival distribution.

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4.3 Conclusions

• In this thesis , the transmuted survival of Rayleigh distribution (TSRD) was derived which is an innovative distribution for the original Rayleigh distribution and transmuted Rayleigh distribution, this model is considered the best distribution in comparison with the original distribution because it is more flexible and accurate in dealing with data and get better results, we found the mathematical and statistical characteristics of the new distribution .

• The transmuted survival of Pareto distribution (TSPD) was derived which is an innovative distribution for the original Pareto distribution and transmuted Pareto distribution, this model is considered the best distribution in comparison with the original distribution because it is more flexible and accurate in dealing with data and get better results.

• Comparison between the two innovative distributions and the original distributions the results were the best is transmuted survival Rayleigh distribution (TSRD) lifetime on the transmuted survival Pareto distribution (TSPD), and their original distributions thus , we obtained a special distribution and has great attributes in dealing with data, by estimating parameters of these distributions .

Future Work

There are many important notes which we take into account :

1-Finding the fourth transmuted survival distribution of our two distribution or other.

2-Study the transmuted survival of other distributions.
3- use other estimation methods to estimate parameters for distributions.

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Algorithms

Algorithms

Appendix

The program plote the pdf of transmuted survival Rayleigh distribution (λ=0.2)

```
syms t b1 b2 b3 b4 K ;

t=0:0.01:6;b1=2.5;b2=3;b3=3.8;b4=4;k=0.2 ;

y1=((2*t./b1.^2)*(1+k).*exp(-(t.^2/b1.^2))-(t./b1.^2)*k.*exp(-
t.^2/2*b1.^2))));

y2=((2*t./b2.^2)*(1+k).*exp(-(t.^2/b2.^2))-(t./b2.^2)*k.*exp(-
t.^2/2*b2.^2))));

y3=((2*t./b3.^2)*(1+k).*exp(-(t.^2/b3.^2))-(t./b3.^2)*k.*exp(-
t.^2/2*b3.^2))));

y4=((2*t./b4.^2)*(1+k).*exp(-(t.^2/b4.^2))-(t./b4.^2)*k.*exp(-
t.^2/2*b4.^2))));

plot(t,y1,'--','linewidth',2.5);hold all;

plot(t,y2,'-.','linewidth',2.5);hold all;

plot(t,y3,':','linewidth',2.5);hold all;

plot(t,y4,'-k','linewidth',2.5);hold all;

legend('b1=2.5','b2=3','b3=3.8','b4=4');

xlabel('x-axis');

ylabel('pdf of TSR');
```

Algorithms

The program plote the CDF of transmuted survival Rayleigh distribution (λ=0.001)

syms t b1 b2 b3 b4 k;

t=0:0.01:6;b1=4;b2=4.5;b3=5;b4=5.5;k=0.001;

y1=((1-(1+k).*exp(-(t.^2/b1.^2)))+k.*exp(-(t.^2/2*b1.^2)));

y2=((1-(1+k).*exp(-(t.^2/b2.^2)))+k.*exp(-(t.^2/2*b2.^2)));

y3=((1-(1+k).*exp(-(t.^2/b3.^2)))+k.*exp(-(t.^2/2*b3.^2)));

y4=((1-(1+k).*exp(-(t.^2/b4.^2)))+k.*exp(-(t.^2/2*b4.^2)));

plot(t,y1,'--','linewidth',1.5);hold all;

plot(t,y2,'-.','linewidth',1.5);hold all;

plot(t,y3,':','linewidth',1.5);hold all;

plot(t,y4,'-k','linewidth',1.5);hold all;

legend('b1=4','b2=4.5','b5','b4=5.5');

xlabel('x-axis');

ylabel('CDF of TSR');

Algorithms

The program plote the hazard function of transmuted survival Rayleigh distribution (λ=-0.1)

syms t b1 b2 b3 b4 k;

t=0:0.01:6;b1=1;b2=2;b3=3;b4=4;k=-0.1;

y1=((2*t./b1.^2)*(1+k).*exp(-(t.^2/b1.^2))-(t./b1.^2)*k.*exp(-t.^2/2*b1.^2))/((1+k).*exp(-(t.^2/b1.^2))-k.*exp(-(t.^2/2*b1.^2)))));

y2=((2*t./b2.^2)*(1+k).*exp(-(t.^2/b2.^2))-(t./b2.^2)*k.*exp(-t.^2/2*b2.^2))/((1+k).*exp(-(t.^2/b2.^2))-k.*exp(-(t.^2/2*b2.^2)))));

y3=((2*t./b3.^2)*(1+k).*exp(-(t.^2/b3.^2))-(t./b3.^2)*k.*exp(-t.^2/2*b3.^2))/((1+k).*exp(-(t.^2/b3.^2))-k.*exp(-(t.^2/2*b3.^2)))));

y4=((2*t./b4.^2)*(1+k).*exp(-(t.^2/b4.^2))-(t./b4.^2)*k.*exp(-t.^2/2*b4.^2))/((1+k).*exp(-(t.^2/b4.^2))-k.*exp(-(t.^2/2*b4.^2)))));

plot(t,y1,'--','linewidth',2.5);hold all;

plot(t,y2,'-.','linewidth',2.5);hold all;

plot(t,y3,':','linewidth',2.5);hold all;

plot(t,y4,'-k','linewidth',2.5);hold all;

legend('b1=1','b2=2','b3=3','b4=4');

xlabel('x-axis');

ylabel('the hazard function of TSR');

Algorithms

The program plote the hazard function of transmuted survival Rayleigh distribution (λ=1)

```
syms t b1 b2 b3 b4 k;

t=0:0.01:6;b1=0.2;b2=0.3;b3=0.4;b4=0.5;k=1;

y1=((2*t./b1.^2)*(1+k).*exp(-(t.^2/b1.^2))-(t./b1.^2)*k.*exp(-
t.^2/2*b1.^2))/((1+k).*exp(-(t.^2/b1.^2))-k.*exp(-
(t.^2/2*b1.^2)))));

y2=((2*t./b2.^2)*(1+k).*exp(-(t.^2/b2.^2))-(t./b2.^2)*k.*exp(-
t.^2/2*b2.^2))/((1+k).*exp(-(t.^2/b2.^2))-k.*exp(-
(t.^2/2*b2.^2)))));

y3=((2*t./b3.^2)*(1+k).*exp(-(t.^2/b3.^2))-(t./b3.^2)*k.*exp(-
t.^2/2*b3.^2))/((1+k).*exp(-(t.^2/b3.^2))-k.*exp(-
(t.^2/2*b3.^2)))));

y4=((2*t./b4.^2)*(1+k).*exp(-(t.^2/b4.^2))-(t./b4.^2)*k.*exp(-
t.^2/2*b4.^2))/((1+k).*exp(-(t.^2/b4.^2))-k.*exp(-
(t.^2/2*b4.^2)))));

plot(t,y1,'--','linewidth',2.5);hold all;

plot(t,y2,'-.','linewidth',2.5);hold all;

plot(t,y3,':','linewidth',2.5);hold all;

plot(t,y4,'-k','linewidth',2.5);hold all;

legend('b1=0.2','b2=0.3','b3=0.4','b4=0.5');

xlabel('x-axis');

                              '); ylabel('the hazard function of TSR
```

Algorithms

The maximum Likelihood of TSRD and estimation of parameters

The initial value

g= .1;L=-.3;

s1=0;s2=0;s3=0;s4=0;

t=[0.012 0.015 0.022 0.024 0.032 0.033 0.034 0.038 0.043
0.044 0.048 0.052 0.053 0.054 0.055 0.056 0.057 0.058
0.059 0.060 0.061 0.062 0.063 0.065 0.067 0.068 0.070
0.072 0.073 0.075 0.076 0.081 0.083 0.084 0.085 0.087
0.091 0.095 0.096 0.098 0.099 0.109 0.110 0.121 0.127
0.129 0.131 0.143 0.146 0.175 0.211 0.233 0.258 0.263
0.297 0.341 0.376 0.030 0.036 0.043 0.061 0.060 0.063
0.063 0.063 0.072 0.081 0.153 0.181 0.260 0.347 0.074];

t=sort(t)

n=length(t);

for i=1:n

   frt(i)=(((t(i)/(g^2))*exp(-t(i)^2/(2*g^2)))*(2*(1+L)*exp(-t(i)^2/(2*g^2))-L);

end

figure

plote(t,frt,'r');

for i=1:n

   )); s1=s1+log(t(i

end

for i=1:n

```
  ); s2=s2+(t(i)^2

 s3=s3+(log(2*(1+L)*exp(-t(i)^2/(2*g^2))-L));

end

LK=s1-n*2*log(g)-(s2/(2*g^2))+s3;

 )]par_mle]=fsolve(@(S) MLE_TSR(t,S),[g L[

 g=par_mle(1);L=par_mle(2);

 s21=0;s31=0;

 for i=1:n

   ); s21=s21+(t(i)^2

)); s31=s31+(log(2*(1+L)*exp(-t(i)^2/(2*g^2))-L

end

 lkh=s1-n*2*log(g)-(s21/(2*g^2))+s31

 for i=1:n

 ftsr(i)=(2*t(i)/g^2)*(1+L)*exp(-t(i)^2/(g^2))-(t(i)/g^2)*L*exp(-
t(i)^2/(2*g^2));

end

for k=1:n

  ))  fr(k)=(t(k)/g^2)*exp(-t(k)^2/(2*g^2

end

 figure

+); plot (t,ftsr,'bo',t,frt,'r--',t,fr,'c

disp('LK=');
```

```
disp(LK)

)); mse_mle=immse( ftsr,(frt

 )); mse_mle=immse( ftsr,(fr

m=2;

; AIC=-2*LK+2*m

=') disp('AIC

disp(AIC)

AICC=AIC+(2*m*(m+1)/(n-m+1));

=') disp('AICC

disp(AICC)

BIC=-2*LK+m*log(n);

disp('BIC=')

disp(BIC)

plot(t,fr,'k',t,ftsr,'r',t,frt,'bo')

for i=1:n

   Ftr(i)=1-(1+L)*exp((-t(i)^2)/(g^2))+L*exp((-t(i)^2)/(2*g^2));

end

for j=1:n

   Str(j)=(1+L)*exp((-t(j)^2)/(g^2))-L*exp((-t(j)^2)/(2*g^2));

end

for i=1:n

  )); Fr(i)=1-exp((-t(i)^2)/(2*g^2
```

end

ff=(1:n)/n;

ks_tsr=max(abs((ff-Ftr)));

ks_r=max(abs((ff-Fr)));

The program of maximum likelihood of the estimation of parameters of TSRD

```
function [FF]=MLE_TSR(t,S)

n=length(t);

g=S(1);

L=S(2);

s1=0;s2=0;s3=0;

for i=1:n

   s1=s1+t(i)^2;

   numg(i)=2*(1+L)*(t(i)^2/g^3)*exp((-t(i)^2)/(2*g^2));

   ; dem(i)=(2*(1+L))*exp((-t(i)^2)/(2*g^2))-L

   numL(i)=2*exp((-t(i)^2)/(2*g^2))-1;

   )); s2=s2+(numg(i)/dem(i

   )); s3=s3+(numL(i)/dem(i

end

dg=(s1/g^3)-((2*n)/g)+s2;

dL=s3;

FF=[dg dL ];
```

The program of plote of TSPD and the estimation of parameters

tt=[0.012 0.015 0.022 0.024 0.032 0.033 0.034 0.038 0.043
0.044 0.048 0.052 0.053 0.054 0.055 0.056 0.057 0.058 0.059
0.060 0.061 0.062 0.063 0.065 0.067 0.068 0.070 0.072 0.073
0.075 0.076 0.081 0.083 0.084 0.085 0.087 0.091 0.095 0.096
0.098 0.099 0.109 0.110 0.121 0.127 0.129 0.131 0.143 0.146
0.175 0.211 0.233 0.258 0.263 0.297 0.341 0.376 0.030 0.036
0.043 0.061 0.060 0.063 0.063 0.063 0.072 0.081 0.153 0.181
0.260 0.347 0.074];

t=sort(tt);

to=min(t);n=length(t);

a1=0.01;a2=0.02;a3=0.04;L1=-0.7;L2=-0.8;L3=-0.9;

for i=1:n

 )); fp1(i)=(a1*to^a1)/(t(i)^(a1+1

  )); fp2(i)=(a2*to^a2)/(t(i)^(a2+1

  )); fp3(i)=(a3*to^a3)/(t(i)^(a3+1

   Fp1(i)=1-(to/t(i))^a1;

   ; Fp2(i)=1-(to/t(i))^a2

  ; Fp3(i)=1-(to/t(i))^a3

 ; sp1(i)=(to/t(i))^a1

  ; sp2(i)=(to/t(i))^a2

```
  ; sp3(i)=(to/t(i))^a3

end

for i=1:n

   ftsp1(i)=((2*a1*(1+L1)*to^(2*a1))/(t(i)^2*(a1+1)))-
2*L1*(to^a1))/t(i)^(a1+1))));

ftsp2(i)=((2*a2*(1+L2)*to^(2*a2))/(t(i)^2*(a2+1)))-
2*L2*(to^a2))/t(i)^(a2+1))));

ftsp3(i)=((2*a3*(1+L3)*to^(2*a3))/(t(i)^2*(a3+1)))-
2*L3*(to^a3))/t(i)^(a3+1))));

Ftsp1(i)=1-(1+L1)*((to/t(i))^2*a1)+L1*(to/t(i))^a1;

Ftsp2(i)=1-(1+L2)*((to/t(i))^2*a2)+L1*(to/t(i))^a2;

Ftsp3(i)=1-(1+L3)*((to/t(i))^2*a3)+L1*(to/t(i))^a3;

stsp1(i)=(1+L1)*((to/t(i))^2*a1)-L1*(to/t(i))^a1;

stsp2(i)=(1+L2)*((to/t(i))^2*a2)-L1*(to/t(i))^a2;

stsp3(i)=(1+L3)*((to/t(i))^2*a3)-L1*(to/t(i))^a3;

htsp1(i)=ftsp1(i)/stsp1(i);

htsp2(i)=ftsp2(i)/stsp2(i);

htsp3(i)=ftsp3(i)/stsp3(i);

end

figure

plot(t,ftsp1,t,ftsp2,t,ftsp3)

title(' the pdf of TSP distibution')

figure
```

plot(t,Ftsp1,t,Ftsp2,t,Ftsp3)

title(' the cdf of TSP distibution')

figure

plot(t,stsp1,t,stsp2,t,stsp3)

title(' the survival function of TSP distibution')

figure

plot(t,htsp1,t,htsp2,t,htsp3)

title(' the hazared function of TSP distibution')

a=0.5;L=-0.7;

s=0;s1=0;s2=0;s3=0;s4=0;

for i=1:n

  )); s=s+log(t(i

end

s1=(3*a+2)*s;

for i=1:n

  s2=s2+log(2*a*(1+L)*(to^(2*a))*((t(i))^(a+1))-a*L*(to^a)*((t(i))^(2*a+1))));

end

logL=s1-s2;

])par_mle]=fsolve(@(S) MLE_TSP(t,S),[a L[

 ah=par_mle(1);Lh=par_mle(2);

] parini=[a L

Algorithms

```
] parhut=[ah Lh

for i=1:n

    ftsph(i)=((2*ah*(1+Lh)*to^(2*ah))/(t(i)^2*(ah+1)))-
2*Lh*(to^ah))/t(i)^(ah+1))));

   ; Ftsph(i)=1-(1+Lh)*((to/t(i))^2*ah)+Lh*(to/t(i))^ah

   ; stsph(i)=(1+Lh)*((to/t(i))^2*ah)-Lh*(to/t(i))^ah

 end

 figure

) plot(t,ftsph

 figure

) plot(t,Ftsph

 figure

) plot(t,stsph

 for i=1:n

 )); fph(i)=(a1*to^ah)/(t(i)^(ah+1

   ; Fph(i)=1-(to/t(i))^ah

     ; sph(i)=(to/t(i))^ah

 end

 figure

 ') plot(t,fp1,'-ro',t,fph,'-.b',t,ftsph,'+k

 ') title(' the pdf of tow distributions befor and after estimations

 ') legend('fp','fph','ftsph
```

```
; sump=0

 for j=1:n

   )); sump=sump+log(t(j

 end

;  LKP=n*log(ah)+n*ah*log(to)-(ah+1)*sump; mp=2

;  AICP=-2*LKP+2*mp

=') disp('AICP

disp(AICP)

AICCP=AICP+(2*mp*(mp+1)/(n-mp+1));

=') disp('AICCP

disp(AICCP)

BICP=-2*LKP+mp*log(n);

disp('BICP =')

disp(BICP)

for j=1:n

   dem4(j)=(2*ah*(1+Lh)*(to^(2*ah))*(t(j)^(ah+1)))-
ah*Lh*(to^ah)*(t(j)^(2*ah+1))));

 end

m1=0;m2=0;

 for k=1:n

m1=m1+log(dem4(k));

 )); m2=m2+  (3*ah+2)*log(t(k
```

```
 end

m=2;

LK=m1-m2;

AIC=-2*LK+2*m;

=') disp('AIC

disp(AIC)

AICC=AIC+(2*m*(m+1)/(n-m+1));

=') disp('AICC

disp(AICC)

BIC=-2*LK+m*log(n);

disp('BIC=')

disp(BIC)

  ; ff=(1:n)/n

ks_tsp=max(abs((ff-Ftsph )))

ks_p=max(abs((ff-Fph )))
```

Algorithms

The program of estimation of parameters of TSPD

```
function [pp]=MLE_TSP(t,S )

n=length(t);to=min(t );

a=S(1);

L=S(2);

sa=0;sL=0;s1=0;s2=0;s3=0;s4=0;dem=0;num=0;num1=0;num2
=0;num3=0;

for i=1:n

  dem(i)=(2*a*(1+L)*(to^(2*a))*(t(i)^(a+1)))-
a*L*(to^a)*(t(i)^(2*a+1))));

num1(i)= (2*(1+L)*(to^(2*a))*(t(i)^(a+1)))-
L*(to^a)*(t(i)^(2*a+1))));

num2(i)=(2*a*(1+L)*(2*log(to)*to^(2*a))*(t(i)^(a+1)))-
a*L*(log(to)*to^a)*(t(i)^(2*a+1))));

 num3(i)= (2*a*(1+L)*(to^(2*a))*(log(t(i))*t(i)^(a+1)))-
2*a*L*(to^a)*(log(t(i))*t(i)^(2*a+1))));

; )num(i)=num1(i)+num2(i)+num3(i

); s1=s1+num(i)/dem(i

)); s2=s2+log(t(i

end

sa=s1-3*s2;

num4=0;dem4=0;

for j=1:n
```

```
    num4(j)=(2*a*(to^(2*a))*(t(j)^(a+1)))-
(a*(to^a)*(t(j)^(2*a+1))));

    dem4(j)=(2*a*(1+L)*(to^(2*a))*(t(j)^(a+1)))-
a*L*(to^a)*(t(j)^(2*a+1))));

  )); sL=sL+(num4(j)/dem4(j

end

pp=[ sa Sl ];
```

المستخلص

الهدف من الرسالة هو أنشاء صيغة رياضية تعتمد في تكوينها على دالة البقاء، وهذه الصيغة تتكون في الأساس من معلمة التحويل. وأختبار هذه الصيغة من خلال توزيعين هما توزيع رايلي وتوزيع باريتو لانتاج توزيعات مختلفة، عملية الانتاج هذه تتضمن تطبيق الصيغة الرياضية على توزيعين أساسيين من توزيعات البقاء.التوزيعات الاخيرة ضرورية للغاية لأنها أكثر مرونه ودقة في تطبيق البيانات حيث توجد أحيانا بيانات لاتتناسب مع التوزيعات القياسية،لذلك فأن التوزيعات الجديدة تعطي المزيد من النتائج الدقيقة في تطبيق البيانات.قمنا بأستخراج دالة الكثافة الاحتمالية والدالةالتراكمية الاحتمالية ،ودالة الفشل.وأيضا تم دراسة أشكال مختلفة لدوال هذه التوزيعات مع دالة الهزار لكل توزيع. بالأضافة الى ذلك تم أشتقاق الخصائص الرياضية والاحصائية للتوزيعات الجديدة مثل(العزوم ،والدالة المولدة للعزوم والوسيط والوسط الحسابي والتوقع.....الخ).تم أختبار أداء كل توزيع من خلال حساب بعض المعايير الاحصائية وأختبار حسن المطابقة لكل توزيع مع أجراء المقارنات لكل توزيع ،تم الاعتماد على طريقة كلاسيكية للتقدير وهي طريقة الامكان الاعظم للتقدير ،وأثبتنا أفضلية توزيعاتنا الجديدة مقارنة بالتوزيعات الاصلية،وكذلك تحديد أي من التوزيعات الجديدة يكون أفضل وذلك من خلال تقدير المعلمات لنفس البيانات المتبعه.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

نماذج البقاء المحول مع التطبيق

رسالة

مقدمة الى مجلس كلية التربية للعلوم الصرفة / جامعة بابل كجزء من متطلبات نيل
درجة الماجستير في التربية / الرياضيات

من قبل

مروة عباس مظلوم شمران

بإشراف

أ.د. كريمة عبد الكاظم مخرب
الخفاجي

1444هـ   2022م