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**Ministry of Higher Education**
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**University of Babylon**
**College of Education for Pure Sciences**

# Extending Semimodules andSome of Their Generalizations

A Dissertation
Submitted to the Council of College of Education forPure Sciences/University of Babylon in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Education / Mathematics.

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# Dedication

***To the soul of my Father*** …… my role model, and my ideal in life; The person who taught me how to live with dignity and glory, may God have mercy on him.

***To my loving mother*** …… I can't find words that can give her right, she is the epic of love and the joy of life, and an example of dedication and giving.

***To my husband*** …… the highest symbols of sincerity, loyalty, and companion on the way.

***To my brothers and sister***…… my support and source of strength in this life.

***To my children*** …… to the souls of my liver.

***To all comrades*** …… I dedicate my PhD thesis to you.

## Acknowledgments

In the name of ALLAH, Most Gracious, Most Merciful:

"**Glory be to Thee! We have no knowledge but that which Thou hast taught us; surely Thou art the Knowing, the Wise**".

(The Holy Qur'an - (Surah Al Baqarah 2:32))

Peace is upon **Muhammad S.A.W.**, and the pure Imams, the messengers who have been sent to guide people truly. All praises and thanks go to almighty **ALLAH** for giving me the patience, the health, and the guidance in completing this work successfully. All praise and glory be to Allah for granting me health, strength, and knowledge to attain this stage of my life journey. Favors and mercy of preserving me are undeniable. I want to thank so many wonderful and talented people for making my time at the University of Babylon possible and the chance to do my study at the Department of Mathematics in the College of Education for Pure Sciences.

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*Samah A. Alhashemi*

*2022*

## Declaration Associated with this Thesis

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v. Samah Alhashemi and Asaad M. A.Alhossaini, "$C_{11}$SemimoduleoverSemiring". **(Submitted)**

# Extending Semimodules over Semirings

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**Abstract.** The objective of our research paper is to introduce as well as to study many essential properties of the concept of extending semimodules. A semimodule S is named extending (CS) if every subsemimodule of S is essential in a direct summand of S. Therefore, extending semimodule behaviour with respect to direct sums and direct summands are examined. Moreover, studying some properties of these semimodules concepts, e.g., every direct summand of a CS-semimodule is a CS-semimodule. While the direct sum of extending semimodules is not necessarily extending.

## 1. Introduction

The A $T$-module $S$ is called an extending module ($CS$–module) based on the extending property as follows: for each submodule $X$ of $S$, there exists a direct summand $N$ of $S$, which is an essential extension of $X$. It is known that a complement submodule need not be a summand, in the class of $CS$–modules any complement is a summand. Originality of $CS$-modules was presented by Von Neumann in 1930 [1]. In 1960, Utumi has studied this condition (identifying it as $C_1$ condition) in his study on self-injective and continuous ring [2]. In fact $C_1$ condition is common generalization of the injective and the semi simple condition, this motivates the name of extending condition. Another name of this condition is $CS$ condition. it has developed in many articles and in at least [3][4][5].
In recent years, the extending modules theory has come to represent an important role and generally major contributions to this theory, through its widely available interesting findings on expanding properties in the theoretical preparation of the module. For background and applications of extending module (see [6]).

In this work, the extending semimodule over a semiring will be introduced and investigated. A semiring can be defined as a set $T$, which is non-empty together with two binary operations multiplication (.) and addition (+) ; as mentioned that $(T, .)$ is a monoid with an identity element $1 \neq 0$; $(T, +)$ is a commutative monoid with identity element 0; $t0 = 0t = 0$ for all $t \in T$; $a_1(a_2 + a_3) = a_1a_2 + a_1a_3$ and $(a_2 + a_3)a_1 = a_2a_1 + a_3a_1$; for all $a_1, a_2, a_3 \in T$. The semiring $T$ is commutative if the monoid $(T, .)$ is commutative [7]. Let $(S, +)$ be an additive abelian monoid with additive identity $0_S$. Then $S$ is named a left T-semimodule if there exists a scalar multiplication $T \times S \to S$ defined by $(t, x) \mapsto tx$, such that $t(x + y) = tx + ty$; $(ts)x = t(sx)$; $(t + s)x = tx + sy$ ; $0_T S = t0_S = 0_S$ for all $x, y \in S$ and for all $t, s \in T$ [7].

Alhashemi and Alhossaini

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# FI-Extending Semimodule and Singularity

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**Abstract**

The main aim of this research is to present and to study several basic characteristics of the idea of FI-extending semimodules. The semimodule $F$ is said to be an FI-extending semimodule if each fully invariant subsemimodule of $F$ is essential in direct summand of $F$. The behavior of the FI-extending semimodule with respect to direct summands as well as the direct sum is considered. In addition, the relationship between the singularity and FI-extending semimodule has been studied and investigated. Finally extending propertywhich is stronger than FI extending, that has some results related to FI-extending and singularity is also investigated.

**Keywords:** Semimodules, Fully Invariant Subsemimodule, FI-Extending Semimodule, Extending Semimodule, Singular Semimodule, Nonsingular Semimodule.

## شبه مقاس التوسع من النمط FI و خاصية الشذوذ

**سماح الهاشمي[1]*, اسعد محمد علي الحسيني[2]**

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الخلاصة

الهدف الرئيسي من البحث هو تقديم ودراسة عدة خصائص لمفهوم شبه مقاس التوسع من النمط FI . شبه المقاس F يسمى شبه مقاس التوسع من النمط FI اذا كان كل شبه مقاس جزئي تام الثبات يكون اساس في مركبة جداء مباشر ل F .كذلك تم دراسة سلوك شبه مقاس التوسع من النمط FI بالنسبة الى مركبة الجداء المباشر و الجداء المباشر . اضافه الى ذلك تم دراسة العلاقة بين الشذوذ و شبه مقاس التوسع من النمط FI .اخيرا درسنا العلاقه بين شبه مقاس التوسع وهذا النوع من شبه المقاسات مع خاصية الشذوذ.

## 1. Introduction

The originality of CS-modules is given by Von Neumann in 1930 [1]. In [2], Utumi in 1960 had identified and studied modules with a C1 condition in his research on the continuous and self-injective rings. The C1 condition is a common generalization of the

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Extending semimodules and singularity

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# T-Extending Semimodule over Semiring

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Samah Alhashemi ; Asaad M.A. Alhossaini All Authors

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Abstract

Document Sections

I. Introduction

II. T-Essential and T-Closed Subsemimodule

III. T-Extending Semimodule

IV. Conclusion

Authors

References

Keywords

Metrics

**Abstract:**

Themain aim of this research is to present and study several basic characteristics of the idea of t-extending semimodules. The semimodule $F$ is said to be a t-extending semimodule if each t-closed sub-semimodule of $F$ is t-essential in a direct summand of $F$. Hence, the behavior of the t-extending semimodule is considered. In addition, the relationship between the t-essential (t-closed) and essential (closed) has been studied and investigated as well. Finally, in this work, there are a number of results related to the t-extending property, which is one of the generalizations of extending property, (every extending is t-extending, while the converse is not true).

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I. Introduction

In this work, the t-extending semimodule over a semiring will be introduced and investigated. Throughout this paper, will denote a commutative semiring with identity, and is a left R-semimodule. A semiring is a non-empty set with two operations of addition (+) and multiplication such that , +) is a commutative ... monoid with identity element for all ; and for every . We say that is a commutative semiring if the monoid is commutative [1]. ... be an additive abelian monoid with additive identity , then is called a left R-semimodule if there exists a scalar multiplication denoted by , such that and for all and all [2]

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# Abstract

Previously, the concept of extending module over ring was studied by some authors. This work will present and study extending semimodule over semiring and investigate the conditions that would be required to obtain properties and characteristicss similar to their counterparts in module theory. Let *R* be a semiring with identity element $1\neq 0$ and *S* a left unitary semimodule, so we say that the semimodule is extending (CS) if every subsemimodule is essential in a direct summand. The main objective of this thesis is to give a comprehensive description and study the basic properties of the extending concept and to discuss the relationship between this concept and the singularity property. In addition to presenting and studying several basic characteristics of the generalizations of extending semimodules" FI-extending semimodule, T-extending semimodule, $C_{11}$-semimodule" parallel to the cases in modules and arriving at some new results for these types of semimodules.

A semimodule *S* is said to be an FI-extending semimodule if each fully invariant subsemimodule of *S* is essential in a summand of *S*, while a semimodule *S* is said to be t-extending if every t-closed subsemimodule of *S* is a summand. Howover, a semimodule*S* is said to satisfy $C_{11}$condition if each subsemimodule of *S* has a complement that is a summand of *S*.It is proved that the CS *R*–semimodule is FI-CS but the converse is not true as well as the direct sum of two CS *R*–semimodules (or FI-CS semimodules) is FI-CS. It is also proved, where *S* is a duo semimodule then *S* is CS if and only if *S* is FI-CS. More results were obtained like, every CS *R*–semimodule is t-extending; every summand of t-extending semimodule is t-extending; and every direct sum of two t-extending semimodules(orCS *R*–semimodule) is t-extending and

everyCS *R*–semimodule is $C_{11}$. Furthermore, by adding the condition subtractive or distributive, more results were obtained like, every summand of distributive $C_{11}$ semimodule is $C_{11}$; the direct sum of two distributive CS *R*–semimodules (or distributive $C_{11}$semimodules) is $C_{11}$; if *S* is a subtractive *R*-semimodule with $C_{11}$ then *S* is FI-CS; if *S* is distributive *R*–semimodule with $C_{11}$, then every subsemimodule of *S* is CS and, any distributive $C_{11}$ semimodule is CS.

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# Listof Symbols

| Symbol | Definition |
|---|---|
| $\boldsymbol{\mathcal{N}} \leq \boldsymbol{S}$ | |
| ***R*** | Semiring |
| ***S*** | Semimodule |
| $\boldsymbol{\mathcal{N}} \leq \boldsymbol{S}$ | $\mathcal{N}$ is a subsemimodule of *S* |
| $\boldsymbol{\mathcal{N}} \leq^{e} \boldsymbol{S}$ | $\mathcal{N}$ is an essential subsemimodule of *S* |
| $\boldsymbol{\mathcal{N}} \leq^{c} \boldsymbol{S}$ | $\mathcal{N}$ is a closed subsemimodule of *S* |
| $\boldsymbol{\mathcal{N}} \subseteq \boldsymbol{S}$ | $\mathcal{N}$ is a subset of *S* |
| $\boldsymbol{\mathcal{N}} \subset \boldsymbol{S}$ | $\mathcal{N}$ is a proper subset of *S* |
| $\cap$ | Intersection |
| **E(*S*)** | An injective hull of a semimodule *S* |
| **Ě**(*S*) | A rational hull of semimodule *S* |
| $\oplus$ | Direct sum |
| **Summand <u>or</u>** $\leq^{\oplus}$ | Direct summand |
| $\in$ | Belong to |
| **End(*S*)** | All endomorphisms of semimodule*S* |
| **Hom($S_1$,$S_2$)** | The set of homomophisms from $S_1$ to $S_2$ |
| **CS-semimodule** | Extending semimodule |
| $\cong$ | Isomorphic |
| **Soc(*S*)** | The socleof *S* |
| ***Z*(*S*)** | The Singular subsemimodule of *S* |
| **$Z_2$(*S*)** | The Second singular subsemimodule of *S* |

| | |
|---|---|
| $\mathbb{N}$ | The set of natural numbers |
| $\mathbb{Z}$ | The set of integer numbers |
| $\mathbb{R}$ | The set of real numbers |
| **ker*f*** | The kernel of a homomorphism *f* |
| **Im*f*** | The image of a homomorphism *f* |
| $\overline{\boldsymbol{S}}$ | Subtractive closure of *S* |
| **FI-CS** | FI-extending |
| $\boldsymbol{\mathcal{N} \trianglerighteq S}$ | $\mathcal{N}$ is a fully invariant subsemimodule of *S* |
| $\boldsymbol{\mathcal{N} \trianglerighteq^{c} S}$ | $\mathcal{N}$ is a fully invariant closed subsemimodule of *S* |
| $\boldsymbol{\mathcal{N} \leq^{tes} S}$ | $\mathcal{N}$ is a t-essential subsemimodule of *S* |
| $\boldsymbol{\mathcal{N} \leq^{tc} S}$ | $\mathcal{N}$ is a t-closed subsemimodule |

## Introduction

Golan [1]mentioned that: “The structure of earliest algebraic, in which mankind calculates the semiring of natural numbers, whether or whether not they include a zero”.

Semimodules over semirings have a long history of research. In 1966, Yusuf [2]pioneered the idea of an inverse semimodule over a semiring and gained some analogues to module theory theorems, the study of semirings invariably entails the study of semimodules over them, just as the study of rings invariably entails the study of modules over them. Because semirings (resp. semimodules) are simply "rings (resp. modules) without subtraction," extending the properties of factor module and submodule from the modules category to the semimodules category, which was prompted many writers to demonstrate most of the result that found in the modules theory are validated for semimodules (for example[3-10]. Semimodules are a natural generalization of modules, having a wide range of applications in computer science's mathematical foundations[11].

In the 1930s, the concept of an extending module dates back to Von Neumann's work. He interests in quantum mechanics which leads him to create "continuous geometry," that is known as lower and upper continuing complete modular lattice. According to the Von Neumann, the regular rings to be continuous if the lattice of principal left ideals is upper and lower continuous. The research was continued by Utumi[12-14].He stated that if the left $R$-module is extending the regular ring $R$ is left continuous, if its lattice of principal left ideals is upper continuous a regular ring $R$ is left continuous. Utumi[15]investigated rings that do not have to be regular but whose left $R$-module $R$ is quasi-continuous or continuous (that is $\pi$-injective), and then these principles to modules have been applied by Takeuchi[16], Mohamed-Bouhy[17]and Jeremy[18].

Various authors investigated continuous and quasi-continuous modules, and a theory was created. Excellent accounts of these monographs have been presented by Wisbauer, Smith, Huynh and Dung [19], Mohamed and Muller[20]. Harada and his students, particularly Oshiro and Muller and his students, particularly Kamal and Rizvi have made significant contributions (see, for example,[21-39]).Of course, there are many more individuals and articles that could be mentioned here.

In his work on quotient rings, Goldie[30][31]considered complements, which encouraged Hajarnavis to consider left CS-rings, i.e. rings *R* for which *R* as an *R*-module is extending, and to publish [32] with Chatters. In [33], Khuri and Chatters collaborated on a study of module endomorphism rings over left CS-rings. Complements, in fact, play a crucial role in the theory. In two respects, our extended explanations are founded on it.

Various authors have used different terms, due to the divergent pattern of theory development. The word "extending module" has been used by Harada[34][35] and his school as a term to "lifting module", which has utilized[19]. The abbreviation "CS" has been used Hajarnav and Chatters for complements are summands, and this term is commonly used. Due to that there is a common property of quasi-continuous and continuous, which is of the submodules extending property, which means that each submodule is essentialin a direct summand; equivalently, every closed submodule is a direct summand, throughout the development of the theory, it has become increasingly clear that more general modules required research (with its various origins). So, the examples of extending modules are quasi-injective, semisimple and uniform modules. In addition, any finite-rank free abelian group is an extending module.

There are numerous generalizations to be discovered. Some instances of studies on property extension are as follows: Birkenmeier, Müller, and Rizvi (2002) investigated and defined the notion of FI-CS[36], which is defined as the direct sum of FI-CS modules. While the authors in [37] investigated some of the conditions that must be met in order for the direct summand of the FI-CS module to be FI-CS module. As a result, leads to the answers to the questions under discussion. Y¨ucel developed the generalized FI-CS module in[38]as well, demonstrating that the class of FI-CS modules is not closed under direct summands but is closed under direct sums.

The concept of t-extending modules was introduced in [39], and it was discovered that the homomorphic image (thus a direct summand) as well as the direct sum of a t-extending module receives the properties.The concept of the $C_{11}$-module was studied by Birkenmeier and Tercan[40], if every submodule has complements, which is direct summands and provided, because of $C_{11}$-modules class is closed under direct sums but not under direct summands, by a certain condition, to make the direct summand of the $C_{11}$-module be the$C_{11}$-module.

*R* will be a semiring with identity in this study, and the term semimodule refers to a unitary left *R*-semimodule. The thesis is organized as follows: it is divided into four chapters. The first chapter is divided into twosections, the firstsectioncontains preliminaries on semimodules, such as definitions, properties, specific classes of semimodules, specific types of subsemimodules, and the majority of the results needed for this project, which were obtained in semimodules literature, the second section includes some general applications of the semimodules and semirings, and does not represent applications of the concepts studied in this work.

The second chapter, devoted to the concept of extending semimodules, is divided into four sections: one for introduction, and the other two for introducing extending semimodules and investigating their characterization with some properties. The following part went through the ideas of extending semimodules with respect to direct sum and direct summand, as well as various characterizations of extending semimodules under specific conditions. In this section, further properties and outcomes were demonstrated. The final section is devoted to extending rational hull, which is an interesting generalization of extending semimodules.

The third chapter is divided into three parts. The second portion verified and studied various features of singular and nonsingular semimodules in greater depth. More aspects of extending semimodules were investigated in the third section studying the relationship between extending semimodule and singularity, and intriguing results were discovered when the idea of quasi-continuous was applied.

The generalization of extending semimodule was introduced and investigated in the fourth Chapter, which comprises four sections. It is divided into three sections, the second of which introduces and investigates the concept of a FI-extending semimodule and the features of the FI-CS semimodule. There are also numerous properties of fully invariant subsemimodules that are relevant. The relationship between the singularity and the FI-extending semimodule for direct summands and the direct sum is also examined and investigated. The notion of a t-extending semimodule and its features are introduced and investigated in the last section. Many properties of t-essential and t-closed subsemimodules are also researched, as well as many properties of nonsingular and $Z_2$-torsion semimodules.

In the fourth section, the $C_{11}$ semimodule was presented and analyzed, with some of its features proved. The behavior of $C_{11}$ in terms of sub-semimodule, direct sum, and direct summand was also investigated.

To take a deeper look at our work, we'll summarize all of our findings in the table below.

| | | |
|---|---|---|
| **Chapter Two: Extending Semimodule over Semiring** | | |
| A ($k$-$c$) $R$-semimodule $S$ is CSif and only if every closed subsemimodule of $S$ is a summand of $S$. | **Proposition2.2.8** | Page22 |
| Any summand of a CS-semimodule is CS. | **Proposition 2.2.22** | Page 29 |
| Let $S= S_1 \oplus S_2$ be acancellative and semisubtractive ($k$-$c$) $R$-semimodule, where $S_1$ and $S_2$ are relative injective semimodules then $S$ is CS-semimodule if and only if $S_1$ and $S_2$ are CS-semimodule. | **Proposition 2.3.3** | Page33 |
| **Chapter Three: Extending semimodule with Singularity** | | |
| Let $S$ be a($k$-$c$) CS$R$-semimodule and $Z_2(S)$ be a group over semiring, then | **Proposition 3.3.1** | Page48 |

| | | |
|---|---|---|
| $S$=$Z_2(S)\oplus\mathcal{K}$, where $Z_2(S)$ and $\mathcal{K}$are CS and $Z_2(S)$ is $\mathcal{K}$-injective. | | |
| Let $S$ be a cancellative and semisubtractive ($k$-$c$) $R$-semimodule such that $S$= $Z_2(S)\oplus\mathcal{K}$, where $Z_2(S)$ and $\mathcal{K}$are CS and $Z_2(S)$ is $\mathcal{K}$-injective, then $S$ is CS. | **Proposition 3.3.2** | Page49 |
| If S is a cancellative and semisubtractive (k-c) quasi-continuous semimodule with injective hull E(S), then<br>1. $S$ is CS.<br>2. For any $\mathcal{N}_i \leq^{\oplus} S$ and $\mathcal{N}_1 \cap \mathcal{N}_2$=0then $\mathcal{N}_1 + \mathcal{N}_2 \leq^{\oplus} S$ . | **Proposition 3.3.11** | Page54 |
| Let $S$be a cancellativesemisubtractive ($k$-$c$) quasi-continuous $R$–semimodule with injective hull $E(S)$, then<br>1. $S$ is CS.<br>2. If $S$ =$S_1\oplus S_2$, where $S_2$ is an $R$–module, then $S_2$is $S_1$ - injective. | **Proposition 3.3.15** | Page57 |
| **Chapter Four:Some Generalizations of Extending Semimodules** | | |
| Let $S$be an $R$–semimoduleand $\mathcal{N} \trianglerighteq S$. If $S$ is FI-CS then $\mathcal{N}$is FI-CS. | **Proposition 4.2.6** | Page63 |

| | | |
|---|---|---|
| Let $S = S_1 \oplus S_2$ be an $R$-semimodule If $S_1$and $S_2$ are FI-CS then $S$ is FI-CS. | **Proposition 4.2.7** | Page63 |
| A $(k\text{-}c)R$-semimodule $S$ is FI-CS if and only if the closure of any fully invariant subsemimodule in $S$ is a summand of $S$. | **Proposition 4.2.15** | Page66 |
| Let $S = S_1 \oplus S_2$ be FI-CS and $S_1 \trianglerighteq S$ then both $S_1$ and $S_2$ are FI-CS. | **Proposition4.2.20** | Page67 |
| AsemimoduleS is FI-CS if and only if $S=Z_2(S)\oplus\mathcal{K}$, where $Z_2(S)$ and $\mathcal{K}$ are FI-CS. | **Proposition4.2.23** | Page69 |
| Let $S$ be an $R$–semimodule. If $S$ is t-extending then $S=Z_2(S)\oplus S'$, where $S'$ is nonsingular CS-semimodule. | **Proposition 4.3.33** | Page83 |
| Every homomorphic image of t-extendingsubtractive $R$-semimoduleis t-extending. | **Proposition 4.3.37** | Page85 |
| A direct sum of t-extending semimodules is t-extending. | **Proposition 4.3.39** | Page85 |
| Every fully invariant t-closed subsemimodule of t-extending semimodule is t-extending. | **Proposition 4.3.43** | Page86 |
| Let $S=S_1\oplus S_2$ be distributive $R$-semimodule. If $S_1$and $S_2$ are $C_{11}$ then $S$ is $C_{11}$. | **Proposition 4.4.8** | Page90 |

| | | |
|---|---|---|
| Any summand of distributive $C_{11}$-semimodule is $C_{11}$. | **Proposition 4.4.12** | Page90 |
| If $S$ is a distributive $C_{11}$-semimodule, then $S=Z_2(S)\oplus S'$ for some nonsingular semimodule $S'$ of $S$ where both $Z_2(S)$ and $S'$ are $C_{11}$. | **Proposition 4.4.13** | Page 91 |
| Let $S$ be $C_{11}R$-semimodule. If $S$ is distributive, then every subsemimodule of $S$ is *CS*. In particular any distributive $C_{11}$ semimodule is CS. | **Proposition 4.4.23** | Page 94 |

# Chapter One:

# Preliminaries and Examples

## 1.1 Basic Definitions and Examples

We will introduce some of the definitions and remarks that will be needed in the main results in the next chapters of this study. Let us start with a definition of a semiring.

**Definition 1.1.1**[41]**:** A semiring is a nonempty set $R$ on which operations of addition and multiplication have been defined such that the following conditions are satisfied:

1. $(R,+)$is a commutative monoid with identity element0;
2. $(R, \cdot)$ is a monoid with identity element $1 \neq 0$ $(1 = 1_R)$;
3. Multiplication distributes over addition,i.e. $a(b+ c) = ab +ac$ and $(a+b)c =ac+ bc$ for all $a, b, c\in R$;
4. The element 0 is an absorbing element of the multiplication, i.e. $r\cdot 0=0\cdot r= 0$ for all $r\in R$.

A semiring $R$ is said to be a commutative if its multiplication is commutative.

**Definition 1.1.2**[42]:A non-empty subset $I$ of a semiring $R$ will be called an ideal of $R$ if $a, b\in I$ and $r\in R$ imply that $a+ b\in I$, $ra$, and $ar\in I$ .

**Definition 1.1.3**[41]: Let $(S,+)$ be an additive commutative monoid with additive identity $0_S$, and $R$ is a semiringwith identity then $S$ is called a left $R$–semimodule if there exists a scalar multiplication $R\times S\longrightarrow S$ denoted by $(r,s)\mapsto rs$ such that $(rr')s=r(r's)$; $r(s+s')=rs+ rs'$; $(r+ r')s=rs+r's$; $r0_S=0_Rs=0_S$ for all $r, r'\in R$ and all $s,s'\in S$.

**Definition 1.1.4**[41]: A subset $\mathcal{N}$ of an $R$–semimodule $S$ is called a left subsemimodule of $S$ if for any $n, n' \in \mathcal{N}$ and $r \in R$, then $n + n' \in \mathcal{N}$ and $rn \in \mathcal{N}$ and write ($\mathcal{N} \leq S$).

**Remark 1.1.5:** It is clear that if $\{S_i \mid i \in \Omega\}$ is a family of subsemimodules of $S$ then $\cap_{i\in\Omega} S_i$ is a subsemimodule of $S$ and if $S_1$ and $S_2$ are subsemimodules of $S$ then$S_1+S_2=\{a+b|a\in S_1,\ b\in S_2\}$ is a subsemimodule of $S$.

**Definition 1.1.6**[43]**:**A nonzero $R$-subsemimodule $\mathcal{K}$ of $S$ is calledessential and denoted by ($\mathcal{K} \leq^e S$) if $\mathcal{K} \cap \mathcal{N} \neq 0$ for every $0 \neq \mathcal{N} \leq S$.Equivalently, for every $0 \neq x \in S$, there exists $r \in R$ such that $0 \neq rx \in \mathcal{K}$.

**Definition 1.1.7**[44]**:**A subsemimodule $\mathcal{N}$ of a semimodule $S$ is said to be closed if $\mathcal{N} \leq^e S' \leq S$ implies $\mathcal{N}=S'$(denoted by $\mathcal{N} \leq^c S$).

**Remark 1.1.8**[44]**:** Every subsemimodule $\mathcal{K}$ of a semimodule $S$ is essential in closed subsemimodule $\mathcal{H}$ of $S$.

**Definition 1.1.9**[44]**:** A subsemimodule $\mathcal{N}$ of a semimodule $S$ is called complement of a subsemimodule $\mathcal{K}$of $S$ if $\mathcal{N} \cap \mathcal{K} = 0$ and $\mathcal{N}$ is a maximal with this property.

**Remark 1.1.10:**Iet$S$ be an $R$ -semimodule then**,**

1. Every subsemimodules of $S$ has a complement.
2. If $\mathcal{N}$ and $\mathcal{K}$ are subsemimodules of S with $\mathcal{N} \cap \mathcal{K} = 0$, then there exists $\mathcal{N}'$complement of $\mathcal{K}$contain $\mathcal{N}$.
3. Every complement subsemimodules of $S$ is closed.

**Proof:**

1. Clear by Zorn's lemma.
2. Clear by Zorn's lemma.
3. Assume that $\mathcal{A}$ is complement of $\mathcal{B}$ in *S*. Suppose $\mathcal{A} \leq^{e} \mathcal{A}' \leq S$. Let $0 \neq x \in \mathcal{A}' \cap \mathcal{B}$,then there exists $r \in R$ such that $0 \neq rx \in \mathcal{A}$, hence$0 \neq rx \in \mathcal{A} \cap \mathcal{B}$, a contradiction, so $\mathcal{A}' \cap \mathcal{B}=0$, which implies$\mathcal{A}' = \mathcal{A}$, therefore $\mathcal{A}$ is closed in *S*. □

**Definition 1.1.11**[44]**:** A subsemimodule $\mathcal{N}$ of a semimodule *S* is said to be closure of a subsemimodule $\mathcal{K}$ in *S* if $\mathcal{N}$is closed and $\mathcal{K}$ essential in $\mathcal{N}$. Equivalently the closure of $\mathcal{K}$ is the smallest closed subsemimodule containing $\mathcal{K}$.

**Definition 1.1.12**[41]**:** An *R*-subsemimodule $\mathcal{K}$ of an *R*-semimodule *S* is said to be subtractive if for each $x, x' \in S$ with $x, x+x' \in \mathcal{K}$ implies $x' \in \mathcal{K}$. Thus, for example the subsemimodule {0} and *S* of a semimodule *S* are always subtractive.

Thus, for example in a semimodule (ℕ,+) over semiring (ℕ,+,.), asubsemimodule$\langle k \rangle = \mathbb{N}k$, is a subtractive subsemimodule of ℕ. But a subsemimodule $\mathcal{K} = \{\mathbb{N}\backslash 1\}$ is not subtractive subsemimodule of *S* , since $10+1 \in \mathcal{K}$, $10 \in \mathcal{K}$ but $1 \notin \mathcal{K}$.

**Definition 1.1.13**[41]**:**An *R*-semimodule *S* is said to be subtractive if all its subsemimodules are subtractive.

To reduce the subtractive condition on the semimodule, we will add the following condition

**Definition 1.1.14:**A semimodule $S$ is said to be a $k$-closed If every closed subsemimodule of $S$ is a subtractive denoted by ($k$-$c$) $R$-semimodule.

**Remark 1.1.15:** A subsemimodule of ($k$-$c$) $R$-semimodule $S$ is contained in a closure subsemimodule.

**Proof:**Assume that$\mathcal{N} \leq S$. If $\mathcal{N} \leq^{e} S$, then S is a closure of $\mathcal{N}$. If not, then there exists $0 \neq \mathcal{K} \leq S$, and $\mathcal{N} \cap \mathcal{K} = 0$, let $\mathcal{K}'$be a complement of $\mathcal{N}$containing $\mathcal{K}$ and $\mathcal{N}'$ is a complement of $\mathcal{K}'$containing $\mathcal{N}$(which exists by Zorn's lemma). Let $x \in \mathcal{N}' \setminus \mathcal{N}$ then $(\mathcal{K}' + Rx) \cap \mathcal{N} \neq 0$, so there exists $0 \neq n = k' + rx'$ where $k' \in \mathcal{K}'$, $r \in R$ and $n \in \mathcal{N}$ since $n \in \mathcal{N} \leq \mathcal{N}'$, $rx \in \mathcal{N}'$and $\mathcal{N}'$ is subtractive (since it is closed by Remark (1.1.10)) so $k' \in \mathcal{N}' \cap \mathcal{K}' = 0$ hence $0 \neq rx \in \mathcal{N}$that is$\mathcal{N} \leq^{e} \mathcal{N}'$, therefore $\mathcal{N}'$is a closure of $\mathcal{N}$. □

**Definition 1.1.16**[41]**:** An $R$-semimodule $S$ is said to be semisubtractive if for all $x, x' \in S$, there exists $h \in S$ such that either$x = x' + h$ or $x + h = x'$.

Thus, for example$\mathbb{Z}$, $\mathbb{N}$ are semisubtractive semimodules over ($\mathbb{N}$ ,+,.).

**Definition 1.1.17**[41]**:**A semimodule $S$ is additively cancellativeif for all $n, n'$and $n'' \in S$, $n + n' = n + n''$ implies $n' = n''$.

**Definition 1.1.18**[45]**:**An $R$–semimodule $S$ is called a direct sum of subsemimodules$S_1$, $S_2$,…, $S_k$ of $S$ if each $s \in S$ can be written uniquely as $s = s_1 + s_2 + \ldots + s_k$, where $s_i \in S_i$. It is denoted by $S = S_1 \oplus S_2 \oplus \ldots \oplus S_k$. In this case each $S_i$ is called a direct summand of $S$(denoted bysummand or $\leq^{\oplus}$).

**Remark 1.1.19:** Let $S$ be a semisubtractive $R$-semimodule, and $\mathcal{K}, \mathcal{H}$ are subtractive subsemimodule of $S$ then $S$=$\mathcal{K} \oplus \mathcal{H}$ if and only if $S$= $\mathcal{K} + \mathcal{H}$ and $\mathcal{K} \cap \mathcal{H}$=0.

***Proof*:**

(⟹)Clear**.**

(⟸)Assume that $x_1+x_2=y_1+y_2 \in \mathcal{K} + \mathcal{H}$, then by semisubtractive either $x_1=y_1+h$ or $y_1=x_1+h$, since $\mathcal{K}$ and $\mathcal{H}$ are subtractive, hence $h \in \mathcal{K} \cap \mathcal{H}$=0, therefore $x_1=y_1$ and similarly, $x_2= y_2$, so the representation of $x_1+x_2$ is unique, hence $S$=$\mathcal{K} + \mathcal{H} = \mathcal{K} \oplus \mathcal{H}$.□

**Remark 1.1.20:** If $S$ is a cancellative and semisubtractivesemimodule then every closed subsemimodules of $S$ is a complement.

***Proof*:**Let $\mathcal{A}$ is closed subsemimodule of $S$, then $\mathcal{A}$ is not essential in $S$ and there existe0≠ $\mathcal{B} \leq S$ such that $\mathcal{A} \cap \mathcal{B}$=0. Let $\mathcal{B}'$be a complement of $\mathcal{A}$ containing $\mathcal{B}$, and $\mathcal{A}'$ be a complement of $\mathcal{B}'$ containing $\mathcal{A}$. If $x \in \mathcal{A}' \setminus \mathcal{A}$ then $(\mathcal{B}' + Rx) \cap \mathcal{A} \neq 0$. Sincs $S$ is a semisubtractive hence by Remark (1.1.19), $\mathcal{B}' + \mathcal{A}' = \mathcal{B}' \oplus \mathcal{A}'$. now, there exists $b' \in \mathcal{B}$and $r \in R$ such that $0 \neq b' + rx = a \in \mathcal{A} \leq \mathcal{A}'$, implies $b'$=0 and $rx=a \in \mathcal{A}$, hence $\mathcal{A} \leq^{e} \mathcal{A}'$ but $\mathcal{A}$ is closed, then $\mathcal{A} = \mathcal{A}'$, that is $\mathcal{A}$ is a complement of $\mathcal{B}'$in $S$. □

In module every summand is closed but in semimodule we must add ($k$-$c$) condition .

**Remark 1.1.21:** Let $S$ be a ($k$-$c$) $R$-semimodule, then any summand of $S$ is closed.

***Proof*:** Assume that $X$ is a summand of $S$, say $S=X\oplus Y$ for some $Y\leq S$. Let $B\leq^{c} S$ such that $X\leq^{e} B$, since $X\cap(B\cap Y)=0$, hence $B\cap Y=0$. Let $b\in B$, then $b=x+y$ for some $x\in X$ and $y\in Y$, since $b\in B$, $x\in X\leq B$ and $B$ is subtractive (since $S$ is ($k$-$c$)) hence $y\in Y\cap B=0$, that is $b=x\in X$, so $B\leq X$, therefore $X=B$ is closed in $S$. □

**Remark 1.1.22:** The converse of Remark (1.1.21) is not true. Thus for example, let $S=\mathbb{Z}_8\oplus\mathbb{Z}_2$, $R=\mathbb{Z}$, Let $S_i\leq S$, where (i = 1, 2, …,8) such that $S_1=0\oplus\mathbb{Z}_2$, $S_2=\mathbb{Z}_8\oplus 0=\langle(\bar{3},\bar{0})\rangle=\langle(\bar{5},\bar{0})\rangle=\langle(\bar{7},\bar{0})\rangle$, $S_3=\langle(\bar{2},\bar{0})\rangle$, $S_4=\langle(\bar{4},\bar{0})\rangle$, and $S_5=\langle(1,\bar{1})\rangle$, $S_6=\langle(\bar{0},\bar{1}),(\bar{2},\bar{0})\rangle=S_1\oplus S_3$ $S_7=\langle(\bar{0},\bar{1}),(\bar{4},\bar{0})\rangle=S_1\oplus S_4$ and $S_8=\langle(\bar{2},\bar{1})\rangle$, $S_1$, $S_2$ and$S_5$ are summand of $S$, $S_2$ and $S_5$are complement of $S_1$, $S_8$is a closed subsemimodule of $S$, but not a summand of $S$.

**Definition 1.1.23**[44]**:** A semimodule $S$ is said to be indecomposable if it is non-zero and the direct summands of$S$are only {0} and it self.

**Definition 1.1.24**[46]**:**Let$\mathcal{A}$and$\mathcal{B}$be $R$-semimodules. A homomorphism from$\mathcal{A}$to $\mathcal{B}$is a map $\beta$: $\mathcal{A}\rightarrow\mathcal{B}$such that

a) $\beta(a+a')=\beta(a)+\beta(a')$

b) $\beta(ra)=r\beta(a)\forall a, a'\in\mathcal{A}$and $r\in R$.

A homomorphism of $R$-semimodules $\beta:\mathcal{A}\rightarrow\mathcal{B}$is said to be:

*a.* monomorphism, if it is 1-1.

*b.* epimorphism if it is onto.

*c.* isomorphism if it is monomorphism and epimorphism.

**Note:**

*a.* $\ker(\beta)=\{a\in\mathcal{A}\mid\beta(a)=0\}$.

*b.* The set of all $R$-homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ is denoted by $\mathrm{Hom}_R(\mathcal{A},\mathcal{B})$. An $R$-homomorphism from an $R$-semimodule $S$ to itself is called an $R$-endomorphism of $S$denoted by $\mathrm{End}_R(S)$, we shall mean the set of all $R$-endomorphisms of $S$, it is clear that $\mathrm{End}_R(S)$ is a semiring[46].

**Remarks 1.1.25**[47]**:**For a homomorphism of $R$-semimodules $\varphi$:$S\rightarrow \mathcal{K}$ we define:

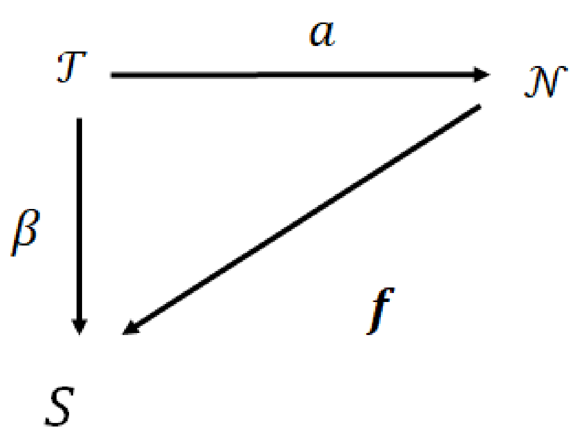
a) $\varphi$ ($S$) ={ $\varphi(s)|s\in S$}.

b) Im ($\varphi$) = {$k\in \mathcal{K}$ | $k + \varphi$ ($s$) = $\varphi$ ($s'$) for some $s$, $s'\epsilon S$}.

It is obvious that ker($\varphi$) is a subtractive subsemimodule of $S$,Im($\varphi$) is a subtractive subsemimodule of $\mathcal{K}$ and $\varphi(S)$ is a subsemimodule of $\mathcal{K}$. In module theory $\varphi(S)$=Im($\varphi$), in semimodule theory it is not true always. It is clear that $\varphi(S)\subseteq$Im($\varphi$), the equality hold if $\varphi(S)$is subtractive subsemimodule of $\mathcal{K}$ [47].

**Definition 1.1.26**[48]**:** An $R$-homomorphism $\varphi$:$\mathcal{A}\longrightarrow\mathcal{B}$is said to be *i*-regular, if$\varphi(\mathcal{A})$ = Im($\varphi$).

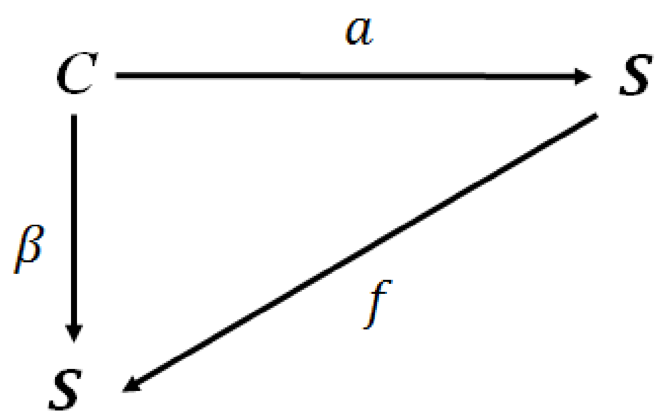
**Definition 1.1.27**[49]**:**Let $\mathcal{N}$be an $R$–semimodule. An $R$–semimodule $S$ is said to be $\mathcal{N}$-*injective*, if for each subsemimodule $\mathcal{T}$ of $\mathcal{N}$, any homomorphism from $\mathcal{T}$ into $S$ can be extended to an $R$-homomorphism from $\mathcal{N}$ into $S$. The $R$–semimodule $S$ is injective if it is injective relative to every $R$–semimodule.



**Remark 1.1.28**[49]**:** A summand of an injective semimodule is injective.

**Remark 1.1.29**[49]**:**A semimodule $S$ is injective if and only if it is a summand of any extension of it.

**Definition 1.1.30**[50]**:**A semimodule $S$ is quasi-injective if it is $S$-injective. As the following diagram, i.e. there exist $f$ such that $f\alpha=\beta$(with $\alpha$ is a monomorphism).




**Remark 1.1.31**[46]**:** Every injective semimodule is quasi-injective but the converse not true, for example$\mathbb{Z}/2\mathbb{Z}$isquasi-injective$\mathbb{Z}$-semimodule but not injective

**Definition1.1.32**[51]**:** A nonzero $R$–semimodule $S$ is called simple if $S$ has no nonzero proper $R$–subsemimodule.

Note that $\mathbb{Z}$ and $\mathbb{Z}_4$ are not simple $\mathbb{Z}$-semimodule but $\mathbb{Z}_p$ is a simple $\mathbb{Z}$ -semimodule for any prime number $p$.

**Definition 1.1.33**[52]**:**A semimodule $S$ is said to be semisimple if it is a direct sum of its simple subsemimodules.

***Thus for example***:

1. Every simple semimodule is semisimple.
2. $\mathbb{Z}$ and $\mathbb{Z}_4$ are not semisimple $\mathbb{Z}$-semimodule.
3. $\mathbb{Z}_6$ is a semisimple $\mathbb{Z}$-semimodule,but not simple.

**Definition 1.1.34**[44]**:** A nonzero semimodule *S* is said to be uniform if any nonzero subsemimodule $\mathcal{N}$ of *S* is essential.

Thus, for example$\mathbb{Z}$ is a uniform semimodule.

**Definition 1.1.35**[53]**:** If *E* is an injective *R*–semimodule, and it is a minimal injective extension of the *R*–semimodule *S*, then *E* is said to be an injective hull of *S* denoted by *E*(*S*).

It is commonly known, however, that if *R* is a ring, the injective hull of *R*-semimodule always exists. But, Golan ([41],Proposition 17.21, p. 198) demonstrated that injective hulls of non-zero *R*- semimodules do not need to exist for every semiring *R*. Every semimodule over an additively idempotent semiring has an injective hull, as Wang[54] shown. For further information on an injective hull of semimodules over semirings See[53].

***Note***: For example asemimodule$(\mathbb{N}, +)$ over $(\mathbb{N}, +, .)$without injective hull [41].

**Definition 1.1.36**[55]**:**A subsemimodule $\mathcal{K}$ of *S* is said to be fully invariant if $g(\mathcal{K}) \subseteq \mathcal{K}$for each *R*-endomorphism $g$on *S*(denoted by$\mathcal{K} \trianglerighteq S$).

**Definition 1.1.37**[56]**:** A semimodule *S* is said to be duo if each subsemimodule of *S* is fully invariant.

**Definition 1.1.38**[48]**:**If *S* is an *R*-semimodule and $X \subseteq S$, thenthe left annihilator of *S* in*X* is $\text{ann}_R(X)=\{r\in R: rx=0 \text{ for every } x\in X\}$.

**Definition 1.1.39**[48]**:** A subsemimodule Z(*S*) of *S* is defined by $Z(S)=\{f\in S|\text{ann } f\leq^{e} R\}$is said to be singular subsemimodule of *S*. If Z(*S*)=*S* then *S* is called singular. If Z(*S*) = 0 then *S* is called nonsingular.

Thus for example $\mathbb{Z}_p$ is singular (when *p* is prime) but $\mathbb{Z}$ is nonsingular (as $\mathbb{Z}$-semimodules).

**Definition 1.1.40** [57]**:**The second singular subsemimodule $Z_2(S)$ of *S* is that subsemimodule of *S*, containing Z(*S*) such that $Z_2(S)/Z(S)$ is the singular subsemimodule of *S/Z*(*S*).

**Definition 1.1.41**[46]**:** The socleof a semimodule*S* is denoted by Soc(*S*)and defined as Soc(*S*)= $\Sigma\{L: L$a simple *R*- semimodule of *S*}.

**Remark 1.1.42**:

1. Soc($\mathbb{Z}$)=0, since $\mathbb{Z}$has no simple semimodule.
2. If $S=S_1\oplus S_2$, then $\text{Soc}(S)=\text{Soc}(S_1)\oplus \text{Soc}(S_2)$.

# Chapter Two:

# Extending Semimodules over Semirings

## 2.1 Introduction

In section two of this chapter. Extending semimodules will be presented as well as investigating some properties of them. Initially for this purpose, some properties of complement subsemimodules that are useful in analyzing the structure of extending semimodule, will be given. In section three of this chapter, the direct summand and direct sum of extending semimodule will be studied as well as conditions that ensure a subsemimodule of extending semimodule to be extending semimodule and supply related properties of extending semimodule property.

## 2.2 Extending-Semimodules

According to[58], the concepts of extending modules will be converted for semimodule in the following.

**Detention 2.2.1:** An $R$-semimodule $S$ is said to be extending (CS) if every subsemimodule of $S$ is essential in a summand of $S$.

**Example 2.2.2:**

1. It is clear that any simple $R$-semimodule is CS. In fact any semisimple or (uniform)$R$-semimodule is CS.
2. A semimodule $S$ in Remark (1.1.22) is not CS, since $S_8$ is closed in $S$ but not summand.

Now, the following lemmas are needed to prove our main results.

**Lemma 2.2.3**[49]**:**Let $S$ be a subtractive $R$-semimodule, If $\mathcal{N}$ is a subsemimodule of $S$ and $C$ is a complement of $\mathcal{N}$ in $S$ then $\mathcal{N}\oplus C \leq^{e} S$.

The condition $S$ is subtractive in Lemma (2.2.3), is strong, in the following the same result will be gotten with weaker conditions.

**Lemma 2.2.4:** Let $S$ be a cancellative and semisubtractive $R$-semimodule if $\mathcal{N}$ and $C$ are subtractive subsemimodules of $S$ with $\mathcal{N} \cap C=0$ then

$\mathcal{N}+C=\mathcal{N}\oplus C$. Moreover if $C$ is a complement of $\mathcal{N}$ and $\mathcal{N}+C$ is subtractive then $\mathcal{N}+C\leq^{e}S$.

**Proof:** Assume $n_1+c_1=n_2+c_2\in \mathcal{N}+C$, then by semisubtractive either $c_1=c_2+h$ or $c_2=c_1+h$**,** since $\mathcal{N}$ and $C$ are subtractive by assumption, hence the two above cases lead to $h\in \mathcal{N}\cap C=0$, hence $n_1=n_2$ and $c_1=c_2$, therefore the representation of $n_1+c_1$ is unique,hence $\mathcal{N}+C=\mathcal{N}\oplus C$. Let $x\notin \mathcal{N}+C$, then $x\notin \mathcal{N}$ and $x\notin C$, and since $C$ is a complement of $\mathcal{N}$ in $S$, then $\mathcal{N}\cap(C+Rx)\neq 0$, let $0\neq n=c+rx\in \mathcal{N}\cap(C+Rx)$, it is clear $rx\neq 0$ (if not $n=c\in \mathcal{N}\cap C=0$, a contradiction**).** Now $n\in \mathcal{N}\leq \mathcal{N}+C$,and $c\in C\leq \mathcal{N}+C$ by subtractive $rx\in \mathcal{N}+C$ therefore $\mathcal{N}+C\leq^{e}S$. □

**Lemma 2.2.5**[44]**:** If $S=S_1\oplus S_2$ with injective hull $E$(S), then $E(S)=E(S_1)\oplus E(S_2)$.

**Lemma 2.2.6** [44]**:** Let $S$ be an $R$–semimodule with injective hull $E$(S). If $\mathcal{K}\leq^{e}S$ then $E(\mathcal{K})=E(S)$.

**Lemma 2.2.7:** Every injective $R$-semimodule is CS.

**Proof:** Let $S$ be injective semimodule and let $\mathcal{N}\leq S$, then $\mathcal{N}\leq^{e} E(\mathcal{N})\leq E(S)=S$, but $E(\mathcal{N})$ is a summand of $S$ (since by Lemma(1.1.29) $S$ is an injective extension of $E(\mathcal{N})$), hence $S$ is *CS*. □

**Proposition2.2.8:** An ($k$-$c$) $R$-semimodule $S$ is CS if and only if every closed subsemimodule of $S$ is a summand of $S$.

**Proof:** Assume $S$ is a $CS$ $R$-semimodule, let $\mathcal{K} \leq^{c} S$ then $\mathcal{K} \leq^{e} S'$, where $S' \leq^{\oplus} S$, then by definition of closed subsemimodule, $\mathcal{K} = S'$, that is $\mathcal{K}$ is a summand of $S$. Conversely, let $\mathcal{K} \leq S$, and let $S'$ be closure of $\mathcal{K}$, then $S' \leq^{\oplus} S$ and $\mathcal{K} \leq^{e} S'$, hence $S$ is $CS$.□

In the example of Remark (1.1.22)the subsemimodule $S_8$ is closed in $S$ but not summand.

**Lemma 2.2.9:** If $\mathcal{K} \leq S$ *and* $L \leq^{e} S$ then $L \cap \mathcal{K} \leq^{e} \mathcal{K}$.

***Proof*:** Clear.

**Lemma 2.2.10:** Let $L \leq \mathcal{K} \leq \mathcal{K}' \leq S$ (where $L$ and $\mathcal{K}$ are subtractive in $S$). If $\mathcal{K}/L \leq^{e} \mathcal{K}'/L$ ,then $\mathcal{K} \leq^{e} \mathcal{K}'$.

***Proof*:** Let $x \in \mathcal{K}' \setminus \mathcal{K}$, then $0 \neq x+L \in \mathcal{K}'/L$, so there exists $r \in R$ such that, $0 \neq r(x+L) \in \mathcal{K}/L$, that is $0 \neq rx \in \mathcal{K}$. Therefore, $\mathcal{K} \leq^{e} \mathcal{K}'$. □

**Lemma2.2.11:** If $\mathcal{N} \leq L \leq^{c} S$, $\mathcal{N}$ and $L$ are subtractive in $S$, then $L/\mathcal{N} \leq^{c} S/\mathcal{N}$.

***Proof*:** Assume that $L/\mathcal{N} \leq^{e} L'/\mathcal{N} \leq S/\mathcal{N}$ and ($L \neq L'$), then $L \leq^{e} L'$, which contradicts the assumption $L \leq^{c} S$, hence $L/\mathcal{N} \leq^{c} S/\mathcal{N}$. □

**Lemma2.2.12:** If $\mathcal{K}$ and $\mathcal{N}$ are subtractive subsemimodules of an $R$-semimodule $S$ with $\mathcal{K}$ is closed in $S$, then $\mathcal{K} \leq \mathcal{N} \leq^{e} S$ if and only if $\mathcal{N}/\mathcal{K} \leq^{e} S/\mathcal{K}$.

***Proof:*** By Lemma (2.2.10), it is enough to prove the necessity condition. Suppose $\mathcal{K} \leq \mathcal{N} \leq^{e} S$ and $\mathcal{K} \leq^{c} S$. Let $L$ be subsemimodule of $S$ such that $\mathcal{K} \leq L$ and $(\mathcal{N}/\mathcal{K}) \cap (L/\mathcal{K})=0$, then $\mathcal{K} = \mathcal{N} \cap L \leq^{e} L$, based on Lemma (2.2.9). Since $\mathcal{K}$ is closed, then $L = \mathcal{K}$ and $L/\mathcal{K} = 0$. So $\mathcal{N}/\mathcal{K} \leq^{e} S/\mathcal{K}$. □

**Lemma 2.2.13:** If $\mathcal{K} \leq^{c} \mathcal{N}$, and $\mathcal{N} \leq^{c} S$, then $\mathcal{K} \leq^{c} S$.

***Proof*:** Assume that $\mathcal{K} \leq^{e} \mathcal{K}' \leq S$, then $\mathcal{K} \leq^{e} \mathcal{K}' \cap \mathcal{N} \leq \mathcal{N}$, therefore $\mathcal{K} = \mathcal{K}' \cap \mathcal{N}$ (since $\mathcal{K} \leq^{c} \mathcal{N}$), and $\mathcal{K}' \cap \mathcal{N} \leq^{e} \mathcal{K}'$ so $\mathcal{N} \leq^{e} \mathcal{N} + \mathcal{K}'$ (if $n+k' \in \mathcal{N} + \mathcal{K}'$ and $k' \in \mathcal{K}'$, then there exists $r \in R$ such that $0 \neq rk' \in \mathcal{K}' \cap \mathcal{N}$, hence $0 \neq r(n+k') \in \mathcal{N}$), since $\mathcal{N} \leq^{c} S$, then $\mathcal{N} = \mathcal{N} + \mathcal{K}'$, and $\mathcal{K}' \subseteq \mathcal{N}$, therefore $\mathcal{K} = \mathcal{K}' \cap \mathcal{N} = \mathcal{K}'$, and $\mathcal{K} \leq^{c} S$. □

In a module there is a modular law, and to deduce its analogue in a semimodule we need an additional condition.

**Lemma 2.2.14(Modular Law for Semimodules):** Let $S$ be a left $R$-semimodule .If $\mathcal{N}, \mathcal{N}'$ and $\mathcal{N}''$ are subsemimodules of $S$ with $\mathcal{N}$ is subtractive and $\mathcal{N}' \leq \mathcal{N}$, then $\mathcal{N} \cap (\mathcal{N}' + \mathcal{N}'') = \mathcal{N}' + (\mathcal{N} \cap \mathcal{N}'')$.

***Proof*:** Assume that $x \in \mathcal{N} \cap (\mathcal{N}' + \mathcal{N}'')$, then $x = n' + n''$, for some $n' \in \mathcal{N}'$ and $n'' \in \mathcal{N}''$, since $\mathcal{N}$ is subtractive and $\mathcal{N}' \leq \mathcal{N}$, then $n'' \in \mathcal{N} \cap \mathcal{N}''$ and $n' + n'' \in \mathcal{N}' + (\mathcal{N} \cap \mathcal{N}'')$, therefore $x \in \mathcal{N}' + (\mathcal{N} \cap \mathcal{N}'')$ and $\mathcal{N} \cap (\mathcal{N}' + \mathcal{N}'') \subseteq \mathcal{N}' + (\mathcal{N} \cap \mathcal{N}'')$. But $\mathcal{N}' + (\mathcal{N} \cap \mathcal{N}'') \subseteq (\mathcal{N}' + \mathcal{N}'') \cap \mathcal{N}$, therefore $\mathcal{N} \cap (\mathcal{N}' + \mathcal{N}'') = \mathcal{N}' + (\mathcal{N} \cap \mathcal{N}'')$ . □

**Remark2.2.15:** For example of semimidule does not satisfy Modular law, in semimodule $(\mathbb{N}, +)$ over semiring $(\mathbb{N}, +, .)$( $\mathbb{N}$ is not subtractive), let $\mathcal{N}=\{0,5,6,7,\ldots\}$, $\mathcal{N}'=3\mathbb{N}$ and $\mathbb{N}''=5\mathbb{N}$, then $8 \in \mathcal{N} \cap (\mathcal{N}' + \mathcal{N}'')$(since $8 \in \mathcal{N}$ and $8=3+5 \in \mathcal{N}' + \mathcal{N}''$ but $8 \notin (\mathcal{N} \cap \mathcal{N}') + (\mathcal{N} \cap \mathcal{N}'') = (\mathcal{N} \cap \mathcal{N}') + \mathcal{N}''$, since $\mathcal{N} \cap \mathcal{N}'=\{0,6,9,12,\ldots\}$, $\mathcal{N}''=\{0,5,10,\ldots\}$, $8 \neq n' + n''$ where $n' \in \mathcal{N} \cap \mathcal{N}'$ and $n'' \in \mathcal{N}''$.

As a special case of Proposition (2.2.8), we obtain.

**Proposition 2.2.16:** Let $S = S_1 \oplus S_2$ be ($k$-$c$) $R$-semimodule then $S$ is CS if and only if every complement subsemimodule of $S_i$, where ($i$ = 1or 2) is CS and a summand of $S$.

***Proof*:** Let $\mathcal{K}$ be a complement of $S_1$ in $S$, then $\mathcal{K} \leq^{c} S$, since $S$ is $CS$-semimodule and by Proposition (2.2.8),$\mathcal{K} \leq^{\oplus} S$. Let $L \leq^{c} \mathcal{K}$, so by Lemma (2.2.13), $L \leq^{c} S$ and $L \cap S_1 = 0$,since $S$ is $CS$-semimodule, $L \leq^{\oplus} S$, so $S = L \oplus L'$, for some $L' \leq S$ and since $\mathcal{K} \leq^{\oplus} S$, hence subtractive, byLemma(2.2.14),$\mathcal{K} = L \oplus (L' \cap \mathcal{K})$ therefore $L \leq^{\oplus} \mathcal{K}$, and $\mathcal{K}$ is $CSR$-semimodule. Conversely, let $\mathcal{N} \leq^{c} S$ then there exists aclosed subsemimodule $\mathcal{H}$ of $\mathcal{N}$such that$\mathcal{N} \cap S_1 \leq^{e} \mathcal{H}$, clearly $(\mathcal{H} \cap S_2)=0$. By Zorn's Lemma,there exists a complement $Q$ of $S_2$ in $S$ with $\mathcal{H} \leq Q$. Also, by Lemma (2.2.13) $\mathcal{H} \leq^{c} S$, hence $\mathcal{H} \leq^{c} Q$, since $Q$ is a complement of $S_2$ then by assumption, it is $CS$-semimodule, hence $\mathcal{H} \leq^{\oplus} S$, therefore $S = \mathcal{H} \oplus \mathcal{H}'$ for some $\mathcal{H}' \leq S$, since $\mathcal{N}$ is subtractive in $S$ then by Lemma (2.2.14), $\mathcal{N} = \mathcal{H} \oplus (N \cap \mathcal{H}')$, since $(\mathcal{N} \cap \mathcal{H}')$ is closed in $S$, and $(\mathcal{N} \cap \mathcal{H}') \cap S_2 = 0$, hence $(\mathcal{N} \cap \mathcal{H}') \leq^{\oplus} S$ and also in$\mathcal{H}'$, $\mathcal{H}' = (\mathcal{N} \cap \mathcal{H}') \oplus \mathcal{H}''$for some $\mathcal{H}'' \leq S$, so $S = \mathcal{N} \oplus \mathcal{H}''$ therefore $\mathcal{N} \leq^{\oplus} S$. □

**Example 2.2.17:**Assume $S = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Let $\mathcal{N}_i \leq S$, where ($i$ = 1, 2, 3, 4,5) such that $\mathcal{N}_1 = \mathbb{Z}_2 \oplus 0$, $\mathcal{N}_2 = 0 \oplus \mathbb{Z}_4$, $\mathcal{N}_3 = \langle (\bar{1}, \bar{1}) \rangle = \langle (\bar{1}, \bar{3}) \rangle$, $\mathcal{N}_4 = \langle (\bar{1}, \bar{2}) \rangle$, and $\mathcal{N}_5 = \langle (0, \bar{2}) \rangle$, $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$and $\mathcal{N}_4$are summands of $S$, and $\mathcal{N}_5$ is essential in $\mathcal{N}_2$ then $S$ is CS-semimodule.

The following Proposition gives an equivalent statement to CS-semimodule under extra condition.

**Proposition 2.2.18:** Acancellative and semisubtractive ($k$-$c$)$R$-semimodule $S$ is CS if and only if for all subtractive subsemimodules

$\mathcal{P}$and$L$of $S$ with $\mathcal{P} \cap L = 0$, there exists $\mathcal{H} \leq^{\oplus} S$ such that $L \leq \mathcal{H}$ andP ∩ H=0, moreover if $\mathcal{P} + \mathcal{H}$ is subtractive, then $\mathcal{P} + \mathcal{H} \leq^{e} S$

***Proof*:** Assume that $S$ is CS, let $\mathcal{P}$ and $L$ be subsemimodules of $S$ with $\mathcal{P} \cap L$ =0, so by Zorn's lemma there exists a complement $\mathcal{H}$ of $\mathcal{P}$ such that $L \leq \mathcal{H}$, since $S$ is $CS$, then $\mathcal{H} \leq^{\oplus} S$, and by Lemma (2.2.4) $\mathcal{P} + \mathcal{H} \leq^{e} S$. Conversely, assume that $S$ satisfies the stated condition and let $\mathcal{A}$ be a complement in $S$, then there exists a subsemimodule $\mathcal{P}$ of $S$ such that $\mathcal{A}$ is a complement of $\mathcal{P}$, by hypotheses there exists $\mathcal{H} \leq^{\oplus} S$such that $\mathcal{A} \leq \mathcal{H}$, and $\mathcal{P} \cap \mathcal{H}$=0, since $\mathcal{A}$ is a complement of $\mathcal{P}$, hence $\mathcal{A} = \mathcal{H}$ and therefore $\mathcal{A} \leq^{\oplus} S$, so $S$ is $CS$. □

Recall that, let $\varphi$:$S$→ $\mathcal{A}$ be a homomorphism of$R$-semimodule.

1. If $\varphi$ is a monomorphism, then ker($\varphi$) = 0.
2. If ker($\varphi$ )=0, $S$ is a semisubtractive and $\mathcal{A}$ is cancellative, then $\varphi$is a monomorphism[47].

In the next , we must add a condition that the semimodule had an injective hull, whenever it is needed.

**Lemma 2.2.19:** Let $S$=$S_1 \oplus S_2$ be a cancellative andsemisubtractive$R$-semimodule (withinjective hull$E(S)$), $\varphi \in$ Hom($S_1$,$E(S_2)$) and$\mathcal{K}$={$s_1$+$\varphi$($s_1$): $s_1 \in \varphi^{-1}(S_2)$} then,

1. $\varphi^{-1}(S_2) \cap \mathcal{K}$ = ker$\varphi$.
2. If $\pi$:$S$⟶$S_1$ is the natural projection then $\pi|_{\mathcal{K}}$ is a monomorphism.

***Proof*:**

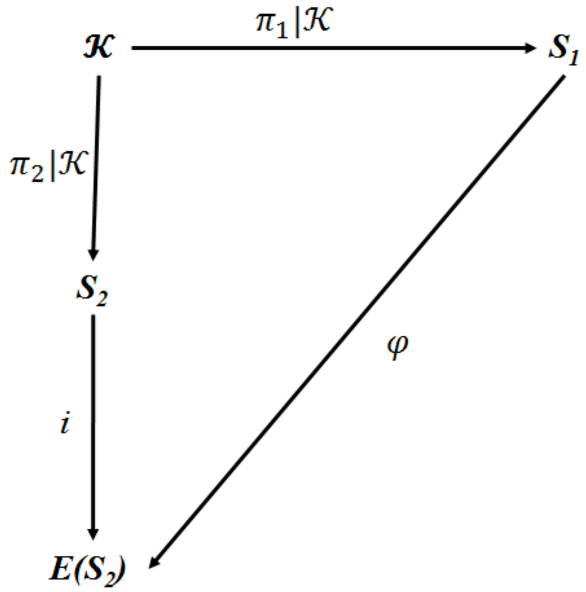
**(1)**First we must prove that $\mathcal{K} \cap S_2$ = 0, let $x \in \mathcal{K} \cap S_2$, then $x$=$s$+$\varphi$($s$), $s \in \varphi^{-1}(S_2) \leq S_1$, $x \in S_2$ and φ($s$) ∈$S_2$implies $s$ ∈$S_2$($S_2$ is subtractive since $S_2 \leq^{\oplus} S$), so $s$ ∈$S_1 \cap S_2$, then $s$ = 0 and $x$ = 0 + φ(0) = 0. Now, let $x \in \varphi^{-1}(S_2) \cap \mathcal{K}$then $x \in S_1$ and $x \in \mathcal{K}$, so $x$=$x_1$+$\varphi$($x_1$) and $\varphi$($x_1$)=0, therefore $x=x_1$ and $x_1\in \ker\varphi$, so $\varphi^{-1}(S_2)\cap \mathcal{K} \subseteq \ker\varphi$, but $\ker\varphi \subseteq \varphi^{-1}(S_2)$. On other hand, $x\in\ker\varphi$, then $\varphi(x)=0$ and $x+\varphi(x)\in \mathcal{K}$, then $\varphi^{-1}(S_2)\cap \mathcal{K} = \ker\varphi$.

**(2)** Since $\ker(\pi|_{\mathcal{K}})=\ker\pi \cap K=S_2\cap K=0$, therefore $\pi|_{\mathcal{K}}$ is a monomorphism. □

In the next lemma , we give a characterization of a complement subsemimodule, in certain cases.

**Lemma 2.2.20:** Let $S = S_1\oplus S_2$ be a $(k\text{-}c)$ $R$-semimodule (with semisubtractive injective hull) and $\mathcal{K}$ is a subsemimodule of $S$, then $\mathcal{K}$ is a complement of $S_2$ in $S$ if and only if $\mathcal{K}=\{s_1+ \varphi(s_1): s_1 \in \varphi^{-1}(S_2)\}$ for some $\varphi \in \mathrm{Hom}(S_1, E(S_2))$.

***Proof*:** Let $\mathcal{K}$ be a complement of $S_2$ in $S$, and $\pi_i:S\rightarrow S_i$, where $(i = 1, 2)$ be the natural projections, since $\ker(\pi_1|_{\mathcal{K}}) = \ker(\pi_1) \cap \mathcal{K} = S_2\cap \mathcal{K}=0$, then $\pi_1|_{\mathcal{K}}$ is a monomorphism, consider the diagram as follow:




Where $i$ is the inclusion map, since $E(S_2)$ is injective, then there exists $\varphi \in\mathrm{Hom}(S_1,E(S_2))$, such that $\varphi(\pi_1|_{\mathcal{K}}) =i(\pi_2|_{\mathcal{K}})$. Let $x\in \mathcal{K}$, then

$x=\pi_1(x)+\pi_2(x)$,since$\varphi(\pi_1(x)=i(\pi_2(x))=\pi_2(x)$, then $x=\pi_1(x)+\varphi(\pi_1(x))$ and $x\in\{s_1+\varphi(s_1):s_1\in\varphi^{-1}(S_2)\}$, hence $\mathcal{K}\subseteq\{s_1+\varphi(s_1): s_1\in\varphi^{-1}(S_2)\}$ and $\{s_1+\varphi(s_1):s_1\in\varphi^{-1}(S_2)\}\cap S_2=0$ [note that $S_2$ is a summand of $S$, hence subtractive], since $\mathcal{K}$ is a complement of $S_2$ by assumption then $\mathcal{K}=\{s_1+\varphi(s_1): s_1\in\varphi^{-1}(S_2)\}$.

*Conversely*, suppose $\varphi\in \mathrm{Hom}(S_1,E(S_2))$, and $\mathcal{K}=\{s_1+\varphi(s_1):s_1\in\varphi^{-1}(S_2)\}$, then $\mathcal{K}\leq S$ and $\mathcal{K}\cap S_2=0$. Now suppose$L$ is a complement of $S_2$in $S$ contain $\mathcal{K}$with$L\cap S_2=0$, and $\mathcal{K}\subseteq L$. Let $u\in L$ satisfy $\pi_2(u)\neq\varphi(\pi_1(u))$, since $E(S_2)$ is semisubtractive then there exists $x$ such that either $\pi_2(u)+x=\varphi(\pi_1(u))$ or $\pi_2(u)=x+\varphi(\pi_1(u))$, it is clear that $0\neq x\in E(S_2)$, hence there exists $r\in R$ such that$0\neq rx\in S_2$(since $S_2\leq^e E(S_2)$) then from case one $\varphi(\pi_1(ru)\in S_2$, hence $ru+rx=\pi_1(ru)+\pi_2(ru)+rx=\pi_1(ru)+\varphi(\pi_1(ru)\in\mathcal{K}\leq L$ (since $L$ is subtractive then $rx\in L\cap S_2=0$. In case two $\pi_2(ru)=rx+\varphi(\pi_1(ru)$. hence $\varphi(\pi_1(ru)\in S_2$(since $S_2$ subtractive) then $ru=\pi_1(ru)+\pi_2(ru)=\pi_1(ru)+\varphi(\pi_1(ru)+rx$, hence $rx\in L\cap S_2=0$, thus for each $u\in L$, $\pi_2(u)=\varphi(\pi_1(u))$, so $u=\pi_1(u)+\pi_2(u)=\pi_1(u)+\varphi(\pi_1(ru)\in\mathcal{K}$, therefore $\mathcal{K}=L$ and $\mathcal{K}$ is a complement of $S_2$ in $S$. □

By Lemma (2.2.20) and Proposition (2.2.16) we have the next result.

**Proposition 2.2.21:** Let $S=S_1\oplus S_2$ be a cancellative and semisubtractive (*k-c*) *R*-semimodule (with semisubtractiveinjective hull) then the following statements are equivalent:

1. $S$ is a CS-semimodule.
2. For every $\varphi\in\mathrm{Hom}(S_1,E(S_2))$, the subsemimodule $\{s_1+\varphi(s_1):s_1\in\varphi^{-1}(S_2)\}$is a CS-semimodule and summand.

***Proof:***

(**1⇒2**) Suppose $\varphi:S_1\to E(S_2)$ and let $\mathcal{K}=\{s_1+\varphi(s_1):\ s_1\in \varphi^{-1}(S_2)\}\leq S$, by Lemma (2.2.20) $\mathcal{K}$ is a complement of $S_2$ in $S$, since $S$ is CS-semimodule and by Proposition (2.2.16) $\mathcal{K}$ is a CS-semimodule and summand.

(**2⇒1**) Let $\mathcal{K}\leq^{c}S$. If $\mathcal{K}\cap S_1=0$ then by Lemma (2.2.20), $\mathcal{K}=\{x+\varphi(x):\ x\in\varphi^{-1}(S_1)\}$ for some $\varphi\in \mathrm{Hom}(S_2,E(S_1))$ and by assumption $\mathcal{K}\leq^{\oplus}S$. If $\mathcal{K}\cap S_1\neq 0$, there exists a closed submodule $\mathcal{N}$ of $\mathcal{K}$ such that $\mathcal{K}\cap S_1\leq^{e}\mathcal{N}$. Clearly $\mathcal{N}\cap S_2=0$. Let $\pi_i:S\to S_i$, where ($i$ = 1, 2) be the natural projections, then $\pi_1|_{\mathcal{N}}$ is a monomorphism and there exists $\varphi\in \mathrm{Hom}(S_1, E(S_2))$ such that $\varphi(\pi_1(n))=\pi_2(n)$ for all $n\in\mathcal{N}$.

If $\mathcal{P}=\{s_1+\varphi(s_1):s_1\in\varphi^{-1}(S_2)\}\leq S$, thenby Lemma (2.2.20) $\mathcal{P}$ is a complement of $S_2$ in $S$, and summand by assumption. Note that if $n\in\mathcal{N}$, then $n=\pi_1(n)+\pi_2(n)=\pi_1(n)+\varphi(\pi_1(n))\in\mathcal{P}$, that is $\mathcal{N}\leq\mathcal{P}$. Since $\mathcal{P}$ is CS-semimodule (by assumption) $\mathcal{N}\leq^{\oplus}\mathcal{P}$, hence $\mathcal{N}\leq^{\oplus}S$, say $S=\mathcal{N}\oplus\mathcal{N}'$, and byLemma(2.2.14), we have $\mathcal{K}=\mathcal{N}\oplus(\mathcal{K}\cap\mathcal{N}')$. Now, $\mathcal{K}\cap\mathcal{N}'\leq^{c}S$, clearly $(\mathcal{K}\cap\mathcal{N}')\cap S_2=0$ by an argument similar to the above $\mathcal{K}\cap\mathcal{N}'\leq^{\oplus}S$ and hence also in $\mathcal{N}'$It follows that $\mathcal{K}\leq^{\oplus}S$. Thus $S$ is CS-semimodule. □

Now we give the following important result.

**Proposition 2.2.22:** Any summand of CS$R$-semimodule is CS.

***Proof*:**Let $S$ be a CS $R$-semimodule and $C$ is asummand of $S$, then $S=C\oplus\mathcal{D}$, for some $\mathcal{D}\leq S$. Let $\mathcal{N}\leq^{c}C$, since $C\leq^{c}S$ then by Lemma (2.2.13) $\mathcal{N}\leq^{c}S$, therefore $\mathcal{N}\leq^{\oplus}S$ (since $S$ is CS-semimodule), so $S=\mathcal{N}\oplus\mathcal{N}'$for some $\mathcal{N}'\leq S$, since $C$ is a subtractive then by Lemma(2.2.14),$C=\mathcal{N}\oplus(C\cap\mathcal{N}')$, and $\mathcal{N}$ is a summand of $C$ therefore $C$ is a CS$R$-semimodule. □

For the next result the following lemmas are required.

**Lemma 2.2.23:** If $\alpha$ ∈Hom($S$,$S'$) is an isomorphism and $\mathcal{N} \leq^{e} S$, then $\alpha(N) \leq^{e} S'$.

***Proof*:** Let $\mathcal{K} \leq S'$ such that $\alpha(\mathcal{N}) \cap \mathcal{K} = 0$, then $\alpha^{-1}(\alpha(\mathcal{N}) \cap \mathcal{K})=0$, thus $\mathcal{N} \cap \alpha^{-1}(\mathcal{K}) = 0$, therefore $\alpha^{-1}(\mathcal{K}) = 0$, and $\mathcal{K} = 0$. □

**Lemma 2.2.24:** If $S \cong S'$, then $S$ is CS-semimodule if and only if $S'$ is CS$R$-semimodule.

***Proof*:** Let $\mathcal{K} \leq S'$ and $S$ is CS-semimodule, then $\alpha^{-1}(\mathcal{K}) \leq S$, since $S$ is CS, then there exist $\mathcal{N}$ is a summand of $S$ such that $\alpha^{-1}(\mathcal{K}) \leq^{e} \mathcal{N}$ therefore $\alpha(\alpha^{-1}(\mathcal{K})) \leq^{e} \alpha(\mathcal{N})$, thus $\mathcal{K} \leq^{e} \alpha(\mathcal{N})$ with $\alpha(\mathcal{N})$is a summand of $S'$(by isomorphism), hence $S'$ is CS. The proof of converse is similar. □

**Proposition 2.2.25:** If $S=S_1 \oplus S_2$ is a cancellative and semisubtractive ($k$-$c$) CS$R$–semimodule (withsemisubtractive injective hull), $S_1$ and $S_2$ are relative injective semimodules and $\varphi \in$ Hom($S_1$, $E(S_2)$), then $\varphi^{-1}(S_2)$ is CS$R$–semimodule.

***Proof*:** Let $\mathcal{N}=\{x+\varphi(x)\colon x \in \varphi^{-1}(S_2)\}$, by (Lemma 2.2.20) $\mathcal{N}$is a complement of $S_2$, also by Proposition (2.2.16) $\mathcal{N}$ is CS$R$–semimodule, let $\pi_1|_{\mathcal{N}} = \alpha$, then$\alpha\colon \mathcal{N} \to S_1$ is a monomorphism. Let $y \in \alpha(\mathcal{N})$,then $y = \pi_1(n)$, $n \in \mathcal{N}$hence y$= \pi_1(x+\varphi(x))=x$, but $x \in \varphi^{-1}(S_2)$, therefore $\alpha(\mathcal{N}) \subseteq \varphi^{-1}(S_2)$. If $x \in \varphi^{-1}(S_2)$, then $x \in S_1$ and $\varphi(x) \in S_2$, but $x+\varphi(x) \in \mathcal{N}$, therefore $\alpha(x+\varphi(x))=\pi_1(x+\varphi(x))=x$, therefore $x \in \alpha(\mathcal{N})$, hence $\alpha(\mathcal{N})=\varphi^{-1}(S_2)$, hence $\alpha'\colon \mathcal{N} \to \varphi^{-1}(S_2)$ defined by $\alpha'(n)= \alpha(n)$ is an isomorohism, Since $\mathcal{N}$ is CS$R$-semimodule by Lemma (2.2.24), $\varphi^{-1}(S_2)$ is CS $R$–-semimodule. □

## 2.3 More on Direct Sum and Summand of Extending Semimodules

In this section, the behavior of extending (CS)$R$–semimodulesaccording to direct sum and summand are studied.

**Proposition 2.3.1:** Let $S=S_1\oplus S_2$ be a ($k$-$c$) $R$-semimodule, where $S_1$ and $S_2$ are CS $R$-semimodule, then $S$ is CS$R$-semimodule if and only if every closed $\mathcal{K} \leq S$ with $\mathcal{K} \cap S_1 = 0$ or $\mathcal{K} \cap S_2 = 0$ ,$\mathcal{K}$is a summand of $S$.

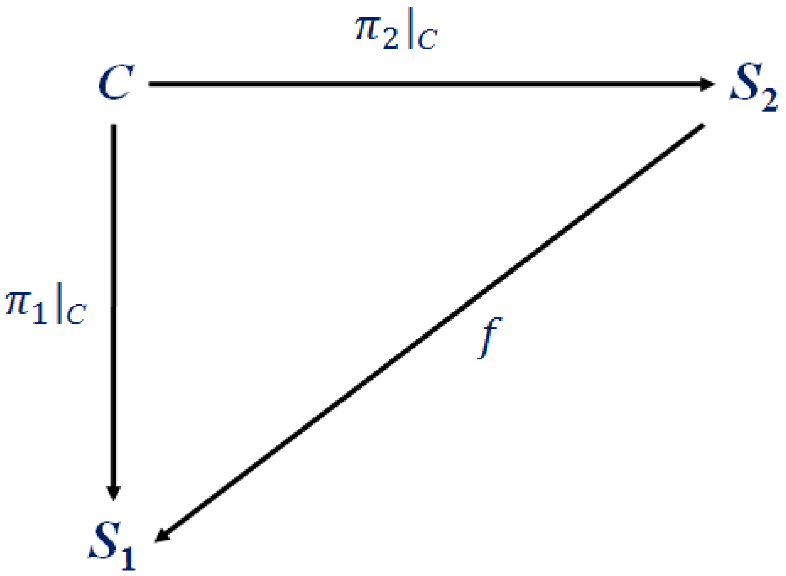
***Proof*:**

(⟹) It is proved by Proposition (2.2.8).

(⟸) Let $\mathcal{B} \leq^c S$, then either$\mathcal{B} \cap S_1 = 0$, then by assumption $\mathcal{B} \leq^{\oplus} S$. Or $\mathcal{B} \cap S_1 \neq 0$, then there exists $\mathcal{D}$ such that $\mathcal{B} \cap S_1 \leq^e \mathcal{D} \leq^c \mathcal{B}$[byRemark (1.1.8)], then$\mathcal{D} \cap S_2 = 0$. Note that $\mathcal{D} \leq^c S$ by Lemma (2.2.13), then by assumption, $\mathcal{D} \leq^{\oplus} S$, that is, $S = \mathcal{D} \oplus \mathcal{D}'$ for some $\mathcal{D}' \leq S$. Let $b \in \mathcal{B}$, then $b=d_1+d_2$, for some $d_1 \in \mathcal{D}$ and $d_2 \in \mathcal{D}'$ since $\mathcal{B}$ is a subtractive and $\mathcal{D} \subseteq \mathcal{B}$ then $d_2 \in \mathcal{B} \cap \mathcal{D}'$ and so $\mathcal{B} \subseteq \mathcal{D} \oplus (\mathcal{B} \cap \mathcal{D}')$ also $\mathcal{D} \oplus (\mathcal{B} \cap \mathcal{D}') \subseteq \mathcal{B}$ therefore $\mathcal{B} = \mathcal{D} \oplus (\mathcal{B} \cap \mathcal{D}')$, but $(\mathcal{B} \cap \mathcal{D}')$ is closed in $S$, with $(\mathcal{B} \cap \mathcal{D}') \cap S_2 = 0$, so by assumption$\mathcal{B} \cap \mathcal{D}' \leq^{\oplus} \mathcal{D}'$,then$\mathcal{D}' = (\mathcal{B} \cap \mathcal{D}') \oplus \mathcal{D}''$for some$\mathcal{D}'' \leq S$, so $S=\mathcal{D} \oplus (\mathcal{B} \cap \mathcal{D}') \oplus \mathcal{D}''=\mathcal{B} \oplus \mathcal{D}''$,therefore $\mathcal{B} \leq^{\oplus} S$ and $S$ is CS semimodule. □

**Lemma2.3.2:** Let $S = S_1 \oplus S_2$, be a cancellative and semisubtractive$R$-semimodule then $S_1$ is $S_2$-injective implies that for every subsemimodule $C$ of $S$ with $C \cap S_1 = 0$, there exists a subsemimodule $S'$ of $S$ such that $S=S_1 \oplus S'$, $C \leq S'$.

***Proof*:** Assume that $S_1$ is $S_2$-injective, let $\pi_i: S\rightarrow S_i$, where ($i$ = 1, 2) be the natural projections, let $C\leq S$ with $C\cap S_1=0$, consider the diagram where $\pi_2|_C$ is a monomorphism. By assumption there exists $f: S_2\rightarrow S_1$ such that $f(\pi_2|_c)=\pi_1|_c$.




Define $S'=\{f(a)+a:a\in S_2\}$, then $S'\leq S$. For $c\in C$, $c=\pi_1(c)+\pi_2(c)=f(\pi_2(c))+\pi_2(c)\in S'$, so $C\leq S'$. For $a\in S$, $a=\pi_1(a)+\pi_2(a)$, if $\pi_1(a)=f(\pi_2(a))$, then $a\in S'\leq S_1+S'$. If $\pi_1(a)\neq f(\pi_2(a))$, then by semisubtractive property either $\pi_1(a)+h=f(\pi_2(a))$ or $\pi_1(a)=f(\pi_2(a))+h$ for some $h\in S$ (in any case $h\in S_1$, since $S_1$ is summand hence a subtractive). So, either $a+h=\pi_1(a)+h+\pi_2(a)=f(\pi_2(a))+\pi_2(a)\in S'\leq S_1+S'$ hence $a\in S_1+S'$. Or $a=\pi_1(a)+\pi_2(a)=f(\pi_2(a))+h+\pi_2(a)=h+f(\pi_2(a))+\pi_2(a)\in S_1+S'$, therefore $S=S_1+S'$.

On other hand, let $a_1+b_1=a_2+b_2\in S_1+S'$ and $b_i=n_i+f(n_i)$ for $n_i\in S_2$, so $a_1+n_1+f(n_1)=a_2+n_2+f(n_2)$.Now, by unique representation in $S_1\oplus S_2$, it follows $n_1=n_2\in S_2$ and $a_1+f(n_1)=a_2+f(n_2)\in S_1$, also $f(n_1)=f(n_2)$ implies $a_1=a_2$. This completes the proof of unique representation in $S_1+S'$. Therefore $S=S_1\oplus S'$.□

For the following result, we give a condition for the direct sum of CS$R$-semimodules to be CS$R$-semimodule.

**Proposition 2.3.3:** Let $S= S_1\oplus S_2$ be acancellative and semisubtractive ($k$-$c$) $R$-semimodule, where $S_1$ and $S_2$ are relative injective semimodules then $S$ is CS-semimodule if and only if $S_1$ and $S_2$ are CS-semimodule.

***Proof*:**

($\Longrightarrow$) It is proved by Proposition (2.2.22).

($\Longleftarrow$) Assume that $S_1$ and$S_2$ are extending$R$-semimoduleand $S_i$ is $S_j$ injective for ($i$, $j$=1, 2 and $i\neq j$).Let$\mathcal{K} \leq^c S$ and $\mathcal{K} \cap S_1 = 0$, by (Lemma 2.3.2) there exists $S'\leq S$ such that $S = S_1\oplus S'$and $\mathcal{K} \leq S'$, it is clear that $S'\cong S_2$and hence $S'$ is extending$R$-semimodule by Lemma (2.2.24). On the other hand,$\mathcal{K} \leq^c S'$ [since $\mathcal{K} \leq^c S$), hence $\mathcal{K} \leq^\oplus S'$, therefore $\mathcal{K} \leq^\oplus S$, similarly for any subsemimodule $\mathcal{H}$ of $S$ with $\mathcal{H} \cap S_2$=0, $\mathcal{H} \leq^\oplus S$, therefore by Proposition (2.3.1), $S$ is CS$R$-semimodule. □

**Proposition 2.3.4:** Let $S=S_1\oplus S_2$ be a cancellative and semisubtractive$R$-semimodule, if $S_1$ is CS$R$-semimodule and $S_2$ is $S_1$ injective then every closed subsemimodule $\mathcal{K}$ of $S$ with $K\cap S_2 = 0$, is a summand of $S$.

***Proof*:**Let $\mathcal{K} \leq^c S$ with $\mathcal{K} \cap S_2$=0, since $S_2$is $S_1$ injective by (Lemma 2.3.2) there exists $S'\leq S$, such that $\mathcal{K} \subseteq S'$ and $S =S' \oplus S_2$, therefore $S'\cong S_1$ since $S_1$ is a CS-semimodule, then $S'$ is a CS-semimodule and $\mathcal{K} \leq^\oplus S'$ (say $S'=\mathcal{K}\oplus\mathcal{K}'$) hence $S =(\mathcal{K}\oplus\mathcal{K}')\oplus S_2=\mathcal{K}\oplus(\mathcal{K}'\oplus S_2)$, that is $\mathcal{K} \leq^\oplus S$. □

**Corollary 2.3.5:**Let $S=S_1\oplus S_2$ be a cancellativesemisubtractive$R$-semimodule, if $S_1$ and $S_2$ are $CS$-semimodules and relatively injective then every closed subsemimodule $\mathcal{K}$ of $S$ with $K\cap S_1$= 0 or $K\cap S_2 = 0$, is a summand of $S$.

***Proof*:** Clear by Proposition (2.3.4).

In the following result, we examine a special case of CS$R$-semimodule.

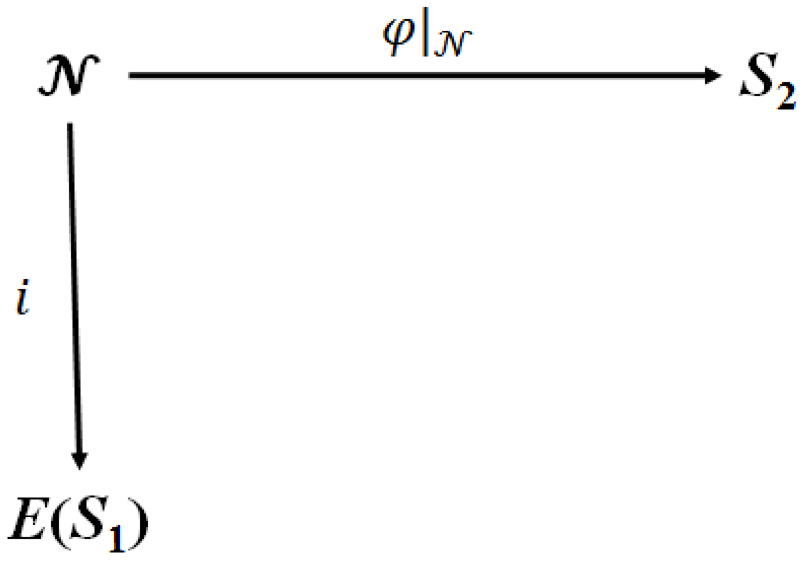
**Corollary 2.3.6:** Let $S_1$ be semisimple $R$-semimodule and $S= S_1\oplus S_2$. If $S$ is ($k$-$c$) semisubtractive andcancellative and $S_2$ is CS then $S$ is CS.

***Proof*:** Let $S = S_1 \oplus S_2$ and $S_2$ is CS, since $S_1$ is CS and $S_2$ is $S_1$-injective, hence by Proposition (2.3.4) and Proposition (2.3.1), $S$ is CS. □

For determining under which condition a subsemimodule has a unique complement we have the following.

**Lemma 2.3.7:** Let $S = S_1 \oplus S_2$ be acancellative and semisubtractive$R$-semimodule(with injective hull$E(S)$), and $\varphi \in \mathrm{Hom}(S_1, E(S_2))$. If $\mathrm{Hom}(S_2, E(S_1))=0$, then $S_2$ is a unique complement of $\mathcal{K}=\{x+\varphi(x): x\in\varphi^{-1}(S_2)\}$.

***Proof*:**Let $Y\leq S$, with $Y\cap\mathcal{K} = 0$. Note that $ker(\varphi)\subseteq\mathcal{K}$. Let $Y\cap\varphi^{-1}(S_2)=\mathcal{N}$, if $\mathcal{N}\neq 0$, then $\varphi|_{\mathcal{N}}$ is a monomorphism [since $\ker(\varphi|_{\mathcal{N}})=\ker\varphi\cap\mathcal{N}\subseteq\mathcal{K}\cap\mathcal{N}\subseteq\mathcal{K}\cap Y=0$]. Consider the diagram:

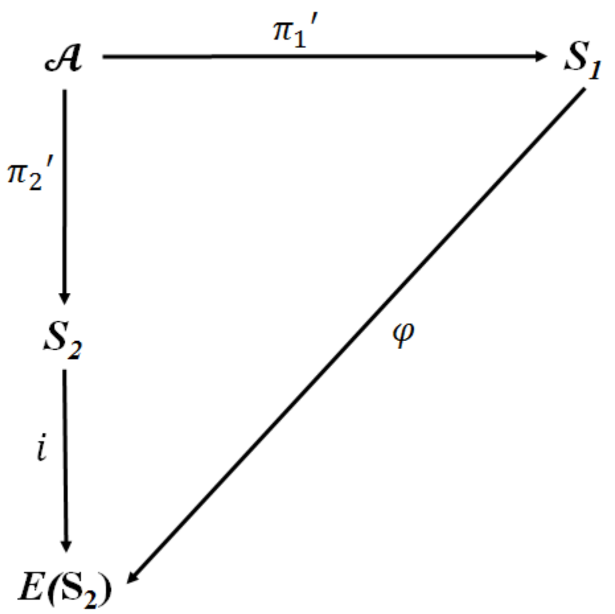


Since $E(S_1)$ is injective, there exists $0 \neq \alpha \in$ Hom $(S_2, E(S_1))$ but this contradicts the assumption, then $\mathcal{N}=0$, therefore $Y \cap \varphi^{-1}(S_2)=0$, but $\varphi^{-1}(S_2) \leq^e S_1$, then $Y \cap S_1 = 0$, hence $\pi_2|_Y$ is a monomorphism and $\pi_1(Y)=0$ therefore, $Y \subseteq S_2$, and $S_2$is a unique complement of $\mathcal{K}$. □

For a specific purpose, we derive a new lemma from Lemma (2.2.20) that will be more generality as follows:

**Lemma 2.3.8:** Let $S = S_1 \oplus S_2$ be acancellative and semisubtractive$R$-semimodule (with injective hull$E(S)$), and $\mathcal{A} \leq S$ with $\mathcal{A} \cap S_2 = 0$, then $\mathcal{A} \leq^c S$ if and only if $\mathcal{A}=\{x+ \varphi(x): x \in X\}$ where $X \leq^c \varphi^{-1}(S_2)$, for some $\varphi \in$Hom$(S_1, E(S_2))$.

***Proof*:** Let $\pi_i : S \rightarrow S_i$, where $(i=1,2)$ be the natural projections, since $\mathcal{A} \cap S_2=0$, then $\pi_1{}'=\pi_1|_{\mathcal{A}} : \mathcal{A} \rightarrow S_1$ is a monomorphism. Consider the diagram where $\pi_2{}'= \pi_2|_{\mathcal{A}}$ and $i$is the inclusion mapping.



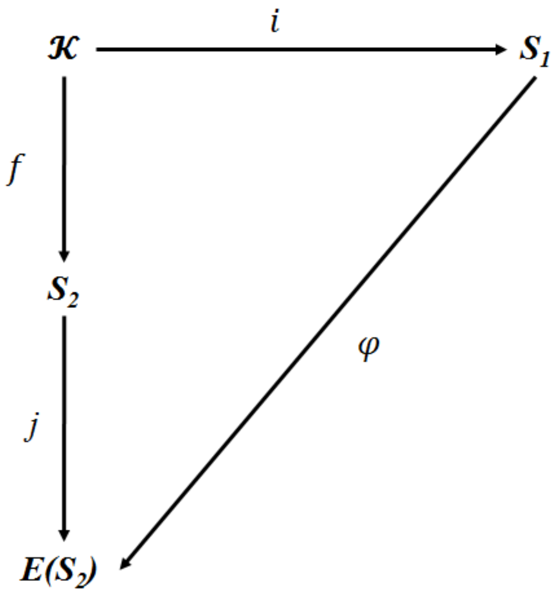

Since $E(S_2)$ is injective, then there exists $\varphi \in$ Hom$(S_1, E(S_2))$ such that $\varphi(\pi_1{}'(a))=\pi_2{}'(a)$ for all $a \in \mathcal{A}$, then$\varphi(\pi_1(a))=\pi_2(a)$. Hence, for each $a \in \mathcal{A}$, $a=\pi_1(a)+\pi_2(a)=\pi_1(a)+\varphi(\pi_1(a))$, so $\mathcal{A}=\{x+\varphi(x): x \in \pi_1(\mathcal{A})\}$, note that $\pi_2(\mathcal{A})=\varphi(\pi_1(\mathcal{A}))\subseteq S_2$, hence $\pi_1(\mathcal{A})\leq \varphi^{-1}(\pi_2(\mathcal{A})) \leq \varphi^{-1}(S_2)$, if $\pi_1(\mathcal{A}) \leq^e Y \leq \varphi^{-1}(S_2)$, then $\mathcal{A} \leq^e \{y+\varphi(y): y \in Y\} \leq S$ (for $y+\varphi(y)\neq 0$, then $y\neq 0$ so there exists $r\in R$ such that $ry\in \pi_1(\mathcal{A})$ i.e $ry=\pi_1(a)$, $a\in \mathcal{A}$, then $\varphi(ry)=\varphi(\pi_1(a))=\pi_2(a)$, thus $ra=r\pi_1(a)+r\pi_2(a)=r(y+\varphi(y)) \in \mathcal{A}$, since $\mathcal{A} \leq^c S$, it follow that $\mathcal{A}=\{y+\varphi(y): y \in Y\}$ then $\pi_1(\mathcal{A})=Y$, and hence $\pi_1(\mathcal{A}) \leq^c \varphi^{-1}(S_2)$. *Conversely*, if $\mathcal{A}=\{x+\varphi(x): x \in X\}$ and $X\leq^c \varphi^{-1}(S_2)$, it is clear that $\mathcal{A} \leq \mathcal{K}=\{x+\varphi(x): x \in \varphi^{-1}(S_2)\}$, and that $\mathcal{A}$ has a proper essential extension in $\mathcal{K}$ if and only if $X$ has a proper essential extension in $\varphi^{-1}(S_2)$, since $X$ is closed in $\varphi^{-1}(S_2)$, it follows that $\mathcal{A} \leq^c \mathcal{K}$ then $\mathcal{A} \leq^c S$. □

**Lemma 2.3.9:** Let $S= S_1\oplus S_2$ be acancellative and semisubtractive$R$-semimodule (with injective hull$E(S)$), where $S_1$ and $S_2$ are subsemimodules of $S$. If $S_2$ is $S_1$-injective then any closed subsemimodule $\mathcal{A}$ in $S$ with $\mathcal{A}\cap S_2=0$ must have the form $\mathcal{A}=\{x+\varphi(x): x\in X\}$, where $X$ is closed subsemimodule of $S_1$ and $\varphi \in$Hom$(S_1, E(S_2))$.

***Proof*:** Let $\mathcal{A}$ be a closed subsemimodule in $S$ with $\mathcal{A}\cap S_2 = 0$, then by Lemma (2.3.8) $\mathcal{A}=\{x+\varphi(x): x \in X\}$, where $X$ is a closed subsemimodule of $\varphi^{-1}(S_2)$, for some $\varphi \in$ Hom$(S_1, E(S_2))$. But $\varphi^{-1}(S_2) \leq^e S_1$, so $X\leq^c S_1$.
□

***Lemma* 2.3.10:** Let $S=S_1\oplus S_2$ be an $R$-semimodule (with injective hull$E(S)$), where $S_1$ and $S_2$ are subsemimodules of $S$. If $\varphi^{-1}(S_2)=S_1$ for each $\varphi \in$Hom$(S_1, E(S_2))$,then $S_2$*is* $S_1$-injective.

***Proof*:**Consider the diagram below, assume $\mathcal{K}$ is a subsemimodule of $S_1$ in $S$ where $i$, $j$ are inclusion maps,$f$ is any homomorphism.




Since $E(S_2)$ is injective there exists $0 \neq \varphi \in \mathrm{Hom}(S_1, E(S_2))$. such that $\varphi i=jf$. Since $\varphi^{-1}(S_2)=S_1$, then $\varphi(S_1)\subseteq S_2$ and $\varphi \in \mathrm{Hom}(S_1, S_2)$.Therefore $S_2$is $S_1$-injective. □

In the following, the condition of Proposition (2.3.4) gives extra results.

**Proposition 2.3.11:** Let $S=S_1\oplus S_2$ be a cancellative and semisubtractive ($k$-$c$)$R$-semimodule (with semisubtractive injective hull$E(S)$), and $\mathrm{Hom}(S_2,E(S_1))=0$, then $S_1$ is a CS-semimodule and $S_2$is $S_1$–injective if and only if every closed subsemimodule $\mathcal{K}$ of $S$ with $\mathcal{K}\cap S_2=0$, is a summand of $S$.

***Proof***:

(⇒) It is proved by Proposition (2.3.4).

(⇐) Suppose $\mathcal{K} \leq^c S_1$, then$\mathcal{K}\cap S_2=0$and$\mathcal{K} \leq^c S$ by Lemma (2.2.13). By assumption $\mathcal{K} \leq^{\oplus} S$, say $S=\mathcal{K}\oplus\mathcal{K}'$, hence by Lemma(2.2.14), $S_1=\mathcal{K}\oplus(\mathcal{K}'\cap S_1)$,therefore $\mathcal{K} \leq^{\oplus} S_1$, and $S_1$ is a CS- semimodule.

Now, let $\alpha \in$Hom($S_1$,$E(S_2)$) be arbitrary, then by (Lemma (2.2.20) $L$={$x$+ $\alpha(x)$: $x\in \alpha^{-1}(S_2)$}is closed in $S$ and it is a complement of $S_2$, by assumption $L \leq^{\oplus} S$. If $\pi_1$is the natural projection of $S= S_1 \oplus S_2$ onto $S_1$, then $y\in L$ implies $y= x+ \alpha(x)$ for some $x\in \alpha^{-1}(S_2)$ and $\pi_1(y) = \pi_1(x)=x$, that is, $\pi_1(L)\subseteq \alpha^{-1}(S_2)$. If $x\in \alpha^{-1}(S_2)$, then $\pi_1(x)=x$ and $x+ \alpha(x) \in L$ hence $x=\pi_1(x)=\pi_1(x+ \alpha(x)) \in \pi_1(L)$, that is, $\pi_1(L) = \alpha^{-1}(S_2)$. Since $\pi_1(L)$ is closed in $S_1$ and $\alpha^{-1}(S_2)$ is essential in $S_1$, it follows $S_1=\alpha^{-1}(S_2)$. Therefore, by (Lemma 2.3.10), $S_2$ is $S_1$-injective. □

For Propositions (2.2.22),( 2.3.11) and(2.3.1) we have the following corollary.

**Corollary 2.3.12:**Let $S = S_1 \oplus S_2$ be a cancellative andsemisubtractive ($k$-$c$)$R$-semimodule (with injective hull$E(S)$), and Hom($S_2$,$E(S_1)$) = 0, then $S$ is CS if and only if $S_1$and $S_2$ are CS-semimoduleand $S_2$ is$S_1$–injective.

## 2.4 Extending of Rational Hull

The properties of the rational hull semimodule are introducing and investigating in this section. It can be seen by analyzing the structure of rational hull semimodule; there are many properties of dense subsemimodules that are also useful.

**Definition 2.4.1:** Let $S$ be an $R$-semimodule with injective hull $E(S)$the rational hull of $S$ denoted by $\check{E}(S)$is a subsemimodule of $E(S)$defined by:
$\check{E}(S)$={$x\in E(S)$|$h(S)$=0 implies $h(x)$=0, for all$h\in$End($E(S)$)}=$\cap${ker$h$|$h\in$End($E(S)$) and $S\subseteq$ker$h$}.

***Note***: It is clear that $S\leq \check{E}(S)$.

**Definition 2.4.2:** A subsemimodule $\mathcal{N}$ of $S$ is said to be dense subsemimodule if for all $x$ and $y$ in $S$ with $x \neq 0$, there exists $r \in R$, such that $rx \neq 0$ and $ry \in \mathcal{N}$. In this case we write $\mathcal{N} \leq_d S$.

**Remark 2.4.3:** Every dense subsemimodule is essential.

***Proof*:**Clear.

***Note***: The converse is not true in general, for example$< \bar{2} >$in $Z_4$ as a $\mathbb{Z}$-semimodule is essential but not dense since 3,2$\in Z_4$, then there exist 2$\in R$, such that 2.3=2$\in Z_4$but 2.2=0.

In the next, there are some properties of dense subsemimodules.

**Lemma 2.4.4:** Let$\mathcal{B}$and $C$ are subsemimodules of $S$.

**(i)** If $\mathcal{B} \leq_d S$ and $C \leq_d S$ then $\mathcal{B} \cap C \leq_d S$.

**(ii)** $\mathcal{B} \leq C \leq S$ then $\mathcal{B} \leq_d S$ if and only if $\mathcal{B} \leq_d C$and $C \leq_d S$.

***Proof:***

**(i)** Let $x, y \in S$ with $x \neq 0$, then there exists $r \in \mathcal{R}$ such that $rx \neq 0$ and $ry \in \mathcal{B}$. Also there exists $r' \in R$ such that $r'rx \neq 0$, and$r'ry \in C$ but$r'ry \in \mathcal{B}$, hence$r'ry \in \mathcal{B} \cap C$ .

**(ii)** Assume that $\mathcal{B} \leq_d C$and $C \leq_d S$, let $x,y \in S$, with $x \neq 0$, then there exists $r \in R$ such that $rx \neq 0$ and $ry \in C$, since $C \leq^e S$ (by Remark(2.4.3)), then there exists $s \in R$, such that $srx \neq 0$ and $sry \in C$ now $\mathcal{B} \leq_d C$, implies there exists $t \in R$such that $t(srx) \neq 0$ and $t(sry) \in \mathcal{B}$. The other direction is clear . □

**Proposition 2.4.5:**Let $S$ be an $R$-semimodule with injective hull $E(S)$, then the following statements are equivalent:

(i) $\mathcal{N} \leq_d S$.

(ii) Hom$_R$ $(S/\mathcal{N}, E(S)) = 0$, where $\mathcal{N}$ is a subtractive .

(iii) Forany subsemimodule $\mathcal{P}$ such that $\mathcal{N}$ is a subtractive and $\mathcal{N} \leq \mathcal{P} \leq S$, Hom$_R(\mathcal{P}/\mathcal{N}, S)=0$.

***Proof*:**

**(i)⇒(ii)** Let $f \in$ Hom $(S/\mathcal{N}, E(S))$, and assume that $f(a+\mathcal{N})=y$, $y\neq 0$, since $S \leq^e E(S)$ and by Remark(2.4.3) $\mathcal{N} \leq^e S$, then $\mathcal{N} \leq^e E(S)$, so there exists $r \in R$ such that $0\neq ry \in S$, and there exists $s \in R$ such that $s(ry) \neq 0$, and $s(ry) \in \mathcal{N}$ therefore $0\neq (sr)y = srf(a+\mathcal{N}) = f(sra+\mathcal{N}) = f(0) = 0$, that is contradiction, therefore $f=0$ hence Hom $(S/\mathcal{N}, E(S))=0$.

**(ii)⇒(iii)** Let $g: \mathcal{P}/\mathcal{N} \to S$ be a homomorphism, such that $g\neq 0$, and let $i: S \to E(S)$ be the inclusion map, then $0\neq ig: \mathcal{P}/\mathcal{N} \to E(S)$, can be extended to $f: S/\mathcal{N} \to E(S)$ by (ii), $f=0$ this a contradiction, hence $g=0$.

**(iii)⇒(i)** Let $0\neq x \in S$ and $y \in S$, assume for all $r \in R$, $0=rx$, $ry \notin \mathcal{N}$, then defined $f: (\mathcal{N} + \mathcal{R}y)/\mathcal{N} \to S$ , by $(n+ry)+\mathcal{N} \to rx$ ($f$ is well defined ) by (iii) $f=0$, this a contradiction since $f(n+y+\mathcal{N})=x\neq 0$, therefore $\mathcal{N} \leq_{d} \mathcal{A}$.

□

**Lemma 2.4.6:** Let $S$ be a subtractive $R$-semimodule with injective hull $E(S)$ and $S \leq \mathcal{B} \leq E(S)$ then $S \leq_d \mathcal{B}$ if and only if $\mathcal{B} \leq \check{E}(S)$.

***Proof*:**Assume that $S \leq_d \mathcal{B}$, and consider $h \in$ End$(E(S))$ such that $h(S)=0$ if $h(\mathcal{B}) \neq 0$, then $0\neq$ Hom$_R(\mathcal{B}/S, E(S))$=Hom$_R(\mathcal{B}/S, E(\mathcal{B}))$, this contradicting by Lemma(2.4.5) therefore $h(\mathcal{B})=0$, which implies $\mathcal{B} \leq \check{E}(S)$. Conversely, consider $h$: $\check{E}(S) \to E(\check{E}(S)) = E(S)$, with $h(S)=0$, since $h$ can be extended to $E(S)$, we can consider it as an endomorphism of $E(S)$, then by definition of $\breve{E}(S)$, $h(\breve{E}(S))$=0, thus Hom($\breve{E}(S)/S$,$E(S)$)=0, and by Lemma(2.4.5) $S \leq_d \breve{E}(S)$, then $S \leq_d B$. □

**Corollary 2.4.7:** Let $S$ be a subtractive and $S \leq \mathcal{B} \leq E(S)$ if $\mathcal{B} \leq \breve{E}(S)$, then $S \leq_d \breve{E}(S)$.

***Proof:*** Immediately by definition and Lemma (2.4.6).

**Lemma 2.4.8:** Let $S$ be an $R$- semimodulewith injective hull $E(S)$.If $S = S_1 \oplus S_2$ then $\breve{E}(S) = \breve{E}(S_1) \oplus \breve{E}(S_2)$.

**Proof:** Let $S = S_1 \oplus S_2$ implies $E(S) = E(S_1) \oplus E(S_2)$, so $x \in \breve{E}(S)$ implies $x = x_1 + x_2$ with $x_i \in E(S_i)$ $(i=1,2)$ and $h(x)=0$, for all $h \in End(E(S))$, with $h(S)=0$. Hence $h(x_i)=0$ $(i=1,2)$. If $g \in End(E(S_1))$, with $g(S_1)=0$, then $g\pi_1 \in End(E(S))$ and $(g\pi_1)(S) = g(S_1) = 0$ (where $\pi_1$ is the natural projection of $E(S)$onto $E(S_1)$, then $g(x_1)=0$, then $x_1 \in \breve{E}(S_1)$. Similarly $x_2 \in \breve{E}(S_2)$, therefore$\breve{E}(S) = \breve{E}(S_1) \oplus \breve{E}(S_2)$ . □

**Theorem 2.4.9:** If $S$ is a CS $R$-semimodule with injective hull $E(S)$, then $\breve{E}(S)$ is a CS $R$-semimodule.

***Proof:*** Let $\mathcal{K} \leq \breve{E}(S)$, then $X = \mathcal{K} \cap S \leq S$, hence $S = S_1 \oplus S_2$ and $X \leq^e S_1$ (since $S$is $CS$). Note that $S_1 \leq^e \breve{E}(S_1)$ implies $X \leq^e \breve{E}(S_1)$. Let $0 \neq k \in K$, if $\pi_2(k) \neq 0$ (where $\pi_2$is the natural projection of $E(S)$ onto $E(S_2)$), then there exists $r \in R$ such that $0 \neq r\pi_2(k) \in S_2$(since $S_2 \leq^e E(S_2)$), so $0 \neq \pi_2(rk) \in S_2$ but $rk \in X \leq^e S_1$ a contradiction, hence $\pi_2(k) = 0$ that is $k \in E(S_1)$ and $K \leq E(S_1)$. Now let $k = k_1 + k_2$ where $k_i \in \breve{E}(S_i)$ but $\pi_2 \in End(E(S))$, so $\pi_2(k)=0$ implies $k_2=0$, hence $K \leq \breve{E}(S_1)$. On the other hand $X \leq^e S_1 \leq^e \check{E}(S_1)$ and $X \leq K \leq \check{E}(S_1)$, implies $K \leq^e \check{E}(S_1) \leq^{\oplus} \check{E}(S)$ by Lemma( 2.4.8), therefore $\check{E}(S)$ is CS. □

# Chapter Three:

# Extending Semimodules with Singularity

## 3.1. Introduction

In this chapter, some properties of singular and second singular subsemimodules of semimodule are studied. In addition discusses the relationship between the above concepts, with CS-semimodules. Furthermore the concept of quasi-continuous property which is stronger than extendingproperty,has some results related to CS and singularity.

## 3.2 Singular and Second Singular Subsemimodules

In this section, some properties of singular and second singular subsemimodules are proved.

**Proposition 3.2.1:** For any $R$-semimodule $S$, $Z(S)$ is a subtractive subsemimodule of $S$.

***Proof*:** Clearly $0\in Z(S)$ since $R0 = 0$, where $R\leq^{e} R$. Let $a$, $b\in Z(S)$, then there exist left ideals$I$, $J\leq^{e} R$, such that $Ia=Jb=0$, since $I\cap J\leq^{e} R$, hence $(I\cap J)(a+b)= (I\cap J)a+(I\cap J)b\leq Ia+Jb=0$, hence $a+b\in Z(S)$. Let $a\in Z(S)$ and $r\in R$, then there exists left$I\leq^{e} R$ such that $Ia=0$ and $r(Ia)=I(ra)=0$, so $ra\in Z(S)$, then $Z(T)$ is a subsemimodule of $T$. Let $(a+b)\in Z(T)$ and $a \in Z(T)$ then there exist left$I,J\leq^{e} R$ such that $I(a+b)=Ja=0$, hence$(I\cap J)(a+b)=0$, so $(I\cap J)a+(I\cap J)b=0$ but $(I\cap J)a=0$ [since$Ja=0$] then $(I\cap J)b=0$, where $(I\cap J)\leq^{e} R$, thus $b\in Z(S)$ and $Z(S)$ is a subtractive subsemimodule of $S$. □

***Note***: If $S_1$ and $S_2$ are$R$–semimodules with $f\in \text{Hom}(S_1, S_2)$. If $S_2'$ is a subtractive subsemimodule of $S_2$ then $f^{-1}(S_2')$ is a subtractive subsemimodule of $S_1$ see ([41],p.171).

**Proposition 3.2.2:** For any $R$-semimodule $S$, $Z_2(S)$ is a subtractive subsemimodule of $S$.

***Proof*:** $Z_2(S)$ is a subsemimodule of $S$ by definition. Let $\pi:S \rightarrow S/Z(S)$ be the natural epimorphism, then since $Z(S/Z(S))=Z_2(S)/Z(S)$ it follows that $Z_2(S)=\pi^{-1}(Z(S/Z(S))$, but $Z(S/Z(S))$ is a subtractive subsemimodule of $S/Z(S)$, based on Proposition (3.2.1),So $Z_2(S)$ is subtractive in $S$. □

It is known in modules that singularity commutes with the direct sum, in the following this property will be proved for semimodules.

**Proposition 3.2.3:** Let $S=S_1\oplus S_2$ be an $R$–semimodule then $Z(S)=Z(S_1)\oplus Z(S_2)$.

***Proof*:** Let $s\in Z(S)$ such that $s=s_1+s_2$, where $s_1\in S_1$ and $s_2\in S_2$, then $Is=0$, for some left ideal$I\leq^e R$, then $Is=Is_1+Is_2=0$, since $Is_1\leq S_1$ and $Is_2\leq S_2$, by unique representation of direct sum then $Is_1=Is_2=0$. Therefore $s_1\in Z(S_1)$, $s_2\in Z(S_2)$ and $Z(S)=Z(S_1)+Z(S_2)$, by unique representation in $S$, then $Z(S)=Z(S_1)\oplus Z(S_2)$. □

There are some properties of a singular submodules, those properties will be converted for subsemimodules.

The next result was proved in [48].

**Proposition 3.2.4:** Let $S$ be an $R$–semimodule and $\mathcal{K}\leq S$, then $Z(\mathcal{K})=\mathcal{K}\cap Z(S)$.

**Proposition 3.2.5:** Let $S$ be an $R$–semimodule and $\mathcal{K}\leq S$, then $Z_2(\mathcal{K})=\mathcal{K}\cap Z_2(S)$.

***Proof*:** Let $x\in Z_2(\mathcal{K})\leq \mathcal{K}$ then $x\in \mathcal{K}$ and$x+Z(\mathcal{K})\in Z(\mathcal{K}/Z(\mathcal{K}))$, hence there exists left$I\leq^e R$ such that $I(x+Z(\mathcal{K}))=\bar{0}$, therefore $Ix\leq Z(\mathcal{K})$, since

$Z(\mathcal{K}) \leq Z(S)$, hence $Ix \leq Z(S)$, and $I(x+Z(S))=\bar{0}$, therefore $x \in Z_2(S)$ and $Z_2(\mathcal{K})=\mathcal{K} \cap Z_2(S)$. □

The next result was proved in[50].

**Lemma 3.2.6:** Let $S$ be an $R$–semimodule. If $S=S_1 \oplus S_2$ and $\mathcal{K}$ is fully invariant in $S$ then $\mathcal{K}=(\mathcal{K} \cap S_1) \oplus (\mathcal{K} \cap S_2)$.

**Proposition 3.2.7:** For any $R$-semimodule $S$, $Z(S)$ is a fully invariant subsemimodule of $S$.

***Proof*:** Let $f \in End(S)$, and $x \in Z(S)$, then there exists left $I \leq^{e} R$, such that $Ix=0$, hence $f(Ix)=0$, therefore $If(x)=0$ and $f(x) \in Z(S)$, hence $f(Z(S)) \subseteq Z(S)$, and $Z(S)$ is fully invariant in $S$. □

**Proposition 3.2.8:** For any $R$-semimodule $S$, $Z_2(S)$is fully invariant in $S$.

***Proof*:** Let $f \in End(S)$ and $x \in Z_2(S)$, then $x+Z(S) \in Z(S/Z(S))$, so there exists left $I \leq^{e} R$, such that $I(x+Z(S))=\bar{0}$, andhence $Ix \leq Z(S)$ so $f(Ix) \leq f(Z(S)) \leq Z(S)$ by Proposition (3.2.7), therefore, $I(f(x)+Z(S))=\bar{0}$, and so $f(x) \in Z_2(S)$. Thus $f(Z_2(S)) \leq Z_2(S)$ and $Z_2(S)$ is fully invariant in $S$. □

It is known in modules that second singular commutes with a direct sum, in the following this property will be proved for semimodule.

**Proposition 3.2.9:** Let $S$ be an $R$–semimodule. If $S=S_1 \oplus S_2$ then $Z_2(S) = Z_2(S_1) \oplus Z_2(S_2)$.

***Proof*:** Clearly by Proposition (3.2.8), $Z_2(S)$ is fully invariant in $S$, hence by Proposition (3.2.6), $Z_2(S)=(Z_2(S) \cap S_1) \oplus (Z_2(S) \cap S_2)$, since $S_1 \leq S$ and $S_2 \leq S$, hence by Proposition (3.2.5) $Z_2(S)=Z_2(S_1) \oplus Z_2(S_2)$. □

***Note*:** Another proof can be given for Proposition (3.2.3), by using the argument of the proof of Proposition (3.2.9).

**Lemma 3.2.10:**If $B\leq^{e}S$ then for every $a\in S$, there exists left$I\leq^{e}R$, such that$Ia\leq B$.

***Proof*:** Assume $a\in S$, and $I = [B{:}a]= \{r\in R{:}\ ra\in B\}$. Let $J$ be any nonzero left ideal of $R$. *If* $J\subseteq I$,then$J\cap I=J\neq 0$, let $j\in J\backslash I$, then $0\neq ja\in S$(for if $ja=0$, then $j\in I$), since $\mathcal{B}\leq^{e}S$, then there exists $r\in R$, such that $0\neq rja\in\mathcal{B}$, so $0\neq rj\in I\cap J$, that is $I\cap J\neq 0$, so $I\leq^{e}R$, and $Ia\leq\mathcal{B}$. □

**Proposition 3.2.11:** For any semimodule $S$, $Z_2(S)\leq^{c}S$.

***Proof*:** Assume that $Z_2(S)\leq^{e}S'\leq S$, let $x\in S'$, $I$=ann($x+Z(S)$) and $J$ be any nonzero left ideal of $R$(not contained in $I$). Let $0\neq j\in J$, such that $0\neq jx\in S'$, then there exists $r\in R$, such that$0\neq r(jx)\in Z_2(S)$[ since $Z_2(S)\leq^{e}S'$] then $(rjx+Z(S))\in Z(S/Z(S))$, so there exists left$K\leq^{e}R$, such that $K(rjx+Z(S))=\bar{0}$. Now since $K\leq^{e}R$, there exists $s\in R$ such that $0\neq s(r\mathrm{j})\in K$, then $s(rj)\in I$, and $s(rj)\in J$, hence $I\cap J\neq 0$, therefore $I\leq^{e}R$, and $x\in Z_2(S)$, which means $Z_2(S)=S'$, hence $Z_2(S)$ is closed in $S$. □

Recall that to consider the quotient $S/\mathcal{B}$, where$S$ is a semimodule,we must have a subtractivesubsemimodule $\mathcal{B}$ of $S$(see[41] p.165 ).

**Lemma 3.2.12:**If $\mathcal{B}\leq^{e}S$ and subtractive then $S/\mathcal{B}$ is singular.

***Proof*:** Let $x\in S$, and let $f{:}R\longrightarrow S$, defined by $r\longmapsto rx$, then $f^{-1}(\mathcal{B})\leq^{e}R$, but

$f^{-1}(\mathcal{B})=\mathrm{ann}_R\,(x+\mathcal{B})$,so $x+\mathcal{B}\in Z(S/\mathcal{B})$, then $Z(S/\mathcal{B}) = S/\mathcal{B}$. □

The following example shows that the converse of Lemma (3.2.12) is not true.

**Example 3.2.13:** Let $B= \mathbb{Z}_2$, and $A$=0, then $\mathcal{B}/A$ is singular but $A$ is not essential in $\mathcal{B}$.

**Proposition3.2.14:**If $\mathcal{B}$ is a subtractive subsemimodule of an $R$-semimodule$S$ and$S/\mathcal{B}$ is nonsingular, then $\mathcal{B} \leq^{c} S$.

***Proof*:** Assume that $\mathcal{B} \leq^{e} \mathcal{B}' \leq S$, then by Lemma (3.2.12) $\mathcal{B}'/\mathcal{B}$ is singular, but $\mathcal{B}'/\mathcal{B} \leq S/\mathcal{B}$, then $\mathcal{B}'/\mathcal{B}=Z(\mathcal{B}'/\mathcal{B}) \leq Z(S/\mathcal{B})$, this is a contradiction, therefore $\mathcal{B}$ has no proper essential extension, and it is closed. □

**Lemma 3.2.15:** Let $S$ be a nonsingular $R$-semimodule and $\mathcal{B}$ is a subtractive subsemimodule of $S$, then $\mathcal{B} \leq^{c} S$ if and only if $S/\mathcal{B}$ is nonsingular.

***Proof*:**let $\mathcal{B} \leq^{c} S$, and $0 \neq x+\mathcal{B} \in S/\mathcal{B}$. If $I(x+\mathcal{B})=\bar{0}$, where left $I \leq^{e} R$, hence $\mathcal{B} \leq^{e} Rx+\mathcal{B} \leq S$, (Let $0 \neq rx+b \in Rx+\mathcal{B}$, if $rx$=0, then $b \in \mathcal{B}$. If $rx \neq 0$, then $r \neq 0$, hence there exists $s \in R$ such that $0 \neq sr \in I$, this implies$0 \neq srx \in \mathcal{B}$, that is, $0 \neq s(rx+b) \in \mathcal{B}$) this is a contradiction, hence$S/\mathcal{B}$ is nonsingular. *Conversely*, assume that $S/\mathcal{B}$ is nonsingular and $\mathcal{B} \leq^{e} S' \leq S$, then by Lemma (3.2.12), $S'/\mathcal{B}$ is singular, which contradicts the assumption $S/\mathcal{B}$ is nonsingular. □

**Lemma 3.2.16:** Let $S$ be an $R$-semimodule and $\mathcal{B} \leq S$ if $S= Z_2(S) \oplus \mathcal{B}$ then $\mathcal{B}$ is nonsingular.

***Proof*:** It is clear by $Z_2(S)=Z_2(S) \oplus Z_2(\mathcal{B})$, hence $Z_2(\mathcal{B})=0$ and $\mathcal{B}$ is nonsingular.□

**Corollary 3.2.17:** If $S$ is nonsingular semimodule, then $S/Z_2(S)$ is nonsingular.

***Proof*:**By Proposition (3.2.11), $Z_2(S)\leq^c S$ then by Lemma (3.2.15), $S/Z_2(S)$ is nonsingular. □

## 3.3 Extending Semimodules with Singularity

In modules theory the following proposition proved by Kamal and Muller [29]. But in semimodule theory we must add the following condition.

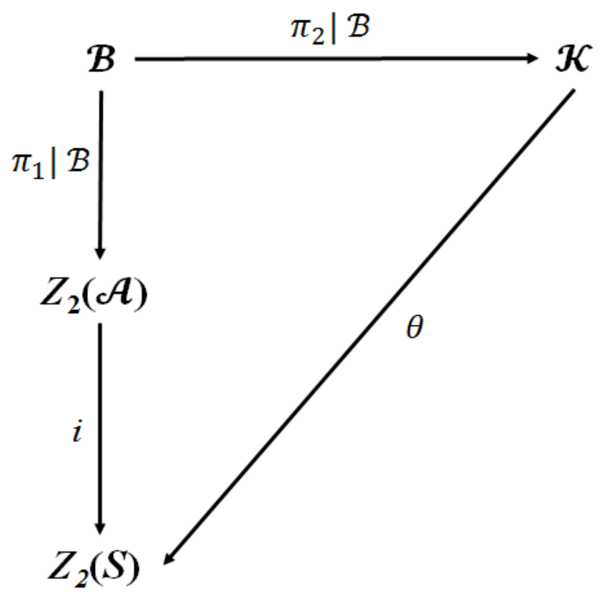
**Proposition 3.3.1:** Let $S$ be a$(k\text{-}c)$CS$R$-semimodule and $Z_2(S)$ be a group over semiring, then $S=Z_2(S)\oplus\mathcal{K}$, where $Z_2(S)$ and $\mathcal{K}$are CS and $Z_2(S)$ is $\mathcal{K}$-injective.

***Proof*:** Since by Proposition (3.2.11)$Z_2(S)$ is closed in $S$ and $S$ is extending, then $Z_2(S)$ is a summand of $S$ therefore, $S=Z_2(S)\oplus\mathcal{K}$, for some $\mathcal{K}\leq S$, so by Lemma (3.2.16) $\mathcal{K}$ is nonsingular. Since $S$ is extending, hence by Proposition (2.2.22), $Z_2(S)$ and $\mathcal{K}$are CS. To show that $Z_2(S)$ is $\mathcal{K}$–injective, let $X\leq\mathcal{K}$and$\varphi:X\rightarrow Z_2(S)$ be any homomorphism.Suppose $X'=\{x\text{-}\varphi(x): x\in X\}$, (note that $\varphi(x)$ has additive inverse since by assumption $Z_2(S)$ is a group).Clearly $X'\leq S$, since $S$ is CS, hence there exists $L\leq^{\oplus}S$ such that $X'\leq^e L$, and $S=L\oplus Y$, for some $Y\leq S$. Since $X'\cap Z_2(S)=0$ ([if $y\in X'\cap Z_2(S)$, then $y=x\text{-}\varphi(x)$ so $x=y+\varphi(x)$, then $x\in\mathcal{K}\cap Z_2(S)=0$, hence $\varphi(x)=y=0$) then $X'\cap Z_2(S)\cap L=0$, but $X'\leq^e L$, so $Z_2(S)\cap L=0$.That is, by Proposition (3.2.5), $Z_2(L)=0$ and $L$ is nonsingular. By Proposition (3.2.9), $Z_2(S)=Z_2(Y)$ and $Z_2(S)$ is a summand of $Y$, so $Y=Z_2(S)\oplus Y'$, for some $Y'\leq Y$.Now $S=L\oplus Z_2(S)\oplus Y'$. Let $\pi:S\rightarrow Z_2(S)$ be the natural projection and $x\in X$, such that $x=\ell+z+y$, where $\ell\in L$, $z\in Z_2(S)$, and $y\in Y'$, by adding $(-\varphi(x))$ to both side we get $x\text{-}\varphi(x)=\ell+z+y\text{-}\varphi(x)$. Since $x\text{-}\varphi(x)\in X'\leq L$, by unique representation we have $y$=z-$\varphi(x)$=0 so $\varphi(x)$=z=$\pi(x)$, hence$\pi|_X=\varphi$, therefore $Z_2(S)$ is $\mathcal{K}$ – injective . □

In the next, we will prove the partial converseto Proposition (3.3.1).

**Proposition 3.3.2:**Let $S$ be a cancellative and semisubtractive($k$-$c$) $R$-semimodule such that $S$= $Z_2(S)\oplus\mathcal{K}$, where $Z_2(S)$ and $\mathcal{K}$are CS and $Z_2(S)$ is $\mathcal{K}$-injective, then $S$ is CS.

***Proof*:** Let $\mathcal{A}\leq^c S$, by Proposition (3.2.11) $Z_2(\mathcal{A})\leq^c\mathcal{A}$, hence by Lemma (2.2.13), $Z_2(\mathcal{A})\leq^c S$. Since $Z_2(\mathcal{A})\leq Z_2(S)$ and $Z_2(S)$ is CS by hypotheses, then $Z_2(\mathcal{A})$ is a summand of $Z_2(S)$, hence $Z_2(S)=Z_2(\mathcal{A})\oplus C$, for some $C\leq Z_2(S)$.So $S=Z_2(\mathcal{A})\oplus C\oplus\mathcal{K}$, by Lemma (2.2.14), $\mathcal{A}=Z_2(\mathcal{A})\oplus((C\oplus\mathcal{K})\cap\mathcal{A})=Z_2(\mathcal{A})\oplus\mathcal{B}$, where $\mathcal{B}=(C\oplus\mathcal{K})\cap\mathcal{A}$, therefore $Z_2(\mathcal{A})$ is a summand of $\mathcal{A}$, and $\mathcal{B}$ is nonsingular subsemimodule of $\mathcal{A}$by Lemma(3.2.16), since $\mathcal{B}\cap Z_2(\mathcal{A})$=0, consider the diagram:




Where $\pi_1:S\to Z_2(\mathcal{A})$, $\pi_2:S\to\mathcal{K}$ are natural projections, and $i$ be an inclusion map. Since $\pi_2|_{\mathcal{B}}$ is one to one and $Z_2(S)$ is $\mathcal{K}$–injective, then there exists $\theta:\mathcal{K}\to Z_2(S)$ such that $\theta(\pi_2|_{\mathcal{B}})=i(\pi_1|_{\mathcal{B}})$It is clear that $\theta(\pi_2|_{\mathcal{B}})(\mathcal{B})\leq Z_2(\mathcal{A})$.Consider $\mathcal{K}'=\{x+\theta(x):x\in\mathcal{K}\}$, let $b\in\mathcal{B}$, then $b=\pi_1(b)+\pi_2(b)=\theta\pi_2(b)+\pi_2(b)\in\mathcal{K}'$, therefore, $\mathcal{B}\leq\mathcal{K}'$. Since $\mathcal{B}$ is a summand of $\mathcal{A}$, so $\mathcal{B}\leq^c\mathcal{K}'$, define $g:\mathcal{K}\to\mathcal{K}'$ by $g(k)=k+\theta(k)$ then $g$ is a well-defined homomorphism and $kerg=0$ [since $k+\theta(k)=0$ such that $0\in Z_2(S)$ and $\theta(k)\in Z_2(S)$ by subtractive $k\in\mathcal{K}\cap Z_2(S)=0$]. Hence $g$ is one to one and clearly ontoso $g$ is an isomorphism and $\mathcal{K}\cong\mathcal{K}'$, since $\mathcal{K}$ is CS by hypotheses, then by Lemma (2.2.24), $\mathcal{K}'$ is CS and $\mathcal{B}$ is a summand of $\mathcal{K}'$so $\mathcal{K}'=\mathcal{B}\oplus\mathcal{D}$, for some$\mathcal{D}\leq\mathcal{K}'$.Therefore $S=Z_2(\mathcal{A})\oplus C\oplus\mathcal{B}\oplus\mathcal{D}=\mathcal{A}\oplus C\oplus\mathcal{D}$, thus $\mathcal{A}$ is a summand of $S$ and $S$ is CS. □

**Remark 3.3.3:** If $S$ is a nonsingular $R$-semimodule, having an injective hull $E(S)$, then $E(S)$ is nonsingular too.

***Proof*:** By Proposition (3.2.4), $Z(S)=S\cap Z(E(S))$. If $S$ is nonsingular, then $Z(S)=0$, but $S\leq^e E(S)$, so $Z(E(S))=0$, that is $E(S)$ is nonsingular. □

The next proposition gives a useful property of extendingsemimodulesand Socalwith nonsingular in terms of homomorphisms.

**Proposition 3.3.4:** Let $S_1$ be a nonsingular $R$-semimodule with injective hull $E(S_1)$ and $S_2$ be an $R$-semimodule with $\mathrm{Soc}(S_2)=0$, if $S_2$ is CS, thenHom$(S_2, E(S_1))=0$.

***Proof*:** Let $f\in\mathrm{Hom}(S_2, E(S_1))$, and $\ker f\leq^e S_2'$ where $S_2'\leq S_2$. Let $s_2\in S_2'$then by Lemma (3.2.10)there exists left$I\leq^e R$, such that $Is_2\leq\ker f$.Hence $If(s_2)=f(Is_2)=0$. Since $E(S_1)$ is nonsingular by Remark (3.3.3) then $f(s_2)=0$, therefore $s_2\in\ker f$, hence $\ker f=S_2'$, so $\ker f$ has no proper essential extension, hence ker$f$ is closed in $S_2$. Since $S_2$ is CS by hypotheses then $S_2$= ker$f$ $\oplus S_2''$, for some $S_2'' \leq S_2$.Since Soc($S_2$)=0 then $S_2''$ = 0 and $f$ = 0. □

For the next result the following lemmas are required.

**Lemma 3.3.5:** If $S$ is a cancellative$R$–semimodule and $\alpha$ is an idempotent endomorphism of $S$, then$\alpha$ is $i$- regular**.**

***Proof*:** Let $y\in$ Im $\alpha$, then $y+\alpha(x)=\alpha(x')$ for some $x$, $x'\in S$, so $\alpha(y)+\alpha(x)=\alpha(x')$**,** thus$\alpha(y+x)=\alpha(x')$**,** therefore$y+\alpha(x)=\alpha(y+x)=\alpha(y)+\alpha(x)$, by cancellative$y=\alpha(y) \in \alpha(S)$, but $\alpha(S)\subseteq$Im $\alpha$, hence Im $\alpha=\alpha(S)$and$\alpha$ is $i$- regular. □

Recall that a subtractive closur of a subsemimodule$L$ of an $R$-semimodule $S$ is defined by $\bar{L}=\{s\in S|\ s+a=a'$, for some $a, a'\in L\}$[59].

**Lemma 3.3.6:** Let $\alpha$ be an idempotent endomorphism of a cancellative$R$–semimodule$S$ then

1. $\alpha(S)\cap\ker\alpha = 0$
2. $\alpha$ is $i$- regular, hence $\alpha(S)$ is subtractive **.**
3. $\alpha(S)+\ker\alpha=\alpha(S)\oplus\ker\alpha$ .
4. $S = \overline{\alpha(S)\oplus\ker\alpha}$, if $S$ is a semisubtractive.

***Proof*:**

**(1)** Let $x\in\alpha(S)\cap\ker\alpha$ then $\alpha(x)=0$ and $x=\alpha(t)$ for some $t\in S$, then,$\alpha(x)=\alpha(\alpha(t))=\alpha(t)$ and hence $x$=0.

**(2)** By assumption and Remark(3.3.5), α is $i$-regular, hence $\alpha(S)$=Imα, but by [49]Im α is a subtractive, therefore, α(S) is a subtractive.

**(3)** Let $\alpha(s_1)+k_1=\alpha(s_2)+k_2\in\alpha(S)+\ker\alpha$, where $\alpha(s_i)\in\alpha(S)$ and $k_i\in\ker\alpha$ then $\alpha(\alpha(s_1))+\alpha(k_1)=\alpha(\alpha(s_2))+\alpha(k_2)$, so $\alpha(s_1)=\alpha(s_2)$. By cancellative $k_1=k_2$, hence the representation unique and the sum is direct sum.

**(4)** Let $x\in S$, and $\alpha(x)=y$, by semisubtractive, there exists $h\in S$ such that $x=y+h$, then $\alpha(x)=\alpha(x)+\alpha(h)$ by cancellative $\alpha(h)=0$ that is $h\in\ker\alpha$, hence $x\in\alpha(S)\oplus\ker\alpha$. Or $x+h=y$, then $h\in\ker\alpha\leq\alpha(S)\oplus\ker\alpha$ and $y\in\alpha(S)\leq\alpha(S)\oplus\ker\alpha$, hence $x\in\overline{\alpha(S)\oplus\ker\alpha}$, that is in any case $x\in\overline{\alpha(S)\oplus\ker\alpha}$, therefore $S=\overline{\alpha(S)\oplus\ker\alpha}$. □

***Note*:** Lemma (3.3.6) is true if $\alpha$ *i*-regular endomorphism on $R$–semimodule $S$.

Recall that a semimodule $S$ is $\pi$-injective if whenever $S_1$ and $S_2$ are subsemimodules of $S$ with $S_1\cap S_2=0$, there exists $\alpha,\beta\in\mathrm{End}(S)$ such that: **(*i*)** $\alpha,\beta$ are idempotent. **(*ii*)** $\alpha+\beta=1_S$. **(*iii*)** $S_1\leq\ker\alpha$, $S_2\leq\ker\beta$. Let $S$ be an $R$-semimodule with injective hull, $S$ is quasi-continuous if for each idempotent $\alpha\in\mathrm{End}(E(S))$, $\alpha(S)\subseteq S$[44].

The above two concepts are equivalent in modules. In semimodules the proof of the equivalence between the two concepts need strong conditions, namely, cancellative, semisubtractive and subtractive.

**Proposition 3.3.7:** Let $S$ be semisubtractive, cancellative semimodule. If $S$ is $\pi$-injective with injective hull $E(S)$, then $S$ is quasi-continuous.

***Proof*:** Assume that $e$ is an idempotent endomorphism of $E(S)$, let $S_1=S\cap e(S)$ and $S_2=\{m\in S|m+e(x)=x$ for some $x\in S\}$, then $S_1$ and $S_2$ are subsemimodules of $S$ and $S_1\cap S_2=0$(since $m\in S_1\cap S_2$ implies $m=e(m_1)$ and $m+e(x)=x$ for some $m_1$, $x\in S$, then $e(m)=e(m_1)=0$ and, hence $e(m)=0$, i.e, $m$=0). Since $S$ is $\pi$-injective, there exists $\alpha, \beta$ idempotentendomorphism of $S$, such that $\alpha+\beta=I_S$,$S_1\subseteq\ker\alpha$, and $S_2\subseteq\ker\beta$. Let $X$={$z\in E(S)|z+\beta(m)=e(m)$ for some $m\in S$}. It is clear that $X\leq E(S)$. Let $y\in S\cap X$, then $y+\beta(m)=e(m)$ for some $m\in S$, $y\in S$and $\beta(m)\in S$, imply $e(m)\in S$, hence $e(m)\in S_1$. Since $S$ is semisubtractive, there exists $h\in S$, such that either $e(m)+h=m$ or $e(m)=m+h$. In any case $h\in\ker e$.On the other hand $h\in S_2$(since $h+e(h)=h$) hence $\beta(h)$=0, $e(m)\in S_1$so $\beta(e(m))=e(m)$ [for $e(m)=\alpha(e(m))+\beta(e(m))$ and $\alpha(e(m))$=0]. Hence either $\beta(e(m))+\beta(h)=\beta(m)$ so $e(m))=\beta(m)$ or $\beta(e(m))=\beta(m)+\beta(h)$ so $e(m))=\beta(m)$. That is in two cases $e(m))=\beta(m)$ which implies $y$=0, then $S\cap X$=0, but $S\leq^e E(S)$, so $X$=0, therefore $e(m))=\beta(m)\in S$ for all $m\in S$, hence $e(S)\subseteq S$, therefore $S$ is quasi-continuous.□

**Remark3.3.8:**

1. Any uniform semimodule is $\pi$-injective.
2. $\mathbb{N}$ as an $\mathbb{N}$-semimodule is $\pi$-injective since it is uniform, but it is not quasi-continuou since it has no injective hull([41], p.198).
3. As an $\mathbb{N}$-semimodule is cancellative and semisubtractive but not subtractive.
4. $\mathbb{Z}$ is $\pi$-injective and quasi-continuous since $\mathbb{Z}$ is a cancellative,semisubtractive and subtractive.
5. The $\mathbb{N}$-semimodule$S=\mathbb{Z}_2\oplus\mathbb{Z}_4$ is nither $\pi$-injectivesee[44] nor quasi-continuous, but in the case of $\mathbb{Z}$-semimodule$S=\mathbb{Z}_2\oplus\mathbb{Z}_4$ is $\pi$-injective and quasi-continuous.

The following result was proved in [44]a different way to the definition of quasi-continuous ($\pi$-injective) semimodule as it need in that paper.

**Proposition 3.3.9:**Let $S$ be an $R$–semimodule with injective hull $E(S)=E(S_1)\oplus E(S_2)$. If $S$ is quasi-continuous then $S=(S\cap E(S_1))+(S\cap E(S_2))$.

***Proof*:** Let $\pi_i:E(S)\rightarrow E(S_i)$ be the natural projections of $E(S)$ onto $E(S_i)$ ($i$= 1, 2), then $\pi_i$ is idempotentendomorphism of $E(S)$ so by Definition of quasi-continuouse$\pi_i(S)\subseteq S$then $\pi_i(S)\subseteq S\cap E(S_i)(i=1,2)$. But $S=\pi_1(S)+\pi_2(S)\subseteq S\cap E(S_1)+S\cap E(S_2)\subseteq S$, therefore $S=S\cap E(S_1)+S\cap E(S_2)$. □

In[44], it is shown that quasi-continuous is stronger than CS property, i.e. CS is the generalization of quasi-continuous.

**Lemma 3.3.10:** If $S$ is cancellative and semisubtractive$R$–semimodule and if $N_i\leq^{\oplus}S$, ($i$=1, 2)with $N_1\cap N_2=0$ then $N_1+N_2=N_1\oplus N_2$.

***Proof*:** Assume $n_1+n_2=x_1+x_2\in N_1+N_2$, then either $n_1=x_1+h$ or $x_1=n_1+h$**,** since $N_1$and $N_2$ are summand, hence subtractive the two above cases lead to $h\in N_1\cap N_2=0$, hence $n_1=x_1$ and $n_2=x_2$, therefore the representation of $n_1+n_2$ is unique,hence $N_1+N_2=N_1\oplus N_2$. □

**Proposition 3.3.11**: If $S$ is a cancellative and semisubtractive ($k$-$c$) quasi-continuous semimodule with injective hull $E(S)$, then

**1-** $S$ is CS.

**2-** For any $\mathcal{N}_i\leq^{\oplus}S$ and $\mathcal{N}_1\cap\mathcal{N}_2=0$then $\mathcal{N}_1+\mathcal{N}_2\leq^{\oplus}S$.

***Proof*:**

**(1)** Let $\mathcal{N}\leq S$ and $\mathcal{K}$is a complement of $\mathcal{N}$ in $S$, then $\mathcal{N}\oplus\mathcal{K}\leq^{e}S$, which implies $E(\mathcal{N})\oplus E(\mathcal{K})=E(S)$[by Lemma (2.2.6)]. Now by

Proposition(3.3.9), $S$=($S$∩ $E(\mathcal{N})$) ⊕($S$∩ $E(\mathcal{K})$), but $\mathcal{N} \leq^{e} E(\mathcal{N})$, it follow that $\mathcal{N} \leq^{e} S \cap E(\mathcal{N}) \leq^{\oplus} S$, hence $S$ is CS.

**(2)** Since $N_1+N_2 \leq S$, then by Lemma(3.3.10) $E(N_1 \oplus N_2) \leq E(S)$, therefore $E(S)=E(N_1) \oplus E(N_2) \oplus E'$, hence by Proposition (3.3.9), $S=S \cap (E(N_1) \oplus E(N_2)) \oplus (S \cap E')$. Now $\mathcal{N}_i \leq^{e} E(\mathcal{N}_i)$, then $\mathcal{N}_i \leq^{e} S \cap E(\mathcal{N}_i)$($i$=1,2), but $\mathcal{N}_i \leq^{\oplus} S$, therefore $\mathcal{N}_i = S \cap E(\mathcal{N}_i)$, hence $S=N_1 \oplus N_2 \oplus (S \cap E')$, i.e. $N_1+N_2 \leq^{\oplus} S$. □

For example theℕ-semimodule$S = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not quasi-continuous but $S$ is $CS$ since every closed subsemimodule is a summand.

The following result is proved in[44], another proof will be given now, according to definition of quasi-continuous.

**Proposition 3.3.12**: Any summand of quasi-continuous semimoduleis quasi- continuous.

***Proof***:Let$S$bequasi-continuous semimodule andlet$\mathcal{A} \leq^{\oplus} S$ and $\alpha \in$End($E(\mathcal{A})$)be an idempotent then α can be extend to $\alpha' \in$End($E(S)$)in this way $S=\mathcal{A} \oplus \mathcal{A}'$ so by Lemma(2.2.5), $E(S)=E(\mathcal{A}) \oplus E(\mathcal{A}')$. Let $\pi:E(S) \rightarrow E(\mathcal{A})$ be natural projection and $j:E(\mathcal{A}) \rightarrow E(S)$ be inclusion map so $\alpha' = j\,\alpha\pi$, hene $\alpha'^2 = (j\alpha\pi)(j\alpha\pi) = j\alpha I_{E(\mathcal{A})}\alpha\pi = j\alpha^2\pi = j\alpha\pi = \alpha'$, hence $\alpha'(S) \subseteq S$ that is $(j\alpha\pi)(S) \subseteq S$ but $(j\alpha\pi)(S) \subseteq E(\mathcal{A})$(since $j(E(\mathcal{A})=E(\mathcal{A}))$, so $(j\alpha\pi)(S) \subseteq S \cap E(\mathcal{A})$. Since $S \cap E(\mathcal{A})=(\mathcal{A} \oplus \mathcal{A}') \cap E(\mathcal{A})=\mathcal{A} \oplus (\mathcal{A}' \cap E(\mathcal{A}))$(byLemma(2.2.14)). But $(\mathcal{A}' \cap E(\mathcal{A})) \cap \mathcal{A}=0$(since $\mathcal{A} \cap \mathcal{A}=0$) which implies $\mathcal{A}' \cap E(\mathcal{A})=0$ since $\mathcal{A} \leq^{e} E(\mathcal{A})$, hence $S \cap E(\mathcal{A})=\mathcal{A}$. Since $\pi$ idempotentendomorphism of $E(S)$ implies $\pi(S) \subseteq S$. Also $\mathcal{A} \subseteq E(\mathcal{A})=\pi(E(S))$ hence $\pi(S)=\mathcal{A}$, therefore $(j\alpha\pi)(S)=\alpha(\mathcal{A}) \subseteq \mathcal{A}$, therefore $\mathcal{A}$ is quasi continuous. □

**Proposition 3.3.13**: Let $S$ be a subtractive $R$-semimodulewith injective hull $E(S)$, then the following statements are equivalent:

1. If $E(S)=E(S_1)\oplus E(S_2)$, then$S=(S\cap E(S_1))\oplus(S\cap E(S_2))$.
2. If $L_1, L_2\leq S$ with $L_1\cap L_2=0$, then there exist $S_i\leq S(i=1, 2)$, $L_i\leq S_i$ such that $S=S_1\oplus S_2$

***Proof*:**

**(1⇒2)** Assume that $L_i\leq S$ $(i=1, 2)$, let $L_2'$ be a complement of $L_1$ containing $L_2$, then by Lemma (2.2.3)$L_1\oplus L_2'\leq^e S$, hence by Lemma (2.2.6)$E(L_1\oplus L_2')=E(S)$, so $E(L_1)\oplus E(L_2')=E(S)$. Now by (1) $S=S\cap(E(L_1))\oplus S\cap(E(L_2'))$, with $L_1\leq S\cap(E(L_1))$ and $L_2\leq S\cap(E(L_2'))$.

**(2⇒1)** Assume that $E(S) = E(S_1)\oplus E(S_2)$, let $L_1=S\cap(E(S_1))$, $L_2=S\cap(E(S_2))$, then $L_1\cap L_2 = 0$ and by (2) $S=S_1'\oplus S_2'$, where $L_i\leq S_i'= S\cap(E(S_i))(i=1, 2)$. □

For a specific purpose, we derive a new lemma from Lemma (2.2.4) that will be special case as follows:

**Lemma 3.3.14:** Let$S=\mathcal{N}\oplus\mathcal{N}'$be a cancellative$R$–semimodule and $\mathcal{N}$ is an $R$–module. If $C$ is a complement of $\mathcal{N}$ in $S$ and subtractive then $\mathcal{N}+C=\mathcal{N}\oplus C\leq^e S$.

***Proof*:** Let $X\leq S$ such that $(\mathcal{N}+C)\cap X=0$, then $\mathcal{N}\cap X=0$ and $C\cap X=0$.

Claim $\mathcal{N}\cap(C+X)=0$, which implies $C+X=C$, hence $X=0$. Let $n=c+x$, where $n\in\mathcal{N}$, $c\in C$ and $x\in X$, by assumption $c=c_1+c_2$ and $x=x_1+x_2$ where $c_1, x_1\in\mathcal{N}$ and $c_2, x_2\in\mathcal{N}'$ hence $n= c_1+ x_1+ c_2+ x_2$ by uniqueness $n=c_1+x_1$, $c_2+ x_2=0$ since $C$ is subtractive so $x_2\in C\cap X=0$, hence$c_2=0$, then$c= c_1\in C\cap\mathcal{N}=0$and $x_1\in\mathcal{N}\cap X=0$, so $n=0$, that is the claim is true, so $\mathcal{N}+C\leq^e S$. It remains proving $\mathcal{N}+C=\mathcal{N}\oplus C$. Assume that $n_1+c_1=n_2+c_2\in\mathcal{N}+C$, hence $c_1=(n_2-n_1)+c_2$( since $\mathcal{N}$ is an $R$-module ) but $C$ is a subtractive so $n_2$-$n_1$∈$C$∩$\mathcal{N}$=0, that is $n_1$=$n_2$ and $c_1$=$c_2$ that is $\mathcal{N}$+$C$=$\mathcal{N}$⊕$C$. □

**Proposition 3.3.15:** Let $S$ be a cancellativesemisubtractive ($k$-$c$) quasi-continuous $R$–semimodule with injective hull $E(S)$ then,

1. $S$ is CS.
2. If $S$ =$S_1$⊕$S_2$, where $S_2$ is an $R$–module, then $S_2$is $S_1$ - injective.

***Proof*:**

**(1)** Is clear by Proposition (3.3.11)(1).

**(2)** Let $S$ = $S_1$⊕$S_2$, where $S_1$ is $R$–semimodule and $S_2$ is an $R$–module, to prove that $S_2$ is $S_1$–injective (the other case is similar). Let $X$≤$S_1$, and $\alpha$:$X$⟶$S_2$ be any homomorphism. Let $\mathcal{B}$={$x$- $\alpha(x)$|$x$∈$X$}, then $\mathcal{B}$ is a subsemimodule of $S$ ( note that $-\alpha(x)$ exists since $S_2$ isan $R$–module). If y∈ $\mathcal{B}$ ∩$S_2$, then y= $x$- $\alpha(x)$, for some $x$∈$X$≤$S_1$. But $S_2$ is subtractive, so $x$∈$S_1$∩$S_2$ which implies $x$=0, hence y=0, therefore $\mathcal{B}$ ∩$S_2$=0. Let $\mathcal{B}$* be a complement of $S_2$containing $B$, then by Lemma(3.3.14) $B$*⊕ $S_2$$\leq^e$$S$, $B$* is a summand of $S$ (since $S$ is CS) hence it subtractive, so $\mathcal{B}$*⊕ $S_2$is a summand of $S$, by Proposition (3.3.11)(2), therefore$\mathcal{B}$*⊕ $S_2$=$S$. Now $x$∈$X$, implies $x$- $\alpha(x)$ ∈ $\mathcal{B}$ ≤ $\mathcal{B}$*. If $\pi$: $S$= $\mathcal{B}$*⊕ $S_2$⟶$S_2$ be the natural projection, then 0= $\pi(x$- $\alpha(x))$= $\pi(x)$- $\pi(\alpha(x))$= $\pi(x)$- $\alpha(x)$. Hence $\pi(x)$=$\alpha(x)$, for each $x$∈$X$, that is $\pi$ is an extension of $\alpha$ to $S$, and $\pi|_{S_1}$ is an extension of $\alpha$ to $S_1$, therefore $S_2$ is $S_1$–injective. □

The next results gives a special case condition for a relative injective.

**Lemma 3.3.16:** Let $S_1$ be a semisimple $R$–semimodule with injective hull$E(S_1)$ and Hom($S_2$, $E(S_1)$)=0 then $S_1$ and $S_2$ are relative injective.

***Proof*:** $Hom(S_2, E(S_1)) = 0$ implies $\mathrm{Hom}(S_2, S_1)=0$ which implies $\mathrm{Hom}(L, S_1)=0$, for each $L \leq S_2$ then, $S_1$ is $S_2$ injective. Since $S_1$ be semisimple, then $S_2$ is $S_1$ injective. Therefore, $S_1$ and $S_2$ are relative injective. □

**Lemma 3.3.17:** If $S$ is a nonsingular $R$-semimodule, $\mathcal{N} \leq S$ and subtractive, $S/\mathcal{N}$ is singular, then $\mathcal{N} \leq^e S$.

***Proof*:** Let $s+\mathcal{N} \in S/\mathcal{N}$, since $S/\mathcal{N}$ is singular then there exists left $I \leq^e R$ such that $I(s+\mathcal{N})=\bar{0}$, that is $Is \leq \mathcal{N}$. Since $S$ is nonsingular then $Is \neq 0$, so $0 \neq Is \leq \mathcal{N}$, therefore $\mathcal{N} \leq^e S$. □

**Lemma 3.3.18:** If $S$ is a nonsingular $R$-semimodule, $\mathcal{N} \leq S$ and $\mathcal{K} \leq^c S$, then $\mathcal{K} \cap \mathcal{N} \leq^c \mathcal{N}$.

***Proof*:** Assume that $\mathcal{K} \cap \mathcal{N} \leq^e \mathcal{D} \leq \mathcal{N}$.

Claim $\mathcal{K} \leq^e \mathcal{D} + \mathcal{K} \leq S$ , let $0 \neq d+k \in (\mathcal{D} + \mathcal{K}) \setminus \mathcal{K}$, then $d \neq 0$, hence there exists left $I \leq^e R$ such that $0 \neq Id \leq \mathcal{K} \cap \mathcal{N}$, then $I(d+k) \leq \mathcal{K}$ since $S$ is nonsingular then $I(d+k) \neq 0$ hence $\mathcal{K} \leq^e \mathcal{D} + \mathcal{K} \leq S$ but $\mathcal{K} \leq^c S$, then $\mathcal{K} = \mathcal{D} + \mathcal{K}$ and $\mathcal{D} \leq \mathcal{K}$ so $\mathcal{D} \leq \mathcal{K} \cap \mathcal{N} \leq \mathcal{D}$, therefore $\mathcal{D} = \mathcal{K} \cap \mathcal{N}$ and $\mathcal{K} \cap \mathcal{N} \leq^c \mathcal{N}$. □

In the following, certain condition will be assumed to a subsemimodule of CS- semimodule that assured the CS property for it.

**Definition 3.3.19:** A subsemimodule $X$ of a semimodule $S$ is said to be IDS subsemimodule, if its intersection with any summand of $S$ is summand of $X$.

**Example 3.3.20:** If $S=\mathbb{R}^2$ as an $\mathbb{R}$-semimodule(where $\mathbb{R}$ is the set of real number), then $S_1=\mathbb{R} \times 0$ is IDS subsemimodule of $S$.

**Proposition 3.3.21:** If $S$ is extending$R$-semimodule and $X$ is IDS subsemimodule of $S$, then $X$ is a CS, too.

***Proof*:** Let $\mathcal{N} \leq X$, so $\mathcal{N} \leq S$ then there exists $\mathcal{D}$ is a summand of $S$ such that $\mathcal{N} \leq^{e} \mathcal{D}$ since$\mathcal{N} \cap X \leq^{e} \mathcal{D} \cap X$, then $\mathcal{N} \leq^{e} \mathcal{D} \cap X$ but $\mathcal{D} \cap X$ is a summand of $X$ (since$X$ is IDS), therefore $X$ is a CS. □

**Proposition 3.3.22:** Let $S$ be a nonsingular $R$-semimodule. If $X$ is extending subsemimodule of $S$, then $X$ is ID.S.

***Proof*:** Let$X$be CS and $\mathcal{D}$isa summand of $S$, say $S$=$\mathcal{D}\oplus\mathcal{D}'$for some$\mathcal{D}' \leq S$, since$X$ is CS, then there exists a summand $C$ of $X$such that $\mathcal{D} \cap X \leq^{e} C$. Claim: $\mathcal{D} \cap X = C$, if$\mathcal{D} \cap X \neq C$, then $\mathcal{D} \neq \mathcal{D} + C$. Let $x+y \in (\mathcal{D} + C) \setminus \mathcal{D}$, where $x \in \mathcal{D}$, $0 \neq y \in C$, since $\mathcal{D} \cap X \leq^{e} C$, then by Lemma (3.2.10) there exists left$I \leq^{e} R$, such that $0 \neq Iy \leq \mathcal{D} \cap X$. But $\mathcal{D}$is nonsingular since $S$ is nonsingular, so $0 \neq I(x+y) \leq \mathcal{D}$. Hence $\mathcal{D} \leq^{e} \mathcal{D}+C$, which implies $(\mathcal{D}+C) \cap \mathcal{D}' = 0$. But $\mathcal{D}$ is a complement of $\mathcal{D}'$in $S$. so$\mathcal{D}+C = \mathcal{D}$, hence C$\leq \mathcal{D} \cap X \leq C$, that is $\mathcal{D} \cap X = C$, and $\mathcal{D} \cap X$ is a summand of $X$. □

**Definition 3.3.23:** A semimodule $S$ has summand intersection property (for short, SIP), if the intersection of any two summand of $S$ is again a summand of $S$.

***Note:*** If $S$ is SIP then any summand of it is IDS subsemimodule of $S$.

By Proposition (3.3.22), we have the following result.

**Corollary 3.3.24:**If $S$is nonsingularCS$R$-semimodule then $S$ has SIP property.

**Corollary 3.3.25:** Let $X$ be an $R$-semimodule with injective hull such that $X \cap \mathcal{D}$ is a summand of $X$, for all injective subsemimodules $\mathcal{D}$ of $E(X)$, then $X$ is *CS.*

***Proof*:** Let $\mathcal{D}$ be an injective subsemimodule of $E(X)$, hence $\mathcal{D}$ is a summand of $E(X)$, since $X \leq E(\mathrm{X})$, and $E$(X) is injective so CS, so by assumption $X$ is IDS, hence by Proposition (3.3.21), $X$ is CS. □

By Proposition (3.3.22), and Corollary (3.3.25) we can prove the necessary and sufficient condition for following corollary.

**Corollary3.3.26:** Let $X$ be a nonsingular$R$-semimodule with injective hull $E(X)$. Then $X$ is CS if and only if $X \cap \mathcal{D}$is a summand of $X$, for all injective subsemimodules $\mathcal{D}$ of $E(X)$.

# Chapter Four:

# Some Generalizations of Extending Semimodules

## 1.1. Introduction

In this chapter some generalization of extending semimodule including FI-CSsemimodule,T-extending semimodule and $C_{11}$ semimodulewill be studied in three sections in addition to the introduction section. In section two, the properties of the FI-CSsemimodule are introduced and investigated. It can be seen by analyzing the structure of FI-CSsemimodule; there are many properties of fully invariant subsemimodules that are also useful, in addition to, the relationship between singularity and FI-CSsemimodule for summand as well as the direct sum are studying and investigating. Section three is devoted to introducing the concept of T-extending semimodule.

The properties of t-extending semimoduleas well as t-essential and t-closed subsemimodules are studied. It can be seen by analyzing the structure of T-extending semimodules; there are many properties of nonsingular and $Z_2$-torsion semimodule that are also useful. In section four, $C_{11}$ semimodulesis introduced and studied, some of its properties are proved, in addition the behavior of $C_{11}$ according to subsemimodule, direct sum, and summand are studied.

## 4.2. FI-ExtendingSemimodules and Singularity

In this section, we will introduce the concept of FI-Extending semimodule, as a kind of generalization of the concept of CS-semimodule. Now all of the following properties of a fully invariant were introduced and prove for submodules. They will be converted for subsemimodules. Recall that a subsemimodule $\mathcal{N}$ of a semimodule $S$ is said to be fully invariant (denoted $\mathcal{N} \trianglerighteq S$) if $f(\mathcal{N}) \subseteq \mathcal{N}$ for every $f \in \mathrm{End}(S)$.

**Lemma 4.2.1:** Let $S$ be an $R$–semimodule. If $\mathcal{N} \rhd S$ and $\mathcal{M} \rhd S$, then

1. $(\mathcal{N} + \mathcal{M}) \rhd S$.
2. $(\mathcal{N} \cap \mathcal{M}) \rhd S$.

***Proof***: Clear.

**Lemma 4.2.2:** Let $S$ be an $R$–semimodule. If $\mathcal{A} \leq \mathcal{B}$, such that $\mathcal{B} \rhd S$ and $\mathcal{A} \rhd \mathcal{B}$, then $\mathcal{A} \rhd S$.

***Proof***: Let $h \in End(S)$, then $g = h|_{\mathcal{B}} \in \mathrm{End}(\mathcal{B})$, since $g(\mathcal{B}) \subseteq \mathcal{B}$.Now $h(\mathcal{A}) = g(\mathcal{A}) \subseteq \mathcal{A}$(since $\mathcal{A} \rhd \mathcal{B}$), therefore $\mathcal{A} \rhd S$. □

The following remark relates the concept of IDS subsemimodule and the concept of fully invariant subsemimodule.

**Remark 4.2.3:**

1. It is clear that if $X \rhd S$, then it is IDS.
2. The converse of (1) is not true for example the subsemimodule in Example (3.3.20) is IDS but not fully invariant.

The concept FI-CS on modules has been introduced and explained by[37]and [36].We will introduce this concept for semimodules and give some results related to this concept.

**Definition 4.2.4:** An $R$–semimodule $S$ is called FI-Extending (FI-CS) if each fully invariant subsemimodule of $S$ is essential in a summand of $S$.

**Remark 4.2.5:**It is clear any CS–semimodule is FI-CS but the converse is not trueingeneral. For example $S=\mathbb{Z}_8 \oplus \mathbb{Z}_2$, $R=\mathbb{Z}$, then $S$ is not CS (Remark(2.2.2)), but $S$ is FI-CS since ($S_8=\langle(\overline{2},\overline{1})\rangle$), the only closed subsemimodule which is not summand and not fully invariant of $S$since $\pi_2\ (S_8) \nsubseteq S_8$ where $\pi_2$:$S\to S_2$be a natural projection of $S$ onto$S_2$.

The next results give useful properties of FI-CSsemimodules.

**Proposition 4.2.6:** Let $S$ be an $R$–semimoduleand $\mathcal{N} \trianglerighteq S$. If $S$ is FI-CS then $\mathcal{N}$is FI-CS.

***Proof*:** Assume that $S$ is FI-CS, and $\mathcal{K} \trianglerighteq N$, by Lemma (4.2.2), $\mathcal{K} \trianglerighteq S$, hence $\mathcal{K} \leq^e \mathcal{D}_1$, where $\mathcal{D}_1$is a summand of $S$, say $S$=$\mathcal{D}_1 \oplus \mathcal{D}_2$, for some$\mathcal{D}_2 \leq S$, since $\mathcal{N} \trianglerighteq S$, hence by Lemma (3.2.6), $\mathcal{N} = (\mathcal{N} \cap \mathcal{D}_1) \oplus (\mathcal{N} \cap \mathcal{D}_2)$, since $\mathcal{K} \leq^e \mathcal{D}_1$, then $\mathcal{K} \leq^e (N \cap \mathcal{D}_1)$, where $(N \cap \mathcal{D}_1)$is a summand of $\mathcal{N}$, therefore $\mathcal{N}$is FI-CS. □

**Proposition 4.2.7:** Let $S = S_1 \oplus S_2$ be an $R$-semimodule If $S_1$and $S_2$ are FI-CS then $S$ is FI-CS.

***Proof*:** Assume that $S_1$and $S_2$ are FI-CS, and let $\mathcal{N} \trianglerighteq S$. Let $\pi_i$:$S\to S_i$be the natural projections of $S$ onto $S_i$($i$=1, 2), then $\mathcal{N} = \pi_1(\mathcal{N}) \oplus \pi_2(\mathcal{N})$, where $\pi_i(\mathcal{N}) \leq S_i$ ($i$=1, 2), since $S_i$ is FI-CS, then there exist summand subsemimodules $\mathcal{D}_i$ of $S_i$such that $\pi_i\ (\mathcal{N}) \leq^e \mathcal{D}_i$, then $S_i$=$\mathcal{D}_i \oplus \mathcal{D}_i{}'$, and $\pi_1(\mathcal{N}) \oplus \pi_2(\mathcal{N}) \leq^e \mathcal{D}_1 \oplus \mathcal{D}_2$, so $\mathcal{N} \leq^e \mathcal{D}_1 \oplus \mathcal{D}_2$,where $S$= $\mathcal{D}_1 \oplus \mathcal{D}_1{}' \oplus \mathcal{D}_2 \oplus \mathcal{D}_2$, and $\mathcal{D}_1 \oplus \mathcal{D}_2$a summand of $S$, therefore $S$ is FI-CS. □

**Corollary 4.2.8:** If $S$=$S_1 \oplus S_2$, where $S_1$ and $S_2$ are CS $R$–semimodulesthen $S$ is *FI-CS*.

***Proof*:**Clear.

The following result immediately by Remark(4.2.5)and Corollary(4.2.8).

**Corollary 4.2.9:** If $S = S_1 \oplus S_2$ where $S_1$ and $S_2$ are uniform or semisimple subsemimodules of $S$, then $S$ is FI-CS.

***Proof*:** Clear.

**Lemma 4.2.10:** If $e$ and $\hbar$ are idempotent endomorphisms of a cancellative semimodule $S$, with $e + \hbar = I_S$, then $S = e(S) \oplus \hbar(S)$.

***Proof*:** It is enough to prove the unique representation, let $e(x) + \hbar(x) = e(t) + \hbar(t)$, then $e(e(x) + \hbar(x)) = e(e(t) + \hbar(t))$, which implies $e(x) = e(t)$ (since $e^2 = e$ and $eh = 0$). Similarly $h(x) = h(t)$, hence $S = e(S) \oplus \hbar(S)$. □

The next two results establish relation between an FI-CS semimodule and its injective hull.

**Proposition 4.2.11:** Let $S$ be a cancellative $R$–semimodule with injective hull, then $S$ is FI-CS if and only if for each $\mathcal{N} \trianglerighteq S$, there exist $e$, $\hbar$ idempotent endomorphisms of $E(S)$, with $e + \hbar = I_{E(S)}$, such that $\mathcal{N} \leq^e e(E(S))$, and $e(S) \subseteq S$.

***Proof*:** Assume that $S$ is FI-CS, and $\mathcal{N} \trianglerighteq S$, then there exists $\mathcal{D} \leq^{\oplus} S$ such that $\mathcal{N} \leq^e \mathcal{D}$, say $S = \mathcal{D} \oplus \mathcal{D}'$ for some $\mathcal{D}' \leq S$, since $S$ has injective hull, then by Lemma (2.2.5), $E(S) = E(\mathcal{D}) \oplus E(\mathcal{D}')$. Now, let $e, \hbar$ be the natural projections of $E(S)$ onto $E(\mathcal{D})$ and $E(\mathcal{D}')$ respectively, then $e$ can be considered as an idempotent endomorphism of $E(S)$. On other hand, $\mathcal{N} \leq^e \mathcal{D}$ and $\mathcal{D} \leq^e E(\mathcal{D})$, imply $\mathcal{N} \leq^e E(\mathcal{D})$, since $S = \mathcal{D} \oplus \mathcal{D}'$, hence $e(S) = e(\mathcal{D}) \oplus e(\mathcal{D}')$, but $\mathcal{D}' \leq E(\mathcal{D}')$, hence $e(\mathcal{D}') = 0$, therefore $e(S) = e(\mathcal{D})$, but $\mathcal{D} \leq E(\mathcal{D})$, then $e(\mathcal{D}) = \mathcal{D}$, so $e(S) = \mathcal{D} \leq S$. On other hand, $\mathcal{N} \leq^e \mathcal{D} \leq^e E(\mathcal{D}) = e(E(S))$, hence $\mathcal{N} \leq^e e(E(S))$. Conversely, assume that $\mathcal{N} \trianglerighteq S$, and $\mathcal{N} \leq^e e(E(S))$, and $e(S) \subseteq S$ by assumption, where

$e$ is an idempotent endomorphism of $E(S)$, then $\mathcal{N} \leq^e S \cap e(E(S)) \subseteq S$, and $\mathcal{N} \leq^e e(S)$ but by Lemma (4.2.10), $e(S)$ is a summand of $S$, hence $S$ is FI-CSsemimodule. □

**Proposition 4.2.12:**Let $S$ be an FI-CSsemimodule with injective hull$E(S)$,and $\mathcal{B} = S \cap \mathcal{P}$ where $\mathcal{P}$is fully invariant summand in $E(S)$, then $\mathcal{B}$is fully invariant summand in $S$.

***Proof***: Assume that $\hbar \in \mathrm{End}(S)$ since $E(S)$ is injective, then there exists $\hbar' \in \mathrm{End}(E(S))$, such that $\hbar'$ extends $\hbar$. Let $s \in \mathcal{B}$, since $\mathcal{B} \subseteq S$, then$s \in S$ and $\hbar(s) \in \hbar(S) \subseteq S$, therefore$\hbar(s) \in S$. Since$\mathcal{B} \subseteq \mathcal{P}$,hence$\hbar'(s) = \hbar(s) \in S \cap \mathcal{P} = \mathcal{B}$, so$\mathcal{B} \trianglerighteq \mathcal{P}$. While $\mathcal{P}$ is injective (since it is a summand of $E(S)$, and $\mathcal{B} \leq S$, then $E(\mathcal{B}) \leq \mathcal{P}$. Since $S$ is FI-CS, then there exists a summand $\mathcal{X}$ of $S$ such that $\mathcal{B} \leq^e \mathcal{X}$, therefore by Lemma (2.2.6), $E(\mathcal{B}) = E(\mathcal{X})$, so $\mathcal{X} \leq S \cap E(\mathcal{X}) = S \cap E(\mathcal{B}) \leq S \cap \mathcal{P} = \mathcal{B}$,therefore $\mathcal{B} = \mathcal{X}$, since $\mathcal{X}$ is a summand of $S$, then $\mathcal{B}$ is a summand of $S$ . □

The next results give a characterization of FI-CS semimodule.

**Proposition4.2.13:** Let $S$ be a subtractive $R$–semimodule. Then $S$ is FI-CS if and only if every fully invariant subsemimodule of $S$ has a complement which is a summand of $S$.

***Proof*:** Assume that$S$ is FI-CS, let $\mathcal{B} \trianglerighteq S$, then there exists a summand subsemimodule $\mathcal{D}$ of $S$ such that $\mathcal{B} \leq^e \mathcal{D}$, say $S = \mathcal{D} \oplus \mathcal{D}'$, for some $\mathcal{D}' \leq S$. Let$e = \pi_{\mathcal{D}}$ and $\hbar = \pi_{\mathcal{D}'}$ be the natural projections of $S$ onto $\mathcal{D}$ *and* $\mathcal{D}'$ respectively, it is clear that $e = e^2$, $\hbar = \hbar^2$, and $e + \hbar = I_S$, so $\mathcal{B} \leq^e e\,S$, since $\mathcal{D}'$ is a complement of $\mathcal{D}$, so $\mathcal{D}'$ is the desired complement. Conversely, assume that $\mathcal{B} \trianglerighteq S$ and $\mathcal{D}'$ is a complement of $\mathcal{B}$in $S$, say $S = \mathcal{D} \oplus \mathcal{D}'$ for some $\mathcal{D} \leq S$. Since $\mathcal{B} \trianglerighteq S$ then $\mathcal{B} = (\mathcal{B} \cap \mathcal{D}) \oplus (\mathcal{B} \cap$

$\mathcal{D}'$)=($\mathcal{B} \cap \mathcal{D}$), therefore $\mathcal{B} \subseteq \mathcal{D}$. If $\mathcal{Z} \subseteq \mathcal{D}$such that$\mathcal{B} \cap \mathcal{Z}= 0$, so $\mathcal{B} \cap [\mathcal{Z} + \mathcal{D}'] = 0$, (if $x \in \mathcal{B}$, and $x = z + u'$, where $z \in \mathcal{Z}$ and $u' \in \mathcal{D}'$ so $u' \in \mathcal{D} \cap \mathcal{D}'$=0, therefore $x = z \in \mathcal{B} \cap \mathcal{Z} = 0$), so $\mathcal{Z} + \mathcal{D}' = \mathcal{D}'$,and$\mathcal{Z} = 0$, hence $\mathcal{B} \leq^e \mathcal{D}$. □

By Proposition (4.2.13) and the definition of complement, the proof of the following corollary is clear.

**Corollary 4.2.14:**A subtractive *R*-semimodule *S* is FI-CS if and only if for any $\mathcal{B} \trianglerighteq S$there exists a summand subsemimodule$\mathcal{D}$of *S* such that $\mathcal{B} \cap \mathcal{D} = 0$, and $\mathcal{B} \oplus \mathcal{D} \leq^e S$. □

**Proposition 4.2.15:**A (*k*-*c*) *R*-semimodule *S* is FI-CS if and only if the closure of any fully invariant subsemimodule in *S* is a summand of *S*.

***Proof*:** Assume *S* is FI-CS, and $\mathcal{N} \trianglerighteq S$, with $\mathcal{N}'$ is closure of $\mathcal{N}$, by definition $\mathcal{N} \leq^e \mathcal{D}$ and $\mathcal{D}$ is a summand of*S*, then $\mathcal{N}' \leq \mathcal{D}$, in fact $\mathcal{N}' \leq^e \mathcal{D}$, therefore $\mathcal{N}' = \mathcal{D}$, which is a summand subsemimodule of *S*. Conversely, assumethe closure of any fully invariant subsemimodule in *S* is a summand of *S*. Let $\mathcal{N} \trianglerighteq S$ and $\mathcal{N}'$ is closure of $\mathcal{N}$, then $\mathcal{N}'$ is a summand of *S*, and $\mathcal{N} \leq^e \mathcal{N}'$, hence *S* is FI-CS. □

Depending on the definition of fully invariant, closed and FI-CS, the following result can be obtained

**Corollary 4.2.16:**If *S* is a FI-CSsemimodule then any fully invariant closed subsemimodule of *S* is a summand of *S*.

***Proof*:**Clear.

The next corollary gives a useful characterization of duoCS semimodules.

**Corollary 4.2.17:** Assume that $S$ is a duo semimodule then $S$ is CS if and only if $S$ is FI-CS.

***Proof*:**Clear.

For proving Proposition (4.2.19), we need the following Lemma.

**Lemma 4.2.18:** Let $S = S_1 \oplus S_2$ be an $R$-semimodule. If $S_1 \trianglerighteq S$ and $\mathcal{P} \trianglerighteq S_2$ then $(S_1 \oplus \mathcal{P}) \trianglerighteq S$.

***Proof*:** Assume $\mathcal{P} \trianglerighteq S_2$, and $\hbar \in \mathrm{End}(S)$, then $\hbar\,(S_1 \oplus \mathcal{P}) = \hbar(S_1) \oplus \hbar(\mathcal{P})$, where $\hbar(S_1) \leq S_1$(since $S_1 \trianglerighteq S$), and $\hbar(\mathcal{P} = \pi_1(\hbar(\mathcal{P}) + \pi_2(\hbar(\mathcal{P})) = \pi_1(\hbar(\mathcal{P})) + \pi_2(\hbar(i(\mathcal{P}))$, and $i$ is the inclusion map from $S_2$ into $S$. Now$\pi_2\,\hbar i \in End(S_2)$, and$\mathcal{P} \trianglerighteq S_2$, hence $\pi_2\,\hbar i(\mathcal{P}) \leq \mathcal{P})$, on other hand $\pi_1(\hbar(\mathcal{P}))\ \leq S_1$.Hence $\hbar\,(S_1 \oplus \mathcal{P}) \leq S_1 \oplus \mathcal{P}$ , and $(S_1 \oplus \mathcal{P}) \trianglerighteq S$. □

**Proposition 4.2.19:** Let $S = S_1 \oplus S_2$ be a FI-CS semimodule. If $S_1 \trianglerighteq S$ and $\mathcal{P} \trianglerighteq S_2$ then both $S_1$ and $(S_1 \oplus \mathcal{P})$ are FI-CS.

***Proof*:**It is clear by Lemma (4.2.18) and Proposition (4.2.6).

**Proposition4.2.20:** Let $S = S_1 \oplus S_2$ be a FI-CSsemimodule and $S_1 \trianglerighteq S$ then both $S_1$ and $S_2$ are FI-CS.

***Proof*:** By Proposition (4.2.6) $S_1$ is FI-CS. Let $\mathcal{P} \trianglerighteq S_2$ by Lemma (4.2.18), $(S_1 \oplus \mathcal{P}) \trianglerighteq S$, since $S$ is *FI-CS*, then there exists a summand subsemimodule $\mathcal{D}$of $S$, such that $S_1 \oplus \mathcal{P} \leq^e \mathcal{D}$, say $S = \mathcal{D} \oplus \mathcal{D}'$, for some $\mathcal{D}' \leq S$. By Lemma(2.2.14), $\mathcal{D} = S_1 \oplus (\mathcal{D} \cap S_2)$ [since $S_1 \leq S_1 \oplus \mathcal{P} \leq \mathcal{D}$]. Now$\mathcal{P} \leq S_1 \oplus \mathcal{P} \leq \mathcal{D}$, and $\mathcal{P} \leq S_2$ imply $\mathcal{P} \leq \mathcal{D} \cap S_2$. Further, $S_1 \oplus \mathcal{P} \leq^e \mathcal{D}$, implies $(S_1 \oplus \mathcal{P}) \cap S_2 \leq^e \mathcal{D} \cap S_2$, but $(S_1 \oplus \mathcal{P}) \cap S_2 = \mathcal{P}$, so $\mathcal{P} \leq^e \mathcal{D} \cap S_2$. Now $S = S_1 \oplus (\mathcal{D} \cap S_2) \oplus \mathcal{D}'$, implies

$S_2=(\mathcal{D}\cap S_2)\oplus(S_1\oplus\mathcal{D}')\cap S_2$by Lemma (2.2.14). So $\mathcal{P}\leq^e\mathcal{D}\cap S_2$, where $\mathcal{D}\cap S_2$ is a summand of $S_2$, therefore$S_2$ is FI-CS. □

**Proposition4.2.21:** Let $S=S_1\oplus S_2$ be a cancellative and semisubtractive$R$–semimodule. If $S_1$ is FI-CS and $S_2$ is $S_1$-injective then for every fully invariant closed subsemimodule $\mathcal{T}$of$S$($\mathcal{T}\trianglerighteq^c S$)with $\mathcal{T}\cap S_2=0$ is a summand of$S$.

***Proof*:** Let $\mathcal{T}\trianglerighteq^c S$ with $\mathcal{T}\cap S_2=0$, since $S_2$ is $S_1$injective then by Lemma (2.3.2), there exists $S'\leq S$, such that $\mathcal{T}\subseteq S'$, and $S=S'\oplus S_2$, therefore $S'\cong S_1$, since $S_1$ is FI-CS, then $S'$ is FI-CS, and so $\mathcal{T}$ is a summand of $S'$ say $S'=\mathcal{T}\oplus S''$, for some, $S''\leq S'$, so $S=\mathcal{T}\oplus S''\oplus S_2$, that is $\mathcal{T}$ is a summand of $S$. □

The following result relates the oncept of semisimple$R$-semimodule and the concept of FI-CS.

**Corollarrly 4.2.22:** Let $S_1$ be a semisimple $R$–semimodule, then $S=S_1\oplus S_2$ is FI-CS for any FI-CS subsemimodule $S_2$.

***Proof*:** Assume $S=S_1\oplus S_2$ and $S_2$is FI-CS. Since $S_1$is semisimple so $S_1$ is CS, therefore by Remark (4.2.5) $S_1$ is FI-CS, hence by Proposition (4.2.7)$S$ is FI-CS. □

## FI-CS with Singularity

In the next the relationship between FI-CS semimodule and singularityfor summand as well as direct sum are studied some of results are proven according to this relation. The next proposition establish relation between FI-CSsemimodule and second singular subsemimodule of FI-CS.

**Proposition4.2.23:** A semimodule $S$ is FI-CS if and only if $S=Z_2(S)\oplus\mathcal{K}$, where $Z_2(S)$ and $\mathcal{K}$ are FI-CS.

***Proof*:** Since by Propositions (3.2.8) and (3.2.11), $Z_2(S)$ is a fully invariant closed in $S$ and $S$ is FI-CS, then by Corollary (4.2.16) $Z_2(S)$ is a summand subsemimodule of $S$, that is $S=Z_2(S)\oplus\mathcal{K}$, for some $\mathcal{K}\leq S$, since $S$ is FI-CS, then by Proposition (4.2.20) both $Z_2(S)$ and $\mathcal{K}$ are FI-CS.□

Conversely, clear by Proposition (4.2.7).

**Proposition 4.2.24:** Let $S_1$ be a nonsingular semisimple $R$–semimodule with injective hull $E(S_1)$ and $S_2$ be an $R$–semimodule with Soc($S_2$)=0, if $S_2$ is FI-CS and $\hbar\in$Hom($S_2$, $E(S_1)$) such that ker$\hbar\trianglerighteq S_2$, then $\hbar=0$.

***Proof*:** Assume that $\hbar\in$Hom($S_2$, $E(S_1)$), and ker$\hbar\leq^e S_2'$ where $S_2'\leq S_2$. Let $t_2\in S_2'$ then by Lemma(3.2.10), there exists left $\mathcal{I}\leq^e R$, such that $\mathcal{I}t_2\leq$ker$\hbar$, hence $\mathcal{I}\hbar(t_2)=\hbar(\mathcal{I}t_2)=0$. Since $E(S_1)$ is nonsingular by Remark(3.3.3), then $\hbar(t_2)=0$, therefore $t_2\in$ ker$\hbar$, hence ker$\hbar=S_2'$, so ker$\hbar$ has no proper essential extension, hence ker$\hbar$ is closed in $S_2$, but by hypotheses ker$\hbar\trianglerighteq S_2$, hence ker$\hbar$ is a fully invariant closed in $S_2$, since $S_2$ is FI-CS then by Corollary (4.2.16), ker$\hbar$ is a summand of $S_2$, say $S_2=$ ker$\hbar\oplus S_2''$, for some $S_2''\leq S_2$. Since Soc($S_2$)=0 then $S_2''=0$, and $\hbar=0$. □

**Corollary 4.2.25:** Let $S_1$ be nonsingular semisimple $R$–semimodule with injective hull $E(S_1)$ and $S_2$ be an $R$–semimodule with Soc($S_2$)=0, if $S_2$ is FI-CS and $0\neq\hbar\in$Hom($S_2$, $E(S_1)$) then ker$\hbar$ is not fully invariant in $S_2$.

***Proof*:** Clear by Proposition (4.2.24).

The following proposition gives the characterization of FI-CS and indecomposable semimodule.

**Proposition 4.2.26:** If $S$ is FI-CS and indecomposable then every fully invariant subsemimodule is essential in $S$.

***Proof*:** Assume that $0 \neq \mathcal{N} \trianglerighteq S$, since $S$ is FI-CS, then there exists a summand $\mathcal{D}$ of $S$, such that $\mathcal{N} \leq^e \mathcal{D}$, since $S$ indecomposable by hypotheses, then either $\mathcal{D} = 0$ or $\mathcal{D} = S$, but $\mathcal{N}$ is nonzero, therefore $\mathcal{N} \leq^e S$. □

**Remark (4.2.27):** If $S_1$ is singular and $S_2$ nonsingular, then Hom($S_1$, $S_2$)=0.

***Proof*:** Assume that $\hbar \in$Hom($S_1$, $S_2$)and $x \in S_1$, since $S_1$ is singular then there exists left $\mathcal{J} \leq^e R$, such that $\mathcal{J}x = 0$, hence $\hbar(\mathcal{J}x) = \mathcal{J}\hbar(x) = 0$, but $\hbar(x) \in S_2$, and $S_2$ is nonsingular then $\hbar(x) = 0$, so $\hbar = 0$. □

The next result give the properties of singular subsemimodule of FI-CS semimodules.

**Proposition(4.2.28):** Let $S$ be a FI-CS semimodule, then $S$=$Z$($S$)⊕$\mathcal{P}$, for some nonsingular subsemimodule $P$ of $S$ and $P$ is $Z$($S$)-injective.

***Proof*:** If $Z$($S$)=0 or $Z$($S$)=$S$, it is trivial. Suppose that Z($S$) ≤$S$, since $Z$($S$) is fully invariant by Proposition(3.2.7), and $S$ is FI-CS, hence there exists a summand $\mathcal{D}$ of $S$, such that $Z(S) \leq^e \mathcal{D}$, and $S$= $\mathcal{D} \oplus \mathcal{P}$, for some $\mathcal{P} \leq S$, so $Z$($S$)=Z($\mathcal{D}$)⊕Z($\mathcal{P}$), but by Proposition(3.2.4), Z($\mathcal{D}$)=$\mathcal{D}$ ∩$Z$($S$)=$Z$($S$) therefore Z($\mathcal{P}$)=0, hence $\mathcal{P}$ is nonsingular), therefore by Remark (4.2.27), Hom($\mathcal{N}, \mathcal{P}$)= 0, for any $\mathcal{N} \leq Z(S)$, so $\mathcal{P}$ is $Z$($S$)-injective. □

## 4.3 T–ExtendingSemimodules

The concept T–extendingfor modules has been introduced and explained by [39].We will introduce this concept for semimodules and gives some results related to this concept.

### 4.3.1 T-essential and T-closed Subsemimodules

In the next, there are some properties of t-essential and t-closed semimodules, those properties will be converted for subsemimodules.

**Definition 4.3.1.** A subsemimodule $\mathcal{A}$ of $S$ is said to be t-essential and write ($\mathcal{A} \leq^{tes} S$) if for any $C \leq S$, $\mathcal{A} \cap C \leq Z_2(S)$ implies $C \leq Z_2(S)$.

**Definition 4.3.2.** A subsemimodule $C$ of $S$ is said to be t-closed and write ($C \leq^{tc} S$) if it has no proper t-essential subsemimodule, i.e if $C \leq^{tes} C' \leq S$ implies $C = C'$.

The next lemma gives a useful note about second singular subsemimodule of nonsingular semimodule.

**Lemma 4.3.3.** If $S$ is a nonsingular $R$-semimodule then $Z_2(S) = 0$.

***Proof*:** Assume $S$ is nonsingular, then $Z(S)=0$, since $Z_2(S)/Z(S)=Z(S/Z(S))=0$, then $Z_2(S)=Z(S)=0$. □

***Note*:** If $S$ is singular then $Z_2(S)=Z(S)=S$.

**Remarks 4.3.4:** Let $S$ be an $R$-semimodule, then:

1. Every essential subsemimodule of $S$ is t-essential.
2. If $\mathcal{A}$ is a complement of $Z_2(S)$ in $S$ then $\mathcal{A}$ is t-essential.
3. Every t-closed subsemimodule of $S$ is closed.

4. If $\mathcal{A} \leq^{c} S$, and $S$ is nonsingular then $\mathcal{A} \leq^{tc} S$.
5. For each $\mathcal{A} \leq S$, $0 \leq^{tes} Z_2(\mathcal{A})$ (in particular $0 \leq^{tes} Z_2(S)$).
6. $S$ is nonsingular if and only if $0 \leq^{tc} S$.
7. $Z_2(S) \leq^{tc} S$.
8. $S/Z_2(S)$ is nonsingular.

***Proof*:**

**(1)** Assume that $\mathcal{A} \leq^{e} S$ and $\mathcal{A} \cap C \leq Z_2(S)$, let $0 \neq x \in C$, so $x \in C+Z_2(S)$. Since $\mathcal{A} \leq^{e} S$, then there exists $r \in R$ such that $0 \neq rx \in \mathcal{A}$ hence $0 \neq rx \in \mathcal{A} \cap C \leq Z_2(S)$. This implies $Z_2(S) \leq^{e} C+Z_2(S)$, but $Z_2(S)$ is closed in $S$, so $Z_2(S) = C+Z_2(S)$, that is, $C \leq Z_2(S)$ and $\mathcal{A} \leq^{tes} S$.

***Note:*** The converse of (1) is not true in general for example in $Z_6$, $\langle\bar{2}\rangle$ and $\langle\bar{3}\rangle$ are t-essential subsemimodules but not essential.

**(2)** Assume that $\mathcal{A}$ is a complement of $Z_2(S)$ and $\mathcal{A} \cap C \leq Z_2(S)$, let $0 \neq x \in C$. If $x \notin Z_2(S)$, then $x \notin \mathcal{A}$ and $\mathcal{A}+Rx$ is a proper extension of $\mathcal{A}$, hence there exists $r \in R$ such that $0 \neq rx \in Z_2(S)$(since $\mathcal{A}$ is a complement of $Z_2(S)$), that is $Z_2(S) \leq^{e} C+Z_2(S)$. Therefore $C \leq Z_2(S)$ and $\mathcal{A} \leq^{tes} S$.

**(3)** Assume $\mathcal{A} \leq^{tc} S$ and let $\mathcal{A} \leq^{e} \mathcal{B} \leq S$, then by (1), $\mathcal{A} \leq^{tes} \mathcal{B}$, but $\mathcal{A} \leq^{tc} S$, then $\mathcal{A} = \mathcal{B}$, and hence $\mathcal{A} \leq^{c} S$.

***Note:*** the converse of (3) is not true for example in $Z_6$, $\langle\bar{2}\rangle$ and $\langle\bar{3}\rangle$ are closed subsemimodules but not t-closed.

**(4)** Assume that $\mathcal{A} \leq^{c} S$, and $\mathcal{A} \leq^{tes} \mathcal{B} \leq S$, then for each $C \leq \mathcal{B}$, $\mathcal{A} \cap C \leq Z_2(S) = 0$ (since $S$ is nonsingular) and by Lemma (4.3.3) implies $C=0$, that is, $\mathcal{A} \leq^{e} \mathcal{B}$, but $\mathcal{A} \leq^{c} S$ by assumption , so $\mathcal{A} = \mathcal{B}$ and $\mathcal{A} \leq^{tc} S$.

**(5)** Clear.

**(6)** Since $0\leq^{c}S$ (always true), by (4), $0\leq^{tc}S$. Conversely, assume that $0\leq^{tc}S$, and $S$ is not nonsingular then $Z_2(S)\neq 0$, and by(5) $0\leq^{\mathbf{tes}}Z_2(S)$, a contradiction, hence $S$ is nonsingular.

**(7)** Assume $Z_2(S)\leq^{tes}S'\leq S$, let $\mathcal{B}\leq S'$ and $\mathcal{B}\nsubseteq Z_2(S)$, then $Z_2(S)\cap\mathcal{B}\leq Z_2(S)$, a contradiction, then $Z_2(S)=S'$.

**(8)** Since $0=(Z_2(S)/Z_2(S))\leq^{tc}(S/Z_2(S))$ then by (6), $S/Z_2(S))$ is nonsingular. □

The following corollary is immediate from Remarks (4.3.4(8)) and Lemma (4.3.3).

**Corollary 4.3.5:** For any *R*-semimodule *S*, $Z_2(S/Z_2(S))=0$.

As in modules the next definition will be given, it is needed to prove analogues results to that in modules.

**Definition 4.3.6:** An *R*-semimodule *S* is said to be $Z_2$–torsion if $Z_2(S)=S$.

The next lemma gives the relation between singular and $Z_2$ –torsion.

**Lemma 4.3.7:** Every singular *R*-semimodule *S* is $Z_2$–torsion.

***Proof*:** Let *S* be a singular semimodule, since $Z(S)\leq Z_2(S)\leq S$, and $Z(S)=S$ , then $Z_2(S)=S$. □

The following lemma connects between $Z_2$-torsion and second singular of the same semimodule.

**Lemma 4.3.8:** A subsemimodule $\mathcal{A}$of $S$ is $Z_2$-torsion if and only if $\mathcal{A} \leq Z_2(S)$.

***Proof:*** Suppose that $\mathcal{A}$ is $Z_2$-torsion, then $Z_2(\mathcal{A})= \mathcal{A}$, but $Z_2(\mathcal{A}) \leq Z_2(S)$ then $\mathcal{A} \leq Z_2(S)$. Conversely, assume that $\mathcal{A} \leq Z_2(S)$, since by Proposition (3.2.5), $Z_2(\mathcal{A})= \mathcal{A} \cap Z_2(S)= \mathcal{A}$, therefore $\mathcal{A}$ is $Z_2$-torsion . □

In the following some results that appeared for modules, will be converted to semimodules by adding suitable condition.

**Lemma 4.3.9:** Let $\mathcal{A}$ be a subtractive subsemimodule of $S$. If $\mathcal{A}$ and $S/\mathcal{A}$ are $Z_2$-torsion, then $S$ is $Z_2$-torsion.

***Proof*:** Let $x \in S$, then $x+\mathcal{A} \in S/\mathcal{A}$, by assumption $x+\mathcal{A} \in Z_2(S/\mathcal{A})$, then $x+\mathcal{A}+Z(S/\mathcal{A}) \in Z(S/\mathcal{A}/Z(S/\mathcal{A}))$, so there exists left$I \leq^{e} R$, such that $I(x+\mathcal{A}+Z(S/\mathcal{A}))=0$, therefore $I(x+\mathcal{A}) \leq Z(S/\mathcal{A})$, hence there exists left $J \leq^{e} R$ such that $(I \cap J)(x+\mathcal{A})=0$, then $(I \cap J)(x) \leq \mathcal{A}=Z_2(\mathcal{A})$[by assumption],since $Z_2(\mathcal{A}) \leq Z_2(S)$then $(I \cap J)x \leq Z_2(S)$, this implies, $x+Z_2(S) \in Z_2(S/Z_2(S))$but by Corollary (4.3.5), $Z_2(S/Z_2(S))=0$, hence $x \in Z_2(S)$, but $Z_2(S) \leq S$, then $Z_2(S)= S$ and $S$ is $Z_2$-torsion. □

As a special case of nonsingular semimodule.

**Proposition 4.3.10:** If $\mathcal{A} \leq S$, and $S$ is a nonsingular $R$-semimodule then $\mathcal{A} \leq^{tes} S$if and only if $\mathcal{A} \leq^{e} S$.

***Proof*:** Assume that $\mathcal{A} \leq^{tes} S$, and let $\mathcal{A} \cap \mathcal{B} =0$, where $\mathcal{B} \leq S$. Since $\mathcal{A} \leq^{tes} S$, then $\mathcal{B} \leq Z_2(S)$ since $S$ is nonsingular then by Lemma (4.3.3), $Z_2(S)=0$, therefore $B=0$, and so $\mathcal{A} \leq^{e} S$. The converseis clear by Remarks (4.3.4(1)). □

***Note***: If $S$ is a singular semimodule then any subsemimodule of $S$ is t-essential.

**Proposition 4.3.11:** For a subtractive subsemimodule $\mathcal{A}$ of $S$. If $(\mathcal{A}+Z_2(S))\leq^{e} S$ then $S/\mathcal{A}$ is $Z_2$-torsion.

***Proof***: Assume that $(\mathcal{A}+Z_2(S))\leq^{e} S$,then byLemma(3.2.12), $S/(\mathcal{A}+Z_2(S))$ singular, and hence by Lemma(4.3.7) $S/(\mathcal{A}+Z_2(S))$ is $Z_2$-torsion. But$\mathcal{A}+Z_2(S)/\mathcal{A} \cong Z_2(S)/\mathcal{A} \cap Z_2(S))=Z_2(S)/Z_2(\mathcal{A})$is singular, hence by Lemma (4.3.7),$\mathcal{A}+Z_2(S)/\mathcal{A}$is $Z_2$-torsion, and $(S/\mathcal{A})/( (\mathcal{A}+Z_2(S)/ \mathcal{A}) \cong S/(\mathcal{A}+Z_2(S))$ is $Z_2$-torsion, then by Lemma (4.3.9), $S/\mathcal{A}$ is $Z_2$-torsion. □

**Proposition 4.3.12:** If $\mathcal{A}$ is a subtractive subsemimodule of $S$ and $S/\mathcal{A}$ is $Z_2$-torsion, then $\mathcal{A} \leq^{tes} S$.

***Proof***: Assume that $S/\mathcal{A}$ is $Z_2$-torsion, since $(S/\mathcal{A})/Z(S/\mathcal{A})=Z_2(S/\mathcal{A})/Z(S/\mathcal{A})=Z((S/\mathcal{A})/Z(S/\mathcal{A}))$, then $(S/\mathcal{A})/Z(S/\mathcal{A})$is singular, but $(S/\mathcal{A})/Z(S/\mathcal{A})\cong S/\mathcal{A}^*$, where $\mathcal{A}^*/\mathcal{A} =Z(S/\mathcal{A})$, so $S/\mathcal{A}^*$ is singular. Now let $\mathcal{A} \cap \mathcal{B} \leq Z_2(S)$, and $b\in \mathcal{B} \leq S$, then $b\in S$, so $b+\mathcal{A}^*\in S/\mathcal{A}^*=Z(S/\mathcal{A}^*)$, then there exists left $I\leq^{e} R$, such that $I(b+\mathcal{A}^*)=0$. Therefore $Ib\leq \mathcal{A}^*$, for every $x\in I$, $xb+\mathcal{A} \in \mathcal{A}^*/\mathcal{A}$, since $\mathcal{A}^*/\mathcal{A}= Z(S/\mathcal{A})$, then there exists left $K\leq^{e} R$, such that $K(xb+\mathcal{A})=0$, so $Kxb\leq \mathcal{A}$, but $Kxb\leq \mathcal{B}$, so $Kxb\leq \mathcal{A} \cap \mathcal{B} \leq Z_2(S)$, thus $xb+Z_2(S) \in Z(S/Z_2(S))=0$ hence $Ib\leq Z_2(S)$, so $b+Z_2(S) \in Z(S/Z_2(S))=0$, hence $b\in Z_2(S)$ and hence $B\leq Z_2(S)$, so $\mathcal{A} \leq^{tes} S$.□

**Proposition 4.3.13:** For any $R$-semimodule $S$. If $\mathcal{A} \leq S$, then the following statements are equivalent:

**1.** $\mathcal{A} \leq^{tes} S$.

**2.** $\mathcal{A}$+$Z_2(S)\leq^{e}S$

**3.** $\mathcal{A}$+$Z_2(S)/Z_2(S)\leq^{e}S/Z_2(S)$.

***Proof*:**

$(\mathbf{1}\Rightarrow\mathbf{2})$ Assume that $(\mathcal{A}+Z_2(S))\cap B=0$, where $B\leq S$. Then $\mathcal{A}\cap B=0$ and $Z_2(S)\cap B=0$. Since $\mathcal{A}\leq^{tes}S$, and $\mathcal{A}\cap B=0\leq Z_2(S)$, so $B\leq Z_2(S)$, but $Z_2(S)\cap B=0$ implies $B=0$. Therefore $\mathcal{A}+Z_2(S)\leq^{e}S$.

$(\mathbf{2}\Rightarrow\mathbf{3})$Assume that $\mathcal{A}+Z_2(S)\leq^{e}S$ (by Proposition (3.2.11), $Z_2(S)\leq^{c}S$ then by Lemma (2.2.12), $\mathcal{A}+Z_2(S)/Z_2(S)\leq^{e}S/Z_2(S)$ .

$(\mathbf{3}\Rightarrow\mathbf{1})$ By Propositions (4.3.11) and (4.3.12) . □

The next Lemma gives the homomorphic image of $Z_2$-torsion is $Z_2$-torsion too.

**Lemma 4.3.14:** Let $S$ and $S'$ be *R-semimodules* and $f:S\to S'$ be an epimorphism. If $S$ is $Z_2$- torsion then $S'$ is $Z_2$- torsion.

***Proof*:** Assume that $f:S\to S'$, is an epimorphism and $S$ is $Z_2$-torsion, since $S'=f(S)=f(Z_2(S))\subseteq Z_2(S')$, therefore $S'$is $Z_2$- torsion. □

***Note:*** If $S$ is $Z_2$-torsion, then $S/Z(S)=Z_2(S)/Z(S)=Z(S/Z(S))$, therefore $S/Z(S)$, is singular hence by Lemma (4.3.7), $S/Z(S)$ is $Z_2$-torsion.

**Proposition 4.3.15:** Let $S$ be an $R$-semimodule. If$\mathcal{A}$is a subtractive t-closed subsemimodule of $S$, then $Z_2(S)\leq\mathcal{A}$.

***Proof*:** Assume that$\mathcal{A}\leq^{tc}S$, since $Z_2(S)$ is $Z_2$-torsion, then by Lemma (4.3.14), $Z_2(S)/Z_2(\mathcal{A})$ is $Z_2$-torsion but$\mathcal{A}+Z_2(S))/\mathcal{A}\cong Z_2(S)/\mathcal{A}\cap Z_2(S)=Z_2(S)/Z_2(\mathcal{A})$, then $\mathcal{A}+Z_2(S))/\mathcal{A}$ is $Z_2$-

torsion, hence by Proposition (4.3.12), $\mathcal{A} \leq^{tes}$ ($\mathcal{A}$+$Z_2(S)$), but $\mathcal{A} \leq^{tc} S$, by assumption, then $\mathcal{A} = (\mathcal{A}+Z_2(S))$, hence $Z_2(S) \leq \mathcal{A}$. □

**Proposition 4.3.16:** For an $R$-semimodule $S$, if $C \leq \mathcal{A} \leq S$ and $C$ is subtractive, then $\mathcal{A} \leq^{tc} S$ if and only if $\mathcal{A}/C \leq^{tc} S/C$.

***Proof*:** Assume that $\mathcal{A} \leq^{tc} S$ and $\mathcal{A}/C$ is not t-closed in $S/C$, then there exists $S'/C \leq S/C$ such that $\mathcal{A}/C \leq^{tes} S'/C$, then by Propositions (4.3.13) $\mathcal{A}/C+Z_2(S'/C) \leq^{e} S'/C$, since $S'/\mathcal{A} \cong (S'/C)/(\mathcal{A}/C)$ then by Propositions (4.3.11), $S'/\mathcal{A}$ is $Z_2$ torsion, hence by Proposition (4.3.12) $\mathcal{A} \leq^{tes} S'$, a contradiction with the assumption that $\mathcal{A} \leq^{tc} S$. Conversely, suppose that $\mathcal{A}/C \leq^{tc} S/C$, and $\mathcal{A}$ is not t-closed in $S$, then there exists $S' \leq S$, such that $\mathcal{A} \leq^{tes} S'$, then by Proposition (4.3.13), $(\mathcal{A}+Z_2(S)) \leq^{e} S'$ hence by Proposition (4.3.11) $S'/\mathcal{A}$ is $Z_2$ torsion, then $(S'/C)/(\mathcal{A}/C)$ is $Z_2$ torsion. By Proposition (4.3.12), $\mathcal{A}/C \leq^{tes} S'/C$, acontradiction, therefore $\mathcal{A} \leq^{tc} S$. □

**Proposition 4.3.17:** Let $S$ be an $R$-semimodule and $\mathcal{A} \leq S$. If there exists a subsemimodule $L$ such that $\mathcal{A}$ maximal with respect to property that $L \cap \mathcal{A}$ is a $Z_2$ torsion, then $\mathcal{A} \leq^{tc} S$.

***Proof*:** Suppose the property of $\mathcal{A}$ hold, and let $\mathcal{A} \leq^{tes} S' \leq S$ then $\mathcal{A} \cap (S' \cap L) \leq Z_2(S)$, implies $(S' \cap L) \leq Z_2(S)$, therefore by Lemma (4.3.8) $S' \cap L$ is $Z_2$ torsion, but $\mathcal{A}$ is maximal with this property then $\mathcal{A} = S'$, and hence $\mathcal{A} \leq^{tc} S$. □

**Proposition 4.3.18:** Let $S$ be an $R$-semimodule and $\mathcal{A} \leq S$. If $\mathcal{A} \leq^{tc} S$, then $\mathcal{A}$ contains $Z_2(S)$, and $\mathcal{A}/Z_2(S) \leq^{c} S/Z_2(S)$.

***Proof*:** Suppose that $\mathcal{A} \leq^{tc} S$ then by Proposition (4.3.15), $Z_2(S) \leq \mathcal{A}$, now let $(\mathcal{A}/Z_2(S)) \leq^{e} S'/Z_2(S) \leq S/Z_2(S)$ then by Lemma(2.2.10), $\mathcal{A} \leq^{e} S'$, but by Remarks (4.3.4)(3),$\mathcal{A} \leq^{c} S$ a contradiction, hence $\mathcal{A}/Z_2(S) \leq^{c} S/Z_2(S)$ . □

**Proposition 4.3.19:** Let $S$ be a subtractive$R$-semimoduleand $\mathcal{A} \leq S$. If $Z_2(S) \leq \mathcal{A}$, and $\mathcal{A}/Z_2(S) \leq^{c} S/Z_2(S)$, then $\mathcal{A} \leq^{c} S$.

***Proof*:** Suppose that $\mathcal{A}/Z_2(S) \leq^{c} S/Z_2(S)$, with $Z_2(S) \leq \mathcal{A}$, and let $\mathcal{A} \leq^{e} S' \leq S$, then by Proposition(2.2.12), $(\mathcal{A}/Z_2(S)) \leq^{e} S'/Z_2(S)$, a contradiction, then$\mathcal{A}=S'$and$\mathcal{A} \leq^{c} S$. □

**Proposition 4.3.20:** Let $S$ be an $R$-semimodule. If $Z_2(S) \leq \mathcal{A}$ and$\mathcal{A} \cap S'=0$, where $S' \leq S$ then $S'$ is nonsingular.

***Proof*:**By Proposition (3.2.5), $Z_2(S')=S' \cap Z_2(S)=0$, then $S'$ is nonsingular. □

***Note*:** In Proposition (3.2.14), if we replace closed by t-closed subsemimodule the condition become necessary and sufficient condition.

**Proposition 4.3.21:** Let $S$ be an $R$-semimodule and$\mathcal{A}$asubtractive subsemimodule of $S$, then $\mathcal{A} \leq^{tc} S$, if and only if $S/\mathcal{A}$ is nonsingular.

***Proof*:** It is clear by Proposition (4.3.16), Remarks (4.3.4)(6).

Recall that, a homomorphism of $R$-semimodules $\varphi:\mathcal{A} \longrightarrow B$ is said to be $k$- *regular* if $\varphi(a)=\varphi(a')$ then $a+k=a'+k'$ for some $a, a' \in \mathcal{A}$ and $k, k' \in \ker(\varphi)$[49].

A result, which appeared for modules, will be converted for semimodules by adding suitable conditions.

**Corollary (4.3.22):** For any semisubtractive, cancellative$R$-semimodule$S$. If$\emptyset$ is a $k$- regular endomorphism of $S$ and $\mathcal{A}$ is a t-closed subtractive subsemimodule of $S$, then $\emptyset^{-1}(\mathcal{A}) \leq^{\mathbf{tc}} S$.

***Proof*:** Let$\theta$:$S/\emptyset^{-1}(\mathcal{A}) \to S/\mathcal{A}$such that $\theta: m + \emptyset^{-1}(\mathcal{A}) \longmapsto \emptyset(\mathrm{m}) + \mathcal{A}$ [$\theta$ is well defined since if $m_1 + \emptyset^{-1}(\mathcal{A}) = m_2 + \emptyset^{-1}(\mathcal{A})$, then $m_1 + h_1 = m_2 + h_2$, for some $h_1, h_2 \in \emptyset^{-1}(\mathcal{A})$, then $\emptyset(m_1) + \emptyset(h_1) = \emptyset(m_2) + \emptyset(h_2)$, where $\emptyset(h_1), \emptyset(h_2) \in \mathcal{A}$, hence$\emptyset(m_1) + \mathcal{A} = \emptyset(m_2) + \mathcal{A}$. On other hand if $\emptyset(m_1) + \mathcal{A} = \emptyset(m_2) + \mathcal{A}$, then $\emptyset(m_1) + a_1 = \emptyset(m_2) + a_2$, where $a_1, a_2 \in \mathcal{A}$. By semi subtractive there exists $t$ such that either $m_1+t=m_2$or $m_1=m_2+t$. Case one: $m_1+t=m_2$,by cancellative, $a_1 = \emptyset(t) + a_2$, so by subtractive $\emptyset(t) \in \mathcal{A}$. Case two: $m_1=m_2+t$, by cancellative, $\emptyset(t) + a_1 = a_2$, by subtractive$\emptyset(t) \in \mathcal{A}$, hence $t \in \emptyset^{-1}(\mathcal{A})$. Therefore $\emptyset(m_1) + \emptyset(t)$=$\emptyset(m_2) + \emptyset(t)$, so $\emptyset(m_1 + t) = \emptyset(m_2 + t)$. Since $\emptyset$ is k- regular, then$m_1 + t + k = m_2 + t + k'$, where $k$,$k'$∈ ker$\emptyset$, since$ker\emptyset \leq \emptyset^{-1}(\mathcal{A})$, hence$m_1 + \emptyset^{-1}(\mathcal{A}) = m_2 + \emptyset^{-1}(\mathcal{A})$, therefore $\theta$ is a monomorphism hence $S/\,\emptyset^{-1}(\mathcal{A}) \cong$ subsemimodule of $S/\mathcal{A}$ (which is nonsingular), then $S/\emptyset^{-1}(\mathcal{A})$ is nonsingular hence by Proposition (4.3.21), $\emptyset^{-1}(\mathcal{A}) \leq^{\mathrm{tc}} S$. □

Our next result gives a characterization of $Z_2$-torsion property**.**

**Corollary 4.3.23:** Let $S$ be an $R$-semimodule. If $\mathcal{A}$is asubtractive t-closed subsemimodule of $S$, then $\mathcal{A}$=$Z_2(S)$ if and only if $\mathcal{A}$ is $Z_2$-torsion.

***Proof*:** Assume that $\mathcal{A} \leq^{\mathbf{tc}} S$, then by Proposition (4.3.15) $Z_2(S) \leq \mathcal{A}$, but $\mathcal{A}$ is $Z_2$-torsion implies$\mathcal{A}$=$Z_2(\mathcal{A})$≤$Z_2(S)$, then $\mathcal{A}$=$Z_2(S)$. *Conversely*, assume that $\mathcal{A}$=$Z_2(S)$, implies $Z_2(\mathcal{A})$=$\mathcal{A} \cap Z_2(S)$=$\mathcal{A}$, that is $\mathcal{A}$ is $Z_2$ torsion. □

**Corollary 4.3.24:** Let $S$ be an $R$-semimodule and $\mathcal{A}$ a subtractive t-closed subsemimodule of $S$, then $\mathcal{A}$ is $Z_2$-torsion if and only if there exists $L \leq^{\mathbf{tes}} S$ for which $\mathcal{A} \cap L \leq Z_2(S)$.

***Proof*:** Assume that $\mathcal{A} \leq^{\mathbf{tc}} S$ and $\mathcal{A}$ is $Z_2$-torsion, then by Corollary (4.3.23) $\mathcal{A}=Z_2(S)$ and $\mathcal{A} \cap S \leq Z_2(S)$, where $S$ is t-essential subsemimodule of $S$. *Conversely*, assume that $L \leq^{tes} S$ and $\mathcal{A} \cap L \leq Z_2(S)$, then $\mathcal{A} \leq Z_2(S)$, but by Proposition (4.3.15) $Z_2(S) \leq \mathcal{A}$ hence $\mathcal{A} = Z_2(S)$,then by Corollary(4.3,23)$\mathcal{A}$ is $Z_2$-torsion. □

**Proposition 4.3.25:** Let $S$ be an $R$-semimodule and $\mathcal{A} \leq \mathcal{N} \leq S$. If $\mathcal{A} \leq^{tc} S$, then $\mathcal{A} \leq^{\mathbf{tc}} \mathcal{N}$.

***Proof*:** Assume that $\mathcal{A} \leq^{\mathbf{tc}} S$. Let $\mathcal{A} \leq^{\mathbf{tes}} \mathcal{B} \leq \mathcal{N}$, then $\mathcal{A} \leq^{\mathbf{tes}} \mathcal{B} \leq S$(since $\mathcal{N} \leq S$), and so $\mathcal{A} = \mathcal{B}$ as required. □

**Proposition 4.3.26:** Let $S$ be an $R$-semimodule, $\mathcal{A} \leq \mathcal{N} \leq S$, with $\mathcal{A}, \mathcal{N}$ are subtractive. If $\mathcal{A} \leq^{\mathbf{tc}} \mathcal{N}$, and $\mathcal{N} \leq^{tc} S$ then $\mathcal{A} \leq^{\mathbf{tc}} S$.

***Proof*:** Assume that $\mathcal{A} \leq^{\mathbf{tc}} \mathcal{N}$, and $\mathcal{N} \leq^{\mathbf{tc}} S$, then by Proposition (4.3.15), $Z_2(\mathcal{N}) \leq \mathcal{A}$, and $Z_2(S) \leq \mathcal{N}$, therefore by Proposition (4.3.18) $\mathcal{A}/Z_2(\mathcal{N}) \leq^{c} \mathcal{N}/Z_2(\mathcal{N})$ and $\mathcal{N}/Z_2(S) \leq^{c} S/Z_2(S)$, since by Proposition (3.2.5), $Z_2(\mathcal{N}) = \mathcal{N} \cap Z_2(S) = Z_2(S)$, then $\mathcal{A}/Z_2(S) \leq^{c} \mathcal{N}/Z_2(S)$ and $\mathcal{N}/Z_2(S) \leq^{c} S/Z_2(S)$, therefore by Lemma(2.2.13), $\mathcal{A}/Z_2(S) \leq^{c} S/Z_2(S)$, so by Proposition (4.3.19), $\mathcal{A} \leq^{c} S$. If $\mathcal{A} \leq^{\mathbf{tes}} S' \leq S$, $\mathcal{A} \cap \mathcal{B}=0$, for some $\mathcal{B} \leq S'$, then $\mathcal{A} \cap \mathcal{B} \leq Z_2(S)$, hence $\mathcal{B} \leq Z_2(S) \leq \mathcal{A}$ then $\mathcal{B} = \mathcal{A} \cap \mathcal{B}=0$, so $\mathcal{A} \leq^{e} S'$, a contradiction, therefore $\mathcal{A} \leq^{\mathbf{tc}} S$. □

**Remark 4.3.27:** For any $R$-semimodule $S$. If $\mathcal{A} \leq^{c} S$ and $\mathcal{A}' \leq^{c} S$, this does not lead to $\mathcal{A} \cap \mathcal{A}' \leq^{c} S$.

***Note*:**In semimodule $S$=$\mathbb{Z}_8 \oplus \mathbb{Z}_2$, $R$= $\mathbb{Z}$, the subsemimodules $\langle(\bar{1},\bar{0})\rangle$and $\langle(\bar{1},\bar{1})\rangle$are summand hence closed, white $\langle(\bar{1},\bar{0})\rangle \cap \langle(\bar{1},\bar{1})\rangle$=$\langle(\bar{2},\bar{0})\rangle \leq^{e} \langle(\bar{1},\bar{0})\rangle$, hencethe intersection of two closed in $S$ is not necessarily closed in $S$.

But if we replace closed subsemimodule by t-closed subsemimodule in Remark(4.3.27), we get the following result.

**Proposition4.3.28:** Let $S$ be a subtractive $R$-semimodule, if $\mathcal{A} \leq S$ and$\mathcal{A}' \leq^{tc} S$, then$\mathcal{A} \cap \mathcal{A}' \leq^{tc} \mathcal{A}$.

***Proof*:** Assume that$\mathcal{A} \cap \mathcal{A}' \leq^{tes} \mathcal{D} \leq \mathcal{A}$, then by Propositions (4.3.11) and (4.3.13) $D/(\mathcal{A} \cap \mathcal{A}')$is $Z_2$-torsion, hence $\mathcal{D}/(\mathcal{D} \cap \mathcal{A}')$ is $Z_2$-torsion (since$\mathcal{D}/(\mathcal{D} \cap \mathcal{A}')$is homomorphic image of$\mathcal{D}/(\mathcal{A} \cap \mathcal{A}')$ ). But $\mathcal{D}/(\mathcal{D} \cap \mathcal{A}') \cong (\mathcal{D} + \mathcal{A}')/\mathcal{A}'$, therefore $(\mathcal{D} + \mathcal{A}')/\mathcal{A}'$ is $Z_2$-torsion, so by Proposition (4.3.12), $\mathcal{A}' \leq^{tes} (\mathcal{D} + \mathcal{A}')$, then$\mathcal{A}' = \mathcal{D} + \mathcal{A}'$, and $\mathcal{D} \leq \mathcal{A}'$. But $\mathcal{A} \cap \mathcal{A}' \leq \mathcal{D}$, then $\mathcal{D} = \mathcal{A} \cap \mathcal{A}'$, and so$\mathcal{A} \cap \mathcal{A}' \leq^{tc} \mathcal{A}$. □

In the next result we assume that a semimodule iscancellativeand semisubtractive to get the same result as in modules.

**Proposition 4.3.29:** Let $S$ be acancellativesemisubtractive $R$-semimodule. An arbitrary intersection of t-closed subtractive subsemimodules is t-closed.

***Proof*:** Assume that $C = \bigcap_{\lambda \in \Lambda} C_\lambda$, where $C_\lambda$ is a t-closed subsemimodule of $S$, for any $\lambda$ in an index set Λ. Let $\theta$:$S/C \to \prod_\lambda (S/C_\lambda)$, defined by $\mathrm{m} + C \mapsto (\mathrm{m} + C_\lambda)$. If $m + C = m' + C$, then $m + c_1 = m' + c_2$, where $c_1$, $c_2 \in C$, hence $c_1, c_2 \in C_\lambda$, for each $\lambda$, hence$(m + C_\lambda) = (m' + C_\lambda) \in \prod_\lambda (S/C_\lambda)$, therefore $\theta$ is well defined . Now let $(m + C_\lambda) = (m' + C_\lambda)$, then $m + C_\lambda = m' + C_\lambda$, for each $\lambda$, and $m + c_\lambda = m' + c'_\lambda$, where $c_\lambda$, $c'_\lambda \in C_\lambda$ for each $\lambda \in \Lambda$. By semi subtractive there exists $t$ such that either $m$+$t$=$m'$ or $m$=$m'$+$t$.

***Case one*:** $m+ t=m'$,by cancellative, $c_\lambda = t + c'_\lambda$, so by subtractive $t \in C_\lambda$, hence for each $\lambda, t \in C$.

***Case two:*** $m=m'+ t$, by cancellative, $c_\lambda + t = c'_\lambda$, by subtractive $t \in C_\lambda$, hence for each $\lambda$, $t \in C$. Therefore $m+ C=m'+C$, hence $\theta$ is a monomorphism. Since $S/C_\lambda$ is nonsingular [by Proposition (4.3.21)], then $\prod_\lambda(S/C_\lambda)$ is nonsingular, and hence $S/C$ is nonsingular. So $C \leq^{tc} S$. □

In the next t-extending semimoduleis introducing and investigating, as well as some properties of t-extending semimodule are also studying a characterization of this concept.

**Definition 4.3.30:** A semimodule $S$ is said to be t-extending if every t-closed subsemimodule of $S$ is a summand.

The following Remark relates the concept of $Z_2$- torsion and CS-semimodules with the concept of t-extending.

**Remark 4.3.31:**

1. Every $Z_2$- torsion semimodule is t-extending.
2. Every *CS R*-semimodule is t- extending.

***Proof*:**

**(1)** Let $S$ be $Z_2$-torsion, then the only t-closed subsemimodule of $S$ is $S$ which is a summand of $S$, then $S$ is t-extending.

**(2)** Assume $S$ is *CS* and let $C \leq^{tc} S$, then $C \leq^{c} S$ by Remark (4.3.4(3)), since $S$ is *CS*, then $C$is a summand of $S$, so $S$ is t-extending. □

***Note***: For example to (2), $Z_6$ is CS and t-extending semimodule. But the converse of (2) is not true for example where $S=\mathbb{Z}_8 \oplus \mathbb{Z}_2$, then $S$is singular, hence each subsemimodule is t-essential, therefore $S$ is t-extending , but by Remark (2.2.2(2)), $S$ is not CS.

The result in the following propositions gives the useful properties of t-extending semimodules.

**Proposition 4.3.32:** Let $S$ be an $R$–semimodule. If $S$ is t-extending then for any subtractive subsemimodule $\mathcal{A}$of $S$, $\mathcal{A}_2$ is a summand of $S$, where $\mathcal{A}_2/\mathcal{A}= Z_2(S/\mathcal{A})$.

***Proof***: Since $S/\mathcal{A}_2 \cong (S/\mathcal{A})/(\mathcal{A}_2/\mathcal{A}) = (S/\mathcal{A})/Z_2(S/\mathcal{A})$, then by Proposition (4.3.21) $\mathcal{A}_2/\mathcal{A} \leq^{\mathbf{tc}} S/\mathcal{A}$ therefore by Proposition (4.3.16),$\mathcal{A}_2 \leq^{tc} S$, since $S$ ist-extending then $\mathcal{A}_2$is a summand of $S$. □

**Proposition 4.3.33:** Let $S$ be an $R$–semimodule. If $S$ is t-extending then $S=Z_2(S)\oplus S'$, where $S'$ is nonsingular $CS$-semimodule.

***Proof***: Since by Remark (4.3.4(7)),$Z_2(S)\leq^{\mathbf{tc}} S$ and $S$ is t-extending then $Z_2(S)$ is a summand of $S$, say $S=Z_2(S) \oplus S'$, for some $S'\leq S$, hence by Lemma(3.2.16) $S'$ is nonsingular. Let $C\leq^{c} S'$, since $S'$ is nonsingular then by Remark (4.3.4(4)), $C\leq^{\mathbf{tc}} S'$ so by Proposition (4.3.21), $S'/C$ is nonsingular. Since $C\leq Z_2(S) \oplus C$then $S'/(Z_2(S) \oplus C)$ is nonsingular, that is $Z_2(S)\oplus C\leq^{\mathbf{tc}} S$ , therefore $Z_2(S)\oplus C$ is a summand of $S$ (since $S$ is t-extending), say $S=Z_2(S)\oplus C\oplus S''$, by Lemma(2.2.14),$S'=C\oplus(Z_2(S) \oplus S'')\cap S'$, so $C$ is a summand of $S'$, and $S'$ is $CS$. □

**Proposition 4.3.34:** Let $S$ be an $R$–semimodule. If $S$ is t-extending then every subtractive subsemimodule of $S$ containing $Z_2(S)$ is essential in a summand of $S$.

***Proof*:** Let $\mathcal{A} \leq S$ such that $Z_2(S) \leq \mathcal{A}$, since $S$ is t–extending then $Z_2(S)$ is a summand of $S$, say $S = Z_2(S) \oplus S'$ for some $S' \leq S$, then by Lemma(2.2.14) $\mathcal{A} = Z_2(S) \oplus (S' \cap \mathcal{A})$, since $(S' \cap \mathcal{A}) \leq S'$ and $S'$ is CS by Proposition (4.3.33) then there exists $\mathcal{D}$ a summand of $S'$, such that $S' = \mathcal{D} \oplus S''$ for some $S'' \leq S'$ and $(S' \cap \mathcal{A}) \leq^e \mathcal{D}$, therefore, $\mathcal{A} = Z_2(S) \oplus (S' \cap \mathcal{A}) \leq^e Z_2(S) \oplus \mathcal{D}$, where $Z_2(S) \oplus \mathcal{D}$ is a summand of $S$ (since $S = Z_2(S) \oplus \mathcal{D} \oplus S''$). □

The next result is a general case of Proposition (4.3.34).

**Proposition 4.3.35:** Let $S$ be an $R$–semimodule. If $S$ is t-extending then everysubtractive subsemimodule of $S$ is t-essential in a summand of $S$.

***Proof*:** Let $\mathcal{A} \leq S$, then by Proposition (4.3.32), $Z_2(S/\mathcal{A}) = \mathcal{D}/\mathcal{A}$, where $\mathcal{D}$ is a summand of $S$, hence $\mathcal{D}/\mathcal{A}$ is $Z_2$–torsion (since $Z_2(\mathcal{D}/\mathcal{A}) = (\mathcal{D}/\mathcal{A}) \cap Z_2(S/\mathcal{A}) = \mathcal{D}/\mathcal{A}$), therefore by Proposition (4.3.12), $\mathcal{A} \leq^{\mathbf{tes}} \mathcal{D}$. □

**Proposition 4.3.36:** Let $S$ be an $R$–semimodule. Then $S$ is t-extending if and only if for everysubtractive subsemimodule $\mathcal{A}$ of $S$ there exists a decomposition $(S/\mathcal{A}) = (\mathcal{D}/\mathcal{A}) \oplus (\mathcal{D}'/\mathcal{A})$ such that $\mathcal{D}$ is a summand of $S$, and $\mathcal{D}' \leq^{tes} S$.

***Proof*:** Let $\mathcal{A} \leq S$ then by Proposition (4.3.35) there exists a decomposition $S = \mathcal{D} \oplus L$, such that $\mathcal{A} \leq^{tes} \mathcal{D}$, then $S/\mathcal{A} = \mathcal{D}/\mathcal{A} \oplus ((L \oplus \mathcal{A})/\mathcal{A})$, since $S/(L \oplus \mathcal{A}) \cong (S/\mathcal{A})/(L \oplus \mathcal{A})/\mathcal{A} \cong \mathcal{D}/\mathcal{A}$ but by Propositions (4.3.11) and (4.3.13), $\mathcal{D}/\mathcal{A}$ is $Z_2$-torsion, so $S/(L \oplus \mathcal{A})$ is $Z_2$-torsion, therefore by Proposition (4.3.12) $L \oplus \mathcal{A} \leq^{tes} S$, if $\mathcal{D}' = L \oplus \mathcal{A}$, then $\mathcal{D}' \leq^{tes} S$. *Conversely*, let $\mathcal{A} \leq^{\mathbf{tc}} S$, then by assumption there exists a decomposition $S/\mathcal{A} = \mathcal{D}/\mathcal{A} \oplus \mathcal{D}'/\mathcal{A}$, since $S/\mathcal{D}' \cong \mathcal{D}/\mathcal{A}$, hence $\mathcal{D}/\mathcal{A}$ is singular, and therefore $\mathcal{A} \leq^{tes} \mathcal{D}$ by Lemma(4.3.7)and Proposition(4.3.12), then $\mathcal{A} = \mathcal{D}$, and so $\mathcal{A}$ is a summand of $S$, that is $S$ is t-extending. □

**Proposition 4.3.37:** Every homomorphic image of t-extendingsubtractive $R$-semimoduleis t-extending.

***Proof*:** Let $S$ be a t-extending semimodule. It is enough to show that $S/\mathcal{A}$ is $t$-extending for any subsemimodule $\mathcal{A}$ of $S$. Let $L/\mathcal{A} \leq^{\mathbf{tc}} S/\mathcal{A}$, since $S$ is t-extending, then by Proposition (4.3.35) there exists $\mathcal{D} \leq^{\oplus} S$ such that $S=\mathcal{D} \oplus S'$, for some $S' \leq S$, and $L \leq^{\mathbf{tes}} \mathcal{D}$, then by Propositions (4.3.13) and (4.3.11),$\mathcal{D}/L$ is $Z_2$-torsion, but $\mathcal{D}/L \cong (\mathcal{D}/\mathcal{A})/(L/\mathcal{A})$, then by Proposition(4.3.12) $L/\mathcal{A} \leq^{\mathbf{tes}} \mathcal{D}/\mathcal{A}$,hence$L/\mathcal{A} = \mathcal{D}/\mathcal{A}$, therefore $S/\mathcal{A}$is t-extending. □

From the previous Proposition, we get the following result.

**Corollary 4.3.38:** Every summand of t-extending semimodule is t-extending.

***Proof*:** It is clear by Proposition (4.3.37).

The class of t-extending semimodule is closed under the direct sum.

**Proposition 4.3.39:**A direct sum of two t-extending semimodules is t-extending.

***Proof*:** Assume that $S=S_1 \oplus S_2$ ,where $S_1$ and $S_2$ are t-extending, and let $\mathcal{A} \leq^{tc} S$. Let $\pi_i$:$S \rightarrow S_i$ be the natural projections from $S$ onto $S_i$ ($i$ =1, 2), then $\mathcal{A} = \pi_1(\mathcal{A}) \oplus \pi_2(\mathcal{A})$, since $S_1$ and $S_2$ are t-extending, then there exists summands $\mathcal{D}_1$ and $\mathcal{D}_2$ of$S_1$ and $S_2$ respectively, such that $\mathcal{A} = \pi_1(\mathcal{A}) \oplus \pi_2(\mathcal{A}) \leq^{tes} \mathcal{D}_1 \oplus \mathcal{D}_2$, but $\mathcal{D}_1 \oplus \mathcal{D}_2$ is a summand of $S$ (since $S_1=\mathcal{D}_1 \oplus \mathcal{D}_1{}'$ and$S_2=\mathcal{D}_2 \oplus \mathcal{D}_2{}'$,therefore $S=\mathcal{D}_1 \oplus \mathcal{D}_1{}' \oplus \mathcal{D}_2 \oplus \mathcal{D}_2{}' = (\mathcal{D}_1 \oplus \mathcal{D}_2) \oplus (\mathcal{D}_1{}' \oplus \mathcal{D}_2{}')$),then $S$ is t-extending. □

As a particular case of Proposition (4.3.39), we get the following corollary.

**Corollary 4.3.40:** Let $S_1$ be a semisimple $R$- semimodule, then $S = S_1 \oplus S_2$ is t-extending for any t-extending subsemimodule $S_2$.

***Proof*:** Since $S_1$ and $S_2$ are t-extending subsemimodules then by Proposition (4.3.39), $S$ is t-extending. □

**Proposition 4.3.41:** Let $S = S_1 \oplus S_2$ be nonsingular ($k$-$c$) $R$-semimodule, then $S$ is t-extending if and only if for every $\mathcal{K} \leq^{tc} S$ with $\mathcal{K} \cap S_1 = 0$ or $\mathcal{K} \cap S_2 = 0$ is a summand of $S$.

***Proof*:**the necessity follow by definition. *Conversely*, Let $\mathcal{B} \leq^{\mathrm{tc}} S$ then either $\mathcal{B} \cap S_1 = 0$, hence by assumption $\mathcal{B}$ is a summand of $S$. Or $\mathcal{B} \cap S_1 \neq 0$, then there exists $\mathcal{D}$ such that $\mathcal{B} \cap S_1 \leq^{tes} \mathcal{D} \leq^{tc} \mathcal{B}$(byRemark (1.1.8),and Remark (4.3.4(1)and(4)), then$\mathcal{D} \cap S_2 = 0$ (since $\mathcal{B} \cap S_1 \cap \mathcal{D} \cap S_2 = 0$). Note that$\mathcal{D} \leq^{tc} S$ by Proposition (4.3.26), then by assumption, $\mathcal{D}$ is a summand of $S$, that is, $S = \mathcal{D} \oplus \mathcal{D}'$ for some $\mathcal{D}' \leq S$, by Lemma(2.2.14), $\mathcal{B} = \mathcal{D} \oplus (\mathcal{B} \cap \mathcal{D}')$ , but $(\mathcal{B} \cap \mathcal{D}')$ is t-closed in $S$, then $(\mathcal{B} \cap \mathcal{D}') \cap S_2 = 0$, also by assumption$(\mathcal{B} \cap \mathcal{D}')$is a summand of $\mathcal{D}'$,then$\mathcal{D}' = (\mathcal{B} \cap \mathcal{D}') \oplus \mathcal{D}''$for some$\mathcal{D}'' \leq \mathcal{D}'$, so $S = \mathcal{D} \oplus (\mathcal{B} \cap \mathcal{D}') \oplus \mathcal{D}'' = \mathcal{B} \oplus \mathcal{D}''$, therefore $\mathcal{B}$ is a summand of $S$ and $S$ is t-extending. □

The following corollary is immediate from Remarks (4.3.31)(2) and Proposition (4.3.39).

**Corollary 4.3.42:** Let $S = S_1 \oplus S_2$, if $S_1$ is a semisimple (uniform), then $S$ is t-extending for any $S_2$ is t-extending. □

**Proposition 4.3.43:** Every fully invariant t-closed subsemimodule of t-extending semimodule is t-extending.

***Proof*:** Let $S$ be t-extending and $\mathcal{N}$ is a fully invariant t-closed subsemimodule of $S$, and let $\mathcal{A} \leq^{\mathrm{tc}} \mathcal{N}$, then by Proposition(4.3.26) $\mathcal{A} \leq^{\mathrm{tc}} S$, since $S$ is t-extending then there exists $S'\leq^{\oplus}S$, say $S=S'\oplus S''$ such that, $\mathcal{A}\leq^{\mathbf{tes}}S'$, acontradiction, hence $\mathcal{A}=S'$, therefore $S=\mathcal{A}\oplus S''$ since $\mathcal{N}\trianglerighteq S$ then by Lemma (3.2.6**)**, $\mathcal{N}=\mathcal{A}\oplus(\mathcal{N}\cap S'')$ so $\mathcal{N}$ is t-extending. □

**Proposition 4.3.44:** Every IDS subsemimodule of a subtractive t-extending semimodule is t-extending.

***Proof*:** Assume that $S$ is t-extending and $N$ be IDS subsemimodule of $S$, let $\mathcal{A}\leq^{tc}\mathcal{N}$, then $\mathcal{A}\leq S$, since $S$ is t-extending then by Proposition(4.3.35) there exists $S'\leq^{\oplus}S$, say $S=S'\oplus S''$ such that, $\mathcal{A}\leq^{tes}S'$, since $N$ is IDS then $\mathcal{N}\cap S'$ is a summand of $\mathcal{N}$. Clearly $\mathcal{A}\leq^{tes}\mathcal{N}\cap S'\leq\mathcal{N}$, acontradiction, hence $\mathcal{A}=\mathcal{N}\cap S'$, therefore $\mathcal{N}$ is t-extending. □

## 4.4 $C_{11}$-Semimodules

The concept of $C_{11}$for modules has been introduced and explained by [40].We will introduce this concept for semimodule and give some results related to this concept.

In order to get some interest results for $C_{11}$-semimodule, the condition of distributive or subtractive are needed.

**Detention 4.4.1:** An $R$-semimodule $S$ is said to satisfy $C_{11}$ if every subsemimodule of $S$ has a complement which is a summand of $S$.

**Definition 4.4.2:** An $R$-semimodule $S$ is said to be distributive if $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are subsemimodules of $S$, then $\mathcal{A}\cap(\mathcal{B}+\mathcal{C})=\mathcal{A}\cap\mathcal{B}+\mathcal{A}\cap\mathcal{C}$.

The next remark gives the relationship betweendistributive semimodule and IDS subsemimodule.

**Remark 4.4.3:** If $S$ is a distributive $R$-semimodule, then any subsemimodule of it is IDS.

**Lemma 4.4.4:** Let $S$be a distributive $R$-semimodule, $\mathcal{A}$ is a subsemimodule of $S$ and $C$ is a closed subsemimodule of $S$ with$\mathcal{A} \cap C$=0, then $C$ is a complement of $\mathcal{A}$ if and only if $\mathcal{A} \oplus C \leq^{e} S$

***Proof:*** Suppose that $C$ is a complement of $\mathcal{A}$ and$x \notin \mathcal{A} \oplus C$, then$x \notin C$, hence $C+Rx \supsetneqq C$, since $C$ is a complement of $\mathcal{A}$ then $\mathcal{A} \cap (C+Rx) \neq 0$, by distributive condition $(\mathcal{A} \cap C)+(\mathcal{A} \cap Rx) \neq 0$, so $(\mathcal{A} \cap Rx) \neq 0$, therefore $(\mathcal{A} \oplus C) \cap Rx \neq 0$, hence $\mathcal{A} \oplus C \leq^{e} S$. *Conversely*, assume that $\mathcal{A} \cap C=0$,$\mathcal{A} \oplus C \leq^{e} S$. If $C \lneqq \mathcal{X}$ and $\mathcal{X} \cap \mathcal{A} = 0$, let$x \in \mathcal{X} \backslash C$, then $(\mathcal{A} \oplus C) \cap Rx \neq 0$, by distributive condition $(\mathcal{A} \cap Rx) \oplus (C \cap Rx) \neq 0$, but $\mathcal{A} \cap Rx = 0$ (since$\mathcal{A} \cap \mathcal{X} = 0$), then$C \cap Rx \neq 0$ and hence$C \leq^{e} S$, but this is a contradiction, hence $C = \mathcal{X}$, and therefore $C$ is a complement of $\mathcal{A}$ in $S$. □

***Note***: The result of Lemma (4.4.4), will be obtained too, if the condition (distributive) was replaced by the condition (subtractive) Lemma (2.2.3).

The following remark relates the concept CS-semimodule and the concept $C_{11}$-semimodule.

**Remarks 4.4.5:**

1. Every CS$R$- semimodule is $C_{11}$.
2. Every uniform or semisimple $R$-semimodule is $C_{11}$.

***Proof***:

**(1)** Assume that $S$ is CS- semimodule, let $C \leq S$, and $\mathcal{B}$ a complement of $C$in $S$ (a complement always exists by Zorn's Lemma), since $S$ is CS, then $\mathcal{B} \leq^{\oplus} S$.

**(2)** Assume that $S$ is a uniform or semisimple $R$-semimodule, hence byRemark ( 2.2.2), $S$ is CS, therefore by (1) $S$ is $C_{11}$. □

***Note*:** The converse of (1) is not true since for example $S=\mathbb{Z}_{\mathbf{8}}\oplus\mathbb{Z}_{\mathbf{2}}$, $R=\mathbb{Z}$, by Remark (2.2.2) is not $CS$, but $S$ is $C_{11}$ semimodule since $[S_{\mathbf{8}}=\langle(\overline{2},\overline{1})\rangle]$, is the only closed subsemimodule which is not summand has a complement $[S_{\mathbf{1}}=\langle(\overline{0},\overline{1})\rangle]$, which is a summand of $S$.

The following Proposition is analogous to one for modules. It is true for modules without distributive condition.

**Proposition 4.4.6:**Let $S$ be distributive $R$-semimodule, then the following statements are equivalent:

1. $S$ has $C_{11}$.
2. Every complement in $S$ has a complement in $S$ which is a summand.
3. If $\mathcal{B}\leq S$ then $\mathcal{B}$ has a complement $\mathcal{D}\leq^{\oplus}S$and $\mathcal{B}\oplus\mathcal{D}\leq^{e}S$.

***Proof:***

$(1\Rightarrow 2)$ Clear by Zorn's lemma and definition.

$(2\Rightarrow 3)$ Let $\mathcal{B}\leq S$, then by (2) there exists $\mathcal{D}\leq^{\oplus}S$ such that $\mathcal{D}$ is a complement of $\mathcal{B}$ in $S$ by Lemma (4.4.4) $\mathcal{B}\oplus\mathcal{D}\leq^{e}S$.

$(3\Rightarrow 1)$ Clear by definition of $C_{11}$. □

**Proposition 4.4.7:** Any indecomposable distributive (or subtractive) $R$-semimodule $S$ with $C_{11}$ is uniform.

***Proof*:** Let $\mathcal{A}$ be nonzero subsemimodule of $S$, since $S$ is $C_{11}$, then by Proposition (4.4.6) there exists a subsemimodule $\mathcal{D}$ of $S$ such that $\mathcal{D}\leq^{\oplus}S$ and $\mathcal{D}$ is a complement of $\mathcal{A}$, so by Lemma (4.4.4)(or Lemma (2.2.3)), $\mathcal{A}\oplus\mathcal{D}\leq^{e}S$, since $S$ is indecomposable, hence either $\mathcal{D}=0$ so $\mathcal{A}\leq^{e}S$, or $\mathcal{D}=S$ (not possible since $\mathcal{A}\neq 0$), therefore $\mathcal{A}\leq^{e}S$. □

In the next the behavior of $C_{11}$ according to subsemimodule, direct sum, and direct summand are studied.

**Proposition (4.4.8):** Let $S=S_1\oplus S_2$ be a distributive $R$-semimodule. If $S_1$and $S_2$ are $C_{11}$ then $S$ is $C_{11}$.

***Proof*:** Let $\mathcal{A} \leq S$, then by distributive $\mathcal{A} = (\mathcal{A} \cap S_1)\oplus(\mathcal{A} \cap S_2)$. Now $\mathcal{A} \cap S_i \leq S_i (i=1, 2)$ and $S_i$ is $C_{11}$ so by Proposition (4.4.6) there exists $\mathcal{D}_i, \mathcal{D}_i' \leq S_i$ such that $S_i = \mathcal{D}_i \oplus \mathcal{D}_i'$, and $\mathcal{D}_i$is a complement of $\mathcal{A} \cap S_i$ furthermore $(\mathcal{A} \cap S_i)\oplus\mathcal{D}_i \leq^e S_i (i=1,2)$ [by Lemma (4.4.4)], then $(\mathcal{A} \cap S_1) \oplus\mathcal{D}_1\oplus(\mathcal{A} \cap S_2) \oplus\mathcal{D}_2 \leq^e S_1\oplus S_2=S$, hence $\mathcal{D}_1\oplus\mathcal{D}_2$ is a complement of $(\mathcal{A} \cap S_1)\oplus(\mathcal{A} \cap S_2)= \mathcal{A}$, but $\mathcal{D}_1\oplus\mathcal{D}_2 \leq^{\oplus} S$ (since $S=S_1\oplus S_2 =\mathcal{D}_1 \oplus\mathcal{D}_1'\oplus\mathcal{D}_2 \oplus\mathcal{D}_2'$), therefore $S$ is $C_{11}$. □

The following corollaries is immediate from Remark (4.4.5) and Proposition (4.4.8).

**Corollary 4.4.9:** Let $S=S_1\oplus S_2$ be a distributive $R$-semimodule. If $S_1$and $S_2$ are CS,then $S$ is $C_{11}$. □

**Corollary 4.4.10:** Let $S=S_1\oplus S_2$ be a distributive $R$-semimodule. If $S_1$and $S_2$ are uniform or semisimplethen $S$ is $C_{11}$.

**Lemma 4.4.11:** If $S=S_1\oplus S_2$ is a distributive $R$-semimodule,$\mathcal{A} \leq S_1$ and $C$ is a complement of $\mathcal{A}$ in $S$, then $C \cap S_1$is a complement of $\mathcal{A}$ in $S_1$.

***Proof*:** It is clear that $\mathcal{A} \cap(C \cap S_1)=0$, if $x\in S_1$ and $x\notin(C \cap S_1)$, then $x\notin C$, hence $\mathcal{A} \cap( C+Rx)\neq 0$, by distributive condition $(\mathcal{A} \cap C)+(\mathcal{A} \cap Rx)\neq 0$, therefore $(\mathcal{A} \cap Rx)\neq 0$ and hence $\mathcal{A} \cap(C \cap S_1+Rx)\neq 0$, that is $C \cap S_1$ is maximal with this property $\mathcal{A} \cap(C \cap S_1)=0$ in $S_1$ . □

**Proposition 4.4.12:** Any summand of distributive $C_{11}$-semimodule is $C_{11}$.

***Proof*:** Assume that $S=S_1\oplus S_2$ is $C_{11}$ and let $\mathcal{A}\leq S_1$, since $S$ is $C_{11}$ then by Proposition (4.4.6) there exist $\mathcal{D}$, $\mathcal{D}'$ such that $S=\mathcal{D}\oplus\mathcal{D}'$, where $\mathcal{D}$ is a complement of $\mathcal{A}$ in $S$, so according to Lemma (4.4.11) $\mathcal{D}\cap S_1$ is a complement of $\mathcal{A}$ in $S_1$, but by distributive condition, $S_1=(S_1\cap\mathcal{D})\oplus(S_1\cap\mathcal{D}')$, so $S_1\cap\mathcal{D}$ a complement of $\mathcal{A}$ in $S_1$ and $S_1\cap\mathcal{D}\leq^{\oplus}S_1$, therefore $S_1$ is $C_{11}$. Similarly $S_2$ is $C_{11}$. □

Now, the following important Proposition which demonstrated for modules will be converted for semimodules by adding a condition.

**Proposition 4.4.13:** If $S$ is a distributive $C_{11}$-semimodule, then $S=Z_2(S)\oplus S'$ for some nonsingular subsemimodule $S'$ of $S$ where both $Z_2(S)$ and $S'$ are $C_{11}$.

***Proof*:** Since $Z_2(S)\leq S$, then by Proposition (4.4.6) and Lemma (4.4.4) there exist $S_1$, $S_2\leq S$, such that $S=S_1\oplus S_2$ and $S_1$ is a complement of $Z_2(S)$ and $S_1\oplus Z_2(S)\leq^{e}S$, by distributive condition $Z_2(S)=Z_2(S)\cap S_2$, hence $Z_2(S)\leq S_2$, so $Z_2(S_2)=Z_2(S)$, but $Z_2(S)=Z_2(S_1)\oplus Z_2(S_2)$, hence $Z_2(S_1)=0$ and $S_1$ is nonsingular. Now $S_1\oplus Z_2(S)\leq^{e}S_1\oplus S_2$, therefore $Z_2(S)\leq^{e}S_2$, so $Z_2(S)=S_2$, therefore $S=S_1\oplus Z_2(S)$ by Proposition (4.4.12) both $S_1$ and $Z_2(S)$ are $C_{11}$. □

**Lemma 4.4.14:** Let $S$ be a distributive $R$-semimodule with $C_{11}$, then $S=S_1\oplus S_2$ where $S_1$ is a subsemimodule of $S$ with $Soc(S_1)\leq^{e}S_1$ and $Soc(S_2)=0$.

***Proof:*** Since $Soc(S)\leq S$ and $S$ is $C_{11}$, then there exist subsemimodules $\mathcal{D}$ and $\mathcal{D}'$ such that $\mathcal{D}$ is a complement of $Soc(S)$ and $S=\mathcal{D}\oplus\mathcal{D}'$. Furthermore $Soc(S)\cap\mathcal{D}=0$ implies $Soc(\mathcal{D})=0$ and hence $Soc(S)=Soc(\mathcal{D}')$, but $Soc(S)\oplus\mathcal{D}\leq^{e}S$, hence $Soc(\mathcal{D}')\leq^{e}\mathcal{D}'$. □

**Proposition 4.4.15:** A distributive $R$-semimodule $S$ is $C_{11}$ if and only if $S=S_1\oplus S_2$, where $S_1$ is $C_{11}$ subsemimodule of $S$ with $Soc(S_1)\leq^{e}S_1$ and $S_2$ is $C_{11}$ subsemimodule of $S$ with $Soc(S_2)=0$.

***Proof:***

(⇒)Clear by Lemma (4.4.14) and Proposition (4.4.12).

(⇐) Clear by Proposition (4.4.8). □

In the next, we need to defined the following subsemimodule $C(\mathcal{P})$ for a subsemimodule $\mathcal{P}$ of $S$ by: $C(\mathcal{P})=\{a\in S|Ia\leq \mathcal{P}$, for some $I\leq^{e}R\}$

**Proposition 4.4.16:** If $S$ is a distributive nonsingular $R$-semimodule with $C_{11}$, then $C(\mathrm{Soc}(S))$ is $C_{11}$.

***Proof*:** Assume $S$ is $C_{11}$, then by Proposition (4.4.15) $S=S_1\oplus S_2$, where $S_1$ and $S_2$ have the stated condition, hence$\mathrm{Soc}(S_1)\leq^{e}S_1$, so by Lemma(3.2.10), if $a\in S_1$ there exists left$I\leq^{e}R$ such that $Ia\leq\mathrm{Soc}(S)$, hence $a\in C(\mathcal{S}oc(S))$ so $S_1\leq C(\mathrm{Soc}(S))$. Now let $a\in C(\mathrm{Soc}(S))$then there exists left$I\leq^{e}R$ such that $Ia\leq\mathrm{Soc}(S)\leq S_1$, hence $S_1= C(\mathrm{Soc}(S))$, and $C(\mathrm{Soc}(S))$ is $C_{11}$. □

**Lemma 4.4.17:** Suppose $S$ is an $R$-semimodule and $S=S_1\oplus S_2$, let $\mathcal{A}\leq S$, and $\mathcal{A}\cap S_1=0$ then $S_1\oplus \mathcal{A}=S_1\oplus\pi_2(\mathcal{A})$,where $\pi_i:S\rightarrow S_i$is a natural projection ($i$=1, 2).

***Proof*:** Since $S_1+ \mathcal{A}=S_1+(\pi_1(\mathcal{A})+\pi_2(\mathcal{A}))=S_1+\pi_2(\mathcal{A})$and $\pi_1(\mathcal{A})\leq S_1$, since $\pi_2(\mathcal{A})\cap S_1=0$ and by unique representation of $S_1\oplus S_2$, then $S_1\oplus \mathcal{A}=S_1\oplus\pi_2(\mathcal{A})$. □

**Lemma 4.4.18:** Let $\mathcal{D}$ be a summand of cancellativeand semisubtractive$R$-semimodule $S$ and $\mathcal{A}\leq S$, $\mathcal{A}\cap\mathcal{D}=0$ with $\mathcal{A}$ is injective, then $\mathcal{D}\oplus\mathcal{A}$ is a summand of $S$.

***Proof*:** Let $\mathcal{D}\leq^{\oplus}S$ then there exists $\mathcal{D}'$ such that $S=\mathcal{D}\oplus\mathcal{D}'$, let $\pi$: $S\longrightarrow\mathcal{D}'$ be the natural projection, $\pi|_{\mathcal{A}}:\mathcal{A}\longrightarrow\mathcal{D}'$is a monomorphism (since $Ker\pi\cap$

$\mathcal{A}$=0), therefore $\pi(\mathcal{A}) \leq^{\oplus} \mathcal{D}$'(since $\pi|_{\mathcal{A}}$ is split), say $\mathcal{D}' = \pi(\mathcal{A}) \oplus \mathcal{D}''$, for some $\mathcal{D}'' \leq \mathcal{D}'$ so $S$=$\mathcal{D} \oplus \pi(\mathcal{A}) \oplus \mathcal{D}''$, hence by Lemma (4.4.17) $\mathcal{D} \oplus \pi(\mathcal{A}) = \mathcal{D} \oplus \mathcal{A}$, therefore $\mathcal{D} \oplus \mathcal{A} \leq^{\oplus} S$. □

**Proposition 4.4.19:** Let $S$ be a subtractive, cancellative and semisubtractive $R$-semimodule with $C_{11}$, and let $\mathcal{D} \leq^{\oplus} S$ such that $S/\mathcal{D}$ is injective, then $\mathcal{D}$ is $C_{11}$.

***Proof:*** Suppose $\mathcal{D} \leq^{\oplus} S$ , then there exists $\mathcal{D}'$ such that $S$= $\mathcal{D} \oplus \mathcal{D}'$, so by hypotheses $\mathcal{D}'$ is injective subsemimodule of $S$. Now let $\mathcal{A} \leq \mathcal{D}$, then $\mathcal{A} \oplus \mathcal{D}' \leq S$, since $S$ is $C_{11}$, then there exists subsemimodule $\mathcal{K}$ such that $\mathcal{K}$ is a complement of $\mathcal{A} \oplus \mathcal{D}'$, by Lemma(2.2.4), $(\mathcal{A} \oplus \mathcal{D}') \oplus \mathcal{K} \leq^{e} S$ and $(\mathcal{A} \oplus \mathcal{D}') \cap \mathcal{K}$=0, so by Proposition (4.4.18) $\mathcal{D}' \oplus \mathcal{K} \leq^{\oplus} S$, but by Lemma (4.4.17), $\mathcal{D}' \oplus \mathcal{K} = \mathcal{D}' \oplus \pi(\mathcal{K})$, where $\pi$:$S \rightarrow \mathcal{D}$denoted natural projection, hence $\pi(\mathcal{K}) \leq^{\oplus} \mathcal{D}$, since $(\mathcal{A} \oplus \mathcal{D}') \oplus \mathcal{K} \leq^{e} S$, then $\mathcal{A} \oplus \pi(\mathcal{K}) \leq^{e} \mathcal{D}$, and $\mathcal{A} \cap \pi(\mathcal{K}) = 0$, therefore $\mathcal{D}$ is $C_{11}$. □

**Corollary 4.4.20:**Let $S$ be a cancellative and semisubtractive distributive direct sum of subsemimodules $S_1$ and injective subsemimodule $S_2$, then $S$ is $C_{11}$ if and only if $S_1$ is $C_{11}$.

***Proof:*** Suppose $S$ is $C_{11}$ then by Proposition (4.4.12)$S_1$ is $C_{11}$. *Conversely,* assume that $S_1$ is $C_{11}$, since $S_2$ is injective by assumption, then by Lemma (2.2.7) and Remark (4.4.5) $S_2$ is $C_{11}$, therefore by Proposition (4.4.8) $S$ is $C_{11}$. □

**Proposition 4.4.21:** Let $S$ be a subtractive $R$-semimodule. If $S$ is $C_{11}$ then $S$ is FI- CS.

***Proof:*** Let $S$ be $C_{11}$, and $\mathcal{A} \trianglerighteq S$, then there exist subsemimodules $\mathcal{D}$, $\mathcal{D}$ ' such that $S$= $\mathcal{D} \oplus \mathcal{D}'$, $\mathcal{A} \cap \mathcal{D}$=0 and $\mathcal{A} \oplus \mathcal{D} \leq^{e} S$, since $\mathcal{A} \trianglerighteq S$ then by

Lemma(3.2.6),$\mathcal{A} = (\mathcal{A} \cap \mathcal{D}) \oplus (\mathcal{A} \cap \mathcal{D}') = \mathcal{A} \cap \mathcal{D}'$, therefore $\mathcal{A} \leq^e \mathcal{D}'$ (since $\mathcal{A} \oplus \mathcal{D} \leq^e \mathcal{D} \oplus \mathcal{D}'$), therefore $S$ is FI- CS. □

In the next, we show special case of a subsemimodule of $C_{11}$ semimodule.

**Proposition 4.4.22:** Let $S$ be a $C_{11}$ distributive (or subtractive) $R$-semimodule and $\mathcal{K}$ subsemimodule of $S$. If $\mathcal{K}$ is IDS, then $\mathcal{K}$ is $C_{11}$.

***Proof*:** Let $\mathcal{A} \leq \mathcal{K} \leq S$, since $S$ is a $C_{11}$, then by Proposition (4.4.6) there exist subsemimodules $\mathcal{D}$ and $\mathcal{D}'$ of $S$ such that $S=\mathcal{D} \oplus \mathcal{D}'$, and $\mathcal{D}$ is a complement of $\mathcal{A}$, furthermore $\mathcal{A} \oplus \mathcal{D} \leq^e S$,[by Lemma(4.4.4) or Lemma (2.2.3)], hence $\mathcal{K} \cap (\mathcal{A} \oplus \mathcal{D}) = \mathcal{A} \oplus (\mathcal{D} \cap \mathcal{K})$, so $\mathcal{A} \oplus (\mathcal{D} \cap \mathcal{K}) \leq^e \mathcal{K}$ [$\mathcal{K} \cap (\mathcal{A} \oplus \mathcal{D}) \leq^e \mathcal{K} \cap S = \mathcal{K}$)]and $\mathcal{A} \cap (\mathcal{D} \cap \mathcal{K})=0$, since $\mathcal{D} \cap \mathcal{K}$ is summand of $\mathcal{K}$ by hypotheses, then $\mathcal{K}$ is $C_{11}$. □

The following result gives me the sufficient condition for Remark(4.4.5)(2) to be true, with an additional condition

**Proposition 4.4.23:** Let $S$ be a $C_{11}R$-semimodule. If $S$ is distributive, then every subsemimodule of $S$ is CS. In particular any distributive $C_{11}$ semimodule is CS.

***Proof*:** let $\mathcal{A} \leq S$, and $\mathcal{B} \leq^c \mathcal{A}$, since $S$ is $C_{11}$ then there exist $\mathcal{D}$ and $\mathcal{D}'$ such that $S = \mathcal{D} \oplus \mathcal{D}', \mathcal{B} \cap \mathcal{D} = 0$, and $\mathcal{B} \oplus \mathcal{D} \leq^e S$, since $\mathcal{B} = \mathcal{B} \cap S$, then $\mathcal{B} \leq \mathcal{D}'$, since $\mathcal{B} \oplus \mathcal{D} \leq^e S$, then $\mathcal{B} \leq^e \mathcal{D}'$, this is a contradiction, hence $\mathcal{B}=\mathcal{D}'$, therefore $\mathcal{A}$ is CS. □

**Proposition 4.4.24:** Let $S=S_1 \oplus S_2$ be a distributive (or subtractive) $R$-semimodule, $S_1$ is $C_{11}$ if and only if for every subsemimodule $\mathcal{B}$ of $S_1$, there exists a summand of $S$ which is a complement of $\mathcal{B}$ in $S$ and containing $S_2$.

***Proof*:** Assume that $S_1$ is $C_{11}$ and let $\mathcal{B} \leq S_1$, then by definition there exists $\mathcal{T} \leq^{\oplus} S_1$, which is a complement of $\mathcal{B}$ in $S_1$, that is $S_1 = \mathcal{T} \oplus \mathcal{T}'$, then $S = \mathcal{T} \oplus \mathcal{T}' \oplus S_2$ so $\mathcal{T} \oplus S_2 \leq^{\oplus} S$ containing $S_2$. Moreover, $\mathcal{T}$ is a complement of $\mathcal{B}$ in $S_1$ implies $\mathcal{T} \oplus \mathcal{B} \leq^{e} S_1$, then $\mathcal{T} \oplus \mathcal{B} \oplus S_2 \leq^{e} S_1 \oplus S_2 = S$, that is $\mathcal{T} \oplus S_2$ is a complement of $\mathcal{B}$ in $S$. *Conversely*, Assume that $\mathcal{B} \leq S_1$, then by assumption there exist subsemimodules $\mathcal{D}$ and $\mathcal{D}'$of $S$ such that $S = \mathcal{D} \oplus \mathcal{D}'$, $S_2 \subseteq \mathcal{D}$, and $\mathcal{D}$ is a complement of $\mathcal{B}$ in $S$, hence $\mathcal{D} \oplus \mathcal{B} \leq^{e} S$. Now $S_1 = (S_1 \cap \mathcal{D}) + (S_1 \cap \mathcal{D}')$ hence $S_1 \cap \mathcal{D} \leq^{\oplus} S_1$ on other hand $S_1 \cap (\mathcal{D} \oplus \mathcal{B}) \leq^{e} S_1$ implies $(S_1 \cap \mathcal{D}) \oplus \mathcal{B} \leq^{e} S_1$ by definition $S_1$ is $C_{11}$. □

# Chapter Five:

# Conclusions and Future Works

## 5.1. Conclusions

In fact, every module is a semimodule, but the converse is not true Thus, we can say that most of the results achieved in module are not achieved in semimodule since semimodule with out subtraction and injective hull.To achieve these results, we need to add all or some of the following conditions: subtractive, semisubtractive and cancellative. Assuming the three conditions is strong, so, we sometimes use a weaker condition like the (*k*-*c*)condition. In certain cases by using different arguments we reached the same result as in modules without extra conditions.

In addition, in some proposition, we need to assume the existence of the injective hull because it is present only in the semimodule overring or over an additively idempotent semiring.The two concepts, $\pi$-injective and quasi-continuousare equivalent in modules. But in semimodules the proof of these equivalence need strong conditions (cancellative, semisubtractive and subtractive).

Finally, we study the generalizations of extending semimodule such that every extending semimodule is FI-extending or (t-extending,$C_{11}$-extending) but the converse is not true only with some condition . In our work, we remark that all semimodules considered in this work are unitary left semimodules oversemirings with identity 1.

## 5.2. Future Works

1. For future work, we plan to find a condition for achieving the relationship between subtractive semimodule, cancellativesemimodule, and semisubtractive that reduces the presence of the part or all of these concepts in our work.
2. We also plan to prove another kind of extending semimodule can

be proved in which every subsemimodule is dense in a summand.

3. Furthermore, our study can be developed by finding a condition for dispensing injective hull.
4. In the other case, our study can be applied and approved in several applications in future work.

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# المستخلــص

في السابق ، تمت دراسة مفهوم مقاس التوسع على الحلقة من قبل بعض الباحثين. هذا العمل سيقدم ويدرس شبه مقاس التوسع على شبه الحلقة والتحقيق في الشروط التي ستكون مطلوبة للحصول على خصائص وصفات مماثله للمقاس. لتكن $R$ شبه حلقه تمتلك عنصر محايد $1\neq0$ و $S$ شبه مقاس يساري وحدوي فنقول ان شبه المقاس هو شبه مقاس توسع اذا كان كل شبه مقاس جزئي ه اساسي في مركبة جداء مباشر.

الهدف الرئيسي من هذه الأطروحة هو إعطاء توصيف ودراسة شاملة للخصائص الأساسية لمفهوم التوسع ودراسة العلاقة بين هذا المفهوم وخاصية الشذوذ. بالإضافة إلى عرض ودراسة العديد من الخصائص الأساسية لتعميمات التوسع لشبه المقاسات: مثل التوسع من النمط FI, التوسع من النمط T و النمط $C_{11}$ والوصول إلى بعض النتائج الجديدة لهذه الانواع من شبه المقاسات .

يقال ان شبه المقاس هوشبه مقاس توسع من النمط FI اذا كان كل شبه مقاس جزئي ثابت تماماهو اساسيفي مركبة جداء مباشر. بينما يسمى $S$ شبه مقاس توسع من النمط T اذا كان كل شبه مقاس جزئي مغلق من النمطt يمثل مركبة جداء مباشر.في حين ان$S$ شبه مقاس توسع من النمط$C_{11}$ اذا كان كل شبه مقاس جزئي له مكمله تكون مركبة جداء مباشر. فقد تم إثبات أن شبه مقاس التوسعهو شبه مقاس توسع من النمط FI ولكن العكس غير صحيح كما أن المجموع المباشر لاثنين من شبه مقاسات توسع (او شبه مقاسات توسع من النمط FI)يكون شبه مقاس توسع من النمط .FI كما وتم اثبات ان شبه المقاس من النوع الثنائي سيكون شبه مقاس توسع اذا وفقط اذا كان شبه مقاس توسع من النمط .FI

علاوة على ذلك تم الحصول على النتائج التالية: ان كل شبه مقاس توسع هوشبه مقاستوسع من النمطt و ان كل مركبة جداء مباشر من شبه مقاس توسع من النمطt هي شبه مقاس توسع من النمط;tان الجمع المباشر لاثنين من شبه مقاسات توسع من النمط t (او شبه مقاسات توسع) يكون شبه مقاس توسع من النمط tوان كل شبه مقاس توسع يكون شبه مقاس توسع من النمط$C_{11}$. من ناحية اخرى عند اضافة الشرط التوزيعي او الطرحي الى شبه المقاس تم الحصول على النتائج التالية: كل مركبة جداء مباشر لشبه مقاس توزيعي من النمط $C_{11}$تكون من شبه مقاس من النمط $C_{11}$; ان الجمع المباشر لاثنين من شبه مقاسات التوسع التوزيعي

(اوشبه المقاس التوزيعي من النمط$C_{11}$ )يكون شبه مقاس من النمط $C_{11}$.  اذا كان شبه مقاس طرحي شبه مقاس توسع من النمط  $C_{11}$ فانه شبه مقاس توسع من النمط ; FIاذا كان شبه مقاس توسع توزيعي و من النمط $C_{11}$ فان كل شبه مقاس جزئي توسع وبشكل عام كل شبه مقاس توزيعي من النمط $C_{11}$ سيكون شبه مقاس توسع .

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# شبه مقاسات التوسع
# مع بعضالتعميمات

**أطروحـــــــة**

**مقدمـــة إلى مجلس كليــة التربيــة للعلـوم الصرفــة / جامعـــة بابل**

**كجزء من متطلبات نيل درجــة الدكتوراه فلسفــةفي التربيــة/ الرياضيات**

**مــن قبـــل**

**سماح عبد الهادي عباس عيسى**

**بإشـــراف**

**أ.د.أسعد محمد علي الحسيني**

**1443هـ 2022م**