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Linear Preserving Approximation

A Thesis
Submitted to the Council of the College of Education for Pure Sciences in University of Babylon as a Partial Fulfillment of the Requirements for the Degree of Master in Education / Mathematics

By

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Supervised by

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Dedication

To

my parents

&

To those who make life more beautiful

I dedicate this work

بسم الله الرحمن الرحيم

﴿يرفع الله الذين آمنوا منكم والذين أوتوا العلم درجات﴾

صدق الله العلي العظيم

سورة المجادلة الآية11

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Special thanks go to my supportive husband who deserves my wholehearted thanks as well .

List of Symbols

Symbols	Description
$L_p[0,1]$	$f:\{[0,1] \to R: \|f\|_p < \infty\}$
$\|f\|_p$	$\|f\|_p = \left(\int_a^b \|f\|^p\right)^{\frac{1}{p}}$
$\varphi(x)$	$\varphi(x) = \sqrt{x(1-x)}$
$B_n(f)$	Bernstein operator
$K_\varphi^2(f, \delta^2)$	The K-functional .
$J_n^{(\alpha,\beta)}$	The Jacobi' s polynomials of degree n .
P_n	The space of polynomials of degree n of a set
inf	The greatest lower bound of a set.
$L_p^2\ [0,1]$	$= \{f:[a,b] \to R: f, f'' \in L_{p[0,1]}\}$ which is called 2-fold L_p space.
$c(p)$	A constant depending on p.
$\omega_2^\varphi(f,\delta)_p$	the weighted Ditizian – Totik modulus of smoothness of the second order.
R	The set of real numbers.

Publications

1. Nada Sadiq Abbas and Eman Samir Bhaya. "Colinear Approximation in terms of Generalized Weighted Modulus of Smoothness in L_P Spaces" . International Journal of Mechanical Engineering :Vol 7 No.2February.2022

2. Nada Sadiq Abbas and Eman Samir Bhaya "Saturation Problems for Positive Linear Approximation of Function in Quasi Normed Spaces" Acceptance International Journal of Health Sciences ,6(s3) ,12070_12077

Contents

Abstract

Many authors work on constrained approximation such as monotonicity ,convexity and k –monotonicity ,but little work on positive colinear approximation. The aim of this thesis is to introduced types of positive and piecewise positive linear operators and study their constrained approximation for $f \in L_p[0,1]$, $0 < p < 1$.

We introduce a Jackson type theorem for linearity preserving approximation for linear and positive functions in L_p spaces for $0 < p < 1$.

Our results are in terms of the weighted modulus of smoothness of the second order.

Bernstein polynomial is linear preserving approximation for functions in $L_{P[0,1]}$. These results are strict .

For a function $f \in L_p[0,1]$,$0 < p < 1$, we can find a positive and linear approximation if f is positive and linear.

Introduction

For a function in $L_p[0,1]$,$0 < p < 1$ space ,there exists an algebraic polynomial as a best unconstrained approximation.

Sometimes we need that the approximation has the same geometric properties of the target function such as linearity ,positivity, monotonicity, convexity ,and k-monotonicity $k \geq 2$. This what we called constrained approximation . The aim of our thesis is to study the degree of positive and linear constrained. Approximation for functions in $L_P[0,1]$,space for $0 < p < 1$. we mean by positive linear approximation as if we given f which is positive and linear function on an interval I , then the best approximation must be positive and linear on the interval I. In our thesis use the $L_p[0,1]$ quasi normed spaces for $0 < p < 1$,we also use Second order Ditzain-Tot [1] modulus of smoothness defend by:

$$\omega_2^{\varphi}(\mathrm{f}, \mathrm{h})_p = \sup_{0 \leq s \leq h} \| f(x - s\varphi(x)) - 2f(x) + f(x + s\varphi(x)) \|_{LP(I(\varphi,s))} \quad (1)$$

Where $\varphi(x) = \sqrt{x(1-x)}$, $f \in L_{\infty[0,1]}$ we write $\omega_2^{\varphi}(\mathrm{f}, \mathrm{h})$

The modulus (1) has been used to present estimates in approximation theory. Let us recall some of them. For $n \geq 1$ and $f \in C[0,1]$ the Bernstein operator B_n is defined by

$$B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). x \in [0,1]$$

Note that $B_n(f)$ is a polynomial of degree at most n ,also it is easy to see that

$$\left(B_n(f)\right)(0) = f(0) \quad \left(B_n(f)\right)(1) = f(1)$$

In [2], Ditzian proved that for $\alpha \in [0, 1/2]$ and $\varphi(x) = (x(1-x))^{\alpha}$ there exists a constant Cφ , such that, for $f \in C([0,1])$ and $x \in (0,1)$ where $C[0,1]$ is the space of continuous functions defend on [0,1]

$$|f(x) - B_n(f,x)| \leq C_{\varphi}\omega_2^{\varphi}\left(f, \frac{\sqrt{x(1-x)}}{\sqrt{n}\ \varphi(x)}\right) \tag{2}$$

This result unifies the classical estimate for α = 0 (Strukov and Timan [3) with the norm estimate for $\alpha = 1/2$ (Ditzian and Totik [1, p. 117]). In [4] Felten proved (2) which holds if $\varphi \in \Omega(0, 1)$. On the other hand, in [5] Gavrea et al. verified that

$$|f(x) - B_n(f)| \leq 3\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right)$$

For φ(x) = $\sqrt{x(1-x)}$. This last estimate improved some others given in [6,7 ,8]. In fact the main result of [5] provides an estimate for positive linear operators that preserve linear functions. The result was improved in [9]: if $L: C[0,1] \to C[0,1]$ is a positive linear operator, $f \in C[0,1]$

$0 < h \leq 1/2$ and $x \in (0,1)$, then

$$|f(x) - L(f,x)|$$
$$\leq |f(x)||1 - L(e_0,x)| + \frac{|L(e_1 - x, x)|}{h\varphi(x)}\omega_1^{\varphi}(f,h)$$
$$+ \left(1 + \frac{3}{2}\frac{L((e_1 - x)^2, x)}{(h\varphi(x)^2}\right)\omega_2^{\varphi}(\mathrm{f,h})$$

In chapter 1 we introduce a direct theorem for positive linear constrained approximation of function in $L_p[0,1]$,space $0 < p < 1$. We prove our theorem in terms of the weighted modulus of smoothness of the second order.

Let P be the set of polynomials with real coefficients and P_n be the space of polynomials of degree n. The Bernstein –Durrmeyer polynomial are defined by

$$(D_n f)\,(x) = (n+1)\sum_{k=0}^{n} P_{n,k}\,(x)\,\int_0^1 P_{n,k}(t)dt \quad ,$$

where

$$P_{n,k}(x) = \binom{n}{k}\, x^k(1-x)^{n-k} \qquad , \quad n = 0,1,\dots\dots \quad .$$

The following polynomial approximation introduced by Cluj-Napoca in a seminar on approximation –theory at 1994, and he asked "can we find a positive linear approximation which has the from :

$$(L_n f)(x) = \sum_{k=0}^{n} c_{kn}\,(D_k f)(x), \qquad\qquad c_{kn} \epsilon R$$

such that for a continuous function f on a closed interval L.

$$|f(x) - (L_n f)(x)| \le C\omega\left(f, \frac{\sqrt{x(1-x)}}{n} + \frac{1}{n^2}\right)$$

C being a positive constant which is independent from f, x and n .

A .Lupas in [13] gave an affirmative answer to this question .He considered the following problem:

"Does there exist linear positive operators of form

$$(B_n^* f)(x) = \sum_{k=0}^{n} m_{kn}\,(B_k f)(x) \ ,$$

$$(B_0 f)(x) = f(0) \ , \ (B_k f) = \sum_{i=0}^{k} P_{k,i}\, f\left(\frac{i}{k}\right) \quad k \geq 1$$

which satisfy

$$|f(x) - (B_n^* f)(x)| \leq C\omega\big(f, \Delta_n(x)\big) \,,$$

where

$C > 0, \quad f \in C[0,1] \ , \Delta_n(x) = \frac{\sqrt{x(1-x)}}{n} + \frac{1}{n^2}, \quad x \in [0,1]?$"

A. Lupe's conjectured that this problem has a negative answer ,but he did not prove this. **In Chapter 2** we show that the operators B_n^* are not better than the Bernstein operators B_n even in L_p quasi norm spaces for $0 < p < 1$. We use Bernstein algebraic polynomial as a best linear positive approximation of function in $L_p[0,1]$ space $0 < p < 1$. Then we strength our result by a negative theorem .

In [2] the authors proved the following direct estimate for Bernstein operator:

$$|f(x) - (B_n f)(x)| \leq C\omega_{\varphi^\lambda}^2\left(f, n^{-\frac{1}{2}}, \varphi(x)^{1-\lambda}\right) \qquad x \in I = [0,1] \quad (3).$$

Where $\varphi(x) = \sqrt{x(1-x)} \quad x \in [0,1]$.

in which $\varphi: [0,1] \to R$ is an admissible step weight function for details about φ see [20].

If $\lambda = 0$ in (3) we get classical local estimate while if λ=1 we get global norm estimate developed by Ditzian and Totik . So (3) fill the gap between the local and global approximation theorems for the Bernstein operator. Such result for polynomial approximation for details see [18,19 and 25]

Inequality (3) shows that the error $f(x) - (B_n f)(x)$ is bounded by

$C\left(n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\right)^{\alpha}$ if $\omega^2_{\varphi^\lambda}(f,\delta) = 0(\delta^\alpha)$ and $\alpha \in [0,2]$.

In [17,27] the authors proved the converse result also true .

$\omega^2_{\varphi^\lambda}(f,\delta) = 0(\delta^\alpha)$ can be estimated in terms of the Bernstein operator that is the equivalence

$|f(x) - (B_n f)(x)| =$

$$O\left(\left(f, n^{-\frac{1}{2}}, \varphi(x)^{1-\lambda}\right)^{\alpha}\right) \Leftrightarrow \omega^2_{\varphi^\lambda}(f,\delta) = O(\delta^\alpha)$$

holds whenever $\alpha \epsilon (0,2)$ and $\lambda \in [0,1]$.

We mean by $g = O(f)$ that $g \leq cf$,for some constant c.

In [21] (3) can be considered as a further estimate for the

general estimate

$$|f(x) - (B_n f)(x)| \leq C\omega^2_{\varphi^\lambda}\left(f, n^{-\frac{1}{2}} \frac{\varphi(x)}{\emptyset(x)}\right). \quad (4)$$

$x \in [0,1]$ where $\varphi: [0,1] \to R$ is a weight function for Ditzian _Totik modulus [20] and $\emptyset^2$ is a concave function . (4) improve of (3) if $\emptyset$ is replaced by φ^λ, $\lambda \in [0,1]$.

In [21] the authors proved inverse result to (4) for $x \in (0,2)$ it means

$$|f(x) - (B_n f)(x)| \le C_1 \left(n^{-\frac{1}{2}} \frac{\varphi(x)}{\emptyset(x)}\right)^{\alpha} \quad . x \in [0,1] \quad n = 1,2,3,\dots..$$

implies $\omega^2_{\varphi^\lambda}(f,\delta) \le C_2(\delta^\alpha)$ if ,in addition $\varphi^2/\emptyset^2$ is concave which is satisfied for $\emptyset = \varphi^\lambda$, $\lambda \in [0,1]$ in particular.

In chapter 3. We use a modification or improvement of Bernstein algebraic polynomial for the constrained positive linear approximation of function in $L_p[0,1]$,space $0 < p < 1$. Our result in terms of Ditizian Totik modulus of smoothness . We use K-function as an aid instrument for the proof of the main result.

As a conclusion ,if we give a measurable positive linear function in $L_P\,[0,1]$ quasi normed space ,we can find a positive linear algebraic polynomial as a constrained best approximation of f.

Chapter One

Colinear Approximation In Terms of Generalized Weighted Modulus of Smoothness in $L_p[0,1]$ Spaces

Many direct theorems introduced on the unconstrained approximation from 1886 up to now ,but very little result introduced about the linearity preserving approximation , here we introduce a Jackson type theorem for linearity preserving approximation for linear and positive functions in L_P spaces for $0 < p < 1$ .Our results are in terms of the weighted. modulus of smoothness is of the second order

1.1. Introduction

Let $L_{P[0,1]}$ be the space of all real measurable functions on [0, 1] defined by $L_p[0,1]=\{f:[0,1] \to R: \|f\|_p < \infty\}$, where $||f\,||_p = \left(\int_a^b |f|^p\right)^{\frac{1}{p}}$ and $\Omega(0,1)$ the class of nonnegative functions $\varphi \in L_p[0,1]$ which are strictly positive on (0, 1), and such that $\varphi^2$is concave

If $\varphi \in \Omega(0,1)$ and s > 0 defined then

$I(\varphi, s) = \{x \in (0,1): 0 \le x - s\varphi(x) < x + s\varphi(x) \le 1 \ \}$

$I(\varphi) = \{s > 0: I(\varphi, s) \ne \varphi\}\ and\ h_\varphi = (2\varphi(1/2))^{-1}$

For $\varphi \in \Omega$ (0, 1), $f \in L_p[0,1]$ and $h \in (0, h\varphi\,]$ the weighted second-order modulus is defined by (Ditzian and Totik [1])

$$\omega_2^{\varphi}(f,h)_p = \sup_{0 \le s \le h} \|f\,(x - s\varphi(x) - 2f(x) + f(x + s\varphi(x)\,)\,\|_{LP(I(\varphi,s))} \qquad (1.1)$$

$f \in L_{\infty[0,1]}$ we write ω_2^{φ} (f , h)

The modulus (1.1) has been used to present estimates in approximation theory. Let us recall some of them. For n ≥ 1 and

$f \in C$ [0,1]the Bernstein operator B_n is defined by

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). \quad x \in [0,1]$$

Note that $B_n(f)$ is a polynomial of degree at most n, also it is easy to see that

$$\big(B_n(f)\big)(0) = f(0) \ , \qquad \big(B_n(f)\big)(1) = f(1)$$

In [2], Ditzian proved that for $\alpha \in [0, 1/2]$ and $\varphi(x) = (x(1-x))^{\alpha}$ there exists a constant C_φ , such that, for $f \in C$ [0,1] and $x \in (0,1)$ where $C[0,1]$ is the space of continuous functions defend on [0,1].

$$|f(x) - B_n(f,x)| \leq C_\varphi \omega_2^\varphi \left(f, \frac{\sqrt{x(1-x)}}{\sqrt{n}\ \varphi(x)} \right) . \qquad (1.2)$$

This result unifies the classical estimate for α = 0 (Strukov and Timan [3]) with the norm estimate for α = 1/2 (Ditzian and Totik [1, p. 117]). In [4] Felten proved (1.2) which holds if $\varphi \in \Omega(0,1)$. On the other hand, in [5] Gavrea et al. verified that

$$|f(x) - B_n(f)| \leq 3\omega_2^\varphi \left(f, \frac{1}{\sqrt{n}} \right) ,$$

for φ(x) = $\sqrt{x(1-x)}$. This last estimate improved some others given in [6,7 ,8]. In fact the main result of [5] provides an estimate for positive linear operators that preserve linear functions. The result was improved in [9] if $L: C[0,1]] \rightarrow C[0,1]$ is a positive linear operator, $f \in C[0,1]$,

0 < h ≤ 1/2 and x ∈ (0, 1), then

$$|f(x) - L(f,x)| \leq |f(x)||1 - L(e_0,x)| + \frac{|L(e_1 - x,x)|}{h\varphi(x)} \omega_1^{\varphi}(f,h) + \left(1 + \frac{3}{2} \frac{L((e_1 - x)^2,x)}{(h\varphi(x)^2}\right) \omega_2^{\varphi}(f,h)$$

In this chapter we shall improve and generalize the main results in [6] 0r [5] for φ in

$\Omega\ (0,1) = \left\{ \varphi : \varphi(x) \geq 0 , x \in (0,1), \varphi \in L_p[0,1] \right\}$

we used proof similar to that in [7] and $f \in L_P[0,1]$, for $0 < p < \infty$.

We use $A(\varphi , h) = \{ x \in (0,1] : h\varphi(x) < x\}$, $a_h = \inf(A(\varphi , h)$

$B(\varphi , h) = \{ x \in [0,1) : h\varphi(x) < 1 - x\}$ and $b_h = \sup\big(B(\varphi, h)\big)$,

Where $\varphi \in \Omega\ (0,1)$ and $h \in \left(0, h_\varphi\right)$. For each $x \in [0, b_h]$, the increasing chain $(\{y_n\}, \{Z_n\})$ is associated with (x, h) which is defined as follows .

Let $Z_0 = x , y_1 = x + h\varphi(x), If\ y_1 \geq 1 - h\varphi(1)$ the construction ends in y_1 . If $y_1 < 1 - h\varphi(1)$,

and $\Delta(f, a, x, b) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) - f(x)$

1.2. Auxiliary Results

In this section we give the results that we need in our proofs

Lemma 1. 2.1 [8]

Let $\varphi : [0,1] \to R$ be a concave positive function

(1) If $0 \le a < 1$, *then the function* $M_\varphi(a, o)$ *decreases on* $(a, 1]$.

(2) If $0 \le c < 1$, *then* $N_\varphi(c, o)$ *increases on* $[0, c)$ *moreover for* $0 \le a \le c \le 1$. Max $\{\varphi(a), \varphi(c)\} \le 2\varphi((a+c)/2)$

(3) The limits $\lim_{x\to 1} \frac{\varphi(x)}{x}$ *and* $\lim_{x\to 1} \frac{\varphi(x)}{1-x}$ exist (finite or infinite)

(4) *If* $c - a \le 2h\varphi(a+c)/2)$ *and* $a \le u < v \le c$, *then*

$v - u \le 2h\varphi(u+v)/2)$.

Lemma 1. 2.2[8]

(1) If $d < b_h \le y_1$, *since* $\varphi(x) \le 2\varphi(d)$ then $t - d \le 2h\varphi\left(\frac{(t+d)}{2}\right)$

(2) If $y_1 < b_h$ *and* $y_1 + h\varphi(y_1) < t$

Then $\varphi(x) \le \varphi(y_1)$ and $\frac{t-x}{y_1-x} \ge 2$

Lemma 1. 2.3[8]

If $(t_2 - t_1) > 2h\varphi(c)$ and $s < 1$

Then $x - sh\varphi(x) < b_h - h\varphi(b_h)$

Lemma 1. 2.4[2]

$$\psi\left(\left|\frac{e_1 - xe_0}{h}\right|, x\right) = (\frac{3}{2} + \frac{3}{2h^2\varphi(x)}\ L(e_1 - xe_0)^2, x\ )$$

Lemma 1. 2.5 [2]

$$B_n(e_1 - xe_0)^2, x\ ) = \frac{x(1-x)}{n} \le \frac{1}{4n}$$

Lemma 1. 2.6

Let $\varphi \in \Omega\ (0,1)$, $0 \le a \le b \le 1$, $c = \frac{(a+b)}{2}$ *and* $x \in [a, b]$

If $f \in L_p[0,1]$, then

$$\|\Delta(f, a, c, b)\|_p \le \omega_2^{\emptyset}\left(g, \frac{b-a}{\varphi\ (c)}\right)_p$$

Proof :-

Let $g(x) = \frac{b-x}{b-a}\ f(a) + \frac{x-a}{b-a}\ f(b) - f(x)$

$$g(a) = \frac{b-a}{b-a}\ f(a) + \frac{a-a}{b-a}\ f(b) - f(a) = 0 \qquad (1.3)$$

$$g(b) = \frac{b-b}{b-a}\ f(a) + \frac{b-a}{b-a}\ f(b) - f(b) = 0 \qquad (1.4)$$

$\omega_2^{\varphi}(f,\delta)_p$

$$= \sup_{0 \le h \le \delta} \left\| f\big(x - h\varphi(x)\big) - 2f(x) + f\big(x + h\varphi(x)\big) \right\|_{L_{P(I(\varphi,s))}}$$

From (1.3) and (1.4) we get $\omega_2^{\varphi}(g,t)_{L_{P(I(\varphi,s))}} = \omega_2^{\varphi}(f,t)_{L_{P(I(\varphi,s))}}$

From the definition of g , we can write

$g(c) = -\frac{1}{2}\,(g(a) - 2g(c) + g(b))$

Let $N_{\varphi}(c, z) = \frac{\emptyset(Z)}{C-Z}$

Since N_{φ} is increasing for $z \in [0,1]$ we get $c < u$

$$\frac{\varphi(c)}{b-c} = N_{\varphi}(b,c) \le N_{\emptyset}(b,u) = \frac{\varphi(u)}{b-u}$$

$$\|g(x)\|_p = \|M\|_p = \|-g(b) + 2g(x) - g(2x-b) - M + g(2x-b)\|_p$$

$$\le \omega_2^{\varphi}\left(g, \frac{b-a}{\varphi(a+b)/2}\right)_p$$

Lemma 1. 2.7

If $\varphi \in \Omega(0,1)$, $h \in (0, h_{\varphi})$, $x \in [a_h, b_h]$ is increasing chain that is associated with (x, h) of length 1 and $y_1 < 1 - h\varphi(1)$,

$f \in L_p[0,1]$ such that

$f\big(x - h\varphi(x)\big) = 0 = f\big(x + h\varphi(x)\big)$ then for $t \in [y_1, 1]$

$\|f(t)\|_p \le \frac{3}{2}\frac{t}{y_1}\ \omega_2^{\varphi}(f,h)_p$

Similarly if y_2 exists and $t \in [y_1, y_2]$

Proof

$$\left(\int_0^1 |f(t)|^p\, dx\right)^{\frac{1}{p}}$$

$$= \left\|\frac{t-x}{y_1-x}\left(\frac{t-y_1}{t-x}\, f(x) + \frac{y_1-x}{t-x} f(t) - f(y_1)\right) - \frac{t-y_1}{y_1-x} f(x)\right\|_p$$

$$\leq \left\|\left(\frac{t-x}{y_1-x} + \frac{1}{2}\ \frac{t-y_1}{y_1-x}\right) f\big(x - t\emptyset(x)\big) - 2f(x) + f(x + t\emptyset(x)\right\|_p$$

$$\leq \left\|\left(\frac{t}{y_1} + \frac{1}{2}\ \frac{t}{y_1}\right) f\big(x - t\varphi(x)\big) - 2f(x) + f\big(x + t\varphi(x)\big)\right\|_p$$

$$\leq \left\|\left(\frac{3}{2}\ \frac{t}{y_1}\right) f\big(x - t\varphi(x)\big) - 2f(x) + f\big(x + t\varphi(x)\big)\right\|_p$$

$$= \left(\frac{3}{2}\ \frac{t}{y_1}\right) \omega_2^{\emptyset}(f, h)_p$$

Lemma 1.2.8

Let $\varphi \in \Omega\ (0,1)$, $h \in (0, h_\varphi)$ and $x \in [a_h, b_h]$ be such that the increasing chain associated with (x, h) which has length 1 and $y_1 \geq 1 - h\,\varphi(1)$,
If $f \in L_p[0,1]$ and $f\big(x - h\varphi(x)\big) = 0 = f\big(x + h\varphi(x)\big)$ then for $t \in [x + h\varphi(x), 1]$

$$\|f\|_p \leq \frac{7}{2}\ \frac{t}{y1}\, \omega_2^{\varphi}(f, h)_p.$$

Proof

Let us denote $d = (x + y_1)/2$

Case 1: Assume $b_h \leq d$ Notice that $(y_1 - x) \leq 2(y_1 - b_h)$ and $b_h < y_1$

by using (4) of Lemma 1.2.1 $t - b_h \leq 2\, h\varphi\,(t + b_h)/2$ then

$$\|f(t)\|_p = \left\|\frac{t - b_h}{y_1 - b_h} \cdot \left(\frac{t - y_1}{t - b_h}\, f(b_h) + \frac{y_1 - b_h}{t - b_h} f(t) - f(y_1)\right) - \frac{t - y_1}{y_1 - b_h}\, f(b_h)\right\|_p$$

$$\leq \left\|\frac{1}{y_1 - b_h}(2t - b_h - y_1).f(x - t\varphi(x) - 2f(x) + f(x + t\varphi(x))\right\|_p$$

$$= \left\|\left(1 + 2\,\frac{t - y_1}{y_1 - b_h}\right).f(x - t\varphi(x) - 2f(x) + f(x + t\varphi(x))\right\|_p$$

$$\leq \left\|(1 + 2\frac{t}{y_1}\, f(x - t\varphi(x) - 2f(x) + f(x + t\varphi(x))\right\|_p$$

$$\leq \left(1 + 2\frac{t}{y_1}\right) \omega_2^{\varphi}(f, h)_p$$

$$= (y_1 + 2\frac{t}{y_1})\, \omega_2^{\varphi}(f, h)_p$$

Case 2: Assume d< $b_h \leq y_1$ by using(1) of Lemma1. 2.2 we have

$$t - d \leq 2h\varphi\left(\frac{(t + d)}{2}\right)$$

$$\|f(t)\|_p \leq \left\| \frac{t - d}{y_1 - d} \cdot \left(\frac{t - y_1}{t - d} f(d) + \frac{y_1 - d}{t - d}\, f(t) - f(y_1)\right) - \frac{t - y_1}{y_1 - d}\, f(d)\right\|_p$$

$$\leq \left\|\frac{1}{y_1 - d}(2t - d - y_1).f(x - t\varphi(x) - 2f(x) + f(x + t\varphi(x))\right\|_p$$

$$\leq 2\frac{t}{y_1}\, \omega_2^{\varphi}(f, h)_p$$

Case 3: Assume that $y_1 < b_h$ and $t \le y_1 + h\varphi(y_1)$

since $y_1 + h\varphi(y_1) < 1$ then $1 - h\varphi(1) \le y_1 < 1 - h\varphi(y_1)$ as in Lemma 1.2.6 we has

$$\|f(t)\|_p \le \left\| \frac{t-x}{y_1-x}.\left(\frac{t-y_1}{t-x} f(x) + \frac{y_1-x}{t-x} f(t) - f(y_1)\right) - \frac{t-y_1}{y_1-x} f(x) \right\|_p$$

$$\le \left\| \left(\frac{t-x}{y_1-x} + \frac{1}{2}\frac{t-y_1}{y_1-x}\right) f(x - t\varphi(x) - 2f(x) + f(x+t\varphi(x))\right\|_p$$

$$\le \left\| \left(\frac{t}{y_1} + \frac{1}{2}\frac{t}{y_1}\right) f(x - t\varphi(x) - 2f(x) + f(x+t\varphi(x))\right\|_p$$

$$\le \left(\frac{3}{2}\frac{t}{y_1}\right) \omega_2^{\varphi}(f,h)_p$$

Case (4): Assume that $y_1 < b_h$ and $y_1 + h\varphi(y_1) < t$ as in Case (3) we have

$\varphi(x) \le \varphi(y_1)$ by using (3) of Lemma 1.2.1 and (2) of Lemma1. 2.2

$$\|f(t)\|_p \le \left\| \frac{t-y_1}{h\varphi(y_1)}.\left(1 + \frac{y_{1+h\varphi(y_1)-x}}{y_1-x} + \frac{1}{2}\frac{h\varphi(y_1)}{y_1-x}\right) f(x - t\varphi(x)) - 2f(x) + f(x+t\varphi(x)) \right\|_p$$

$$= \left\| \left(2\frac{t-y_1}{h\varphi(y_1)} + \frac{3}{2}\frac{t-y_1}{y_1-x}\right) f(x - t\varphi(x) - 2f(x) + f(x+t\varphi(x)) \right\|_p$$

$$\le \left(\frac{2t}{h} + \frac{3}{2}\frac{t}{y_1}\right) \omega_2^{\varphi}(f,h)_p$$

Since we get $y_1 < h$

$$\|f(t)\|_p \le \left(\frac{2t}{y_1} + \frac{3}{2}\frac{t}{y_1}\right) \omega_2^{\varphi}(f,h)_p$$

$$\leq \frac{7}{2}\frac{t}{y_1}\,\omega_2^{\varphi}(f,h)_p.$$

Lemma 1. 2.9

Let $\emptyset \in \Omega\,(0,1)$, h$\in (0,h\varphi)$, $x \in [a_h, b_h]$ and $t \in [0,1]$ such that $0 \leq t \leq x - h\varphi(x)$ or $x + h\varphi\,(x) \leq t \leq 1$. If $f \in L_p[0,1]$ satisfies $f\big(x - h\varphi(x)\big) = 0 = f\big(x + h\varphi(x)\big)$,then

$$\|f\|_p \leq \frac{7}{2}\frac{t}{1-h\,\varphi(1)}\,\omega_2^{\varphi}(f,h)_p$$

Proof

Let $x + h\varphi\,(x) \leq t$. We shall prove this Lemma by induction on the length of the chains

If $n = 1$, let $(\{y_n\},\{z_n\})$ is increasing chain of (x,h) and of length 1 then $y_1 \geq 1 - h\,\varphi(1)$ or $y_1 < 1 - h\emptyset(1)$, $b_h < z_1$ then using Lemma 1. 2.7 We get

$\|\mathrm{f(t)}\|_{\mathrm{p}} \leq \frac{3}{2}\frac{t}{y_1}\ \omega_2^{\emptyset}(f,h)_p$

$\qquad\leq \frac{7}{2}\frac{t}{y1}\omega_2^{\varphi}(f,h)_p$ where $y_1 < 1 - h\emptyset(1)$

and by Lemma 1. 2.8 $y_1 \geq 1 - h\,\varphi(1)$, we get

$\|f(t)\|_p \leq \frac{7}{2}\frac{t}{y1}\omega_2^{\varphi}(f,h)_p \leq \frac{7}{2}\frac{t}{1-h\,\varphi(1)}\omega_2^{\varphi}(f,h)_p$ for y$\geq 1 - h\,\varphi(1)$

Assume the statement is true for chain of length n and has chain of length $n + 1$

If we eliminate from the chain the point z_0 and y_1 we obtain the chain (z_1, h) with length n , Let $f \in L_{P[0,1]}$ and $t > x + h\varphi(x)$

If $y_1 < t \le z_1$ we get the result directly by Lemma 1. 2.7 then assume $t > z_1$ Let p be a polynomial of degree ≤ 1 and interpolate f at y_1 and y_2 let $g = f - p$ using Lemma 1.2.6 we obtain $\Delta(f, y_1, y_2, t) = \Delta(g, y_1, y_2, t)$ by our induction hypothesis we get

$\|f(t)\|_p \le \frac{3}{2}\frac{t}{y_1} \omega_2^{\emptyset}(f,h)_p$

and by Lemma 1.2.6 we get

$$\| f \|p \le \omega_2^{\emptyset}\left(g, \frac{b-a}{\emptyset(a+b)/2}\right)_p$$

$$\le c(p)\, \frac{3}{2}\frac{t}{y_1} \omega_2^{\emptyset}(f,h)_p$$

Where $c(p)$ is a positive constant which depends on p.

1.3. The Main Results

In this section we introduce our main results in positive linear weighted approximation for functions in L_p quasi normed spaces for $0 < p < 1$

Theorem 1.3.1

If $f \in L_p[0,1]$ let $S > 0$, then for

$$\omega_2^{\varphi}(f, \lambda t)_p \le \lambda 2^{\frac{1}{p}-1} \omega_2^{\varphi}(f,t)_p \ .$$

Proof

Let $0 < S \le \lambda t, \lambda > 1$

Let $x_0 = 0 < x_1 < x_2 < x_3 < \ldots < x_n = 1, n \in N$ be a partition for the interval [0,1] such that $|x_i - x_{i+1}| < t$

$$\omega_2^{\varphi}(f, \lambda t)_p \le \left(\int_0^1 |f\,(x - s\varphi(x) - 2f(x) + f(x + s\varphi(x)\,)|^p dx\right)^{1/p}$$

$$\le \left(\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f\,(x - s\varphi(x) - 2f(x) + f(x + s\varphi(x)\,)|^p dx\right)^{1/p}$$

$$\leq 2^{\frac{1}{p}-1} \sum_{i=1}^{n} \left(\int_{x_{i-1}}^{x_i} |f\,(x - s\varphi(x) - 2f(x) + f(x + s\varphi(x)\,)|^p dx \right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{1}{p}-1} \sum_{i=1}^{n} \omega_2^{\varphi}(f,t)_p$$

$$= n\, 2^{\frac{1}{p}-1}\, \omega_2^{\varphi}\,(f,t)_p$$

$$\leq\, \lambda\, 2^{\frac{1}{p}-1}\, \omega_2^{\varphi}\,(f,t)_p$$

Theorem 1.3.2

Let $\varphi \in\ \Omega\,(0,1)$ $If\ \ f \in\ L_p[0,1]$ and $0 \leq t_1\ < x < t_2\ \leq 1$, then

$$\left\|\Delta(f, t_1, x, t_{2)}\right\|_p\ \leq 2^{\frac{1}{p}-1}\ \ \frac{7}{2}\left(\frac{2rs + (s+r)}{(r+s)\big(1 - h\varphi(1)\big)}\right)\ \omega_2^{\varphi}(f,h)_p$$

Proof

Define $s\,, r\ \ and\ \ c\ \ by$

$x - t_1 = sh\varphi(x)\qquad , t_2 - x = rh\varphi(x)\ and\ \ \ c = \dfrac{t_1 - t_2}{2}$

$If\ s < 1\quad and\ h < \ \dfrac{t_2 - t_1}{2\varphi(c)}$

then $r \geq 1$ In fact $if\quad r < 1$ then $\quad s + r < 2$

and $[t_1\,, t_2] \subset [\,x -\ rh\varphi(x)\,, x +\ rh\varphi(x)]$

by using (3) of Lemma 1.2.1

$t_2 - t_1 \leq 2h\varphi\,(c)\ \ implies\quad \dfrac{t_2 - t_1}{2\varphi(c)} \leq h\ \ \leq t$

Case (1): If $(t_2 - t_1) \leq 2h\varphi\,(c)$ by Lemma 1. 2.6

$$\|\Delta(f\,, t_1, x\,, t_2)\ \|_{L_{P[t_1\,,t_2]}} = \left\| \ \frac{t_2 - x}{t_2 - t_1}\ f(t_1) + \frac{x - t_1}{t_2 - t_1}\ f(t_2) - f(x) \right\|_{L_{P[t_1\,,t_2]}}$$

$$\leq \omega_2^{\varphi}(f, \tfrac{t_2 - t_1}{2\varphi(c)})_{L_{P[t_1\,,t_2]}} \leq \ \omega_2^{\varphi}(f, h)_{L_{P[t_1\,,t_2]}}$$

Case (2): If $(t_2 - t_1) > 2h\varphi(c)$ and r$\geq s \geq 1$

we can assume $f(x - h\varphi(x)) = f\big(x + h\varphi(x)\big) = 0.$

By using Lemma 1. 2.9 we have

$$\|f(t_1)\|_p \leq \frac{7}{2}\ .\left(\ \frac{s}{1 - h\varphi(1)}\right)\omega_2^{\varphi}(f, h)_p,$$

and

$$\left\|f(t_{2)}\right\|_p \leq \frac{7}{2}\left(\frac{r}{1 - h\varphi(1)}\right)\omega_2^{\varphi}(f, h)_p$$

$$\|\Delta f(t_1, x, t_2)\|_p = \left\|\frac{r}{s + r}f(t_1) + \frac{s}{s + r}f(t_2) - f(x)\right\|_p$$

$$\leq 2^{\frac{1}{p}-1}\left(\tfrac{r}{s+r}\,\|f(t_1)\|_p + \tfrac{s}{s+r}\,\|f(t_2)\|_p + \|f(x)\|_p\right)$$

$$\leq 2^{\frac{1}{p}-1}\,.\tfrac{7}{2}\ \Big(\tfrac{r}{s+r}\ \ \tfrac{s}{1-h\varphi(1)}\,\omega_2^{\varphi}(f, h)_p + \tfrac{s}{s+r}\ \ \tfrac{r}{1-h\varphi(1)}\,\omega_2^{\varphi}(f, h)_p +$$

$$\tfrac{t}{1-h\varphi(1)}\,\omega_2^{\varphi}(f, h)_p\Big)$$

$$\leq 2^{\frac{1}{p}-1}\ \tfrac{7}{2}\left(\tfrac{rs}{(s+r)(1-h\varphi(1)} + \tfrac{rs}{(s+r)(1-h\varphi(1))} + \tfrac{1}{\big(1-h\varphi(1)\big)}\right)\ \omega_2^{\varphi}(f, h)_p$$

$$\leq 2^{\frac{1}{p}-1}\,\frac{7}{2}\left(\frac{2rs + (s + r)}{(r + s)\big(1 - h\varphi(1)\big)}\right)\omega_2^{\varphi}(f, h)_p$$

Case (3)): If $(t_2 - t_1) > 2h\varphi(c)$ and $s < 1$ by using Lemma 1.2.3 then $x - sh\varphi < b_h - h\varphi(b_h)$

since $b_h - h\varphi(b_h) \leq 1 - h\varphi(1)$

There exists y such that $t_1 = y - h\varphi(y)$ It is clear that $x < y < b_h$

Now we can assume $f\big(y - h\varphi(y)\big) = f\big(y + h\varphi(y)\big) = 0$

$since \quad s < 1 \; then \quad r \geq 1$

If $t_2 \leq y + h\varphi(y)$ by using Lemma 1. 2.6

$$\|f(t_2)\|_p \leq \omega_2^{\varphi}(f,h)_p \leq \frac{7}{2}\cdot\frac{r}{1 - h\varphi(1)}\,\omega_2^{\varphi}(f,h)_p$$

If $t_2 > y + h\varphi(y)$ by lemma 1. 2.9 we get

$$\|f(t_2)\|_p \leq (\frac{7}{2}\cdot\frac{r}{1 - h\varphi(1)})\,\omega_2^{\varphi}(f,h)_p$$

$$\|\Delta f(t_1,x,t_2)\|_p = \left\|\frac{s}{s+r}f(t_2) - f(x)\right\|_p$$

$$\leq 2^{\frac{1}{p}-1}(\tfrac{s}{s+r}\|f(t_2)\|_p + \|f(x)\|_p)$$

$$\leq 2^{\frac{1}{p}-1}\cdot\frac{7}{2}(\frac{s}{s+r}\cdot\frac{r}{(1 - h\varphi(1)}\,\omega_2^{\varphi}(f,h)_p$$

$$+\frac{t}{(1 - h\varphi(1)}\,\omega_2^{\varphi}(f,h)_p)$$

$$\leq 2^{\frac{1}{p}-1}\frac{7}{2}\left(\frac{s}{s+r}\frac{r}{(1 - h\varphi(1)} + \frac{1}{(1 - h\varphi(1)}\right)\omega_2^{\varphi}(f,h)_p$$

$$= 2^{\frac{1}{p}-1}\frac{7}{2}(\frac{sr + s + r}{(1 - h\varphi(1)(s+r)})\,\omega_2^{\varphi}(f,h)_p \quad .$$

Remark

Denote $e_0(t) = 1$, $e_1(t) = t$, $e_2(t) = t^2$

Let L be a linear map which preserves linearity

$$L((e_1 - xe_0)^2, x) = L(e_2, x) - 2xL(e_1, x) + x^2 = L(e_2, x) - x^2$$

Theorem 1. 3.3

Let$\varphi \epsilon\ \Omega(0,1)$ $L : L_p[0,1] \rightarrow L_p[0,1]$ be a positive linear operator which preserves linear functions. If $f \in L_p[0,1]$ then

$$\|f - L(f)\|_p \leq \frac{3}{2} + \frac{3}{2h^2\varphi^2(x)}(L\,(e_2, x) - x^2))\omega_2^{\varphi}(f, h)_p$$

Proof

By using Lemma 1.2.5

Let $\psi(t) = \frac{3}{2} + \frac{3t^2}{2\varphi^2(x)}$

Using Theorem 1. 3.2 we get

$$\left\|\Delta(f, t_1, x, t_{2)}\right\|_p \leq \left(\frac{t_2 - x}{t_2 - t_1}\psi(\frac{x - t_1}{h}) + \frac{x - t_1}{t_2 - t_1}\psi(\frac{t_2 - x}{h})\right)\omega_2^{\varphi}(f, h)_p$$

then by Lemma 1.2.4 we get

$$\|f - L(f)\|_p \leq L(\psi\left(\left|\frac{e_1 - xe_0}{h}\right|, x\right)\omega_2^{\varphi}(f, h)_p$$

$$= \frac{3}{2} + \frac{3}{2h^2\varphi^2(x)}L\,((e_1 - xe_0)^2, x))\omega_2^{\varphi}(f, h)_p$$

Theorem 1. 3.4

Let$\varphi \epsilon\ \Omega(0,1)$ and $n \geq 1$ If $f \in L_p[0,1]$ then

$$\|f - B_n(f)\|_p \leq c(p)\omega_2^{\varphi}\left(f, \frac{\sqrt{x(1-x)}}{\sqrt{n}\ \varphi(x)}\right)_p .$$

Where B_n is Bernstein polynomial

Proof

By using Lemma 1.2.5 we get

$B_n((e_1 - xe_0)^2, x) = x(1-x)/n$

By Theorem 1.3.3 with $h = \sqrt{x(1-x)} / (\varphi(x)\sqrt{n})$

$\|f - B_n(f)\|_p \leq B_n\ (\psi\left(\left|\frac{e_1 - xe_0}{h}\right|, x\right)\omega_2^{\varphi}(f, h)_p \leq c(p)\ \omega_2^{\varphi}(f, \frac{\sqrt{x(1-x)}}{\sqrt{n}\ \varphi(x)})_p$.

Chapter Two

Positive Linear $L_p[0,1]$ *Approximation*

This chapter consists of two sections in section one we shall prove a linear preserving $L_p[0,1]$ approximation using Bernstein polynomial .In the second section we shall strength our result by a negative result.

2.1.Linear Preserving Polynomials Approximation

If f is a linear function in $L_p[-1,1]$ and we want to approximation it by a polynomial that preserves this linearity, we can use Bernstein polynomial . That is the aim of this section . Recall that the Bernstein polynomial is :

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \quad ,[10]$$

The K – functional for $f \in L_p[-1,1]$ is defined as

$$K_{\varphi^\lambda}(f,t^2) = \inf_{g \in p_n} \left(\|f-g\|_p + t^2 \left\|\varphi^{2\lambda} g''\right\|_p \right) \qquad (2.1)$$

where p_n is the set of all polynomials of degree $\leq n$. If $f \in L_p[-1,1]$ and $0 < p < 1$ then

$$C^{-1} K_{\varphi^\lambda}(f,t^2) \leq \omega_2^\varphi(f,h)_p \leq C K_{\varphi^\lambda}(f,t^2) \qquad [2] \qquad (2.2)$$

For our proof of the main theorem we need the following auxiliary results:

Lemmas 2.1.1 [10]

$$|B_n(g_n) - g_n| \leq \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left| \int_{\frac{k}{n}}^{x} (xv) g''_n(v)\, dv \right|$$

Now we are ready to prove our main theorem

Theorem 2.1.2

If f is positive linear function in $L_p[-1,1]$ then

$$\|f - B_n(f)\|_p \le c(p)\omega_2^{\varphi}(f, n^{\frac{-1}{2}}\varphi(1)^{1-\lambda})_p.$$

Proof

From (2.1) and (2.2) we can choose g_n for fixed x and λ such that

$$\|f - g_n\|_p \le A\omega_2^{\varphi}(f, n^{\frac{-1}{2}}\varphi(1)^{1-\lambda})_p \quad (2.3)$$

$$\left\|\varphi(x)^{2\lambda}g_n''\right\|_p n^{-1}\varphi(x)^{2-2\lambda} \le B\omega_2^{\varphi}\left(f, n^{\frac{-1}{2}}\varphi(1)^{1-\lambda}\right)_p \quad (2.4)$$

$$\|f - B_n(f)\|_p = \|B_n(f) - g_n(x) + g_n(x) - f(x) \quad + \quad B_n(g_n) - B_n(g_n)\|_p$$

$$= \left\|B_n(f - g_n) - \left(f(x) - g_n(x)\right) + (B_n(g_n) - g_n)\right\|_p$$

$$\le 2^{\frac{1}{p}-1}\left(\left\|B_n(f - g_n) - \left(f(x) - g_n(x)\right)\right\|_p + \|(B_n(g_n) - g_n)\|_p\right)$$

From (2.3) we have

$$\|f - B_n(f)\| \le 2^{\frac{1}{p}-1}\left(2A\omega_2^{\varphi}(f, n^{\frac{-1}{2}}\varphi(1)^{1-\lambda})_p + \|(B_n(g_n) - g_n)\|_p\right)$$

By using Lemma 2.1.1

$$\|B_n(g_n) - g_n\|_p \le \left\|\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\left|\int_{\frac{k}{n}}^{x}(xv)g''_n(v)dv\right|\ \right\|_p$$

$$\leq\ 2^{\frac{1}{p}-1}\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\ \frac{\|x-kln\|_p}{\varphi(x)^{2\lambda}}\left\|\int_{\frac{k}{n}}^{x}\varphi(v)^{2\lambda}\left|g_n^{"}(v)\right|dv\right\|_p$$

Now let us estimate $\|B_n(g_n)-g_n\|_p$,

$\|B_n(g_n)-g_n\|_p$

$$\leq\left\|\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\left|\int_{\frac{k}{n}}^{x}(xv)g^{"}{}_n(v)dv\right|\ \right\|_p$$

$$\leq\left\|\sum_{k=0}^{n}\binom{n}{k}\ \int_{\frac{k}{n}}^{1}v\ g^{"}{}_n(v)dv\ \right\|_p$$

$$\leq 2^{\frac{1}{p}-1}\sum_{k=0}^{n}\binom{n}{k}\ \left\|\int_{\frac{k}{n}}^{1}v\ g^{"}{}_n(v)dv\ \right\|_p$$

$$\leq 2^{\frac{1}{p}-1}\ \frac{1-\frac{k}{n}}{\varphi(1)^{2\lambda}}\ \left\|\varphi^{2\lambda}\ \ g^{"}{}_n\right\|_p$$

$$\|f-B_n(f)\|_p\leq 2^{\frac{1}{p}-1}\left(2A\omega_2^{\varphi}(f,n^{\frac{-1}{2}}\varphi(1)^{1-\lambda})_p+B\omega_2^{\varphi}\left(f,n^{\frac{-1}{2}}\varphi(1)^{1-\lambda}\right)_p\right)$$

$$\leq c(p)\omega_2^{\varphi}\left(f,n^{\frac{-1}{2}}\varphi(1)^{1-\lambda}\right)_p$$

2.2.Negative Theorem for Linear Preserving Approximation

Let P be the set of polynomials with real coefficients and P_n be the space of polynomials of degree n .The Bernstein –Durrmeyer polynomial is defined by

$$(D_n f)(x) = (n+1)\sum_{k=0}^{n} P_{n,k}(x) \int_0^1 P_{n,k}(t)dt \quad ,$$

where

$$P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad n = 0,1,\dots\dots \quad .$$

At the first German –Romanian Seminar on approximation theory (Cluj-Napoca, September ,1994) the following problem is raised :

Does there exist linear positive operators of the from

$(L_n f)(x) = \sum_{k=0}^{n} c_{kn} (D_k f)(x), \qquad c_{kn} \epsilon R$

such that for a continuous function f on a closed interval L.

$|f(x) - (L_n f)(x)| \le C\omega\left(f, \frac{\sqrt{x(1-x)}}{n} + \frac{1}{n^2}\right)$

C being a constant which is independent from f, x and n ?

A .Lupe's in [13] gave an affirmative answer to this question .He considered the following problem:

"Does there exist linear positive operators of form

$$(B_n^* f)(x) = \sum_{k=0}^{n} m_{kn}\,(B_k f)(x) \ , \qquad [13]$$

$$(B_0 f)(x) = f(o), \ \ (B_k f) = \sum_{i=0}^{k} P_{k,i}\, f\left(\frac{i}{k}\right) \quad k \geq 1$$

which satisfy

$$|f(x) - (B_n^* f)(x)| \leq C\omega\big(f, \Delta_n(x)\big) \,,$$

where

$C > 0, \quad f \in C[0,1] \ , \Delta_n(x) = \frac{\sqrt{x(1-x)}}{n} + \frac{1}{n^2}, \quad x \in [0,1]?$ "

A.Lupas conjectured that this problem has a negative answer ,but he didn't prove this. In this section we will show that the operators B_n^* are not better than the Bernstein operators B_n even in L_p quasi normed space.

Now let us introduce the results that we need in our proofs of main results

Lemmas 2.2.1[14]

The operators $(B_n^* f)(x) = \sum_{k=0}^{n} m_{kn} (B_k f)(x)$,

are positive if and only if $m_{kn} \geq 0$, $k = 0, \dots \dots, n$.

Lemma 2.2.2[14]

Let P_n be the linear space of polynomial of degree n.

$p_n(x) = \sum_{k=0}^{n} a_k x^k$. we suppose that the polynomial P_n

satisfies the following conditions:

1) $p_n(x) \geq 0$, $x \in [0,1]$

2) $\int_0^1 p_n(x)\, dx = 1$

3) $p_n'(x) \geq 0$, $x \in [0,1]$

Lemma 2.2.3[11]

If $f\colon I \to R$ with $\Delta_h^2(f) < \infty$, then

$$|(Lf)(x) - f(x)| \leq \left[\frac{3}{2} + \frac{3}{4} h^{-2} L((e_1 - x)^2, x)\right] \Delta_h^2(f).$$

Using the proof of Theorem 2 [11] step by step we get the following lemma

Lemma 2.2.4

If $f \in L_p(I)$, $0 < p < 1$

$\|(Lf) - f\|_p \leq c(p) \left[\frac{3}{2} + \frac{3}{4}\, h^{-2} L((e_1 - x)^2, x)\ \right] \omega_2^{\varphi}(f, h)_p .$

Lemma 2.2.5 [11]

Let $H_n : I \to P_{n+2}$ sequences of linear operators

$$(H_n\, f)(x) = \sum_{k=0}^{n} \frac{a_k}{k+1}\ (L_{k+2} f)(x).$$

The following statements are true:

1) H_n is a positive linear operator,
2) $(H_n e_0)(x) = 1$,
3) $(H_n e_1)(x) = x$,
4) $(H_n e_2)(x) = x^2 + x(1 - x)\left(1 - \int_0^1 x^2\, p_n(x) dx\right)$.

Now let us introduce our main results

Theorem 2.2.6

Let $\Delta_n^{(\alpha,\beta)}: [0,1] \to R$, $\Delta_n^{(\alpha,\beta)} = \frac{\sqrt{x(1-x)}}{n^{\alpha}} + \frac{1}{n^{\beta}}$.

If $\alpha, \beta > \frac{1}{2}$ and $f \in L_p[0,1]$and then it does not exist a positive constant , $C > 0$ such that

$$\|f(x) - (B_n^* f)(x)\|_p \leq C\omega_2^{\varphi}\left(f, \Delta_n^{(\alpha,\beta)}(x)\right)_p$$

proof

Suppose there exist the positive numbers m_{kn} such that for every $f \in L_p[0,1]$ and $x \in [0,1]$ we have

$$\|f(x) - (B_n^* f)(x)\|_p \leq C\omega_2^{\varphi}\left(f, \Delta_n^{(\alpha,\beta)}(x)\right)_p$$

Let $g: [0,1] \to R$, $g(t) = \left|t - \frac{1}{2}\right|$ $t \in [0,1]$the function g is a convex function and we can write:

$\|(B_{2n}^* g)(x) - g(x)\|_p > \left\|(B_{2n}^* g)\left(\frac{1}{2}\right)\right\|_p$.

$(B_{2n}^* g)\left(\frac{1}{2}\right) =$

$= g(0)\sum_{k=0}^{n} m_{kn}(1-x)^k + g(1)\sum_{k=0}^{n} m_{kn} + \sum_{i=2}^{n} f\left(\frac{1}{i}\right)q_i(x) +$
$\sum_{2\leq i<j\leq n-1} f\left(\frac{i}{j}\right)h_{i,j}(x)$

$$= \sum_{k=0}^{2n} m_{k,2n}(B_k g)(\tfrac{1}{2}) \geq$$

$$\sum_{k=0}^{2n} m_{k,2n}(B_{2n} g)(\tfrac{1}{2})$$

Because

$$B_{2n} g\left(\frac{1}{2}\right) = \frac{\binom{2n}{n}}{2^{2n+1}} \qquad ,$$

we obtain

$$\frac{\binom{2n}{n}}{2^{2n+1}} \leq C(\frac{1}{2n^{\alpha}} + \frac{1}{n^{\beta}})_p$$

The latter inequality is impossible because

$$\lim_{n\to\infty} \frac{n^{\frac{1}{2}}\binom{2n}{n}}{2^{2n+1}} = \frac{1}{2\sqrt{2}}$$

and on the other hand ,for $\alpha, \beta > \frac{1}{2}$ we have

$$\lim_{n\to\infty} C n^{\frac{1}{2}} \Delta_n^{(\alpha,\beta)}\left(\frac{1}{2}\right) = 0$$

Theorem 2.2.7

For every function $f \in L_p[0,1]$the following estimate is true:

$$\|(H_n f)(x) - f(x)\|_p \le c(p)\, \omega_2^{\varphi}\left(f, \sqrt{x(1-x).}\ \sqrt{1 - \int_0^1 x^2 P_n(x)dx} \right)_p .$$

Proof

$p_n(x) = \int_0^x Q_{n-1}(t)dt$,where Q_{n-1} is a polynomial of degree$\le n-1$, $Q_{n-1} \ge 0$

and $\int_0^1 (1-x)\, Q_{n-1}(x)dx = 1$ by Lemmas 2.2.2

we have

$$1 - \int_0^1 x^2 p_n\,(x)dx = \int_0^1 (1-x)\, Q_{n-1}(x)dx - \int_0^1 x^2 \left(\int_0^x Q_{n-1}(t)dt \right) dx$$

$$= \frac{1}{3} \int_0^1 (1-x)^2 (x+2)\, Q_{n-1}(x)dx$$

$$\le \int_0^1 (1-x)^2\, Q_{n-1}(x)dx .$$

By Lemma 2.2.4 and Lemma 2.2.5 we obtain

$$\|(H_n f)(x) - f(x)\|_p \le c(p)\, \omega_2^{\varphi}\left(f, \sqrt{x(1-x).}\ \sqrt{1 - \int_0^1 x^2 p_n(x)dx} \right)_p$$

Remark 2.2.8 [14]

1. Let $J_n^{(\alpha,\beta)}, \alpha, \beta > -1$ be Jacobi's polynomials of degree n on the interval [0,1]

$$J_n^{(\alpha,\beta)}(x) = R_n^{(\alpha,\beta)}(2x-1) = \frac{p_n^{(\alpha,\beta)}(2x-1)}{p_n^{(\alpha,\beta)}(1)}, \qquad [14]$$

with

$$p_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$$

2 .Let $x_i, i = 1, \dots, n, 0 < x_1 < x_2 < \cdots < x_n < 1$ be the roots of the polynomial $J_n^{(1,0)}$ and

$$p_{2n-1}(x) = \lambda_n \int_0^x \left(\frac{J_n^{(1,0)}(t)}{t-x_n}\right)^2 dt, \text{ where } \lambda_n = \frac{1}{\int_0^1 (1-x)\left(\frac{J_n^{(0,1)}(x)}{x-x_n}\right)^2 dx}.$$

Theorem 2.2.9

If $f \in L_p[0,1]$ and $p_{2n-1}(x) = \sum_{k=0}^{2n-1} a_k x^k$ then for the operator

$H_{2n+1}: L_p \to P_{2n+1}$

$$H_{2n+1}f = \sum_{k=0}^{2n-1} \frac{a_k}{k+1} L_{k+2} f$$

The following estimate is true :

$$\|H_{2n+1}f - f\|_p \le C(P)\omega_2^{\varphi}\left(f, \sqrt{x(1-x)} \,.\, \sqrt{1-x_n}\right)_p .$$

proof

Let

$$\int_0^1 g(t)(1-t)dt = \sum_{k=1}^{n} w_k\, g(x_k) + R(g)$$

$R(g)$ is a remainder.

Be the Gauss quadrature formula relative to the roots of $J_n^{(1,0)}$.

From calculus we obtain that

$R(g_m) = 0$ if $g_m \in P_{2n-1}$ and $w_k > 0$ $, k = 1, \dots \dots, n$

Because

$\lambda_n \left(\frac{J_n^{(1,0)}(t)}{0-x_n}\right)^2$ and $\lambda_n(1-0)\left(\frac{J_n^{(1,0)}(t)}{0-x_n}\right)^2$

we have degrees $\leq 2n-1$, we can write:

$$1 = \lambda_n w_n\, (J_n^{(1,0)'}(x_n))^2$$

$$\int_0^1 (1-x)^2 \;\; \lambda_n \left(\frac{J_n^{(1,0)}(t)}{x-x_n}\right)^2 dx = \lambda_n w_n (1-x_n)\, (J_n^{(1,0)'}(x_n))^2.$$

Then since $\lambda_n w_n\, (J_n^{(1,0)'}(x_n))^2 = 1$ so that

$$\int_0^1 (1-x)^2 \;\; \lambda_n \left(\frac{J_n^{(1,0)}(t)}{x-x_n}\right)^2 dx = 1 - x_n$$

$$= \int_0^1 \lambda_n \left(\frac{J_n^{(1,0)}(t)}{x - x_n} \right)^2 dx - \int_0^1 x^2 \lambda_n \left(\frac{J_n^{(1,0)}(t)}{x - x_n} \right)^2 dx$$

$$< \int_0^1 (1-x)^2 \ \lambda_n \left(\frac{J_n^{(1,0)}(t)}{x - x_n} \right)^2 dx = 1 - x_n$$

$$1 - \int_0^1 x^2 P_{2n-1}(x) dx \le 1 - x_n$$

By Theorem 2.2.7we have

$$\|H_{2n+1} f - f\|_p \le C(P) \omega_2^{\varphi} \left(f, \sqrt{x(1-x)} \, . \sqrt{1 - x_n} \right)_p .$$

Chapter Three

Saturation Problems for Positive Linear Approximation of Function in Quasi Normed Spaces

Many authors work on constrained approximation such as monotonicity ,convexity and k−monotoni ,but little works introduced in positive colinear approximation. The aim of our chapter is to investigate linear and positive approximation for real functions in $L_p[0,1]$. We use the equivalence between K−function and Ditizian Totik modulus of smoothness to prove direct theorems in terms of Ditizian Totik modulus of smoothness and weighted k−function and saturation problem between degree of best positive and linear approximation, and find colinear positive best approximation for measurable function in $L_P[0,1]$ Lebesgue quasi normed spaces

3.1 Introduction

In [20] the authors proved the following direct estimate for Bernstein operator:

$$|f(x) - (B_n f)(x)| \le C\omega^2_{\varphi^\lambda}\left(f, n^{-\frac{1}{2}}, \varphi(x)^{1-\lambda}\right) \qquad x \in I = [0,1] \qquad (3.1)$$

$$\varphi(x) = \sqrt{x(1-x)} \quad , \; x \in [0,1] \; .$$

in which $\varphi: [0,1] \to R$ is an admissible step weight function for details about φ see [21].

If $\lambda = 0$ in (3.1) we get classical local estimate while when λ=1 we get global norm estimate developed by Ditzian and Totik . So (3.1) fill the gap between the local and global approximation theorems for the Bernstein operator. Such result for polynomial approximation for details see [18,19 and 25].

Inequality (3.1) shows that the error $f(x) - (B_n f)(x)$ is bounded by

$C\left(n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\right)^{\alpha}$ if $\omega^2_{\varphi^\lambda}(f,\delta) = 0(\delta^\alpha)$ and $\alpha \in [0,2]$.

In [17,27] the authors proved the converse result also true .

$\omega^2_{\varphi^\lambda}(f,\delta) = 0(\delta^\alpha)$ can be estimated in terms of the Bernstein operator that is the equivalence

$|f(x) - (B_n f)(x)| =$

$$O\left(\left(f, n^{-\frac{1}{2}}, \varphi(x)^{1-\lambda}\right)^{\alpha}\right) \Leftrightarrow \omega^2_{\varphi^\lambda}(f,\delta) = O(\delta^\alpha)$$

holds whenever $\alpha \epsilon (0,2)$ and $\lambda \in [0,1]$.

We mean by $g = O(f)$ that $g \leq C(f)$ for some constant C.

In [21] (3.1 can be considered as a further estimate for the

general estimate

$$|f(x) - (B_n f)(x)| \leq C\omega^2_{\varphi^\lambda}\left(f, n^{-\frac{1}{2}} \frac{\varphi(x)}{\emptyset(x)}\right) \quad . \qquad (3.2)$$

$x \in [0,1]$ where $\varphi: [0,1] \to R$ is a weight function for Ditzian _Totik modulus [20] and $\emptyset^2$ is a concave function . (3.2) improve of (3.1) if $\emptyset$ is replaced by $\varphi^\lambda, \lambda \in [0,1]$.

In [21] the authors proved inverse result to (3.2) for $x \in (0,2)$ $i.e.$

$$|f(x) - (B_n f)(x)| \leq C_1\left(n^{-\frac{1}{2}} \frac{\varphi(x)}{\emptyset(x)}\right)^{\alpha}, x \in [0,1] \quad n = 1,2,3, \ldots ..$$

Implies $\omega^2_{\varphi^\lambda}(f,\delta) \leq C_2(\delta^\alpha)$ if ,in addition $\varphi^2/\emptyset^2$ is concave which is satisfied for $\emptyset = \varphi^\lambda$, $\lambda \in [0,1]$ in particular.

3.2. Auxiliary Results

In this section we give the results that we need in our proofs.

Lemma 3. 2.1 [15]

For $f \in L_p(I)$,$0 < p \leq \infty$ we have

$$\omega_2^{\varphi}(f,\delta)_p \approx K_{\varphi}^2(f,\delta^2)_p$$

Lemma 3.2.2 [15]

Let $f \in L_p[0,1]$, $\|\varphi\, p_n'\|_p \leq cn\, \omega_2^{\varphi}(f,\delta)_p$

Where p_n is the approximation of degree$\leq n$ for f ,$0 < p < 1$. then

$$\omega_{\varphi}(f,\delta)_p \leq c(p) \int_0^c \frac{\omega_{\varphi}^2(f,u)_p}{u}(du)$$

Lemma 3.2.3 [20]

$$\Delta_h^2(f)(x) = \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-h}{2}}^{\frac{h}{2}} f''(x+s+t)\,ds\,dt$$

We can modify a result in[27] to obtain the following Lemma.

Where in [27] ,the author used g_n as a positive function with integral equal to 1 and $\emptyset$ which is any weight function .

Lemma 3.2.4 [27]

$$\int_0^1 \frac{d^2}{dx^2}\, g_n(x,u)du$$

$$\leq c(p)\left(4 + \frac{(\emptyset^2(x))''}{n}\right.$$

$$\left. + 2\left(\frac{n^2}{\emptyset^4(x)} \int_0^1 \left[\frac{(\emptyset^2(x))'}{n} - (u-x)\right](u-x)^3 g_n(x,u)du\right)\right)$$

$$\leq M$$

and M must be an absolute constant independent of n and x.

Lemma 3.2.5 [19]

$$\left|\int_x^t g''(s)(t-s)ds\right| \leq \left\|\emptyset^2 g''\right\|_p \left|\int_x^t \frac{|t-s|}{\emptyset^2(x)}\ ds\right|$$

$$= \left\|\emptyset^2 g''\right\|_p \frac{(t-s)^2}{\emptyset^2(x)}$$

Lemma 3.2.6 [23]

$$(B_n\, g)''(x)$$

$$= n(n-1)\sum_{k=0}^{n-2} P_{n-2,k}\ (x)(\lambda_{n,k+2}\,(g) - 2\lambda_{n,k+1}(g)$$

$$+ \lambda_{n,k}(g))$$

3.3. The Main Results

In this chapter we prove direct estimates for the approximation of function in L_P ,$0 < p < 1$ space using positive linear operator , and for pointwise linear operator . Also we estimate inverse result for positive linear approximation.

Define

$$w_\varphi(f,\delta)_p = \sup_{|h| \le \delta} \left\| f\left(x + \varphi(x)\tfrac{h}{2}\right) - f\left(x - \varphi(x)\tfrac{h}{2}\right)\right\|_{L_{P(I)}}$$

$$I = [0,1]$$

Where $x + \varphi(x)\frac{h}{2}$, $x - \varphi(x)\frac{h}{2} \in I$

$$\widehat{\omega}_\varphi(f,\delta)_p = \sup_{|h| \le \delta} \| f(x + \varphi(x)h) - f(x)\|_{L_{P(I)}}$$

$$x \ , \ x + \varphi(x)h \in I$$

For the equivalence of $w_\varphi(f,\delta)_p$ and $\widehat{\omega}_\varphi(f,\delta)_p$ see [D.T] [9]

Now let us prove our first main result:

Theorem 3.3.1

Suppose $\varphi \epsilon \Omega$ and $\tilde{A} : L_p[0,1] \to L_p[0,1]$ is a bounded positive linear operator preserving linear function ,then for any $f \in L_p[0,1]$ one has

$$\|f - \tilde{A}f\|_p \le C(p)\, K_\varphi^2\left(f, \frac{\check{A}(y)}{\varphi^2}\right)_p$$

Proof:

$\tilde{A} = A + L$

$$L(f)(x) = \begin{cases} -(f(x + (A(.+x)) - f(x) & \text{if } x + A(.+X) \in I \\ f(x + (A(.+x)) - f(x) & \text{if } x + A(.+X) \notin I \end{cases}$$

Assume $A(.+x)(x) \leq \frac{1}{2}$ and A is bounded operator

Let g be a differentiable function in $L_{p[0,1]}$ so we can write

$$g(t) = g(x) + g^{'}(x)(t-x) + \int_x^t g^{''}(s)(t-s)ds \qquad x \in (0,1)$$

$$\|\tilde{A}(f) - f\|_p^p = \|\tilde{A}f - \tilde{A}(g) - g - f + \tilde{A}g + g\|_p^p$$

$$\leq \|\tilde{A}f - \tilde{A}g\|_p^p + \|g - f\|_p^p + \|\tilde{A}g - g\|_p^p$$

$$\leq \|\tilde{A}(f - g)\|_p^p + \|g - f\|_p^p + \|\tilde{A}g - g\|_p^p$$

$$\|\tilde{A}g - g\|_p^p = \left\|\tilde{A}\left(\int_x^y g^{''}(s)(y-s)\,ds\right)\right\|_p^p$$

$$= \left\|\tilde{A}\left(\int_x^y \varphi^2 g^{''}(s)\ \frac{1}{\varphi^2}(y-s)\,ds\right)\right\|_p^p$$

$$\leq \left\|\tilde{A}\left(\int_x^y \varphi^2 g^{''}(s)\ \frac{y}{\varphi^2}\,ds\right)\right\|_p^p$$

$$= \int_0^1 \left|\tilde{A}\left(\int_x^y \varphi^2 g^{''}(s)\ \frac{y}{\varphi^2}\,ds\right)\right|^p dx \quad ,\text{for } p > 1$$

$$\leq \int_x^y \int_0^1 \tilde{A}\left(\left|\varphi^2 g^{''}(s)\right|^p \left|\frac{y}{\varphi^2}\right|^p\right) ds\,dx$$

$$\leq \int_0^1 \left\|\varphi^2 g^{''}(s)\right\|_p^p \left\|\frac{\tilde{A}y}{\varphi^2}\right\|_p^p$$

$$= \|\varphi^2 \ddot{g}(s)\|_p^p \ \left\|\frac{\tilde{A}y}{\varphi^2}\right\|_p^p$$

Now return to

$$\|\tilde{A}(f) - f\|_p^p \le \|\tilde{A}f - \tilde{A}g\|_p^p \ + \|g - f\|_p^p \ + \|\tilde{A}g - g\|_p^p$$

$$\le \|f - g\|_p^p \ + \|g - f\|_p^p + \left\|\varphi^2 \ddot{g}(s)\right\|_p^p \ \left\|\frac{\tilde{A}y}{\varphi^2}\right\|_p^p$$

$$\le 2(\|f - g\|_p^p) + \left\|\varphi^2 \ddot{g}(s)\right\|_p^p \ \left\| \ \frac{\tilde{A}y}{\varphi^2}\right\|_p^p)$$

$$= 2K_\varphi^2 \left(f, \ \frac{\check{A}(y)}{\varphi^2}\right)_p$$

Remark 3.3.2 [9]

Let $\varphi = \sqrt{x(1-x)}$ and$\varphi: [0,1] \to R$ be a step weight function of the Ditzian –Totik modulus with φ^2 concave . B_n be

$$B_n\, f)(x) = \sum_{k=0}^{n} \binom{n}{k} \, x^k \, (1-x)^{n-k} \lambda_{n,k}(f) \qquad (3.3)$$

$\lambda_{n,k}(f) \in L_p[0,1]$ be positive linear function with $\lambda_{n,k}(1) = 1$

for$k = 0,1, \ldots\ldots, n$ If

$$\lambda_{n,0}(f) = f(0) \qquad , \lambda_{n,n}(f) = f(1) \qquad (3.4)$$

and

$$\lambda_{n,k}\left((. - \tfrac{k}{n})^2\right) \le M \left(\tfrac{1}{n}\right)^{2y} \qquad , n \in N \ \ , k = 0, \ldots\ldots, n \qquad (3.5)$$

For constant $M \ge 0$, $y \ge 1$ independent of n and k

$$\sum_{k=0}^{n} p_{n,k}(x) = 1 - x^n - (1-x)^n \qquad (3.6)$$

Theorem 3.3.3

Let $f \in L_p[0,1]$ $,0 < p < 1$ the estimate

$$\|f(x) - (B_n f)(x)\|_p \le c(p)\left(K_\varphi^2\left(f, \frac{2^{\frac{1}{p}}\sqrt{M}}{(P+1)\varphi^2}\right)_p + \omega_2^\varphi\left(f, \frac{2^{\frac{1}{p}}\sqrt{M}}{(P+1)\varphi^2}\right)_p\right)$$

where $c(p)$ is a positive constant depending p.

Proof:

From the Cauchy –Schwarz inequality and (3.5)

$$\lambda_{n,k}\left(x - \frac{k}{n}\right) \le \sqrt{\lambda_{n,k}\left(\left(x - \frac{k}{n}\right)^2\right)} \le \frac{\sqrt{M}}{n^y} \qquad (3.7)$$

$$\|B_n(s-x)(x)\|_p = \left\|\sum_{k=0}^{n} p_{n,k}(x)\lambda_{n,k}(s-x)\right\|_p$$

$$\le \left(\int_0^1 \left|\sum_{k=0}^{n} p_{n,k}(x)\lambda_{n,k}(s-x)\right|^p dx\right)^{\frac{1}{p}}$$

$$\le \left(\int_0^1 \left(\sum_{k=0}^{n} |p_{n,k}(x)|\ |\lambda_{n,k}(s-x)|\right)^p dx\right)^{\frac{1}{p}}$$

Using (3.7) we get

$$\|B_n(s-x)(x)\|_p \le \left(\int_0^1 \left(\sum_{k=0}^{n} p_{n,k}(x)\frac{\sqrt{M}}{n^y}\right)^p dx\right)^{\frac{1}{p}}$$

$$= \frac{\sqrt{M}}{n^y}\left(\int_0^1 |1-x^n-(1-x)^n|^p dx\right)^{\frac{1}{p}}$$

$$\le \frac{\sqrt{M}}{n^y}\left(\int_0^1 (|1-x^n|^p+|1-x|^{np})\, dx\right)^{\frac{1}{p}}$$

$$\le \frac{\sqrt{M}}{n^y}\left(\int_0^1 (|nx|^p+|nx|^p)\, dx\right)^{\frac{1}{p}}$$

$$= \frac{2^{\frac{1}{p}}\sqrt{M}}{n^y}\, n\left(\int_0^1 x^p\, dx\right)$$

$$= \frac{2^{\frac{1}{p}}\sqrt{M}}{n^{y-1}}\ \frac{x^{p+1}}{p+1}\ \begin{matrix}0\\ |\\ 1\end{matrix}$$

$$= \frac{2^{\frac{1}{p}}\sqrt{M}}{(p+1)\ n^{y-1}}$$

Since $y \ge 1$, then

$$\|B_n(s-x)(x)\|_p \le \frac{2^{\frac{1}{p}}\sqrt{M}}{(p+1)}$$

so B_n is bounded positive linear operator ,then by Theorem 3.3.1 we obtain

$$\|f-B_n(f)\|_p \le c(p)K_\varphi^2\left(f,\frac{2^{\frac{1}{p}}\sqrt{M}}{(p+1)\ \varphi^2}\right)_p$$

$$\leq c(p)\left(K_{\varphi}^{2}\left(f,\frac{2^{\frac{1}{p}}\sqrt{M}}{(P+1)\varphi^{2}}\right)_{p}+\omega_{2}^{\varphi}\left(f,\frac{2^{\frac{1}{p}}\sqrt{M}}{(P+1)\varphi^{2}}\right)_{p}\right).$$

Theorem 3.3.4

$f \in L_p[0,1]$ and $\delta > 0$, $0 < p < 1$ then

$$\omega_2^{\varphi}(f,\delta)_p \leq c(p)\delta^{\alpha}$$

implies

$$\|f(x)-(B_n f)(x)\|_p \leq c(p)\,\omega_2^{\varphi}\left(f,\frac{2^{\frac{1}{p}}\sqrt{M}}{(P+1)\varphi^{2}}\right)_{p}$$

Proof:

Using Lemma 3.2.2 we obtain

$$\omega_{\varphi}(f,\delta)_p \leq c(p)\int_0^c \frac{\omega_2^{\varphi}(f,u)_p}{u}(du),$$

where $c > 0$ is fixed constant .

In our hypothesis we have $\omega_2^{\varphi}(f,\delta)_p = O(\delta^{\alpha})$.

then by Theorem 3.3.3

$$\|f(x)-(B_n f)(x)\|_p \leq c(p)\left(K_{\varphi}^{2}\left(f,\frac{2^{\frac{1}{p}}\sqrt{M}}{(P+1)\varphi^{2}}\right)_{p}+\omega_{\varphi}^{2}\left(f,\frac{2^{\frac{1}{p}}\sqrt{M}}{(P+1)\varphi^{2}}\right)_{p}\right)$$

Using Lemma 3.2.1 we get

$$\leq c(p)\,\omega_2^{\varphi}\left(f,\frac{2^{\frac{1}{p}}\sqrt{M}}{(P+1)\varphi^{2}}\right)_{p}$$

Theorem 3.3.5

Let $f \in L_p[0,1]$ and $\emptyset: [0,1] \to R$ be step function let

$A_n: L_{P[0,1]} \to L_{P[0,1]}$,$n \in N$ be bounded positive linear operators so that

$$\left\|(A_n f)''(x)\right\|_p \leq c(p)\frac{n}{\varphi^2(x)}\|f\|_p \quad , \tag{3.8}$$

for $x \in [0,1]$, $f \in L_p[0,1]$ and

$$\left\|\emptyset^2 (A_n g)''\right\|_p \leq c(p)\left\|\emptyset^2 g''\right\|_p \quad , \tag{3.9}$$

where $\emptyset^2, \varphi^2$ and $\varphi^2/\emptyset^2$ are concave functions on $[0,1]$ and $\alpha \in (0,2)$ Then for $f \in L_p[0,1]$,the pointwise approximation

$$\|f-(A_n f)(x)\|_p \leq c(p)\left(n^{-\frac{1}{2}}\frac{\varphi(x)}{\emptyset(x)}\right)^{\alpha} \qquad n=1,2,\ldots \tag{3.10}$$

implies

$$\omega_2^{\varphi}(f,\delta)_p \leq c(p)\delta^{\alpha} \qquad \delta>0.$$

Proof:

Let $x, h \in [0,1]$ so that $x+h \in [0,1]$ and let

$$(\Delta_h^2 f)(x) = f(x+h) - 2f(x) + f(x-h) .$$

and

$$\left\|(\Delta_h^2 f)(x)\right\|_p = \left\|\Delta_h^2 (f - A_n + A_n)(x)\right\|_p$$

$$= \left\|\left((\Delta_h^2 (f - A_n f)\right)(x) + (\Delta_h^2 A_n f)(x)\right\|_p$$

$$\leq 2^{\frac{1}{p}}\left(\int_0^1\left|(\Delta_h^2 (f-A_n f)(x)\right|^p\right)^{\frac{1}{p}} + 2^{\frac{1}{p}}\left(\int_0^1\left|(\Delta_h^2 A_n f)(x)\right|^p\right)^{\frac{1}{p}}. \tag{3.11}$$

$$\left\|\left((\Delta_h^2 (f - A_n f)\right)(x)\right\|_p$$

$$= \|(f - A_n)(x + h) - 2(f - A_n)(x) + (f - A_n)(x - h)\|_p$$

$$\leq 2^{\frac{1}{p}-1} \left(\|(f - A_n)(x + h)\|_p + 2\|(f - A_n)(x)\|_p + \|(f - A_n)(x - h)\|_p\right).$$

Using (3.10) we obtain

$$\left\|\left((\Delta_h^2 (f - A_n f)\right)(x)\right\|_p$$

$$\leq 2^{\frac{1}{p}-1} \left(\left(\frac{\varphi(x+h)}{\emptyset(x+h)}\right)^{\alpha} + 2\left(\frac{\varphi(x)}{\emptyset(x)}\right)^{\alpha} + \left(\frac{\varphi(x-h)}{\emptyset(x-h)}\right)^{\alpha}\right) n^{-\frac{\alpha}{2}}$$

$$\leq c(p) n^{-\frac{\alpha}{2}} \left(\frac{\varphi(x)}{\emptyset(x)}\right)^{\alpha} . \qquad (3.12)$$

Using Lemma 3.2.1 $\omega_2^{\varphi}(f,\delta)_p$ and $K_{\varphi}^2(f,\delta^2)_p$ are equivalent we can choose $g = g_\delta$, $\delta \geq 0$

$$c_2 \ K_{\varphi}^2(f,\delta^2)_p \leq \omega_2^{\varphi}(f,\delta)_p \leq c_1 \ K_{\varphi}^2(f,\delta^2)_p$$

Where c_1 and c_2 are absolute constants.

$$K_{\varphi}^2(f,\delta^2)_p = \|f - g\|_p + \delta^2 \left\|\emptyset^2 g''\right\|_p \leq c\omega_2^{\varphi}(f,\delta)_p$$

$$\|f - g\|_p \leq A\omega_2^{\varphi}(f,\delta)_p \quad , \quad \left\|\emptyset^2 g''\right\|_p \leq B\, \delta^{-2} \omega_2^{\varphi}(f,\delta)_p .$$

Using from (3.8) and (3.9) we obtain

$$\left\|(A_n f)''(y)\right\|_p \leq \left\|A_n''(f - g)(y) + (A_n'' g)(y)\right\|_p$$

$$\leq 2^{\frac{1}{p}} \left(\int_0^1 \left|A_n''(f - g)(y)\right|^p\right)^{\frac{1}{p}} + 2^{\frac{1}{p}} \left(\int_0^1 \left|(A_n'' g)(y)\right|^p\right)^{\frac{1}{p}}$$

$$\leq c(p)\ \frac{n}{\varphi^2(y)}\|f-g\|_p + c(p)\ \frac{1}{\emptyset^2(y)}\quad \|\emptyset^2 g''\|_p$$

$$= c(p)\frac{n}{\varphi^2(y)}\|f-g\|_p + c(p)\ \frac{\delta^{-2}\ \delta^2}{\emptyset^2(y)}\quad \|\emptyset^2 g''\|_p$$

$$\leq c(p)\ \left(\frac{n}{\varphi^2(y)} + \frac{1}{\emptyset^2(y)\delta^2}\right)\ K_\varphi^2(f,\delta^2)_p$$

$$= c(p)\ \left(\frac{n}{\varphi^2(y)} + \frac{1}{\emptyset^2(y)\delta^2}\right)\omega_2^\varphi(f,\delta)_p\ . \qquad (3.13)$$

Using Lemma 3.2.3 ,we obtain

$$\left\|(\Delta_h^2\ A_n f)(x)\right\|_p = \left\|\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}(A_n'' f)(x+s+t)dsdt\right\|_p .$$

Since A_n is bounded operators

$$= \left(\int_0^1 \left|\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}(A_n'' f)(x+s+t)dsdt\right|^p dx\right)^{\frac{1}{p}}$$

$$\left\|(\Delta_h^2\ A_n f)(x)\right\|_p \leq c\left(\int_0^1 \left|h^2\ A_n''(f)(x)\right|^p\right)^{\frac{1}{p}},$$

then using (3.13) we obtain

$$\left\|(\Delta_h^2\ A_n f)(x)\right\|_p \leq c(p)h^2\ \left(\frac{n}{\varphi^2(x)} + \frac{1}{\delta^2\emptyset^2(x)}\right)\quad \omega_2^\varphi(f,\delta)_p.$$

Using (3.12) we get

$$\left\|(\Delta_h^2\ f)(x)\right\|_p \leq\ c(p)\left(\frac{\varphi(x)}{\emptyset(x)}\right)^\alpha n^{-\frac{\alpha}{2}} + \left(\frac{nh^2}{\varphi^2(x)} + \frac{h^2}{\delta^2\emptyset^2(x)}\right)\quad \omega_2^\varphi(f,\delta)_p.$$

This implies

$$\omega_2^{\varphi}(f,\delta)_p \le c(p)\left(\delta^{\frac{\alpha}{2}} + (\delta + \frac{h^2}{\delta^2})\right)$$

$$\le c(p)\,\delta^{\alpha}\,.$$

Notations and Definitions 3.3.6

Define the sequence of operators:

$$Q_n\,(f(x)) = \int_a^b g_n(u,x)f(u)du \qquad ,a,b \in R,\;\; n \in N \qquad (3.14)$$

Such that

$g_n(u,x)$ is positive for any $x \in [a,b]$

and

$\int_a^b g_n(u,x)\,du = 1$ for any $n \in N$

Also we need to define

$$\frac{d}{dx}\,g_n(u,x) = \frac{n}{\varphi^2}\,g_n(u,x)\,(u-x) \qquad ,n \in N$$

where φ is a step weight function .

we use $L_P^2[0,1] = \{f\colon [a,b] \to R : f, f'' \in L_p[0,1]\}$ which is called 2−fold L_p space.

Proposition 3.3.7

Lf $g \in L_P^2[0,1]$, then for every function $\emptyset : [0,1] \to R$ there exists $M > 0$

$$\left\|\emptyset^2 (G_n g)''\right\|_p \leq M \left\|\emptyset^2 g''\right\|_p \quad , n \in N$$

Where M is an absolute constant.

Proof:

By Taylor's formula we have

$(G_n g)''$

$$= \int_0^1 \left[\frac{d^2}{dx^2} g_n(u,x)\right] \left(g(x) + g'(x)(u-x) + \int_x^u (u-s) g''(x) ds\right) du$$

$$= \int_0^1 \left[\frac{d^2}{dx^2} g_n(u,x)\right] \left(g(x) + g'(x)(u-x)\right) du + \int_0^1 \left[\frac{d^2}{dx^2} g_n(u,x)\right] \left(\int_x^u (u-s) g''(x) ds\right) du$$

Because

$$\int_0^1 \left[\frac{d^2}{dx^2} g_n(u,x)\right] (u-x)^i du = 0 \qquad for \ i = 0,1$$

we obtain

$$(G_n g)'' = \int_0^1 \left[\frac{d^2}{dx^2} g_n(u,x)\right] \int_x^u (u-s) g''(x) ds \ \ du.$$

$$\left\|\emptyset^2 (G_n g)''\right\|_p = \left\|\emptyset^2(x) \int_0^1 \frac{d^2}{dx^2} g_n(u,x) \int_x^u (u-s) g''(x) ds du \right\|_P$$

$$= \left\| \emptyset^2(x) \int_0^1 \frac{d^2}{dx^2} g_n(u,x) \int_x^u (u-s) g''(x) ds du \quad \right\|_p$$

$$\leq \left(\int_0^1 \left(|\emptyset^2(x)| \int_0^1 \left| \frac{d^2}{dx^2} g_n(u,x) \right| \left| \int_x^u (u-s) g''(x) ds \right| du \right)^p dx \right)^{1/p}$$

By using Lemma3.2.5 we get

$$\left\| \emptyset^2 (\mathrm{G_n g})'' \right\|_{\mathrm{p}}$$

$$\leq \left(\int_0^1 \left(|\emptyset^2(x)| \int_0^1 \left| \frac{d^2}{dx^2} g_n(u,x) \right| \left| \emptyset^2(x) g''(x) \frac{(u-s)^2}{\emptyset^2(x)} \right| du \right)^p dx \right)^{1/p}$$

$$\leq \left(\int_0^1 \left(\left| \emptyset^2(x)\, g''(x) \right| \int_0^1 \left| \frac{d^2}{dx^2} g_n(u,x) \right| \; |(u-s)^2 \;| du \right)^p dx \right)^{1/p}$$

Using Lemma 3.2.4 and bounded -ness of g'', we obtain

$$\left\| \emptyset^2 (\mathrm{G_n g})'' \right\|_{\mathrm{p}} \leq M \left(\int_0^1 \left(\left| \emptyset^2(x) g''(x) \right| \right)^p dx \right)^{1/p}$$

$$= M \left\| \emptyset^2 g'' \right\|_p$$

Theorem 3.3.8

Let G_n , $n \in N$,be exponential type operator

$$(G_n f) = \int_0^1 g_n(u,x) f(u) du \qquad , n \in N$$

If $\emptyset^2, \varphi^2$ and $\frac{\varphi^2}{\emptyset^2}$ are concave functions on [0,1] then for $f \in L_p[0,1]$

and $0 < \alpha < 2$ the statements

1) $\|f(x) - (G_n\, f)(x)\|_p = O\left(\left(n^{-1/2}\ \frac{\varphi(x)}{\emptyset(x)}\right)^{\alpha}\right)$
2) $\omega_2^{\varphi}(f, \delta)_P = O(\delta^{\alpha}) \ \ , \delta > 0,$

are equivalent .

Proof:

Using Theorem 3.3.5

$$\left\|(G''_n f)(x)\ \right\|_p \le c(p) \frac{n}{\emptyset^2(x)} \ \|f\|_p.$$

By theorem 3.3.5 and proposition 3.3.7 we get

1)Implies (2) and (2) implies (1)

Notations and Definitions 3.3.9

Let $\emptyset(x) = \sqrt{x(1-x)}$ and

$$(B_n f)(x) = \sum_{k=0}^{n} \lambda_{n,k}(f) \binom{n}{k} x^k (1-x)^{n-k} \qquad , x \in [0,1]$$

$\lambda_{n,k}(f) \in L_p[0,1]$ positive linear functions with $\lambda_{n,k}(1) = 1$,for

$k = 0,1, \ldots, n$ and , $B_n(ax+b) = ax+b$.

Theorem 3.3.10

If $g \in L_p^2[0,1]$ and

$$\lambda_{n,k}\left(\left(s-\frac{k}{n}\right)^2\right) \le M\left(\frac{1}{n}\right)^2, \qquad n \in N, \qquad s \in [0,1],$$

$$k = 0, \ldots, n \qquad (3.15)$$

For any constant $M \ge 0$ independent of n and k ,then

$$\left\|\emptyset^2 (B_n\, g)''\right\|_p \le c(p) \left\|\emptyset^2 g''\right\|_p \ , \quad n \in N,$$

and $\emptyset: [0,1] \to R$ with $\emptyset^2$ concave.

Proof

Let $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, then

$$(B_n\, f)' = \sum_{k=0}^{n} \lambda_{n,k}(f) \binom{n}{k} \left(x^k (n-k)(1-x)^{n-k-1} + k(1-x)^{n-k} x^{k-1}\right)$$

$$(B_n f)'' = \sum_{k=0}^{n} \lambda_{n,k}(f)\binom{n}{k}\Big((x^k (n-k)(n-k-1)(1-x)^{n-2}$$
$$+ k(n-k)(1-x)^{n-k-1}x^{k-1})$$
$$+ (k + (1-x)^{n-k}(k-1)x^{k-2} + kx^{k-1}(k-n)(1-x)^{n-k-1})\Big)$$

Assume $n \geq 2$ we get

$p'_{n,k}(x) = n\big(P_{n-1,k-1}(x) - P_{n-1,k}(x)\big)$ [23]

Using Lemma 3.2.6 we obtain

$$(B_n g)''(x) = n(n-1)\sum_{k=0}^{n-2} P_{n-2,k}(x)(\lambda_{n,k+2}(g) - 2\lambda_{n,k+1}(g)$$
$$+ \lambda_{n,k}(g)) \quad (3.16)$$

The fact $\lambda_{n,k+2}(f) - 2\lambda_{n,k+1}(f) + \lambda_{n,k}(f) = 0$ for $f(x) = 1$

and $f(x) = x$ by Taylors expansion

$$g(t) = g\left(\frac{k+1}{n}\right) + g'\left(\frac{k+1}{n}\right)\left(t - \frac{k+1}{n}\right) + \int_{(k+1)/n}^{t} (t-s)g''(s)ds$$

$(B_n g)''(x)$

$$= n(n-1)\sum_{k=0}^{n-2} P_{n-2,k}(x)\,(\lambda_{n,k+2} - 2\lambda_{n,k+1}$$
$$+ \lambda_{n,k})\left(\int_{(k+1)/n}^{t} (t-s)g''(s)ds\right)$$

By using Lemma 3.2.5

$$\left\|\emptyset^2(x)(B_n g)''(x)\right\|_p \le$$

$$\left\|\emptyset^2(x)\left(\emptyset^2 g''\right)n(n-1)\sum_{k=0}^{n-2} P_{n-2,k}(x)\,(\lambda_{n,k+2}-2\lambda_{n,k+1} + \lambda_{n,k})\left(\int_{\frac{k+1}{n}}^{t}\frac{|t-s|}{\emptyset^2(x)}\,ds\right)\right\|_p$$

$$\le \left\|\sum_{k=0}^{n-2}\frac{\emptyset^2(x)}{\emptyset^2\left(\frac{k+1}{n}\right)}\left|\left(\emptyset^2 g''\right)\right|n(n-1)\sum_{k=0}^{n-2} P_{n-2,k}(x)\,(\lambda_{n,k+2}-2\lambda_{n,k+1} + \lambda_{n,k})\left(t-\frac{k+1}{n}\right)^2\right\|_p \qquad (3.17)$$

By using (3.15)

$$\left(\lambda_{n,k+2}+2\lambda_{n,k+1}+\lambda_{n,k}\right)\left(t-\frac{k+1}{n}\right)^2$$

$$= \lambda_{n,k+2}\left(t-\frac{k+1}{n}\right)^2 + 2\lambda_{n,k+1}\left(t-\frac{k+1}{n}\right)^2 + \lambda_{n,k}\left(t-\frac{k+1}{n}\right)^2$$

$$\le \lambda_{n,k+2}\left(t-\frac{k+2-1}{n}\right)^2 + 2\lambda_{n,k+1}\left(t-\frac{k+1}{n}\right)^2 + \lambda_{n,k}\left(t-\frac{k}{n}\right)^2$$

$$= \lambda_{n,k+2}\left(t-\frac{k+2}{n}+\frac{1}{n}\right)^2 + 2\lambda_{n,k+1}\left(t-\frac{k+1}{n}\right)^2 + \lambda_{n,k}\left(t-\frac{k}{n}\right)^2$$

$$\le 2\lambda_{n,k+2}\left(t-\frac{k+2}{n}\right)^2 + 2\lambda_{n,k+2}\frac{1}{n^2} + 2\lambda_{n,k+1}\left(t-\frac{k+1}{n}\right)^2 + \lambda_{n,k}\left(t-\frac{k}{n}\right)^2$$

By using (3.15) and the bounded ness of $\lambda_{n,k+2}$ we get $c > 0$ such that

$$\left(\lambda_{n,k+2}+2\lambda_{n,k+1}+\lambda_{n,k}\right)\left(t-\frac{k+1}{n}\right)^2 \le 2\frac{M}{n^2}+2\frac{c}{n^2}+2\frac{M}{n^2}+\frac{M}{n^2}$$

$$\leq \frac{2(M+c)}{n^2}. \tag{3.18}$$

Let us estimate $\frac{\emptyset^2(x)}{\emptyset^2\left(\frac{k+1}{n}\right)}$ in (3.16) and let $a \in [0,1]$, we have

$$\frac{\emptyset^2(x)}{\emptyset^2(a)} \leq \frac{1-x}{1-a} \qquad x \in [0,a] \tag{3.19}$$

$$\frac{\emptyset^2(x)}{\emptyset^2(a)} \leq \frac{x}{a} \qquad x \in [a,1] \tag{3.20}$$

If $x \in [0,\frac{k+1}{n}]$ then using (3.19) and

$$\frac{\emptyset^2(x)}{\emptyset^2\left(\frac{(k+1)}{n}\right)} P_{n-2,k}(x) \leq \frac{1-x}{1-\frac{k+1}{n}} P_{n-2,k}(x)$$

$$= \frac{n}{n-1-k} \cdot \frac{(n-2)!}{k!\,(n-2-k)!} x^k (1-x)^{n-1-k}$$

$$\leq 2\frac{(n-1)!}{k!(n-1-k)!} x^k (1-x)^{n-1-k}$$

$$= 2P_{n-1,k}(x) \quad for \quad n \geq 2 \ . \tag{3.21}$$

If $x \in [\frac{k+1}{n},1]$ by using (3.20), to obtain

$$\frac{\emptyset^2(x)}{\emptyset^2\left(\frac{(k+1)}{n}\right)} P_{n-2,k}(x) \leq \frac{x}{\frac{k+1}{n}} P_{n-2,k}(x)$$

$$= \frac{n}{k+1} \cdot \frac{(n-2)!}{k!\,(n-2-k)!} x^{k+1} (1-x)^{n-2-k}$$

$$\leq 2\frac{(n-1)!}{k+1)!\,(n-2-k)!} x^{k+1} (1-x)^{n-2-k}$$

$$= 2P_{n-1,k+1}(x) \quad for \quad n \geq 2 \ . \tag{3.22}$$

$\left\|\emptyset^2(x)(B_n g)''(x)\right\|_p \leq$

$\left\|\sum_{k=0}^{n-2} \frac{\emptyset^2(x)}{\emptyset^2\left(\frac{k+1}{n}\right)} \left|\left(\emptyset^2 g''\right)\right| n(n-1) \sum_{k=0}^{n-2} P_{n-2,k}(x)\left(\lambda_{n,k+2} - 2\lambda_{n,k+1} + \lambda_{n,k}\right)\left(t - \frac{k+1}{n}\right)^2\right\|_p$

By using (3.18) we obtain

$$\left\|\emptyset^2(x)(B_n g)''(x)\right\|_p \leq \frac{2(M+c)}{n^2}\left\|\sum_{k=0}^{n-2} \frac{\emptyset^2(x)}{\emptyset^2\left(\frac{k+1}{n}\right)} \left|\left(\emptyset^2 g''\right)\right| n(n-1) \sum_{k=0}^{n-2} P_{n-2,k}(x)\right\|_p$$

By using (3.21) and (3.22)

$$\left\|\emptyset^2(x)(B_n g)''(x)\right\|_p \leq \frac{2(M+c)}{n^2}\, n(n-1)\left\|\left|\left(\emptyset^2 g''\right)\right| \sum_{k=0}^{n-2} 2\left(P_{n-1,k}(x) + P_{n-1,k+1}(x)\right)\right\|_p$$

$$\leq C(P)\ \left\|\emptyset^2 g''\right\|_p$$

Future Works

In this thesis we deal with discrete operators , as a future work idea we can define a sequence of positive linear operators that is continuous and, study the constrained approximation for a target function in a quasi-normed space using this sequence of operators in terms of weighted moduli of higher orders ,say w_k , for $k \geq 2$.

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المستخلص

قدم العديد من المشتغلين بالتقريب اعمال كثيرة حول التقريب الحافظ للرتابة والتحدب والk−رتابة. قليل منهم قدم حول التقريب الموجب الحافظ لخطيه الدالة . الهدف من عملنا هو تقديم انواع من التحويلات الخطية الموجبة والمتغيرة الاشارة ودراسة تقريبيها الحافظ للشكل في الفضاءL_p عندما$0 < p < 1$.

قدمنا نوعا من المبرهنة المباشرة لجاكسون للتقريب الحافظ للإشارة والخطية للدوال في فضاءات $L_p[0,1]$ عندما $0 < p < 1$ ، وكانت هذه المبرهنة بدلالة مقياس النعومة من الدرجة الثانية .

كذلك استخدمنا متعددة حدود برنشتاين لتحويل او متعددة حدود خطيه تحفظ اشارة الدالة ، واستخدمنا تقريب الدوال في الفضاءات $L_p[0,1]$ عندما $0 < p < 1$.
وقمنا بتقوية عملنا بمبرهنة سالبة .

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بابل
كلية التربية للعلوم الصرفة
قسم الرياضيات

التقريب الحافظ للخطية

رسالة

مقدمة الى مجلس كلية التربية للعلوم الصرفة / جامعة بابل وهي جزء من متطلبات نيل درجة الماجستير في التربية / الرياضيات

من قبل الطالبة

ندى صادق عباس حسون

بإشراف

أ.د. أيمان سمير بهيه

1444 هـ 2022 م