Republic of Iraq
Ministry of Higher Education
and Scientific Research
University of Babylon
College of Education for Pure Sciences

Approximation

in

Statistics and Probability

A Thesis

Submitted to College of Education for Pure Sciences- University of Babylon in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Education/Mathematics

By

Nadiah Abed Habeeb Ali

Supervised by

Prof. Dr. Eman Samir Bhaya

2022 A.D. 1443 A.H.

بسم الله الرحمن الرحيم

﴿ اقْرَأْ بِاسْمِ رَبِّكَ الَّذِي خَلَقَ *خَلَقَ الْإِنْسَانَ مِنْ عَلَقٍ *اقْرَأْ وَرَبُّكَ الْأَكْرَمُ *الَّذِي عَلَّمَ بِالْقَلَمِ *عَلَّمَ الْإِنْسَانَ مَا لَمْ يَعْلَمْ ﴾

صدق الله العلي العظيم

[العلق: 1 – 5]

Dedication

I dedicate this humble work to my mother's soul.

Declaration

Aware of legal liability I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

Signature:

Date: 22/ 2 /2022

Name: Nadiah Abed Habeeb

Committee Certification

We, the examining committee, after reading this dissertation, " *Approximation in Statistics and Probability*" and examining the student " *Nadiha Abed Habeeb* ", in its content, find that it is adequate as a dissertation for the Degree of Doctor of Philosophy in Education / Mathematics.

Signature:

Name: Dr ***Ali Hussein Battor***

Title: Professor

Date: / /2022

Signature:

Name: Dr. ***Iftichar M. T.AL-Shara'a***

Title: Professor

Date: / /2022

Signature:

Name: Dr. ***Kareema Abed AL-Kadim***

Title: Professor

Date: / /2022

Signature:

Name: Dr. ***Salwa Salman Abed***

Title: Professor

Date: / /2022

Signature:

Name: Dr. ***Najah Rasoul Dakhel Aljaberi***

Title: Assist. Professor

Date: / /2022

Signature:

Name: Dr. ***Eman S. Bhaya***

Title: Professor

Date: / /2022

Approved by the Dean of College Committee for Post Graduate Studies

Signature:

Name: Prof. Dr. Bahaa Hussein Salih Rabee

Date: / / 2022

Supervisor's Certification

*I certify that this dissertation "**Approximation in Statistics and Probability**" by student **Nadiha Abed Habeeb** was prepared under my supervision at the University of Babylon, College of Education for Pure Sciences, in a partial fulfillment of requirements for the degree of Doctor of Philosophy in Education/ Mathematics*

Signature:

Name: ***Dr. Eman Samir Bhaya***

Title: Professor

Date: / /2022

In view of available recommendations, I forward this thesis for debate by the examining committee

Signature:

Name: ***Dr. Azal Jaafar Musa***

Title: Asst. Prof.

Head of Mathematics Department, Faculty of Education for Pure Sciences, University of Babylon

Date: / /2022

Certification of Linguistic Expert

I certify that I have read this dissertation entitled "Approximation in Statistics and Probability" and corrected its grammatical mistakes; therefore, it has qualified for debate.

Signature:

Name: Dr.

Title:Asst. Prof.

Date: / /2022

Certification of Scientific Expert

*I certify that I have read the scientific content of this dissertation entitled "**Approximation in Statistics and Probability**" and I have approved this dissertation is qualified for debate.*

Signature:

Name: ***Dr.***

Titl Asst. Prof.

Date: / /2022

Certification of Scientific Expert

I certify that I have read the scientific content of this dissertation entitled "Approximation in Statistics and Probability" and I have approved this dissertation is qualified for debate.

Signature:

Name: Dr.

Titl Asst. Prof.

Date: / /2022

Acknowledgments

First and foremost I would like to thank God for giving me the strength and determination to continue this message.

I would also like to express my special thanks and gratitude to my supervisor, Prof. Eman Samir Bhaya, for his real efforts in supervising and guiding me to come up with this work.

We also extend our thanks to all the faculty members of the Mathematics Department at the University of Babylon, College of Education for Pure Sciences.

Finally, I would like to express my appreciation to my dear husband for his continuous support and I extend my thanks to my family for their help and patience during my studies. My thanks and gratitude are due to my friend, brother and cousin Abu Raed for his continuous efforts to support and motivate throughout the study period.

Nadiah Abed Habeeb

Abstract

In our thesis we focus on the applications of functional analysis and approximation in probability theory and statistics.

As a contribution of functional analysis and approximation on theorem we introduce a new formula of Ostrowski theorem for function in quasi normed spaces.

We applied our version of Ostrowski theorem in random variable whose probability density function and cumulative distribution function in quasi normed spaces. Also we applied our Ostrowski version theorem in $b$eta and normal distributions.

We generalize Pre-Gruss inequality for functionsin quasi normed spaces and applied it to estimate the expectation, variance and dispersion.

We prove a generalization refinement of the Chebyshev inequality to quasi normed spaces. Then we applied it for expectation of cumulative distribution function of random variable with probability density function and its derivative in quasi normed spaces.

In approximation theory we use Taylor formula to approximate expressions written in terms of expectation and variance simultaneously with probability density functions in quasi normed spaces. We prove a type of Ostrowski inequality and applied it to cumulative density function, βeta and Normal distribution

Publications

1. Nadiha Abed Habeeb, Eman Samir Bhaya, A Modified Ostrowski inequality with Random variable Application on $L_p, 0 < p < 1$,Spaces, international of Mechanical Engineering:ISSN:0974-5823,Vol.7 No3 March,2022.

2. Nadiha Abed Habeeb, Eman Samir Bhaya, Approximation of Expectation and Varaince in $L_p, 0 < p < 1$, international of Mechanical Enginereing:ISSN:0974-5823,Vol.7 No3 March,2022.

3. Eman Samir Bhaya, Nadiha Abed Habeeb, Inequalities For Expectation, Variance and Dispersion, Accepted in AIP Conference Proceeding,2021.

4. Eman Samir Bhaya, Nadiha Abed Habeeb, Application of Ostrowski inequality in Beta and Normal Distribution, Accepted in AIP Conference Proceeding,2021.

5. Eman Samir Bhaya, Nadiha Abed Habeeb, An $L_p, p < 1$ Application of a modified chebyshev inequality, Accepted in AIP Conference Proceeding,2021.

6. Eman Samir Bhaya, Nadiha Abed Habeeb, An $L_p, p < 1$ Ostrowski inequality, Accepted in IOP Published,2021.

List of Symbol

symbol	Description
$\|f\|_p$	L_p-norm
$\|f\|_\infty$	Infinity norm
C(p,q)	Constant depended upon p and q only
C(p)	An absolute constant depending on p only
F(X)= $P_r(X \leq x)$	Cumulative distribution function
pdf	Probability denisty function
E(X)	Expectation value of x
$X \sim B(p,q)$	Xdistributed Beta distribution with p and q parameter
$X \sim N(\mu, \sigma^2)$	X distributed normal distribution
μ	The mean value of X
σ^2	Variance of X
Ω	Sample space
$\mathbb{R}$	The set of real numbers
$\mathbb{R}_+$	The set of positive real numbers
$f(\acute{X})$	The derivative of *f(x)*
I°	The interior of an interval I

Table of Contents

Subjects	*Page No.*
Introduction	1
Chapter One An $L_{p,}$ $p < 1$ Ostrowski inequality	8
Ostrowski inequality for functions in $L_p, 0 < p < 1$	11
Chapter Two Application Of the Ostrowski Inequality	23
Application of the Ostrowski Inequality in Random Variable	23
Applications of Ostrowski Inequality in Beta and Normal Distributions	36
Chapter Three Inequalities for Expectation, Variance and Dispersion	47
Pre Gruss inequality for Expectation, Variance Dispersion	54
Chapter Four An $L_p, p < 1$ Application of a Modified Chebyshev Inequality	87
Application of a Modified Chebyshev Inequality	89
Chapter Five Approximation of Expectation and Variance on $[a, b]$Interval, with Probability Density Function in $L_p[a, b], 0 < p < 1$	108
Expectation and Variance with probability density function in $L_p, 0 < p < 1$	110

Chapter Six A Modified Ostrowski Inequality with Random Variable Application on $L_p[a, b]$, $0 < p < 1$,Spaces	131
A Modified Ostrowski Inequality	134
Applications for Distribution Function and Random Variable.	142
Conclusions	147
Future Works	148
References	149

Introduction

Our thesis is a contribution of approximation theory and functional analysis to probability theory.

Many researchers prove many types of Ostrowski inequality for continuous functions whose derivative bounded on (a,b).

such as the researcher Ostrowski in 1938 introduced that in

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|\acute{f}\|_\infty. \qquad (1)$$

For all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.where

$\|\acute{f}\|_\infty = sup_{t\in(a,b)}|\acute{f}| < \infty.$

The Fink׳s result is also used to find another type of Ostrowski inequality, which is as follows [1].

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} \|\acute{f}\|_p \qquad (2)$$

when$\left(p > 1, \frac{1}{p} + \frac{1}{q} = 1\right)$.Later on, some generalizations were found for this inequality were proved. In 1995 G.A.Anastassiou studied Ostrowski inequalities [2].In 1997, the same researcher studied multivariate Ostrowski type inequalities [3]. In 1998 N.S.Barentt and others studied Ostrowski inequalitu for double integrals and applications for cubature formula [4].In the same year, S.S.Dagomir and others as stated in [5]which has applications in numerical integrals as well as having a special meaning mentioned in [6].As for its applications on the derivative functions whose derivate belongs to the L_p spaces, for $1 \leq p \leq \infty$ the researchers here used another type on Ostrowski_inequality as mentioned in [4].In 1999, S.S.Dagomir and others also studied An n-dimensional version of Ostrowski inequality for mapping of the Holder type [7].In 2000, B.G.Pachpatte studied On multivariate Ostrowski Type inequality [8].In 2001 N.S.Barentt and others study On weighted Ostrowski Type inequalities for operators and vector –valued functions [9].In the same year N.S.Barentt and others studied inequality for double integrals and applications for cubature formula [10].In 2002, P.Gerone study A new Ostrowski type inequality involving integral Means over end intervals [11].In 2004, P.Gerone and others studied Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions [12].In 2006, B.G.Pachpatte studied A new Ostrowski Type inequality for double integrals [13].In the same year Arif Rsfiq and others studied weighted Ostrowski Type inequality for differentiable mappings whose first derviatives belong to $L_p(a, b), p > 1$ [14].In 2008, Shiow-Ru Hwang and others studied A note on multivariate Ostrowski Type inequality [15].In 2010, M.Z.Sarikaya studied on the Ostrowski Type integral inequality [16].In 2011, ANA Maria and others studied The Mean value theorem and inequalities of Ostrowski type [17]. In the same year M.Emin ozdermir and others studied inequalities for convex and s-convex functions on $\Delta = [a, b] \times [c, d]$ [18].In the same year M.Z.Sarikaya studied on the weigthed Ostrowski type integral inequalities for S-convex [19]. In 2012, Mevlut Tunc study Ostrowski type inequalities for $(\propto, m)$-Geometrically convex functions via Riemann-Louville fractional integrals [20].In 2013, S.S.Dragomir studied Ostrowski type inequalities for functions whose derivatives are h-covex in absoulte value [21].In 2014, M..E.Ozdemir and others studied Ostrowski type inequalities for convex functions [16].In the same year A.Qayyum and others studied on new generalized Ostrowski type inequalities [22].In 2016, Samet and others studied An Ostrowski type inequalities for twice differentiable mappings and applications [24].In 2017, Silvestru Dragomir studied Ostrowski type inequalities for Lebesgue integral A survey of recent results [23].In 2019, Samet Erden and others studied some inequalities for double integrals and applications for cubature formula[30]. In 2020, year Huseyin and others studied weighted Ostrowski trapezoid and midpoint type inequalities for Riemann-Liouville fractional integrals[25].In same year Naila and others studied Ostrowski type inequalities via some exponentially convex functions with application [26].In 2021, Praveen and others studied new Ostrowski type inequalities for generalized S-convex functions with applications to some special means of real numbers and to midpoint formula[27].

In Chapter One we find a new formula for the Ostrowski inequality in the space L_p for $0 < p < 1$.

Many articles on Ostrowiski Inequality [41],[43] and [47] are choose β and Γ distribution (Γ is a special case of β distribution) because the p.d.f of these distributions in terms of p (the shape parameter) and q (the distribution parameter). The sample space of the random variable of β and Γ belongs to (0,∞). The above properties make the applications of these distributions very easy.

The first section in Chapter Two includes the application of Ostrowski inequality in random variable whose probability density function and cumulative distribution function belong to L_p. While the second section of Chapter Two concerns with the application of Ostrowski's inequality in the Beta distribution and the normal distribution. The normal distribution don't have the above properties. The p.d.f of the normal distribution in terms of μ and σ. The sample space of the normal distribution subset of R. So that no one work on the Ostrowiski inequality appear for the expectation of normal distribution.

In the last decade few authors introduced Pre-Gruss inequality for functions in $L_p[a, b]$ space, and used it to estimate the error bounds of the reminders of Taylor-Like formula and quadrature formula.

In [45],[46], the authors introduced the so called Pre-Gruss inequality at the form for function in $L_1[a,b]$ normed space.

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt\,\frac{1}{b-a}\int_a^b g(t)dt\right|$$

$$\leq \frac{1}{2}(\varphi-\gamma)\left[\frac{1}{b-a}\int_a^b g^2(t)dt - \left(\frac{1}{b-a}\int_a^b g(t)dt\right)^2\right]^{\frac{1}{2}} \qquad (3)$$

M.Maice and others in 1999, used (3) to estimate the bound of Taylor-Like formula.[9]. P.Gerone and others in 2000, used the (3) for estimating the remainder in three point quadrature formula.In the same year Nenad study A Genralization of the Pre-Gruss inequality and applications to some Quadrature formula [49].In 2009, A.Vukelic study estimation of the error for general simpson type formula via Pre-Gruss inequality [29]. In 2019, Samet Erdan and others studied Pre-Gruss inequality involving conformable fractional intgrals and applications for random variables [30].In 2020, Silver Dragomir studied A refinement of Pre-Gruss inequality for the complex integral [31].

In Chapter Three we extend and generalize the above results by introducing a type of pre-Gruss inequality for formula in $L_p[a,b]$ quasi- normed space for theorems related to expection, variance and Dispersion.Some authors proved a Chebyshev inequality for absolutely continuous functions.In 1867, The Chebyshev inequality is famous result from probability theory and has been studied in the most of sciences[32].In 1984, Saw and others studied Chebyshev inequality with estimated Mean and variance [33]. In 1995, Smith .T studied generalized Chebyshev inequalities in his book Theory and application in decision analysis [34] .In 1999, D.S, Mitrinovic and others proved classical inequalities in analysis such as Chebyshev inequality for absolutely continuous function[50]. In 2004, Hogg and others studied Chebyshev inequality in his book Introduction to mathematical statistics [35].In 2011, Chen ,X studied Anew generalization of Chebyshev inequality for random vector [36]. In 2013, Navarro, J studied A very simple proof of the multivariate Chebyshev inequality [38]. The same researcher in 2014, study a note on confidence Chebyshev inequality [39], also studied Can the bounds in the multivariate Chebyshev inequality be attained?[40]

In Chapter Four a generalize and refinement of this Chebyshev inequality for functions in the spaces L_p,for $0 < p < 1$ are introduced. Then we apply it to estimate expectation of cumulative distribution function of random variable with probability density function, such that $f, \acute{f} \in L_p[a,b], 0 < p < 1$.

In Chapter Five we use the our inequalities in pervious Chapters to approximate (estimate) expectation and variance with measurable probability density functions, in the aid of Taylor formula.

In Chapter Six we improve the results in Chapter One by introducing the best results for function in $L_p[a,b]$. Then we applied them to $beta$ and normal distribution and cumulative density function.

Chapter One

An L_p, $p < 1$ *Ostrowski inequality*

An L_p,$p < 1$ Ostrowski inequality

Many researchers prove many types of Ostrowski inequality for continues functions but in this chapter there is a new formula for the Ostrowski inequality in the L_p space when it is $0 < p < 1$.

1.1.Introduction

We need to recall the definition of the spaces L_p.

$$L_p(I)\ \ L_p(I) = \left\{ f: I \to R: \|f\|_{L_{p(I)}} = \|f\|_p = \left(\int_I \ |f|^p \right)^{\frac{1}{p}} < \infty \right\}$$

$\|\bullet\|_p$ is a norm for $1 \le p \le \infty$. Characteristic for L_p space, $p \ge 1$ are the inequalities of Holder and Minkowski, where $0 < p < \infty$. If $\|f\|_p$ is in the interval $\|f\|_p(J)$. For the mean interval we shall write $\|f\|_p$. The Ostrowski integral inequality was first proved by the researcher Ostrowski in 1938 [41], by using the following Theorem.

Theorem 1.1.1

Let $f: [a, b] \to \mathbb{R}$ be continuous on [a,b] and differentiable on (a,b) whose derivative $\acute{f}: (a, b) \to R$ is bounded on (a,b).

Then,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2} \right)^2}{(b-a)^2} \right] (b-a) \|\acute{f}\|_\infty$$

For all $x \in [a, b]$ and the constant $\frac{1}{4}$ is the best possible.Where

$\|\acute{f}\|_{\infty} = sup_{t\in(a,b)}|\acute{f}| < \infty.$

Later on, some generalizations were found for this inequality as stated in [44] Which has applications in numerical integrals as well as having a special meaning mentioned in [5].

As for its applications on the derivative functions whose derivate belongs to the $L_p$spaces, for $1 \leq p \leq \infty$ the researchers here used another type on Ostrowski_inequality as mentioned in [6].

The Fink׳s result is also used to find another type of Ostrowski inequality, which is as follows Theorem.

Theorem1.1.2

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on $I^{\circ}$and $a, b \in I^{\circ}$ with $a < b$. If $\acute{f} \in L_p(a,b)$, when$\left(\mathrm{p} > 1, \frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1\right)$, then we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} \|\acute{f}\|_p,$$

for all $x \in [a,b]$.

For instant Theorem(1) in [1], where appropriate calculations are found when $\mathrm{n} = 1$.

In addition , there are other types of Ostrowski inequalities reported in research [6], [45] and [44].

In our work we will find a new formula for the Ostrowski inequality in the Lp space when it is $0 < p < 1$.

1.2.Ostrowski inequality for functions in $L_p, 0 < p < 1$

Here we introduce our main results

Theorem 1.2.1

Let $a = x_\circ < x_1 < x_2 < x_3 < \cdots < x_n = b$ be a division of interval [a, b] with $x_i = \alpha_{i+1}\frac{b-a}{n}$, such that $[x_i, x_{i+1}] < 1$ with $n > b - a$ and $\propto_i$ $(i = 0,1,..,k+1)$ has $k+2$ of points so that $\propto_0 = a, \propto_i \in [x_{i-1}, x_i]$, $(i = 1, \ldots, k)$and $\propto_{i+1} = b$, if $f, \acute{f} \in L_p[a,b]$, when $f: [a,b] \to \mathbb{R}$, then

$$\left| \int_a^b f(t)dt - \sum_{i=0}^{n} (\alpha_{i+1} - \alpha_i) f(x_i) \right|$$

$$\leq C(p,q) \sum_{i=0}^{n} [(\alpha_{i+1} - x_i)^q + (x_{i+1} - \alpha_{i+1})^q]^{\frac{1}{q}} \|\acute{f}\|_{p[x_i, x_{i+1}]}.$$

Proof:

Define the mapping g: [a, b] → ℝ giving by

$$g(t) = \begin{cases} t - \alpha_1, t \in [a, x_{1)} \\ t - \alpha_2, t \in [x_1, x_{2)} \\ \vdots \\ t - \alpha_k, t \in [\alpha_{k-1}, b]. \end{cases}$$

Now, let us integral $\int_a^b g(t)\ \acute{f}(t)dt$ by part.

Let $u = g(t) = (t - \propto_{i+1})$, then $\frac{du}{dt} = dt$, and $dv = f(\acute{t})$, then,$v = f(t)$.

Then ,

$$\int_a^b g(t)\ \acute{f}(t)dt = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} g(t)\acute{f}(t)dt$$

$$= \sum_{i=0}^{n-1} \left[(t-\alpha_{i+1})f(t)|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} f(t)dt \right]$$

$$= \sum_{i=0}^{n-1} \left[(x_{i+1} - \propto_{i+1})f(x_{i+1}) - (x_i - \propto_{i+1})f(x_i) - \int_{x_i}^{x_{i+1}} f(t)dt \right]$$

$$= \sum_{i=0}^{n-1} (x_{i+1} - \propto_{i+1})f(x_{i+1})$$

$$+ \sum_{i=0}^{n-1} (\propto_{i+1} - x_i)f(x_i) - \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(t)dt$$

$$= \sum_{i=0}^{n-1} \Bigg[(\propto_{i+1} - x_i)f(x_i) + (x_{i+1} - \propto_{i+1})f(x_{i+1})$$

$$- \int_{x_i}^{x_{i+1}} f(t)dt \Bigg]$$

$$= \sum_{i=0}^{n-1} (x_{i+1} - \propto_{i+1})f(x_{i+1})$$

$$+ \sum_{i=0}^{n-1} (\propto_{i+1} - x_i)f(x_i) - \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(t)dt \qquad (1.1)$$

Since $x_\circ = a$, then,

$$\sum_{i=0}^{n-1}(\alpha_{i+1} - x_i)\, f(x_i)$$

$$= (\propto_1 - a)f(a) + \sum_{i=1}^{n-1}(\alpha_{i+1} - x_i)\, f(x_i). \quad (1.2)$$

Also, $x_n = b$,then,

$$\sum_{i=0}^{n-1}(x_{i+1} - \propto_i)f(x_{i+1}) = (b - \propto_n)f(b) + \sum_{i=0}^{n-2}(x_{i+1} - \propto_{i+1})\, f(x_{i+1})\,. \quad (1.3)$$

Also

$$\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(t)dt = \int_a^b f(t)dt$$

Now,

Put (1.2) and (1.3) in (1.1), we get

$$\int_a^b g(t)\ \acute{f}(t)$$

$$= (\propto_1 - a)f(a) + \sum_{i=1}^{n-1}(\alpha_{i+1} - x_i)\, f(x_i) + (b - \alpha_n)f(b)$$

$$+ \sum_{i=0}^{n-2}(x_{i+1} - \alpha_{i+1})\, f(x_{i+1}) - \int_a^b f(t)dt$$

This implies,

$$\int_a^b g(t)\ \acute{f}(t) = (\propto_1 - a)f(a) + \sum_{i=1}^{n-1}(\alpha_{i+1} - x_i)\, f(x_i) + (b - \alpha_n)f(b)$$

$$+\sum_{i=1}^{n-1}(x_i - \alpha_i)\, f(x_i) - \int_a^b f(t)dt. \qquad (1.4)$$

Since,

$$(\propto_1 - a)f(a) + \sum_{i=1}^{n-1}(\alpha_{i+1} - x_i)\, f(x_i) + (b - \alpha_n)f(b)$$

$$+\sum_{i=1}^{n-1}(x_i - \alpha_i)\, f(x_i)$$

$$= (\propto_1 - a)f(a) + (b - \alpha_n)f(b) + \sum_{i=1}^{n-1}(\alpha_{i+1} - \alpha_i)\, f(x_i)$$

$$= \sum_{i=0}^{n}(\alpha_{i+1} - \alpha_i)\, f(x_i). \qquad (1.5)$$

Put (1.5) in (1.4), we get,

$$\int_a^b g(t)\ \acute{f}(t)dt = \sum_{i=0}^{n}(\alpha_{i+1} - \alpha_i)\, f(x_i) - \int_a^b f(t)dt. \qquad (1.6)$$

So,

$$\int_a^b f(t)dt = \sum_{i=0}^{n}(\propto_{i+1} - \alpha_i)f(x_i) - \int_a^b g(t)\acute{f}(t)\, dt. \qquad (1.7)$$

But,

$$\left|\int_a^b g(t)\ \acute{f}(t)dt\right| = \left|\sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} g(t)\acute{f}(t)dt\right|$$

$$\leq \sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} |g(t)|\left|\acute{f}(t)\right|dt$$

$$= \sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}|\left|\acute{f}(t)\right|dt\,. \tag{1.8}$$

By using Holder′s inequality we get,

$$\sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}|\left|\acute{f}(t)\right|dt$$

$$\leq \sum_{i=0}^{n}\left[\left(\int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}|^{\acute{q}}\,dt\right)^{\frac{1}{\acute{q}}}\left(\int_{x_i}^{x_{i+1}} \left|\acute{f}(t)\right|^{\acute{p}}\,dt\right)^{\frac{1}{\acute{p}}}\right] \tag{1.9}$$

When $1 \leq \acute{p} \leq \acute{q} < \infty$ and $\frac{1}{\acute{p}} + \frac{1}{\acute{q}} = 1.$

Since,

$$|t-\alpha_{i+1}| = \begin{cases} (\alpha_{i+1}-t) & if\ t \in (x_i, \alpha_{i+1}) \\ \sum(t-\alpha_{i+1}) & if\ t \in (\alpha_{i+1}, x_{i+1}). \end{cases}$$

This implies,

$$\int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}|^{\acute{q}}\, dt$$

$$= \int_{x_i}^{x_{i+1}} \left[|\alpha_{i+1}-t|^{\acute{q}} + \sum_{\alpha_{i+1}}^{x_{i+1}} (t-\alpha_{i+1})^{\acute{q}} \right] dt.$$

$$= \int_{x_i}^{\alpha_{i+1}} |\alpha_{i+1}-t|^{\acute{q}} dt + \int_{\alpha_{i+1}}^{x_{i+1}} |t-\alpha_{i+1}|^{\acute{q}} dt. \qquad (1.10)$$

Then,

$$\int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}|^{\acute{q}} = \frac{1}{\acute{q}+1} (\alpha_{i+1}-t)^{\acute{q}+1}\Big|_{x_i}^{\alpha_{i+1}}$$

$$+ \frac{1}{\acute{q}+1} (t-\alpha_{i+1})^{\acute{q}+1}\Big|_{\alpha_{i+1}}^{x_{i+1}}$$

$$= \frac{1}{\acute{q}+1} [(\alpha_{i+1}-\alpha_{i+1}) - (\alpha_{i+1}-x_i)]^{\acute{q}+1}$$

$$+ \frac{1}{\acute{q}+1} [(x_{i+1}-\alpha_{i+1}) - (\alpha_{i+1}-\alpha_{i+1})]^{\acute{q}+1}$$

$$= \frac{1}{\acute{q}+1} [(x_i-\alpha_{i+1}) + (x_{i+1}-\alpha_{i+1})]^{\acute{q}+1}. \qquad (1.11)$$

Put (1.11) in (1.9) we get,

$$\sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}| \left|\acute{f}(t)\right|$$

$$\leq \sum_{i=0}^{n} \left[\left(\frac{1}{\acute{q}+1} [(x_i-\alpha_{i+1}) + (x_{i+1}-\alpha_{i+1})]^{\acute{q}+1} \right)^{\frac{1}{\acute{q}}} \left(\int_{x_i}^{x_{i+1}} \left|\acute{f}(t)\right|^{\acute{p}} dt \right)^{\frac{1}{\acute{p}}} \right]$$

Now, by using (1.8) we get,

$$\left|\int_a^b g(t)\ \acute{f}(t)dt\right|$$

$$\leq \sum_{i=0}^{n}\left[\left(\frac{1}{\acute{q}+1}[(x_i-\alpha_{i+1})+(x_{i+1}-\alpha_{i+1})]^{\acute{q}+1}\right)^{\frac{1}{\acute{q}}}\left(\int_{x_i}^{x_{i+1}}\left|\acute{f}(t)\right|^{\acute{p}}dt\right)^{\frac{1}{\acute{p}}}\right]$$

Since $\acute{q} < \acute{q} + 1$,

then,

$$\sum_{i=0}^{n}\left[\left(\frac{1}{\acute{q}+1}[(x_i-\alpha_{i+1})+(x_{i+1}-\alpha_{i+1})]^{\acute{q}+1}\right)^{\frac{1}{\acute{q}}}\left(\int_{x_i}^{x_{i+1}}\left|\acute{f}(t)\right|^{\acute{p}}dt\right)^{\frac{1}{\acute{p}}}\right]$$

$$\leq \sum_{i=0}^{n}\left[\left(\frac{1}{\acute{q}+1}[(x_i-\alpha_{i+1})+(x_{i+1}-\alpha_{i+1})]^{\acute{q}}\right)^{\frac{1}{\acute{q}}}\left(\int_{x_i}^{x_{i+1}}\left|\acute{f}(t)\right|^{\acute{p}}dt\right)^{\frac{1}{\acute{p}}}\right].$$

This implies,

$$\left|\int_a^b g(t)\ \acute{f}(t)dt\right|$$

$$\leq \sum_{i=0}^{n}\left[\left(\frac{1}{\acute{q}+1}[(x_i-\alpha_{i+1})+(x_{i+1}-\alpha_{i+1})]^{\acute{q}}\right)^{\frac{1}{\acute{q}}}\left(\int_{x_i}^{x_{i+1}}\left|\acute{f}(t)\right|^{\acute{p}}dt\right)^{\frac{1}{\acute{p}}}\right].$$

Let $0 < q < 1$, then for $q < \acute{q}$, this implies

$$\sum_{i=0}^{n}\left[\left(\frac{1}{\acute{q}+1}[(x_i-\alpha_{i+1})+(x_{i+1}-\alpha_{i+1})]^{\acute{q}}\right)^{\frac{1}{\acute{q}}}\left(\int_{x_i}^{x_{i+1}}\left|\acute{f}(t)\right|^{\acute{p}}dt\right)^{\frac{1}{\acute{p}}}\right]$$

$$\leq C(q)\sum_{i=0}^{n-1}\left[\left[(x_i-\alpha_{i+1})^q+(x_{i+1}-\alpha_{i+1})^q\right]^{\frac{1}{q}}\left(\int_{x_i}^{x_{i+1}}\left|\acute{f}(t)\right|^{\acute{p}}dt\right)^{\frac{1}{\acute{p}}}\right].(1.12)$$

Now, by using the result

$$\int_n^m f(a)da=\frac{m-n}{j}\sum_{i=0}^{j}f(a_i), [50]$$

when, $a_i = {n+(m-n)(2i-1)}/_{2j}$.

So that,

$$\int_{x_i}^{x_{i+1}}\left|\acute{f}(t)\right|^{\acute{p}}dt=\frac{1}{n}\sum_{i=1}^{n}\left|\acute{f}(t_i)\right|^{\acute{p}}\ ,1<\acute{p}\leq\infty.$$

Let $0<p<1$,then $p<\acute{p}$,this implies,

$$\int_{x_i}^{x_{i+1}}\acute{f}(t)^{\acute{p}}dt=\frac{1}{n}\sum_{i=1}^{n}\left|\acute{f}(t_i)\right|^{\acute{p}}\ \leq\frac{1}{n}\sum_{i=1}^{n}\left|\acute{f}(t_i)\right|^{p}\ ,0<p<1.$$

So that,

$$\left(\int_{x_i}^{x_{i+1}}\left|\acute{f}(t)\right|^{\acute{p}}dt\right)^{\frac{1}{\acute{p}}}\leq\left(C(p)\int_{x_i}^{x_{i+1}}\left|\acute{f}(t)\right|^{p}dt\right)^{\frac{1}{\acute{p}}}.$$

Since,

$$\left(C(p)\int_{x_i}^{x_{i+1}}\left|\acute{f}(t)\right|^{p}dt\right)^{\frac{1}{\acute{p}}}$$

$$= \left(C(p) \int_{x_i}^{x_{i+1}} \left| \acute{f}(t) \right|^p dt \right)^{\frac{1}{p}} \left(C(p) \int_{x_i}^{x_{i+1}} \left| \acute{f}(t) \right|^p dt \right)^{\frac{1}{\acute{p}} - \frac{1}{p}}.$$

Also, Since, $\acute{f} \in L_p$, then $\int_{x_i}^{x_{i+1}} \left| \acute{f}(t) \right|^p dt < \infty$,so that

$$\left(\int_{x_i}^{x_{i+1}} \left| \acute{f}(t) \right|^p dt \right)^{\frac{1}{\acute{p}}} \leq \left(C(p) \int_{x_i}^{x_{i+1}} \left| \acute{f}(t) \right|^p dt \right)^{\frac{1}{p}}$$

$$\leq C(p) \left\| \acute{f} \right\|_{p[x_i, x_{i+1}]} . \qquad (1.13)$$

$C(p)$is an absolute constant depending on p and it not the same at each step it is different from one step to another, also $C(p_1, p_2, \dots, p_n)$ is a positive constant depending on $(p_1, p_2, \dots, p_n)$.We shall use this notation in all our work.

Put (1.13) in (1.12), we get

$$\left| \int_a^b g(t) \acute{f}(t) dt \right|$$

$$\leq C(p,q) \left(\sum_{i=o}^{n-1} (\alpha_{i+1} - x_i)^q + (x_{i+1} - \alpha_{i+1})^q \right)^{\frac{1}{q}} \left\| \acute{f} \right\|_{P[x_i, x_{i+1}]}. \qquad (1.14)$$

When $0 < p < 1$ and $0 < q < 1$.

Now , by using (1.8), we get,

$$\left|\int_a^b f(t)dt - \sum_{i=0}^{n}(\alpha_{i+1}-\alpha_i)f(x_i)\right|$$

$$\leq C(p,q)\sum_{i=0}^{n-1}[(\alpha_{i+1}-x_i)^q + (x_{i+1}-\alpha_{i+1})^q]^{\frac{1}{q}}\|\acute{f}\|_{p[x_i,x_{i+1}]}. \quad ■$$

Corollary 1.2.2

For $f, \acute{f} \in L_p[a,b] o < p < 1$, and

$a = x_\circ < x_1 < x_2 < x_3 < \cdots < x_k = b$, be a partition of the interval [a,b] with $I_n = [x_{n-1}, x_n]$.Then,

$$\left|\int_a^b f(x)dx - \sum_{i=0}^{k}(\alpha_{i+1-}\alpha_i)f(x_i)\right| \leq C(p) max_i h_i (b-a)\|\acute{f}\|_p$$

Where, $h_i = x_{i+1} - x_i, (i = 0,1,\ldots,k-1)$

Proof:

Since,

$$\left[\sum_{i=0}^{n-1}(\alpha_{i+1}-x_i)^{q+1} + (x_{i+1}-\alpha_{i+1})^{q+1}\right]^{\frac{1}{q}} \leq \left(\sum_{i=0}^{k-1} h_i^{q+1}\right)^{\frac{1}{q}},$$

where $0 < q < 1$

$$\leq \left(\sum_{i=0}^{k-1} h_i^q \ h_i\right)^{\frac{1}{q}}$$

$$\leq \left(\sum_{i=0}^{k-1} (max h_i)^q \right) h_i^{\frac{1}{q}}$$

$$\leq {}^{max}_{i} h_i \sum_{i=0}^{k-1} h_i^{\frac{1}{q}}, 0 < q < 1.$$

Let C be a constant such that $h_i^{\frac{1}{q}} < ch_i$

Then,

$$\left[\sum_{i=0}^{n-1} (\alpha_{i+1} - x_i)^{q+1} + (x_{i+1} - \alpha_{i+1})^{q+1} \right]^{\frac{1}{q}}$$
$$\leq C(p)^{max}{}_{i} h_i (b-a). \qquad (1.15)$$

By using Theorem 1.2.1 and (1.15), we get,

$$\left| \int_a^b f(t)dt - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \leq C(p)^{max}{}_{i} h_i (b-a) \|\acute{f}\|_p ,$$

$$0 < p < 1.$$ ■

Chapter Two

Application of the Ostrowski Inequality

Application Of the Ostrowski Inequality

In this chapter we introduce two sections. The first section includes the application of Ostrowski inequality in random variable whose probability density function and cumulative distribution function belong to L_p.While the second section concerns with the application of Ostrowski's inequality in the beta distribution and the normal distribution.

2.1 Application of the Ostrowski Inequality in Random Variable

Here we begin our applications theorems for a random variable whose probability densityt function and cumulative distribution function belongs to $L_p[a,b]$, $0 < p < 1$. Let us start our main results with Theorem 2.1.1.

Theorem 2.1.1

Let X be a random variable with probability density function

$f:[a,b] \subset \mathbb{R} \to \mathbb{R}_+$, and the cumulative distribution function

$F(x) = P(X \leq x)$.If f and F belongs to $L_p[a,b]$, $0 < p < 1$ then

$$\left|P_r(X \leq x) - \frac{b - E(x)}{h}\right|$$

$$\leq \frac{1}{h} C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \; \|f\|_{p[a,b]},$$

where $h = b - a, 0 < p < 1$ and $0 < q < 1$ and $C(p,q)$ is a constant depend upon p and q.

Proof:

Since $\left|\int_a^b f(t)dt - \sum_{i=0}^{k}(\alpha_{i+1} - \alpha_i)f(x_i)\right|$

$$= \left|\sum_{i=0}^{k}(\alpha_{i+1} - \alpha_i)f(x_i) - \int_a^b f(t)dt\right|$$

Set $\alpha_{i+1} = b, \alpha_i = a$, then

$$\left|\sum_{i=0}^{k}(\alpha_{i+1} - \alpha_i)f(x_i) - \int_a^b f(t)dt\right| = \left|(b-a)f(x) - \int_a^b f(t)dt\right|$$

By using Theorem1.2.1

$$\left|(b-a)f(x) - \int_a^b f(t)dt\right|$$

$$\leq C(p,q)\sum_{i=0}^{n-1}[(\alpha_{i+1} - x_i)^q + (x_{i+1} - \alpha_{i+1})^q]^{\frac{1}{q}}\left\|\acute{f}\right\|_{p[x_i,x_{i+1}]} \quad (2.1)$$

Put the cumulative distribution function F instead of f in (2.1),

we get

$$\left|(b-a)F(x) - \int_a^b F(t)dt\right|$$

$$\leq C(p,q)\sum_{i=0}^{n-1}[(\alpha_{i+1} - x_i)^q + (x_{i+1} - \alpha_{i+1})^q]^{\frac{1}{q}}\left\|\acute{F}\right\|_{p[x_i,x_{i+1}]}. \quad (2.2)$$

Now we need to find the expectation, since x ∈ $[a, b]$ then using the define of Expectation we get:

$E(x) = \int_a^b t dF(t).$

Now using integration by part, we obtain

$E(x) = bF(b) - aF(a) - \int_a^b F(t)dt.$

Since the cumulative function F(x) , x ∈ [a, b] always bounded by 0 and 1, then we have F(a)=0 and F(b)=1. So that

$$\int_a^b F(t)dt = b - E(x). \tag{2.3}$$

Since $\acute{F} = f$, $h = b - a$, $p_r(X \leq x) = F(x)$. (2.4)

Put (2.3) and (2.4)in (2.1) we get

$$\left| P_r(X \leq x) - \frac{b - E(x)}{h} \right|$$

$$\leq \frac{1}{h} C(p, q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \, \|f\|_{p[a,b]}.$$

when $0 < p < 1$ and $0 < q < 1$ ■

Before we introduce Theorem 2.1.2 we need the concept equivalent:

A equivalent to B iff ∃ two constants c_1 and c_2 such that

$c_2 B \leq A \leq c_1 B$

Theorem 2.1.2

The upper bound of $\left|P_r(X \le x) - \frac{b-E(x)}{h}\right|$ and $\left|P_r(X \ge x) - \frac{b-E(x)}{h}\right|$

are equivalent.

Proof:

By using Theorem 2.2.1

$$\left|P_r(X \le x) - \frac{b-E(x)}{h}\right|$$

$$\le \frac{1}{h} C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \|f\|_{p[a,b]}, 0 < p < 1.$$

Now, since $P_r(X \ge x) = 1 - P_r(X \le x)$.

Then $P_r(X \le x) = 1 - P_r(X \ge x)$.

So that,

$$\left|P_r(X \le x) - \frac{b-E(x)}{h}\right| = \left|1 - P_r(X \ge x) - \frac{b-E(x)}{h}\right|.$$

Since $h = b - a$, then

$$\left|\frac{-a}{h} - P_r(X \ge x) + \frac{E(x)}{h}\right| = \left|-\left(\frac{a}{h} + P_r(X \ge x) - \frac{E(x)}{h}\right)\right|$$

$$= \left|P_r(X \ge x) - \frac{E(x)-a}{h}\right|$$

This implies

$$\left|P_r(X \geq x) - \frac{E(x) - a}{h}\right|$$

$$\leq \frac{1}{h} C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \|f\|_{p[a,b]}, 0 < p < 1.$$

Hence, the upper bound of $\left|P_r(X \leq x) - \frac{b-E(x)}{h}\right|$ equivalent to $\left|P_r(X \geq x) - \frac{E(x)-a}{h}\right|$, for all $x \in [a,b]$ ■

Theorem 2.1.3

Let X be a random variable with density function $f: [a,b] \subset \mathbb{R} \to \mathbb{R}_+$ and with cumulative distribution $F(X)$, then the expectation has the following form when $0 < p, q < 1$

$$b - C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \|f\|_{p[a,b]}$$

$$\leq E(X) \leq$$

$$a + C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \|f\|_{p[a,b]}.$$

Proof:

By using Theorem 2.1.2

$$\left|P_r(X \leq x) - \frac{b - E(x)}{h}\right|$$

$$\leq \frac{1}{h} C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \|f\|_{p[a,b]}.$$

When $0 < p, q < 1$. (2.5)

Since $x \in [a,b]$, then $a \leq E(x) \leq b$

(i) if $x = a$, then (2.5) become,

$$\left|\frac{b - E(x)}{h}\right| \leq \frac{1}{h} C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \|f\|_{p[a,b]}.$$

This implies,

$$b - E(x) \leq C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \|f\|_{p[a,b]}.$$

Then we get,

$$E(x) \geq b - C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \|f\|_{p[a,b]}. \quad (2.6)$$

(ii) if $x = b$, then (2.5) become,

$$\left|1 - \frac{b - E(x)}{h}\right| \leq \frac{1}{h} C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \|f\|_{p[a,b]}.$$

So,

$$\left|\frac{h-b+E(x)}{h}\right| \leq \frac{1}{h}C(p,q)\sum_{i=1}^{n-1}(\alpha_{i+1}-a)^{\frac{1}{q}+1}+(b-\alpha_{i+1})^{\frac{1}{q}+1}\ \|f\|_{p[a,b]}.$$

Since $h=b-a$,then

$$-a+E(x) \leq C(p,q)\sum_{i=1}^{n-1}(\alpha_{i+1}-a)^{\frac{1}{q}+1}+(b-\alpha_{i+1})^{\frac{1}{q}+1}\ \|f\|_{p[a,b]}.$$

This implies,

$$E(x) \leq a+C(p,q)\textstyle\sum_{i=1}^{n-1}(\alpha_{i+1}-a)^{\frac{1}{q}+1}+(b-\alpha_{i+1})^{\frac{1}{q}+1}\ \|f\|_{p[a,b].} \quad (2.7)$$

From (2.6) and (2.7) we get

$$b-C(p,q)\sum_{i=1}^{n-1}(\alpha_{i+1}-a)^{\frac{1}{q}+1}+(b-\alpha_{i+1})^{\frac{1}{q}+1}\ \|f\|_{p[a,b]} \leq E(x) \leq$$

$$a+C(p,q)\textstyle\sum_{i=1}^{n-1}(\alpha_{i+1}-a)^{\frac{1}{q}+1}+(b-\alpha_{i+1})^{\frac{1}{q}+1}\ \|f\|_{p[a,b]} \qquad \blacksquare$$

Theorem 2.1.4

Let X be a random variable whose probability density function $f(X)$ and cumulative distribution function $F(X)$ belongs to $L_p[a,b]$, $0 < p < 1, 0 < q < 1$we have

$$\left|E(X)-\frac{a+b}{2}\right|$$

$$\leq C(p,q)\|f\|_p\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}-\frac{a+b}{2}.$$

Proof:

By using Theorem 2.1.3we get,

$$b - C(p,q)\|f\|_p \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1} \leq E(x) \leq$$

$$a + C(p,q)\|f\|_p \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1}.$$

Then, we have

$$b - \frac{a+b}{2} - C(p,q)\|f\|_p \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1}$$

$$\leq E(x) - \frac{a+b}{2} \leq$$

$$a - \frac{a+b}{2} + C(p,q)\|f\|_p \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1}.$$

This implies

$$\frac{b-a}{2} - C(p,q)\|f\|_p \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1}$$

$$\leq E(x) - \frac{a+b}{2}$$

$$\leq \left(\frac{a-b}{2}\right) + C(p,q)\|f\|_p \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1}.$$

This implies

$$-\left[\left(\frac{a-b}{2}\right)+C(p,q)\|f\|_p\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}\right]$$

$$\leq E(x)-\frac{a+b}{2}\leq$$

$$\left(\frac{a-b}{2}\right)+C(p,q)\|f\|_p\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}.$$

Then we obtain

$$\left|E(x)-\frac{a+b}{2}\right|$$

$$\leq C(p,q)\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}\|f\|_p+\frac{a-b}{2}\ \blacksquare$$

Corollary 2.1.5

Let X be a random variable and f be a density probability function belong to $L_p[a,b]$, $o<p<1$ whose expectation $E(X)$ close to the midpoint of interval ${}^{(a+b)}/_{2}$.

If

$$\|f\|_p \leq \left[\frac{b-a}{2C(p,q)\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}} + \frac{\in}{C(p,q)\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}}\right].$$

For a given $\in > 0$, then we have $\left|E(X)-\frac{a+b}{2}\right| \leq \in$.

Proof:

By using Theorem 2.1.4,we have

$$\left|E(x)-\frac{a+b}{2}\right|$$

$$\leq C(p,q)\|f\|_p \sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}+\frac{a-b}{2}.$$

Now from our assumption

$$\|f\|_p \leq \left[\frac{b-a}{2C(p,q)\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}} + \frac{\in}{C(p,q)\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}}\right].$$

Then

$$\left|E(x)-\frac{a+b}{2}\right| \leq C(p,q)\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}$$

$$\left[\frac{b-a}{2C(p,q)\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}} + \frac{\in}{C(p,q)\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}}\right]+\frac{a-b}{2}.$$

Then

$$\left|E(x)-\frac{a+b}{2}\right| \leq \left(\frac{b-a}{2}+\in\right)+\frac{a-b}{2},$$

This completes the proof. ■

Theorem 2.1.6

Let X be a random variable whose probability density function $f(X)$ belongs to $L_p[a,b], 0<p<1$ if $x={a+b}/{2}$,then

$$\left|P_r\left(X\leq\left(\frac{a+b}{2}\right)\right)-\frac{1}{2}\right|$$

$$\leq\frac{1}{h}\left[C(p,q)\|f\|_p\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}+\left|E(x)-\frac{a+b}{2}\right|\right].$$

where h belongs to $[a,b]$.

Proof:

Put $x = \frac{a+b}{2}$ in Theorem 2.1.1, we get

$$\left|P_r\left(X \leq \left(\frac{a+b}{2}\right)\right) - \frac{b-E(X)}{h}\right|$$

$$\leq \frac{1}{h} C(p,q)\|f\|_p \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1}.$$

So,

$$\left|P_r\left(X \leq \left(\frac{a+b}{2}\right)\right) - \frac{1}{2} + \frac{1}{2} - \frac{b-E(X)}{h}\right|$$

$$\leq \frac{1}{h} C(p,q)\|f\|_p \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1}.$$

Using triangle inequality $|x-y| > |x| - |y|$, we get

$$\left|P_r\left(X \leq \left(\frac{a+b}{2}\right)\right) - \frac{1}{2}\right| - \left|\frac{1}{2} - \frac{b-E(X)}{h}\right| <$$

$$\left|P_r\left(X \leq \left(\frac{a+b}{2}\right)\right) - \frac{1}{2} + \frac{1}{2} - \frac{b-E(X)}{h}\right| \leq$$

$$\leq \frac{1}{h} C(p,q)\|f\|_p \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1}.$$

Implies,

$$\left|P_r\left(X \le \left(\frac{a+b}{2}\right)\right) - \frac{1}{2}\right|$$

$$\le \frac{1}{h} C(p,q)\|f\|_p \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1} + \left|\frac{1}{2} - \frac{b - E(X)}{h}\right|.$$

Then

$$\left|P_r\left(X \le \left(\frac{a+b}{2}\right)\right) - \frac{1}{2}\right|$$

$$\le \frac{1}{h} C(p,q)\|f\|_p \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1} + \left|\frac{2E(x) - (a+b)}{2h}\right|.$$

Then

$$\left|P_r\left(X \le \left(\frac{a+b}{2}\right)\right) - \frac{1}{2}\right|$$

$$\le \frac{1}{h} C(p,q)\|f\|_p \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1}$$
$$+ \frac{1}{h}\left|E(x) - \frac{(a+b)}{2}\right|.$$

Now,

$$\left|\mathrm{P_r}\left(\mathrm{X} \le \left(\frac{\mathrm{a+b}}{2}\right)\right) - \frac{1}{2}\right|$$

$$\le \frac{1}{\mathrm{h}}\left[\mathrm{C(p,q)}\|\mathrm{f}\|_{\mathrm{p}} \sum_{\mathrm{i}=1}^{\mathrm{n}-1} (\propto_{\mathrm{i}+1} - \mathrm{a})^{\frac{1}{\mathrm{q}}+1} + (\mathrm{b} - \propto_{\mathrm{i}+1})^{\frac{1}{\mathrm{q}}+1} + \left|\mathrm{E(x)} - \frac{\mathrm{a+b}}{2}\right|\right] \quad \blacksquare$$

2.2 Applications of Ostrowski Inequality in Beta and Normal Distributions

In our work in this section we estimate the probability density function of the beta distributions, and normal distribution. We use the results in previous section for estimating the distance between the cumulative density function and the expectation of a random variable. Many articles on Ostrowiski inequality [41],[1] ,[44]and [43] are choose β and Γ distribution (Γ is a special case of β distribution) because the p.d.f of these distributions in terms of p (the shape parameter) and q (the distribution parameter). The sample space of the random variable of β and Γ belongs to (0,∞).The above properties make the applications of these distributions very easy.

The normal distribution don't have the above properties. The p.d.f of the normal distribution in terms of μ and σ. The sample space of the normal distribution subset of R. So that no one work on the Ostrowiski inequality appear for the expectation of Normal distribution.

Here we approximate the expectation of the normal and βeta distributions.

Now we shall introduce our main results as we mentioned above.

We mean by $X\sim D$, as X distributed as D distribution. Firstly we begin with the beta distribution.

Theorem 2.2.1

Let X be a random variable and $X\sim B_{\alpha,\beta}$ whose the density function f belongs to the L_p where $0 < p < 1$,then

$$\left|P_r(X \le x) - \frac{\beta}{\alpha+\beta}\right| \le \frac{1}{h} C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1}$$

$$+(b-\alpha_{i+1})^{\frac{1}{q}+1} \frac{1}{B(\alpha,\beta)} [B(P(\alpha-1)+1, P(\beta-1)+1]^{\frac{1}{P}},$$

where $B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du$.

Proof:

Recall that

$$f(X,\alpha,\beta) = \frac{X^{(\alpha-1)}(1-X)^{(\beta-1)}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du}$$

$$= \frac{1}{B(\alpha,\beta)} X^{(\alpha-1)}(1-X)^{(\beta-1)}.$$

The probability density function of beta distribution.

Then$\|f(X,\alpha,\beta)\|_p = \frac{1}{B(\alpha,\beta)} (\int_0^1 X^{p(\alpha-1)}(1-X)^{p(\beta-1)} dX)^{\frac{1}{p}}$

$$= \frac{1}{B(\alpha,\beta)} (\int_0^1 X^{p(\alpha-1)+1-1}(1-X)^{p(\beta-1)+1-1} dX)^{\frac{1}{p}}.$$

This implies,

$$\|f(X,\alpha,\beta)\|_p$$
$$=\frac{1}{B(\alpha,\beta)}(B(P(\alpha-1)+1,P(\beta-1)+1)^{\frac{1}{p}}. \quad (2.8)$$

Now

$P(\alpha-1)+1>0$, then $\alpha>1-\frac{1}{p}$, also $P(\beta-1)+1>0$, then $\beta>1-\frac{1}{p}$. By using Theorem 2.1.1, we get for a random variable X with parameter (α,β) where $\alpha>1-\frac{1}{p}$, $\beta>1-\frac{1}{p}$ and $0<p<1$,

$$\left|P_r(X\le x)-\frac{b-E(x)}{h}\right|$$
$$\le\frac{1}{h}C(p,q)\sum_{i=1}^{n-1}(\alpha_{i+1}-a)^{\frac{1}{q}+1}+(b-\alpha_{i+1})^{\frac{1}{q}+1}\|f\|_p.$$

For all $X\in[0,1]$,then $E(X)=\frac{\alpha}{\alpha+\beta}$.

So,

$$\left|P_r(X\le x)-\frac{b-E(x)}{h}\right|=\left|P_r(X\le x)-\frac{1-\frac{\alpha}{\alpha+\beta}}{1}\right|$$
$$=\left|P_r(X\le x)-\frac{\alpha+\beta-\alpha}{\alpha+\beta}\right|.$$

This implies

$$\left|P_r(X \le x) - \frac{\beta}{\alpha+\beta}\right| \le \frac{1}{h} C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1}$$

$$+(b-\alpha_{i+1})^{\frac{1}{q}+1} \frac{1}{B(\alpha,\beta)} [B(P(\alpha-1)+1, P(\beta-1)+1]^{\frac{1}{P}} \blacksquare$$

Now let us turn the light to the normal distribution.

Theorem 2.2.2

Let X be a random variable with parameters $(\mu, \delta^2) \in \Omega$, with the probability density function $f(\mathrm{X}, \mu, \delta^2) = \frac{1}{\delta\sqrt{2\pi}} \mathrm{e}^{\frac{-(\mathrm{X}-\mu)^2}{2\delta^2}}$, where

$\Omega = \{(\mu, \delta^2); -\infty < \mu < \infty, 0 < \delta^2 < \infty\}$, then

$$\left|P_r(X \le x) - \frac{b-\mu}{h}\right|$$

$$\le \frac{1}{h} C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b-\alpha_{i+1})^{\frac{1}{q}+1} \left[\left(\frac{2}{\pi}\right)^{\frac{1}{2p}} \left(\mathrm{e}^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}}\right]$$

Proof:

Since $X \sim N(\mu, \delta^2)$,then $f(X, \mu, \delta^2) = \frac{1}{\delta\sqrt{2\pi}} \mathrm{e}^{\frac{-(\mathrm{X}-\mu)^2}{2\delta^2}}$.

So, $\|f\|_{p(-\infty,\infty)} = \left(\frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\frac{-(\mathrm{X}-\mu)^2 p}{2\delta^2}} dX\right)^{\frac{1}{p}}$.

Now , if $\mu > 0$, we have

$$\int_{-\infty}^{\infty} \mathrm{e}^{\frac{-(\mathrm{X}-\mu)^2 p}{2\delta^2}}\, dX$$

$$= \int_{-\infty}^{-\mu} \mathrm{e}^{\frac{-(\mathrm{X}-\mu)^2 p}{2\delta^2}}\, dX + \int_{-\mu}^{\mu} \mathrm{e}^{\frac{-(\mathrm{X}-\mu)^2 p}{2\delta^2}}\, dX + \int_{\mu}^{\infty} \mathrm{e}^{\frac{-(\mathrm{X}-\mu)^2 p}{2\delta^2}}\, dX.$$

Since in the normal distribution we have

$\int_{-\infty}^{-\mu} \mathrm{e}^{\frac{-(\mathrm{X}-\mu)^2 p}{2\delta^2}}\, dX$ and $\int_{\mu}^{\infty} \mathrm{e}^{\frac{-(\mathrm{X}-\mu)^2 p}{2\delta^2}}\, dX$ approaches to zero, then

$$\int_{-\infty}^{\infty} \mathrm{e}^{\frac{-(\mathrm{X}-\mu)^2 p}{2\delta^2}}\, dX \le \int_{-\mu}^{\mu} \mathrm{e}^{\frac{-1}{2}\left(\frac{X-\mu}{\delta}\right)}\, dX \le \int_{-\mu}^{\mu} \mathrm{e}^{\frac{\mu}{\delta}} dX = \frac{\delta}{\mu}\left(2\mu e^{\frac{\mu}{\delta}}\right).$$

Similarly if $\mu = m < 0$.

This implies

$$\|f\|_p \le \left(\frac{1}{\delta\sqrt{2\pi}}\ 2\delta e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}} \le \left(\sqrt{\frac{2}{\pi}}\, e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}}.$$

Then

$$\|f\|_p \le \left(\frac{2}{\pi}\right)^{\frac{1}{2\mathrm{p}}} \left(\mathrm{e}^{\frac{\mu}{\delta}}\right)^{\frac{1}{\mathrm{p}}}. \tag{2.9}$$

By using Theorem 2.1.1,we obtain

$$\left|p_r(X \le x) - \frac{b - E(x)}{h}\right|$$

$$\leq \frac{1}{h} C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \left[\left(\frac{2}{\pi}\right)^{\frac{1}{2p}} \left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}} \right]$$

Since $X \sim N(\mu, \delta^2)$, then $E(x) = \mu$.

so that

$$\left| P_r(X \leq x) - \frac{b - \mu}{h} \right|$$

$$\leq \frac{1}{h} C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \left[\left(\frac{2}{\pi}\right)^{\frac{1}{2p}} \left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}} \right] \qquad \blacksquare$$

Theorem 2.2.3

Let X be a random variable with parameters $(\mu, \delta^2) \in \Omega$, with the probability density function belong to $L_p[a,b], o < p < 1$, where

$\Omega = \{(\mu, \delta^2); -\infty < \mu < \infty, 0 < \delta^2 < \infty\}$, if $x = {a+b}/{2}$.

Then we have

$$\left| P_r\left(X \leq \left(\frac{a+b}{2}\right)\right) - \frac{1}{2} \right|$$

$$\leq \frac{1}{h} \left[C(p,q) \left(\frac{2}{\pi}\right)^{\frac{1}{2p}} \left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}} \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1} \right.$$

$$\left. + \left| \mu - \frac{a+b}{2} \right| \right]$$

Proof:

By using Theorem 2.1.6 and (2.9),we obtain

$$\left|P_r\left(X \le \left(\frac{a+b}{2}\right)\right) - \frac{1}{2}\right|$$

$$\le \frac{1}{h}\left[C(p,q)\left(\frac{2}{\pi}\right)^{\frac{1}{2p}}\left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}}\sum_{i=1}^{n-1}(\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1}\right.$$

$$\left. + \left|E(x) - \frac{a+b}{2}\right|\right]$$

Since $X \sim N(\mu, \delta^2)$, then $E(x) = \mu$. This implies,

$$\left|P_r\left(X \le \left(\frac{a+b}{2}\right)\right) - \frac{1}{2}\right|$$

$$\le \frac{1}{h}\left[C(p,q)\left(\frac{2}{\pi}\right)^{\frac{1}{2p}}\left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}}\sum_{i=1}^{n-1}(\propto_{i+1} - a)^{\frac{1}{q}+1} + (b - \propto_{i+1})^{\frac{1}{q}+1}\right.$$

$$\left. + \left|\mu - \frac{a+b}{2}\right|\right] \quad \blacksquare$$

Theorem 2.2.4

Let X be a random variable with parameters $(\mu, \delta^2) \in \Omega$, has the probability density function belong to $L_p[a,b], o < p < 1$, if

$-1 \le \mu < 0$ then variance bounded below.

$$\sigma^2 \geq \left(b + h\left|P_r\left(X \leq \left(\frac{a+b}{2}\right)\right) - \frac{1}{2}\right| - \left|\frac{a+b}{2}\right|\right)$$

$$- C(p,q)\left(\left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}} + \|f\|_{p[a,b]}\right)\sum_{i=1}^{n-1}(\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1}.$$

Proof :

Since

$$\sigma^2 = E(X^2) - \mu^2. \tag{2.10}$$

Also $X^2 \geq X$, implies $E(X^2) \geq E(X)$. (2.11)

By using Theorem 2.1.4 and using (2.11), we get

$$\boldsymbol{E(X^2) \geq b - C(p,q)\sum_{i=1}^{n-1}(\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1}\,\|f\|_{p[a,b]}}. \tag{2.12}$$

Since $\left|\boldsymbol{\mu} - \frac{\boldsymbol{a+b}}{\boldsymbol{2}}\right| \leq |\boldsymbol{\mu}| + \left|\frac{\boldsymbol{a+b}}{\boldsymbol{2}}\right|$, where $-\infty \leq \boldsymbol{\mu} \leq \infty$.

By using Theorem 2.1.6, we get

$$\left|\boldsymbol{P_r}\left(\boldsymbol{X} \leq \left(\frac{\boldsymbol{a+b}}{\boldsymbol{2}}\right)\right) - \frac{\boldsymbol{1}}{\boldsymbol{2}}\right|$$

$$\leq \frac{\boldsymbol{1}}{\boldsymbol{h}}\left[\boldsymbol{C(p,q)}\left(\frac{\boldsymbol{2}}{\boldsymbol{\pi}}\right)^{\frac{1}{2p}}\left(\boldsymbol{e^{\frac{\mu}{\delta}}}\right)^{\frac{1}{p}}\sum_{i=1}^{n-1}(\propto_{i+1} - \boldsymbol{a})^{\frac{1}{q}+1} + (\boldsymbol{b} - \propto_{i+1})^{\frac{1}{q}+1} + \left|\boldsymbol{\mu} - \frac{\boldsymbol{a+b}}{\boldsymbol{2}}\right|\right]$$

$$\leq \frac{1}{h}\left[C(p,q)\left(\frac{2}{\pi}\right)^{\frac{1}{2p}}\left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}}\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}+|\mu|\right.$$
$$\left.+\left|\frac{a+b}{2}\right|\right]$$

We obtain,

$$h\left|P_r\left(X\leq\left(\frac{a+b}{2}\right)\right)-\frac{1}{2}\right|-C(p,q)\left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}}\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}$$
$$+(b-\propto_{i+1})^{\frac{1}{q}+1}-\left|\frac{a+b}{2}\right|\leq|\mu|=\mu. \qquad (2.13)$$

If $-1\leq\mu<0$,then we get $\mu^2\leq-\mu$

Therefore $-\mu^2\geq\mu$ (2.14)

By using (2.13) and (2.14), we get

$$-\mu^2\geq h\left|P_r\left(X\leq\left(\frac{a+b}{2}\right)\right)-\frac{1}{2}\right|$$
$$-C(p,q)\left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}}\sum_{i=1}^{n-1}(\propto_{i+1}-a)^{\frac{1}{q}+1}+(b-\propto_{i+1})^{\frac{1}{q}+1}$$
$$-\left|\frac{a+b}{2}\right| \qquad (2.15)$$

Put (2.12) and (2.15) in (2.10) we get

$$\sigma^2 \geq \left[b - C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \, \|f\|_{p[a,b]} \right]$$

$$+ \left[h \left| P_r \left(X \leq \left(\frac{a+b}{2} \right) \right) - \frac{1}{2} \right| - C(p,q) \left(e^{\frac{\mu}{\delta}} \right)^{\frac{1}{p}} \sum_{i=1}^{n-1} (\propto_{i+1} - a)^{\frac{1}{q}+1} \right.$$

$$\left. + (b - \propto_{i+1})^{\frac{1}{q}+1} - \left| \frac{a+b}{2} \right| \right].$$

This implies

$$\sigma^2 \geq \left(b + h \left| P_r \left(X \leq \left(\frac{a+b}{2} \right) \right) - \frac{1}{2} \right| - \left| \frac{a+b}{2} \right| \right)$$

$$- C(p,q) \sum_{i=1}^{n-1} (\alpha_{i+1} - a)^{\frac{1}{q}+1} + (b - \alpha_{i+1})^{\frac{1}{q}+1} \left(\left(e^{\frac{\mu}{\delta}} \right)^{\frac{1}{p}} + \|f\|_{p[a,b]} \right) \qquad \blacksquare$$

Chapter Three

Inequalities for Expectation, Variance and Dispersion

Inequalities for Expectation, Variance and Dispersion

In the last decade few authors introduced Pre-Gruss Inequality for functions in $L_p[a,b]$ space, and used it to estimate the error bounds of the reminders of Taylor-Like formula and quadrature formula. In this work we generalize and make an extension of Pre-Gruss inequality for functions in $\mathrm{L_p[a,b]}$,quasi-normed spaces and used it to estimate the expectation, variance and dispersion.

3.1 Introduction

In [41] and [1] the authors introduced the so called Pre-Gruss Inequality at the form for function in $L_1[a,b]$ normed space.

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{b-a}\int_a^b f(t)dt\,\frac{1}{b-a}\int_a^b g(t)dt\right|$$

$$\leq\frac{1}{2}(\varphi-\gamma)\left[\frac{1}{b-a}\int_a^b g^2(t)dt-\left(\frac{1}{b-a}\int_a^b g(t)dt\right)^2\right]^{\frac{1}{2}} \qquad (3.1)$$

In [1],the authors used (3.1) to estimate the bounded Taylor – Like formula. While in [5], the authors used the (3.1) for estimating the remainder in three point quadrature formula.

Here we extend and generalize the results of [41],[1]and[5] by introducing a type of pre-Gruss inequality for formula in $L_p[a,b]$ quasi- normed space in the following theorems.

Theorem 3.2.1

Let $f\colon [a,b] \to \mathbb{R}, f^p$ be a differentiable mapping on (a, b) whose derivative is bounded on (a, b) and $\|f\|_p = \left(\int_a^b |f|^p\right)^{\frac{1}{p}} < \infty$ and $(f^p)' \in L_p[a,b]$, where $0 < p < 1$, further that $\alpha\colon [a,b] \to \mathbb{R}$ and $\beta\colon [a,b] \to \mathbb{R}$ are two polynomials functions, then for all $x \in [a,b]$ we have the inequality

$$\left|\int_a^b g(x,t)(f^p(t))'\,dt\right|$$

$$\leq \frac{1}{2}[(\alpha(x)-a)^2 + (\alpha(x)+x)^2 + (\beta(x)-b)^2 + (\beta(x)+x)^2]\left\|(f^p(t))'\right\|_p.$$

and a direct consequence of Theorem 3.2.1 we have

Corollary 3.2.2

$|\mathrm{E(X)}| \leq \left[\left(X + \frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2\right]\|h(t)\|_p$,where $X \in [a,b]$.

Theorem 3.2.3

Let X be a random variable having the probability density function $h\colon [a,b] \to \mathbb{R}$, then

$$\left|\frac{1}{b-a}\int_a^b h(t)g(x,t)dt - \int_a^b h(t)\,dt\,\frac{1}{b-a}\int_a^b g(x,t)dt\right|$$
$$\leq \frac{1}{2}[(a+x)^2+(b+x)^2]\|h(t)\|_p$$
$$+\left|\int_a^b h(t)dt\,\frac{1}{b-a}\int_a^b g(x,t)dt\right|$$

Using Theorem 3.2.3 and Corollary 3.2.2 we introduce important Inequality in probability theory for estimating expectation and variance. As in the following theorems

Theorem 3.2.4

Let X be a random variable having the probability density function $h: [a,b] \to \mathbb{R}$,then

$$\left|E(X)-\frac{a+b}{2}\right|$$
$$\leq \frac{1}{2}\left[((a+X)^2+(b+X)^2)\|h(t)\|_p+(a+b)\right],$$

where $E(X)$ is the expectation of the random variable X.

Theorem 3.2.5

If $E_P(X)$ is the p-moment of X and M_P is p-logarithm mean, then,

$$|E_P(X)-M_P^P(a,b)|$$
$$\leq \frac{1}{2}[(a+X)^2+(b+X)^2]\|h(t)\|_p+M_P^P(a,b).$$

Theorem 3.2.6

Let X be a random variable having the probability density belong to $L_p[a,b]$, then

$$\left|\frac{1}{(b-a)}\sigma_\mu^2(X)-\left(\mu-\frac{a+b}{2}\right)^2-\frac{(b-a)^2}{12}\right|$$

$$\leq\left[X^2+(a+b)X+\frac{a^2+b^2}{2}\right]\|h(t)\|_p-\left[\left(\mu-\frac{a+b}{2}\right)^2-\frac{(b-a)^2}{12}\right]$$

Corollary 3.2.7

Let X be a random variable and h(t) belong to $L_P[a,b], 0<p<1$ is a probability density function,$\left(\mu=\frac{a+b}{2}\right)$,then we obtain the best inequality of Theorem 3.2.1

$$\left|\frac{1}{(b-a)}\sigma_\mu^2(X)-\frac{(b-a)^2}{12}\right|$$

$$\leq\left[X^2+(a+b)X+\frac{a^2+b^2}{2}\right]\|h(t)\|_p-\frac{(b-a)^2}{12}.$$

Theorem 3.2.8

Let X be a random variable having the probability density belong to $L_P[a,b]$, if $\mathrm{A}_\mu(\mathrm{t})=\int_a^b|t-\mu|h(t)dt,\ \mu\epsilon[a,b]$

then we have the inequality

$$\left|A_\mu(X)-\frac{1}{b-a}\left[\frac{(b-a)^2}{4}+\left(\mu-\frac{a+b}{2}\right)^2\right]\right|$$

$$\leq\frac{(b-a)}{2}[(a+X)^2+(b+X)^2]\|h(t)\|_p+\left[\frac{(b-a)^2}{4}+\left(\mu-\frac{a+b}{2}\right)^2\right].$$

Corollary 3.2.9

The best inequality of the Theorem 3.2.8 as following

$$\left|A_{\mu_\circ}(X) - \frac{b-a}{4}\right|$$
$$\leq \frac{(b-a)}{2}[(a+X)^2 + (b+X)^2]\|h(t)\|_p + \frac{(b-a)^2}{4}.$$

where $\mu = \mu_\circ = \frac{a+b}{2}$

Theorem 3.2.10

Let X be a random variable whose the probability density belong to $L_p[a,b]$ defined on the finite interval $[a,b]$and $\sigma(X) < \infty$, then we get

$$0 \leq \sigma_\mu^2(X) - (E(X) - \mu)^2$$
$$\leq (4 + 2M_1)b^2 + 2b - \alpha(X) - \beta(X) + \big(2b - \alpha(X) - \beta(X)\big)^2$$
$$\forall\, \mu \in [a,b]$$

Theorem 3.2.11

Let X be a random variable whose the probability density $h(t)$ belong to $L_p[a,b]$ and $E(X)$ is expectation of X, then

$$\left|E(X) + (b-a)F(X) - X - \frac{b-a}{2}\right|$$
$$\leq \frac{(b-a)}{2}[(a+X)^2 + (b+X)^2]\|h(t)\|_p + (b-a)(X - \frac{a+b}{2})$$

Theorem 3.2.14

Let X be a random variable whose the probability density $h(t)$ belong to $L_p[a,b]$ and $H(t)$ is a cumulative distribution, then

$$\left|\mathrm{E(X)} + \frac{b-a}{2}\mathrm{H(X)} - \frac{b+x}{2}\right|$$

$$\leq \frac{(b-a)}{2}\left[((a+X)^2 + (b+X)^2)\|h(t)\|_p - H(t) + 1\right]$$

Corollary 3.2.16

If $X = a$ or $X = b$ in $|(b-a)H(X) + E(X) - b| = \left|E(X) - \frac{a+b}{2}\right|$

We have the same upper bound. $\forall X \in [a,b]$

Corollary 3.2.17

If $X = \frac{a+b}{2}$ in Theorem 3.2.15, then we have the best inequality

$$\left|E(X) + \left(\frac{b-a}{2}\right)P_r\left(X \leq \frac{a+b}{2}\right) - \frac{a+3b}{2}\right| \leq$$

$$\frac{(b-a)}{2}\left[\left(\left(\frac{3a+b}{2}\right)^2 + \left(\frac{3b+a}{2}\right)^2\right)\|h(t)\|_p - P_r\left(X \leq \frac{a+b}{2}\right) + 1\right].$$

Corollary 3.2.18

If $X = \frac{a+b}{2}$ in Theorem 3.2.15, then we have the best inequality

$$\left|E(X) + \left(\frac{b-a}{2}\right)P_r\left(X \leq \frac{a+b}{2}\right) - \frac{a+3b}{2}\right| \leq$$

$$\frac{(b-a)}{2}\left[\left(\left(\frac{3a+b}{2}\right)^2 + \left(\frac{3b+a}{2}\right)^2\right)\|h(t)\|_p - P_r\left(X \leq \frac{a+b}{2}\right) + 1\right].$$

3.2.Pre Gruss inequality for Expectation, Variance Dispersion

Here we introduce our main results.

Theorem 3.2.1

Let $f:[a,b] \to R, f^p$ be a differentiable mapping on (a,b) whose derivative is bounded on (a, b) and $f, (f^p(t))' \in L_p[a,b]$, where $0 < p < 1$, further that $\alpha:[a,b] \to R$ and $\beta:[a,b] \to \mathbb{R}$ are two polynomials functions, then for all $x \in [a,b]$ we have the inequality

$$\left| \int_a^b g(x,t)(f^p(t))' \, dt \right|$$

$$\leq \frac{1}{2}[(\alpha(x)-a)^2 + (\alpha(x)+x)^2 + (\beta(x)-b)^2 + (\beta(x)+x)^2] \left\| (f^p(t))' \right\|_p \tag{3.2}$$

Proof:

Let $g(x,t) = \begin{cases} t - \propto(x), & t \in [a,x] \\ t - \beta(x), & t \in [x,b] \end{cases}$

and consider $\int_a^b g(x,t)(f^p(t))' \, dt$, this implies

$$\int_a^b g(x,t)(f^p(t))' \, dt$$

$$= \int_a^x (t - \propto(x))(f^p(t))' \, dt + \int_x^b (t - \beta(x))(f^p(t))' \, dt. \tag{3.3}$$

Now compute $\int_a^x (t-\propto(x)(f^p(t))' \, dt$ by parts

let $u = (t-\propto(x))$ and $dv = (f^p(t))' \, dt$, then

$$\int_a^x (t-\propto(x))(f^p(t))' \, dt = (t-\propto(x)(f^p(t))\Big|_a^x - \int_a^x f^p(t)dt$$

$$= (x-\alpha(x))f^p(x) - (a-\alpha(x))f^p(a) - \int_a^x f^p(t)dt. \qquad (3.4)$$

Also $\int_x^b (t-\beta(x)(f^p(t))' \, dt$

let $u = (t-\beta(x))$ and $dv = (f^p(t))' \, dt$, then

$$\int_x^b (t-\beta(x)(f^p(t))' \, dt = (t-\beta(x))f^p(t)|_x^b - \int_x^b f^p(t)dt$$

$$= (b-\beta(x))f^p(b) - (x-\beta(x))f^p(x) - \int_x^b f^p(t)dt. \qquad (3.5)$$

Put (3.4) and (3.5) in (3.3),we get

$$\int_a^b g(x,t)(f^p(t))' \, dt$$

$$= (x-\alpha(x))f^p(x) - (a-\alpha(x))f^p(a) - \int_a^x f^p(t)dt$$

$$+\big(b-\beta(x)\big)f^p(b)-\big(x-\beta(x)\big)f^p(x)-\int_x^b f^p(t)dt.$$

This implies,

$$\int_a^b g(x,t)(f^p(t))^{\prime}\,dt$$

$$=\big(\beta(x)-\alpha(x)\big)f^p(x)+(\alpha(x)-a)f^p(a)$$
$$+\big(b-\beta(x)\big)f^p(b)-\int_a^b f^p(t)dt.$$

So that

$$\left|\int_a^b g(x,t)(f^p(t))^{\prime}\,dt\right|$$

$$=\Bigg|\big(\beta(x)-\alpha(x)\big)f^p(x)+(\alpha(x)-a)f^p(a)$$
$$+\big(b-\beta(x)\big)f^p(b)-\int_a^b f^p(t)dt\Bigg|$$

Since

$$\left|\int_a^b g(x,t)(f^p(t))^{\prime}\,dt\right|$$

$$\leq Sup_{t\in[a,b]}\left|\int_a^b g(x,t)dt\right|\left\|(f^p(t))^{\prime}\right\|_p$$

$$\leq \int_a^b |g(x,t)|dt \, \left\| (f^p(t))' \right\|_p . \tag{3.6}$$

Now, we need to find $\int_a^b |g(x,t)|\, dt$

$$\int_a^b |g(x,t)|\, dt = \left| \int_a^b g(x,t)dt \right|$$

$$= -\int_a^{\alpha(x)} \big(t-\alpha(x)\big)dt + \int_{\alpha(x)}^{x} \big(t-\alpha(x)\big)dt$$

$$- \int_x^{\beta(x)} \big(t-\beta(x)\big)dt + \int_{\beta(x)}^{b} \big(t-\beta(x)\big)dt$$

$$= -\left[\frac{t^2}{2} - \alpha(x)t\right]_a^{\alpha(x)} + \left[\frac{t^2}{2} - \alpha(x)t\right]_{\alpha(x)}^{x}$$

$$- \left[\frac{t^2}{2} - \beta(x)t\right]_x^{\beta(x)} + \left[\frac{t^2}{2} - \beta(x)t\right]_{\beta(x)}^{b}$$

$$= -\frac{\big(\alpha(x)\big)^2}{2} + \big(\alpha(x)\big)^2 + \frac{a^2}{2} - a\alpha(x) + \frac{x^2}{2} - \alpha(x)x$$

$$-\frac{\big(\alpha(x)\big)^2}{2} + \big(\alpha(x)\big)^2 - \frac{\big(\beta(x)\big)^2}{2} + \big(\beta(x)\big)^2 + \frac{x^2}{2} - \beta(x)x$$

$$+\frac{b^2}{2} - \beta(x)b - \frac{\big(\beta(x)\big)^2}{2} + \big(\beta(x)\big)^2$$

$$= -2\left[\frac{(\alpha(x))^2}{2}\right] + 2(\alpha(x))^2 + \frac{a^2}{2} - a\alpha(x) - x\alpha(x) + \frac{x^2}{2}$$

$$-2\left[\frac{(\beta(x))^2}{2}\right] + 2(\beta(x))^2 + \frac{b^2}{2} - b\beta(x) - x\beta(x) + \frac{x^2}{2}$$

$$= (\alpha(x))^2 + \frac{a^2}{2} - a\alpha(x) + x\alpha(x) + \frac{x^2}{2} + (\beta(x))^2 + \frac{b^2}{2}$$

$$-b\beta(x) + x\beta(x) + \frac{x^2}{2}$$

$$= \frac{1}{2}\Big[\Big(2(\alpha(x))^2 + a^2 - 2a\alpha(x) + 2x\alpha(x) + x^2\Big) +$$

$$\Big(2(\beta(x))^2 + b^2 - 2b\beta(x) + 2x\beta(x) + x^2\Big)\Big]$$

$$= \frac{1}{2}\Big[\Big((\alpha(x))^2 - 2a\alpha(x) + a^2\Big) + \Big((\alpha(x))^2 + 2x\alpha(x) + x^2\Big) +$$

$$\Big((\beta(x))^2 - 2b\beta(x) + b^2\Big) + \Big((\beta(x))^2 + 2x\beta(x) + x^2\Big)\Big].$$

By using (3.6), we get

$$\left|\int_a^b g(x,t)(f^p(t))' \, dt\right|$$

$$\leq \frac{1}{2}[(\alpha(x) - a)^2 + (\alpha(x) + x)^2 + (\beta(x) - b)^2$$

$$+ (\beta(x) + x)^2] \left\|(f^p(t))'\right\|_p \qquad \blacksquare$$

Corollary 3.2.2

, where $X \in [a,b]$$|\mathrm{E(X)}| \leq \left(\frac{a+b}{2} - X\right)^2 + \left(\frac{a-b}{2}\right)^2 \|h(t)\|_p$

Proof:

Setting $h(t) = (f^p(t))^{\prime}$ and $g(x,t) = t$ in the Theorem 3.2.1 , and let $\alpha(X)$, $\beta(X)$ are two identity polynomials, such that $\alpha(x) = a$ and $\beta(x) = b$, then we get

$$\left|\int_a^b th(t)dt\right| \leq \frac{1}{2}[(a+x)^2 + +(b+x)^2] \left\|(f^p(t))^{\prime}\right\|_p . \quad (3.7)$$

Since $\int_a^b th(t)dt = E(X)$, and we use the identity

$$\frac{X^2+Y^2}{2} = \left(\frac{X+Y}{2}\right)^2 + \left(\frac{X-Y}{2}\right)^2 . \quad (3.8)$$

Then, we get

$$|E(X)| \leq \left(\frac{(x+a)+(x+b}{2}\right)^2 + \left(\frac{(x+a)-(x+b}{2}\right)^2 \|h(t)\|_p .$$

This implies,

$$|E(X)| \leq \left(\frac{a+b}{2} - X\right)^2 + \left(\frac{a-b}{2}\right)^2 \|h(t)\|_p \qquad \blacksquare$$

Theorem 3.2.3

Let X be a random variable having the probability density function $h: [a,b] \to R$,then

$$\left|\frac{1}{b-a}\int_a^b h(t)g(x,t)dt - \int_a^b h(t)\,dt\,\frac{1}{b-a}\int_a^b g(x,t)dt\right|$$
$$\leq \frac{1}{2}[(a+x)^2 + (b+x)^2]\|h(t)\|_p$$
$$+\left|\int_a^b h(t)dt\,\frac{1}{b-a}\int_a^b g(x,t)dt\right|$$

Proof:

By using pre-Gruss inequality in (3.1), we put $(f^p(t))$ instead of $(f^p(t))'$, we get

$$\left|\frac{1}{b-a}\int_a^b (f^p(t))'\,g(x,t)dt\right.$$
$$\left.-\frac{1}{b-a}\int_a^b (f^p(t))'\,dt\frac{1}{b-a}\int_a^b g(x,t)dt\right|$$

We denote $(f^p(t))' = h(t)$, then $f^p(t) = \int_a^b h(t)dt$

This implies,

$$\left|\frac{1}{b-a}\int_a^b\int_a^b h(t)\,g(x,t)dt - \frac{1}{b-a}\int_a^b\int_a^b h(t)\,dt\frac{1}{b-a}\int_a^b g(x,t)dt\right|$$

$$= \left| \frac{b-a}{b-a} \int_a^b h(t)g(x,t)dt - \frac{b-a}{b-a} \int_a^b h(t)\,dt \frac{1}{b-a} \int_a^b g(x,t)dt \right|$$

by using the triangle inequality and Theorem 3.2.1, we get

$$\left| \int_a^b h(t)g(x,t)dt - \int_a^b h(t)\,dt \frac{1}{b-a} \int_a^b g(x,t)dt \right|$$

$$\leq \frac{1}{2}[(\alpha(x)+x)^2 + (\beta(x)+x)^2]\|h(t)\|_p + \left| \int_a^b h(t)dt \frac{1}{b-a} \int_a^b g(x,t)dt \right|.$$

When $\alpha(x) = a$ and $\beta(x) = b$ in Theorem 3.2.1, then we get

$$\left| \int_a^b h(t)g(x,t)dt - \int_a^b h(t)\,dt \frac{1}{b-a} \int_a^b g(x,t)dt \right|$$

$$\leq \frac{1}{2}[(a+x)^2 + (b+x)^2]\|h(t)\|_p + \left| \int_a^b h(t)dt \frac{1}{b-a} \int_a^b g(x,t)dt \right| \qquad \blacksquare$$

Using Theorem 3.2.3 we estimate the expectation of the random variable X

Theorem 3.2.4

Let X be a random variable having the probability density function $h: [a,b] \to R$,then

$$\left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{2}\left[((a+x)^2 + (b+x)^2)\|h(t)\|_p + (a+b)\right],$$

where E(X) is the expectation of the random variable X.

Proof:

If we put $g(x,t) = t$ in Theorem 3.2.3, we get

$$\left|\int_a^b th(t)dt - \int_a^b h(t)\,dt \frac{1}{b-a}\int_a^b tdt\right|$$

$$\leq \frac{1}{2}[(a+x)^2 + (b+x)^2]\|h(t)\|_p$$
$$+\left|\int_a^b h(t)dt \frac{1}{b-a}\int_a^b tdt\right|. \qquad (3.9)$$

Since $\int_a^b th(t)dt = E(X)$, $X \in [a,b]$, also

$$\int_a^b h(t)dt = 1\,, \frac{1}{b-a}\int_a^b tdt = \frac{a+b}{2},$$

by using (3.10), this implies

$$\left|E(X) - \frac{a+b}{2}\right|$$

$$\leq \frac{1}{2}\left[((a+x)^2 + (b+x)^2)\|h(t)\|_p + (a+b)\right] \qquad \blacksquare$$

Now let us introduce estimation for variance

Let $E_P(X)$ be the P-moment of X, defined as follows:

$$E_P(X) = \int_a^b t^p h(t)dt$$

When P-moment of the random variable X,$P \in \mathbb{R}/\{-1,0\}$, then the P-logarithm mean, defined as follows,

$\mu_P(a,b) = \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(a-b)}\right]^{\frac{1}{p}}$, where $0 < a < b$.

In the following Theorem we study the distance between the P-moment of the random variable X and the the P-logarithm mean.

Theorem 3.2.5:

If $E_P(X)$ is the p-moment of X and M_P is p-logarithm mean, then,

$$|E_P(X) - M_P^P(a,b)|$$
$$\leq \frac{1}{2}[(a+x)^2 + (b+x)^2]\|h(t)\|_p + M_P^P(a,b).$$

Proof:

If we put g(x,t)= t^{p} in Theorem 3.2.3, we get

$$\left|\int_a^b t^p h(t)dt - \int_a^b h(t)\,dt \frac{1}{b-a}\int_a^b t^p dt\right|$$
$$\leq \frac{1}{2}[(a+x)^2 + (b+x)^2]\|h(t)\|_p$$
$$+\left|\int_a^b h(t)dt \frac{1}{b-a}\int_a^b t^p dt\right|. \quad (3.10)$$

Since

$$\int_a^b t^p h(t)dt = E_p(X)\,, X \in [a,b].$$

Also,

$$\int_a^b h(t)dt = 1\,, \frac{1}{b-a}\int_a^b t^p dt = \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} = \mu_p^p(a,b)$$

by using (3.2.10), this implies,

$|E_P(X) - M_P^P(a,b)|$

$$\leq \frac{1}{2}[(a+X)^2 + (b+X)^2]\|h(t)\|_p + M_P^P(a,b) \qquad \blacksquare$$

In the following let us estimate the variance of the random variable X.

Theorem 3.2.6

Let X be a random variable and h(t) belong to $L_p[a,b], 0 < p < 1$ is a probability density function, then

$$\left|\frac{1}{(b-a)}\sigma_\mu^2(X) - \left(\mu - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}\right|$$

$$\leq \left[X^2 + (a+b)X + \frac{a^2+b^2}{2}\right]\|h(t)\|_p - \left[\left(\mu - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}\right]$$

Proof:

If we put $g(x,t) = (t-\mu)^2$ in Theorem 3.2.3, we get

$$\left|\int_a^b (t-\mu)^2 h(t)dt - \int_a^b h(t)\,dt \frac{1}{b-a}\int_a^b (t-\mu)^2 dt\right|$$

$$\leq \frac{1}{2}[(a+x)^2 + (b+x)^2]\|h(t)\|_p$$
$$+\left|\int_a^b h(t)dt \frac{1}{b-a}\int_a^b (t-\mu)^2 dt\right|. \qquad (3.11)$$

Since

$$\int_a^b (\mathrm{t}-\mathrm{\mu})^2 h(t)dt = \sigma_\mu^2(X)\,, \mu \in [a,b].$$

Also, $\int_a^b h(t)dt = 1$, and

$$\frac{1}{b-a}\int_a^b (\mathrm{t}-\mathrm{\mu})^2 dt = \left.\frac{(\mathrm{t}-\mathrm{\mu})^3}{3(b-a)}\right|_a^b = \frac{(b-\mu)^3 - (a-\mu)^3}{3(b-a)}$$

$$= \frac{(b-a)[(b-\mu)^2 + (a-\mu)(b-\mu) + (a-\mu)^2]}{3(b-a)}$$

$$= \frac{1}{3}\left[(b^2 - 2b\mu + \mu^2) + \left(ab - a\mu - b\mu + \mu^2 + (a^2 - 2a\mu + \mu^2)\right)\right]$$

$$= \frac{1}{3}(3\mu^2 - 3a\mu + 3b\mu) + \frac{1}{3}(a^2 + ab + b^2)$$

$$= (\mu^2 - (a+b)\mu + \frac{1}{3}\left[\frac{3}{4}(a^2 + 2ab + b^2)\right] + \frac{1}{3}\left[\frac{1}{4}(a^2 - 2ab + b^2)\right]$$

$$=\left(\mu - \left(\frac{a+b}{2}\right)\right)^2 + \frac{(a-b)^2}{12}.$$

Also

$$(a+x)^2+(b+x)^2 = 2X^2+2(a+b)X+(a^2+b^2)$$

This implies

$$\left|\frac{1}{(b-a)}\sigma_\mu^2(X)-\left(\mu-\frac{a+b}{2}\right)^2-\frac{(b-a)^2}{12}\right|$$

$$\leq\left[X^2+(a+b)X+\frac{a^2+b^2}{2}\right]\|h(t)\|_p$$

$$-\left[\left(\mu-\frac{a+b}{2}\right)^2-\frac{(b-a)^2}{12}\right] \qquad \blacksquare$$

Corollary 3.2.7

Let X be a random variable and h(t) belong to $L_p[a,b], 0<p<1$ is a probability density function, $\left(\mu=\frac{a+b}{2}\right)$,then we obtain the best inequality of Theorem 3.2.6

$$\left|\frac{1}{(b-a)}\sigma_\mu^2(X)-\frac{(b-a)^2}{12}\right|$$

$$\leq\left[X^2+(a+b)X+\frac{a^2+b^2}{2}\right]\|h(t)\|_p-\frac{(b-a)^2}{12}.$$

Proof:

By using Theorem 3.2.6

$$\left|\frac{1}{(b-a)}\sigma_\mu^2(X) - \left(\mu - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}\right|$$

$$\leq \left[X^2 + (a+b)X + \frac{a^2+b^2}{2}\right]\|h(t)\|_p - \left[\left(\mu - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}\right]. \qquad (3.12)$$

If we put $\left(\mu = \frac{a+b}{2}\right)$, in (3.12), we get

$$\left|\frac{1}{(b-a)}\sigma_\mu^2(X) - \frac{(b-a)^2}{12}\right|$$

$$\leq \left[X^2 + (a+b)X + \frac{a^2+b^2}{2}\right]\|h(t)\|_p - \frac{(b-a)^2}{12}$$

Is the best inequality of Theorem 3.2.6 ■

Theorem 3.2.8

Let X be a random variable having the probability density belong to $L_P[a,b]$, if $A_\mu(t) = \int_a^b |t-\mu| h(t)dt, \mu \epsilon [a,b]$,

then we have the inequality

$$\left|A_\mu(X) - \frac{1}{b-a}\left[\frac{(b-a)^2}{4} + \left(\mu - \frac{a+b}{2}\right)^2\right]\right|$$

$$\leq \frac{(b-a)}{2}\left[(a+X)^2 + (b+X)^2\right]\|h(t)\|_p + \left[\frac{(b-a)^2}{4} + \left(\mu - \frac{a+b}{2}\right)^2\right]$$

Proof:

If we put $g(X,t) = |t-\mu|$ in Theorem 3.2.3, we get

$$\left|\frac{1}{b-a}\int_a^b h(t)|t-\mu|dt - \int_a^b h(t)\,dt\frac{1}{b-a}\int_a^b |t-\mu|dt\right|$$
$$\leq \frac{1}{2}[(a+X)^2+(b+X)^2]\|h(t)\|_p$$
$$+\left|\int_a^b h(t)dt\frac{1}{b-a}\int_a^b |t-\mu|dt\right| . \qquad (3.13)$$

For $\frac{1}{b-a}\int_a^b |t-\mu|dt,$

Since $|t-\mu| = \begin{cases}(t-\mu), & if \quad t \geq \mu \\ -(t-\mu), & if \ t \leq \mu\end{cases}$,

then ,we have

$$\frac{1}{b-a}\int_a^b |t-\mu|dt = \left[\frac{-1}{b-a}\int_a^\mu |t-\mu|dt + \frac{1}{b-a}\int_\mu^b |t-\mu|dt\right]$$

$$=\frac{-1}{b-a}\left[\left(\frac{t^2}{2}-\mu t\right)\Big|_a^\mu + \left(\frac{t^2}{2}-\mu t\right)\Big|_\mu^b\right]$$

$$=\frac{1}{b-a}\left[\left(\mu^2-\frac{\mu^2}{2}-a\mu+\frac{a^2}{2}\right)+\left(\frac{b^2}{2}-b\mu-\frac{\mu^2}{2}+\mu^2\right)\right]$$

$$=\frac{1}{b-a}\left[\left(\frac{1}{2}\mu^2-a\mu+\frac{a^2}{2}\right)+\left(\frac{1}{2}\mu^2-b\mu+\frac{b^2}{2}\right)\right]$$

$$=\frac{1}{b-a}\left[\frac{1}{2}(\mu^2-2a\mu+a^2)+\frac{1}{2}(\mu^2-2b\mu+b^2)\right]$$

$$= \frac{1}{b-a}\left[\frac{(\mu-a)^2+(b-\mu)^2}{2}\right]$$

By using (3.8),we obtain

$$\frac{1}{b-a}\int_a^b |t-\mu|dt = \frac{1}{b-a}\left[\frac{(b-a)^2}{4}+\left(\mu-\frac{a+b}{2}\right)^2\right].$$

Since $\int_a^b h(t)dt = 1$, so that (3.13) become

$$\left|\frac{1}{b-a}A_\mu - \frac{1}{b-a}\left[\frac{(b-a)^2}{4}+\left(\mu-\frac{a+b}{2}\right)^2\right]\right|$$
$$\leq \frac{1}{2}\left[(a+x)^2+(b+x)^2\|h(t)\|_p\right]$$
$$+\frac{1}{b-a}\left[\frac{(b-a)^2}{4}+\left(\mu-\frac{a+b}{2}\right)^2\right].$$

This implies

$$\left|A_\mu(X) - \frac{1}{b-a}\left[\frac{(b-a)^2}{4}+\left(\mu-\frac{a+b}{2}\right)^2\right]\right|$$
$$\leq \frac{(b-a)}{2}\left[(a+X)^2+(b+X)^2\right]\|h(t)\|_p + \left[\frac{(b-a)^2}{4}+\left(\mu-\frac{a+b}{2}\right)^2\right] \quad \blacksquare$$

Corollary 3.2.9

The best inequality of the Theorem 3.2.8 as following

$$\left|A_{\mu_\circ}(X) - \frac{b-a}{4}\right|$$
$$\leq \frac{(b-a)}{2}\left[(a+X)^2+(b+X)^2\right]\|h(t)\|_p + \frac{(b-a)^2}{4},$$

where $\mu = \mu_\circ = \frac{a+b}{2}$

Proof:

By using Theorem 3.2.8

$$\left|A_\mu(X) - \frac{1}{b-a}\left[\frac{(b-a)^2}{4} + \left(\mu - \frac{a+b}{2}\right)^2\right]\right|$$

$$\leq \frac{(b-a)}{2}\left[(a+X)^2 + (b+X)^2\right]\|h(t)\|_p + \left[\frac{(b-a)^2}{4} + \left(\mu - \frac{a+b}{2}\right)^2\right]$$

Since

$$\frac{(b-a)}{2}\left[(a+X)^2 + (b+X)^2\right]\|h(t)\|_p + \left[\frac{(b-a)^2}{4} + \left(\mu - \frac{a+b}{2}\right)^2\right]$$

$$\leq \frac{(b-a)}{2}\left[(a+X)^2 + (b+X)^2\right]\|h(t)\|_p + \frac{(b-a)^2}{4} + \frac{1}{2}\left(\mu - \frac{a+b}{2}\right)^2 - \frac{1}{(b-a)^2}\left(\mu - \frac{a+b}{2}\right)^4.$$

Let

$$K(\mu) = \frac{(b-a)}{2}\left[(a+X)^2 + (b+X)^2\right]\|h(t)\|_p + \frac{(b-a)^2}{4} + \frac{1}{2}\left(\mu - \frac{a+b}{2}\right)^2 - \frac{1}{(b-a)^2}\left(\mu - \frac{a+b}{2}\right)^4.$$

Then , we get

$$\frac{dK(\mu)}{d\mu} = \left(\mu - \frac{a+b}{2}\right) - \frac{4}{(b-a)^2}\left(\mu - \frac{a+b}{2}\right)^3$$

$$= \left(\mu - \frac{a+b}{2}\right)\left[1 - \frac{4}{(b-a)^2}\left(\mu - \frac{a+b}{2}\right)^2\right].$$

(i) when $\frac{dK(\mu)}{d\mu} = 0$, this implies, $\mu = a$ or $\mu = \frac{(a+b)}{2}$ or $\mu = b$

(ii) when $\frac{dK(\mu)}{d\mu} < 0$, then $\mu \in \left(a, \frac{a+b}{2}\right)$

(iii) when $\frac{dK(\mu)}{d\mu} > 0$, then $\mu \in \left(\frac{a+b}{2}, b\right)$

we deduce that, if $\mu = \frac{a+b}{2}$ then we get

$$\left|A_{\mu_\circ}(X) - \frac{b-a}{4}\right|$$

$$\leq \frac{(b-a)}{2}\left[(a+X)^2 + (b+X)^2\right]\|h(t)\|_p + \frac{(b-a)^2}{4}$$

is The best inequality of the Theorem 3.2.8 ■

Lemma 3.2.10:[48]

Let $g, p: [a, b] \to \mathbb{R}$ be measurable functions such that

$, p \geq 0\ a.e\ on\ [a, b]$and $\int_a^b p(X)dX > 0 \alpha < g < \beta\ a.e$

$$0 < \frac{\int_a^b p(X)g^2(X)dX}{\int_a^b p(X)\,dX} - \left(\frac{\int_a^b p(X)g^2(X)dX}{\int_a^b p(X)\,dX}\right)^2 \leq \frac{1}{4}(\beta - \alpha)^2$$

Theorem 3.2.11

Let X be a random variable whose the probability density function belongs to $L_p[a,b]$ defined on the finite interval $[a,b]$and $\sigma(X)<\infty$, then we get

$$0 \le \sigma_\mu^2(X) - (E(X)-\mu)^2$$
$$\le (4+2M_1)b^2 + 2b - \alpha(X) - \beta(X)$$
$$+ \left(2b - \alpha(X) - \beta(X)\right)^2 \quad . \qquad \forall\, \mu \in [a,b]$$

Proof:

By using Theorem 3.2.3 and Lemma 3.2.10 we get

$$\left| \frac{1}{b-a}\int_a^b h(t)g(x,t)dt - \int_a^b h(t)\,dt \frac{1}{b-a}\int_a^b g(x,t)dt \right|$$
$$\le \frac{1}{2}[(a+x)^2+(b+x)^2]\|h(t)\|_p$$
$$+ \left| \int_a^b h(t)dt \frac{1}{b-a}\int_a^b g(x,t)dt \right| + \frac{1}{b-a}\int_a^b g(x,t)^2 dt.$$

Let

$$I_1 + I_2 + I_3 = \frac{1}{2}[(a+x)^2+(b+x)^2]\|h(t)\|_p$$
$$+ \left| \int_a^b h(t)dt \frac{1}{b-a}\int_a^b g(X,t)\,dt \right| + \frac{1}{b-a}\int_a^b g(X,\mathrm{t})^{\,2} dt.$$

Now

$$I_1 = \frac{1}{2}[(a+x)^2 + (b+x)^2]\|h(t)\|_p \le \frac{1}{2}((a+b)^2 + (2b)^2\mu_1)$$

$$\le (a+b)^2 + (2b)^2\mu_1$$

$$\le 4b^2 + 2\mu_1 b^2 = (4+2\mu_1)b^2,$$

where $\mu_1 = \|h(X)\|_p$.

Let

$$I_2 = \left|\int_a^b h(t)dt \frac{1}{b-a}\int_a^b g(X,t)\, dt\right|.$$

Since

$\int_a^b h(X)dt = 1$,then

$$I_2 = \left|\frac{1}{b-a}\int_a^b g(X,t)\, dt\right| \le \frac{1}{b-a}\left|\int_a^b g(X,t)\, dt\right|$$

$$\le \frac{1}{b-a}\int_a^b \left[\left(b-\alpha(X)\right) + \left(b-\beta(X)\right)\right]dt$$

$$= 2b - \alpha(X) - \beta(X).$$

Let

$$I_3 = \frac{1}{b-a}\int_a^b g(X,t)^{\,2} dt \le \left[\left(b-\alpha(X)\right) + \left(b-\beta(X)\right)\right]^2$$

$$= \left(2b - \alpha(X) - \beta(X)\right)^2.$$

Thus

$$I_1 + I_2 + I_3 = 4b^2 + 2\mu_1 b^2$$
$$= (4 + 2\mu_1)b^2 + 2b - \alpha(X) - \beta(X)$$
$$+ \left(2b - \alpha(X) - \beta(X)\right)^2.$$

This implies,

$$\left| \frac{1}{b-a} \int_a^b h(t)g(x,t)dt - \int_a^b h(t)\, dt \frac{1}{b-a} \int_a^b g(x,t)dt \right|$$
$$\leq (4 + 2\mu_1)b^2 + 2b - \alpha(X) - \beta(X)$$
$$+ \left(2b - \alpha(X) - \beta(X)\right)^2. \qquad (3.14)$$

We put $g(X,t) = (X - \mu), h(t) = p(X)$ in Lemma 3.2.10, and using (3.14) we get

$$0 \leq \int_a^b (X-\mu)^2\, p(X)dX - \left(\int_a^b (X-\mu)p(X)dX \right)^2$$

$$\leq (4 + 2\mu_1)b^2 + 2b - \alpha(X) - \beta(X) + \left(2b - \alpha(X) - \beta(X)\right)^2 \quad (3.15)$$

Since

$$\int_a^b p(t)(X-\mu)dt = \sigma_\mu^2(X).$$

And

$$\int_a^b (X-\mu)p(X)dX$$

$$= \int_a^b Xp(X)dX - \int_a^b \mu\, p(X)dX = E(X) - \mu.$$

By using (3.15) we get,

$$0 \le \sigma_\mu^2(X) - (E(X)-\mu)^2$$
$$\le (4+2M_1)b^2 + 2b - \alpha(X) - \beta(X)$$
$$+ \big(2b - \alpha(X) - \beta(X)\big)^2 \qquad \forall\, \mu \in [a,b] \qquad \blacksquare$$

We will relate connect the expectation $E(X)$ to cumulative distribution function $H(t) = \int_a^b h(t)dt$, where $h(t)$is the probability density function.

Theorem 3.2.12

Let X be a random variable whose the probability density h(t) belong to $L_p[a,b]$ and E(X) is expectation of X, then

$$\left|E(X) + (b-a)F(X) - X - \frac{b-a}{2}\right|$$

$$\le \frac{(b-a)}{2}[(a+X)^2 + (b+X)^2]\|h(t)\|_p + (b-a)\left(X - \frac{a+b}{2}\right).$$

Proof:

By using the inequality of Barnett and Dragomir in [52]

$$(b-a)H(X)+E(X)-b$$
$$=\int_a^b K(X,t)dH(t)=\int_a^b K(X,t)h(t) \qquad (3.16)$$

Now put $g(X,t)=K(X,t)$ in Theorem 3.2.3

Where $K(X,t)=\begin{cases} t-a & if\ a\le t\le X\le b, \\ t-b & if\ a\le X\le t\le b, \end{cases}$

then, we get

$$\left|\frac{1}{b-a}\int_a^b h(t)K(X,t)dt-\int_a^b h(t)\,dt\,\frac{1}{b-a}\int_a^b K(X,t)dt\right|$$

$$\le\frac{1}{2}[(a+x)^2+(b+x)^2]\|h(t)\|_p$$
$$+\left|\int_a^b h(t)dt\,\frac{1}{b-a}\int_a^b K(X,t)dt\right| \qquad (3.17)$$

Now

$$\frac{1}{b-a}\int_a^b K(X,t)dt=\frac{1}{b-a}\left[\int_a^x (t-a)dt+\int_x^b (t-b)dt\right]$$

$$=\frac{1}{b-a}\left[\left(\left(\frac{t^2}{2}-at\right)\Big|_a^x\right)+\left(\left(\frac{t^2}{2}-bt\right)\Big|_x^b\right)\right]$$

$$=\frac{1}{b-a}\left[\left(\frac{x^2}{2}-ax-\frac{a^2}{2}+a^2\right)+\left(\frac{b^2}{2}-b^2-\frac{x^2}{2}+bx\right)\right]$$

$$=\frac{1}{b-a}\left[(b-a)x-\left(\frac{b^2-a^2}{2}\right)\right]$$

$$= \frac{1}{b-a}\left[(b-a)x - \frac{(b-a)(b+a)}{2}\right]$$

$$= \left(X - \frac{a+b}{2}\right).$$

This implies

$$\frac{1}{b-a}\int_a^b K(X,t)dt = \left(X - \frac{a+b}{2}\right) \ . \tag{3.18}$$

then, by using (3.16) ,(3.17) and (3.18) , we get

$$\left|\frac{1}{b-a}\left[E(X) + (b-a)F(X) - b - \left(X - \frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{1}{2}[(a+x)^2 + (b+x)^2]\|h(t)\|_p + \left(X - \frac{a+b}{2}\right)$$

This implies,

$$\left|\frac{1}{b-a}\left[E(X) + (b-a)F(X) - X - \frac{b-a}{2}\right]\right|$$

$$\leq \frac{1}{2}[(a+x)^2 + (b+x)^2]\|h(t)\|_p + \left(X - \frac{a+b}{2}\right)$$

Where $\left(b - \frac{a+b}{2}\right) = \frac{b-a}{2}$.Then, the proof is completed ■

Corollary 3.2.13

If $X = a$ or $X = b$ in Theorem 3.2.12

We have the same upper bound of $\left|E(X) - \frac{a+b}{2}\right|$.

Proof:

(1) if $X = a$

By using Theorem 3.2.12, we get

$$\left|E(X) + (b-a)F(a) - a - \frac{b-a}{2}\right|$$

$$\leq \frac{(b-a)}{2}[(a+b)^2 + (2a)^2]\|h(t)\|_p + (b-a) + (b-a)\left(\frac{a+b}{2}\right)$$

Since $F(a) = 0$, when a is constant then,

$$\left|E(X) + (b-a)F(X) - X - \frac{b-a}{2}\right| = \left|E(X) - \frac{a+b}{2}\right|$$

$$\leq \frac{(b-a)}{2}[(a+b)^2 + (2b)^2]\ \|h(t)\|_p + \left(\frac{b^2-a^2}{2}\right) \qquad (3.18)$$

(2) if $X = b$

By using Theorem 3.2.11, we get

$$\left|E(X) + (b-a)F(b) - b - \frac{b-a}{2}\right|$$

$$\leq \frac{(b-a)}{2}[(a+b)^2 + (2a)^2]\|h(t)\|_p + (b-a) + (b-a)\left(\frac{a+b}{2}\right)$$

Since $F(b) = 1$, when b is constant then

$$\left|E(X) + (b-a)F(X) - X - \frac{b-a}{2}\right| = \left|E(X) - \frac{a+b}{2}\right|$$

$$\leq \frac{(b-a)}{2}[(a+b)^2 + (2b)^2]\|h(t)\|_p + \left(\frac{b^2-a^2}{2}\right) \qquad (3.20)$$

From (3.19) and (3.20) we have the upper bound of $\left|E(X)-\frac{a+b}{2}\right|$ are equivalent ■

Corollary 3.2.14

If $X=\frac{a+b}{2}$ in Theorem 3.2.12, then we have the best inequality

$$\left|E(X)+(b-a)P_r\left(X\leq\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)}{2}\left[\left(\left(\frac{3a+b}{2}\right)^2+\left(\frac{3b+a}{2}\right)^2\right)\|h(t)\|_p\right].$$

Proof:

By using Theorem 3.2.12, we get

$$\left|E(X)+(b-a)H\left(\frac{a+b}{2}\right)-\frac{a+b}{2}-\frac{b-a}{2}\right|$$
$$\leq\frac{(b-a)}{2}\left[\left(\frac{a+b}{2}+a\right)^2+\left(\frac{a+b}{2}+b\right)^2\right]\|h(t)\|_p+$$
$$(b-a)\left(\frac{a+b}{2}-\frac{a+b}{2}\right) \qquad (3.21)$$

Since

$H\left(\frac{a+b}{2}\right)=P_r\left(X\leq\frac{a+b}{2}\right)$ and $\frac{a+b}{2}-\frac{b-a}{2}=0$, also

$\left(\frac{a+b}{2}+a\right)^2=\left(\frac{3a+b}{2}\right)^2$ and $\left(\frac{a+b}{2}+b\right)^2=\left(\frac{3b+a}{2}\right)^2$.

Then by using (3.21), we obtain

$$\left|E(X)+(b-a)P_r\left(X\leq\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)}{2}\left[\left(\left(\frac{3a+b}{2}\right)^2+\left(\frac{3b+a}{2}\right)^2\right)\|h(t)\|_p\right]$$ ■

Theorem 3.2.15

Let X be a random variable whose the probability density h(t) belong to $L_p[a,b]$ and H(t) is a cumulative distribution ,then

$$\left|E(X)+\frac{b-a}{2}H(X)-\frac{b+x}{2}\right|$$

$$\leq\frac{(b-a)}{2}\left[((a+X)^2+(b+X)^2)\|h(t)\|_p-H(t)+1\right]$$

$\forall X\in[a,b]$

Proof:

Since

$$(b-a)H(X)+E(X)-b$$

$$=\int_a^x (t-a)dH(t)+\int_x^b (t-b)dH(t)$$

$$=\int_a^x (t-a)h(t)+\int_x^b (t-b)h(t)\ ,\forall X\in[a,b]\ . \qquad (3.22)$$

By using Theorem 3.2.3, put $g(X,t)=(t-a)$,we get

$$\left|\frac{1}{x-a}\int_a^x h(t)(t-a)dt-\frac{1}{x-a}\int_a^x h(t)\,dt\frac{1}{x-a}\int_a^x (t-a)dt\right|$$

$$\leq\frac{1}{2}[(a+x)^2+(b+x)^2]\|h(t)\|_p$$

$$+\left|\int_a^x h(t)dt\frac{1}{x-a}\int_a^x (t-a)dt\right|. \qquad (3.23)$$

Now

$$\frac{1}{x-a}\int_a^x (t-a)dt = \frac{1}{x-a}\left(\left(\frac{t^2}{2}-at\right)\right|_a^x$$

$$= \frac{1}{x-a}\left[\frac{x^2}{2}-ax-\frac{a^2}{2}+a^2\right]$$

$$= \frac{1}{2(x-a)}[x^2-2ax+a^2]$$

$$= \frac{1}{2(x-a)}(x-a)^2 = \frac{(x-a)}{2},$$

also, $\int_a^x h(t)\,dt = H(X)$

So by above (3.22) become

$$\left|\int_a^x h(t)(t-a)dt - \frac{x-a}{2}H(t)\right|$$

$$\leq \frac{x-a}{2}[(a+x)^2+(b+x)^2]\|h(t)\|_p + \left|\frac{x-a}{2}H(x)\right| \quad (3.24)$$

and similarly, put $g(X,t) = (t-b)$in Theorem 3.2.3, we get

$$\left|\frac{1}{b-x}\int_x^b (t-b)h(t)dt - \frac{1}{b-x}\int_x^b h(t)\,dt\,\frac{1}{b-x}\int_x^b (t-a)dt\right|$$

$$\leq \frac{1}{2}[(a+x)^2+(b+x)^2]\|h(t)\|_p + \left|\int_x^b h(t)dt\,\frac{1}{b-x}\int_x^b (t-a)dt\right| \quad (3.25)$$

Since

$$\frac{1}{b-x}\int_x^b (t-b)dt = \frac{1}{b-x}\left((\frac{t^2}{2} - bt)\right)\Big|_x^b$$

$$= \frac{1}{b-x}\left[\frac{b^2}{2} - b^2 + bx - \frac{x^2}{2}\right]$$

$$\frac{1}{b-x}\left[-\left(\frac{x^2}{2} - bx + \frac{b}{2}\right)\right]$$

$$= \frac{-1}{2(b-x)}[x^2 - 2bx + b^2]$$

$$= \frac{-1}{2(b-x)}(b-x)^2 = \frac{(x-b)}{2},$$

$$also, \int_x^b h(t)\, dt = (H(x))^c = 1 - H(x).$$

So by above (3.25) become

$$\left|\frac{1}{b-x}\int_x^b (t-b)h(t)dt - \frac{(x-b)}{2}\frac{1}{b-x}(1-H(x))\right|$$
$$\leq \frac{1}{2}[(a+x)^2 + (b+x)^2]\|h(t)\|_p + \left|\frac{1}{2}(1-H(t)\right|.$$

This implies,

$$\left|\int_x^b (t-b)h(t)dt + \frac{b-x}{2} - \frac{b-x}{2}H(t)\right|$$
$$\leq \frac{b-x}{2}[(a+x)^2 + (b+x)^2]\|h(t)\|_p$$
$$+ \left|\frac{b-x}{2} - \frac{b-x}{2}H(t)\right| \qquad (3.26)$$

by summing (3.24) and(3.26) ,also using the triangle inequality , we get

$$\left|\int_a^x (t-a)h(t)dt - \frac{x-a}{2}H(x) + \int_x^b (t-b)h(t)dt + \frac{b-x}{2} - \frac{b-x}{2}H(x)\right|$$

$$\leq \frac{(x-a)}{2}[(a+x)^2+(b+x)^2]\|h(t)\|_p + \frac{(x-a)}{2}H(x) + \frac{(b-x)}{2}[(a+x)^2+(b+x)^2]\|h(t)\|_p + \left|\frac{b-x}{2} - \frac{b-x}{2}H(x)\right|.$$

This implies

$$\left|\int_a^x h(t)(t-a)dt + \int_x^b h(t)(t-b)dt - \frac{b-a}{2}H(x) + \frac{b-x}{2}\right|$$

$$\leq \frac{(b-a)}{2}[(a+x)^2+(b+x)^2]\|h(t)\|_p + \frac{(b-a)}{2} - \frac{(b-a)}{2}H(x)$$

$$\leq \frac{(b-a)}{2}\Big[[(a+x)^2+(b+x)^2]\|h(t)\|_p - H(x) + 1\Big], \qquad (3.27)$$

by using (3.22) and (3.27), we get,

$$\left|(b-a)H(X) + E(X) - b - \frac{b-a}{2}H(X) + \frac{b-x}{2}\right| \leq \frac{(b-a)}{2}\Big[[(a+x)^2+(b+x)^2]\|h(t)\|_p - H(t) + 1\Big].$$

This implies,

$$\left|E(X) + \frac{b-a}{2}H(X) - \frac{b+x}{2}\right|$$
$$\leq \frac{(b-a)}{2}\left[((a+x)^2 + (b+x)^2)\|h(t)\|_p - H(x) + 1\right] \blacksquare$$

Corollary 3.2.16

If $X = a$ or $X = b$ in $|(b-a)H(X) + E(X) - b| = \left|E(X) - \frac{a+b}{2}\right|$

We have the same upper bound.

Proof:

(1) if $X = a$ in Theorem 3.2.15, we get

$$\left|\frac{b-a}{2}H(a) + E(X) - \frac{b+x}{2}\right|$$
$$\leq \frac{(b-a)}{2}\left[((2a)^2 + (b+a)^2)\|h(t)\|_p - H(t) + 1\right]$$

Since $\leq \frac{(b-a)}{2}\left[((2b)^2 + (b+a)^2)\|h(t)\|_p - H(t) + 1\right]$

$H(a) = 0$, when a is constant then,

(2) if $X = b$ in Theorem 3.2.15, we get

$$\left|(b-a)H(X) + E(X) - \frac{b+x}{2}\right|$$
$$\leq \frac{(b-a)}{2}\left[((2a)^2 + (b+a)^2)\|h(t)\|_p - H(t) + 1\right] \qquad (3.28)$$

From (3.27) and (3.28), we get

$\left|E(X) - \frac{a+b}{2}\right|$ has the same upper bound

when $X = a$ and $X = b$ $\blacksquare$

Corollary 3.2.18

If $X = \frac{a+b}{2}$ in Theorem 3.2.15, then we have the best inequality

$$\left|E(X) + \left(\frac{b-a}{2}\right) P_r\left(X \leq \frac{a+b}{2}\right) - \frac{a+3b}{2}\right| \leq$$

$$\frac{(b-a)}{2}\left[\left(\left(\frac{3a+b}{2}\right)^2 + \left(\frac{3b+a}{2}\right)^2\right) \|h(t)\|_p - P_r\left(X \leq \frac{a+b}{2}\right) + 1\right].$$

Proof:

By using Theorem 3.2.15, when $X = \frac{a+b}{2}$ we get

$$\left|E(X) + \left(\frac{b-a}{2}\right) H\left(\frac{a+b}{2}\right) - \frac{b + \left(\frac{a+b}{2}\right)}{2}\right|$$

$$\leq \frac{(b-a)}{2}\left[\left(\left(a + \frac{a+b}{2}\right)^2 + \left(b + \frac{b+a}{2}\right)^2\right) \|h(t)\|_p - H\left(\frac{a+b}{2}\right) + 1\right],$$

Since,

$$H\left(\frac{a+b}{2}\right) = P_r\left(X \leq \frac{a+b}{2}\right).$$

Then,

$$\left|E(X) + \left(\frac{b-a}{2}\right) P_r\left(X \leq \frac{a+b}{2}\right) - \frac{a+3b}{2}\right|$$

$$\leq \frac{(b-a)}{2}\left[\left(\left(\frac{3a+b}{2}\right)^2 + \left(\frac{3b+a}{2}\right)^2\right)\|h(t)\|_p - P_r\left(X \leq \frac{a+b}{2}\right) + 1\right] \qquad \blacksquare$$

Chapter Four

An $L_p, p < 1$ *Application of A Modified Chebyshev Inequality*

Chapter Four

Application of the Ostrowski inequality

An $L_p, p < 1$ Application of a Modified Chebyshev Inequality

Some authors proved a Chebyshev inequality for absolutely continuous functions. We prove a generalization and refinement of this Chebyshev inequality for functions in the spaces L_p, for $0 < p < 1$. Call it a version of pre- Chebyshev inequality. As an application of pre- Chebyshev inequality are prove fundamental inequalities for Expectation of Cumulative distribution function of random variable with probability density function f, such that $f, \acute{f} \in L_p[a,b], 0 < p < 1, a, b \in \mathbb{R}$.

4.1.Introduction:

In the following type of Chebyshev inequality is well known.

Theorem 4.1.1[50]

Let $f, g: [a,b] \to \mathbb{R}$ be two absolutely continuous mapping on $[a,b]$ whose derivatives $\acute{f}, \acute{g}: [a,b] \to \mathbb{R}$ $b$elong to the Lebesgue space $L_\infty[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)\,dx \frac{1}{b-a}\int_a^b g(x)dx\right|$$

$$\leq \frac{1}{12}(b-a)^2 \left\|\acute{f}\right\|_\infty \|\acute{g}\|_\infty \qquad (4.1)$$

Using the same lines step by step used in [48]and [50] we can prove the following refinement and generalization of Theorem 4.1.2, which is called pre- Chebyshev inequality.

Theorem 4.1.2[50]

If $f, g: [a, b] \rightarrow \mathbb{R}$ are in $L_p[a, b]$, $\acute{f} \in L_p[a, b]$,then

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)\,dx\,\frac{1}{b-a}\int_a^b g(x)dx\right|$$

$$\leq (b-a)\left\|\acute{f}\right\|_p\left|\frac{1}{(b-a)}\int_a^b |g(x)|^2dx - \left(\frac{1}{(b-a)}\int_a^b |g(x)|\,dx\right)^p\right|^{\frac{1}{p}}$$

In this chapter we use same approach, to obtain functional inequalities for expectation and cumulative distribution function of a random variable having probability density function , $f: [a, b] \rightarrow \mathbb{R}$, that is $\acute{f}, f \in L_p[a, b]$, $0 < p < 1$.

4.2. Application of a Modified Chebyshev Inequality

Let us begin our main results with the following theorem.

Theorem 4.2.1

Let X be a random variable having the probability density function $f:[a,b] \to \mathbb{R}$, if $f', f \in L_p[a,b], 0 < p < 1$,then

$$\left|E(X) - \frac{a+b}{2}\right|$$

$$\leq (b-a)^2 \,\|f'\|_p \left|\frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^p\right|^{\frac{1}{p}}$$

Proof:

By using Theorem 4.1.2, put $g(x) = t$,we get

$$\left|\frac{1}{b-a}\int_a^b tf(t)dt - \frac{1}{b-a}\int_a^b f(t)\,dt\,\frac{1}{b-a}\int_a^b tdt\right| \leq$$

$$(b-a)\,\|f'\|_p \left|\frac{1}{(b-a)}\int_a^b |t|^2 dt - \left(\frac{1}{(b-a)}\int_a^b |t|\,dt\right)^p\right|^{\frac{1}{p}}, \qquad (4.2)$$

since

$$\int_a^b th(t)dt = E(X)\,, X \in [a,b]\,, \int_a^b f(t)dt = 1,$$

also

$$\frac{1}{b-a}\int_a^b t dt = \frac{1}{b-a}\frac{b^2-a^2}{2} = \frac{a+b}{2},$$

and

$$\frac{1}{(b-a)}\int_a^b |t|^2 dt = \frac{1}{(b-a)}\frac{b^3-a^3}{3} = \frac{a^2+ab+b^2}{3},$$

where$|t|^p = \begin{cases} t^p, & t \geq 0 \\ -t^p, & t < 0 \end{cases}$'

by using (4.2),we get

$$\left|\frac{1}{b-a}E(X) - \frac{1}{b-a}\left(\frac{a+b}{2}\right)\right|$$

$$\leq (b-a)\,\|\acute{f}\|_p \left|\frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^p\right|^{\frac{1}{p}}.$$

This implies

$$\left|E(X) - \frac{a+b}{2}\right|$$

$$\leq (b-a)^2\,\|\acute{f}\|_p \left|\frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^p\right|^{\frac{1}{p}} \blacksquare$$

Theorem 4.2.2

Let X be a random variable having the probability density function $f:[a,b]\to\mathbb{R}$, If $f',f\in L_p[a,b], 0<p<1$. Then

$$\left|\sigma_\mu^2(X)-\left(\mu-\frac{a+b}{2}\right)^2-\frac{1}{12}(b-a)^2\right|$$

$$\leq (b-a)^2\,\|f'\|_p\left|\left(\frac{a+b}{2}-\mu\right)\right.$$

$$\left.-\left[\left(\mu-\frac{a+b}{2}\right)+\frac{1}{12}(b-a)\right]^p\right|^{\frac{1}{p}}$$

Proof

by using Theorem 4.1.2, put $g(x)=(t-\mu)^2$,we get

$$\left|\frac{1}{b-a}\int_a^b(t-\mu)^2f(t)dt-\frac{1}{b-a}\int_a^b f(t)\,dt\,\frac{1}{b-a}\int_a^b(t-\mu)^2dt\right|\leq$$

$$(b-a)\,\|f'\|_p\left|\frac{1}{(b-a)}\int_a^b|(t-\mu)^2|^2dt\right.$$

$$\left.-\left(\frac{1}{(b-a)}\int_a^b|(t-\mu)^2|\,dt\right)^p\right|^{\frac{1}{p}}. \qquad (4.3)$$

Since,

$$\frac{1}{b-a}\int_a^b (t-\mu)^2 dt = \left(\mu - \frac{a+b}{2}\right) + \frac{1}{12}(b-a)^2,$$

and,

$$\frac{1}{(b-a)}\int_a^b |(t-\mu)^2|^2 dt = \frac{1}{(b-a)}\int_a^b (t-\mu) dt$$

$$= \frac{1}{(b-a)}\left(\frac{b^2-a^2}{2} - \mu(b-a)\right) = \left(\frac{a+b}{2} - \mu\right),$$

also

$$\int_a^b (t-\mu)^2 f(t) dt = \sigma_\mu^2(X), \text{and} \int_a^b f(t)\, dt = 1$$

by using(4.3), we get

$$\left|\frac{1}{b-a}\sigma_\mu^2(X) - \frac{1}{b-a}\left[\left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{12}(b-a)^2\right]\right| \le$$

$$(b-a)\|\acute{f}\|_p \left|\left(\frac{a+b}{2} - \mu\right) - \left[\left(\mu - \frac{a+b}{2}\right) + \frac{1}{12}(b-a)^2\right]^p\right|^{\frac{1}{p}}.$$

This implies

$$\left|\sigma_\mu^2(X) - \left(\mu - \frac{a+b}{2}\right)^2 - \frac{1}{12}(b-a)^2\right|$$

$$\leq (b-a)^2 \left\|\acute{f}\right\|_p \left|\left(\frac{a+b}{2} - \mu\right) - \left[\left(\mu - \frac{a+b}{2}\right) + \frac{1}{12}(b-a)^2\right]^p\right|^{\frac{1}{p}} \qquad \blacksquare$$

Collorary 4.2.3

Let X be a random variable having the probability density function $f: [a,b] \to \mathbb{R}$, If $\acute{f}, f \in L_p[a,b], 0 < p < 1$, then

$$\left|\sigma_\circ^2(X) - \frac{1}{12}(b-a)^2\right| \leq (b-a)^2 \left\|\acute{f}\right\|_p \left|\left[\frac{1}{12}(b-a)^2\right]^p\right|^{\frac{1}{p}}$$

Proof:

By using Theorem 4.2.2, put $\mu = \frac{a+b}{2}$,

where $\sigma_\mu^2(X) = \sigma_{\frac{a+b}{2}}^2(X) = \sigma_\circ^2(X)$, we get

$$\left|\sigma_{\frac{a+b}{2}}^2(X) - \left(\frac{a+b}{2} - \frac{a+b}{2}\right)^2 - \frac{1}{12}(b-a)^2\right|$$

$$\leq (b-a)^2 \left\|\acute{f}\right\|_p \left|\left(\frac{a+b}{2} - \frac{a+b}{2}\right)\right.$$

$$\left. - \left[\left(\frac{a+b}{2} - \frac{a+b}{2}\right) + \frac{1}{12}(b-a)^2\right]^p\right|^{\frac{1}{p}}.$$

This implies

$$\left|\sigma_{\circ}^{2}(X)-\frac{1}{12}(b-a)^{2}\right| \leq (b-a)^{2}\,\|\acute{f}\|_{p}\left|\left[\frac{1}{12}(b-a)^{2}\right]^{p}\right|^{\frac{1}{p}} \qquad \blacksquare$$

Now, we will connects the expectation *E(X)* with the cumulative distribution $F(x)$of a random variable X, when $\acute{f}, f \in L_p[a,b]$, $0 < p < 1$.

To prove our next result, we need the so called Branett and Dragomir inequality [10].

$$(b-a)F(x)+E(X)-b=\int_{a}^{b} K(X,t)\,dF(t)$$

$$=\int_{a}^{b} K(X,t)f(t) \qquad (4.4)$$

Theorem 4.2.4

Let X be a random variable whose probability density function $f:[a,b]\to\mathbb{R}$. If $\acute{f}, f \in L_p[a,b]$, $0<p<1$, and $E(X)$ is the expectation of X,then

$$\left|E(X)+(b-a)F(x)-x-\frac{b-a}{2}\right|$$

$$\leq (b-a)^{2}\,\|\acute{f}\|_{p}\left[\left|\frac{(x-a)^{2}+(b-x)^{2}}{3}\right|+\left|x-\frac{a+b}{2}\right|^{p}\right]^{\frac{1}{p}}$$

Proof:

Let $K(X,t)=\begin{cases}t-a & if\ a\leq t\leq X\leq b\\ t-b & if\ a\leq X\leq t\leq b'\end{cases}$

now by using Theorem 4.1.2, for $g(t)=K(X,t)$, we get

$$\left|\frac{1}{b-a}\int_a^b K(X,t)f(t)dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b K(X,t)dt\right|\leq$$

$$(b-a)\,\|\acute{f}\|_p\left|\frac{1}{(b-a)}\int_a^b |K(X,t)|^2dt-\left(\frac{1}{(b-a)}\int_a^b |K(X,t)|\,dt\right)^p\right|^{\frac{1}{p}}, \qquad (4.5)$$

since,

$$\frac{1}{b-a}\int_a^b K(X,t)dt=\frac{1}{b-a}\left[\int_a^x (t-a)dt+\int_x^b (t-b)dt\right]$$

$$=\frac{1}{b-a}\left[\left(\frac{t^2}{2}-at\right)\Big|_a^x+\left(\frac{t^2}{2}-bt\right)\Big|_x^b\right]$$

$$=\frac{1}{b-a}\left[\left(\frac{X^2}{2}-aX\right)-\left(\frac{a^2}{2}-a^2\right)+\left(\frac{b^2}{2}-b^2\right)-\left(\frac{X^2}{2}-bX\right)\right]$$

$$=\frac{1}{b-a}\left[a^2-ax-\frac{a^2}{2}-b^2+bx+\frac{b^2}{2}\right]$$

$$= \frac{1}{b-a}\left[(b-a)x - \frac{b^2-a^2}{2}\right],$$

this implies,

$$\frac{1}{b-a}\int_a^b K(X,t)dt = \left(x - \frac{a+b}{2}\right),$$

also,

$$\frac{1}{(b-a)}\int_a^b |K(X,t)|^2 dt$$

$$= \frac{1}{b-a}\left[\int_a^x (t-a)^2 dt + \int_x^b (t-b)^2 dt\right]$$

$$= \frac{1}{b-a}\left[\left(\frac{(t-a)^3}{3}\right)\Bigg|_a^x + \left(\frac{(t-b)^3}{3}\right)\Bigg|_x^b\right]$$

$$= \frac{1}{b-a}\left[\frac{(x-a)^3}{3} + \frac{-(x-b)^3}{3}\right]$$

$$= \frac{1}{b-a}\left[\frac{(x-a)^3}{3} + \frac{(b-x)^3}{3}\right],$$

then

$$\frac{1}{(b-a)}\int_a^b |K(X,t)|^2 dt = \frac{(x-a)^3 + (b-x)^3}{3(b-a)},$$

$$\int_a^b f(t)\,dt = 1.$$

By using (4.4) and(4.5), we get

$$\left|E(X) + (b-a)F(x) - b - \left(x - \frac{a+b}{2}\right)\right|$$

$$\leq (b-a)^2 \left\|\acute{f}\right\|_p \left|\frac{(x-a)^3 + (b-x)^3}{3(b-a)} - \left(x - \frac{a+b}{2}\right)^p\right|^{\frac{1}{p}},$$

this implies

$$\left|E(X) + (b-a)F(x) - x - \frac{b-a}{2}\right|$$

$$\leq (b-a)^2 \left\|\acute{f}\right\|_p \left|\frac{(x-a)^3 + (b-x)^3}{3(b-a)} - \left(x - \frac{a+b}{2}\right)^p\right|^{\frac{1}{p}}$$

$$\leq (b-a)^2 \left\|\acute{f}\right\|_p \left[\left|\frac{(x-a)^2 + (b-x)^2}{3}\right| + \left|x - \frac{a+b}{2}\right|^p\right]^{\frac{1}{p}} \quad \blacksquare$$

Corollary 4.2.5

If $x = a$ or $x = b$ in $\left|E(X) + (b-a)F(x) - \frac{2x}{2} - \left(\frac{b-a}{2}\right)\right|$

$$= \left|E(X) - \frac{a+b}{2}\right|.$$

Then $\left|E(X) - \frac{a+b}{2}\right|$ have the same upper bound in Theorem 4.2.4

Proof:

(1) If $x = a$

By using Theorem 4.2.4 , we get

$$\left|E(X) + (b-a)F(a) - a - \frac{b-a}{2}\right|$$

$$\leq (b-a)^2 \left\|\acute{f}\right\|_p \left[\left|\frac{(b-a)^2}{3}\right| + \left|a - \frac{a+b}{2}\right|^p\right]^{\frac{1}{p}}$$

Since $F(a) = 0$, when a is a constant, then

$$\left|E(X) - a - \left(\frac{b-a}{2}\right)\right|$$

$$\leq (b-a)^2 \left\|\acute{f}\right\|_p \left[\left|\frac{(b-a)^2}{3}\right| + \left|a - \frac{a+b}{2}\right|^p\right]^{\frac{1}{p}},$$

this implies

$$\left|E(X) - \frac{a+b}{2}\right|$$

$$\leq (b-a)^2 \left\|\acute{f}\right\|_p \left|\frac{(b-a)^2}{3} - \left(\frac{a-b}{2}\right)^p\right|^{\frac{1}{p}}$$

$$\leq (b-a)^2 \left\|\acute{f}\right\|_p 2^{\frac{1}{p}-1} \left(\frac{(b-a)^{\frac{2}{p}}}{3^{\frac{1}{p}}} + \frac{b-a}{2}\right), n > 1$$

(2) $x = b$

By using Theorem 4.2.4, we get

$$\left|E(X) + (b-a)F(b) - b - \frac{b-a}{2}\right|$$

$$\leq (b-a)^2 \left\|\acute{f}\right\|_p \left[\left|\frac{(b-a)^2}{3}\right| + \left|b - \frac{a+b}{2}\right|^p\right]^{\frac{1}{p}}.$$

Since $F(b) = 0$, when b is a constant, this implies,

$$\left|E(X) - b - \frac{b-a}{2}\right|$$

$$\leq (b-a)^2 \left\|\acute{f}\right\|_p \left[\left|\frac{(b-a)^2}{3}\right| + \left|b - \frac{a+b}{2}\right|^p\right]^{\frac{1}{p}},$$

thus

$$\left|E(X) - \frac{a+b}{2}\right|$$

$$\leq (b-a)^2 \left\|\acute{f}\right\|_p \left|\frac{(b-a)^2}{3} - \left(\frac{b-a}{2}\right)^p\right|^{\frac{1}{p}}$$

$$\leq (b-a)^2 \left\|\acute{f}\right\|_p 2^{\frac{1}{p}-1} \left(\frac{(b-a)^{\frac{2}{p}}}{3^{\frac{1}{p}}} + \frac{b-a}{2}\right), n > 1 \qquad \blacksquare$$

Corollary 4.2.6

If $x=\frac{a+b}{2}$ in Theorem 4.2.4 , then we have the best inequality

$$\left|E(X)+(b-a)P_r(X\leq\frac{a+b}{2})-b\right|$$

$$\leq (b-a)^2\left\|\acute{f}\right\|_p\left[\left|\frac{(b-a)}{6}\right|\right]^{\frac{1}{p}}$$

Proof:

By using Theorem 4.2.4, we get

$$\left|E(X)+(b-a)F\left(\frac{a+b}{2}\right)-\frac{a+b}{2}-\frac{b-a}{2}\right|$$

$$\leq (b-a)^2\left\|\acute{f}\right\|_p\left[\left|\frac{\left(\frac{a+b}{2}-a\right)^2+\left(b-\frac{a+b}{2}\right)^2}{3}\right|\right.$$

$$\left.+\left|\frac{a+b}{2}-\frac{a+b}{2}\right|^p\right]^{\frac{1}{p}}$$

Since $F\left(\frac{a+b}{2}\right)=P_r\left(X\leq\frac{a+b}{2}\right)$,

also $\left(\frac{a+b}{2}-a\right)=\frac{b-a}{2}$ then $\left(\frac{a+b}{2}-a\right)^2=\left(\frac{b-a}{2}\right)^2$

and $\left(b-\frac{a+b}{2}\right)=\frac{a-b}{2}=-\left(\frac{b-a}{2}\right)$,then $\left(b-\frac{a+b}{2}\right)^2=\left(\frac{b-a}{2}\right)^2$,

this implies,

$$\left|E(X)+(b-a)P_r\left(X\le\frac{a+b}{2}\right)-\frac{a+b}{2}-\frac{b-a}{2}\right|$$

$$\le (b-a)^2\,\|\acute{f}\|_p\left[\left|\frac{\left(\frac{b-a}{2}\right)^2+\left(-\left(\frac{b-a}{2}\right)\right)^2}{3}\right|+\left|\frac{a+b}{2}-\frac{a+b}{2}\right|^p\right]^{\frac{1}{p}}$$

$$\le (b-a)^2\,\|\acute{f}\|_p\left[\left|\frac{(b-a)^2}{6}\right|\right]^{\frac{1}{p}}\ .$$

Then, we get

$$\left|E(X)+(b-a)P_r(X\le\frac{a+b}{2})-b\right|$$

$$\le (b-a)^2\,\|\acute{f}\|_p\left[\left|\frac{(b-a)}{6}\right|\right]^{\frac{1}{p}}$$ ■

Theorem 4.2.7

Let X be a random variable with probability density function $f(t)$ and its derivative belong to $L_p[a,b], 0 < p < 1$ .Then,

$$\left|E(X) + \frac{(b-a)}{2}F(x) - \frac{b+x}{2}\right|$$

$$\leq (b-a)^2 \, \|\acute{f}\|_p \left[\left|\frac{x^2+ax+a^2}{3} - \left(\frac{x-a}{2}\right)^p\right|^{\frac{1}{p}} + \left|\frac{x^2+bx+b^2}{3} - \left(\frac{x-b}{2}\right)^p\right|^{\frac{1}{p}} \right]$$

Proof:

By using (4.4), we get

$$(b-a)F(x) + E(X) - b$$

$$= \int_a^x (t-a)dF(t) + \int_x^b (t-a)dF(t)$$

$$= \int_a^x (t-a)f(t)dt + \int_x^b (t-b)f(t)dt \qquad (4.6)$$

Now, by using Theorem 4.1.2, put $g(x) = t - a$, we get

$$\left|\frac{1}{x-a}\int_a^x (t-a)f(t)dt - \frac{1}{x-a}\int_a^x f(t)\,dt\,\frac{1}{x-a}\int_a^x (t-a)dt\right|$$

$$\leq (b-a)\,\|\acute{f}\|_p \left|\frac{1}{x-a}\int_a^x |(t-a)|^2 dt\right.$$

$$\left. -\left(\frac{1}{(x-a)}\int_a^x |(t-a)|\,dx\right)^p\right|^{\frac{1}{p}}. \qquad (4.7)$$

Since

$\int_a^x f(t)dt = F(x)$, and

$$\frac{1}{(x-a)}\int_a^x (t-a)dt = \frac{x-a}{2},$$

$$\frac{1}{(x-a)}\int_a^t |(t-a)|^2 dt = \frac{x^2+ax+a^2}{3},$$

then (4.7) become,

$$\left|\frac{1}{x-a}\int_a^x (t-a)f(t)dt - \frac{F(x)}{2}\right| \leq$$

$$(b-a)\,\|\acute{f}\|_p \left|\frac{x^2+ax+a^2}{3} - \left(\frac{x-a}{2}\right)^p\right|^{\frac{1}{p}}$$

$$\left|\int_a^x (t-a)f(t)dt - \frac{x-a}{2}F(x)\right| \leq$$

$$(b-a)(x-a)\,\|\acute{f}\|_p \left|\frac{x^2+ax+a^2}{3} - \left(\frac{x-a}{2}\right)^p\right|^{\frac{1}{p}} \qquad (4.8)$$

Similarly, by using Theorem 4.1.2, put $g(x) = t - b$, we get,

$$\left|\frac{1}{b-x}\int_x^b (t-b)f(t)dt - \frac{1}{b-x}\int_x^b f(t)\,dt\,\frac{1}{b-x}\int_x^b (t-b)dt\right| \leq$$

$$(b-a)\,\|\acute{f}\|_p \left|\frac{1}{b-x}\int_x^b |(t-b)|^2 dt\right.$$

$$\left. - \left(\frac{1}{(b-x)}\int_x^b |(t-b)|\,dt\right)^p\right|^{\frac{1}{p}} \qquad (4.9)$$

since $\int_x^b f(t)dt = (F(x))^c = 1 - F(x)$, and

$$\frac{1}{(b-x)}\int_x^b (t-b)dt = \frac{x-b}{2} \quad ,$$

$$\frac{1}{(b-x)}\int_x^b |(t-b)|^2 dt = \frac{x^2+bx+b^2}{3},$$

then (4.9) become,

$$\left|\frac{1}{b-x}\int_x^b (t-b)f(t)dt - \frac{x-b}{2}\frac{1}{b-x}(1-F(x))\right| \le$$

$$(b-a)\,\|\acute{f}\|_p\left|\frac{x^2+bx+b^2}{3} - \left(\frac{x-b}{2}\right)^p\right|^{\frac{1}{p}}.$$

we obtain,

$$\left|\frac{1}{b-x}\int_x^b (t-b)f(t)dt + \frac{1}{2}(1-F(x))\right| \le$$

$$(b-a)\,\|\acute{f}\|_p\left|\frac{x^2+bx+b^2}{3} - \left(\frac{x-b}{2}\right)^p\right|^{\frac{1}{p}} \quad .$$

This implies

$$\left|\int_x^b (t-b)f(t)dt + \frac{b-x}{2} - \frac{b-x}{2}F(x)\right| \le$$

$$(b-a)(b-x)\|\acute{f}\|_p\left|\frac{x^2+bx+b^2}{3} - \left(\frac{x-b}{2}\right)^p\right|^{\frac{1}{p}} \qquad (4.10)$$

From (4.8) and (4.10), we get

$$\left|\int_a^x (t-a)f(t)dt + \int_x^b (t-b)f(t)dt - \frac{x-b}{2} - \frac{x-a}{2}F(x) + \frac{x-b}{2}F(x)\right| \le$$

$$(b-a)(x-a)\ \|\acute{f}\|_p \left|\frac{x^2+ax+a^2}{3} - \left(\frac{x-a}{2}\right)^p\right|^{\frac{1}{p}}$$

$$+(b-a)(b-x)\|\acute{f}\|_p \left|\frac{x^2+bx+b^2}{3} - \left(\frac{x-b}{2}\right)^p\right|^{\frac{1}{p}},$$

this implies

$$\left|E(X) + \frac{(b-a)}{2}F(x) - \frac{a+b}{2}\right|$$

$$\le (b-a)\ \|\acute{f}\|_p \left[(x-a)\left|\frac{x^2+ax+a^2}{3} - \left(\frac{x-a}{2}\right)^p\right|^{\frac{1}{p}} + (b-x)\left|\frac{x^2+bx+b^2}{3} - \left(\frac{x-b}{2}\right)^p\right|^{\frac{1}{p}}\right] \qquad \blacksquare$$

Chapter Five

Approximation of Expectation and Variance on $[a, b]$ Interval, with Probability Density Function in $L_P[a, b], 0 < p < 1$.

Approximation of Expectation and Variance on $[a, b]$Interval, with Probability Density Function in $L_p[a, b], 0 < p < 1$

In this chapter we use Taylor׳s formula to approximate expression in terms of expectation and variance simultaneously with probability density function in $L_p, 0 < p < 1$.

5.1.Introduction

If X is a random variable,have probability density function

$f: [a, b] \to \mathbb{R}$. We know that the expectation of the random variable X is

$$E(X) = \int_{\infty} tf(t)dt.$$

Therefore, the variance of the random vaiable X is

$$\sigma^2 = \int_a^b \big(t - E(X)\big)^2 f(t)dt = E(X^2) - (E(X))^2$$

In the previous chapters we prove types pre-Gruss inequality and pre-chebychev inequality in terms of measurable probability density functions. In this chapter, we use these inequalities to approximate (estimate) expectation and variance with measurable probability density functions, in the aid of Taylor׳s formula.

5.2. Expectation and Variance with probability density function in $L_p, 0 < p < 1$

To prove our main theorem we need the following auxiliary Lemmas

Lemma 5.2.1:[51]

$$\int_a^b (b-t)\,(t-a)f(t)dt = |b-E(X)||E(X)-a| - \sigma^2(X), t \in [a,b]$$

Lemma 5.2.2.[52]

If $p < q$, then for$x_i \in R5$

$$\left(\sum_{i=1}^{\infty} |x_i|^q\right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}.$$

Now let us introduce our main Theorem.

Theorem 5.2.3

Let X be a random variable defined on $[a,b]$ with the probability density function $f: [a,b] \to \mathbb{R}$ belongs to $L_p[a,b], p < 1$.Then we have,

$$|b-E(X)||E(X)-a| - \sigma^2(X) \leq c(b-a)^{2+p-\frac{1}{p}}\|f\|_p,$$

$0 < p < 1$,where c is a positive constant.

Proof

Since

$$\int_a^b (b-t)\,(t-a)f(t)dt \le \int_a^b (b-a)^2\,|f(t)|dt$$

$$= (b-a)^2 \int_a^b |f(t)|dt, \qquad (5.1)$$

Now

Let $t_1 < t_2 < t_3 < \cdots < t_n$ be a partition for $[a,b], a = t_\circ$, $\Delta = \frac{b-a}{n}$,

$$t_1 = a + \frac{b-a}{n}, t_2 = a + \frac{2(b-a)}{n}, \dots, t_n = a + \frac{i(b-a)}{n}.$$

This implies,

$$\int_a^b f(t)dt \cong \sum_{i=1}^{n} f(t_i)\frac{b-a}{n}, [54] \qquad (5.2)$$

If $p < 1$, then by using Lemma 5.2.2 and (5.2), we get

$$\int_a^b f(t)dt \le \left(\int_a^b |f(t)|^{\frac{1}{p}}\right)^p \le c\left(\sum_{i=1}^{n} |f(t_i)|^{\frac{1}{p}} \frac{b-a}{n}\right)^p$$

By using Holder inequality when $q > 1, k > 1$ and $\frac{1}{q} + \frac{1}{k} = 1$, we get

$$\int_a^b f(t)dt \le c(b-a)^p \left(\left(\sum_{i=1}^{n} |f(t_i)|^{\frac{q}{p}} \right)^{\frac{p}{q}} \left(\sum_{i=1}^{n} \left|\frac{1}{n}\right|^{k} \right)^{\frac{p}{k}} \right)$$

$$\le c(b-a)^p \left(\left(\sum_{i=1}^{n} |f(t_i)|^{\frac{q}{p}} \right)^{\frac{p}{q}} \sum_{i=1}^{n} \frac{1}{n^k} \right), k > 1$$

$$\le c(b-a)^p \left(\left(\sum_{i=1}^{n} |f(t_i)|^{p} \right)^{\frac{1}{p}} \frac{1}{n^{k-1}} \right)$$

Assume $\frac{1}{p} = k - 1$, then

$$\int_a^b f(t)dt \le \frac{c(b-a)^p}{(b-a)^{\frac{1}{p}}} \left(\left(\sum_{i=1}^{n} |f(t_i)|^{p} \frac{(b-a)}{n} \right)^{\frac{1}{p}} \right)$$

$$\le \frac{c(b-a)^p}{(b-a)^{\frac{1}{p}}} \left(\left(\int_a^b |f(t_i)|^{p} \right)^{\frac{1}{p}} \right)$$

$$\le \frac{c(b-a)^p}{(b-a)^{\frac{1}{p}}} \|f\|_p \, . \tag{5.3}$$

Thus

$$\int_a^b (b-t)\,(t-a) f(t) dt \le c(b-a)^{2+p-\frac{1}{p}} \, \|f\|_p$$

Then by repeating of using of (5.2), we get,

$$\int_a^b (b-t)\,(t-a)f(t)dt \leq (b-a)^{2+p-\frac{1}{p}}\|f\|_{p}.$$

Then by using Lemma 5.2.1 , we get,

$$|b-E(X)||E(X)-a|-\sigma^2(X) \leq (b-a)^{2+p-\frac{1}{p}}\ \|f\|_p,$$

$0 < p < 1$ ■

Theorem 5.2.4

Let X be a random variable defined in [a, b] with the probability density function $f:[a,b] \to \mathbb{R}$ belongs to $L_p[a,b], p < 1$.Then we have,

$$\left||b-E(X)||E(X)-a|-\sigma^2(X)-\frac{(b-a)^3}{6}\right|$$
$$\leq \frac{(b-a)}{2}\left((a+x)^2+(b+x)^2\|f(t)\|_p\right)+\frac{(b-a)^3}{6}$$

Proof

Recall pre-Gruss inequality when $0 < p < 1$

$$\left|\frac{1}{b-a}\int_a^b h(t)g(x,t)dt - \int_a^b h(t)\,dt\,\frac{1}{b-a}\int_a^b g(x,t)dt\right|$$

$$\leq \frac{1}{2}\{(a+x)^2 + (b+x)^2\|h(t)\|_p\} + \left|\int_a^b h(t)dt\,\frac{1}{b-a}\int_a^b g(x,t)dt\right| \quad (5.4)$$

Put $h(t) = f(t), g(x,t) = (b-t)(t-a)$, in (5.4), we get.

$$\left|\frac{1}{b-a}\int_a^b f(t)(b-t)(t-a)dt - \int_a^b f(t)\,dt\,\frac{1}{b-a}\int_a^b (b-t)(t-a)dt\right|$$

$$\leq \frac{1}{2}\left((a+x)^2 + (b+x)^2\|f(t)\|_p\right)$$

$$+\left|\int_a^b f(t)dt\,\frac{1}{b-a}\int_a^b (b-t)(t-a)dt\right|. \quad (5.5)$$

Now let us compute $\int_a^b (b-t)\,(t-a)dt$

$$\int_a^b (b-t)\,(t-a)dt$$

$$= \int_a^b (bt - ab - t^2 + at)dt = \frac{bt^2}{2} - abt - \frac{t^3}{3} + \frac{at^2}{2}\Big|_a^b$$

$$= \left(\frac{b^3}{2} - ab^2 - \frac{b^3}{3} + \frac{ab^2}{2}\right) - \left(\frac{a^2b}{2} - ba^2 - \frac{a^3}{3} + \frac{a^3}{2}\right)$$

$$=\left(\frac{3b^3-6ab^2-2b^3+3ab^2}{6}\right)-\left(\frac{3a^2b-6ba^2-2a^3+3a^3}{6}\right)$$

$$=\frac{\left(b^3-3ab^2\right)-\left(a^3-3ba^2\right)}{6}=\frac{b^3-3ab^2+3ba^2-a^3}{6}=\frac{(b-a)^3}{6}. \qquad (5.6)$$

Then using (5.6) to complete our estimate in (5.5). Also we have $\int_a^b f(t)dt=1$, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(t)(b-t)(t-a)dt-\frac{(b-a)^3}{6}\right|$$

$$\leq\frac{1}{2}\left((a+x)^2+(b+x)^2\|f(t)\|_p\right)+\frac{(b-a)^2}{6}.$$

Then,

$$\left|\int_a^b f(t)(b-t)(t-a)dt-\frac{(b-a)^3}{6}\right|$$

$$\leq\frac{(b-a)}{2}\left((a+x)^2+(b+x)^2\|f(t)\|_p\right)+\frac{(b-a)^3}{6},$$

now by using Lemma 5.2.1, we get,

$$\left||b-E(X)||E(X)-a|-\sigma^2(X)-\frac{(b-a)^3}{6}\right| \leq$$

$$\frac{(b-a)}{2}\left[((a+x)^2+(b+x)^2)\|f(t)\|_p\right]$$
$$+\frac{(b-a)^3}{6} \qquad \blacksquare$$

Corollary 5.2.5

$$\left||b-E(X)||E(X)-a|-\sigma^2(X)-\frac{(b-a)^3}{6}\right|$$

$$\leq (b+a)^3\|f\|_p+(b+a)^3$$

$$\leq (b+a)^3\left(1+\|f\|_p\right)$$

Proof:

By using Theorem 5.2.4, we get

$$\left||b-E(X)||E(X)-a|-\sigma^2(X)-\frac{(b-a)^3}{6}\right|$$

$$\leq \frac{(b-a)}{2}\left[((a+x)^2+(b+x)^2)\|f(t)\|_p\right]+\frac{(b-a)^3}{6}$$
$$\leq (b+a)^3\left(1+\|f\|_p\right) \qquad \blacksquare$$

Theorem 5.2.6

Let X be a random variable defined in $[a, b]$ with the probability density function $f:[a,b] \rightarrow \mathbb{R}$.If $f, \acute{f} \in L_p[a,b], p < 1$,then we have,

$$|b - E(X)||E(X) - a| - \sigma^2(X) \leq 2^{\frac{1}{p}+3} b^{\frac{4}{p}} \|\acute{f}\|_p, 0 < p < 1$$

Proof:

Recall the Pre-Chebychev inequality

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)\, dx \frac{1}{b-a} \int_a^b g(x)dx \right| \leq$$

$$(b-a) \|\acute{f}\|_p \left| \frac{1}{(b-a)} \int_a^b |g(x)|^2 dx - \left(\frac{1}{(b-a)} \int_a^b |g(x)|\, dx \right)^p \right|^{\frac{1}{p}}. \quad (5.7)$$

In (5.7) put $g(x) = (t-a)(b-t)$, we get

$$\left|\frac{1}{b-a}\int_a^b f(t)(t-a)(b-t)dt - \frac{1}{b-a}\int_a^b f(t)\,dt\,\frac{1}{b-a}\int_a^b (t-a)(b-t)dt\right|$$

$$\leq (b-a)\,\|\acute{f}\|_p \left|\frac{1}{(b-a)}\int_a^b |(t-a)(b-t)|^2 dt - \left(\frac{1}{(b-a)}\int_a^b |(t-a)(b-t)|\,dt\right)^p\right|^{\frac{1}{p}} \qquad (5.8)$$

Since

$$\int_a^b |(t-a)(b-t)|\,dt = \frac{(b-a)^3}{6}, \int_a^b f(t)dt = 1.$$

And

$$\int_a^b |(t-a)(b-t)|^2 dt = \int_a^b (t-a)^2(b-t)^2 dt = \frac{(b-a)^5}{30}.$$

Then (5.8) become,

$$\left|\frac{1}{b-a}\int_a^b f(t)(t-a)(b-t)dt-\frac{(b-a)}{6}\right|\leq$$

$$(b-a)\,\|\acute{f}\|_p\left|\frac{1}{(b-a)}\int_a^b\frac{(b-a)^5}{30}dt-\left(\frac{(b-a)^2}{6}\right)^p\right|^{\frac{1}{p}}.$$

Then,

$$\left|\int_a^b f(t)(t-a)(b-t)dt-\frac{(b-a)^2}{6}\right|\leq$$

$$(b-a)^2\,\|\acute{f}\|_p\left|\left[\frac{(b-a)^4}{30}-\left(\frac{(b-a)^2}{6}\right)^p\right]\right|^{\frac{1}{p}}.$$

By using Lemma 5.2.1, we get

$$\left||b-E(X)||E(X)-a|-\sigma^2(X)-\frac{(b-a)^2}{6}\right|\leq$$

$$(b-a)^2\,\|\acute{f}\|_p\left|\left[\frac{(b-a)^4}{30}-\left(\frac{(b-a)^2}{6}\right)^p\right]\right|^{\frac{1}{p}},$$

this implies,

$$\left| |b-E(X)||E(X)-a| - \sigma^2(X) - \frac{(b-a)^2}{6} \right| \le$$

$$(b-a)^2 \left\|\acute{f}\right\|_p 2^{\frac{1}{p}-1} \left(\left(\frac{(b-a)^4}{30} \right)^{\frac{1}{p}} + \frac{(b-a)^2}{6} \right)$$

$$\le (b-a)^2 \left\|\acute{f}\right\|_p 2^{\frac{1}{p}-1} \left(\frac{2^{\frac{4}{p}-1}\left(b^{\frac{4}{p}} + a^{\frac{4}{p}}\right)}{30^{\frac{1}{p}}} + \frac{2}{6}(b^4 + a^4) \right)$$

$$\le (b-a)^2 \left\|\acute{f}\right\|_p 2^{\frac{1}{p}-1} \left(b^{\frac{4}{p}} + a^{\frac{4}{p}} + b^4 + a^4 \right)$$

$$\le (\mathrm{b}-\mathrm{a})^2 \left\|\acute{\mathrm{f}}\right\|_\mathrm{p} 2^{\frac{1}{\mathrm{p}}-1} \left(\mathrm{b}^{\frac{4}{\mathrm{p}}} + \mathrm{a}^{\frac{4}{\mathrm{p}}} + \mathrm{b}^{\frac{4}{\mathrm{p}}} + \mathrm{a}^{\frac{4}{\mathrm{p}}} \right)$$

$$\le (b-a)^2 \left\|\acute{f}\right\|_p 2^{\frac{1}{p}} \left(b^{\frac{4}{p}} + a^{\frac{4}{p}} \right)$$

$$\le 2(b^2 + a^2) \left\|\acute{f}\right\|_p 2^{\frac{1}{p}} \left(b^{\frac{4}{p}} + a^{\frac{4}{p}} \right)$$

$$\le \ 2^{\frac{1}{p}+2} \left(b^{\frac{4}{p}} + a^{\frac{4}{p}} \right) \left\|\acute{f}\right\|_p$$

$$\le \ 2^{\frac{1}{p}+3} b^{\frac{4}{p}} \left\|\acute{f}\right\|_p.$$

Then ,we get

$$\left| |b - E(X)||E(X) - a| - \sigma^2(X) - \frac{(b-a)^2}{6} \right| \leq 2^{\frac{1}{p}+3} b^{\frac{4}{p}} \|\acute{f}\|_p \quad \blacksquare$$

Lemma 5.2.7[52]

If $g, h, \acute{h} \in L_1[a,b]$, then

$$\left| \frac{1}{b-a} \int_a^b h(t)g(t)dt - \frac{1}{b-a} \int_a^b h(t)\,dt \frac{1}{b-a} \int_a^b g(t)dt \right|^2 \leq$$

$$\left(\frac{b-a}{\pi}\right)^2 \int_a^b |\acute{h}|^2 \left[\frac{1}{b-a} \int_a^b g(t)^2 dt - \left(\frac{1}{b-a} \int_a^b g(t)\,dt \right)^2 \right].$$

Is Pre-Lupas inequality, when g, h and $\acute{h} \in L_2[a,b]$.

Now let us generalize Pre-Lupas inequality for $L_p[a,b], p < 1$ spaces.

Theorem 5.2.8

If $g, h, \acute{h} \in L_p[a,b], 0 < p < 1$, then

$$\left|\frac{1}{b-a}\int_a^b h(t)g(t)dt - \frac{1}{b-a}\int_a^b h(t)\,dt\,\frac{1}{b-a}\int_a^b g(t)dt\right| \le$$

$$\frac{c(b-a)^{p-\frac{1}{p}}}{\pi}\left\|\acute{h}\right\|_p\left(\frac{1}{b-a}\int_a^b g(t)^2dt - \int_a^b g(t)dt\right)$$

Where c is a positive constant.

Proof:

By using Lemma 5.2.7 and (5.2) we get,

$$\left|\frac{1}{b-a}\int_a^b h(t)g(t)dt - \frac{1}{b-a}\int_a^b h(t)\,dt\,\frac{1}{b-a}\int_a^b g(t)dt\right| \le$$

$$\frac{c(b-a)}{\pi}\left(\sum_{i=1}^{n}\frac{b-a}{n}\left|\acute{h}(t_i)\right|^2\right)^{\frac{1}{2}}\left(\frac{1}{b-a}\int_a^b g^2(t)dt - \int_a^b g(t)dt\right)$$

Then using (5.3) secondly, we get

$$\left|\frac{1}{b-a}\int_a^b h(t)g(t)dt - \frac{1}{b-a}\int_a^b h(t)\,dt\,\frac{1}{b-a}\int_a^b g(t)dt\right| \le$$

$$\frac{c(b-a)^{p-\frac{1}{p}}}{\pi}\left\|\acute{h}\right\|_p\left(\frac{1}{b-a}\int_a^b g(t)^2dt - \int_a^b g(t)dt\right) \qquad \blacksquare$$

Collorally 5.2.9

$$\left||b-E(X)||E(X)-a| - \sigma^2(X) - \frac{(b-a)^2}{6}\right|$$

$$\le \frac{c(b-a)^{4+p-\frac{1}{p}}}{6\pi}\left(\frac{b-a}{5}-1\right)\left\|\acute{f}\right\|_p ,$$

where c is a positive constant.

Proof:

Put $h(t) = f(t), g(x) = (t-a)(b-t)$ in Theorem 5.2.8, we get

$$\left|\frac{1}{b-a}\int_a^b f(t)(t-a)(b-t)dt - \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b (t-a)(b-t)dt\right| \le$$

$$\frac{c(b-a)^{p-\frac{1}{p}}}{\pi}\|\acute{f}\|_p \left(\frac{1}{b-a}\int_a^b [(t-a)(b-t)]^2 dt - \int_a^b (t-a)(b-t)dt\right)$$

$$\left|\int_a^b (t-a)(b-t)f(t)dt - \int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b (t-a)(b-t)dt\right| \le$$

$$\frac{c(b-a)^{p-\frac{1}{p}+1}}{\pi}\|\acute{f}\|_p \left(\frac{1}{b-a}\int_a^b [(t-a)(b-t)]^2 dt - \int_a^b (t-a)(b-t)dt\right) \tag{5.9}$$

By using Lemma 5.2.1, we get

$$\int_a^b (b-t)\,(t-a)f(t)dt = |b-E(X)||E(X)-a| - \sigma^2(X),$$

Since

$\int_a^b (t-a)^2(b-t)^2\,dt = \frac{(b-a)^5}{30}$ and $\int_a^b f(t)dt = 1.$

Also,

$$\int_a^b (b-t)\,(t-a)dt = \frac{(b-a)^3}{6},$$

So, (5.9) implies,

$$\left| |b-E(X)||E(X)-a| - \sigma^2(X) - \frac{(b-a)^2}{6} \right|$$

$$\leq \frac{c(b-a)^{p-\frac{1}{p}+1}}{\pi} \left\| \acute{f} \right\|_p \left(\frac{(b-a)^4}{30} - \frac{(b-a)^3}{6} \right).$$

Then, we get,

$$\left| |\mathrm{b}-\mathrm{E}(\mathrm{X})||\mathrm{E}(\mathrm{X})-\mathrm{a}| - \sigma^2(\mathrm{X}) - \frac{(b-a)^2}{6} \right|$$

$$\leq \frac{c(b-a)^{4+p-\frac{1}{p}}}{6\pi} \left(\frac{b-a}{5} - 1 \right) \left\| \acute{f} \right\|_p ,$$

where c is a positive constant ■

Theorem 5.2.10

Let X be a random variable with the probability density function $f:[a,b] \to \mathbb{R}$.If $f \in L_p^n[a,b] = \{f:[a,b] \to \mathbb{R}, f, f^{(n)} \in L_p^n[a,b], 0 < p < 1\}$,then we have

$$\left| |b - E(X)||E(X) - a| - \sigma^2(X) - \sum_{i=0}^{n} \frac{(i+1)(b-a)^{i+1} f^{(i)}(a)}{(i+3)!} \right|$$

$$\leq \frac{c}{n!} \left\| f^{(n+1)} \right\|_p \frac{(b-a)^{(n+1)p+3}}{(np+2)(np+3)},$$

where c is an absolute constant.

Proof:

The Taylor׳s formula with integral remainder [13] is

$$f(t) = \sum_{i=0}^{n} \frac{(t-a)^n}{i!} f^{(i)}(a) + \frac{1}{n!} \int_a^t (t-s)^n f^{(n+1)}(s)\, ds$$

$t \in [a,b]$ (5.10)

By using Lemma 5.2.1 and (5.10), we have

$|b - E(X)||E(X) - a| - \sigma^2(X)$

$$= \int_a^b (b-t)\,(t-a)\left[\sum_{i=0}^{n} \frac{(t-a)^n}{i!} f^{(i)}(a) + \frac{1}{n!}\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right] dt,$$

$$= \sum_{i=0}^{n} \frac{(t-a)^i}{i!} f^{(i)}(a) \int_a^b (b-t)\,(t-a)dt + \frac{1}{n!}\left[\int_a^b (b-t)\,(t-a)\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right] dt \qquad (5.11)$$

Using the transformation $t = (1-u)a + ub,\ t \in [a,b]$

If $t = a$, then

$a = (1-u)a + ub$

$a = a - au + ub$

$0 = u(b-a)$.

This implies, $u = 0$ where $t = a$.

Similarly,

If $t = b$, we obtain $u = 1$. Also, $dt = (b-a)du$.

Now

$$\int_a^b (t-a)^{i+1}\,(b-t)dt = (b-a)^{i+3}\int_0^1 u^{i+1}(1-u)\,du$$

$$= \frac{1}{(i+2)(i+3)} \qquad (5.12)$$

By using (5.11), we deduce that,

$$|b-E(X)||E(X)-a|-\sigma^2(X)$$

$$=\sum_{i=0}^{n}\frac{1}{(i+2)(i+3)}\frac{(b-a)^{i+1}f^{(i)}(a)}{i!}+\frac{1}{n!}\left[\int_a^b (b-t)\,(t-a)\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right]dt$$

This implies,

$$\left||b-E(X)||E(X)-a|-\sigma^2(X)-\sum_{i=0}^{n}\frac{(i+1)(b-a)^{i+1}f^{(i)}(a)}{(i+3)!}\right|$$

$$\leq \frac{1}{n!}\int_a^b (b-t)\,(t-a)\left|\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right|dt \qquad (5.13)$$

Since

$$\left|\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right| \leq \int_a^t |t-s|^n\left|f^{(n+1)}(s)\right|ds.$$

Then by using (5.3),this implies,

$$\left|\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right|$$

$$\leq c(b-a)^{p-\frac{1}{p}}\left(\int_a^t |t-s|^{pn}\left|f^{(n+1)}(s)\right|^p ds\right)^{\frac{1}{p}},$$

Then , we get,

$$\left|\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right| \leq c(b-a)^{p-\frac{1}{p}}|t-a|^{pn}\left\|f^{(n+1)}\right\|_p,$$

$0 < p < 1.$ (5.14)

Put (5.14) in (5.13), we get

$$\left||b-E(X)||E(X)-a| - \sigma^2(X) - \sum_{i=0}^{n}\frac{(i+1)(b-a)^{i+1}f^{(i)}(a)}{(i+3)!}\right|$$

$$\leq \frac{c(b-a)^{p-\frac{1}{p}}}{n!}\left\|f^{(n+1)}\right\|_p \int_a^b (b-t)\,(t-a)^{pn+1}dt. \qquad (5.15)$$

And by using (5.12), we get

$$\left||b-E(X)||E(X)-a| - \sigma^2(X) - \sum_{i=0}^{n}\frac{(i+1)(b-a)^{i+1}f^{(i)}(a)}{(i+3)!}\right|$$

$$\leq \frac{c(b-a)^{p-\frac{1}{p}}}{n!}\left\|f^{(n+1)}\right\|_P (b-a)^{np+3}\int_0^1 u^{np+1}(1-u)\,du$$

$$\leq \frac{c(b-a)^{p-\frac{1}{p}}}{n!}\left\|f^{(n+1)}\right\|_p \frac{(b-a)^{np+3}}{(nP+2)(nP+3)}$$

$$= \frac{c}{n!}\left\|f^{(n+1)}\right\|_p \frac{(b-a)^{(n+1)p+3-\frac{1}{p}}}{(nP+2)(nP+3)}$$

This implies,

$$\left| |b-E(X)||E(X)-a| - \sigma^2(X) - \sum_{i=0}^{n} \frac{(i+1)(b-a)^{i+1} f^{(i)}(a)}{(i+3)!} \right|$$

$$\leq \frac{c}{n!}\left\|f^{(n+1)}\right\|_p \frac{(b-a)^{(n+1)p+3-\frac{1}{p}}}{(nP+2)(nP+3)}$$

$$\leq \frac{c}{n!}\left\|f^{(n+1)}\right\|_p \frac{(b-a)^{(n+1)p+3}}{(nP+2)(nP+3)}$$

where c is absolute constant. ■

Chapter Six

A Modified Ostrowski Inequality with Random Variable Application on $L_p[a,b], 0 < p < 1$, Spaces

A Modified Ostrowski Inequality with Random Variable Application on $L_p[a,b], 0 < p < 1$,Spaces

Many outhers proved types of Ostrowski inequality. We improve their inequalities and then applied it to cumulative density function and ***b*eta** and normal distribution:

6.1. Introduction

In 1938 Ostrowski introduced his famous inequality in [41]

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \left(\frac{x - {(a+b)}/{2}}{b-a}\right)^2\right](b-a)M \qquad (6.1)$$

For a differentiable function f with bounded derivative on (a,b).

In [44] Drangomir and Wang proved the of Ostrowski's inequality

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\right| \leq \frac{1}{4}(b-a)(C-D) \qquad (6.2)$$

For a function with bounded derivative, such that $C \leq \acute{f} \leq D$ on (a,b). C, D are positive constants in $\mathbb{R}$.

In [48] Matid, Decarce and Ujevic proved (6.2) with $\frac{1}{4\sqrt{3}}$ constant with twice diffentiable formula on (a, b).Using Chebychev's operator we improve the results in[11], by introducing a best result for functions in $L_p[a, b]$ for $0 < p < 1$. And we assume that $\acute{f}$ is also in $L_p[a, b]$. Then we applied our inequality to $beta$ and normal distribution and cumulative density function.

6.2. A Modified Ostrowski Inequality

To prove our main theorem we need the following auxiliary result.

Lemma 6.2.1[54]

$$f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)$$

$$=\frac{1}{2(b-a)^2}\int_a^b\int_a^b (P(x,t)-P(x,s))\big(f'(t)-f'(s)\big)dtds\,,$$

where

$$p(x,t)=\begin{cases} t-a & if \quad t\in[a,x) \\ t-b & if \quad t\in(x,b]\end{cases}$$

Theorem 6.2.2

Let X be a random variable $f, f' \in L_p[a,b]$, $0<p<1$, then,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|$$

$$\le\left(\frac{c(p)(b-a)^{p-1}}{2(p+1)}\right)\left[\|f'\|_p+\left(\left(\frac{f(b)-f(a)}{b-a}\right)^p\right)^{\frac{1}{p}}\right],$$

Proof:

By using the Gaushy-Schwars inequality for double integrals, we get,

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))\big(\acute{f}(t)-\acute{f}(s)\big)dtds$$

$$\leq\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))dtds\right)^{\frac{1}{2}}\circ$$

$$\left(\int_a^b\int_a^b (\acute{f}(t)-\acute{f}(s))dtds\right)^{\frac{1}{2}},$$

Since any two norms on a finite dimensional space are equivalent, then we get,

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))\big(\acute{f}(t)-\acute{f}(s)\big)dtds$$

$$\le\left(\frac{c}{2(b-a)^2}\int_a^b\int_a^b |p(x,t)-p(x,s)|^p\,dtds\right)^{\frac{1}{p}}\circ$$

$$\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left|\acute{f}(t)-\acute{f}(s)\right|^2 dtds\right)^{\frac{1}{2}}, \tag{6.3}$$

where c is a positive constant.

Now, using (5.3), we get

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))\big(\acute{f}(t)-\acute{f}(s)\big)dtds$$

$$\le\left(\frac{c(p)}{2(b-a)^2}\int_a^b\int_a^b |p(x,t)-p(x,s)|^p\,dtds\right)^{\frac{1}{p}}\circ$$

$$\left(\frac{c(p)}{2(b-a)^2}\int_a^b\int_a^b \left|\acute{f}(t)-\acute{f}(s)\right|^p dtds\right)^{\frac{1}{p}}$$

$$= \mathrm{I}_1 \circ \mathrm{I}_2. \tag{6.4}$$

Now,

$$I_1 = \left(\frac{cc(p)}{2(b-a)^2} \int_a^b \int_a^b |p(x,t) - p(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq \left[\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |p(x,t)|^P + |p(x,s)|^P \, dtds \right]^{\frac{1}{p}}.$$

This implies,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |p(x,t) - p(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq \left[\frac{c(p)}{2(b-a)^2} \left(\int_a^b \int_a^b |p(x,t)|^p \, dtds + \int_a^b \int_a^b |p(x,s)|^p \, dtds \right) \right]^{\frac{1}{p}}.$$

Then,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |p(x,t) - p(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq \left[\frac{c(p)}{2(b-a)^2} \left((b-a) \int_a^b |p(x,t)|^p \, dt + (b-a) \int_a^b |p(x,s)|^p \, ds \right) \right]^{\frac{1}{p}}.$$

This implies,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |p(x,t) - p(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq \left[\frac{c(p)}{2(b-a)} \left(\int_a^b |p(x,t)|^p \, dt + \int_a^b |p(x,s)|^p \, ds \right) \right]^{\frac{1}{p}}.$$

By using assumption in Lemma 6.2.1, we get,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |p(x,t) - p(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq \left[\frac{c(p)}{2(b-a)} \left[\left(\int_a^x (t-a)^p \, dt + \int_x^b (t-b)^p \, dt \right) + \left(\int_a^x (s-a)^p \, ds + \int_x^b (s-b)^p \, ds \right) \right] \right]^{\frac{1}{p}}.$$

Therefore,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |p(x,t) - p(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq \left[\frac{c(p)}{2(b-a)} \left(\frac{1}{p+1} (t-a) \right)^{p+1} \Bigg|_a^x + \left(\frac{1}{p+1} (t-b) \right)^{p+1} \Bigg|_x^b \right]^{\frac{1}{p}}.$$

Then,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |p(x,t) - p(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq c(p) \left[\frac{c(p)}{2(b-a)(p+1)} \left((x-a)\right)^{p+1} - \left((x-b)\right)^{p+1} \right]^{\frac{1}{p}}$$

$$\leq \left(\frac{2c(p)}{2(b-a)(p+1)} (b-a)^{p+1} \right)^{\frac{1}{p}}.$$

This implies,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |p(x,t) - p(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq \left(\frac{c(p)(b-a)^p}{(p+1)} \right)^{\frac{1}{p}}. \tag{6.5}$$

Also,

$$I_2 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b \left|\acute{f}(t) - \acute{f}(s)\right|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq \left[\frac{c(p)}{2(b-a)^2} \left(\iint_a^b \left|\acute{f}(t)\right|^p \, dtds + \iint_a^b \left|\acute{f}(s)\right|^p \, dtds \right) \right]^{\frac{1}{p}},$$

since $\|\bullet\|_p < \|\bullet\|_1$, we get,

$$I_2 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b \left| \acute{f}(t) - \acute{f}(s) \right|^p dtds \right)^{\frac{1}{p}}$$

$$\leq \left[c(p) \frac{c}{2(b-a)^2} \left((b-a) \int_a^b \left| \acute{f}(t) \right|^p dt + (b-a) \int_a^b \left| \acute{f}(s) \right|^p ds \right) \right]^{\frac{1}{p}}$$

$$\leq \frac{c(p)}{2(b-a)^2} (b-a) \left(\left\| \acute{f} \right\|_p + \int_a^b \left| \acute{f}(s) \right|^p ds \right)$$

Therefore,

$$I_2 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b \left| \acute{f}(t) - \acute{f}(s) \right|^p dtds \right)^{\frac{1}{p}}$$

$$\leq \left(\frac{c(p)}{2(b-a)} \right)^{\frac{1}{p}} \left(\left\| \acute{f} \right\|_p + \frac{f(b) - f(a)}{b-a} \right). \qquad (6.6)$$

Put (6.5) and (6.6) in (6.4), we get,

$$\left| \frac{1}{2(b-a)^2} \int_a^b \int_a^b (p(x,t) - p(x,s)) \left(\acute{f}(t) - \acute{f}(s) \right) dtds \right|$$

$$\leq \left(\frac{c(p)(b-a)^p}{(p+1)} \right)^{\frac{1}{p}} \left(\frac{c(p)}{2(b-a)} \right)^{\frac{1}{p}} \left(\left\| \acute{f} \right\|_p + \frac{f(b) - f(a)}{b-a} \right). \qquad (6.7)$$

By using Lemma 6.2.1 and (6.7), we get,

$$\left|\boldsymbol{f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)}\right|$$

$$\leq\left(\frac{c(p)(b-a)^{p-1}}{2(p+1)}\right)^{\frac{1}{P}}\left[\|\acute{f}\|_p+\left(\frac{f(b)-f(a)}{b-a}\right)^p\right],$$

where $\boldsymbol{c(p)}$ is a positive constant depending on p only ■

Collorary 6.2.3

Let $\boldsymbol{X}$ be a random variable $\boldsymbol{f},\boldsymbol{f}' \in \mathbf{L_p}[\mathbf{a},\mathbf{b}]$, $\mathbf{0}<\boldsymbol{p}<\mathbf{1}$, then

$$\left|\boldsymbol{f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt}\right|$$

$$\leq\left(\frac{\boldsymbol{c(p)(b-a)^{p-1}}}{\boldsymbol{2(p+1)}}\right)^{\frac{1}{\boldsymbol{p}}}\left[\|\acute{\boldsymbol{f}}\|_{\boldsymbol{p}}+\left(\frac{\boldsymbol{f(b)-f(a)}}{\boldsymbol{b-a}}\right)^{\boldsymbol{p}}\right].$$

Proof:

Take $X=\frac{\mathrm{a+b}}{2}$ is the midpoint,

By using Theorem 6.2.2, we get,

$$\left|\boldsymbol{f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt}\right|$$

$$\leq\left(\frac{\boldsymbol{c(p)(b-a)^{p-1}}}{\boldsymbol{2(p+1)}}\right)^{\frac{1}{\boldsymbol{P}}}\left[\|\acute{\boldsymbol{f}}\|_{\boldsymbol{p}}+\left(\frac{\boldsymbol{f(b)-f(a)}}{\boldsymbol{b-a}}\right)^{\boldsymbol{p}}\right],$$

where *c(p)* is a positive constant depending upon p.

6.3.Applications for Distribution Function and Random Variable.

Theorem 6.3.1

Let X be a random variable with probability density function

$f:[a,b] \to R_+$ and having the cumulative density function

$F:[a,b] \to [0,1]$and $f, F \in L_p[a,b]$, $0 < p < 1$, then,

$$\left| F(x) - \frac{b-E(X)}{b-a} - \frac{1}{b-a}\left(x - \frac{a+b}{2}\right) \right|$$

$$\leq \left(\frac{c(p)(b-a)^{p-1}}{2(p+1)} \right)^{\frac{1}{P}} \left[\|f\|_p + \left(\frac{1}{b-a} \right)^p \right],$$

where $\boldsymbol{c(p)}$ is a positive constant depending on p only .

Proof:

Put F(x) instead of f in Theorem 6.2.2, we get,

$$\left| \boldsymbol{F(x) - \frac{1}{b-a}\int_a^b F(t)dt - \frac{F(b)-F(a)}{b-a}\left(x - \frac{a+b}{2}\right)} \right|$$

$$\leq \left(\boldsymbol{\frac{c(p)(b-a)^{p-1}}{2(p+1)}} \right)^{\frac{1}{P}} \left[\|\acute{F}\|_p + \left(\boldsymbol{\frac{F(b)-F(a)}{b-a}} \right)^p \right] \qquad (6.8)$$

Since $F(a) = 0$ and $F(b) = 1$. (6.9)

Also, since $F(x) = \int_a^X f(t)dt,\ X \in [a, b]$.

So, $\acute{F} = f$ and E(X) $\int_a^b tdF(t)dt$.

Using (2.3) in Chapter Two. Its mean that,

$$\int_a^b F(t)dt = b - \mathrm{E(X)}. \quad (6.10)$$

Put (6.9) and (6.10) in (6.8), we get,

$$\left|\mathrm{F(x)} - \frac{\mathrm{b - E(X)}}{\mathrm{b - a}} - \frac{1}{b-a}\left(\mathrm{x} - \frac{\mathrm{a+b}}{2}\right)\right|$$

$$\leq \left(\frac{c(\mathrm{p})(b-a)^{p-1}}{2(p+1)}\right)^{\frac{1}{P}} \left[\|f\|_p + \left(\frac{1}{b-a}\right)^p\right] \quad \blacksquare$$

Colloraly 6.3.2

Let X be a random variable with probability density function

f and having the cumulative density function F ,such that

$f, F \in L_p[a, b], 0 < p < 1$, then,

$$\left|P_r\left(X \leq \frac{a+b}{2}\right) - \frac{b - E(X)}{b-a}\right|$$

$$\leq \left(\frac{c(p)(b-a)^{p-1}}{2(p+1)}\right)^{\frac{1}{P}} \left[\|f\|_p + \left(\frac{1}{b-a}\right)^p\right]$$

Proof:

Take $X = \frac{a+b}{2}$. Since $P_r(X \leq x) = F(X)$, then by using Theorem 6.3.1, we get

$$\left|P_r\left(X \leq \frac{a+b}{2}\right) - \frac{b - E(X)}{b-a}\right|$$

$$\leq \left(\frac{c(p)(b-a)^{p-1}}{2(p+1)}\right)^{\frac{1}{P}}\left[\|f\|_p + \left(\frac{1}{b-a}\right)^p\right] \qquad \blacksquare$$

The following theorem related to application for βeta distribution;

Theorem 6.3.3

Let X be a random variable and $X \sim B_{\alpha,\beta}$ whose the density function belongs to the L_p where $0 < p < 1$ then

$$\left|P_r(X \leq x) + \frac{\beta}{\alpha+\beta} - X + \frac{1}{2}\right|$$

$$\leq \left(\frac{c(p)}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{1}{B(\alpha,\beta)}(B(P(\alpha-1)+, P(\beta-1)+1)^{\frac{1}{p}} + 1\right]$$

Proof:

Since $X \sim \beta_{\alpha,\beta}$, then the probability density function as follows,

$$f(X,\alpha,\beta) = \frac{X^{\alpha-1}(1-X)^{\beta-1}}{\beta(\alpha,\beta)} \qquad 0 < X < 1,$$

where, $B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$.Then by using (2.8) in Chapter Two, we get,

$$\|f(.,\alpha,\beta)\|_p=\frac{1}{B(\alpha,\beta)}(B(P(\alpha-1)+,P(\beta-1)+1)^{\frac{1}{p}}.$$

Also, $E(X)=\frac{\alpha}{\beta+\alpha}$.

Now, by using Theorem 6.3.1, we get,

$$\left|F(x)-\frac{b-\frac{\alpha}{\beta+\alpha}}{b-a}-\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)\right|$$

$$\leq\left(\frac{c(p)(b-a)^{p-1}}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{1}{B(\alpha,\beta)}(B(P(\alpha-1)+,P(\beta-1)+1)^{\frac{1}{P}}+\left(\frac{1}{b-a}\right)^{p}\right].$$

Since $P_r(\mathrm{X}\leq \mathrm{x})=F(X)$, and b=1, then

$$\left|P_r(\mathrm{X}\leq x)+\frac{\beta}{\alpha+\beta}-x+\frac{1}{2}\right|$$

$$\leq\left(\frac{c(p)}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{1}{B(\alpha,\beta)}(B(P(\alpha-1)+,P(\beta-1)+1)^{\frac{1}{p}}+1\right]$$ ■

The next theorem is an application for normal distribution.

Theorem 6.3.4

Let X be a random variable with parameters $(\mu, \delta^2) \in \Omega$, with the probability density function

$$f(\mathrm{X}, \mu, \delta^2) = \frac{1}{\delta\sqrt{2\pi}} e^{\frac{-(\mathrm{X}-\mu)^2}{2\delta^2}},$$

where $\Omega = \{(\mu, \delta^2); -\infty < \mu < \infty, 0 < \delta^2 < \infty\}$.

Then

$$\left| P_r\left(X \leq \left(\frac{a+b}{2}\right)\right) + \mu - \mathrm{X} - \frac{1}{2} \right|$$
$$\leq \left(\frac{c(P)}{2(p+1)}\right)^{\frac{1}{P}} \left[\left(\frac{2}{\pi}\right)^{\frac{1}{2p}} \left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}} + 1 \right]$$

Proof:

Since $X \sim N(\mu, \delta^2)$,in L_p where $0 < p < 1$ then by using(2.9) in Chapter Two, we get,$\|f\|_p \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2p}} \left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}}$.Also, $E(X) = \mu$, $F(X) = P_r(\mathrm{X} \leq \mathrm{x})$, $\mathrm{X} \in [0,1]$. By using Theorem 6.3.1 , we get

$$\left|P_r\left(X \leq \left(\frac{a+b}{2}\right)\right) - (1-\mu) - \left(X - \frac{1}{2}\right)\right|$$
$$\leq \left(\frac{c(P)}{2(p+1)}\right)^{\frac{1}{P}} \left[\left(\frac{2}{\pi}\right)^{\frac{1}{2p}} \left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}} + 1\right].$$

This implies,

$$\left|P_r\left(X \leq \left(\frac{a+b}{2}\right)\right) + \mu - X - \frac{1}{2}\right|$$
$$\leq \left(\frac{c(p)}{2(p+1)}\right)^{\frac{1}{p}} \left[\left(\frac{2}{\pi}\right)^{\frac{1}{2p}} \left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}} + 1\right] \quad \blacksquare$$

Conclusions

1. We can generalize Ostrowski theorem to quasi normed spaces.
2. We can applied the theorem in 1 for random variable whose probability density function, and cumulative distribution in quasi normed spaces.
3. We can apply our Ostrowski theorem for beta and normal distribution.
4. There is a generalization for Pre-Gruss inequality to quasi normed spaces.
5. There is an application of the generalization in estimation expectation variances and dispersion.
6. We can generalize chebyshev inequality to quasi normed spaces.
7. The theorem in 6 can be applied in expectation of cumulative distribution for quasi normed spaces.
8. We can approximate the terms written in terms of expectation and variances with probability density function in quasi normed spaces.

Future Works

1. In Chapter One , we proved type of Ostrowski inequality in terms of the quasi norm of the first derivative . We can generalize this inequality for norms in terms of fractional derivatives.
2. We can generalize Hadmared inequality to the spaces of fractional $L_p, 1 \leq p \leq \infty$ spaces.

Refrences:

[1] A.M. Fink J.E. Pecaric and D.S. Mitrinovid, "Inequalities involving Functions and Their Derivatives," *Kluwer Academic, Dordrecht,* (1994).

[2] G.A. Anastassiou, "Ostrowski type inequalities," *Proc. Amer. Math.Soc.,* vol. 123(12), p. 3775–3781, (1995).

[3] G.A. Anastassiou,"Multivariate Ostrowski type inequalities *Acta Math. Hungar.,* vol. 76, p. 267–278, (1997).

[4] N.S. Barnett andS.S Dragomir, "An Ostrowski type inequality for double integrals and applications for cubature for mulae," *RGMIA Res. Rep. Coll.,* , vol. 1(1), p. 13–22, (1998).

[5] S.S.Dragomir and S.Wang, "Applications of Ostrowsiki inequality to the estimation of error bounds for some special means and some numerical quadrature rules.," *Apple.Mth.Lett.,* vol. Vol. 11, pp. pp.105-109, (1998).

[6] S.S.Dragomir and S.Wang., "A new inequality of Ostrowski type in Lp- norm. ," *Indian.Math.* , vol. Vol. 40, 3, pp. pp. 299-304, (1998).

[7] N.S. Barnett and P. Cerone, S.S Dragomir, "An n−dimensional version of Ostrowski's inequality for mappings of the Hölder type," *RGMIA Res. Rep. Coll* , vol. 2(2), p. 169–180, (1999).

[8] B.G. Pachpatte, "On multivaiate Ostrowski type inequalities , *Journal of Inequalities in Pure and Applied Mathematics*", no. ISSN (electronic): 1443-5756, 2000.

[9] P. Cerone, S.S. Dragomir and N.S. Barnett, "On weighted Ostrowski type inequalities for operators and vector -valued functions," *Mathematics Subject Classification*, May 17, 2001.

[10] N. S. Barnett and S. S. Dragomir, "An Ostrowski type inequality for double integrals and applications for cubature formulae," Soochow J. Math., vol. 27(1), pp. 109-114, (2001).

[11] P. Cerone, "A new Ostrowski type inequality involving integral means over end intervals," *Tamkang journal of mathematics* Volume 33, Number 2, 2002.

[12] P. Cerone and S.S. Dragomir, "Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions, ," Demonstratio Math., vol. 37, no. no. 2, 299– 308, (2004).

[13] B. G. Pachpatte, "A new Ostrowski type inequality for double integrals," Soochow journal of mathematics, Volume 32, No. 2, pp. pp. 317-322, April 2006.

[14] Nazir Ahmad Mir and Farooq Ahmad Arif Rafiq, "Weighted Ostrowski Type Inequality for Differentiable Mappings1 whose first derivatives belong to Lp(a, b)," *General Mathematics* , vol. Vol. 14, No. 3 , p. 91–102, (2006).

[15] Shiow-Ru Hwangy, "A Note on Multivariate Ostrowski Type Inequalities," *Tamsui Oxford Journal of Mathematical Sciences* , vol. 24(1) , pp. 97-108, (2008).

[16] H. Kavurmaci and M. Avci M. E. Ozdemir,"Ostrowski type inequalities for convex functions," Tamkang J. Math., vol. 45(4), pp. 335-340, 2014.

[17] Ana maria acu, Alina babos, Florin sofonea"The mean value theorems and inqualities of Ostrowski type," *S*eries Mathematics and Informatics, vol. Vol. 21 No. 1, 5 - 16, (2011).

[18] Havva Kavrmaci ,Ahmet Ocak Akdemir and Merve Avcim. Emin Ozdemir, "Inequalities for convex and s−convex functions on Δ = [a, b] × [c, d]," Math. CA, 15 June 2011.

[19] M. Z. Sarikaya and H. Ogunmez, "On the weighted Ostrowski type integral inequality for double integrals," The Arabian Journal for Science and Engineering (AJSE)-Mathematics, vol. 36:, pp. 1153-1160, (2011).

[20] Mevluttunc, "Ostrowski type inequalities for m− and geometrically convex functions via Riemann-Louville fractional integrals," math.C A, 2012.

[21] S. S. Dragomir, "Ostrowski type inequalities for functions whose derivatives are h-convex in absolute value," RGMIA Research Report Collection, , vol. 16Article 71, , p. 15 pp, (2013).

[22] M. Shoaib,A. E. MatoukandM. A. Latif A. Qayyum, "On New Generalized Ostrowski Type Integral Inequalities," Academic Editor: Elena Berdysheva, 26 February 2014.

[23] Silestru Sever Dragomir, " Ostrowski Type Inequalities for Lebesgue integral asurvey of resent results," The Australian Journal of Mathematical Analysis and Applications, vol. Volume 14, no. Issue 1, Article 1, pp. 1-287, 2017.

[24] Huseyin Budak & Mehmet Zeki Sarikaya Samet Erden, "An Ostrowski Type Inequality for Twice Differentiable Mappings and Applications," Mathematical Modelling and Analysis, no. ISSN: 1392-6292, pp. 1648-3510, 23 Jun 2016.

[25] Huseyin Budak and Ebru Pehlivan, "Weighted Ostrowski, trapezoid and midpoint type inequalities for Riemann-Liouville fractional integrals," AIMS Mathematics, , vol. 5(3), p. 1960–1984, 20 February 2020.

[26] Naila Mehreen and Matloob Anwar, "Ostrowski type inequalities via some exponentially convex functions with applications," AIMS Mathematics, vol. 5(2), pp. 1476–1483., 21 January 2020.

[27] Miguel Vivas-Cortez, Yenny Rangel-Oliveros and Muhammad AamirAli Praveen Agarwal1, "New Ostrowski type inequalities for generalized s-convex functions with applications to some special means of real numbers and to midpoint formula," AIMS Mathematics, vol. 7(1), p. 1429–1444., 26 October 2021.

[28] Nenad Ujevic, "A generalization of thePre-Gruss inequality and applications to some quadrature formulae," Journal of Inequalities in Pure and Applied Mathematics, pp. 1443-5756, 2000.

[29] A. Vukelic, "Estimations of the error for general Simpson type formulae viaPre–Gruss inequality," Journal of Mathematica Inequalities,vol. Volume 3, no. Number 4 , pp. 559–566, (2009).

[30] Samet Erden and Mehmet Zeki Sarikaya, "Pre-Grüss inequality involving conformable fractional integrals and its applications for random variables," Journal of Interdisciplinary Mathematics, no. ISSN:0972-0502 , November 2019.

[31] Silvestru Sever Dragomir, "A refinement of Gruss inequality for the complex integral," General Mathematics, vol. Vol. 28, , no. No. 1, pp. 67-83, (2020).

[32] P. Chebyshev, "Des valeurs moyennes.," Journal de Mathematiques pures et Appliquees , vol. 12 (2), pp. 177-184., (1867).

[33] J. G., M. C. K. Yang, and T. C. Mo Saw, "Chebyshev Inequality With Estimated Mean and Variance," The American Statistician , vol. 38 (2), , pp. 130-132, (1984).

[34 J. Smith, "Generalized Chebyshev inequalities," Theory and applications in decision analysis., vol. 43(5), pp. 807–825, (1995).

[35] R.V., Craig, A., McKean, J.W Hogg, Introduction to mathematical statistics. New York, Prentice Hall , 2004.

[36] X.M. Sim and P. Sun Chen, "A Robust Optimization Perspective on Stochastic Programming.," Operations Research , vol. 55 (6), pp. 1058-1071., December, (2007).

[37] X. Chen, "A New Generalization of Chebyshev Inequality for Random Vectors.," arXiv.org., (2011).

[38] J. Navarro, "A very simple proof of the multivariate Chebyshev's inequality.," To appear Communications in Statistics - Theory and Methods., (2013).

[39 J. Navarro, "A note on confidence regions based on the bivariate chebyshev inequality Applications to order statistics and data sets.," Journal of the Turkish Statistical.,vol. Association 7(2014).

[40] J. Navarro, "Can the bounds in the multivariate Chebyshev inequality be attained?," Statistics and Probability, vol. Letters 91, pp. 1-5, August 2014.

[41] A. Ostrowski, "Uber die Absolutabweichung einer differentien-baren Funktionen yon ihren Integralmittelwert," Comment. Math., vol. Hel. 10, (1938).

[42] J.E. Pecarid and A.M. Fink, D.S. Mitrinovid, "Inequalities for Functions and Their Integrals and Derivatives," Kluwer Academic, Dordrecht, (1994).

[43] S.S. Dragomir and S. Wang, "A new inequality of Ostrowski's type in L1 norm and applications to somespecial means and to some numerical quadrature rules," Tamkang J. of Math. , vol. 28, pp. 239-244, (1997).

[44] S.S. Dragomir and S. Wang, "An inequality of Ostrowski-Gruss type and its applications to the estimation oferror bounds for some special means and for some numerical quadrature rules," Computers Math. Applic. , vol. (11), 15 20, no. 33, ((1997).

[45] dorothy J.Musseleith and Brian C.Wesolowski, "the SAGE Encyclopedia of Education," Resarch, Measurment, and Evalution., vol. Thouson Oak ,S.n., 2018.

[46] C.D.Kemp, "distributions, statistical special and discrete. ," international Encyclopedia of the social and Behavioral Sciences., 2001.

[47] J.E. Pecarid and N. Ujevid, M. Matid, "Improvement and further generalisation of some inequalities of Ostrowski-Gruss type" Computers Math. Applic., vol. 39 (3/4), pp. 111-175, (2000).

[48] J.E.Pecari and N. Ujevid, M. Matid, "On new estimation of the remainder in generalized Taylor's formula," Mathematical Inequalities and Applications, vol. 2 (3), pp. 343 361, (1999).

[49] P. Cerone and S.S. Dragomir, "Three point quadrature rules involving, at most, a first derivative," *RGMIA Res. Rep. Coll.*, vol. 2 (4), , no. Article 8, h t t p : / / m e l b a , vu. edu. au/- rgmia/v2n4.html., (1999).

[50] J.E. Pecari and A.M. Fink, D.S. Mitrinovid, "Classical and New Inequalities in Analysis," *Kluwer Academic,Dordrecht,* (1999).

[51] N.S. Barnett and S.S. Dragomir, "Some elementary inequalities for the expectation and variance of a random variable whose PDF is defined on a finite interval," Article 12, RGMIA Res. Rep. Coll., vol. 2 (7), (1999).

[52] Eman Samir Bhaya, "A study on approximation of bounded measurable functions with some discrete series in Lp spaces (0<p)Thesis, 1999.

[53] David C.Lay Angus E.Taylor, Introduction to functional analysis. New Yourk, Chichester Brisbane Toronto: John Wiely and Sons, Inc, 1980.

[54] J.E. Pecarid and A.M. Fink, D.S. Mitrinovid, "Inequalities for Functions and Their Integrals and Derivatives," Kluwer Academic, Dordrecht, (1994).

[55] P. Cerone and S.S. Dragomir, "Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions," *Demonstratio Math.,* vol. 37 , pp. no. 2, 299– 308., (2004).

المستخلص

ان الهدف الاساسي من عملنا هو تطبيق للتحليل الدالي ونظرية التقريب في النظرية الاحتمالية والاحصاء.

كمساهمة للتحليل الدالي ونظرية التقريب قدمنا صيغة جديدة لنظرية اوستروسكي للدوال في الفضاءات المعيارية الكاذبة وطبقناها في المتغيرات العشوائية التي تكون دالة كثافة الاحتمال لها والدالة التوزيعية دوال في الفضاءات المعيارية الكاذبة واضفنا تطبيقا لتلك النظرية لتوزيعات بيتا والتوزيع الطبيعي.

عممنا متراجحة بري كروس للدوال في الفضاءات المعيارية الكاذبة وقمنا بتطبيقها على التوقع والتباين والانحراف.

برهنا تعميما لمتراجحة شبشف على الفضاءات المعيارية الكاذبة وطبقنا هذا التعميم على توقع الدالة التوزيعية التي متغيرها العشوائي بدالة كثافة احتمالية هي ومشتقتها الى فضاءات معيارية كاذبة.

في نظرية التقريب استخدمنا صيغة تايلر لتقريب مقادير بدلالة التوقع والتباين لدالة كثافة الاحتمال ينتمي الى فضاءات معيارية كاذبة.

برهنا نوع من متراجحة اوستروسكي وطبقناها في دالة الكثافة التوزيعية وتوزيعات بيتا والتوزيع الطبيعي.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

التقريب
في
الاحصاء والاحتمالية

أطروحة

مقدمة إلى مجلس كلية التربية للعلوم الصرفة في جامعة بابل

كجزء من متطلبات نيل درجة دكتوراه فلسفة في التربية / الرياضيات

من قبل

نادية عبد حبيب علي

بأشراف

أ. د. ايمان سمير بهيه

1443هـ 2022م