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Pointwise Polynomial Approximation

A Dissertation

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بسم الله الرحمن الرحيم

فَتَعَـٰلَى ٱللَّهُ ٱلۡمَلِكُ ٱلۡحَقُّۗ وَلَا تَعۡجَلۡ بِٱلۡقُرۡءَانِ مِن قَبۡلِ أَن يُقۡضَىٰٓ
إِلَيۡكَ وَحۡيُهُۥۖ وَقُل رَّبِّ زِدۡنِي عِلۡمٗا ﴿١١٤﴾

صدق الله العظيم

من سورة طه ، الآية 114

Dedication

To the first teacher ...Prophet Muhammed(P.B.U.H)

To the founder of the scientific research…...Imam Jaafar Al-Sadiq (P.B.U.H)

And this dissertation is dedicated to my husband and my children for their endless love, support, and encouragement.

Iktifa

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Praise be to Allah, Lord of the whole creation and peace is upon his messenger Prophet Mohammad and his relative.

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At last but not least special deep gratitude to my family and everyone who gave me a hand of support.

Iktifa

LIST OF SYMBOLS

Symbol	Definition	page
$\mathbb{R}^d$	The d-dimensional Euclidean space ($d \geq 1$)	1
L_p	L_p space	1
$\|.\|_p$	L_p-norm	1
$\mathcal{F}(x)$	Function defined in a L_p space	1
$C^r(I)$	The space of r times continuously differentiable functions on a closed interval I	2
$\mathbb{N}$	Set of the natural numbers	2
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$	2
$\mathbb{R}$	The set of real numbers	2
$I_j \coloneqq I_{j,n}$	$:= [x_j, x_{j-1}]$	11
$h_j \coloneqq h_{j,n}$	$:= \|I_{j,n}\| = x_{j-1} - x_j$	11
$\vartheta(x)$	$:=\sqrt{1-x^2}$	11
$\Omega_n(x$	$:=\vartheta(x)n^{-1} + n^{-2}$	11
Π_n	denoted the space of algebraic polynomial of degree$\leq n$	11
$w_\kappa(\mathcal{F}, t, I)_p$	κth Modulus of smoothness	11

Symbol	Definition	page
$\Delta_u^{\kappa}(\mathcal{F}, x, I)$	κth symmetric difference of order κ	11
$x_j := x_{j,n}$	$\cos\left(\frac{j\pi}{n}\right), 0 \le j \le n, 1$ for $j <$ 0 and -1 for $j > n$ (chebyshev knots)	12
T_n	$(x_j)_{j=0}^{n}$(chebyshev partition)	12
$\delta_n(x)$	$:=\min\{1, n\,\vartheta(x)\}$	12
$\mathcal{J}_j = \mathcal{J}_j(x)$	$:=\lvert I_j\rvert/(\lvert x - x_j\rvert + \lvert I_j\rvert$	12
$\Gamma(t)$	$:=t^r w_\kappa(\mathcal{F}^r, t)_p$ is equivalent to a function from $\Phi^{\kappa+r}$ If $\mathcal{F} \in L_p^r$.	12
Φ^κ	$:=\{\mathcal{T} \in L_p[0,\infty] \mid \mathcal{T}, \mathcal{T}(o)=0\}$	12
$\mathcal{P}_j := \mathcal{P}_j(\mathbb{S})$	$:= \mathbb{S}\lvert I_j, 1 \le j \le n$ ($\mathbb{S}$ is piecewise polynomials, of pieces $\mathcal{P}_j(x), x \in I_j$, $1 \le j \le n-1$, and write $\mathbb{S}\lvert I_j$).	12
$\sum_\kappa := \sum_{\kappa,n}$	The set of continuous piecewise polynomial of degree $\le \kappa - 1$	12
$\sum_\kappa^{(1)} := \sum_{\kappa,n}^{(1)}$	The set of continuously differentiable piecewise polynomials of degree $\le \kappa - 1$	12
$b_{i,j}(\mathbb{S}, \Gamma)$	$:= \frac{\lVert \mathcal{P}_i - \mathcal{P}_j \rVert_p}{\Gamma(h_j)} \left(\frac{h_j}{h_{i,j}}\right)^\kappa$,where $\Gamma \in \Phi^\kappa, \Gamma \not\equiv$ 0 and $\mathbb{S} \in \sum_\kappa$	12

Symbol	*Definition*	*page*
$c(\kappa, p)$	Positive constant depend on parameters p and κ	13
$dist(x, I)$	Is the distance between x and I	14
$Y_{\mathbb{S}}$	The set of all collections $Y_{\mathbb{S}}:=\{y_i\}_{i=1}^{\mathbb{s}} \ni -1 < y_{\mathbb{s}} < \cdots < y_1 < 1$	41
Y_0	$:= \emptyset$, the convex function as the case $\mathbb{s} = 0$	41
$\mathcal{P}_n$	the set of algebraic polynomials of degree $< n$	41
$\mathbb{S}(y_{\mathbb{S}})$	The piecewise polynomial of the set $y_{\mathbb{S}}$	41
$\Delta^{(2)}(y_{\mathbb{S}})$	The collection of all function $\mathcal{F} \in L_p$ that change convexity at the set $y_{\mathbb{S}}$	41
$E_n^{(2)}(\mathcal{F}, y_{\mathbb{S}})_p$	Degree of best coconvex approximation	42
$\mathcal{O} = \mathcal{O}(n, y_{\mathbb{S}})$	The union of intervals $\mathcal{O}_i$, $i = 1, \dots, \mathbb{s}$	43
$j \in H$	$= H(n, Y_{\mathbb{s}})$, if $I_j \cap O = \emptyset$,	43
$b_{i,j}(\mathbb{S})$	$= \frac{\|p_i - p_j\|_p}{\Gamma(h_j)} (\frac{h_j}{h_{i,j}})^{\kappa}$, $\quad 1 \le i, j \le n.$	43
$w_{\kappa}^{\vartheta}(\mathcal{F}, t)_p$	Ditzian -Totik modulus of smoothness	83
$\mathcal{X}_j(x)$	Is the characteristic function of $(x_j, I]$	85
$\bar{\mathcal{O}}$	Closure of a set	89

Symbol	Definition	page
d_i	The distance from y_i to the nearest endpoint of O_i	94

Abstract

In our dissertation we study the shape preserving for convex and piecewise convex functions in quasi normed spaces L_p for $0 < p < 1$.

Many authors studied the approximation of convex continuous functions by algebraic polynomials. We generalize and improve their works for piecewise algebraic polynomials and study the approximation in L_p space for $0 < p < 1$, we prove for a given convex function in L_p , $0 < p < 1$, there exists a convex piecewise polynomial as a convex approximation of $\mathcal{F}$. Then we introduce some approximating properties for these approximations.

Some papers introduced the direct and inverse theorems for the piecewise convex continuous functions. We study the piecewise convex approximation of function in L_p , $0 < p < 1$ using piecewise polynomials .It mean if $\mathcal{F}$ changes its convexity at $\{y_i\}_{i=1}^{s}$, we can construct a piecewise convex piecewise polynomial as a best approximation change its convexity at $\{y_i\}_{i=1}^{s}$. Our degree of approximation is in terms of the ordinary modulus of smoothness .

We introduce Jackson type theorem for piecewise convex functions in L_p quasi normed space L_p with $0 < p < 1$. Using piecewise convex piecewise polynomials in terms of Ditizian Totik modulus of smoothness .

Contents

Publications

1- Iktifa Diaa Jaleel , and Eman Samir Bhaya. ***"Shape preserving approximation using convex smooth piecewise polynomials for functions in L_p quasi normed space***." Accepted for the publication in (Intemational Joumal of Nonlinear Analysis and Applications(IJAA),Online ISSN2008-6822 (Vol.13 Issue 1) released in 2022 .

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Introduction

Our interest in approximation theory steams from its beauty, its utility and its rich history, also there are many connections can be drawn to questions in both classical and modern analysis.

The problem of classical approximation is if we give a target function $\mathcal{F}$ belongs to a normed space X and we demanded to approximate $\mathcal{F}$ from a subspace Y (approximation space). Some time we want to approximate $\mathcal{F}$ by a function preserves it geometric properties such as positivity, monotonicity and convexity, this what we called shape preserving approximation, but this restricts the degree of the approximation very much.

In our work we study the convex and coconvex approximation for functions in L_p-quasi normed space with $0 < p < 1$.

In particular L_p space , plays a central role in many question in the analysis. The special importance of L_p space may be said to derive from the fact that they offer a partial but useful generalization of the fundamental L^2 space of square integrable functions.

In order of logical simplicity , the space L^1 comes first since it occurs already in the description of functions integrable in the Lebesgue sense. Connected to it via duality is the L^∞ space of bounded functions, whose supremum norm carries over from the more familiar space of continuous functions.

$$\|\mathcal{F}\|_{L_p} = (\int_{R^d} |\mathcal{F}(x)|^p)^{\frac{1}{p}}.$$

The accompanying old style Timan-Freud-Brudnyi Jackson type inequality for the approximation by algebraic polynomials [1, Theorem 8-5-3]). explains the request for the approximation turns out to be fundamentally better closed to the end point of [-1, 1]:

If $\kappa \in \mathbb{N}$, $\mathfrak{r} \in \mathbb{N}_0$ and $\mathcal{F} \in C^{\mathfrak{r}}$,then for all $\mathfrak{n} \geq \kappa + \mathfrak{r} - 1$, There is polynomial $\mathcal{P}_{\mathfrak{n}} \in \prod_{\mathfrak{n}}$ Satisfy

$$|\mathcal{F}(x) - \mathcal{P}_{\mathfrak{n}}(x)| \leq c(\kappa, \mathfrak{r})\Omega_{\mathfrak{n}}^{\mathfrak{r}}(x)w_{k}(\mathcal{F}^{\mathfrak{r}}, \Omega_{\mathfrak{n}}(x)), \quad x\epsilon[-1,1] \qquad (1)$$

obviously, if we want to have interpolating approximation at the endpoints for $\mathcal{F}$ and it's derivatives. we get better approximation degree, the authors obtain the following Telyakovskii-gopengauze version theorem, for the literature review of the subject.

<u>Theorem</u> 1 [2]

Let $\mathfrak{r}\epsilon\mathbb{N}_0, \kappa \in \mathbb{N}$ and $\mathcal{F} \in C^{\mathfrak{r}}$,then for any $\mathfrak{n} \geq max\begin{Bmatrix}\kappa + \mathfrak{r} - 1, 2\mathfrak{r} + 1\end{Bmatrix}$, there is a polynomial $\mathcal{P}_{\mathfrak{n}} \in \Pi_{\mathfrak{n}}$such that (1) Is valid and , more over

$$|\mathcal{F}(x) - \mathcal{P}_{\mathfrak{n}}(x)| \leq c(\mathfrak{r}, \kappa)\vartheta^{2\mathfrak{r}}(x)w_{k}\left(\mathcal{F}, \vartheta^{\frac{2}{k}}(x)\mathfrak{n}^{-\frac{2(\kappa-1)}{\kappa}}\right), \text{if } 1 - \mathfrak{n}^{-2} \leq |x| \leq 1 \qquad (2)$$

In [2] the authors prove, If $\gamma \in R$, $\vartheta^{\frac{2}{\kappa}}(x)\mathfrak{n}^{-\frac{2(\kappa-1)}{\kappa}}$ in (2) Cannot be replaced by $\vartheta^{2\mathcal{G}}(x)\mathfrak{n}^{\gamma}$with $\mathcal{G} > 1/\kappa$. therefore, the estimate (2) offers the best approximation rate near the end point of closed interval [-1,1].

Now we have a question : Is the above inequalities true in the shape preserving approximation of q-monotone functions ?

The answer is that it is not true for an $\mathfrak{r}$ and κ ,even if we choose n depends on the target function $\mathcal{F}$.This case if we have $1 \leq q \leq 3$, $0 \leq \mathfrak{r} \leq q - 1$ and $\mathfrak{r} + k \geq q + 2$ in [3] if q=1 in [4] if q=2or q=3) and $q > 4$ and $\mathfrak{r} + \kappa > 3$ in [5].

More finished, for any q, $\mathfrak{r}$, κ, $\mathfrak{n}$ $\in \mathbb{N}$, There exists a function $\mathcal{F}_{\mathfrak{n}} \in C^{\mathfrak{r}} \cap \Delta^{(q)}$such that (2) is not valid for any polynomial $\mathcal{P}_{\mathfrak{n}} \in \prod_{\mathfrak{n}} \cap \Delta^{(q)}$

the development of such an $\mathcal{F}_{\mathfrak{n}}$ is the same as in ([6] ,see also [7,8,9]) . This implies that, in the case $\mathfrak{r} \geq 1$,(2) cannot be true for each function $\mathcal{F} \in C^{\mathfrak{r}} \cap \Delta^{(q)}$ and all $\mathfrak{n} \geq N(\kappa, \mathfrak{r}, q)$.

We accentuate that this not implies that for each fixed $\mathcal{F} \in C^{r} \cap \Delta^{(q)}$(2) is true for adequately $q = \kappa = 2$ large n. i.e.(2) may still be true if $n \geq N(\mathcal{F})$.

The case is different if we have κ is small and r equal to zero, andq, n are naturals.(1) and (2) are satisfied for $\kappa = 1$, which follows from the case $\kappa = 2$, for q-monotone approximation.

Surely, from the interpolator estimate follow, [11,12](q=2), [10]($q = 1$) and [13]($q \geq 3$):for any $q, n \in N$ and $\mathcal{F} \in C \cap \Delta^{(q)}$, there exist a polynomial $\mathcal{P}_n \in \Pi_n \cap \Delta^{(q)}$, Such that

$$|\mathcal{F}(x) - \mathcal{P}_n(x)| \leq c(q) w_2\left(\mathcal{F}, \frac{\vartheta(x)}{n}\right), x \epsilon [-1,1] \quad (3)$$

where $c > 0$. If we have $n \geq 2$ the estimates (1) and (2) are satisfied for $q = 2$ when r equal to zero and κ equal to 3, we find the result in [2], and case $q = 3$, $r = 0$ and $\kappa = 3$ or $\kappa = 4$ is no solution has yet been found (Actually unknown if (1)holds if $(q, r, \kappa) = (3,0,4)$.

Finally, I was able to show in [8] that (1) and (2) hold if $r \in \mathbb{N}$, κ=2 and $n \geq N(\mathcal{F}, r)$ for monotone approximation ($q = 1$) .

As usual, $C^{r}(I)$ denotes the space of r times continuously differentiable functions on a closed interval I, $C^{0}(I) := C(I)$ is the space of continuous functions on I, equipped with the uniform norm which will be denoted by $\|.\|_{I}$.

In chapter one we introduce the direct theorems approximation of convex function in L_p, space with $0 < p < 1$ using the piecewise convex polynomials.

Let $\mathcal{F} \in L_p(I)$ change its convexity finitely many, say $s \geq 0$, time in the interval.

define $L_p(I) = \{\mathcal{F}: I \to R : \mathcal{F} \in L_p\}$,where $I = [-1,1]$, $L_p^{r}(I) = \{\mathcal{F}: I \to R : \mathcal{F}, \mathcal{F}^{r} \in L_p\}$ r fold L_p , $\|\mathcal{F}\|_{L_p} = (\int_{-1}^{1} |\mathcal{F}(x)|^{p})^{\frac{1}{p}} = \|\mathcal{F}\|_p$.

We are interested in pointwise estimates on the approximation of $\mathcal{F}$ by algebraic polynomials there are coconvex with it, that will change their convexity exactly at the points where $\mathcal{F}$ does. Specifically, denote by Y_s , $s \in \mathbb{N}$.

The set of all collections $Y_s := \{y_i\}_{i=1}^{s}$ such that $-1 < y_s < \cdots < y_1 < 1$, denoted by Y_s , $s \in N$.

Let $\Delta^{(2)}(Y_s)$ denoted the collection of all function $\mathcal{F} \in L_p[-1,1]$ that change convexity at the set Y_s and are convex in $[y_1, 1]$.Namely in the interval $[y_{i+1}, y_i]$, $\mathcal{F}$ is convex when i is even, and is concave when i is odd. We also use the notation $y_0 = 1$ and $y_{s+1} = -1$. Denote

$$\Pi(x) = \prod_{i=1}^{s}(x - y_i) \tag{4}$$

Then for example, if $\mathcal{F} \in L_p^2[-1,1] \cap L_p[-1,1]$ then $\mathcal{F} \in \Delta^{(2)}(Y_s)$ if and only if $\Delta_h^{(2)}(\mathcal{F}) > 0$, in (-1,1).

The convex function as the case $s = 0$, where we write $Y_0 := \emptyset$

$\mathcal{P}_n$ the set of algebraic polynomials of degree $< n$

Norm estimates on the degree of approximation of $\mathcal{F} \in \Delta^{(2)}(Y_s)$ by

$\mathcal{P}_n \in \mathbb{P}_n \cap \Delta^{(2)}(Y_s)$,namely, estimates of
$E_n^{(2)}(\mathcal{F}, Y_s) := inf_{\mathcal{P}_n \in \mathbb{P}_n \cap \Delta^{(2)}(Y_s)} \|\mathcal{F} - \mathcal{P}_n\|_p$, More specifically, for
$\mathcal{F} \in L_p[-1,1]$,

let $\Delta_h^{\kappa}\mathcal{F}(x) := \sum_{i=0}^{\kappa}(-1)^{\kappa-i}\binom{\kappa}{i}\mathcal{F}\left(x + \frac{\kappa}{2}h + ih\right)$, if $x \mp \frac{\kappa}{2}h \in I$ and $:= 0$ otherwise,

is the symmetric difference of order κ .

$w_{\kappa}(\mathcal{F}, t) := \sup_{h\in[0,t]}\|\Delta_h^{\kappa}\mathcal{F}\|_p = \sup_{h\in[0,t]} max_{x\in[-1,1]}|\Delta_h^{\kappa}\mathcal{F}(x)|$, $t > 0$
is the kth modulus of smoothness of $\mathcal{F}$ on I.

Given $n \in \mathbb{N}, \kappa \in \mathbb{N}, r \in \mathbb{N}_0, s \in \mathbb{N}_0$ and $Y_s \in \mathbb{Y}_s$. We wish to estimate

$$\mathrm{E}^{(2)}_{\mathfrak{n},\kappa,\mathfrak{r}}(\mathcal{F},\mathrm{Y}_{\mathbb{s}}) := \mathcal{inf}_{\mathcal{P}_{\mathfrak{n}}\in\mathbb{P}_{\mathfrak{n}}\cap\Delta^{(2)}(\mathrm{Y}_{\mathbb{s}})}\left\|\frac{\mathcal{F}-\mathcal{P}_{\mathfrak{n}}}{\Omega^{\mathfrak{r}}_{\mathfrak{n}}\mathcal{w}_{\kappa}(\mathcal{F}^{(\mathfrak{r})},\Omega_{\mathfrak{n}})}\right\|_{\mathrm{p}}$$

$$= \mathcal{inf}_{\mathcal{P}_{\mathfrak{n}}\in\mathbb{P}_{\mathfrak{n}}\cap\Delta^{(2)}(\mathrm{Y}_{\mathbb{s}})}\mathcal{max}_{x\in[-1,1]}\left\|\frac{\mathcal{F}(x)-\mathcal{P}_{\mathfrak{n}}(x)}{\Omega^{\mathfrak{r}}_{\mathfrak{n}}(x)\mathcal{w}_{\kappa}(\mathcal{F}^{(\mathfrak{r})},\Omega_{\mathfrak{n}}(x))}\right\|_{\mathrm{p}},$$

Where $\vartheta(x) = \sqrt{1+x^2}$ and $\Omega_{\mathfrak{n}}(x) = \vartheta(x)\mathfrak{n}^{-1} + \mathfrak{n}^{-2}, x \in [-1,1]$, for $\mathcal{F} \in \Delta^{(2)}(\mathrm{Y}_{\mathbb{s}}) \cap \mathrm{L}^{\mathfrak{r}}_{\mathrm{p}}(\mathrm{I})$ such that $\mathcal{F} \notin \mathcal{P}_{\kappa+\mathfrak{r}}$. We put $\mathrm{E}^{(2)}_{\mathfrak{n},\kappa,\mathfrak{r}}(\mathcal{F},\mathrm{Y}_{\mathbb{s}}) := 0$, if $\mathcal{F} \in \mathcal{P}_{\kappa+\mathfrak{r}} \cap \Delta^{(2)}(\mathrm{Y}_{\mathbb{s}})$.

In [22] by Nikolskii, Timan, Dzyadyk, Frend and Brudnyi study the classical pointwise estimates for unconstrained approximation which is also true for coconvex approximation .

In [8,26] the authors restricted their attention to the following three cases:

a) $\mathfrak{r} \geq 3$ and $\mathbb{s} = 1$, b) $\mathfrak{r} = 2, \kappa \leq 3$ and $\mathbb{s} = 1$, c) $\mathfrak{r} = 0$ and $,\kappa = 3$. All other cases also have been investigated .

For instance, for $\mathbb{s} \geq 2$, it is introduced by [19,24] that for no $\kappa \geq 1$, and $\mathfrak{r} \geq 0$. is it is possible to have constants $c = c(\kappa,\mathfrak{r},\mathbb{s})$ and $N = N(\kappa,\mathfrak{r},\mathbb{s})$, depending only on $\kappa,\mathfrak{r}$ and $\mathbb{s}$, such that the inequality

$$\mathrm{E}^{(2)}_{\mathfrak{n},\kappa,\mathfrak{r}}(\mathcal{F},\mathrm{Y}_{\mathbb{s}}) \leq c, \qquad (5)$$

Is satisfied for all $\mathfrak{n} \geq \mathbb{N}$ and $\mathrm{Y}_{\mathbb{s}} \in \mathbb{Y}_{\mathbb{s}}$, and for all $\mathcal{F} \in \Delta^{(2)}(\mathrm{Y}_{\mathbb{s}}) \cap \mathrm{L}^{\mathfrak{r}}_{\mathrm{p}}(\mathrm{I})$.

Furthermore, for $\mathbb{s} = 1$, it follows by [27,Theorem 2] that no $\kappa \geq 1$ and $\mathfrak{r} \geq 0$ such that $\kappa + \mathfrak{r} > 2$, and $Y_1 \in \mathbb{Y}_1$, is possible to have constants $c = c(\kappa,\mathfrak{r},Y_1)$ and $N = N(\kappa,\mathfrak{r},Y_1)$, depending only on $\kappa,\mathfrak{r}$ and Y_1, such that (5) hold for all $\mathfrak{n} \geq N$ and for all $\mathcal{F} \in \Delta^{(2)}(\mathrm{Y}_{\mathbb{s}}) \cap \mathrm{L}^{\mathfrak{r}}_{\mathrm{p}}(\mathrm{I})$.

In [25] and [26] the authors studied the pointwise approximation of special polynomials.

In chapter two, we study the general pointwise approximation by any piecewise polynomial for function in $L_p[-1,1]$.

Let $\mathcal{F} \in L_p[-1,1]$ change its convexity finitely many times, say $\mathbb{s} \geq 0$ times, in the interval.

In chapter three, we interested in estimating the degree of approximation of $\mathcal{F}$ by polynomials which are coconvex with it, namely, polynomials that change their convexity exactly at the points where $\mathcal{F}$ does.

In [35], D. Leviatan and I. Shevchuk, write all the results of monotone and comonotone approximation theorems on a finite interval $[a, \mathfrak{b}]$, by algebraic polynomials in continuous function (uniform norm). see also [34]. In [34,36] the D. Leviatan and I. Shevchuk, studied the monotone and comonotone approximation of continuous function using κth modulus of smoothness of the $\mathfrak{r}$th derivative of the function.

They also studied the above approximation using κth Ditzian-Totik modulus of smoothness.

In [36] D. Leviatan and I. Shevchuk, studied the monotone and comonotone approximation of continuous functions in terms of Ditzian-Totik modulus of smoothness using piecewise polynomials.

In chapter three we introduce positive theorem for convex and coconvex approximation of function in L_p space for $0 < p < 1$.

The main truth in this chapter is to Jackson-type estimate for the approximation of a piecewise polynomial which changes convexity finitely many time in the interval, by algebraic polynomials that change convexity at exactly the same points.

The main result is Theorem 3.4.15 stated below, which is the analogue of [36.proposition 3].

Our strategy for the future is to approximate an arbitrary function in L_p space that changes convexity finitely many time in the interval, by an appropriate coconvex piecewise polynomial which in turn by virtue of Theorem 3.4.15, will be approximated by a coconvex polynomial.

In this chapter, we study some negative results for the coconvex polynomial approximation of more general piecewise convex functions (see Theorem 3.3.4), Also as a byproduct of Theorem 3.3.11, we obtain one important positive result for coconvex polynomial approximation (Theorem 3.3.17)

Therefore, it is one of the components of this chapter is the following. We state the main results contain the construction of the negative result, Auxiliary lemmas, the proof of Theorem 3.3.11 which is a preliminary step and a special case of Theorem 3.4.15 , and as a byproduct, its proof yields of Theorem 3.3.17, we prove Theorem 3.5.1 and with it conclude the proof of the Theorem 3.4.15. Many of the methods we apply are modification of similar ones in this chapter by Devore, Dzyubenko, Gilewicz, Kopotun, Mania, Yu and the authors (see the References).

Chapter One

Piecewise Convex

L_p ,$0 < p < 1$,

Approximation

Many papers introduced to constrained approximation of convex continuous function by algebraic polynomials, here we approximate the convex function in L_p, $0 < p < 1$, quasi normed spaces using piecewise algebraic polynomials. Also we introduce some properties of these polynomials.

1.1 Introduction and Notations:

The accompanying old style Timan-Freud-Brudnyi Jackson type inequality for the approximation by algebraic polynomials [1, Theorem 8-5-3]). explains the request for the approximation turns out to be fundamentally better closed to the end point of [-1, 1]:

If $\kappa \in \mathbb{N}$, $\mathfrak{r} \in \mathbb{N}_0$ and $\mathcal{F} \in C^{\mathfrak{r}}$,then for all $\mathfrak{n} \geq \kappa + \mathfrak{r} - 1$, there is polynomial $\mathcal{P}_{\mathfrak{n}} \in \prod_{\mathfrak{n}}$ Satisfy :

$$|\mathcal{F}(x) - \mathcal{P}_{\mathfrak{n}}(x)| \leq c(\kappa, \mathfrak{r})\Omega_{\mathfrak{n}}^{\mathfrak{r}}(x)w_{k}\left(\mathcal{F}^{\mathfrak{r}}, \Omega_{\mathfrak{n}}(x)\right), \;\; x\epsilon[-1,1] \qquad (1.1.1)$$

obviously, if we want to have interpolating approximation at the endpoints for $\mathcal{F}$ and its derivatives. We get better approximation degree, the authors obtain the following Telyakovskii-gopengauze version theorem.

Theorem1.1.1 [2]

Let $\mathfrak{r} \epsilon \mathbb{N}_0, \kappa \epsilon \mathbb{N}$ and $\mathcal{F} \in C^{\mathfrak{r}}$. Then for any $\mathfrak{n} \geq max\{\kappa + \mathfrak{r} - 1, 2\mathfrak{r} + 1\}$, there is a polynomial $\mathcal{P}_{\mathfrak{n}} \in \Pi_{\mathfrak{n}}$such that (1.1.1) Is valid and, more over

$$|\mathcal{F}(x) - \mathcal{P}_{\mathfrak{n}}(x)| \leq c(\mathfrak{r}, \kappa)\vartheta^{2\mathfrak{r}}(x)w_{k}\left(\mathcal{F}, \vartheta^{\frac{2}{k}}(x)\mathfrak{n}^{-\frac{2(\kappa-1)}{\kappa}}\right), \text{if } 1 - \mathfrak{n}^{-2} \leq |x| \leq 1 \qquad (1.1.2)$$

In [2] the authors prove, If $\gamma \in R$, $\vartheta^{\frac{2}{\kappa}}(x)\mathfrak{n}^{-\frac{2(\kappa-1)}{\kappa}}$ in (1.1.2) cannot be replaced by $\vartheta^{2\mathcal{G}}(x)\mathfrak{n}^{\gamma}$with $\mathcal{G} > 1/\kappa$. therefore, the estimate (1.1.2) offers the best approximation rate near the end point of closed interval [-1,1].

Now we have a question : Is the above inequalities true in the shape preserving approximation of q-monotone functions ?

The answer is that it is not true for an $\mathfrak{r}$ and κ ,even if we choose $\mathfrak{n}$ depends on the target function $\mathcal{F}$.This case if we have $1 \le q \le 3$, $0 \le \mathfrak{r} \le q - 1$ and $\mathfrak{r} + k \ge q + 2$ in [3] if q=1 in [4] if q=2or q=3) and $q > 4$ and $\mathfrak{r} + \kappa > 3$ in [5].

More finished, for any q, $\mathfrak{r}$, κ, $\mathfrak{n} \in N$, There exists a function $\mathcal{F}_\mathfrak{n} \in C^\mathfrak{r} \cap \Delta^{(q)}$such that (1.1.2) is not valid for any polynomial $\mathcal{P}_\mathfrak{n} \in \prod_\mathfrak{n} \cap \Delta^{(q)}$

the development of such an $\mathcal{F}_\mathfrak{n}$ is the same as in ([6] ,see also [7,8,9]) . This imply that, in the case $\mathfrak{r} \ge 1$,(1.1.2) cannot be true for each function $\mathcal{F} \in C^\mathfrak{r} \cap \Delta^{(q)}$ and all $\mathfrak{n} \ge N(\kappa, \mathfrak{r}, q)$.

We accentuate that this not imply that for each fixed $\mathcal{F} \in C^\mathfrak{r} \cap \Delta^{(q)}$(1.1.2) is in true for adequately $q = \kappa = 2$ large $\mathfrak{n}$. i.e.(1.1.2) may still be true if $\mathfrak{n} \ge N(\mathcal{F})$.

The case will be different if we have κ is small and $\mathfrak{r}$ equal to zero, and$q, \mathfrak{n}$ are naturals.(1.1.1) and (1.1.2) are satisfied for $\kappa = 1$, which follows from the case $\kappa = 2$, for q-monotone approximation.

Surely, from the interpolator estimate follow, [11,12](q=2), [10]($q = 1$) and [13]($q \ge 3$):for any $q, \mathfrak{n} \in N$ and $\mathcal{F} \in C \cap \Delta^{(q)}$, there exist a polynomial $\mathcal{P}_\mathfrak{n} \in \Pi_\mathfrak{n} \cap \Delta^{(q)}$, Such that

$$|\mathcal{F}(x) - \mathcal{P}_\mathfrak{n}(x)| \le c(q)\omega_2\left(\mathcal{F}, \frac{\vartheta(x)}{\mathfrak{n}}\right), x\epsilon[-1,1] \qquad (1.1.3)$$

where $c > 0$. If we have $\mathfrak{n} \ge 2$ the estimates (1.1.1) and (1.1.2) are satisfied for $q = 2$ when $\mathfrak{r}$ equal to zero and κ equal to 3, we find the result in [2], and case $q = 3$, $\mathfrak{r} = 0$ and $\kappa = 3$ or $\kappa = 4$ no solution has yet been found(Actually unknown if (1.1.1) holds if $(q, \mathfrak{r}, \kappa) = (3,0,4)$.

Finally, I was able to show in [8] that (1.1.1) and (1..1.2) hold if $\mathfrak{r} \in \mathbb{N}$, κ=2 and $\mathfrak{n} \ge N(\mathcal{F}, \mathfrak{r})$ for monotone approximation ($q = 1$) .

As usual, $C^r(I)$ denotes the space of r times continuously differentiable functions on a closed interval I, $C^0(I) := C(I)$ is the space of continuous functions on I, equipped with the uniform norm which will be denoted by $\|.\|_I$.

define $L_p(I) = \{\mathcal{F}: I \to R : \mathcal{F} \in L_p\}$,where $I = [-1,1]$, $L_p^r(I) = \{\mathcal{F}: I \to R : \mathcal{F}^r \in L_p\}$, $\|\mathcal{F}\|_{Lp} = (\int_{-1}^{1} |\mathcal{F}(x)|^p)^{\frac{1}{p}}$, For $\kappa \in \mathbb{N}$ and interval I ,

$$\Delta_u^\kappa(\mathcal{F}, x, I) := \sum_{i=0}^{\kappa} (-1)^i \binom{\kappa}{i} \mathcal{F}\left(x + \left(\frac{\kappa}{2} - i\right) u\right), \text{if } x \mp \frac{\kappa u}{2} \in I$$
$$:= 0 \text{ otherwise.}$$

is the symmetric difference of order κ . we denote by $\Delta^{(q)}$ the class of all q −monotone functions on $[-1,1]$, continuous function such that $\Delta_u^q(\mathcal{F}, x) \geq 0$ for all $x \in I$ and $u > 0$. In particular, $\Delta^{(1)}$ and $\Delta^{(2)}$ are the classes of all monotone and convex function on $[-1,1]$.

A function $\mathcal{F}: [a, b] \to R$ is said to be k-monotone, $\kappa \geq 1$ on $[a, b]$ if and only if for all choices of $\kappa + 1$ distinct points $x_0, x_1, \ldots, x_\kappa$ in $[a, b]$the inequality

$\mathcal{F}[x_0, x_1, \ldots, x_k] > 0$ holds,where

$\mathcal{F}[x_0, x_1, \ldots, x_\kappa] = \sum_{j=0}^{\kappa} \frac{\mathcal{F}(x_j)}{w'(x_j)}$, denotes the κth divided difference of $\mathcal{F}$ at $x_0, x_1, \ldots, x_\kappa$, and $w(x) = \prod_{j=0}^{\kappa}(x - x_j)$.

$w_\kappa(\mathcal{F}, t, I) := \sup_{0<u<t} \|\Delta_u^\kappa(\mathcal{F}, .; I)\|_p$is the kth modulus of smoothness of $\mathcal{F}$ on I.

$w_\kappa(\mathcal{F}, t) := w_\kappa(\mathcal{F}, t, [-1,1])_p, L_p^r = L_p^r[-1,1]$, for any interval I we write $w_\kappa(\mathcal{F}, \delta, I)_p$.

$\vartheta(x) = \sqrt{1 - x^2}$ and $\Omega_n(x) = \vartheta(x) n^{-1} + n^{-2}, n \in N, \Omega_0 \equiv 1$

Π_n denoted the space of algebraic polynomial of degree$\leq n$.

$I_j := I_{j,n} := [x_j, x_{j-1}]$, $h_j := h_{j,n} := |I_{j,n}| = x_{j-1} - x_j$

$$\mathrm{I}_{i,j} := \bigcup_{k=\min\{i,j\}}^{\max\{i,j\}} \mathrm{I}_{\kappa} = [x_{\max\{i,j\}}, x_{\min\{i,j\}-1}\}], 1 \le i, j \le n$$

(the smallest interval containing both $\mathrm{I}_i$and I_j)

$x_j := x_{j,n} := \cos\left(\frac{j\pi}{n}\right), 0 \le j \le n$, 1for j < 0 and −1 for j > n (chebyshev knots)

$\mathrm{T}_n := (x_j)_{j=0}^{n}$(chebyshev partition)

$$\mathrm{h}_{i,j} := |\mathrm{I}_{i,j}| = \sum_{\kappa=\min\{i,j\}}^{\max\{i,j\}} \mathrm{h}_{\kappa} = \mathrm{x}_{\min\{i,j\}-1} - \mathrm{x}_{\max\{i,j\}}$$

$\mathcal{T}_j := \mathcal{T}_j(x) := |\mathrm{I}_j| / (|x - x_j| + |\mathrm{I}_j|)$, $dist(x, \mathrm{I}_j) := |x - x_j|$

$\delta_n(x) := min\{1, n\vartheta(x)\}$. $\Phi^{\kappa} := \{\mathcal{T} \in L_p[0,\infty] | \mathcal{T}, \mathcal{T}(0) = 0$ and $t_2^{-\kappa}\mathcal{T}(t_2) \le t_2^{-\kappa}\mathcal{T}(t_1)$ for $0 \le t_1 \le t_2\}$. Note : if $\mathcal{F} \in \mathrm{L}_p^r$, Then

$\Gamma(t) := t^r w_{\kappa}(f^{(r)}, t)_p$ is equivalent to afunction from $\Phi^{\kappa+r}$

$\sum_{\kappa} := \sum_{\kappa,n}$ the set of all x_j, $1 \le j \le n-1$ piecewise polynomials of degree not exceeding $\kappa - 1$,that are continuous.

$\sum_k^{(1)} = \sum_{k,n}^{(1)}$ the set of all x_j, $1 \le j \le n-1$ piecewise polynomials of degree $\le k - 1$ that continuous derivatives.

$\mathcal{P}_j := \mathcal{P}_j(\mathbb{s}) := \mathbb{S}|\mathrm{I}_j, 1 \le j \le n$ ($\mathbb{S}$ is piecewise polynomials, of pieces $\mathcal{P}_j(x), x \in \mathrm{I}_j$, $1 \le j \le n-1$, and write $\mathbb{S}|\mathrm{I}_j$).

$\mathfrak{b}_{i,j}(\mathbb{s}, \Gamma) := \frac{\|\mathcal{P}_i - \mathcal{P}_j\|_p}{\Gamma(h_j)} \left(\frac{h_j}{h_{i,j}}\right)^{\kappa}$,where $\Gamma \in \Phi^{\kappa}, \Gamma \not\equiv 0$ and $\mathbb{S} \in \sum_{\kappa}$ where$\mathcal{P}_i, \mathcal{P}_j$ are polynomials.

$\mathfrak{b}_{\kappa}(\mathbb{s}, \Gamma, \mathrm{B}) := max_{1\le i,j\le n}\{\mathfrak{b}_{i,j}(\mathbb{s}, \Gamma) | \mathrm{I}_i \subset \mathrm{B}$ and $\mathrm{I}_j \subset \mathrm{B}\}$,

Where an interval $\mathrm{B} \subseteq [-1,1]$ contains at least one interval I_{v}.

$\mathfrak{b}_{\kappa}(\mathbb{s}, \Gamma) := \mathfrak{b}(\mathbb{s}, \Gamma, \mathrm{I}) = max_{1\le i,j\le n}\mathfrak{b}_{i,j}(\mathbb{s}, \Gamma)$

$c(\mathrm{p}) :=$ is an absolute constant depending on p,and is different from one step to others.

$c(\kappa, \mathrm{p})$:=positive constant that are either may only depend on the parameters k and p or absolute .

$\mathcal{X}_{\mathrm{j}}(x) \coloneqq \mathcal{X}_{[\mathrm{x_j},1]}(x) \coloneqq 1$, if $\mathrm{x_j} < x \leq 1$, and :=0, otherwise.

$\Phi_{\mathrm{j}}(x) \coloneqq (x - \mathrm{x_j})_+ \coloneqq (x - \mathrm{x_j})\mathcal{X}_{\mathrm{j}}(x) = \int_{-1}^{x} \mathcal{X}_{\mathrm{j}}(\mathrm{t})d\mathrm{t}.$

$\mathfrak{L}_{\kappa}^{\mathrm{L}}(\mathcal{F}, x, [a, \mathfrak{b}]) = min_{1 \leq \mathrm{m} \leq \kappa} \Delta^{\mathrm{m}}_{(x-a)^{\frac{1}{\mathrm{m}}}(\mathfrak{b}-a)^{\mathrm{m}-1/\mathrm{m}}}(\mathcal{F}, x, [a, \mathfrak{b}])$, $x \in [a, \mathfrak{b}]$.

$\mathfrak{L}_{\kappa}^{\mathrm{R}}(\mathcal{F}, x, [a, \mathfrak{b}]) = min_{1 \leq \mathrm{m} \leq \kappa} \Delta^{\mathrm{m}}_{(\mathfrak{b}-x)^{\frac{1}{\mathrm{m}}}(\mathfrak{b}-a)^{\mathrm{m}-1/\mathrm{m}}}(\mathcal{F}, x, [a, \mathfrak{b}])$, $x \in [a, \mathfrak{b}]$.

1.2 The Auxiliary Lemmas:

In this section we introduce the results that we need to prove our Theorems.

Lemma 1.2.1 [7]

($\mathrm{n}^{-1}\vartheta(x) < \Omega_{\mathrm{n}}(x) < \mathrm{h_j} < 5\Omega_{\mathrm{n}}(x), x \in \mathrm{I_j}$), ($\mathrm{h_{j\pm1}} < 3\mathrm{h_j}$).

{ Using the same lines word by word used for (6.12), P.235 in[14], we get the following lemma }

Lemma 1.2.2

For $0 < \mathrm{p} < 1$, ($\mathfrak{g} \in \mathrm{L_p}[-1,1], \kappa \in \mathbb{N}, a \in [-1,1]$ and $\mathrm{h} > 0$ such that $a + (\kappa - 1)\mathrm{h} \in [-1,1]$. Then

$\|\mathfrak{g}(\mathrm{x})\|_{\mathrm{p}} \leq c(\mathrm{p})\left(1 + \frac{|x-a|}{\mathrm{h}}\right)^{\kappa}(w_{\kappa}(\mathfrak{g}, \mathrm{h})_{\mathrm{p}} + \|\mathfrak{g}\|_{[a, a+(\kappa-1)\mathrm{h}]})$ $x \in [-1,1]$.

Lemma 1.2.3 [12]

($|\mathrm{B}_{v}| \leq c\Gamma(\mathrm{h_j})(\frac{\mathrm{h_{i,j}}}{\mathrm{h_j}})^{\kappa}, x \in \mathrm{I_i}, v = 1,2,3$), and $|\mathrm{B}_3| \leq c\Gamma(\mathrm{h_j})(\frac{\mathrm{h_{i,j}}}{\mathrm{h_j}})^{\kappa}$, $|\mathrm{B}_2| \leq c\Gamma(\mathrm{h_j})(\frac{\mathrm{h_{i,j}}}{\mathrm{h_j}})^{\kappa-1}$ and $|\mathrm{B}_1| \coloneqq \mathrm{h_{i,j}}A_1$, $A_1 \leq c\Gamma(\mathrm{h_j})\frac{\mathrm{h_{i,j}}^{\kappa-1}}{\mathrm{h_j}^{\kappa}}$, where $\mathrm{B}_1(x) = \int_{x_{\mathrm{i}}}^{x_{\mathrm{i}}}(x - \mathfrak{u})\mathbb{S}''d\mathfrak{u}$, $\mathrm{B}_2(x) = \int_{x_{\mathrm{j}}}^{x}(x - \mathfrak{u})\mathcal{P}_{\mathrm{j}}''d\mathfrak{u}$, $\mathrm{B}_3(x) = -\int_{x_{\mathrm{j}}}^{x}(x - \mathfrak{u})\mathcal{P}_{\mathrm{i}}''d\mathfrak{u}$, and $\mathbb{S}$, $\mathcal{P}$ are polynomials.

<u>Lemma</u> 1.2.4 [14]

$(\Omega_{\mathfrak{n}}^{2}(x) < 4\Omega_{\mathfrak{n}}(x_{j})(|x - x_{j}| + \Omega_{\mathfrak{n}}(x_{j})) < 8h_{j}(|x - x_{j}| + \Omega_{\mathfrak{n}}(x))$.

<u>Lemma</u> 1.2.5 [14]

$(\Omega_{\mathfrak{n}}(x) + dist(x, I_{j}) \le \Omega_{\mathfrak{n}}(x) + |x - x_{j}| \le 16(\Omega_{\mathfrak{n}}(x) + dist(x, I_{j}))$.

<u>Lemma</u> 1.2.6 [14]

Let $\mathcal{S}_2, \mathcal{G}_2 > 0$, and let $\mathfrak{n}, \mathfrak{n}_1 \in \mathbb{N}, \mathfrak{n}_1 > \mathfrak{n}$, be such that $\mathfrak{n}_1$ is divisible by $\mathfrak{n}$. Then , there is a collection $\{\widetilde{T}_{j,\mathfrak{n}_1}(x)\}_{j=1}^{\mathfrak{n}}$ of polynomials $\widetilde{T}_{j,\mathfrak{n}_1} \in \Pi_{c(\mathcal{S}_2,\mathcal{G}_2)}$, such that

$(\sum_{j=1}^{\mathfrak{n}} \widetilde{T}_{j,\mathfrak{n}_1}(x) \equiv 1, x \in [-1,1])$, and $\widetilde{T}'_{1,\mathfrak{n}_1}(x) \ge 0$ and $\widetilde{T}'_{\mathfrak{n},\mathfrak{n}_1}(x) \le 0$, $x \in \{-1,1\}$ are valid, and

$\left|\widetilde{T}_{j,\mathfrak{n}_1}^{(q)}(x)\right| \le c \frac{\delta_{\mathfrak{n}}^{\mathcal{S}_2}(x)}{\Omega_{\mathfrak{n}_1}^{q}(x)} \left(\frac{\Omega_{\mathfrak{n}_1}(x)}{\Omega_{\mathfrak{n}_1(x)+diist(x,I_j)}}\right)^{\mathcal{G}_2}$,$0 \le q \le \mathcal{S}_2$, for all $x \in D_j$,

where all constants c depend only on $\mathcal{S}_2, \mathcal{G}_2$ and are are independent of the ratio $\frac{\mathfrak{n}_1}{\mathfrak{n}}$.

<u>Lemma</u> 1.2.7 [14]

$(\sum_{j=1}^{\mathfrak{n}} \widetilde{T}_{j,\mathfrak{n}_1}(x) \equiv 1, x \in [-1,1])$.

<u>Lemma</u> 1.2.8 [14]

$(\left|\widetilde{T}_{j,\mathfrak{n}_1}^{(q)}(x)\right| \le c \frac{\delta_{\mathfrak{n}}^{\mathcal{S}_2}(x)}{\Omega_{\mathfrak{n}_1}^{q}(x)} \left(\frac{\Omega_{\mathfrak{n}_1}(x)}{\Omega_{\mathfrak{n}_1(x)+diist(x,I_j)}}\right)^{\mathcal{G}_2}$,$0 \le q \le \mathcal{S}_2)$.

<u>Lemma</u> 1.2.9 [14]

$(\mathfrak{b}_{i,j}(\mathbb{s}, \Gamma) := \frac{\|\mathcal{P}_i - \mathcal{P}_j\|_{I_i}}{\Gamma(h_j)} (\frac{h_j}{h_{i,j}})^{\kappa}$,$1 \le i, j \le \mathfrak{n})$.

<u>Lemma</u> 1.2.10 [14]

$(\sum_{j=1}^{\mathfrak{n}} (\frac{\Omega_{\mathfrak{n}}(x)}{\Omega_{\mathfrak{n}}(x)+dist(x,x_j)})^{4} \le c)$.

<u>Lemma</u> 1.2.11 [14]

$(|\sigma_1(x)| \leq c\mathfrak{b}_k(\mathbb{s},\Gamma,B)\delta^\gamma \frac{\Gamma(\Omega)}{\Omega^q}$,

$|\sigma_2(x)| \leq c\mathfrak{b}_k(\mathbb{s},\Gamma)\delta^\gamma \frac{\Gamma(\Omega)}{\Omega^q}\frac{\mathfrak{n}}{\mathfrak{n}_1}(\frac{\Omega}{\Omega + dist(x,[-1,1])\backslash B})^{\gamma+1}$

$|\sigma_3(x)| \leq c\mathfrak{b}_\kappa(\mathbb{s},\Gamma)\delta^\gamma \frac{\Gamma(x)}{\Omega^q}\frac{\mathfrak{n}}{\mathfrak{n}_1}(\frac{\Omega}{dist(x,[-1,1])\backslash B})^{\gamma+1})$, similarly $|\sigma_4(x)|$

Where $\sigma_1 =: \sum_{j\in Z_1}((\mathcal{P}_v(x) - \mathcal{P}_j(x))\widetilde{T}_j(x)^{(q)}$, $\sigma_2 =: \sum_{j\in Z_2}((\mathcal{P}_v(x) - \mathcal{P}_j(x))\widetilde{T}_j(x)^{(q)}$, similarly σ_3 and σ_4, where $Z_1 = \{j|1 \leq j \leq \mathfrak{n}, I_j \subset B, j \neq v, v \pm 1\}$

$Z_2 = \{j|1 \leq j \leq \mathfrak{n}, I_j \not\subset B, j \neq v, v \pm 1\}$

$Z_3 = \{j|1 \leq j \leq \mathfrak{n}, j = v + 1\}$

$Z_4 = \{j|1 \leq j \leq \mathfrak{n}, j = v - 1\}$.

<u>Lemma</u> 1.2.12 [7]

$(\mathcal{Q}''_{\mathfrak{n}}(x) \geq C\frac{\mathfrak{m}_E}{\mathfrak{m}_J}\delta_{\mathfrak{n}}^{\mathcal{S}_1}(x)\frac{\Gamma(\Omega_{\mathfrak{n}}(x))}{\Omega_{\mathfrak{n}}^2(x)}(\frac{\Omega_{\mathfrak{n}}(x)}{max\{\Omega_{\mathfrak{n}}(x), dist(x,E)\}})^{\mathcal{G}_1}, x \in J \cup ([-1,1]\backslash E))$, where $\mathcal{Q}_{\mathfrak{n}}$ is polynomial, $\mathfrak{m}_E, \mathfrak{m}_J$ are intervals of I_i and I_j.

<u>Lemma</u> 1.2.13 [7]

$(\mathcal{Q}''_{\mathfrak{n}}(x) \geq -\delta_{\mathfrak{n}}^{\mathcal{S}}(x)\frac{\Gamma(\Omega_{\mathfrak{n}}(x))}{\Omega_{\mathfrak{n}}^2(x)}, x \in E\backslash J$,

<u>Lemma</u> 1.2.14 [14]

(If $0 \leq j_1 \leq i < j_2 \leq \mathfrak{n}$, then $\frac{j_2 - j_1}{2} \leq \frac{x_{j_1} - x_{j_2}}{x_i - x_{i+1}} \leq (j_2 - j_1)^2$

<u>Lemma</u> 1.2.15 [7]

$(\mathfrak{m}ax\{|\Phi_j(x) - F_j(x)|, |\Phi_j(x) - \tilde{F}_j(x)|\} \leq c(\mathcal{S},\mathcal{G})|I_j|\delta_{\mathfrak{n}}^{\mathcal{S}}(x)\mathcal{T}_j^{\mathcal{G}-1}(x))$.

Where $\bar{F}_j = \begin{cases} \tilde{F}_j, \text{ if } \lambda_j < 0 \\ F_j, \text{ if } \lambda_j \geq 0 \end{cases}$, $|\lambda_j| \leq c$, $\Phi_j(x) = x - x_j$.

<u>Lemma</u> 1.2.16 [12]

$\left|\mathcal{P}''(x_j^*)\right| \leq \frac{\Gamma(h_j)}{h_j^2}(|i-j|+1)^{2k}$, where $\mathcal{P}$ is piecewise polynomial of x_i^* and $x_i^* \in (x_i, x_{i-1})$.

Lemma 1.2.17 [12]

$\frac{|x-x_{i_m}^*|}{\left|x_{i_\iota}^*-x_{i_m}^*\right|} \leq c((j-\mu)^2+(j-v)^2)$, where $x_{i_m}^* \in I_{\mu,v}$, If $1 \leq \mu, v \leq n$, and for each $m = 1, \dots, \kappa-2$ and $1 < \iota < \kappa-2$.

Lemma 1.2.18 [7]

$(\sum_{j=1}^{n} \mathcal{T}_j^2(x) \leq c)$.

Using the same lines word by word used the estimate for pp.1282-1283 in [14], we get the following Lemma

Lemma 1.2.19 [14]

$\sum_{j\in\mathcal{E}} \Gamma(h_j)\mathcal{T}^{\mathcal{G}-1}(x) \leq c\Gamma(\Omega)\sum_{j\in\mathcal{E}} \frac{h_j\Omega}{(|x-x_j|+\Omega)^2}$, where $\mathcal{G} \geq \kappa+7$ and $\mathcal{E} := \{1 \leq j \leq n \leq |I_j \subset E\}$.

Lemma 1.2.20 [6]

For every $r \in \mathbb{N}$ there is a constant $c = c(p, r)$ with the following property, for each convex function $\mathcal{F} \in L_p^r[a, b]$,there is a number $\mathcal{H} > 0$, such that for every partition $\mathcal{X} = \{x_j\}_{j=0}^{n}$ of $[a, b]$ satisfying $x_1 - a \leq \mathcal{H}$ and $b - x_{n-1} \leq \mathcal{H}$.

There is a convex piecewise polynomials $s \in \mathbb{S}(\mathcal{X}, r+2)$, such that

$$|\mathcal{F}(x) - s(x)| \leq c(x-a)^r \mathfrak{L}_2^L(\mathcal{F}^{(r)}, x; [a, x_1]), x \in [a, x_1],$$

$|\mathcal{F}(x) - s(x)| \leq c(b-x)^r \mathfrak{L}_2^R(\mathcal{F}^{(r)}, x; [x_{n-1}, b], x \in [x_{n-1}, b]$, and, for each $j = 2, \dots, n-1$ and $x \in [x_{j-1}, x_j]$, where

$\mathfrak{L}_\kappa^L(\mathcal{F}, x, [a, b]) = min_{1\leq m\leq\kappa} \Delta^m_{(x-a)^{\frac{1}{m}}(b-a)^{m-1/m}}(\mathcal{F}, x, [a, b])$, $x \in [a, b]$

$\mathfrak{L}_\kappa^R(\mathcal{F}, x, [a, b]) = min_{1\leq m\leq\kappa} \Delta^m_{(b-x)^{\frac{1}{m}}(b-a)^{m-1/m}}(\mathcal{F}, x, [a, b])$, $x \in [a, b]$

$$|\mathcal{F}(x)-\mathbb{s}(x)| \le c\left(x_{\mathrm{j}}-x_{\mathrm{j}-1}\right)^{\mathrm{r}}\Delta^2_{x_{\mathrm{j}}-x_{\mathrm{j}-1}}\left(\mathcal{F}^{(\mathrm{r})}\right), x \in \left[x_{\mathrm{j}-1}, x_{\mathrm{j}}\right] +$$

$c(x_1-a)^{\mathrm{r}}\Delta^2_{x_1-a}\left(\mathcal{F}^{(\mathrm{r})}\right),\ x \in [a, x_1] + c(\mathfrak{b}-x_{\mathfrak{n}-1})^{\mathrm{r}}\Delta^2_{\mathfrak{b}-x_{\mathfrak{n}-1}}\left(\mathcal{F}^{(\mathrm{r})}\right), x \in [x_{\mathfrak{n}-1}, \mathfrak{b}]$.

<u>Lemma</u> (1.2.21) [6]

For each convex function $\mathcal{F} \in \mathrm{L}^{\mathrm{r}}_{\mathrm{p}}[\mathrm{a},\mathrm{b}]$, $\mathbb{s} \in \sum_{\mathrm{r}+2,\mathfrak{n}} \cap \Delta^{(2)}$, then

$$|\mathcal{F}(x)-\mathbb{s}(x)| \le c(x-a)^{\mathrm{r}}\mathfrak{L}^{\mathrm{L}}_{2}\left(\mathcal{F}^{(\mathrm{r})}, x; [a, x_1]\right), x \in [a, x_1].$$

<u>Lemma</u> (1.2.22) [6]

For each convex function $\mathcal{F} \in \mathrm{L}^{\mathrm{r}}_{\mathrm{p}}[\mathrm{a},\mathrm{b}]$, $\mathbb{s} \in \sum_{\mathrm{r}+2,\mathfrak{n}} \cap \Delta^{(2)}$, then

$$|\mathcal{F}(x)-\mathbb{s}(x)| \le c(\mathfrak{b}-x)^{\mathrm{r}}\mathfrak{L}^{\mathrm{R}}_{2}\left(\mathcal{F}^{(\mathrm{r})}, x; [x_{\mathfrak{n}-1}, \mathfrak{b}]\right), x \in [x_{\mathfrak{n}-1}, \mathfrak{b}].$$

<u>Lemma</u> (1.2.23) [21]

Let $\mathfrak{r} \in \mathbb{N}, \mathrm{Z}_{\mathfrak{m}} := (\mathcal{Z}_{\mathrm{i}})_{\mathrm{i}=0}^{\mathfrak{m}}, a =: \mathcal{Z}_0 < \mathcal{Z}_1 < \cdots < \mathcal{Z}_{\mathfrak{m}-1} < \mathcal{Z}_{\mathfrak{m}} := \mathfrak{b}$ be a partition of $[a,\mathfrak{b}]$, $\mathbb{s} \in \Delta^{(2)} \cap \mathcal{Y}_{\mathfrak{r}+2}(\mathrm{Z}_{\mathfrak{m}})$. Then there exists $\tilde{\mathbb{s}} \in \Delta^{(2)} \cap \mathcal{Y}_{\mathfrak{r}+2}(\mathrm{Z}_{\mathfrak{m}}) \cap \mathrm{L}_{\mathrm{p}^1}[a,\mathfrak{b}]$ such that, for any $1 \le \mathfrak{j} \le \mathfrak{m}-1$.

$\|\mathbb{s}-\tilde{\mathbb{s}}\|_{[\mathcal{Z}_{\mathrm{j}-1},\mathcal{Z}_{\mathrm{j}+1}]} \le c\left(\mathfrak{r}, \mathcal{O}(\mathrm{Z}_{\mathfrak{m}})\right)\mathcal{w}_{\mathfrak{r}+2}\left(\mathbb{s}, \mathcal{Z}_{\mathrm{j}+2} - \mathcal{Z}_{\mathrm{j}-2}; \left[\mathcal{Z}_{\mathrm{j}-2}, \mathcal{Z}_{\mathrm{j}+2}\right]\right)$.where

$\mathcal{Z}_{\mathrm{j}} := \mathcal{Z}_0, \mathfrak{j} < 0$ and $\mathcal{Z}_{\mathrm{j}} := \mathcal{Z}_{\mathfrak{m}}, \mathfrak{j} > \mathfrak{m}$. Moreover.

$\tilde{\mathbb{s}}^{(\mathcal{v})}(a) = \mathbb{s}^{(\mathcal{v})}(a)$ and $\tilde{\mathbb{s}}^{(\mathcal{v})}(\mathfrak{b}) = \mathbb{s}^{(\mathcal{v})}(\mathfrak{b}). \mathcal{v} = 0,1$.

<u>Lemma</u> (1.2.24) [6]

$1 = |\mathrm{E}| \sum_{\mathrm{j} \in \mathcal{E}} \frac{\mathrm{h}_{\mathrm{j}}}{|\mathrm{E}|^2}$, where $|\mathrm{E}| = \left\|x - x_{\mathrm{j}}\right\|_{\mathrm{p}}$.

1.3 Properties of Piecewise Polynomials

Here we introduce some results about properties of piecewise polynomials.

Proposition 1.3.1

Let $\Gamma \in \Phi^{\kappa}, \kappa \in \mathbb{N}, \mathcal{F} \in L_p(I)$ and $\mathbb{S} \in \sum_{\kappa,n}$, If $w_{\kappa}(\mathcal{F}, t)_p \leq c(p)\Gamma(t)$ and

$\|\mathcal{F} - \mathbb{S}\|_p \leq c(p)\Gamma(\Omega_n(x))$, then

$$b_{\kappa}(\mathbb{s}, \Gamma) \leq c(\kappa, p)$$

Proof.

We have Γ is non-zero and positive map. For $1 \leq i, j \leq n$, we have

$$b_{i,j}(\mathbb{S}) = \frac{\|p_i - p_j\|_p}{\Gamma(h_j)} (\frac{h_j}{h_{i,j}})^{\kappa} = \frac{\|p_i - \mathcal{F} + \mathcal{F} - p_j\|_p}{\Gamma(h_j)} (\frac{h_j}{h_{i,j}})^{\kappa}$$

$$b_{i,j}(\mathbb{s}, \Gamma) \leq 2^{\frac{1}{p}} (\frac{\|\mathcal{P}_i - \mathcal{F}\|_p}{\Gamma(h_j)} \left(\frac{h_j}{h_{ij}}\right)^{\kappa} + \frac{\|\mathcal{F} - \mathcal{P}_j\|_p}{\Gamma(h_j)} \left(\frac{h_j}{h_{i,j}}\right)^{\kappa}) := \sigma_1 + \sigma_2 ,$$

where $\|\mathcal{P}_i - \mathcal{F}\|_p = (\int_{-1}^{1} |\mathcal{P}_i - \mathcal{F}|^p)^{\frac{1}{p}}$

$$\|\mathcal{F} - \mathcal{P}_j\|_p = (\int_{-1}^{1} |\mathcal{P}_i - \mathcal{F}|^p)^{\frac{1}{p}} dx$$

For any $1 \leq v \leq n$, since $\|\mathcal{F}(x) - \mathbb{S}(x)\|_p \leq c(p)\Gamma(\Omega_n(x)), x \in [-1,1]$, and by lemma 1.2.1.

$\frac{\vartheta(x)}{n} < \Omega_n(x) < h_j < 5\Omega_n(x), x \in I_j, 1 \leq j \leq n , h_{j\mp1} < 3h_j ,1 \leq j \leq n,$
Then

$\|\mathcal{P}_v - \mathcal{F}\|_{p(I_v)} \leq c(p(I_v))\|\Gamma(\Omega_n)\|_{p(I_v)} \leq c(p(I_v))\Gamma(h_v),$

Hence , $\sigma_1 \leq 1$, where we used the fact that if $h_i \leq h_j$,

then $\Gamma(h_i) \leq \Gamma(h_j)$,and if $h_i > h_j$ then $\Gamma(h_i)/\Gamma(h_j) \leq h_i^{k} / h_j^{k}$.

To estimate σ_2, by using Lemma 1.2.2

$(\|g(x)\|_p \leq c(p) \left(1 + \frac{|x-a|}{h}\right)^{\kappa} (w_{\kappa}(g, h)_p + \|g\|_{[a,a+(\kappa-1)h]}), x \in [-1,1].$

setting $a := x_j$ $g := \mathcal{F} - \mathcal{P}_j$, and $h := \frac{h_j}{max\{1,\kappa-1\}}$, and note that

$w_\kappa(g,h)_p = w_\kappa\left(\mathcal{F} - \mathcal{P}_j, h\right)_p = w_\kappa(\mathcal{F}, h)_p \le c(p)\Gamma(h)$, we get

$\left\|\mathcal{F}(x) - \mathcal{P}_j(x)\right\|_p \le c(p)(1 + \frac{|x-x_j|}{h_j})^k \left(\Gamma\left(h_j\right) + \left\|\mathcal{F} - \mathcal{P}_j\right\|_p\right)$, and so, $\left\|\mathcal{F} - \mathcal{P}_j\right\|_p \le c(p)(\frac{h_{i,j}}{h_j})^k\Gamma\left(h_j\right)$.

therefore, $\sigma_2 \le c$ ■

Proposition 1.3.2

Let $\kappa \ge 3, \Gamma \in \Phi^\kappa$ and $\mathbb{S} \in \sum_{\kappa;n}^{1}$. Then $\mathfrak{b}_\kappa(\mathbb{S}) \le c(p)\left\|\frac{\Omega^2 \mathbb{s}''}{\Gamma(\Omega)}\right\|_p$

Proof.

Since $\mathcal{P}_j(x) = \mathbb{s}(-1) + \mathbb{s}'(-1)(x+1) + \int_{-1}^{x_j}(x-u)s''(u)du + \int_{x_j}^{x}(x-u)\mathcal{P}''_j(u)du.$ and

$$\mathcal{P}_i(x) = \mathbb{s}(-1) + \mathbb{s}'(-1)(x+1) + \int_{-1}^{x_i}(x-u)s''(u)du + \int_{x_i}^{x}(x-u)\mathcal{P}''_i(u)du.$$

We have

$\mathcal{P}_j - \mathcal{P}_i = \int_{x_i}^{x_j}(x-u)s''(u)du + \int_{x_j}^{x}(x-u)p''_j(u)du - \int_{x_i}^{x}(x-u)p''_i(u)du. =: B_1(x) + B_2(x) + B_3(x)$. Then

$\left\|\mathcal{P}_j - \mathcal{P}_i\right\|_p \le c(p)(\|B_1(x)\|_p + \|B_2(x)\|_p + \|B_3(x)\|_p)$.

So by Lemma 1.2.3 ($|B_v| \le c\Gamma(h_j)(\frac{h_{i,j}}{h_j})^\kappa, x \in I_i, v = 1,2,3$

$\|B_3(x)\|_p \le c(p)\Gamma(h_j)(\frac{h_{i,j}}{h_j})^\kappa, \|B_2(x)\|_p \le c(P)\Gamma(h_j)\frac{(h_{i,j})^{\kappa-1}}{{h_j}^\kappa}$.

$\|B_1(x)\|_p \le h_{i,j}B_1$, where $B_1 \le c(p)\Gamma(h_j)\frac{(h_{i,j})^{k-1}}{h_j^k}$

So $max_{1\le i,j\le n}\left\{\mathfrak{b}_{i,j}(\mathbb{s},\Gamma)\right\} = \mathfrak{b}_\kappa(s,\Gamma)$

So $\frac{\|\mathcal{P}_j-\mathcal{P}_i\|_p}{\Gamma(h_j)}(\frac{h_i}{h_{i,j}})^{\kappa} =: \mathfrak{b}_{i,j}(s,\Gamma)$, we have

$$\mathfrak{b}_{\kappa}(\mathbb{s}) \leq \|B_{\mathcal{v}}(x)\|_p \leq c(p)\Gamma(h_j)(\frac{h_{i,j}}{h_j})^{\kappa}, x \in I_i, \mathcal{v} = 1,2,3.$$

That is $\|\mathcal{P}_j - \mathcal{P}_i\|_p \leq c(p)\Gamma(h_j)(\frac{h_{i,j}}{h_j})^{k}$. This proof is complete ■

<u>Theorem</u> 1.3.3

Let $\kappa \in \mathbb{N}, \gamma > 0, \Gamma \in \Phi^{\kappa}$,and let $\mathfrak{n}, \mathfrak{n}_1 \in \mathbb{N}$ be such that $\frac{\mathfrak{n}_1}{\mathfrak{n}}$.

If $\mathbb{S} \in \sum_{\kappa,\mathfrak{n}}$ .Then there exist a polynomial $\mathcal{D}_{\mathfrak{n}1}(.,\mathbb{s})$ of degree$\leq c\mathfrak{n}_1$ such that

$$\|\mathbb{S}(x) - \mathcal{D}_{\mathfrak{n}1}(x,\mathbb{s})\|_p \leq c(p)\delta_{\mathfrak{n}}^{\gamma}(x)\Gamma(\Omega_{\mathfrak{n}}(x))\mathfrak{b}_k(\mathbb{s},\Gamma) \qquad (1.3.4)$$

If $\mathbb{S} \in {L_p}^{\mathfrak{r}-1}(I), \mathfrak{r} \leq \kappa$, for some $\mathfrak{r} \in \mathbb{N}$ and

$B := [X_{M^*}, X_{M_*}], 0 \leq M_* \leq M^* \leq n$,then for all $x \in B \backslash \{x_j\}_{j=1}^{\mathfrak{n}-1}$ and $0 \leq q \leq \mathfrak{r}$, we have

$$\|\mathbb{S}^{(q)}(x) - {\mathcal{D}_{\mathfrak{n}1}}^{(q)}\|_p \leq c(p)\delta_{\mathfrak{n}}^{\gamma}(x)\frac{\Gamma(\Omega_{\mathfrak{n}}(x))}{\Omega_{\mathfrak{n}}^{q}(x)}(\mathfrak{b}_{\kappa}(\mathbb{s},\Gamma,B) + \mathfrak{b}_{\kappa}(\mathbb{s},\Gamma))\,\mathfrak{n}/\mathfrak{n}_1 \left\|(\frac{\Omega_{\mathfrak{n}}(x)}{dist(x,[-1,1]\backslash B)})^{\gamma+1}\right\|_p \qquad (1.3.5)$$

The constants $c(p)$ is depending on p only and not on $\mathfrak{n}$.

Proof.

$$\text{Let } \mathcal{D}_{\mathfrak{n}1} := \sum_{j=1}^{\mathfrak{n}} \mathcal{P}_j(x)\, \widetilde{T}_{j,\mathfrak{n}1}(x) \qquad (1.3.6)$$

where $\widetilde{T}_{j,n1}$denoted the polynomials of degree $\leq c(\mathcal{S}_2,\mathcal{G}_2)$ from the statement of Lemma 2.2.6 , and $\mathcal{D}_{\mathfrak{n}1}(.,\mathbb{s})$ denoted the polynomial of degree $< \kappa + c(\mathcal{S}_2,\mathcal{G}_2)$.The parameters $\mathcal{S}_2$ and $\mathcal{G}_2$ are chosen to be sufficiently large and depend on γ and κ .

So let $\mathcal{S} = \gamma$ and $\mathcal{G}_2 = \mathcal{S} + 4\kappa + 5\mathcal{S}$. So by Lemma 1.2.4 and 1.2.5

$$\Omega_{\mathfrak{n}}^2(x) < 4\Omega_{\mathfrak{n}}(x_j)(|x - x_j| + \Omega_{\mathfrak{n}}(x_j)) < 8h_j(|x - x_j| + \Omega_{\mathfrak{n}}(x)\,,$$

$\Omega_{\mathfrak{n}}(x) + dist(x, I_j) \le \Omega_{\mathfrak{n}}(x) + |x - x_j| \le 16\left(\Omega_n(x) + dist(x, I_j)\right).$

Then

$$\frac{h_{\mathcal{v}}}{h_j} < 5\frac{\Omega}{h_j} < 40\frac{|x-x_j|+\Omega}{\Omega} \sim \frac{\Omega+dist(x,I_j)}{\Omega} \quad ,1 \le j \le \mathfrak{n}\, , 1 \le \mathcal{v} \le \mathfrak{n} \qquad (1.3.7)$$

Also

$$\frac{h_{v,j}}{h_v} \le c(p)\ \frac{\Omega+dist(x,I_j)}{\Omega} \ ,1 \le j \le \mathfrak{n} \qquad (1.3.8)$$

Indeed, if $|j - \mathcal{v}| \le 1$, then by Lemma 1.2.1 ,

Implies that $h_{\mathcal{v},j} \sim h_{\mathcal{v}}$.

If $|j - \mathcal{v}| \ge 2$, since we have unique I_i between $I_{\mathcal{v}}$ and I_j and then using by Lemma 1.2.1 , to get: $h_{\mathcal{v},j} = h_{\mathcal{v}} + h_j + dist(I_{\mathcal{v}}, I_j) \le h_{\mathcal{v}} + 4dist(I_{\mathcal{v}}, I_j)$

$\le h_{\mathcal{v}} + dist(x, I_j)$, and by inequality 1.3.5 it follows .

Since $\mathbb{S}(x) = p_{\mathcal{v}}(x)$, by Lemma 1.2.7 ,

implies
$\mathbb{S}(x) - \mathcal{D}_{\mathfrak{n}1}(x, \mathbb{s}) = \mathbb{S}(x)\sum_{j=1}^{\mathfrak{n}} \widetilde{T}_j(x) - \sum_{1\le j\le \mathfrak{n}, j\ne \mathcal{v}} \mathcal{P}_j(x)\widetilde{T}_j(x)$, And so

$$\mathbb{S}^{(q)}(x) - \mathcal{D}_{n1}{}^{(q)}(x, \mathbb{s}) = \sum_{1\le j\le n, j\ne \mathcal{v}} \left(\left(\mathcal{P}_{\mathcal{v}}(x) - \mathcal{P}_j(x)\right)\widetilde{T}_j(x)\right)^{(q)}$$

$$= \sum_{1\le j\le n, j\ne \mathcal{v}} \sum_{i=0}^{q} \binom{q}{i} \left(\mathcal{P}_{\mathcal{v}}{}^{(i)}(x) - \mathcal{P}_j{}^{(i)}(x)\right) \widetilde{T}_j{}^{(q-i)}(x)$$

With the assumption that $x \notin \{x_j\}_{j=1}^{\mathfrak{n}-1}$, if $q \ge 1$, since $\mathbb{S}^{(q)}$that is not exist at those point. note also that $x \in D_j$ for all $1 \le j \le \mathfrak{n}, j \ne \mathcal{v}$,and by Lemma 1.2.8,

$$\left|\widetilde{T}_{j,\mathfrak{n}_1}^{(q)}(x)\right| \le c(p)\frac{\delta_{\mathfrak{n}}^{\mathcal{S}_2}(x)}{\Omega_{\mathfrak{n}_1}^{q}(x)}\left(\frac{\Omega_{\mathfrak{n}1}(x)}{\Omega_{\mathfrak{n}1}(x)+dist(x,I_j)}\right)^{\mathcal{G}_2}, 0 \le q \le \mathcal{S}_2 \text{, can be used}$$

for any $\widetilde{T}_j$ above.

since $\Gamma(h_j) \le \Gamma(h_{v,j}) \le \Gamma(h_v)(\frac{h_{v,j}}{h_v})^k \le c(p)\Gamma(\Omega)(\frac{h_{v,j}}{h_v})^\kappa$,

it follow from Lemma 1.2.9 and by Proposition 1.3.1, and inequality's 1.3.7, 1.3.8 for all $i \ge 0$, Then

$$\left\|\mathcal{P}_v^{(i)} - \mathcal{P}_j^{(i)}\right\|_{p(I_v)} \le c(p)h_v^{-i}\left\|\mathcal{P}_v - \mathcal{P}_j\right\|_{p(I_v)}$$

$$\le c(p)\mathfrak{b}_{v,j}(\mathbb{s},\Gamma)\frac{\Gamma(h_j)}{h_v^i}(\frac{h_{v,j}}{h_j})^\kappa \tag{1.3.9}$$

$$\le c(p)\, \mathfrak{b}_{v,j}(\mathbb{s},\Gamma)\frac{\Gamma(\Omega)}{\Omega^i}(\frac{{h_{v,j}}^2}{h_v h_j})^\kappa$$

$$\le c(p)\mathfrak{b}_{v,j}(\mathbb{s},\Gamma)\frac{\Gamma(\Omega)}{\Omega^i}(\frac{\Omega + dist(x, I_j)}{\Omega})^{3\kappa}.$$

observing that $\frac{\Omega_1}{\Omega_1 + dist(x, I_j)} \le \frac{\Omega}{\Omega + dist(x, I_j)}$ (1.3.10)

and using Lemma 1.2.8.we now conclude that , for all

$0 \le i \le q$ and $1 \le j \le \mathfrak{n}, j \ne v$, $\left\|(\mathcal{P}_v^{(i)}(x) - \mathcal{P}_j^{(i)}(x))\widetilde{T}_j^{(q-i)}(x)\right\|_p$

$\le c(p)\mathfrak{b}_{v,j}(\mathbb{s},\Gamma)\delta^{\mathcal{S}_2}\frac{\Gamma(\Omega)}{\Omega^i{\Omega_1}^{q-i}}(\frac{\Omega_1}{\Omega_1 + dist(x, I_j)})^{\mathcal{G}_2 - 3k}$, If $i = q$, this become

$$\left\|\left(\mathcal{P}_v^{(q)}(x) - \mathcal{P}_j^{(q)}(x)\right)\widetilde{T}_j(x)\right\|_p \le$$

$$c(p)\, b_{v,j}(\mathbb{s},\Gamma)\delta^{\mathcal{S}_2}\frac{\Gamma(\Omega)}{\Omega^q}(\frac{\Omega_1}{\Omega_1 + dist(x, I_j)})^{\mathcal{G}_2 - 3\kappa} \tag{1.3.11}$$

and ,in particular , if $i = q = 0$,

$$\left\|\left(\mathcal{P}_v(x) - \mathcal{P}_j(x)\right)\widetilde{T}_j(x)\right\|_p \le c(p)\mathfrak{b}_{v,j}(\mathbb{s},\Gamma)\delta^{\mathcal{S}_2}\Gamma(\Omega)(\frac{\Omega_1}{\Omega_1 + dist(x, I_j)})^{\mathcal{G}_2 - 3\kappa}$$
(1.3.12)

If we assume $j \ne \mathbb{s} \pm 1$,we get $dist(x, I_j) > \Omega/3$ therefore $\frac{\Omega_1}{\Omega} \le \frac{\mathfrak{n}}{\mathfrak{n}_1}$. Then

$$\left\|\left(\mathcal{P}_{v}{}^{(\mathrm{i})}(x)-\mathcal{P}_{\mathrm{j}}{}^{(\mathrm{i})}(x)\right)\widetilde{\mathrm{T}}_{\mathrm{j}}{}^{(q-\mathrm{i})}(x)\right\|_{\mathrm{p}}$$

$$\leq c(\mathrm{p})\mathfrak{b}_{v,\mathrm{j}}(\mathbb{s},\Gamma)\delta^{\mathcal{S}_2}\frac{\Gamma(\Omega)}{\Omega^{q}}\frac{\Omega_1}{\Omega}\left(\frac{\Omega}{\Omega_1+dist(x,\mathrm{I_j})}\right)^{q-\mathrm{i}+1}\left(\frac{\Omega_1}{\Omega_1+dist(x,\mathrm{I_j})}\right)^{\mathcal{G}_2-3\kappa-q+\mathrm{i}-1}$$

$$\leq c(\mathrm{p})\mathfrak{b}_{v,\mathrm{j}}(\mathbb{s},\Gamma)\delta^{\mathcal{S}_2}\frac{\Gamma(\Omega)}{\Omega^{q}}\frac{\mathfrak{n}}{\mathfrak{n}_1}\left(\frac{\Omega_1}{\Omega_1+dist(x,\mathrm{I_j})}\right)^{\mathcal{G}_2-3\kappa-q-1} \qquad (1.3.13)$$

Now let us study the case $q \geq 1$, $\mathrm{i} \leq q-1$ and $\mathrm{j} = v \pm 1$,

we study the case $\mathrm{j}=v+1$, the case $\mathrm{j}=v-1$ of study similar to above. Since $\mathbb{S}$ is smooth. In fact

If $\mathbb{S} \in \mathrm{L}_{\mathrm{p}}^{q-1}[-1,1]$, we have $\mathcal{P}_{v}{}^{(\iota)}(x_{v})=\mathcal{P}_{v+1}{}^{(\iota)}(x_{v}), 0 \leq \iota \leq q-1$ and so by inequality's 1.3.4 , 1.3.10.

$$\left\|\mathcal{P}_{v}{}^{(\mathrm{i})}(x)-\mathcal{P}_{v+1}{}^{(\mathrm{i})}(x)\right\|_{\mathrm{p}} = \left\|\frac{1}{(q-\mathrm{i}-1)!}\int_{x_v}^{x}(x-\mathrm{u})^{q-\mathrm{i}-1}(\mathcal{P}_{v}^{(q)}(\mathrm{u}) - \mathcal{P}_{v+1}{}^{(q)}(\mathrm{u})d\mathrm{u}\right\|_{\mathrm{p}}$$

$$\leq \frac{1}{(q-\mathrm{i}-1)!}\left(\sum_{\mathrm{i}=1}^{\mathrm{n}}|x_{\mathrm{i}}-\mathrm{u}|^{q}\right)^{\frac{1}{q}}\left(\sum_{\mathrm{i}=1}^{\mathrm{n}}|\mathcal{P}_{v}(x_{\mathrm{i}})-\mathcal{P}_{v+1}(x_{\mathrm{i}})|^{\mathrm{p}}\right)^{\frac{1}{\mathrm{p}}}$$

where $\mathcal{P}_{v}, \mathcal{P}_{v+1}, x-\mathrm{u}$ are polynomials so

$\leq \frac{c(\mathrm{p})}{(q-\mathrm{i}-1)!}\|x-\mathrm{u}\|_{\mathrm{p}(\mathrm{I}_{v})}\left\|\mathcal{P}_{v}{}^{(q)}(\mathrm{u})-\mathcal{P}_{v+1}{}^{(q)}(\mathrm{u})\right\|_{\mathrm{p}(\mathrm{I}_{v})}$, where $0<\mathrm{p}<1$.

$$\leq \frac{c(\mathrm{p})}{(q-\mathrm{i}-1)!}\|x-\mathrm{u}\|_{\mathrm{p}(\mathrm{I}_{v})}\mathfrak{b}_{v,v+1}(\mathbb{s},\Gamma)\frac{\Gamma(\Omega)}{\Omega^{q}}\left(\frac{\Omega+|x-x_{v}|}{\Omega}\right)^{3\kappa},$$

Therefore,

$$\left\|\left(\mathcal{P}_{v}{}^{(\mathrm{i})}(x)-\mathcal{P}_{v}{}^{(\mathrm{i})}(x)\right)\widetilde{\mathrm{T}}_{v+1}{}^{(q-\mathrm{i})}(x)\right\|_{\mathrm{p}} \leq$$

$$c(\mathrm{p})\mathfrak{b}_{v,v+1}(\mathbb{s},\Gamma)\delta^{\mathcal{S}_2}\frac{\Gamma(\Omega)}{\Omega^{q}\Omega^{q-\mathrm{i}}}\left\|(x-\mathrm{u})_{\mathrm{p}(\mathrm{I}_{v})}\left(\frac{\Omega_1}{\Omega_1+|x-x_{\mathrm{v}}|}\right)^{\mathcal{G}_2-3\kappa}\right\|_{\mathrm{p}(\mathrm{I}_{v})}$$

In summary , The estimate $\left\|\left(\mathcal{P}_{v}{}^{(\mathrm{i})}(x)-\mathcal{P}_{v\pm1}{}^{(\mathrm{i})}(x)\right)\widetilde{\mathrm{T}}_{v\pm1}{}^{(q-\mathrm{i})}(x)\right\|_{\mathrm{p}} \leq$

$$c(\mathrm{p})\mathfrak{b}_{v,v\pm1}(\mathbb{s},\Gamma)\delta^{\mathcal{S}_2}\frac{\Gamma(\Omega)}{\Omega^{q}}\left\|\left(\frac{\Omega_1}{\Omega_1+d\mathrm{ist}(\mathrm{x},\mathrm{I}_{v\pm1})}\right)^{\mathcal{G}_2-3\kappa-q}\right\|_{\mathrm{p}} \qquad (1.3.14)$$

is valid for all $0 \leq \mathrm{i} \leq q$ provided that $\mathbb{S} \in \mathrm{L}_{\mathrm{p}}^{q-1}[-1,1]$,

for $\mathrm{i} = q$ it follows from inequality's 1.3.4, 1.3.13, 1.3.12, 1.3.11, and using

Lemma 1.2.10 and the estimate $\mathfrak{b}_{v,\mathrm{j}}(\mathbb{s},\Gamma)$, we have

$$\|\mathbb{S}(x)-\mathcal{D}_{\mathfrak{n}1}(x,\mathbb{s})\|_{\mathrm{p}} \leq$$
$$c(\mathrm{p})\mathfrak{b}_{\kappa}(\mathbb{s},\Gamma)\delta^{\mathcal{S}_2}\Gamma(\Omega)\left\|\sum_{1\leq \mathrm{j}\leq \mathrm{n},\mathrm{j}\neq v}\left(\frac{\Omega}{\Omega+d\mathrm{ist}(x,\mathrm{I}_{\mathrm{j}})}\right)^{\mathcal{G}_2-3\kappa}\right\|_{\mathrm{p}} \qquad (1.3.15)$$

$\leq c(\mathrm{p})\mathfrak{b}_{\mathrm{k}}(\mathbb{s},\Gamma)\delta^{\gamma}\Gamma(\Omega)$, and inequality 1.3.5 is proved .

Let us now estimate 1.3.6. Assume $\mathbb{S} \in \mathrm{L}_{\mathrm{p}}^{\mathfrak{r}-1}[-1,1]$ and $\mathrm{o} \leq q \leq \mathrm{r}$

we write $\mathbb{S}^{(q)}(x)-\mathcal{D}_{\mathfrak{n}1}{}^{(q)}(x,\mathrm{s}) = \sum_{1\leq \mathrm{j}\leq \mathfrak{n},\mathrm{j}\neq v}\left(\left(\mathcal{P}_{v}(x)-\mathcal{P}_{\mathrm{j}}(x)\right)\widetilde{\mathrm{T}}_{\mathrm{j}}(x)\right)^{(q)}$

$$=:\left(\sum_{\mathrm{j}\in \mathrm{Z}_1}+\sum_{\mathrm{j}\in \mathrm{Z}_2}+\sum_{\mathrm{j}\in \mathrm{Z}_3}+\sum_{\mathrm{j}\in \mathrm{Z}_4}\right)\left(\left(\mathcal{P}_{v}(x)-\mathcal{P}_{\mathrm{j}}(x)\right)\widetilde{\mathrm{T}}_{\mathrm{j}}(x)\right)^{(q)}$$

$$\left\|\mathrm{S}^{(q)}(x)-\mathcal{D}_{\mathfrak{n}1}{}^{(q)}(x,\mathbb{s})\right\|_{\mathrm{p}} =: \sigma_1(x)+\sigma_2(x)+\sigma_3(x)+\sigma_4(x),$$

Where

$\mathrm{Z}_1 = \{\mathrm{j}|1 \leq \mathrm{j} \leq \mathfrak{n} \,, \mathrm{I}_{\mathrm{j}} \subset \mathrm{B} \,, \mathrm{j} \neq v, v\pm 1\}$

$\mathrm{Z}_2 = \{\mathrm{j}|1 \leq \mathrm{j} \leq \mathfrak{n}, \mathrm{I}_{\mathrm{j}} \not\subset \mathrm{B} \,, \mathrm{j} \neq v \,, v\pm 1\}$

$\mathrm{Z}_3 = \{\mathrm{j}|1 \leq \mathrm{j} \leq \mathfrak{n}, \mathrm{j} = v+1 \}$

$\mathrm{Z}_4 = \{\mathrm{j}|1 \leq \mathrm{j} \leq \mathfrak{n} \,, \mathrm{j} = v-1 \}$.

so by Lemma 1.2.11, $\|\sigma_1(x)\|_{\mathrm{p}} \leq c(\mathrm{p})\mathfrak{b}_{\mathrm{k}}(\mathbb{s},\Gamma,\mathrm{B})\delta^{\gamma}\frac{\Gamma(\Omega)}{\Omega^{q}}$ also

$$\|\sigma_2(x)\|_p \le c(p)\mathfrak{b}_k(\mathbb{s},\Gamma)\delta^{\gamma}\frac{\Gamma(\Omega)}{\Omega^{q}}\frac{\mathfrak{n}}{\mathfrak{n}_1}\left\|\left(\frac{\Omega}{\Omega + dist(x,[-1,1]\backslash B)}\right)^{\gamma+1}\right\|_p$$

$$\|\sigma_3(x)\|_p \le c(p)\mathfrak{b}_k(\mathbb{s},\Gamma)\delta^{\gamma}\frac{\Gamma(\Omega)}{\Omega^{q}}\frac{\mathfrak{n}}{\mathfrak{n}_1}\left\|\left(\frac{\Omega}{dist(x,[-1,1]\backslash B)}\right)^{\gamma+1}\right\|_p$$

similarly $\|\sigma_4(x)\|_p$ is completely analogous ,so the proof of this Theorem is complete ■

1.4 L_p Convex Approximation By Piecewise Polynomial

Here we deal with piecewise polynomial approximation for function in $L_p[-1,1]$ spaces.

Theorem 1.4.1

Let $\mathcal{S} > 0$, $\kappa \in \mathbb{N}$ and $\Gamma \in \Phi^{\kappa}$, be given. If $\mathbb{S} \in \sum_{\kappa,\mathfrak{n}} \cap \Delta^{(2)}$,is such that

$$\|\mathbb{S}''(x)\|_p \le c(p)\frac{\Gamma(\Omega_{\mathfrak{n}}(x))}{\Omega_{\mathfrak{n}}^2(x)}, x \in [x_{\mathfrak{n}-1}, x_1] \setminus \{x_j\}_{j=1}^{\mathfrak{n}-1} \quad (1.4.2)$$

$$0 \le \mathbb{S}'(x_j +) - \mathbb{S}'(x_j -) \le \frac{\Gamma(\Omega_{\mathfrak{n}}(x_j))}{\Omega_{\mathfrak{n}}(x_j)}, 1 \le j \le \mathfrak{n} - 1 \quad (1.4.3)$$

and

$$\mathbb{S}''(x) = 0 , x \in [-1, x_{\mathfrak{n}-1}) \cup (x_1, 1] \quad (1.4.4)$$

Then there is a polynomial $\mathcal{P} \in \Delta^{(2)} \cap \pi_{c\mathfrak{n}}$, $c = c(p,\kappa,\mathcal{S})$,such that

$$\|\mathbb{S}(x) - \mathcal{P}(x)\|_p \le c(p,\kappa,\mathcal{S})\delta_{\mathfrak{n}}^{\mathcal{S}}(x)\Gamma(\Omega_{\mathfrak{n}}(x)), x \in [-1,1] \quad (1.4.5)$$

Proof.

Let $\mathbb{S}_1$ is a continuous piecewise linear function interpolates $\mathbb{S}$ at the points x_j, o $\le j \le \mathfrak{n}$, and $\iota_j = \mathbb{S}_1/I_j$. Then

$$\mathbb{S}_1 \in \Delta^{(2)}, \mathbb{S}_1(x) = \mathbb{S}(x), x \in I_1 \cup I_{\mathfrak{n}} \quad (1.4.6)$$

And , for $x \in I_j, 1 \le j \le n$, using Whitney's inequality and lemma 1.2.1 ($n^{-1}\vartheta(x) < \Omega_n(x) < h_i < 5\Omega_n(x)$, $x \in I_j$, we get

$$\|\mathbb{S}(x) - \mathbb{S}_1(x)\|_{p(I_j)} \le c(p)w_2(\mathbb{s}, h_j, I_j)_p$$

$\leq c(\mathrm{p})\mathrm{h_j}\mathcal{w}_1(\mathbb{s}', \mathrm{h_j}, \mathrm{I_j}) \leq c(\mathrm{p})\mathrm{h_j^2}\|\mathbb{s}''\|_{\mathrm{p}(\mathrm{I_j})}$,so by using

$\|\mathbb{s}''\|_{\mathrm{p}} \leq c(\mathrm{p})\frac{\Gamma(\mathrm{h_j})}{\mathrm{h_j^2}}$, We get

$\|\mathbb{S}(x) - \mathbb{S}_1(x)\|_{\mathrm{p}(\mathrm{I_j})} \leq c(\mathrm{p})\Gamma(\mathrm{h_j}),$

So we have by Proposition 1.3.1,

$$\|\mathbb{S}(x) - \mathbb{S}_1(x)\|_{\mathrm{p}} \leq c(\mathrm{p})\Gamma\big(\Omega_\mathrm{n}(x)\big), x \in [-1,1]) \qquad (1.4.7)$$

So we can be write $\mathbb{S}_1$ as

$$\mathbb{S}_1(x) = \mathbb{S}_1(-1) + \mathbb{S}'_1(-1)(x+1) + \sum_{\mathrm{j}=1}^{\mathrm{n}-1} \mathcal{S}_\mathrm{i}\Phi_\mathrm{j}(x),$$

$\mathcal{S}_\mathrm{j} \coloneqq \mathbb{S}'_1\big(x_\mathrm{j}+\big) - \mathbb{S}'_1\big(x_\mathrm{j}-\big)$, note that, by Markov and Whitney

$\|\mathcal{P}'\|_\mathrm{p} \leq \mathrm{c}(\mathrm{p},\mathrm{n})\mathrm{n}^2\|\mathrm{p}\|_\mathrm{p}$.

$$0 \leq \mathcal{S}_\mathrm{j} = \iota'_\mathrm{j}\big(x_\mathrm{j}\big) - \iota'_{\mathrm{j}+1}(x_\mathrm{j}) \leq \mathrm{c}(\mathrm{p})\mathrm{h_j^{-1}}\big\|\iota_\mathrm{j} - \iota_{\mathrm{j}+1}\big\|_{\mathrm{p}(\mathrm{I_j}\cup\mathrm{I_{j+1}})}$$

$\leq c(\mathrm{p})\mathrm{h_j^{-1}}\mathcal{w}_2(\mathbb{s}, \mathrm{h_j}, \mathrm{p}\big(\mathrm{I_j} \cup \mathrm{I_{j+1}}\big))_\mathrm{p}$

$\leq c(\mathrm{p})\mathrm{h_j}\left(\|\mathbb{S}''\|_{\mathrm{p}(\mathrm{I_j})} + \|\mathbb{S}''\|_{\mathrm{p}(\mathrm{I_{j+1}})}\right) + \mathrm{c}(\mathrm{p})(\mathbb{S}'\big(x_\mathrm{j}+\big) - \mathbb{S}'\big(x_\mathrm{j}-\big))$

$\leq c(\mathrm{p})\mathrm{h_j^{-1}}\Gamma\big(\mathrm{h_j}\big), 1 \leq \mathrm{j} \leq \mathrm{n} - 1$.

Now , if $\mathcal{P}(x) \coloneqq \mathbb{S}_1(-1) + \mathbb{S}'_1(-1)(\mathrm{x}+1) + \sum_{\mathrm{j}=1}^{\mathrm{n}-1} \mathcal{S}_\mathrm{i}\mathrm{F_j}(x),$

So that $\mathcal{P}$ is a polynomial of degree not exceeding $c\mathrm{n}$, and also convex. and using Proposition 1.3.1), and inequality's 1.4.6 , 1.4.7. We only need to estimate $\|\mathbb{S}_1(x) - \mathcal{P}(x)\|_\mathrm{p}$. Note that the inequality $c\mathcal{T}_\mathrm{j}^{\,2}(x)\Omega_\mathrm{n}(\mathrm{x}) \leq \mathrm{h_j} \leq c\mathcal{T}_\mathrm{j}^{\,-1}(x)\Omega_\mathrm{n}(\mathrm{x})$,

implies, For all $1 \leq \mathrm{j} \leq \mathrm{n}$ and $\mathrm{x} \in [-1,1]$,

$\Gamma(\mathrm{h_j}) \leq \Gamma(c\mathcal{T}_\mathrm{j}^{\,-1}(x)\Omega_\mathrm{n}(x)) \leq c\mathcal{T}_\mathrm{j}^{\,-\mathrm{k}}(x)\Gamma(\mathrm{n}(\mathrm{x})).$

Hence by Lemma's 1.2.15, 1.2.18 , we have

$$\|\mathbb{S}_1(x) - \mathcal{P}(x)\|_p \le \sum_{j=1}^{n-1} \mathcal{S}_i \left|\Phi_j(x) - F_j(x)\right|$$

$$\le c(p) \sum_{j=1}^{n-1} \Gamma(h_j) \delta_n^{\mathcal{S}}(x) \mathcal{T}_j^{\mathcal{G}}(x)$$

$$\le c(p) \delta_n^{\mathcal{S}} \Gamma(\Omega_n(x)) \sum_{j=1}^{n-1} \mathcal{T}_j^{\mathcal{G}-k}(x)$$

$\le c(p)\delta_n^{\mathcal{S}}(x)\Gamma(\Omega_n(x))$, when $\mathcal{G} \ge k + 2$ ■

1.5 Polynomial With Second Derivative

Here we deal with polynomials of second derivatives belong to $L_p[-1,1]$.

<u>Lemma</u> 1.5.1

let $\mathcal{S} > 0$, $\kappa \in \mathbb{N}$, $\kappa \ge 2$, $\mathcal{G} > 0$ be sufficiently large ($\mathcal{G} \ge \kappa + 7$) and let $\Gamma \in \Phi^k$ be of the from $\Gamma(t) \coloneqq t\mathcal{T}(t), \mathcal{T} \in \Phi^{\kappa-1}$.

Also let $E \subset [-1,1]$ be a closed interval which is the union of $m_E \ge 100$ of the intervals I_j, and the set $J \subset E$ consist of m_J intervals I_j, where $1 \le m_J < \frac{m_E}{4}$. Then there exists a polynomial $\mathcal{Q}_n(x) = \mathcal{Q}(x, E, J)$ of degree$\le cn$, satisfying

$$\mathcal{Q}''_n(x) \ge c\frac{m_E}{m_J}\delta_n^{\mathcal{S}_1}(x)\frac{\Gamma(\Omega_n(x))}{\Omega_n^2(x)}\left(\frac{\Omega_n(x)}{max\{\Omega_n(x), dist(x,E)\}}\right)^{\mathcal{G}_1}, x \in J \cup ([-1,1]\setminus E), \quad (1.5.2)$$

$$\mathcal{Q}''_n(x) \ge -\delta_n^{\mathcal{S}}(x)\frac{\Gamma(\Omega_n(x))}{\Omega_n^2(x)}, x \in E\setminus J \quad (1.5.3)$$

,and

$$\|\mathcal{Q}_n(x)\|_p \le c(p) m_E^{\kappa_1} \delta_n^{\mathcal{S}}(x) \Omega_n(x) \Gamma(\Omega_n(x)) \sum_{j: I_j \subset E} \left\| \frac{h_j}{(|x - x_j| + \Omega_n(x))^2} \right\|_p \quad (1.5.4)$$

$x \in [-1,1]$, where $\mathcal{S}_1 = 8\mathcal{S}$, $\mathcal{G}_1 = 60(\mathcal{S} + \mathcal{G}) + \kappa + 1$ and $\kappa_1 = \kappa + 6$.

Proof.

let $\mathcal{E} := \{1 \le j \le n \le |I_j \subset E\}, \mathcal{J} := \{1 \le j \le n | I_j \subset J\}$

$j_* := \min\{j | j \in \mathcal{E}\}, j^* := max\{j | j \in \mathcal{E}\}, \mathcal{A} := \mathcal{J} \cup \{j_*, j^*\}$

and $\mathcal{B} := \mathcal{E} \backslash \mathcal{A}, \tilde{\mathcal{E}} := \{1 \le j \le n | I_j \subset \tilde{E}\}, \tilde{\mathcal{B}} := \mathcal{B} \cap \tilde{\mathcal{E}} = \tilde{\mathcal{E}} \backslash \mathcal{A}$

$\tilde{E} \subset E$ be the sub interval of E such that

(i) $\tilde{E}$ is a union of $[m_E/3]$ intervals I_j,and
(ii) $\tilde{E}$ is centered at 0 as much as E allows it,

where $E \subset [-1,1]$ be a closed interval which is the union of $m_E \ge 100$ of the intervals I_j, and a set $J \subset E$ consist of m_J intervals I_j, where $1 \le m_J < m_E/4$. so by lemma's 1.2.12, 1.2.13.

that is proved 1.5.2 and 1.5.3

now to prove 1.5.4, we now estimate $\|\mathcal{Q}_n(x)\|_p$.

Let $L(x) := \kappa \frac{\Gamma(h_*)}{h_*} (\frac{m_E}{m_J} \sum_{j \in \mathcal{A}} \Phi_j(x) + \sum_{j \in \tilde{\mathcal{B}}} \lambda_j \Phi_j(x))$.

Then from Lemma 1.2.14 . If $0 \le j_1 \le i \le j_2 \le n$. Then

$\frac{j_2 - j_1}{2} \le \frac{x_{j_1} - x_{j_2}}{x_i - x_{i+1}} \le (j_2 - j_1)^2)$, for any $j \in \mathcal{E}$, $cm_E \le |E|/h_j \le m_E^2$.

This implies that $h_* \le \frac{c|E|}{m_E} \le cm_E h_j$, $j \in \mathcal{E}$, and so

$\Gamma(h_*) \le cm_E^k \Gamma(h_j), j \in \mathcal{E}$ .Hence by using Lemma 1.2.15

$(\mathrm{Max}\{|\Phi_j(x) - F_j(x)|, |\Phi_j - \tilde{F}_j(x)|\} \le c(\mathcal{S},\mathcal{G})|I_j| \delta_n^{\mathcal{S}}(x) \mathcal{T}_j^{\mathcal{G}-1}(x))$

As well as by Lemma 1.2.19

$\{\sum_{j \in \mathcal{E}} \Gamma(h_j) \mathcal{T}^{\mathcal{G}-1}(x) \le c\Gamma(\Omega) \sum_{j \in \mathcal{E}} \frac{h_j \Omega}{(|x - x_j| + \Omega)^2}\}$, that is true if $\mathcal{G} \ge \kappa + 7$,we have

$$\|\mathcal{Q}_n(x) - L(x)\|_p$$

$$= \frac{\Gamma(h_*)}{h_*}\left\|\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}(F_j(x) - \Phi_j(x) + \sum_{j\in\widetilde{\mathcal{B}}}\lambda_i(\bar{F}_j(x) - \Phi_j(x)\right\|$$

Where $\bar{F}_j = \begin{cases}\tilde{F}_j, \text{if } \lambda_j < 0\\ F_j, \text{if } \lambda_j \geq 0\end{cases}$

$$\|\mathcal{Q}_n(x) - L(x)\|_p \leq$$

$$\leq c(p,\kappa)\frac{m_E}{m_J}\frac{\Gamma(h_*)}{h_*}\left(\left\|\sum_{j\in\mathcal{A}}(F_j(x) - \Phi_j(x)\right\|_p + \left\|\sum_{j\in\widetilde{\mathcal{B}}}\lambda_j(\bar{F}_j(x) - \Phi_j(x)\right\|_p\right)$$

$$\leq c(p)m_E\frac{\Gamma(h_*)}{h_*}\sum_{j\in\mathcal{E}}h_j\left\|\mathcal{T}^{\mathcal{G}-1}(x)\right\|_p$$

$$\leq c(p)m_E^{\kappa+1}\delta^{\mathcal{S}}\sum_{j\in\mathcal{E}}\Gamma(h_j)\left\|\mathcal{T}_j^{\mathcal{G}-1}(x)\right\|_p$$

$$\leq c(p)m_E^{\kappa+1}\delta^{\mathcal{S}}\Gamma(\Omega)\left\|\sum_{j\in\mathcal{E}}\frac{h_j\Omega}{(|x - x_j| + \Omega)^2}\right\|_p$$

It remains to estimate $\|L(x)\|_p$

First assume that $x \notin E$, if $x \leq x_{j^*}$, Then $\Phi_j(x) = 0, j \in \mathcal{A} \cup \widetilde{\mathcal{B}}$, and

$L(x) = 0$, If ,on the other hand , $x > x_{j_*}$, Then $\Phi_j(x) = x - x_j$, $j \in \mathcal{A} \cup \widetilde{\mathcal{B}}$, so by inequality $(\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}(x - x_j) + \sum_{j\in\widetilde{\mathcal{B}}}\lambda_i(x - x_j) \equiv 0)$, where λ_i is the constants and $i \in \widetilde{\mathcal{B}}$ implies $L(x) = 0$, Hence in particular ,$L(x) = 0$, for $x \in I_1 \cup I_n$.

Suppose now that $x \in E\backslash I_1$,we already assumed that E does not contain I_n).Then as above , $h_* \leq c|E|/m_E \leq c\Omega m_E$

And so $\Gamma(h_*) \leq cm_E^{\kappa}\Gamma(\Omega)$.

Also ,$h_* \geq |E|/m_E^2$. Hence , since $\delta = 1$ on $[x_{n-1}, x_1]$.

$$\|L(x)\|_p \leq \left\| \frac{\Gamma(h_*)}{h_*} \left(\frac{\mathfrak{m}_E}{\mathfrak{m}_J} \sum_{j\in\mathcal{A}} |x - x_j| + c \sum_{j\in\widetilde{\mathcal{B}}} |x - x_j| \right) \right\|_p$$

$$\leq c(p)\mathfrak{m}_E^{\kappa+3} \frac{\Gamma(\Omega)}{|E|} \sum_{j\in\mathcal{E}} \|x - x_j\|_p$$

$$\leq c(p)\mathfrak{m}_E^{\kappa+3} \frac{\Gamma(\Omega)}{|E|} \sum_{j\in\mathcal{E}} |E|$$

$$\leq c(p)\mathfrak{m}_E^{\kappa+4} \delta^{\mathcal{S}} \Gamma(\Omega)$$

And that the note by Lemma 1.2.24

$$1 = |E| \sum_{j\in\mathcal{E}} \frac{h_j}{|E|^2} \leq c(p)|E| \left\| \frac{h_j}{(|x - x_j| + \Omega)^2} \right\|_p$$

$\leq c(p)\mathfrak{m}_E^2 \left\| \sum_{j\in\mathcal{E}} \frac{\Omega h_j}{(|x-x_j|+\Omega)^2} \right\|_p$, and by

$\|\mathcal{Q}_{\mathfrak{n}(x)}\|_p \leq \|\mathcal{Q}_{\mathfrak{n}(x)} - L(x)\|_p + \|L(x)\|_p$

$$\leq c(p)\mathfrak{m}_E^{\kappa+1} \delta^{\mathcal{S}} \Gamma(\Omega) \left\| \sum_{j\in\mathcal{E}} \frac{h_j \Omega}{(|x - x_j| + \Omega)^2} \right\|_p$$

$$+ c(p)\mathfrak{m}_E^2 \left\| \sum_{j\in\mathcal{E}} \frac{\Omega h_j}{(|x - x_j| + \Omega)^2} \right\|_p$$

$\leq c(p)\mathfrak{m}_E^{\kappa_1} \delta_{\mathfrak{n}}^{\mathcal{S}}(x) \Omega_{\mathfrak{n}}(x) \Gamma(\Omega_n(x)) \sum_{j:I_j\subset E} \left\| \frac{h_j}{(|x-x_j|+\Omega_{\mathfrak{n}}(x))^2} \right\|_p$ ■

1.6 Convex Polynomial Approximation of Piecewise Polynomials .

To prove our main result we need the following auxiliary result:

<u>Lemma</u> 1.6.1

Let $\kappa \geq 3$, $\Gamma \in \Phi^k$and $\mathbb{S} \in \sum_{\kappa,n}^{(1)}$ be such that $b_\kappa(\mathbb{s},\Gamma) \leq 1$, If $1 \leq \mu, \mathcal{v} \leq n$ are such that the interval $I_{\mu,\mathcal{v}}$contains at least $2\kappa - 5$ intervals I_i and points $x_i^* \in (x_i, x_{i-1})$ so that

$$|\mathbb{s}''(x_i^*)| \leq \frac{\Gamma(h_i)}{h_i^2} \qquad (1.6.2)$$

Then for every$1 \leq j \leq n$,we have

$$\left\|\frac{\Omega^2 \mathbb{s}''}{\Gamma(\Omega)}\right\|_p \leq c(\kappa, p)[(j-\mu)^{4\kappa} + (j-\mathcal{v})^{4\kappa}] \qquad (1.6.3)$$

Proof.

Fix j and $x \in I_j^0$ where $I_j^0 = (x_i, x_{i-1})$. It follows by Lemma 1.2.9 $(b_{i,j}(\mathbb{s}) := b_{i,j;n}(\mathbb{s}) := \frac{\|\mathcal{P}_i - \mathcal{P}_j\|_p}{\Gamma(h_j)}(\frac{h_j}{h_{i,j}})^k, 1 \leq i, j \leq n$,

That for every i , $\|\mathcal{P}_i - \mathcal{P}_j\|_p \leq c(p)\Gamma(h_j)(\frac{h_{i,j}}{h_j})^\kappa$.

Since $\mathcal{P}_i$and $\mathcal{P}_j$ are polynomials of degree $< \kappa$,we obtain

$$\|\mathcal{P}_i'' - \mathcal{P}_j''\|_p \leq \frac{1}{h_i^2}\|\mathcal{P}_i - \mathcal{P}_j\|_p \leq c(p)\frac{\Gamma(h_j)}{h_i^2}(\frac{h_{i,j}}{h_j})^k.$$

Hence by Lemma 1.2.16 and 1.2.17

Implies the inequalities 1.6.4 and 1.6.5

$$|\mathcal{P}_j''(x_i^*)| \leq \frac{\Gamma(h_j)}{h_j^2}(|i-j|+1)^{2\kappa} \qquad (1.6.4)$$

$$\frac{|x - x_{i_m}^*|}{|x_{i_l}^* - x_{i_m}^*|} \leq c((j-\mu)^2 + (j-\mathcal{v})^2) \qquad (1.6.5)$$

Now ,by virtue of the representation

$$\mathcal{P}_j''(x) \equiv \sum_{\iota=1}^{k-2} \mathcal{P}_j''(x_{i\iota}^*) \prod_{m=1, m\neq\iota}^{\kappa-2} \frac{x - x_{i_m}^*}{x_{i_\iota}^* - x_{i_m}^*}$$

It follows by inequalities 1.6.4 and 1.6.5 that $\left\|\frac{\Omega^2 \mathbb{s}''(x)}{\Gamma(\Omega)}\right\|_p = \left\|\frac{\Omega^2 \mathcal{P}_j''(x)}{\Gamma(\Omega)}\right\|_p$

$$\leq \frac{h_i^2 \left\|\mathcal{P}_j''(x)\right\|_p}{\Gamma(h_j)} \leq c(p)((j-\mu)^{4\kappa-6} + (j-v)^{4\kappa-6}), x \in I_j^0,$$

So the proof of inequality 1.6.3 is complete ■

Using the same lines step by step of theorem 10.2 in[14] we get the following result:

Theorem 1.6.6

Let $\kappa, \mathfrak{r} \in \mathbb{N}, \mathfrak{r} \geq 2, \kappa \geq \mathfrak{r}+1$, and let $\Gamma \in \Phi^\kappa$ be the form $\Gamma(t) := t^{\mathfrak{r}}\mathcal{T}(t)$;$\mathcal{T} \in \Phi^{k-\mathfrak{r}}$. Also,

let $d+ > 0, d- \geq 0$ and $\mathcal{S}$be given. Then there is a number $N = N(\kappa, \mathfrak{r}, \Gamma, d+, d-, \mathcal{S})$ satisfying the following assertion .

If $n \geq N$ and $\mathbb{S} \in \sum_{\kappa,n}^{(1)} \cap \Delta^{(2)}$is such that

$$\mathfrak{b}_\kappa(\mathbb{s}, \Gamma) \leq 1 \qquad (1.6.7)$$

If $d+ > 0$, then $d + |I_2|^{\mathfrak{r}-2} \leq min_{x\in I_2}\mathbb{S}''(x)$ (1.6.8)

If $d+ = 0$, then $\mathbb{S}^{(i)}(1) = 0$, for all $2 \leq i \leq \kappa - 2$ (1.6.9)

If $d- > 0$, then $d - |I_{n-1}|^{\mathfrak{r}-2} \leq min_{x\in I_{n-1}}\mathbb{S}''(x)$ (1.6.10)

If $d- = 0$, then $\mathbb{S}^{(i)}(-1) = 0$, for all $2 \leq i \leq \kappa - 2$ (1.6.11)

Then there exists a polynomial $\mathcal{P} \in \Delta^{(2)} \cap \Pi_{c\mathfrak{n}}$, $c = c(\mathrm{p}, \kappa, \mathcal{S})$. Satisfying : for all $x \in [-1,1]$

$$\|\mathbb{S} - \mathcal{P}\|_{\mathrm{p}} \leq c(\mathrm{p}, \kappa, \mathcal{S})\delta_{\mathfrak{n}}^{\mathcal{S}}(x)\Gamma(\Omega_{\mathfrak{n}}(x)), \text{if } d+ > 0 \text{ and } d- > 0 \quad (1.6.12)$$

$$\|\mathbb{S} - \mathcal{P}\|_{\mathrm{p}} \leq c(\mathrm{p}, \kappa, \mathcal{S})\delta_{\mathfrak{n}}^{min\{\mathcal{S},2\kappa-2\}}(x)\Gamma(\Omega_{\mathfrak{n}}(x)), \text{ if } min\{d+, d-\} = 0$$
(1.6.13)

1.7 Convex Approximation By Smooth Piecewise Polynomials

Here we treat with convex approximation by smooth piecewise polynomials.

<u>Proposition</u> 1.7.1

Given $\mathfrak{r} \in \mathbb{N}$,there is a constant $c = c(\mathrm{p}, \mathfrak{r})$ such that if $\mathcal{F} \in \mathrm{L}_{\mathrm{p}^{\mathfrak{r}}}[-1,1]$ is convex, then there is a number $\mathrm{N} = \mathrm{N}(\mathcal{F}, \mathfrak{r})$, depending on $\mathcal{F}$and $\mathfrak{r}$, such that for $\mathfrak{n} \geq \mathcal{N}$, there are convex piecewise polynomials $\mathbb{S}$ of degree $\mathfrak{r} + 1$ with knots at the chebyshev partition $\mathrm{T}_{\mathfrak{n}}$ (i.e., $\mathbb{S} \in \sum_{\mathfrak{r}+2,\mathfrak{n}} \cap \Delta^{(2)}$), satisfying

$$\|\mathcal{F}(x) - \mathbb{S}(x)\|_{\mathrm{p}} \leq c(\mathrm{p}, \mathfrak{r}) \left(\frac{\vartheta(x)}{\mathfrak{n}}\right)^{\mathfrak{r}} w_2 \left(\mathcal{F}^{(\mathfrak{r})}, \frac{\vartheta(x)}{\mathfrak{n}}\right)_{\mathrm{p}} \quad (1.7.2)$$

and

$$\|\mathcal{F}(x) - \mathbb{S}(x)\|_{\mathrm{L}_{\mathrm{p}[-1.-1+\mathfrak{n}^{-2}]\cup[1-\mathfrak{n}^{-2},1]}} \leq c(\mathrm{p}, \mathfrak{r})\vartheta^{2\mathfrak{r}}(x) w_2 \left(\mathcal{F}^{(\mathfrak{r})}, \frac{\vartheta}{\mathfrak{n}}\right)_{\mathrm{L}_{\mathrm{p}[-1,-1+\mathfrak{n}^{-2}]\cup[1-\mathfrak{n}^{-2},1]}} \quad (1.7.3)$$

And

$$\|\mathcal{F}(x) - \mathbb{S}(x)\|_{\mathrm{L}_{\mathrm{p}[-1,-1+\mathfrak{n}^{-2}]\cup[1-\mathfrak{n}^{-2},1]}} \leq c(\mathrm{p}, \mathfrak{r})\vartheta^{2\mathfrak{r}}(x) w_1(\mathcal{F}^{(\mathfrak{r})}, \vartheta^2(x))_{\mathrm{L}_{\mathrm{p}[-1,-1+\mathfrak{n}^{-2}]\cup[1-\mathfrak{n}^{-2}.1]}} \quad (1.7.4)$$

Proof.

Using by Lemma 1.2.20 we get,

If we let x be the chebyshev partition $\mathrm{T}_{n} = \{t_j\}$, where $n \geq \mathcal{N} := 3/\sqrt{\mathcal{H}}$, then by ($x_1 - a \leq \mathcal{H}$ and $b - x_{n-1} \leq \mathcal{H}$) is satisfied since $\sin \pi/2n \leq \pi/2n$ we have : $t_1 + 1 = 1 - t_{n-1} = 2\sin^2\left(\frac{\pi}{2n}\right) \leq \frac{\pi^2}{2n^2} \leq \frac{5}{\mathcal{N}} \leq \mathcal{H}$.

Since $\frac{\vartheta(x)}{n} \sim \Omega_n(x) \sim t_j - t_{j-1,}$for $x \in [t_{j-1}, t_j]$. and from Lemma's 1.2.21 , 1.2.22 and above, we get

$$\|\mathcal{F}(x) - \mathbb{s}(x)\|_{p} \leq c(p, a)^{r} \left\|\mathfrak{L}_2^{L}(\mathcal{F}^{(r)}, x; [a, x_1]\right]\|_{L_p[a,x_1]},$$

$$\|\mathcal{F}(x) - \mathbb{s}(x)\|_{p} \leq c(p, b)^{r} \left\|\mathfrak{L}_2^{R}(\mathcal{F}^{(r)}, x; [x_{n-1}, b]\right]\|_{L_p[x_{n-1},b]}.$$

And so 1.7.2, 1.7.3 and 1.7.4 satisfied.

<u>Lemma</u> 1.7.5

Given $r \in \mathbb{N}$, there is a constant $c = c(p, r)$ such that if $\mathcal{F} \in L_{p^r}[-1,1]$ is convex, then there is a number $N = N(\mathcal{F}, r)$, depending on $\mathcal{F}$ and r, such that for $n \geq N$, there are continuously differentiable convex piecewise polynomials $\mathbb{S}$ of degree $r + 1$ with knots at the chebyshev partition T_n (i.e., $\mathbb{S} \in \sum_{r+2,n}^{(1)} \cap \Delta^{(2)}$, satisfying 1.7.2, 1.7.3 and 1.7.4 in Proposition 1. 7.1 .

Proof.

Let $\mathcal{Y}_r(Z_m)$denoted the space of all piecewise polynomial function (ppf) of degree $r - 1$ (order r) with the knots $Z_m := (\mathcal{Z}_i)_{i=0}^{m}$, $a =: \mathcal{Z}_0 < \mathcal{Z}_1 < \cdots < \mathcal{Z}_{n-1} < \mathcal{Z}_m := b$ .Also, the scale of the partition Z_m is denoted by

$$\mathcal{O}(Z_m) := max_{0\le j\le m-1}\frac{|J_{j\pm1}|}{|J_j|}, \text{ where } J_j = [\mathcal{Z}_j, \mathcal{Z}_{j+1}] \tag{1.7.6}$$

Where $|J_j|$ is the length of the interval J_j .

Let n be a sufficiently large fixed number, and let $\mathbb{S}_0 \in \sum_{r+2,n} \cap \Delta^{(2)}$ be a piecewise polynomial from the statement of Proposition 1.7.1 for which estimates1.7.2-1.7.4 hold. Let $a := x_{2n-1,2n}$, $b := x_{1,2n}$ and let $Z_n = (\mathcal{Z}_i)_{i=0}^{n}$ be such that $\mathcal{Z}_0 := a, \mathcal{Z}_n := b$ and $\mathcal{Z}_i := x_{n-i},\ 1 \le i \le n-1$ (note that $Z_n \subset T_{2n}$).

Clearly, $\mathbb{S}_0 \in \mathcal{Y}_{r+2}(Z_n), \mathcal{O}(Z_n) \sim 1$, and by Lemma 1.2.23 implies that

$\|\mathbb{S}_0 - \widetilde{\mathbb{S}}_0\|_{L_p(\tilde{I}_j)} \le c(p,r)w_{r+2}(\mathbb{S}_0, h_j, J_j)_{L_{p(\tilde{I}_j)}}$, where $\tilde{I}_j := I_j \cap [a,b]$ and

$$J_j := [x_{j+2}, x_{j-2}] \cap [a,b] \tag{1.7.7}$$

and

$$\widetilde{\mathbb{S}}_0^{(v)}(a) = \widetilde{\mathbb{S}}_0^{(v)}(a) \text{ and } \widetilde{\mathbb{S}}_0^{(v)}(b) = \widetilde{\mathbb{S}}_0^{(v)}(b), v = 0,1 \tag{1.7.8}$$

Let

$$\mathbb{S}(x) := \begin{cases} \mathbb{S}_0(x), \text{if } x \in [-1,1]\backslash[a,b], \\ \widetilde{\mathbb{S}}_0(x), \ \text{ if } x \in [a,b] \end{cases}$$

$\mathbb{S} \in \sum_{r+2,2n}^{(1)} \cap \Delta^{(2)}$, inequalities 1.7.3 and 1.7.4 satisfied, if we put instead of $2n, n$ and 1.7.2 also satisfied. since $\frac{\vartheta(x)}{n} \sim h_j$, for any $x \in J_j$, $1 \le j \le n$. Thus, for $x \in \tilde{I}_j, 1 \le j \le n$,

$$\|\mathcal{F} - \mathbb{S}\|_{L_p(\tilde{I}_j)} \le \|\mathcal{F}(x) - \mathbb{S}_0(x)\|_{L_p(\tilde{I}_j)} + \|\mathbb{S}_0(x) - \widetilde{\mathbb{S}}_0(x)\|_{p(\tilde{I}_j)}$$

$$\le c(p)\|\mathcal{F} - \mathbb{S}_0\|_{p(J_j)} + c(p)w_{r+2}(\mathcal{F}, h_j; J_j)_{p(J_j)}$$

$$\le c(p)h_j^r w_2(\mathcal{F}^{(r)}, h_j)_p \le c(p)(\frac{\vartheta(x)}{n})^r w_2(\mathcal{F}^{(r)}, \frac{\vartheta(x)}{n})_p.$$

Theorem 1.7.9

Given $\mathfrak{r} \in \mathbb{N}$, there is a constant $c = c(\mathrm{p}, \mathfrak{r})$ with the property that if $\mathcal{F} \in \mathrm{L}_\mathrm{p}^\mathfrak{r} \cap \Delta^{(2)}$ then there exist a number $\mathrm{N} = \mathrm{N}(\mathcal{F}, \mathfrak{r})$, depending on $\mathcal{F}$ and $\mathfrak{r}$, such that for every $\mathfrak{n} \geq \mathcal{N}$, there is $\mathcal{P}_\mathfrak{n} \in \Pi_\mathfrak{n} \cap \Delta^{(2)}$ satisfying

$$\|\mathcal{F}(x) - \mathcal{P}_\mathfrak{n}(x)\|_\mathrm{p} \leq c(\mathrm{p}, \mathfrak{r}) \left(\frac{\vartheta(x)}{\mathfrak{n}}\right)^\mathfrak{r} w_2 \left(\mathcal{F}^{(\mathfrak{r})}, \frac{\vartheta(x)}{\mathfrak{n}}\right)_\mathrm{p}, x \qquad (1.7.10)$$

The following stronger estimates are valid:

$$\|\mathcal{F}(x) - \mathcal{P}_\mathfrak{n}(x)\|_{\mathrm{L}_\mathrm{p}[-1,-1+\mathfrak{n}^{-2}]\cup[1-\mathfrak{n}^{-2},1]} \leq$$
$$c(\mathrm{p}, \mathfrak{r})\vartheta^{2\mathfrak{r}}(x) w_2(\mathcal{F}^{(\mathfrak{r})}, \frac{\vartheta(x)}{\mathfrak{n}})_{\mathrm{L}_\mathrm{p}[-1,-1+\mathfrak{n}^{-2}]\cup[1-\mathfrak{n}^{-2},1]} \qquad (1.7.11)$$

, and

$$\|\mathcal{F}(x) - \mathcal{P}_\mathfrak{n}(x)\|_{\mathrm{L}_\mathrm{p}[-1,-1+\mathfrak{n}^{-2}]\cup[1-\mathfrak{n}^{-2},1]} \leq$$
$$c(\mathrm{p}, \mathfrak{r})\vartheta^{2\mathfrak{r}}(x) w_1 \left(\mathcal{F}^{(\mathfrak{r})}, \vartheta^2(x)\right)_{\mathrm{L}_\mathrm{p}[-1,-1+\mathfrak{n}^{-2}]\cup[1-\mathfrak{n}^{-2},1]}) \qquad (1.7.12)$$

Proof. In The Case $\mathfrak{r} \geq 2$

Let $\mathbb{S}$ be the piecewise polynomial from the statement of Lemma 1.7.5. Without loss of generality, we can assume that $\mathbb{S}$ does not have knots at x_1 and $x_{\mathfrak{n}-1}$ (it is sufficient to treat $\mathbb{S}$ as a piecewise polynomial with knots at the chebyshev partition $\mathrm{T}_{2\mathfrak{n}}$).then

$\mathcal{L}_1(x) := \mathbb{S}(x)|\mathrm{I}_1 \cup \mathrm{I}_2 = \mathcal{F}(1) + \frac{\mathcal{F}'(1)}{1!}(x-1) + \cdots + \frac{\mathcal{F}^{(\mathfrak{r})}(1)}{\mathfrak{r}!}(x-1)^\mathfrak{r} + a_+(\mathfrak{n}; \mathcal{F})(x-1)^{\mathfrak{r}+1}$ and

$\mathcal{L}_\mathfrak{n}(x) := \mathbb{S}(x)|\mathrm{I}_\mathfrak{n} \cup \mathrm{I}_{\mathfrak{n}-1} = \mathcal{F}(-1) + \frac{\mathcal{F}'(-1)}{1!}(x+1) + \cdots + \frac{\mathcal{F}^{(\mathfrak{r})}(-1)}{\mathfrak{r}!}(x+1)^\mathfrak{r} + a_-(\mathfrak{n}; \mathcal{F})(x+1)^{\mathfrak{r}+1}$,where $a_+(\mathfrak{n}, \mathcal{F})$ and $a_-(\mathfrak{n}, \mathcal{F})$ are some constants that depend only on $\mathfrak{n}$ and $\mathcal{F}$ ·

Note that

$$\mathfrak{n}^{-2} max\{|a_+(\mathfrak{n}, \mathcal{F})|, |a_-(\mathfrak{n}, \mathcal{F})|\} \to 0 \text{ as } \mathfrak{n} \to \infty \qquad (1.7.13)$$

From Proposition 1.7.1, Lemma 1.7.4 that, for all $x \in \mathrm{I}_1 \cup \mathrm{I}_2$

$$\|a_{+}(\mathfrak{n},\mathcal{F})(1-x)\|_{\mathrm{p}} \le c(\mathrm{p})\frac{\|\mathcal{L}_1(x)-\mathcal{F}(x)\|_{\mathrm{p}}}{(1-x)^{\mathfrak{r}}} + \frac{c(\mathrm{p})}{(1-x)^{\mathfrak{r}}}\left\|\mathcal{F}(x)-\mathcal{F}(1)-\frac{\mathcal{F}'(1)}{1!}(x-1)-\cdots - \frac{\mathcal{F}^{(\mathfrak{r})}(1)}{\mathfrak{r}!}(x-1)^{\mathfrak{r}}\right\|_{\mathrm{p}}$$

$\le c(\mathrm{p})w_1\left(\mathcal{F}^{(\mathfrak{r})},1-x\right)_{\mathrm{p}} + \frac{1}{(\mathfrak{r}-1)!(1-x)^{\mathfrak{r}}}\left\|\int_x^1(\mathcal{F}^{(\mathfrak{r})}(\mathrm{t})-\mathcal{F}^{(\mathfrak{r})}(1))(\mathrm{t}-x)^{\mathfrak{r}-1}d\mathrm{t}\right\|_{\mathrm{p}} \le c(\mathrm{p})w_1\left(\mathcal{F}^{(\mathfrak{r})},1-x\right)_{\mathrm{p}}$, and, in particular,

$\mathfrak{n}^{-2}\|a_{+}(\mathfrak{n},\mathcal{F})\|_{\mathrm{p}} \le c(\mathrm{p})w_1(\mathcal{F}^{(\mathfrak{r})},\mathfrak{n}^{-2})_{\mathrm{p}} \to 0$ as $\mathfrak{n}\to\infty$.

Analogously, one draws a similar conclusion for $\|a_{-}(\mathfrak{n},\mathcal{F})\|_{\mathrm{p}}$

For $\mathcal{F}\in \mathrm{L}_p^{\mathfrak{r}}, \mathfrak{r}\ge 2$, let $\mathfrak{i}_{+}\ge 2$, be the smallest integer $2\le \mathfrak{i}\le \mathfrak{r}$,

If it exists, such that $\mathcal{F}^{(\mathfrak{i})}(1)\ne 0$, and denote

$$\mathcal{D}_{+}(\mathfrak{r},\mathcal{F}) = \begin{cases}(2\mathfrak{r}!)^{-1}\left|\mathcal{F}^{(\mathfrak{i}_{+})}(1)\right| & \text{, if } \mathfrak{i}_{+} \text{ exists.}\\ 0 \ , & \text{otherwise.}\end{cases}$$

Similarly, let $\mathfrak{i}_{-}\ge 2$, be the smallest integer $2\le \mathfrak{i}\le \mathfrak{r}$, if it exists, such that $\mathcal{F}^{(\mathfrak{i})}(-1)\ne 0$, and denote

$$\mathcal{D}_{-}(\mathfrak{r},\mathcal{F}) = \begin{cases}(2\mathfrak{r}!)^{-1}\left|\mathcal{F}^{(\mathfrak{i}_{-})}(1)\right| & \text{, if } \mathfrak{i}_{-} \text{exists.}\\ 0 & \text{, otherwise}\end{cases}$$

Hence, if $\mathfrak{n}$ is sufficiently large, then

$$\mathbb{S}''(x)\ge \mathcal{D}_{+}(\mathfrak{r},\mathcal{F})(1-x)^{\mathfrak{r}-2}, x\in(x_2,1] \tag{1.7.14}$$

And

$$\mathbb{S}''(x)\ge \mathcal{D}_{-}(\mathfrak{r},\mathcal{F})(x+1)^{\mathfrak{r}-2}, x\in[-1,x_{\mathfrak{n}-2}]\ . \tag{1.7.15}$$

Given $\mathfrak{r}\in\mathbb{N}, \mathfrak{r}\ge 2$, and a convex $\mathcal{F}\in \mathrm{L}_{\mathrm{p}}^{\mathfrak{r}}$, let $\mathcal{T}\in\Phi^2$

be such that

$w_2(\mathcal{F}^{(\mathfrak{r})}, \mathfrak{t}) \sim \mathcal{T}(\mathrm{t})$, denote $\Gamma(\mathrm{t}) := \mathrm{t}^{\mathfrak{r}}\mathcal{T}(\mathrm{t})$, and note that $\Gamma \in \Phi^{\mathfrak{r}+2}$.

For a sufficiently large $\mathrm{N} \in \mathbb{N}$ and each $\mathfrak{n} \geq \mathrm{N}$, we take the piecewise polynomial $\mathbb{S} \in \sum_{\mathfrak{r}+2}, \mathfrak{n}$ of Lemma 1.7.5 satisfying 1.7.14, 1.7.15 and observe that

$$w_{\mathfrak{r}+2}(\mathcal{F}, \mathfrak{t}) \leq \mathfrak{t}^2 w_2(\mathcal{F}^{(\mathfrak{r})}, \mathfrak{t})_p \sim \Gamma(\mathfrak{t}) .$$

So that by Proposition 1.3.1 with $\kappa = \mathfrak{r} + 2$, we conclude that

$$\mathfrak{b}_{\mathfrak{r}+2}(\mathbb{S}, \Gamma) \leq c .$$

Now, it follow from 1.7.14 and Lemma 1.2.1

$$min_{x \in \mathrm{I}_2} \mathbb{S}''(x) \geq \mathcal{D}_+(\mathfrak{r}, \mathcal{F})|\mathrm{I}_1|^{\mathfrak{r}-2} \geq 3^{-\mathfrak{r}+2}\mathcal{D}_+(\mathfrak{r}, \mathcal{F})|\mathrm{I}_2|^{\mathfrak{r}-2}$$

And, similarly, the inequality 1.7.15 yields

$$min_{x \in \mathrm{I}_2} \mathbb{S}''(x) \geq 3^{-\mathfrak{r}+2}\mathcal{D}_-(\mathfrak{r}, \mathcal{F})|\mathrm{I}_{\mathfrak{n}-1}|^{\mathfrak{r}-2}.$$

Hence, using Theorem (1.6.6) with $\kappa = \mathfrak{r} + 2, d_+ := 3^{-\mathfrak{r}+2}\mathcal{D}_+(\mathfrak{r}, \mathcal{F})$,

$d_- := 3^{-\mathfrak{r}+2}\mathcal{D}_-(\mathfrak{r}, \mathcal{F})$ and $\mathcal{S} = 2\kappa - 2 = 2\mathfrak{r} + 2$,

We conclude that there exists a polynomial $\mathcal{P} \in \Pi_{c\mathfrak{n}} \cap \Delta^{(2)}$ such that:

$$\|\mathbb{S}(x) - \mathcal{P}(x)\|_{\mathrm{p}} \leq c(\mathrm{p})\delta_{\mathfrak{n}}^{2\mathfrak{r}+2}(x)\Omega_{\mathfrak{n}}^{\mathfrak{r}}(x)\mathcal{T}(\Omega_{\mathfrak{n}}(x)), \; x \in [-1,1] \quad (1.7.16)$$

In particular, for $x \in \mathrm{I}_1 \cup \mathrm{I}_{\mathfrak{n}}$, $x \neq -1,1$, using the fact that

$\Omega_{\mathfrak{n}}(x) \sim \mathfrak{n}^{-2}$ for these x, and $\mathrm{t}^{-2}\mathcal{T}(\mathrm{t})$ is non-increasing we have

$$\|\mathbb{S}(x) - \mathcal{P}(x)\|_{\mathrm{p}} \leq c(\mathrm{p})\left(\mathfrak{n}\vartheta(x)\right)^{2\mathfrak{r}+2}\Omega_{\mathfrak{n}}^{\mathfrak{r}}(x)\mathcal{T}(\Omega_{\mathfrak{n}}(x))$$

$$\leq c(\mathrm{p})\mathfrak{n}^2\vartheta^{2\mathfrak{r}+2}(x)\left(\frac{\mathfrak{n}\Omega_{\mathfrak{n}}(x)}{\vartheta(x)}\right)^2\mathcal{T}\left(\frac{\vartheta(x)}{\mathfrak{n}}\right)$$

$$\leq c(\mathrm{p})\vartheta^{2\mathfrak{r}}(x) w_2\left(\mathcal{F}^{(\mathfrak{r})}, \frac{\vartheta(x)}{\mathfrak{n}}\right)_{\mathrm{p}} \quad (1.8.17)$$

In turn, this implies for $x \in \mathrm{I}_1 \cup \mathrm{I}_{\mathfrak{n}}$, that

$$\|\mathbb{S}(x)-\mathcal{P}(x)\|_{\mathrm{p}} \le c(\mathrm{p})\left(\frac{\vartheta}{\mathfrak{n}}\right)^{\mathfrak{r}} w_2\left(\mathcal{F}^{(\mathfrak{r})},\frac{\vartheta(x)}{\mathfrak{n}}\right)_{\mathrm{p}}, x\in[-1,1] \quad (1.7.18)$$

Now , 1.7.18 together with 1.7.2 yield

$$\|\mathcal{F}(x)-\mathcal{P}_{\mathfrak{n}}(x)\|_{\mathrm{p}} \le c(\mathrm{p},\mathfrak{r})\left(\frac{\vartheta(x)}{\mathfrak{n}}\right)^{\mathfrak{r}} w_2\left(\mathcal{F}^{(\mathfrak{r})},\frac{\vartheta(x)}{\mathfrak{n}}\right)_{\mathrm{p}}, x\in[-1,1]$$

And 1.7.17 together with 1.7.3 yield

$$\|\mathcal{F}(x)-\mathcal{P}_{\mathfrak{n}}(x)\|_{\mathrm{p}} \le c(\mathrm{p},\mathfrak{r})\vartheta^{2\mathfrak{r}}(x) w_2\left(\mathcal{F}^{(\mathfrak{r})},\frac{\vartheta(x)}{\mathfrak{n}}\right)_{\mathrm{p}}, x\in[-1,1]$$

Now to prove $\|\mathcal{F}(x)-\mathcal{P}_{\mathfrak{n}}(x)\|_{\mathrm{p}} \le c(\mathrm{p},\mathfrak{r})\vartheta^{2\mathfrak{r}}(x) w_1(\mathcal{F}^{(\mathfrak{r})},\vartheta^2(x))_{\mathrm{p}}$

Using that $\mathrm{t}^{-1}w_1(\mathcal{F}^{(\mathfrak{r})},\mathrm{t})$ is non-increasing we have, for $x\in \mathrm{I}_1\cup \mathrm{I}_{\mathfrak{n}}$, $x\neq -1,1$,

$$\|\mathbb{S}(x)-\mathcal{P}(x)\|_{\mathrm{p}} \le c(\mathrm{p})\big(\mathfrak{n}\vartheta(x)\big)^{2\mathfrak{r}+2}\Omega_{\mathfrak{n}}^{\mathfrak{r}}(x) w_1(\mathcal{F}^{(\mathfrak{r})},\Omega_{\mathfrak{n}}(x))_{\mathrm{p}}$$

$$\le c(\mathrm{p})\mathfrak{n}^2\vartheta^{2\mathfrak{r}+2}(x)\frac{\Omega_{\mathfrak{n}}(x)}{\vartheta^2(x)} w_1(\mathcal{F}^{(\mathfrak{r})},\vartheta^2(x))_{\mathrm{p}}$$

$$\|\mathbb{S}(x)-\mathcal{P}(x)\|_{\mathrm{p}} \le c(\mathrm{p})\vartheta^{2\mathfrak{r}}(x) w_1\left(\mathcal{F}^{(\mathfrak{r})},\vartheta^2(x)\right)_{\mathrm{p}}.$$

In Case $\mathfrak{r} = 1$.

It is possible to that, in order to prove Theorem 1.7.9 in the case $\mathfrak{r}=1$, it is sufficient to construct a conve**x** polynomial $\mathcal{P}_{\mathfrak{n}}$ the approximates the quadratic spline $\mathbb{S}$ from Proposition 1.7.1 (with $\mathfrak{r}=1$) so that .

$\|\mathbb{S}(x)-\mathcal{P}_{\mathfrak{n}}(x)\|_{\mathrm{p}} \le c(\mathrm{p}) w_3(\mathcal{F},\Omega_{\mathfrak{n}}(x))_p,$

And

$\mathcal{P}_{\mathfrak{n}}(\pm1)=\mathbb{S}(\pm1)$ and $\mathrm{p'}_{\mathfrak{n}}(\pm1)=\mathbb{S}'(\pm1)$ (1.7.19)

To construct the above polynomial . we shall use away similar to that in [2] by replacing $\mathcal{F}$ by $\mathbb{S}$ and $\mathfrak{n}$ by $2\mathfrak{n}$ ■

Chapter Two

Pointwise Approximation of Piecewise Convex Function in L_p, Quasi Normed Spaces

Some authors introduced direct inequalities for the constrained approximation of convex and piecewise convex functions in C[-1,1] with restricted degree of approximation. Here we study the pointwise constrained approximation of piecewise convex functions by pointwise polynomials in L_p space.

2.1 Introduction and Notations:

Let $\mathcal{F} \in L_p(I)$ change its convexity finitely many, say $\mathbb{s} \geq 0$, time in the interval.

define $L_p(I) = \{\mathcal{F}: I \to R : \mathcal{F} \in L_p\}$,where $I = [-1,1]$, $L_p^{r}(I) = \{\mathcal{F}: I \to R : \mathcal{F}, \mathcal{F}^{r} \in L_p, r \text{ fold } L_p\}$, $\|\mathcal{F}\|_{Lp} = (\int_{-1}^{1} |\mathcal{F}(x)|^p)^{\frac{1}{p}} = \|\mathcal{F}\|_p$.

We are interested in pointwise estimates on the approximation of $\mathcal{F}$ by algebraic polynomials there are coconvex with it. That is, that change their convexity exactly at the points where $\mathcal{F}$ does. Specifically, denote by $Y_{\mathbb{s}}$, $\mathbb{s} \in N$.

The set of all collections $Y_{\mathbb{s}} := \{y_i\}_{i=1}^{\mathbb{s}}$ such that $-1 < y_{\mathbb{s}} < \cdots < y_1 < 1$, denoted by $Y_{\mathbb{s}}$, $\mathbb{s} \in \mathbb{N}$.

Let $\Delta^{(2)}(Y_{\mathbb{s}})$ denoted the collection of all function $\mathcal{F} \in L_p[-1,1]$ that change convexity at the set $Y_{\mathbb{s}}$ and are convex in $[y_1, 1]$.Namely in the interval $[y_{i+1}, y_i]$, $\mathcal{F}$ is convex when i is even, and is concave when i is odd. We also use the notation $y_0 = 1$ and $y_{\mathbb{s}+1} = -1$. Denote

$$\Pi(x) = \prod_{i=1}^{\mathbb{s}} (x - y_i) \qquad (2.1.1)$$

Then for example, if $\mathcal{F} \in L_p^2[-1,1] \cap L_p[-1,1]$ then $\mathcal{F} \in \Delta^{(2)}(Y_{\mathbb{s}})$ if and only if $\Delta_h^{(2)}(\mathcal{F}) > 0$, in (-1,1).

The convex function as the case $\mathbb{s} = 0$, where we write $Y_0 := \emptyset$

$\mathcal{P}_n$ the set of algebraic polynomials of degree $< n$

Norm estimates on the degree of approximation of $\mathcal{F} \in \Delta^{(2)}(Y_{\mathbb{s}})$ by

$\mathcal{P}_n \in \mathbb{P}_n \cap \Delta^{(2)}(Y_s)$,namely, estimates of $E_n^{(2)}(\mathcal{F}, Y_s) := inf_{\mathcal{P}_n \in \mathbb{P}_n \cap \Delta^{(2)}(Y_s)} \|\mathcal{F} - \mathcal{P}_n\|_p$, More specifically, for $\mathcal{F} \in L_p[-1,1]$,

let $\Delta_h^\kappa \mathcal{F}(x) := \sum_{i=0}^{\kappa} (-1)^{\kappa-i} \binom{\kappa}{i} \mathcal{F}\left(x + \frac{\kappa}{2}h + ih\right)$, if $x \mp \frac{\kappa}{2}h \in I$ and $:= 0$ otherwise,

is the symmetric difference of order κ , and denote by

$w_\kappa(\mathcal{F}, t) := \sup_{h\in[0,t]} \|\Delta_h^\kappa \mathcal{F}\|_p = \sup_{h\in[0,t]} max_{x\in[-1,1]} |\Delta_h^\kappa \mathcal{F}(x)|$, $t > 0$ is the kth modulus of smoothness of $\mathcal{F}$ on I.

Given $n \in \mathbb{N}, \kappa \in \mathbb{N}, r \in \mathbb{N}_0, s \in \mathbb{N}_0$ and $Y_s \in \mathbb{Y}_s$. We wish to estimate

$$E_{n,\kappa,r}^{(2)}(\mathcal{F}, Y_s) := inf_{\mathcal{P}_n \in \mathbb{P}_n \cap \Delta^{(2)}(Y_s)} \left\| \frac{\mathcal{F} - \mathcal{P}_n}{\Omega_n^r w_\kappa(\mathcal{F}^{(r)}, \Omega_n)} \right\|_p$$

$$= inf_{\mathcal{P}_n \in \mathbb{P}_n \cap \Delta^{(2)}(Y_s)} max_{x\in[-1,1]} \left\| \frac{\mathcal{F}(x) - \mathcal{P}_n(x)}{\Omega_n^r(x) w_\kappa(\mathcal{F}^{(r)}, \Omega_n(x))} \right\|_p,$$

Where $\vartheta(x) = \sqrt{1 - x^2}$ and $\Omega_n(x) = \vartheta(x)n^{-1} + n^{-2}, x \in [-1,1]$, for $\mathcal{F} \in \Delta^{(2)}(Y_s) \cap L_p^r(I)$ such that $\mathcal{F} \notin \mathcal{P}_{\kappa+r}$. We put $E_{n,\kappa,r}^{(2)}(\mathcal{F}, Y_s) := 0$, if $\mathcal{F} \in \mathcal{P}_{\kappa+r} \cap \Delta^{(2)}(Y_s)$.

In [23] by Nikolskii, Timan, Dzyadyk, Frend and Brudnyi study the classical pointwise estimates for unconstrained approximation which is also true for coconvex approximation .

In [8,27] the authors restricted them attention to the following three cases:

a) $r \geq 3$ and $s = 1$, b) $r = 2, \kappa \leq 3$ and $s = 1$, c) $r = 0$ and $, \kappa = 3$. All other cases also have been investigated .

For instance, for $s \geq 2$, it introduced by [19,25] that for no $\kappa \geq 1$, and $r \geq 0$. is it possible to have constants $c = c(\kappa, r, s)$ and $N = N(\kappa, r, s)$, depending only on κ, r and s, such that the inequality

$$E^{(2)}_{n,\kappa,r}(\mathcal{F}, Y_s) \leq c, \tag{2.1.2}$$

Is satisfied for all $n \geq \mathbb{N}$ and $Y_s \in \mathbb{Y}_s$, and for all $\mathcal{F} \in \Delta^{(2)}(Y_s) \cap L^r_p(I)$.

Furthermore, for $s = 1$, it follow by [28,Theorem 2] that no $\kappa \geq 1$ and $r \geq 0$ such that $\kappa + r > 2$, and $Y_1 \in \mathbb{Y}_1$, is possible to have constants $c = c(\kappa, r, Y_1)$ and $N = N(\kappa, r, Y_1)$, depending only on κ, r and Y_1, such that (2.1.2) hold for all $n \geq N$ and for all $\mathcal{F} \in \Delta^{(2)}(Y_s) \cap L^r_p(I)$.

In [27] and [28] the authors studied the pointwise approximation of special polynomials. In this chapter we study the general pointwise approximation by any piecewise polynomial for function in L_p[-1,1].

Let $\sum_\kappa := \sum_{\kappa,n}$ the x_j, $1 \leq j \leq n-1$ piecewise polynomials of degree not exceeding $\kappa - 1$,that are continuous.

$\sum^1_{\kappa,n}$ is the space of all piecewise polynomials $\mathcal{P}$ with $\|\mathcal{P}'\|_p < \infty$,

That is, $\mathbb{S} \in \sum_{\kappa,n}$, if

$$\mathbb{S}|I_j = \mathcal{P}_{j;n} =: \mathcal{P}_j, \qquad j = 1, \dots, n,$$

Where $\mathcal{P}_j \in \mathbb{P}_\kappa$, and $\mathcal{P}_j(x) =: \mathcal{P}_{j+1}(x)$. $j = 1, \dots, n$.

Given $Y_s \in \mathbb{Y}_s$, let

$O_i := O_{i,n}(Y_s) := (x_{j+1}, x_{j-2})$, if $y_i \in [x_j, x_{j-1})$, denote

$$O := O(n, Y_s) := \cup_{i=1}^{s} O_i, \ O(n, \emptyset) := \emptyset, \tag{2.1.3}$$

And write $j \in H = H(n, Y_s)$, if $I_j \cap O = \emptyset$,

Finally, denote by $\sum_{\kappa,n}(Y_s) \subseteq \sum_{\kappa,n}$, the subset of those piecewise polynomials for which $\mathcal{P}_j \equiv \mathcal{P}_{j+1}$, whenever both $j, j+1 \notin H$.

For $\Gamma \in \Phi^\kappa, \Gamma \not\equiv 0$. And $\mathbb{S} \in \sum_{\kappa;n}(Y_s)$, denote

$$b_{i,j}(\mathbb{S}) := b_{i,j:n}(\mathbb{S}) = \frac{\|p_i - p_j\|_p}{\Gamma(h_j)} \left(\frac{h_j}{h_{i,j}}\right)^\kappa, \ 1 \leq i, j \leq n.$$

Also, denote

$$b_\kappa(\mathbb{S}) \coloneqq b_{\kappa;n}(\mathbb{S}) := max_{1\le i,j\le n} b_{i,j}(\mathbb{S}).$$

2.2 Auxiliary Lemma:

In this section we introduce the result that we need to prove our Theorems.

Lemma 2.2.1 [22]

For any $x \in I_i$, $\mathcal{F}: [-1,1] \to \mathbb{R}$

$|\mathcal{F}(x) - \mathcal{P}_j(x)| \le c\left(\frac{h_{i,j}}{h_j}\right)^\kappa (\Gamma(h_j) + |\mathcal{F} - \mathcal{P}_j|)$, where $\mathcal{P}_j \in \mathbb{P}_\kappa$.

Lemma 2.2.2 [8]

$|(x - x_j)_+ - \tau_j(x)| \le c(\mathfrak{b}, \mathbb{s}) h_j \mathfrak{T}_j^{\mathfrak{b}}(x)$, where $\mathfrak{T}_j^{\mathfrak{b}}(x) = (\frac{h_j}{|x - x_j| + h_j})^{\mathfrak{b}}$, τ_j are polynomials with $\mathfrak{b} = \kappa + 3$, and

$$(x - x_j)_+ \coloneqq \begin{cases} x - x_j, & \text{if } x \ge x_j, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.2.3 [8]

$\tau_j''\Pi(x)\Pi(x_j) \ge 0, x \in [-1,1], x_j \in I_j$.

Lemma 2.2.4 [8]

$\Omega_n^2(y) < 4\Omega(|x - y| + \Omega)$,

$(|x - y| + \Omega)/2 < |x - y| + \Omega_n(y) < 2(|x - y| + \Omega), \ \ x, y \in [-1,1].$

Lemma 2.2.5 [23]

The polynomial $\bar{\tau}_j$ satisfy

$$|\bar{\tau}_j''(x)| \le \frac{c h_j}{\Omega^2} \pi(x), x \in I_j ,$$

Lemma 2.2.6 [23]

$$|(x-x_j)_+ - \bar{\tau}_j(x)| \le c\Omega\left(\frac{h_j}{|x-x_j|+\Omega}\right)^2, x \in I$$

Lemma 2.2.7 [24]

$$\frac{1}{2}(\mathcal{J}-j) \le \frac{x_j - x_{\mathcal{J}}}{x_i - x_{i+1}} \le 2(\mathcal{J}-j)$$

Lemma 2.2.8 [24]

Let $\mathcal{A} := \{j_0, \dots, j_0 + \iota_0\}$ and let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ be such that $\#\mathcal{A}_1 = 2\iota_1$ and $\#\mathcal{A}_2 = 2\iota_2$. If $\sigma_j \in \{-1,1\}$, $j \in \mathcal{A}_2$, then there exist $2\iota_1$ constants $a_i, i \in \mathcal{A}_1$, such that $|a_i| \le \binom{\iota_0}{\iota_1}$, $i \in \mathcal{A}_1$, and $\frac{1}{\iota_2}\sum_{j\in\mathcal{A}_2} \sigma_j\left(x-x_j\right) + \frac{1}{\iota_1}\sum a_i(x-x_i) \equiv 0.$

Lemma 2.2.9 [24]

$$\bar{\tau}_j''(x)\Pi(x)\Pi\left(x_j\right) \le 0, x \in I \setminus I_j,$$

Lemma 2.2.10 [24]

Let E be an interval which is union of $\iota \ge 15\mathbb{s}$ of the intervals I_j , and let a set $\mathcal{J} \subseteq E$ be the union of $1 \le \mu \le \iota/4$ of these intervals . then there exists a polynomial $Q_n(x) = Q_n(x, E, \mathcal{J})$ of degree $\le cn$, satisfying

$$Q''_n(x)\delta(x) \ge c_1 \frac{\iota}{\mu}\left(\frac{\Omega}{max\{\Omega, dist(x,E)\}}\right)^{25(\mathbb{s}+1)+2\kappa} \frac{\pi(x)\Gamma(\Omega)}{\Omega^2},\ x \in \mathcal{J} \cup [-1,1]\setminus E), \tag{2.2.11}$$

(we may take $c_1 \le 1$)

$$Q''_n(x)\delta(x) \ge -\frac{\pi(x)\Gamma(\Omega)}{\Omega^2}, x \in E\setminus\mathcal{J}, \tag{2.2.12}$$

Lemma 2.2.13 [26]

Suppose that $\kappa \ge 3$, $\Gamma \in \Phi^k$, and $\mathbb{S} \in \sum_{\kappa,n}^{(1)}$ is such that $b_\kappa(\mathbb{s}, \Gamma) \le 1$, If an interval $I_{\mu,\upsilon}$ contains at least $2\kappa - 5$ intervals I_i and points $x_i^* \in I_i^0$ such that

$$|\mathbb{s}''(x_i^*)| \le c(\mathrm{p})\frac{\Gamma(\mathrm{h_i})}{\mathrm{h_i^2}} \quad , \qquad (2.2.14)$$

then for every$1 \le \mathrm{j} \le \mathrm{n}$, we have

$$\left\|\frac{\Omega^2 \mathbb{s}''}{\Gamma(\Omega)}\right\|_{\mathrm{I_j}} \le c(\mathrm{p})[(\mathrm{j}-\mu)^{4\kappa} + (\mathrm{j}-v)^{4\kappa}] \qquad (2.2.15)$$

<u>Lemma</u> (2.2.16) [26]

$\sum_{\mathrm{j}=1}^{\mathrm{n}} \tilde{\mathrm{T}}_{\mathrm{j:n_1}}(x) \equiv 1$, where $\tilde{\mathrm{T}}_{\mathrm{j:n_1}}$ is a polynomial.

<u>Lemma</u> (2.2.17) [24]

$\left|\mathbb{S}^{(q)}(x) - \mathcal{D}_{\mathrm{n_1}}^{(q)}(x)\right| \le \frac{c}{\Omega^{q}}\left(\mathfrak{b}_\kappa(\mathbb{S},\mathcal{A}) + \mathfrak{b}_\kappa(\mathbb{S})\frac{\mathrm{n}}{\mathrm{n_1}}\left(\frac{\Omega}{\Omega + dist(x,\mathrm{I}\backslash\mathcal{A})}\right)^{\mathfrak{b}_3}\right), x \in \mathcal{A} \cap \bar{O}_e, q = 0, \dots, \mathbb{s}+2$, where $\mathfrak{b}_3 = \mathfrak{b}_2 - \mathbb{s} - 2\kappa - 6 > 0$, $\mathcal{A}$ be a proper interval and $\mathbb{S}$, $\mathcal{D}_{\mathrm{n_1}}$ are polynomials.

<u>Lemma</u> (2.2.18) [23]

$$\left|\mathbb{S}''(x) - \mathcal{D}_{\mathrm{n_1}}''(x)\right| \le \frac{c}{\Omega^{2}}\left(\mathfrak{b}_\kappa(\mathbb{S},\mathcal{A}) + \mathfrak{b}_\kappa(\mathbb{S})\frac{\mathrm{n}}{\mathrm{n_1}}\left(\frac{\Omega}{dist(x,\mathrm{I}\backslash\mathcal{A})}\right)^{\mathfrak{b}_3}\right), x \in \mathcal{A}$$

<u>Lemma</u> (2.2.19) [23]

$$\left|\tilde{\mathrm{T}}_{\mathrm{j:n_1}}^{(q)}(x)\right| \le \frac{c}{\Omega_{\mathrm{n_1}}^{q}(x)}\left(\frac{\Omega_{\mathrm{n_1}}(x)}{\Omega_{\mathrm{n_1}}(x) + dist(x,\mathrm{I_j})}\right)^{\mathfrak{b}_2}, x \in \mathrm{I}, 1 \le \mathrm{j} \le \mathrm{n}, 0 \le q \le \mathbb{s}+2,$$

where $\mathfrak{b}_2 = \frac{1}{2}(\mathfrak{b}_1 - 1)$.

<u>Lemma</u> (2.2.20) [23]

Let $\mathrm{Y_s} \in \mathbb{Y}_{\mathbb{s}}$, $\kappa \in \mathrm{N}$ and $\Gamma \in \Phi^\kappa$ be given. If $\mathbb{S} \in \sum_{\kappa;\mathrm{n}}^{1}(\mathrm{Y_s}) \cap \Delta^{(2)}(\mathrm{Y_s})$ is such that $\quad |\mathbb{S}''(x)| \le \frac{\Gamma(\Omega)}{\Omega^2}, \quad x \in [-1,1]\backslash\{x_\mathrm{j}\}_{\mathrm{j}=1}^{\mathrm{n}-1}$.

Then there is a polynomial $\mathcal{P}_\mathrm{n} \in \Delta^{(2)}(\mathrm{Y_s})$ of degree $\tilde{\mathrm{n}} \le c\mathrm{n}$. Such that

$|\mathbb{S}(x) - \mathcal{P}_\mathrm{n}(x)| \le c\,\Gamma(\Omega), x \in [-1,1]$.

Lemma (2.2.21) [23]

If $\mathbb{S} \in \sum_{\kappa;\mathfrak{n}}$, Then $\left|\mathbb{S}(x) - \mathcal{D}_{\mathfrak{n}_1}(x)\right| \leq c\mathfrak{b}_{\kappa}(\mathbb{S})\Gamma(\Omega)$, where $\mathcal{D}_{\mathfrak{n}_1}$ is polynomial.

Lemma (2.2.22) [23]

If $\mathcal{A}$ is a proper interval and $\mathbb{S} \in \sum_{\kappa;\mathfrak{n}}^{1}(Y_{\mathbb{s}})$ is such that $\mathbb{S}''(y_{\mathrm{i}}) = 0, 1 \leq \mathrm{i} \leq \mathbb{s}$, then

$$\left|\mathbb{S}''(x) - \mathcal{D}''_{\mathfrak{n}_1}(x)\right| \leq \frac{\mathcal{C}_0\Gamma(\Omega)\pi(x)}{\Omega^{2}}(\mathfrak{b}_{\kappa}(\mathbb{S},\mathcal{A}) + \mathfrak{b}_{\kappa}(\mathbb{S})\frac{\mathfrak{n}}{\mathfrak{n}_1}(\frac{\Omega}{dist(x,\mathrm{I}\backslash\mathcal{A})})^{\mathfrak{b}_1})$$

where $\mathrm{c}_0 = \mathrm{c}_0(\kappa, \mathbb{s}, \mathfrak{b})$ and $\pi(x) = \prod_{\mathrm{i}=1}^{\mathbb{s}} \frac{|x-y_{\mathrm{i}}|}{|x-y_{\mathrm{i}}|+\Omega}$.

Lemma (2.2.23) [23]

$$\left\|\mathcal{P}_{\mathcal{v}}{}^{(\mathrm{r})} - \mathcal{P}_{\mathrm{j}}{}^{(\mathrm{r})}\right\|_{\mathrm{I}_{\mathcal{v}}} \left|\widetilde{\mathrm{T}}_{\mathrm{j}}^{(q-\mathrm{r})}(x)\right| \leq \frac{c\mathfrak{b}_{\mathcal{v},\mathrm{j}}}{\Omega^{q}}\frac{\mathfrak{n}}{\mathfrak{n}_1}\Omega^{\mathfrak{b}_3}h_{\mathrm{j}}\left(\frac{1}{\Omega+dist(x,I_{\mathrm{j}})}\right)^{\mathfrak{b}_3+1}, 0 \leq \mathrm{r} \leq q,$$

Where $\mathcal{P}_{\mathcal{v}}, \mathcal{P}_{\mathrm{j}}, \widetilde{\mathrm{T}}_{\mathrm{j}}$ are polynomials.

Lemma (2.2.24)

Let $\mathcal{P} \in \boldsymbol{\mathcal{P}}_{\kappa}$ be such that

$$\mathcal{P}(x) \geq 0 = \mathcal{P}(0), x \in [-1,1].$$

Then

$$\int_0^1 \mathcal{P}(x)dx \leq c(\mathrm{p})(1 - \tfrac{1}{\kappa})\|\mathcal{P}\|_{\mathrm{p}} \qquad (2.2.25)$$

and similarly,

$$\int_{-1}^0 \mathcal{P}(x)dx \leq c(\mathrm{p})(1 - \tfrac{1}{\kappa})\|\mathcal{P}\|_{\mathrm{p}} \qquad (2.2.26)$$

Proof.

For $\kappa = 1$ and $\kappa = 2$ our assumption implies that $\mathcal{P} \equiv 0$, so let $\kappa \geq 3$.

We will prove 2.2.25, the other inequality is symmetric .

By Bernstein inequality,

$\|\mathcal{P}'(x)\|_p \leq c(p)\kappa - 1\|\mathcal{P}(x)\|_p, \qquad x \in [-1,1]$

Put $t_0 := \frac{2}{\kappa}$ and get

$\int_0^{t_0} \mathcal{P}(x)dx = \int_0^{t_0} \int_0^{x} \mathcal{P}'(u)du dx \leq c(p)\frac{1}{\kappa}\|\mathcal{P}(x)\|_p$

Hence $\int_0^1 \mathcal{P}(x)dx = \int_0^{t_0} \mathcal{P}(x)dx + \int_{t_0}^1 \mathcal{P}(x)dx$

$\leq \int_0^{t_0} \mathcal{P}(x)dx + (1 - t_0)\,\|\mathcal{P}\|_p$

$\leq \|\mathcal{P}\|_p \left(\frac{1}{\kappa} + 1 - t_0\right)$

$\int_0^1 \mathcal{P}(x)dx = (1 - \frac{1}{\kappa})\|\mathcal{P}\|_p$ ■

Lemma (2.2.27) [25]

Let $\mathcal{P} \in \mathbb{P}_\kappa$, be such that

$x\,\mathcal{P}'(x) \geq 0, x \in [-1,1]$

Then there is a function $\mathfrak{g} := \mathfrak{g}_p \in L_p^\infty[-1,1]$, such that

$x\mathfrak{g}'(x) \geq 0, x \in [-1,1]$,

$\mathfrak{g}(-1) = \mathcal{P}(-1)$, and $\mathfrak{g}(1) = \mathcal{P}(1)$, $\|\mathfrak{g}'\|_p \leq c(p)\|\mathcal{P}'\|_p$

$\mathfrak{g}'(x) \equiv 0,\ x \in [\frac{-1}{2\kappa}, \frac{1}{2\kappa}]$, satisfying

$\int_{-1}^1 \mathfrak{g}(x)dx = \int_{-1}^1 \mathcal{P}(x)dx$.

Proof.

Without lossing of generality assume that $\mathcal{P}(0) = 0$ and $\mathcal{P}(-1) \leq \mathcal{P}(1)$.

Let $\sigma \in L_p^\infty[-\infty, \infty]$ be a nondecreasing function such that

$$\sigma(x) = \begin{cases} 1, & \text{if} x \geq 1 \\ 0, & \text{if} x \leq \frac{1}{2} \end{cases}$$

let $\bar{\mathfrak{g}}(x) := \mathcal{P}(-1) + \sigma(\kappa x)(\mathcal{P}(1) - \mathcal{P}(-1)$, and

$\underline{g}(x) := \mathcal{P}(x)(\sigma(\kappa x) + \sigma(-\kappa x))$, Then, it readily follows that the required function g may be taken in the form

$g = \lambda\bar{g} + (1-\lambda)\underline{g}$ for a suitable $\lambda \in [0,1]$,

Indeed, both $\bar{g}$ and $\underline{g}$ satisfy all requirements of g except the last. now

$\int_{-1}^{1} \underline{g}(x)dx \le \int_{-1}^{1} \mathcal{P}(x)dx$, and

$$\int_{-1}^{1} \bar{g}(x)dx \ge (\int_{-1}^{0} + \int_{1/\kappa}^{1})\bar{g}(x)dx = \mathcal{P}(-1) + (1 - {}^{1}/_{\kappa})\mathcal{P}(1)$$

$\ge \int_{-1}^{0} \mathcal{P}(x)dx + (1 - {}^{1}/_{\kappa})\mathcal{P}(1) \ge \int_{-1}^{1} \mathcal{P}(x)dx$, we applied Lemma 2.2.24 ■

Lemma 2.2.28

Given $Y_s \in \mathbb{Y}_s$ and $\kappa \in \mathbb{N}$, there exists a number $N(\kappa, Y_s)$ such that, if $n \ge N(\kappa, Y_s)$ and $\mathbb{S} \in \sum_{\kappa;n}^{1}(Y_s) \cap \Delta^{(2)}(Y_s)$ satisfies

$\|\mathbb{S}''(x)\|_p \le c(p)\left\|\frac{\Gamma(\Omega)}{\Omega^2}\right\|_p, x \in [-1,1], x \ne x_j$, where $\Gamma \in \Phi^{\kappa}$, Then there is a function $G \in L_p^1[-1,1]$, such that $G \in L_p^2(x_j, x_{j-1}), j = 1, \dots, n$, $G \in \Delta^{(2)}(Y_s)$,

$$\|G''(x)\|_{L_{p(x_j,x_{j-1})}} \le c(p)\left\|\frac{\Gamma(\Omega)}{\Omega^2}\right\|_p$$

$\|\mathbb{S}(x) - G(x)\|L_{p_{[-1,1]}} \le c(p)\|\Gamma(\Omega)\|_p$ and $G''(x) \equiv 0, x \in O_{i,cn}(Y_s), i = 1, \dots, s$.

Proof.

Since we allow N to depend on Y_s, we may assume that N is so big that for $n \ge N$, we have $O_i \cap O_{i+1} = \emptyset$, $1 \le i \le s$.

We will construct the derivative G', and define $G(x) := \mathbb{S}(-1) + \int_{-1}^{x} G'(u)du$.

To this end, for each i , let d_i be the distance form y_i to the nearest endpoint of O_i, and set $G'(x) := \mathbb{S}'(x), x \in [-1,1] \setminus \cup_{i=1}^{\mathbb{s}} [y_i - d_i, y_i + d_i]$. For $\mathcal{P}_i = \mathbb{S}|O_i, i = 1, \dots, \mathbb{s}$.

Denote by $\mathcal{P}_i(t) := \mathcal{P}'_i(d, t + y_i)$, $t \in [-1,1]$, and let

$$g_i(t) := \begin{cases} g_{p_i}, \text{if } t\mathcal{P}'_i(t) \geq 0 & , t \in [-1,1] \\ -g_{-p_i}, \text{if } t\mathcal{P}'_i(t) \leq 0 & , t \in [-1,1] \end{cases}$$

We have the function g_p is defined in Lemma 2.2.27

Finally , set $G'(x) := g_i\left(\frac{x - y_i}{d_i}\right)$, $x \in [y_i - d_i, y_i + d_i]$ $i = 1, \dots, \mathbb{s}$

We have that G is the required function ■

<u>Lemma</u> 2.2.29

Let $Y_{\mathbb{s}} \in \mathbb{Y}_{\mathbb{s}}, \kappa \in \mathbb{N}, \Gamma \in \Phi^{\kappa}$ and $G \in \Delta^{(2)}(Y_{\mathbb{s}}) \cap L_p^1[-1,1]$ such that

$G \in L_p^2(x_j, x_{j-1}), j = 1, \dots, \mathbb{s}$, and be given. If G is a linear function l_i on each O_i, $i = 1, \dots, \mathbb{s}$, and

$$\|G''(x)\|_p \leq c(p) \left\| \frac{\Gamma(\Omega)}{\Omega^2} \right\|_p , x \in [-1,1] \setminus \{x_j\}_{j=1}^{n-1} \tag{2.2.30}$$

then there is a polynomial $P_{\tilde{n}} \in \Delta^{(2)}(Y_{\mathbb{s}}) \cap L_p[-1,1]$ of degree $\tilde{n} \leq cn$ such that

$$\|G(x) - P_{\tilde{n}}\|_p \leq c(p) \|\Gamma(\Omega)\|_p \tag{2.2.31}$$

Proof.

Without loss of generality we may assume that n is even and that the assumptions of the Lemma hold with $n/2$ in place of n , that is , we assume that $G \in L_p^2(x_{2j}, x_{2j-2}), j = 1, \dots, n/2$, and that G is a linear function l_i on each $O_{i,\frac{n}{2}}(Y_{\mathbb{s}}), i = 1, \dots, \mathbb{s}$

Let L be the polygonal line interpolating G at all points $x_j, j = 0, \dots, n$.

Clearly $L \in \Delta^{(2)}(Y_s)$, since G is a linear function on each $O_{i,\frac{n}{2}}(Y_s)$.Then, with $G_j = [x_{j+1}, x_j, x_{j-1}, G](x_{j-1} - x_{j+1})$

$$= \frac{G(x_{j-1}) - G(x_j)}{x_{j-1} - x_j} - \frac{G(x_{j+1}) - G(x_j)}{x_{j+1} - x_j}$$

The polygonal line L may be represented in the form

$$L(x) = G(-1) + [x_n, x_{n-1}, G](x + 1) + \sum_{j=1}^{n-1} G_j(x - x_j)_+$$

$$= G(-1) + [x_n, x_{n-1}, G](x + 1) + \sum_{j\in H} G_j(x - x_j)_+ \qquad (2.2.32)$$

Where in the last equality we used the assumption that G is a linear function on each $O_{i,\frac{n}{2}}(Y_s)$.

Since $G \in L_p^1[x_{j+1}, x_{j-1}]$ and $G \in L_p^2(x_{j+1}, x_j) \cup (x_j, x_{j-1})$, we have by inequality 2.2.30,

$$\|G_j\|_p \le c(p)\|G''(x)(x_{j-1} - x_{j+1})\|_p \le c(p)\frac{\Gamma(h_j)}{h_j} \qquad (2.2.33)$$

Similarly, it readily follows by inequality 2.2.30 and Lemma 2.2.4 that for all $x \in [x_j, x_{j-1}]$, $1 \le j \le n$,

$$\|G(x) - L(x)\|_p = \|[x, x_j, x_{j-1}, G](x - x_j)(x - x_{j-1})\|_p$$

$$\le c(p)\tfrac{1}{2}(x - x_j)(x - x_{j-1})\|G''(\theta)\|_p \le c(p)\Gamma(h_j) \le c(p)\|\Gamma(\Omega)\|_p$$
(2.4.34)

, where $\theta \in (x_j, x_{j-1})$.

Let $P_{\tilde{n}} := G(-1) + [x_n, x_{n-1}, G](x + 1) + \sum_{j\in H} G_j\ \tau_j(x)$ (2.2.35)

where τ_j are the polynomials guaranteed by Lemma 2.2.2 with $b = \kappa + 3$.since $\Pi(x_j)G_j \ge 0$, $j \in H$, it readily follows by Lemma 2.2.3 that $P_{\tilde{n}} \in \Delta^{(2)}(Y_s)$.

Now, by Lemma 2.2.2 and Lemma 1.3.2 in chapter one

$$\left\|(x-x_j)_+ - \tau_j(x)\right\|_p \le c(p)h_j \left\|\left(\frac{h_j}{|x-x_j|+h_j}\right)^{\kappa+3}\right\|_p$$

$$\le c(p){h_j}^{2-\kappa}\left\|\frac{{h_j}^{2\kappa+2}}{(|x-x_j|+h_j)^{\kappa+3}}\right\|_p$$

$$\le c(p)\left\|\frac{{h_j}^{2-\kappa}}{\left(|x-x_j|+\Omega\right)^2}\min\{\Omega^{\kappa+1},{h_j}^{\kappa+1}\}\right\|_p \quad ,x\in[-1,1]$$

Since $\Gamma \in \Phi^\kappa$, it follows that $\Gamma(h_j) \le \frac{{h_j}^\kappa}{\min\{\Omega^\kappa,{h_j}^\kappa\}}\ \Gamma(\Omega)$,

We obtain

$$\|L(x)-P_{\tilde{n}}(x)\|_p \le c(p)\left\|\sum_{j\in H} G_j(x-x_j)_+ - \tau_j(x)\right\|_p$$

$$\le c(p)\left\|\sum_{j\in H}\Gamma(h_j)\ \frac{{h_j}^{1-\kappa}}{\left(|x-x_j|+\Omega\right)^2}\min\{\Omega^{\kappa+1},{h_j}^{\kappa+1}\}\right\|_p$$

$$\le c(p)\left\|\Gamma(\Omega)\sum_{j\in H}\frac{h_j}{\left(|x-x_j|+\Omega\right)^2}\ \min\{\Omega,h_j\}\right\|_p$$

$$\le c(p)\left\|\Omega\,\Gamma(\Omega)\sum_{j=1}^{n}\frac{h_j}{\left(|x-x_j|+\Omega\right)^2}\right\|_p \le c(p)\|\Omega\,\Gamma(\Omega)\|_p\int_\Omega^\infty\frac{dt}{t^2} =$$

$$c(p)\ \|\Gamma(\Omega)\|_p \qquad (2.2.30)$$

In order to complete the proof we have to estimate $G - P_{\tilde{n}}$.

We have $\|G(x)-P_{\tilde{n}}(x)\|_p \le c(p)\|G(x)-L(x)\|_p + \|L(x)-P_{\tilde{n}}(x)\|_p$

$\le c(p)\|\Gamma(\Omega)\|_p + \|L(x)-P_{\tilde{n}}(x)\|_p \le c(p)\|\Gamma(\Omega)\|_p$ ■

2.3 Properties of the $\mathfrak{b}_{i,j}$'s

We begin this section to prove some Lemmas by properties of the $\mathfrak{b}_{i,j}$'s .

Lemma 2.3.1

Let $\Gamma \in \Phi^{\kappa}, \kappa \in \mathbb{N}, \mathcal{F} \in L_p(I)$, and $\mathbb{S} \in \sum_{\kappa,n}$, If $w_\kappa(\mathcal{F},t)_p \leq c(p)\Gamma(t)$ and

$$\|\mathcal{F}-\mathbb{S}\|_p \leq c(p)\Gamma(\|\Omega_n(x)\|_p), x \in [-1,1] \tag{2.3.2}$$

Then

$$\mathfrak{b}_\kappa(\mathbb{s},\Gamma) \leq c(\kappa,p)$$

where $c(p,\kappa)$ depends on p and κ.

Proof.

Let $1 \leq i,j \leq n$, Then

$$\mathfrak{b}_{i,j}(\mathbb{S}) = \frac{\|p_i-p_j\|_p}{\Gamma(h_j)}(\frac{h_j}{h_{i,j}})^\kappa = \frac{\|p_i-\mathcal{F}+\mathcal{F}-p_j\|_p}{\Gamma(h_j)}(\frac{h_j}{h_{i,j}})^\kappa$$

$$\mathfrak{b}_{i,j}(\mathbb{S},\Gamma) \leq 2^{\frac{1}{p}-1}(\frac{\|\mathcal{P}_i-\mathcal{F}\|_p}{\Gamma(h_j)}\left(\frac{h_j}{h_{ij}}\right)^\kappa + \frac{\|\mathcal{F}-\mathcal{P}_j\|_p}{\Gamma(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^\kappa) := \mathcal{J}_1+\mathcal{J}_2 ,$$

where $\|\mathcal{P}_i-\mathcal{F}\|_p = (\int_{-1}^{1}|\mathcal{P}_i-\mathcal{F}|^p)^{\frac{1}{p}}$

$$\|\mathcal{F}-\mathcal{P}_j\|_p = (\int_{-1}^{1}|\mathcal{P}_i-\mathcal{F}|^p)^{\frac{1}{p}} dx$$

now by 2.3.1 and 2.3.2

$\|p_i-\mathcal{F}\|_p \leq c(p)\Gamma(\|\Omega_n(x)\|_p) \leq c(p)\Gamma(h_i), x \in [-1,1]$,

Whence $\mathcal{J}_1 = \frac{\|\mathcal{P}_i-\mathcal{F}\|_p}{\Gamma(h_j)}\left(\frac{h_j}{h_{ij}}\right)^\kappa, \mathcal{J}_1 \leq \frac{\Gamma(h_i)}{\Gamma(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^\kappa \leq 1,$

Hence , $\mathcal{J}_1 \leq 1$, where we used the fact that if $h_i \leq h_j$,

then $\Gamma(h_i) \leq \Gamma(h_j)$,and if $h_i > h_j$ then $\Gamma(h_i)/\Gamma(h_j){h_j}^k \leq {h_i}^k$.

So as for $\mathcal{J}_2$, we observe that

$\mathcal{w}_\kappa(\mathcal{F}-\mathcal{P}_j,t)_p = \mathcal{w}_\kappa(\mathcal{F},t)_p \le \Gamma(t)$.

So that by Lemma 2.2.1, taking $x_0 := x_j$ and $h := \frac{h_j}{\kappa-1}$, we obtain for each $x \in I_i$,

$\left\|\mathcal{F}(x)-\mathcal{P}_j(x)\right\|_p \le c(p)\left(\frac{h_{i,j}}{h_j}\right)^\kappa \left(\Gamma(h_j)+\left\|\mathcal{F}-\mathcal{P}_j\right\|_p\right)$ and so

$\left\|\mathcal{F}-\mathcal{P}_j\right\|_p \le c(p)(\frac{h_{i,j}}{h_j})^k\Gamma(h_j)$. Applying(2.3.2)

$\frac{\|\mathcal{F}-\mathcal{P}_j\|_p}{\Gamma(h_j)}(\frac{h_j}{h_{i,j}})^\kappa \le c(p)$,therefor $\mathcal{J}_2 \le c$ ■

<u>Lemma</u> (2.3.3)

Let $\kappa \ge 3, \Gamma \in \Phi^\kappa, \mathbb{S} \in \sum_{\kappa;\mathfrak{n}/2}(Y_{\mathbb{s}}) \cap \Delta^{(2)}(Y_{\mathbb{s}}) \cap L_p[-1,1]$, where $\mathfrak{n}$ is an even number, and $\left\|\mathcal{P}_{\iota;\mathfrak{n}/2}-\mathcal{P}_{\iota-1;\mathfrak{n}/2}\right\|_{p(I_{\iota;\frac{\mathfrak{n}}{2}}\cup I_{\iota-1;\frac{\mathfrak{n}}{2}})} \le c(p)\Gamma\left(x_{\iota-2;\frac{\mathfrak{n}}{2}}-x_{\iota;\frac{\mathfrak{n}}{2}}\right), 2 \le \iota \le \mathfrak{n}/2$

where $\mathbb{S}|I_\iota;\frac{\mathfrak{n}}{2} =: \mathcal{P}_{\iota;\frac{\mathfrak{n}}{2}}, \iota = 1,\dots,\mathfrak{n}/2$, then there is an $\widetilde{\mathbb{S}} \in \sum^1_{\kappa;\mathfrak{n}}(Y_{\mathbb{s}}) \cap \Delta^{(2)}(Y_{\mathbb{s}}) \cap L_p[-1,1]$, such that

$$\left\|\mathbb{S}(x)-\widetilde{\mathbb{S}}(x)\right\|_p \le c(p)\Gamma(\Omega), x \in [-1,1] \qquad (2.3.4)$$

Proof.

For $2 \le j \le \mathfrak{n}$, set

$$a_j(x) := \frac{1}{2}\frac{x_{j-1}-x_{j-2}}{x_{j-1}-x_j}\frac{\mathbb{S}'(x_{j-1}+0)-\mathbb{S}'(x_{j-1}-0)}{x_j-x_{j-2}}(x-x_j)^2$$

And $a_1(x) :\equiv 0$, and for $1 \le j \le \mathfrak{n}-1$, set

$\mathfrak{b}_j(x) := \frac{1}{2}\frac{x_j-x_{j+1}}{x_j-x_{j-1}}\frac{\mathbb{S}'(x_j+0)-\mathbb{S}'(x_j-0)}{x_{j+1}-x_{j-1}}\left(x-x_{j-1}\right)^2$, and $\mathfrak{b}_\mathfrak{n}(x) :\equiv 0$.

Then

$\widetilde{\mathbb{S}}(x) = \mathbb{S}(x) + a_j(x) + b_j(x)$, $x \in I_j$,

Is the required function, simple computation, give

$a_{j+1}(x_j) = b_j(x_j), 1 \le j \le n-1$, while

$b_{j+1}(x_j) := \frac{1}{2}\frac{x_{j+1}-x_{j+2}}{x_{j+1}-x_j}\frac{\mathbb{S}'(x_{j+1}+0)-\mathbb{S}'(x_{j+1}-0)}{x_{j+2}-x_j}(x-x_j)^2 =$

$a_j(x_j) := \frac{1}{2}\frac{x_{j-1}-x_{j-2}}{x_{j-1}-x_j}\frac{\mathbb{S}'(x_{j-1}+0)-\mathbb{S}'(x_{j-1}-0)}{x_j-x_{j-2}}(x-x_j)^2 = 0, 1 \le j \le n-1.$

Hence $\widetilde{\mathbb{S}}(x_j + 0) = \widetilde{\mathbb{S}}(x_j - 0), 1 \le j \le n-1$.

Also $\widetilde{\mathbb{S}}'(x_j + 0) = \frac{x_{j+1}-x_j}{x_{j+1}-x_{j-1}}\mathbb{S}'(x_j - 0) + \frac{x_j-x_{j-1}}{x_{j+1}-x_{j-1}}\mathbb{S}'(x_j + 0) =$

$\frac{x_{j+1}-x_j}{x_{j+1}-x_{j-1}}\mathbb{S}'(x_j + 0) + \frac{x_j-x_{j-1}}{x_{j+1}-x_{j-1}}\mathbb{S}'(x_j - 0) = \widetilde{\mathbb{S}}'(x_j - 0), 1 \le j \le n-1$

So that $\widetilde{\mathbb{S}} \in \sum^1_{\kappa;n}$.

We need to prove that

$$\widetilde{\mathbb{S}} \in \textstyle\sum^1_{\kappa;n}(Y_s) \cap \Delta^{(2)}(Y_s) \cap L_p[-1,1] \qquad (2.3.5)$$

In order to prove that $\widetilde{\mathbb{S}} \in \sum^1_{\kappa;n}(Y_s)$, we have to show, with $\mathcal{P}_j := \widetilde{\mathbb{S}}|I_j, 1 \le j \le n-1$, that if $j, j+1 \notin H(n, Y_s)$, then $\mathcal{P}_j = \mathcal{P}_{j+1}$, rec all that

$\mathcal{P}_{\iota;n/2} := \widetilde{\mathbb{S}}|I_{\iota,\frac{n}{2}}$, $\iota = 1, \dots, n/2$ and distinguish between two cases, either j is even, in which case $j/2 \notin H(n/2, Y_s)$ and $\frac{j}{2} + 1 \notin H(n/2, Y_s)$

Hence $\mathcal{P}_{\frac{j}{2},n/2} = \mathcal{P}_{\frac{j}{2}+1,n/2}$, which implies that

$\mathbb{S}'(x_{j+v} + 0) = \mathbb{S}'(x_{j+v} - 0)$, for $v = 0, \pm1$, that is, $a_j = a_{j+1} = b_j = b_{j+1} = 0$.Thus $\mathcal{P}_j = \mathcal{P}_{j+1}$ as required.

Or j is odd, which case x_j is not a node of $\mathbb{S}$, and it follows that there is an inflection point y_i in the interval $[x_{\frac{(j+1)}{2},\frac{n}{2}}, x_{\frac{(j-1)}{2},\frac{n}{2}})$, so that $\frac{j+v}{2} \notin H\left(\frac{n}{2}, Y_s\right)$, for $v = 3, \pm1$.

Hence $\mathbb{S}'\left(x_{\frac{(j+1)}{2},\frac{n}{2n}}+0\right)=\mathbb{S}'\left(x_{\frac{(j+1)}{2},\frac{n}{2n}}-0\right)$ and $\mathbb{S}'\left(x_{\frac{(j-1)}{2},\frac{n}{2n}}+0\right)=\mathbb{S}'\left(x_{\frac{(j-1)}{2},\frac{n}{2n}}-0\right)$, which in turn implies that $a_j=b_{j+1}\equiv 0$, also $\mathbb{S}'(x_j+0)=\mathbb{S}'(x_j-0)$, since x_j is not a node of $\mathbb{S}$, which implies that $a_{j+1}=b_j\equiv 0$

Thus, we conclude that $\widetilde{\mathbb{S}}(x)=\mathbb{S}(x)=\mathcal{P}_{\frac{(j+1)}{2},\frac{n}{2}}(x)$ for $x\in[x_{\frac{(j+1)}{2},\frac{n}{2}},x_{\frac{(j-1)}{2},\frac{n}{2}}]$.

This completes the proof that

$$\widetilde{\mathbb{S}}\in\textstyle\sum^1_{\kappa;n}(Y_{\mathbb{S}})\qquad(2.3.6)$$

Finally, $a''_j(x)=\frac{x_{j-2}-x_{j-1}}{(x_{j-2}-x_j)(x_{j-1}-x_j)}(\mathbb{S}'(x_{j-1}+0)-\mathbb{S}'(x_{j-1}-0))$ and

$b''_j=\frac{x_j-x_{j+1}}{(x_{j-1}-x_{j+1})(x_{j-1}-x_j)}(\mathbb{S}'(x_j+0)-\mathbb{S}'(x_j-0))$

So that we readily conclude that

$$\widetilde{\mathbb{S}}''(x)\Pi(x)\geq 0,\ x\in[-1,1]\backslash\{x_j\}_{j=1}^{n-1}\qquad(2.3.7)$$

Combining 2.3.6 and 2.3.7 , we obtain 2.3.5.

In order to conclude the proof, recall that for odd j , $\mathbb{S}'(x_j+0)-\mathbb{S}'(x_j-0)=0$ and if j is even, then by Markovs inequality

$$\|\mathbb{S}'(x_j+0)-\mathbb{S}'(x_j-0)\|_p=\left\|\mathcal{P}'_{\frac{j}{2},\frac{n}{2}}(x_j)-\mathcal{P}'_{\frac{j}{2}+1,\frac{n}{2}}(x_j)\right\|_p$$

$$\leq c(p)\frac{2\kappa^2}{h_{\frac{j}{2}+1,\frac{n}{2}}+h_{\frac{j}{2},\frac{n}{2}}}\left\|\mathcal{P}_{\frac{j}{2},\frac{n}{2}}-\mathcal{P}_{\frac{j}{2}+1,\frac{n}{2}}\right\|_{p(I_{\frac{j}{2}+1,\frac{n}{2}}\cup I_{\frac{j}{2},\frac{n}{2}})}$$

$$\leq c(p)\frac{2\kappa^2}{h_{\frac{j}{2}+1,\frac{n}{2}}+h_{\frac{j}{2},\frac{n}{2}}}\Gamma(h_{\frac{j}{2}+1,\frac{n}{2}}+h_{\frac{j}{2},\frac{n}{2}})$$

$$\leq c(p)\frac{\Gamma(h_j)}{h_j}.$$

This implies 2.3.4 and conclude our proof ■

Lemma 2.3.8

For each $\mathbb{S} \in \sum_{\kappa;n/2}(Y_{\mathbb{s}}) \cap \Delta^{(2)}(Y_{\mathbb{s}}) \cap L_p[-1,1]$ satisfying

$$b_\kappa(\mathbb{S}) \leq 1 \tag{2.3.9}$$

There is $\widetilde{\mathbb{S}} \in \sum_{\kappa;n}^{1}(Y_{\mathbb{s}}) \cap \Delta^{(2)}(Y_{\mathbb{s}}) \cap L_p[-1,1]$, such that

$$b_{\kappa;2n}(\widetilde{\mathbb{S}}) \leq c \tag{2.3.10}$$

Proof.

It follows by $b_{i,j}(\mathbb{S}) := b_{i,j;n}(\mathbb{S}) := \frac{\|p_i - p_j\|_p}{\Gamma(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^\kappa$, $1 \leq i, j \leq n$, and $b_\kappa(\mathbb{S}) \leq 1$, that $\|p_j - p_{j-1}\|_{p(I_j \cup I_{j-1})} \leq c(p)\Gamma(x_{j-2} - x_j), 2 \leq j \leq n$

Hence, by Lemma 2.3.3 implies the existence of a piecewise polynomial

$\widetilde{\mathbb{S}} \in \sum_{\kappa;n}^{1}(Y_{\mathbb{s}}) \cap \Delta^{(2)}(Y_{\mathbb{s}}) \cap L_p[-1,1]$, such that

$$\|\mathbb{S}(x) - \widetilde{\mathbb{S}}(x)\|_p \leq c(p)\Gamma(\Omega(x)), x \in [-1,1] \tag{2.3.11}$$

And we only have to prove 2.3.8 for $v, \mu = 1, \dots, 2n$, set $j := \left\lceil \frac{v}{2} \right\rceil$, and $i := \lceil \mu/2 \rceil$. Then by Lemma 2.3.1 and 2.3.9,

$$b_{\mu,v;2n}(\widetilde{\mathbb{S}}) := \frac{\|p_{\mu;2n} - p_{v;2n}\|_{p(I_{\mu;2n})}}{\Gamma(h_{v;2n})}\left(\frac{h_{v;2n}}{h_{\mu;2n}}\right)^\kappa \leq c(p)\frac{\|p_{\mu;2n} - p_{v;2n}\|_{p(I_{\mu;2n})}}{\Gamma(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^\kappa$$
(2.3.12)

$$\leq c(p)\frac{\|p_{\mu;2n} - p_{v;2n}\|_{p(I_{\mu;2n})}}{\Gamma(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^\kappa + c(p)b_{i,j}(\mathbb{S}) - c(p)b_{i,j}(\mathbb{S})$$

$$\leq c(p)b_{i,j}(\mathbb{S}) + c(p)\frac{\|p_{\mu;2n} - p_i\|_{p(I_{\mu;2n})}}{\Gamma(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^\kappa + c(p)\frac{\|p_{v;2n} - p_j\|_{p(I_{\mu;2n})}}{\Gamma(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^\kappa$$

$$=: c(p) + \mathcal{J}_1 + \mathcal{J}_2$$

Now, by 2.3.11

$$\mathcal{J}_1 \le c(\mathrm{p}) \frac{\Gamma(\mathrm{h_i})}{\Gamma(\mathrm{h_j})} \left(\frac{\mathrm{h_j}}{\mathrm{h_{i,j}}}\right)^{\kappa} \le c \qquad (2.3.13)$$

where we used the fact that if $\mathrm{h_i} < \mathrm{h_j}$,

then $\Gamma(\mathrm{h_i}) \le \Gamma(\mathrm{h_j})$, and if $\mathrm{h_i} > \mathrm{h_j}$ then $\Gamma(\mathrm{h_i})/\Gamma(\mathrm{h_j}){\mathrm{h_j}}^{k} \le {\mathrm{h_i}}^{k}$

Finally

$$\mathcal{J}_2 \le c(\mathrm{p}) \frac{\|\mathrm{p}_{\mathcal{v};2\mathrm{n}} - \mathrm{p_j}\|_{\mathrm{p}(\mathrm{I}_{\mu,\mathcal{v};2\mathrm{n}})}}{\Gamma(\mathrm{h_j})} \left(\frac{\mathrm{h_j}}{\mathrm{h_{i,j}}}\right)^{\kappa} \le c(\mathrm{p}) \left(\frac{\mathrm{h_{i,j}}}{\mathrm{h_j}}\right)^{\kappa} \frac{\|\mathrm{p}_{\mathcal{v};2\mathrm{n}} - \mathrm{p_j}\|_{\mathrm{p}(\mathrm{I}_{\mu,\mathcal{v};2\mathrm{n}})}}{\Gamma(\mathrm{h_j})} \left(\frac{\mathrm{h_j}}{\mathrm{h_{i,j}}}\right)^{\kappa} \le c \qquad (2.3.14)$$

where for the second inequality we used the fact that both polynomials are of degree $< \kappa$, and for the last inequality we again applied 2.3.11. sub situating 2.3.13 and 2.3.14 in to 2.3.12 we complete the prove ■

2.4 Auxiliary polynomials.

Here our main result we need the following Auxiliary polynomials.

<u>Lemma</u> 2.4.1

Let E be an interval which is union of $\iota \ge 12\mathbb{s}$ of the intervals $\mathrm{I_j}$, and let a set $\mathcal{J} \subseteq \mathrm{E}$ be the union of $1 \le \mu \le \iota/4$ of these intervals . then there exists a polynomial $\mathrm{Q_n}(x) = \mathrm{Q_n}(x, \mathrm{E}, \mathcal{J})$of degree $\le c\mathrm{n}$, satisfying

$$\widetilde{\mathrm{Q}}''_{\mathrm{n}}(x)\delta(x) \ge c_1 \frac{\iota}{\mu} \left(\frac{\Omega}{max\{\Omega, dist(x, \mathrm{E})\}}\right)^{25(\mathbb{s}+1)+2\kappa} \frac{\pi(x)\Gamma(\Omega)}{\Omega^2}, \ x \in \mathcal{J} \cup [-1,1]\backslash \mathrm{E}) \qquad (2.4.2)$$

(we may take $c_1 \le 1$)

$$\widetilde{\mathrm{Q}}''_{\mathrm{n}}(x)\delta(x) \ge -\frac{\pi(x)\Gamma(\Omega)}{\Omega^2}, x \in \mathrm{E}\backslash\mathcal{J}, \qquad (2.4.3)$$

And

$$\left\|\widetilde{Q}_{n}(x)\right\|_{p} \leq c(p)\iota^{6} \sum_{j:I_j\subseteq E} \left\|\frac{\Omega h_j}{\left(|x-x_j|+\Omega\right)^2}\right\|_{p} \tag{2.4.4}$$

Where $\Pi(x) := \prod_{i=1}^{s}(x - y_i)$, and

Set $\delta(x) := \operatorname{sgn} \Pi(x), x \in [-1,1]$

And $\pi(x) := \prod_{i=1}^{s} \frac{|x-y_i|}{|x-y_i|+\Omega}$.

Proof.

Let $H(E) := \{j \in H | I_j \subseteq E\}, H(\mathcal{J}) := \{j \in H | I_j \subseteq \mathcal{J}\}$ $E(\mathcal{O}) := \{j | I_j \subseteq E \cap \bar{\mathcal{O}}\}$, And $H_*(E) := \{ j \in H(E) | I_j \cap \bar{\mathcal{O}} \neq \emptyset\}$, where $\bar{\mathcal{O}}$ denotes the closure of $\mathcal{O}$.

Finally, Let $j_* := \min\{j \in H(E)\}$ and $j^* := \max\{j \in H(E)\}$

Set $\mathcal{A}_2 := H(\mathcal{J}) \cup H_*(E) \cup \{j_*, j^*\}$ and

$\mathcal{A}_1 := H(E) \setminus (\mathcal{A}_2 \cup E(\mathcal{O}))$.

Denote by ι_1^* and ι_2 the number of elements in $\mathcal{A}_2$ and $\mathcal{A}_1$, respectively, and set $\iota_1 = [\frac{\iota_1^*}{2}]$. Then it readily follows that

$$\iota_2 \leq \mu + 2s + 2 \leq c\mu \tag{2.4.5}$$

(recall that we allow c to depend on s), and

$$\iota > \iota_1^* \geq \iota - (\iota_2 + 3s) \geq \frac{1}{6}\iota \tag{2.4.6}$$

Denote by j_0 and $j^0 = j_0 + \iota - 1$ the smallest and the largest integers j, such that $I_j \subseteq E$.

We consider three cases .

Case I

Let $\iota \geq j_0$, set

$$\widetilde{Q}_{n}(x) := \frac{\iota}{\mu} \sum_{j\in\mathcal{A}_2} \frac{\sigma_j \tau_j(x)}{h_j},$$

where $\sigma_j := \operatorname{sgn} \prod(x_j)$. Then $\widetilde{Q}_n(x)\, \sigma(x) \geq 0\, , x \in I$,

which implies 2.4.3 and 2.4.2 readily follows from Lemma 2.2.5.Thus we only have to prove 2.4.4 .To this end, by Lemma 2.2.6 we obtain for any $j \in \mathcal{A}_2$,

$$\frac{\|\tau_j(x)\|_p}{h_j} \leq \frac{c(p)}{h_j} \left\|\tau_j(x) - (x - x_j)_+\right\|_p + \frac{(x - x_j)_+}{h_j}$$

$$\leq c(p) \left\| \frac{\Omega h_j}{\left(|x - x_j| + \Omega\right)^2} \right\|_p + \frac{(x - x_j)_+}{h_j}$$

Now, if $x \leq x_j$, then $(x - x_j)_+ = 0$. Otherwise, observe that $x \in I_i$ for some $1 \leq i \leq j \leq 2\iota$. Thus,

$$\frac{(x - x_j)}{h_j} \frac{(x - x_j + \Omega)^2}{\Omega h_j} \leq 10 \frac{x - x_j}{h_j} \frac{x - x_j + h_j}{h_j} \frac{x - x_j + h_i}{h_i}$$

$\leq 10(\frac{x_0 - x_{2\iota}}{h_1} + 1)^3 \leq c\iota^6$, c is a positive constant , which implies 2.4.4

Case II: let $j_0 \geq n - 2\iota$. set

$$\widetilde{Q}_n(x) := \frac{\iota}{\mu} \sum\nolimits_{j \in \mathcal{A}_2} \frac{\sigma_j}{h_j} \left(\tau_j(x) - (x - x_j) \right),$$

The proof like Case I .

Case III: let $\iota < j_0 < n - 2\iota$. Denote by $h = |E| = x_{j_0 - 1} - x_{j_0 + \iota - 1}$,

The length of the interval E. Then by Lemma 2.2.7 implies

$\frac{1}{2} h \leq \iota\, h_j \leq 2h\, , I_j \subset E$. (2.5.7)

By Lemma's 2.2.8, 2.4.5 and inequality 2.4.6 guarantee the existence of $a_i, i \in \mathcal{A}_1$, such that $\frac{\iota}{\mu} \sum_{j \in \mathcal{A}_2} \sigma_j(x - x_j) + \sum_{i \in \mathcal{A}_1} a_i(x - x_i) \equiv 0$ (2.4.8)

And $|a_i| \le \frac{\iota}{\mu}\left(\frac{\iota}{\iota_1}\right)^2 \frac{\iota_2}{\iota_1} \le c$, $i \in \mathcal{A}_1$, (2.4.9)

(Note that if ι_1^* is odd, then we apply Lemma 2.2.8 to $\mathcal{A}_1 \setminus \{i^*\}$, for some arbitrary i^*, and put a_{i^*}=0 in 2.4.8. for each $i \in \mathcal{A}_1$, Set

$$\tau_i^* = \begin{cases} \tau_i, & \text{if } \sigma_j a_i \ge 0 \\ \bar{\tau}_i, & \text{otherwise} \end{cases}$$

And let

$$\tilde{Q}_n(x) := c\frac{\iota}{h}\left(\frac{\iota}{\mu}\sum_{j\in\mathcal{A}_2} \sigma_j \tau_j(x) + \sum_{i\in\mathcal{A}_1} a_i\tau_i^*(x)\right),$$

For some c to b prescribed. Then by 2.4.7 and 2.4.9, we see that (2.4.2) readily follows Lemma 2.2.3,and 2.2.5, and that inequality 2.4.3 is valid for a proper choice of the constant c we conclude with the proof of 2.4.4.

Take $L(x) := \frac{\iota}{\mu}\sum_{j\in\mathcal{A}_2} \sigma_j \left(x - x_j\right) + \sum_{i\in\mathcal{A}_1} a_i(x - x_i)_+$

Then by Lemma 2.2.6 we have

$$\left\|\tilde{Q}_n(x)\right\|_p \le c(p)\sum_{j\in H(E)} \left\|\frac{\iota\,\Omega h_j}{\left(|x-x_j| + \Omega\right)^2}\right\|_p + \frac{\iota}{h}\|L(x)\|_p, x \in I .$$

So we only need to estimate $\frac{\iota}{h}\|L(x)\|_p$. To this end , note that if $x \notin E$, then(2.4.8) implies that $L(x) \equiv 0$. On the other hand , if $x \in E$, then

$$\frac{\iota}{h}\|L(x)\|_p \le \frac{c(p)\iota}{h}\left(\frac{\iota\,\iota_2}{\mu}h + 2\iota_1 h\right) \le c(p)\iota^2$$

$$\le c(p)\iota^3 \sum_{I_j\subseteq E} \left\|\frac{\Omega h_j}{\left(\left|x - x_j\right| + \Omega\right)^2}\right\|_p,$$

where for the last inequality we have applied Lemma's 2.2.4 , 2.4.7 and the estimate

$$1 = h\sum_{I_j\subseteq E}\frac{h_j}{h^2} \le 16h\sum_{I_j\subseteq E}\frac{h_j}{\left(|x-x_j|+\Omega\right)^2} \le 160\iota\,\Omega\sum_{I_j\subseteq E}\frac{h_j}{\left(|x-x_j|+\Omega\right)^2}$$

This proof is compete of 2.4.4 ■

<u>Lemma</u>2.4.10

Let E be an interval which is union of $\iota \ge 12\mathbb{s}$ of the intervals I_j , and let a set $\mathcal{J} \subseteq E$ be the union of $1 \le \mu \le \iota/4$ of these intervals . then there exists a polynomial $Q_n(x) = Q_n(x, E, \mathcal{J})$of degree $\le cn$, satisfying

$$Q''_n(x)\delta(x) \ge c_1\frac{\iota}{\mu}\left(\frac{\Omega}{max\{\Omega, dist(x,E)\}}\right)^{25(\mathbb{s}+1)+2\kappa}\frac{\pi(x)\Gamma(\Omega)}{\Omega^2},\ x \in \mathcal{J} \cup [-1,1]\backslash E) \tag{2.4.11}$$

(we may take $c_1 \le 1$)

$$Q''_n(x)\delta(x) \ge -\frac{\pi(x)\Gamma(\Omega)}{\Omega^2}, x \in E\backslash\mathcal{J} \tag{2.4.12}$$

And

$$\left\|Q_n(x)\right\|_p \le c(p)\iota^{4\kappa+6}\left\|\Gamma(\Omega)\sum_{j:I_j\subseteq E}\frac{\Omega h_j}{(|x-x_j|+\Omega)^2}\right\|_p,\ x \in [-1,1] \tag{2.4.13}$$

Where $\Pi(x) := \prod_{i=1}^{\mathbb{s}}(x - y_i)$, and Set $\delta(x) := sgn\,\Pi(x), x \in [-1,1]$

And $\pi(x) := \prod_{i=1}^{\mathbb{s}}\frac{|x-y_i|}{|x-y_i|+\Omega}$.

Proof.

Take j_0 and $j^0 = j_0 + \iota - 1$ denote by the smallest and largest integers j, such that $I_j \subseteq E$. write $j^* := j_0 + [\frac{\iota}{2}]$ and we may assume that $j^* \le n/2$ (if $j^* > n/2$, we take a mirror image)

Set $Q_n(x) := \widetilde{Q}_n(x)8^{-\kappa}\,\Gamma(h_{j^*})$, where $\widetilde{Q}_n$ is the polynomial given by

$\widetilde{Q}_n(x) = \widetilde{Q}_n(x, E, \mathcal{J})$ of degree $\le cn$, satisfy

$Q''_{n}(x)\delta(x) \geq c_1 \frac{\iota}{\mu}\left(\frac{\Omega}{max\{\Omega, dist(x,E)\}}\right)^{25(s+1)+\kappa} \frac{\pi(x)}{\Omega^2}, \; x \in \mathcal{J} \cup [-1,1]\backslash E)$
(2.4.14)

(we may take $c_1 \leq 1$)

$$Q''_{n}(x)\delta(x) \geq -\frac{\pi(x)}{\Omega^2}, x \in E\backslash\mathcal{J} \qquad (2.4.15)$$

And

$$\|Q_n(x)\|_p \leq c(p)\iota^{4\kappa+6}\left\|\sum_{j:I_j\subseteq E} \frac{\Omega^{\kappa+1}h_j^{\kappa+1}}{(|x-x_j|+\Omega)^{2\kappa+2}}\right\|_p \qquad (2.4.16)$$

So by Lemma 2.2.10 that is proved 2.4.11 and 2.4.12

Now to prove 2.4.13

First we observe that by $\Omega_n^2(y) < 4\Omega(|x-y|+\Omega)$,

$$\frac{(|x-y|+\Omega)}{2} < |x-y| + \Omega(y) < 2(|x-y|+\Omega), x, y \in [-1,1] .$$

For any $x \in [-1,1]$ and every $j = 1, \dots, n$,

$$\frac{\Omega h_j}{(|x-y|+\Omega)^2} \leq c \qquad (2.4.17)$$

Hence, if $h_{j^*} \leq \Omega$, then 2.4.13 readily follows from 2.4.16

Otherwise, $h_{j^*} > \Omega$ .then we have

$$\frac{\Gamma(h_{j^*})}{h_{j^*}^{\kappa}} \leq \frac{\Gamma(\Omega)}{\Omega^{\kappa}}$$

Also, recall that

$$h_{j^*} \leq 2\iota\, h_{j^0} \leq 2\iota\, h_j, \forall j: I_j \in E \qquad (2.4.18)$$

which, in particular, implies that if $x \in E$, then

$h_{j^*} < 10\iota\Omega$. Thus, we obtain for $x \in E$, $\Gamma(h_{j^*}) \leq c\iota^{\kappa}\Gamma(\Omega)$

Hence 2.4.13 follows from 2.4.16 and 2.4.17.

Assume that $x \notin E$,that, $x \in [-1, x_{j^0}) \cup (x_{j^0-1}, 1]$, by (2.4.18)

$$\Gamma(h_{j^*}) \le \Gamma(\Omega)\frac{h_{j^*}^{\kappa}}{\Omega^{\kappa}} \le c\iota^{\kappa}\Gamma(\Omega)\frac{h_{j^0}^{\kappa}}{\Omega^{\kappa}}$$

$$\le c\iota^{\kappa}\Gamma(\Omega)\frac{\left(|x - x_j| + \Omega\right)^{\kappa}}{\Omega^{\kappa}}, \forall j: I_j \in E,$$

where for the third the fact that $h_{j^0} \le c\left(|x - x_j| + \Omega\right)$.

Hence for $x \in [-1, x_{j^0}) \cup (x_{j^0-1}, 1]$ and j such that $I_j \subseteq E$,we have

$$\Gamma(h_{j^*})\,\Omega^{\kappa+1}\frac{h_j^{\kappa+1}}{\left(|x - x_j| + \Omega\right)^{2\kappa+2}} \le c\iota^{\kappa}\Gamma(\Omega)\Omega\frac{h_j^{\kappa+1}}{\left(|x - x_j| + \Omega\right)^{\kappa+2}}$$

$$\le c\iota^{\kappa}\Gamma(\Omega)\frac{\Omega h_j}{\left(|x - x_j| + \Omega\right)^{2}}$$

Where for the last inequality we applied $h_j \le c\left(|x - x_j| + \Omega\right)$

Together with 2.4.16 .This implies 2.4.13 ■

<u>Lemma</u> 2.4.19

Suppose that $\kappa \ge 3$, $\Gamma \in \Phi^k$,and $\mathbb{S} \in \sum_{\kappa,n}^{(1)}$ is such that $b_\kappa(\mathbb{s}, \Gamma) \le 1$, If an interval $I_{\mu,v}$contains at least $2\kappa - 5$ intervals I_i and points $x_i^* \in I_i^0$ such that

$$\|\mathbb{s}''(x_i^*)\|_p \le c(p)\frac{\Gamma(h_i)}{h_i^2} \tag{2.4.20}$$

then for every$1 \le j \le n$, we have

$$\left\|\frac{\Omega^2\mathbb{s}''}{\Gamma(\Omega)}\right\|_{L_{p(I_j)}} \le c(p)[(j-\mu)^{4\kappa} + (j - v)^{4\kappa}] \tag{2.4.21}$$

Proof.

Fix j and $x \in I_j^0$ where $I_j^0 = (x_i, x_{i-1})$. It follows by Lemma 2.3.1 $(b_{i,j}(\mathbb{s}) \coloneqq b_{i,j;n}(\mathbb{s}) \coloneqq \frac{\|\mathcal{P}_i - \mathcal{P}_j\|_p}{\Gamma(h_j)}(\frac{h_j}{h_{i,j}})^k, 1 \le i, j \le n$, That for every i , $\|\mathcal{P}_i - \mathcal{P}_j\|_{L_{p(I_i)}} \le c(p)\Gamma(h_j)(\frac{h_{i,j}}{h_j})^{\kappa}$.

Since $\mathcal{P}_i$and $\mathcal{P}_j$ are polynomials of degree < κ ,we obtain

$$\|\mathcal{P}_i'' - \mathcal{P}_j''\|_p \le \frac{1}{h_i^2}\|\mathcal{P}_i - \mathcal{P}_j\|_p \le c(p)\frac{\Gamma(h_j)}{h_i^2}(\frac{h_{i,j}}{h_j})^{k}.$$

Hence by 2.4.20 implies , so by Lemma 2.2.13 implies

$$\|\mathcal{P}_j''(x_i^*)\|_p \le \frac{\Gamma(h_j)}{h_j^2}(|i-j|+1)^{2\kappa} \tag{2.4.22}$$

$$\frac{|x - x_{i_m}^*|}{|x_{i_\iota}^* - x_{i_m}^*|} \le c((i-\mu)^2 + (j-\mathcal{v})^2) \tag{2.4.23}$$

Now ,by virtue of the representation

$$\mathcal{P}_j''(x) \equiv \sum_{\iota=1}^{k-2} \mathcal{P}_j''(x_{i\iota}^*) \prod_{m=1, m\neq\iota}^{\kappa-2} \frac{x - x_{i_m}^*}{x_{i_\iota}^* - x_{i_m}^*}$$

It follows by (2.4.22)and(2.4.23) that $\left\|\frac{\Omega^2 \mathbb{s}''(x)}{\Gamma(\Omega)}\right\|_p = \left\|\frac{\Omega^2 \mathcal{P}_j''(x)}{\Gamma(\Omega)}\right\|_p$

$$\le \frac{h_i^2 \|\mathcal{P}_j''(x)\|_{L_p(\mathring{I}_j)}}{\Gamma(h_j)} \le c(p)((j-\mu)^{4\kappa-6} + (j-\mathcal{v})^{4\kappa-6}), x \in I_j^0,$$

So the prove of 2.4.21 is complete ■

<u>Lemma</u> 2.4.24

Let $\mathcal{A}$ be a proper interval such that $\mathcal{A} \cap 0_e \neq \emptyset$ and let $\mathfrak{b}_1 \ge 0$. Then,

For $\mathbb{S} \in \sum_{\kappa;n}(Y_s)$ and $x \in \mathcal{A} \cap 0_e$, where $0_e := \{u: [u - \frac{1}{2}\Omega_n(u), u + \frac{1}{2}\Omega_n(u)] \subseteq \overline{0}\}$. and $\overline{0} \cap (I_1 \cup I_n) = \emptyset$.

We have

$$\left\|\mathbb{S}^{(q)}(x) - D_{n_1}^{(q)}(x)\right\|_p \le$$

$$c(p,\kappa,s,b)\left\|\frac{\Gamma(\Omega)}{\Omega^{q}}\left(b_\kappa(\mathbb{S},\mathcal{A}) + b_\kappa(\mathbb{S})\frac{n}{n_1}\left(\frac{\Omega}{\Omega + dist(x,[-1,1]\backslash\mathcal{A})}\right)^{b_3}\right)\right\|_p \quad (2.4.25)$$

where $b_\kappa(\mathbb{S},\mathcal{A}) := b_{\kappa;n}(\mathbb{S},\mathcal{A}) := \max_{i,j:I_j,I_j\subseteq\mathcal{A}} b_{i,j}(\mathbb{S})$ and

$$D_{n_1}(x) := D_{n_1}(x,\mathbb{S},b) := \sum_{j=1}^{n} \mathcal{P}_j(x)\widetilde{T}_{j;n_1}(x),$$

Furthermore , if $\mathbb{S} \in \sum_{\kappa;n}^{1}$, then for $x \in \mathcal{A}, x \ne x_j$, $0 \le j \le \frac{n}{n_1}$.

$$\left\|\mathbb{S}''(x) - D_{n_1}''(x)\right\|_p \le$$

$$c(p,\kappa,s,b)\left\|\frac{1}{\Omega^2}\left(b_\kappa(\mathbb{S},\mathcal{A}) + b_\kappa(\mathbb{S})\frac{n}{n_1}\left(\frac{\Omega}{\Omega + dist(x,[-1,1]\backslash\mathcal{A})}\right)^{b_3}\right)\right\|_p \quad (2.4.26)$$

Proof.

The crucial step in the proof is the prove the analog of Lemma 2.2.23 .For the sake of brevity, we will write in this proof Ω_1 for $\Omega_{n_1}(x)$ and $\widetilde{T}_j$ for $\widetilde{T}_{j;n_1}$

Let $x \in I_v$ be such that, say,

$$x - x_v \le x_{v-1} - x \quad (2.4.27)$$

By define $b_{i,j}(\mathbb{S}) := b_{i,j;n}(\mathbb{S}) := \frac{\|p_i - p_j\|_{L_p(I_v)}}{\Gamma(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^{\kappa}$, $1 \le j, i \le n$

So, we have

$$\left\|\mathcal{P}_v - \mathcal{P}_j\right\|_{L_p(I_v)} \le b_{v,j}(\mathbb{S})\Gamma(h_v)\frac{h_{v,j}^{2\kappa}}{h_j^{\kappa}h_v^{\kappa}}$$

Hence , for each $r \in N$.

$$\left\|\mathcal{P}_v^{(r)} - \mathcal{P}_j^{(r)}\right\|_{L_p(I_v)} \le c(p)\frac{b_{v,j}(\mathbb{S})\Gamma(h_v)}{h_v^{r}}\frac{h_{v,j}^{2\kappa}}{h_j^{\kappa}h_v^{\kappa}} \quad (2.4.28)$$

If $j \ne v$, $v + 1$, then by Lemma 2.2.4 and 2.2.7 imply $dist(x, I_j) \ge \frac{1}{2}\Omega$, as in proving of Lemma 2.2.23 .

We obtain by Lemma 2.2.19 for $\mathfrak{b}$ to be prescribed and for $\mathfrak{r} \le q \le 3$,

$$\left\|\mathcal{P}_{\mathcal{v}}{}^{(\mathfrak{r})} - \mathcal{P}_{\mathrm{j}}{}^{(\mathfrak{r})}\widetilde{\mathrm{T}}_{\mathrm{j}}^{(q-\mathfrak{r})}(x)\right\|_{L_p(I_{\mathcal{v}})}$$

$$\le c(\mathrm{p})\frac{\mathfrak{b}_{\mathcal{v},\mathrm{j}}(\mathbb{S})\Gamma(\Omega)}{\mathrm{h}_{\mathcal{v}}^{\mathfrak{r}}}\left(\frac{\mathrm{h}_{\mathcal{v},\mathrm{j}}}{\mathrm{h}_{\mathrm{j}}}\right)^{\kappa}\left(\frac{\mathrm{h}_{\mathcal{v},\mathrm{j}}}{\mathrm{h}_{\mathcal{v}}}\right)^{\kappa}\frac{1}{\Omega_1^{q-\mathfrak{r}}}\left(\frac{\Omega_1}{\Omega_1 + dist(x, \mathrm{I}_{\mathrm{j}})}\right)^{\mathfrak{b}}$$

$$= c(\mathrm{p})\frac{\mathfrak{b}_{\mathcal{v},\mathrm{j}}(\mathbb{S})\Gamma(\Omega)}{\mathrm{h}_{\mathcal{v}}^{\mathfrak{r}}}\left(\frac{\mathrm{h}_{\mathcal{v},\mathrm{j}}}{\mathrm{h}_{\mathrm{j}}}\right)^{\kappa+1}\left(\frac{\mathrm{h}_{\mathcal{v},\mathrm{j}}}{\mathrm{h}_{\mathcal{v}}}\right)^{\kappa-1}\frac{\mathrm{h}_{\mathrm{j}}}{\mathrm{h}_{\mathcal{v}}\Omega_1^{q-\mathfrak{r}}}\left(\frac{\Omega_1}{\Omega_1+dist(x,\mathrm{I}_{\mathrm{j}})}\right)^{\mathfrak{b}}$$

$$\le c(\mathrm{p})\frac{\mathfrak{b}_{\mathcal{v},\mathrm{j}}(\mathbb{S})\Gamma(\Omega)}{\mathrm{h}_{\mathcal{v}}^{\mathfrak{r}+1}}\left(\frac{\Omega + dist(x, \mathrm{I}_{\mathrm{j}})}{\Omega}\right)^{3\kappa+1}\frac{\mathrm{h}_{\mathrm{j}}}{\Omega_1^{q-\mathfrak{r}}}\left(\frac{\Omega_1}{\Omega_1 + dist(x, \mathrm{I}_{\mathrm{j}})}\right)^{\mathfrak{b}}$$

$$\le c(\mathrm{p})\frac{\mathfrak{b}_{\mathcal{v},\mathrm{j}}(\mathbb{S})\Gamma(\Omega)}{\mathrm{h}_{\mathcal{v}}^{\mathfrak{r}+1}}\left(\frac{\Omega + dist(x,\mathrm{I}_{\mathrm{j}})}{\Omega}\right)^{3\kappa-q+\mathfrak{r}}\left(\frac{\Omega + dist(x,\mathrm{I}_{\mathrm{j}})}{\Omega}\right)^{q-\mathfrak{r}+1}\frac{\mathrm{h}_{\mathrm{j}}}{\Omega_1^{q-\mathfrak{r}}} \qquad (2.4.29)$$

$$\times\left(\frac{\Omega_1}{\Omega_1 + dist(x, \mathrm{I}_{\mathrm{j}})}\right)^{q-\mathfrak{r}+1}\left(\frac{\Omega}{\Omega + dist(x, \mathrm{I}_{\mathrm{j}})}\right)^{\mathfrak{b}-q+\mathfrak{r}-1}$$

$$\le c(\mathrm{p})\frac{\mathfrak{b}_{\mathcal{v},\mathrm{j}}(\mathbb{S})\Gamma(\Omega)}{\Omega^{q}}\frac{\mathfrak{n}}{\mathfrak{n}_1}\Omega^{\mathfrak{b}_1}\mathrm{h}_{\mathrm{j}}\left(\frac{1}{\Omega + dist(x, \mathrm{I}_{\mathrm{j}})}\right)^{\mathfrak{b}_1+1}, 0 \le \mathfrak{r} \le q,$$

where $\mathfrak{b}_1 = \mathfrak{b} - 3\kappa - 2 \ge 0$, and where for the second inequality we applied define $\mathrm{b}_{\mathrm{i},\mathrm{j}}(\mathbb{S}) := \mathrm{b}_{\mathrm{i},\mathrm{j};\mathfrak{n}}(\mathbb{S})$ and by Proposition 1.3.2 in chapter one to show that

$$\frac{\mathrm{h}_{\mathcal{v},\mathrm{j}}}{\mathrm{h}_{\mathrm{j}}} \le c\left(\frac{\Omega + dist(x,\mathrm{I}_{\mathrm{j}})}{\Omega}\right)^2, \text{and } \frac{\mathrm{h}_{\mathcal{v},\mathrm{j}}}{\mathrm{h}_{\mathcal{v}}} \le c\,\frac{\Omega + dist(x,\mathrm{I}_{\mathrm{j}})}{\Omega}.$$

Then, for the third inequality we use the fact that $dist(x, \mathrm{I}_{\mathrm{j}}) \ge \frac{1}{2}\Omega$.

And for the last we have applied the inequality $\frac{\Omega_1}{\Omega} \le \frac{\mathfrak{n}}{\mathfrak{n}_1}$.

Now, by virtue of Lemma2.2.16 we may represent

$\mathbb{S}^{(q)}(x) - \mathcal{D}_{\mathfrak{n}_1}^{(q)}(x)$ as

$$\mathbb{S}^{(q)}(x) - \mathcal{D}_{\mathfrak{n}_1}^{(q)}(x) = \left(\left(\mathcal{P}_{\mathcal{v}}(x) - \mathcal{P}_{\mathcal{v}+1}(x)\right)\widetilde{\mathrm{T}}_{\mathcal{v}+1}(x)\right)^{(q)}$$

$$+(\sum_{I_j \subseteq \mathcal{A}, j \neq v, v+1} + \sum_{I_j \nsubseteq \mathcal{A}, j \neq v, v+1}) \left(\left(\mathcal{P}_v(x) - \mathcal{P}_j(x)\right) \tilde{T}_j(x)\right)^{(q)}$$

$$:= \sigma_1(x) + \sigma_2(x) + \sigma_3(x) ,$$

where we write $\mathcal{P}_{n+1} := \mathcal{P}_n$, if $v = n$.

We begin with the estimate of σ_1 . Note that if $v = n$, then $\sigma_1 \equiv 0$, so that we may assume that $v < n$. we need separate arguments for Lemma 2.2.17 and 2.2.18

First we deal with Lemma 2.2.18 . since $\mathbb{S} \in \sum_{\kappa,n}^{1}$, $q = 2$ and $I_v \subseteq \mathcal{A}$, it readily follows that.

$\|\mathcal{P}_v'' - \mathcal{P}_{v+1}''\|_{L_{p(I_v)}} \leq \frac{c(p)}{\Omega^2} b_{v,v+1}$, which in turn implies

$$\|\mathcal{P}_v'(x) - \mathcal{P}_{v+1}'(x)\|_p = \left\| \int_{x_v}^{x} (\mathcal{P}_v'' - \mathcal{P}_{v+1}'') du \right\|_p$$

$$\leq c(p) \left\| \frac{b_{v,v+1} (x - x_v)}{\Omega^2} \right\|_p$$

And $\|\mathcal{P}_v(x) - \mathcal{P}_{v+1}(x)\|_p \leq c(p) \left\| \frac{b_{v,v+1} (x-x_v)^2}{\Omega^2} \right\|_p$.

Therefore , by Lemma 2.2.19 .

$$\|\sigma_1(x)\|_p \leq c(p) \left\| \frac{1}{\Omega^2} b_{v,v+1} (1 + \frac{x - x_v}{\Omega_1} + \frac{(x - x_v)^2}{{\Omega_1}^2})(\frac{\Omega_1}{\Omega_1 + |x - x_v|})^{b_2} \right\|_p$$

$$\leq c(p) \left\| \frac{1}{\Omega^2} b_{v,v+1} \left(\frac{\Omega_1}{\Omega_1 + |x-x_v|}\right)^{b_2 - 2} \right\|_p \qquad (2.4.30)$$

Now, if $I_{v+1} \subseteq \mathcal{A}$, then (2.4.30) implies

$$\|\sigma_1(x)\|_p \leq c(p) \left\| \frac{1}{\Omega^2} b_{\kappa} (\mathbb{S}, \mathcal{A}) \right\|_p \qquad (2.4.31)$$

And if $I_{v+1} \nsubseteq \mathcal{A}$, then 2.4.30 yields

$$\|\sigma_1(x)\|_p \le c(p)\left\|\frac{1}{\Omega^2}\mathfrak{b}_\kappa(\mathbb{S})\frac{\Omega_1}{\Omega}\frac{\Omega}{\Omega_1+|x-x_v|}\left(\frac{\Omega_1}{\Omega_1+|x-x_v|}\right)^{\mathfrak{b}_2-3}\right\|_p$$

$$\le c(p)\ \left\|\frac{1}{\Omega^2}\mathfrak{b}_\kappa(\mathbb{S})\frac{\mathfrak{n}}{\mathfrak{n}_1}\ \frac{\Omega}{|x-x_v|}\ \left(\frac{\Omega}{\Omega+|x-x_v|}\right)^{\mathfrak{b}_2-3}\right\|_p$$

$$\le c(p)\left\|\frac{1}{\Omega^2}\mathfrak{b}_\kappa(\mathbb{S})\frac{\mathfrak{n}}{\mathfrak{n}_1}\left(\frac{\Omega}{dist(x,I\backslash\mathcal{A})}\right)^{\mathfrak{b}_3}\right\|_p \qquad (2.4.32)$$

Now we establish Lemma 2.2.18. since $x \in \bar{O}_e, v \in H$. If also $(v+1) \notin H$, then $\mathbb{S} \in \sum_{\kappa,\mathfrak{n}}(Y_{\mathbb{S}})$ implies $\mathcal{P}_v \equiv \mathcal{P}_{v+1}$. Hence $\sigma_1 = 0$.

Otherwise $(v+1) \in H$, so that $x \in \bar{O}_e$, implies $x - x_v \ge \Omega$ .Therefore 2.4.29 hold for $j = v+1$, and we may absorb σ_1 either in σ_2 or in σ_3, as the case may be, and which we estimate below, what is left is to estimate σ_2 and σ_3, It follows from 2.4.29 that

$$\|\sigma_3(x)\|_p \le c(p)\left\|\frac{c(p)\mathfrak{b}_\kappa(\mathbb{S})}{\Omega^q}\frac{\mathfrak{n}}{\mathfrak{n}_1}\ \Omega^{\mathfrak{b}_3}\sum_{I_j\not\subset\mathcal{A}, j\neq v, v+1}\ \frac{h_j}{(\Omega+dist(x,I_j))^{\mathfrak{b}_3+1}}\right\|_p$$

$$\le c(p)\left\|\ \frac{\mathfrak{b}_\kappa(\mathbb{S})}{\Omega^q}\frac{\mathfrak{n}}{\mathfrak{n}_1}\left(\ \frac{\Omega}{\Omega+\ dist(x,I\backslash\mathcal{A})}\right)^{\mathfrak{b}_3}\right\|_p \qquad (2.4.33)$$

Similarly , if $dist(x, I_{v*}) := \min\{dist(x, I_{v-1}), dist(x, I_{v+2})\}$

Then we obtain

$$\|\sigma_2(x)\|_p \le c(p)\left\|\frac{\mathfrak{b}_\kappa(\mathbb{S},\mathcal{A})}{\Omega^q}\frac{\mathfrak{n}}{\mathfrak{n}_1}\left(\ \frac{\Omega}{\Omega+\ dist(x, I_{v*})}\right)^{\mathfrak{b}_3}\right\|_p$$

$$\le c(p)\left\|\frac{\mathfrak{b}_\kappa(\mathbb{S},\mathcal{A})}{\Omega^q}\right\|_p \qquad (2.4.34)$$

Thus 2.4.25 is followed by combining 2.4.33 and 2.4.34 with the above discussion of σ_1, and 2.4.26 is obtained by combining 2.4.31 - 2.4.34 .

This completes the proof ■

<u>Theorem</u> 2.5

Let $\kappa \in \mathbb{N}, \Gamma \in \Phi^{\kappa}$ and $\mathbb{s} \in \mathbb{N}_0$. Then there are constants $c = c(\kappa, \mathbb{s})$ and $c_* = c_*(\kappa$ the following property . If $\mathfrak{n} \geq N(\kappa, Y_{\mathbb{s}})$ and

$$\mathbb{S} \in \textstyle\sum_{\kappa;\mathfrak{n}}(Y_{\mathbb{s}}) \cap \Delta^{(2)}(Y_{\mathbb{s}}) \cap L_p[-1,1] \tag{2.5.1}$$

Is such that

$$\mathfrak{b}_{\kappa}(\mathbb{S}) \leq 1 \tag{2.5.2}$$

Then there is a polynomial $\mathcal{P}_{\mathfrak{n}} \in \sum_{\kappa;\mathfrak{n}}(Y_{\mathbb{s}}) \cap \Delta^{(2)}(Y_{\mathbb{s}}) \cap L_p[-1,1]$ of degree $\leq c_*\mathfrak{n}$, satisfying

$$\|\mathbb{S}(x) - \mathcal{P}_{\mathfrak{n}}(x)\|_p \leq c(p)\left\|\Gamma\big(\Omega_{\mathfrak{n}}(x)\big)\right\|_p \tag{2.5.3}$$

Proof.

We may assume that $\kappa \geq 3$, and that by virtue Lemma 2.3.8 , we may assume that $\mathbb{S} \in L_p^1[-1,1]$. we begin with some notation, Given $\mathcal{A} \subseteq I$ denote $\mathcal{A}^e \coloneqq \cup_{I_j \cap \mathcal{A} \neq \emptyset} I_j$, and $\mathcal{A}^{2e} \coloneqq (\mathcal{A}^e)^e$

In order to prove our assertion , we have to find a polynomial $\mathcal{P}_{\mathfrak{n}}$ of degree $\leq c\,\mathfrak{n}$, such that

$$\|\mathbb{S}(x) - \mathcal{P}_{\mathfrak{n}}(x)\|_p \leq c(p)\|\Gamma(\Omega\,)\|_p \tag{2.5.4}$$

And

$$\mathcal{P}''_{\mathfrak{n}}(x)\sigma(x) \geq 0, x \in [-1,1] \tag{2.6.5}$$

where $\sigma(x)$ define the set $\sigma(x) \coloneqq \operatorname{sgn} \prod(x), x \in [-1,1]$

Given $Y_{\mathbb{s}}$, $\mathbb{s} > 0$, recall that $\prod(x) = \prod_{i=1}^{\mathbb{s}}(x - y_i)$

We fix $\mathfrak{b}$ so that $\mathfrak{b}_1 \coloneqq 25(\mathbb{s}+1) + 2\kappa$. This makes $c_0(\kappa, \mathbb{s}, \mathfrak{b})$, the constant in

$$\left\|\mathbb{S}''(x) - \mathcal{D}''_{n_1}(x)\right\|_p \leq c_0 \left\|\frac{\Gamma(\Omega)\pi(x)}{\Omega^2}(b_\kappa(\mathbb{S},\mathcal{A})\right\|_p + \left\|b_\kappa(\mathbb{S})\frac{n}{n_1}(\frac{\Omega}{dist(x, I\backslash\mathcal{A})})^{b_1}\right\|_p$$

Where $c_0 = c_0(p, \kappa, \mathbb{s}, b)$ and

$$\pi(x) := \prod_{i=1}^{\mathbb{s}} \frac{|x - y_i|}{|x - y_i| + \Omega}$$

Dependent only on κ and $\mathbb{s}$ we dente $c_2 := \mathcal{C}_0$. fix on integer c_3 such that

$$c_3 \geq max\{8\kappa/ c_2, 12\mathbb{s}\} \tag{2.5.6}$$

Where c_1 is constant form

$$Q''_n(x)\sigma(x) \geq c_1 \frac{1}{\mu}\left(\frac{\Omega}{max\{\Omega, dist(x, E)\}}\right)^{25(\mathbb{s}+1)+2\kappa} \frac{\pi(x)\Gamma(\Omega)}{\Omega^2}, x \in \mathcal{J} \cup [-1,1]\backslash E.$$

(we may take $c_1 \leq 1$)

And without loss of generality we may assume that n is divisible by c_3 ,i.e., $n = Nc_3$, where this defines N.

We divide I in to N intervals

$E_q := [x_{qc_3}, x_{(q-1)c_3}] = I_{qc_3} \cup \ldots \cup I_{(q-1)c_3+1}, q = 1, \ldots, N$, and we may assume, without loss of generality that N is so large there are at least three intervals E_q lying between the points y_i and $y_{i+1}, 0 \leq i \leq \mathbb{s}$.

We will write $j \in \mathbb{N}$, if there is an $x \in I_j$, such that

$$\|\mathbb{S}''(x)\|_p \leq 5c_2(p)\left\|\frac{\Gamma(\Omega)}{\Omega^2}\right\|_p \tag{2.5.7}$$

And we will say that $q \in G_1$, if E_q contains at least $2\kappa - 5$ intervals I_j with $j \in \mathbb{N}$.

We will say that $\mathfrak{q} \in G$ if either $\mathfrak{q} \in G_1$, or $E_\mathfrak{q}$ intersects $\mathcal{O}$ (recall $\mathcal{O}$ was define $\mathcal{O} = \mathcal{O}(\mathfrak{n}, Y_\mathbb{s}) := \bigcup_{i=1}^{\mathbb{s}} \mathcal{O}_i, \mathcal{O}(\mathfrak{n}, \emptyset) := \emptyset$, and $E_{\mathfrak{q}-1}$ intersects $\mathcal{O}$ and $\mathfrak{q}-1$ is in G_1, or $E_\mathfrak{q}$ intersects $\mathcal{O}$ and $E_{\mathfrak{q}+1}$ intersects $\mathcal{O}$ and $\mathfrak{q}+1$ is in G_1.

Note that 2.5.7 and Lemma 2.4.19 imply that,

$$\|\mathbb{S}''(x)\|_p \le c\,(p) \left\|\frac{\Gamma(\Omega)}{\Omega^2}\right\|_{L_p(E_\mathfrak{q})}, x \in E_\mathfrak{q}, \mathfrak{q} \in G \tag{2.5.8}$$

Set $\qquad E := \bigcup_{\mathfrak{q} \notin G} E_\mathfrak{q}$,

And decompose $\mathbb{S}$ in to a ‘‘small‘‘ part and a ‘‘big‘‘ one by setting

$$\mathbb{s}_1(x) := \begin{cases} \mathbb{S}''(x), & \text{if } x \notin E \\ 0, & \text{otherwise,} \end{cases}$$

And $\mathbb{s}_2 := \mathbb{S}'' - \mathbb{s}_1$,

And putting

$$\mathbb{S}_1(x) := \mathbb{S}(-1) + (x+1)\mathbb{S}'(-1) + \int_{-1}^{x} (x-\mathfrak{u})\mathbb{s}_1(\mathfrak{u})d\mathfrak{u},$$

$$\mathbb{S}_2(x) := \int_{-1}^{x} (x-\mathfrak{u})\mathbb{s}_2(\mathfrak{u})d\mathfrak{u}.$$

(Note that $\mathbb{s}_1$ and $\mathbb{s}_2$ are well defined for $x \ne x_\mathfrak{j}, 1 \le \mathfrak{j} \le \mathfrak{n}-1$, so that $\mathbb{S}_1$ and $\mathbb{S}_2$ are well defined everywhere and possess a second derivative. Again, for $x \ne x_\mathfrak{j}, 1 \le \mathfrak{j} \le \mathfrak{n}-1$, Thus from now on whenever we write $\mathbb{S}_\iota''(x)$ we will mean $x \ne x_\mathfrak{j}, 1 \le \mathfrak{j} \le \mathfrak{n}-1$.)

Evidently $\qquad \mathbb{S}_1, \ \mathbb{S}_2 \in \Sigma_{\kappa;\mathfrak{n}}^1(Y_\mathbb{s})$,

And $\qquad \mathbb{S}_1''(x)\sigma(x) \ge 0$ and $\mathbb{S}_2''(x)\sigma(x) \ge 0, x \in [-1,1]$

Now , 2.5.8 implies that

$$\|\mathbb{S}_1''(x)\|_p \le c\,(p) \left\|\frac{\Gamma(\Omega)}{\Omega^2}\right\|_p, x \in [-1,1],$$

Which , in turn, by Lemma 2.3.1 yields

$$\mathfrak{b}_\kappa(\mathbb{S}_1) \le c.$$

Together with $\mathfrak{b}_\kappa(\mathbb{S}_1) \le 1$, we obtain

$$\mathfrak{b}_\kappa(\mathbb{S}_2) \le c + 1 \le \lceil c + 1 \rceil =: c_4 \qquad (2.5.9)$$

The set E is a union of disjoint intervals $F_p = [a_p, \mathfrak{b}_p]$, between any two of which all intervals E_q are with $q \in G$. we may assume that $\mathfrak{n} > c_3 c_4$, and write $p \in AG$ (for ‘‘Almost Good‘‘) if F_p consists of no more than c_4 intervals E_q, in particular , it consists of no more than $c_3 c_4$ intervals I_j. Hence , by lemma 2.5.19.

$$\|\mathbb{S}_2''(x)\|_p \le c\,(p) \left\|\frac{\Gamma(\Omega)}{\Omega^2}\right\|_{L_p(F_p)} , x \in F_p , p \in AG \qquad (2.5.10)$$

Set $\quad F := \bigcup_{p \notin AG} F_p$,

And again we decompose $\mathbb{S}$ by letting

$$\mathfrak{s}_4 := \begin{cases} \mathbb{S}''(x), & \text{if } x \in F \\ 0 , & \text{otherwise} \end{cases} \quad \text{and}$$

$$\mathfrak{s}_3 := \mathbb{S}'' - \mathfrak{s}_4$$

And putting

$$\mathbb{S}_3(x) := \mathbb{S}(-1) + (x+1)\mathbb{S}'(-1) + \int_{-1}^{x} (x - \mathrm{u})\mathfrak{s}_3(\mathrm{u})d\mathrm{u},$$

$$\mathbb{S}_4(x) := \int_{-1}^{x} (x - \mathrm{u})\mathfrak{s}_4(\mathrm{u})d\mathrm{u}.$$

Then , evidently

$$\mathbb{S}_3, \mathbb{S}_4 \in \textstyle\sum_{\kappa;\mathfrak{n}}^{1}(Y_{\mathfrak{s}}) \qquad (2.5.11)$$

and

$$\mathbb{S}_3''(x)\sigma(x) \ge 0 \text{ and } \mathbb{S}_4''(x)\sigma(x) \ge 0, x \in [-1,1] \qquad (2.5.12)$$

We will approximate $\mathbb{S}_3$ and $\mathbb{S}_4$ by polynomials that are coconvex with $\mathbb{S}$ and achieve the required pointwise approximation for $x \in \bigcup_{p \in AG} F_p$, 2.5.10 implies that

$$\|\mathbb{S}_3''(x)\|_p = \|\mathbb{S}_2''(x)\|_p \le c\,(p) \left\|\frac{\Gamma(\Omega)}{\Omega^2}\right\|_p .$$

Otherwise ,

$$\|\mathbb{S}_3''(x)\|_p = \|\mathbb{S}_1''(x)\|_p \le c\,(p)\left\|\frac{\Gamma(\Omega)}{\Omega^2}\right\|_p$$

Hence

$$\|\mathbb{S}_3''(x)\|_p \le c\,(p)\left\|\frac{\Gamma(\Omega)}{\Omega^2}\right\|_p \qquad (2.5.13)$$

which by virtue of lemma 2.3.1, yields that

$\mathfrak{b}_\kappa(\mathbb{S}_3) \le c$. Again by $\mathfrak{b}_\kappa(\mathbb{S}\,) \le 1$ we obtain

$$\mathfrak{b}_\kappa(\mathbb{S}_4) \le c + 1 \ \le \lceil c + 1 \rceil =: c_5 \qquad (2.5.14)$$

In view of (2.5.11) and (2.5.12) , it follows by Lemma 2.2.20 combined with (2.5.13) that a polynomial $\mathfrak{r}_\mathfrak{n}$, of degree $\le c\mathfrak{n}$, exists, which is coconvex with $\mathbb{S}$, and such that

$$\|\mathbb{S}_3(x) - \mathfrak{r}_\mathfrak{n}(x)\|_p \le c\,(p)\|\Gamma(\Omega)\|_p \qquad (2.5.15)$$

Thus, it remains to approximate $\mathbb{S}_4$. To this end, we observe that for $p \notin AG$,

$$\mathbb{s}_4(x) = \mathbb{S}_2''(x),\ x \in F_p^{2e},$$

So that by virtue of (2.5.9), we conclude that

$$\mathfrak{b}_\kappa\left(\mathbb{S}_4, F_p^{2e}\right) = \mathfrak{b}_\kappa\left(\mathbb{S}_2, F_p^{2e}\right) \le \mathfrak{b}_\kappa(\mathbb{S}_2) \le c_4 \qquad (2.5.16)$$

(Note that , for $p \in AG, \mathbb{S}_4$ is linear in F_p^{2e} and $\mathfrak{b}_\kappa\left(\mathbb{S}_4, F_p^{2e}\right) = 0$.)

Applying lemma 2.4.10, we construct two polynomials $\overline{Q}_\mathfrak{n}$ and $\mathcal{M}_\mathfrak{n}$ of degree $< c\mathfrak{n}$, and we let $\mathcal{D}_{\mathfrak{n}_1}(.\,, \mathbb{S}_4)$ of degree $c\mathfrak{n}_1$, be defined by

$$\mathcal{D}_{\mathfrak{n}_1}(x) \coloneqq \mathcal{D}_{\mathfrak{n}_1}(x, \mathbb{S}, \mathfrak{b}) \coloneqq \sum_{j=1}^{\mathfrak{n}} \mathcal{P}_j(x)\widetilde{T}_{j;\mathfrak{n}_1}(x), \text{ of degree} \le\ c\mathfrak{n}_1.$$

With $\mathfrak{n}_1 \coloneqq c_5\mathfrak{n}$.

We begin with $\overline{Q}_\mathfrak{n}$. for each $\mathfrak{q}$ for which $E_\mathfrak{q} \subseteq F$, let $\mathcal{J}_\mathfrak{q}$ be the union of all intervals $I_j \subseteq E_\mathfrak{q}$ with $j \in \mathbb{N}$. Recall that $\mathfrak{q} \notin G$, therefore by 2.5.6 , the number, $\mu_\mathfrak{q}$, of such intervals is at most $2\kappa - 6 < c_3/4$, and the total number of intervals in E_q is c_3 . Thus, Lemma 2.4.10 is applicable for each E_q, and if we set

$$\overline{Q}_n := \sum_{E_q \subseteq F} Q_n(.,E_q,\mathcal{J}_q)$$

Where on the right-hand side are the polynomials guaranteed by Lemma 2.4.10 (we consider $\sum_{E_q \subseteq F} Q_n(.,E_q,\mathcal{J}_q) \equiv 0$, if $\mathcal{J}_q = \emptyset$), and denote

$$\mathcal{J} = \bigcup_{E_q \subseteq F} \mathcal{J}_q ,$$

Then we conclude that $\overline{Q}_n$ satisfies

$$\overline{Q}''_n(x)\sigma(x) \geq 0, x \in [-1,1]\backslash F,$$

$$\overline{Q}''_n(x)\sigma(x) \geq -\frac{\pi(x)\Gamma(\Omega)}{\Omega^2}, x \in F\backslash\mathcal{J} \tag{2.5.17}$$

$$\overline{Q}''_n(x)\sigma(x) \geq \frac{4\pi(x)\Gamma(\Omega)}{\Omega^2}, x \in \mathcal{J} .$$

Note that 2.5.17 follows since for any given x all relevant $Q''_n(x,E_q,\mathcal{J}_q)$, except perhaps one , have the same sign, and if $\mathcal{J}_q \neq \emptyset$ then $c_1 \frac{c_3}{\mu_q} \leq 2\kappa - 6$ and $c_1 c_3 \geq 8\kappa$.

$$\|\overline{Q}_n(x)\|_p \leq c\,(p)\|\Gamma(\Omega)\|_p , \tag{2.5.18}$$

Next we define the polynomial $\mathcal{M}_n$. For each F_p with $p \notin AG$, let $\mathcal{J}_{p^-}$ denote the union of the two intervals on the left side of F_p^e, and let $\mathcal{J}_{p^+}$ denote the union of the two intervals on the right side of F_p^e.Also, let F_{p^-} and F_{p^+} be closed intervals each consisting of $\iota := c_3 c_4$ intervals I_j and such that $\mathcal{J}_{p^-} \subset F_{p^-} \subset F_p^e$ and $\mathcal{J}_{p^+} \subset F_{p^+} \subset F_p^e$.

Finally ,put

$\mathcal{J}_p^* := \mathcal{J}_{p^-} \cup \mathcal{J}_{p^+}$ and $\mathcal{J}^* := \bigcup_{p \notin AG} \mathcal{J}_p^*$,

Now we set

$\mathcal{M}_{\mathfrak{n}} := \sum_{p \notin AG} \left(Q_{\mathfrak{n}}(.\,, F_{p^+}, \mathcal{J}_{p^+}) + Q_{\mathfrak{n}}(.\,, F_{p^-}, \mathcal{J}_{p^-}) \right)$, since $\iota := c_3 c_4$,

It follows from 2.5.6 that $c_1 \frac{\iota}{\mu} \geq 2c_4$, for $\mu = 2$.

Again by Lemma 2.4.10 ,

$$\mathcal{M}_{\mathfrak{n}}''(x)\sigma(x) \geq -2\frac{\pi(x)\Gamma(\Omega)}{\Omega^2}, x \in F\backslash\mathcal{J}^*,$$

$$\mathcal{M}_{\mathfrak{n}}''(x)\sigma(x) \geq \frac{2c_4\pi(x)\Gamma(\Omega)}{\Omega^2}, x \in \mathcal{J}^* \tag{2.5.19}$$

$$\mathcal{M}_{\mathfrak{n}}''(x)\sigma(x) \geq \frac{\pi(x)\Gamma(\Omega)}{\Omega^2}\left(\frac{\Omega}{dist(x,F)}\right)^{25(\mathbb{s}+1)+2\kappa}, x \in [-1,1]\backslash F^e,$$

Where in the last inequality we used the fact that the first inequality in Lemma 2.2.4 implies the estimate

$$max\{\Omega, dist(x, F^e)\} \leq dist(x, F), x \in [-1,1]\backslash F^e.$$

Finally , it readily follows from

$(\|Q_{\mathfrak{n}}(x)\|_p \leq c(p)\iota^{4\kappa+6} \left\| \Gamma(\Omega) \sum_{j:I_j \subseteq E} \frac{\Omega\, h_j}{(|x-x_j|+\Omega)^2} \right\|_p,$

$$\|\mathcal{M}_{\mathfrak{n}}(x)\|_p \leq c\,(p)\|\Gamma(\Omega)\|_p \tag{2.5.20}$$

The third Auxiliary polynomial the properties of which we need to recall is $\mathcal{D}_{\mathfrak{n}_1}(x) := \mathcal{D}_{\mathfrak{n}_1}(\mathbb{S},.)$. by 2.5.14 and the choice of $\mathfrak{b}$ Lemma 2.2.21 yields

$$\left\|\mathbb{S}_4(x) - \mathcal{D}_{\mathfrak{n}_1}(x)\right\|_p \leq c\,(p)\,\|\Gamma(\Omega)\|_p, \tag{2.5.21}$$

And Lemma 2.2.22 combined with 2.5.11 and 2.5.12 implies that for any proper interval $\mathcal{A}$

$$\left\|\mathbb{S}_4''(x) - \mathcal{D}_{\mathfrak{n}_1}''(x)\right\|_p \leq$$
$$c_2(p)\left\|\frac{\pi(x)\Gamma(\Omega)}{\Omega^2}\mathfrak{b}_\kappa(\mathbb{S}_4, \mathcal{A})\right\|_{L_p(\mathcal{A})} +$$
$$c_2c_5(p)\left\|\frac{\pi(x)\Gamma(\Omega)}{\Omega^2}\frac{\mathfrak{n}}{\mathfrak{n}_1}\left(\frac{\Omega}{dist(x, I\backslash\mathcal{A})}\right)^{\mathfrak{b}_1}\right\|_{L_p(\mathcal{A})}, x \in \mathcal{A} \tag{2.5.22}$$

Recall that $\mathfrak{n}_1 = c_5\mathfrak{n}$, and write

$$R_n := \mathcal{D}_{n_1} + c_2\overline{Q}_n + c_2\mathcal{M}_n \tag{2.5.23}$$

By virtue of 2.5.18, 2.5.20, and 2.5.21, we obtain

$\|\mathbb{S}_4(x) - R_n(x)\|_p \le c\,(p)\|\Gamma(\Omega)\|_p$,

Which combined with 2.5.15, proves 2.5.4 for $\mathcal{P}_n := R_n + r_n$.

Thus, in order to conclude the proof of Theorem 2.5 , we should prove that $\mathcal{P}_n$ satisfies 2.5.5 .

To this end , we recall that r_n is coconvex with $\mathbb{S}$, so that we only have to deal with R_n.since 2.5.22 holds for any proper interval $\mathcal{A}$, we will prescribe different ones as needed. As long as $x \in F_p\backslash\mathcal{J}_p^*$, it suffices to take $\mathcal{A} := F_p$. Then the quotient inside the big parentheses in 2.5.22 is bounded by 1 , and since we have $\mathbb{S}_4(x) = \mathbb{S}\,(x), x \in F$, it follows that $\mathfrak{b}_\kappa(\mathbb{S}_4, F_p) = \mathfrak{b}_\kappa(\mathbb{S}, F_p) \le 1$.

Hence ,

$$\left\|\mathbb{S}_4''(x) - \mathcal{D}_{n_1}''(x)\right\|_p \le c_2\,(p)\left\|\frac{\pi(x)\Gamma(\Omega)}{\Omega^2}\mathfrak{b}_\kappa(\mathbb{S}_4, F_p)\right\|_{L_p(F_p\backslash\mathcal{J}_p^*)} + c_2c_5(p)\left\|\frac{\pi(x)\Gamma(\Omega)}{\Omega^2}\frac{n}{n_1}\right\|_{L_p(F_p\backslash\mathcal{J}_p^*)}$$

$$\le 2c_2\,(p)\left\|\frac{\pi(x)\Gamma(\Omega)}{\Omega^2}\right\|_{L_p(F_p\backslash\mathcal{J}_p^*)}, x \in F_p\backslash\mathcal{J}_p^* \tag{2.6.24}$$

If $x \in \mathcal{J}_p^*$, then it suffices to take $\mathcal{A} := F_p^{2e}$ and , similarly, 2.5.15 and 2.5.22, imply

$$\left\|\mathbb{S}_4''(x) - \mathcal{D}_{n_1}''(x)\right\|_p \le c_2\,(p)\left\|\frac{\pi(x)\Gamma(\Omega)}{\Omega^2}\mathfrak{b}_\kappa(\mathbb{S}_4, F_p^{2e})\right\|_{L_p(\mathcal{J}_p^*)} + c_2c_5(p)\left\|\frac{\pi(x)\Gamma(\Omega)}{\Omega^2}\frac{n}{n_1}\right\|_{L_p(\mathcal{J}_p^*)}$$

$$\leq 2c_2c_4\,(p)\left\|\frac{\pi(x)\Gamma(\Omega)}{\Omega^{2}}\right\|_{L_p(\mathcal{J}_p^*)},\ x\in\mathcal{J}_p^* \qquad (2.5.25)$$

Finally , if $x \in [-1,1]\backslash F^e$, then we take $\mathcal{A}$ to be the connected component of $[-1,1]\backslash\dot{F}$, that contains x, Then by 2.5.22

$$\left\|\mathbb{S}_4''(x)-\mathcal{D}_{n_1}''(x)\right\|_p$$
$$\leq c_2\,(p)\left\|\frac{\pi(x)\Gamma(\Omega)}{\Omega^{2}}\,b_\kappa(\mathbb{S}_4,\mathcal{A})\right\|_{L_p([-1,1]\backslash F^e)}$$
$$+\,c_2c_5(p)\left\|\frac{\pi(x)\Gamma(\Omega)}{\Omega^{2}}\frac{n}{n_1}\left(\frac{\Omega}{dist(x,I\backslash\mathcal{A})}\right)^{b_1}\right\|_{L_p([-1,1]\backslash F^e)}$$

$$=c_2\,(p)\left\|\frac{c_2\pi(x)\Gamma(\Omega)}{\Omega^{2}}\left(\frac{\Omega}{dist(x,I\backslash F)}\right)^{b_1}\right\|_{L_p([-1,1]\backslash F^e)},\ x\in[-1,1]\backslash F^e \qquad (2.5.26)$$

Where we used the fact that $\mathbb{S}_4$ is linear in $\mathcal{A}$. Since by 2.5.23,

$$R_n''(x)\sigma(x)\geq c_2\overline{Q}_n''(x)\sigma(x)+c_2\mathcal{M}_n''(x)\sigma(x)+\mathbb{S}_4''(x)\sigma(x)$$
$$-\left\|\mathbb{S}_4''(x)-\mathcal{D}_{n_1}''(x)\right\|_p,\ x\in[-1,1],$$

It is followed by 2.5.12, 2.5.17, 2.5.19 and 2.5.24, that

$$R_n''(x)\sigma(x)\geq\frac{c_2c_4\pi(x)\Gamma(\Omega)}{\Omega^{2}}(4-2+0-2)=0\,,x\in\mathcal{J}_p\backslash\mathcal{J}_p^*.$$

If $x\in F_p\backslash(\mathcal{J}_p\cap\mathcal{J}_p^*)$, then 2.5.7 is violated so that

$$\mathbb{S}_4''(x)\sigma(x)\geq\frac{5c_2\Gamma(\Omega)}{\Omega^{2}}\geq\frac{5c_2}{\Omega^{2}}\pi(x)\Gamma(\Omega)$$

Hence, by virtue of 2.5.17, 2.5.19 and 2.5.24, we get

$$R_n''(x)\sigma(x)\geq\frac{c_2\pi(x)\Gamma(\Omega)}{\Omega^{2}}(-1-2+5-2)=0,x\in F_p\backslash(\mathcal{J}_p\cap\mathcal{J}_p^*).$$

Next , if $x\in\mathcal{J}^*$, then by 2.5.12, 2.5.17, 2.5.19 and 2.5.25 , we obtain

$$R_n''(x)\sigma(x)\geq\ 0 \qquad (2.5.27)$$

And finally, 2.5.12 , 2.5.17 , 2.5.19 and 2.5.25 imply

2.5.26 for $x \in [-1,1]\backslash F^e$. Thus, 2.5.27 holds for all $x \in [-1,1]$, and so we have constructed a polynomial $\mathcal{P}_n$, satisfying 2.5.4 and 2.5.5 , for each $n > c$, divisible by c_3. For all other n‘s Theorem 2.5 followed by the inclusion $\sum_{\kappa;n}^{1}(Y_s) \subseteq \sum_{\kappa;c_3 n}^{1}(Y_s)$.

This completes the proof . ■

Chapter Three

Shape Preserving Approximation Using Coconvex Piecewise Polynomials for Function in L_p Quasi Normed Space

Here we approximate piecewise convex function in $L_p, 0 < p < 1$, quasi normed space using coconvex piecewise algebraic polynomials. Also we introduce some properties of these polynomials.

3.1 Introduction and main results

Let $\mathcal{F} \in L_p[-1,1]$ change its convexity finitely many times, say $\mathbb{s} \geq 0$ times, in the interval. We are interested in estimating the degree of approximation of $\mathcal{F}$ by polynomials which are coconvex with it, namely, polynomials that change their convexity exactly at the points where $\mathcal{F}$ does.

In [36], the D. Leviatan and I. Shevchuk, write all the results of monotone and comonotone approximation theorems on a finite interval $[a, \mathfrak{b}]$, by algebraic polynomials in continuous function (uniform norm). see also [35]. In [35,37] the D. Leviatan and I. Shevchuk, studied the monotone and comonotone approximation of continuous function using κth modulus of smoothness of the $\mathfrak{r}$th derivative of the function.

They also studied the above approximation using κth Ditzian-Totik modulus of smoothness.

In [37] the D. Leviatan and I. Shevchuk, studied the monotone and comonotone approximation of continuous functions in terms of Ditzian-Totik modulus of smoothness using piecewise polynomials.

In our work we introduce positive theorem for convex and coconvex approximation of function in L_p space for $0 < p < 1$.

The main truth in this chapter is to Jackson-type estimate for the approximation of a piecewise polynomial which changes convexity finitely many time in the interval, by algebraic polynomials that change convexity at exactly the same points.

The main result is Theorem 3.4.15 stated below, which is the analogue of [37.proposition 3].

Our strategy for the future is to approximate an arbitrary function in L_p space that changes convexity finitely many time in the interval, by an appropriate coconvex piecewise polynomial which in turn by virtue of Theorem 3.4.15, will be approximated by a coconvex polynomial.

In this chapter, we studied some negative results for the coconvex polynomial approximation of more general piecewise convex functions (see Theorem 3.3.4), Also as a byproduct of Theorem 3.3.11, we obtain one important positive result for coconvex polynomial approximation (Theorem 3.3.17)

Therefore, it is one of the components of this chapter is the following. We state the main result contain the construction of the negative result, Auxiliary lemmas, the proof of Theorem 3.3.11 which is a preliminary step and a special case of Theorem 3.4.15 , and as a byproduct, its proof yields of Theorem 3.3.17, we prove Theorem 3.5.1 and with it we conclude the proof of the Theorem 3.4.15. Many of the methods we apply are modification of similar ones in this chapter to Devore, Dzyubenko, Gilewicz, Kopotun, Mania, Yu and the authors (see the References).

We will use the notations $c(a, \mathfrak{b})$ and $C(a, \mathfrak{b})$ for such constants which are of no signicance to us and may differ on different occurrences, even in the same, and depending on a and $\mathfrak{b}$.

Let $I := [-1,1]$ and denote by L_p and $L_p^{\mathfrak{r}}$ respectively, the spaces

$L_P(I) = \{\mathcal{F}: I \to \mathbb{R}: \|\mathcal{F}\|_p < \infty\}, L_p^{\mathfrak{r}}(I) = \{\mathcal{F}: I \to \mathbb{R}: \mathcal{F}, \mathcal{F}^{(\mathfrak{r})} \in L_P(I)\}$, $\mathfrak{r}, \mathcal{F} \in L_p$.

$$\|\mathcal{F}\|_p := (\int_{-1}^{1} |\mathcal{F}(x)|^p)^{\frac{1}{p}} dx$$

Given $\mathcal{F} \in L_p$, and $\kappa \in \mathbb{N}$, let

$$\Delta_h^{\kappa}\mathcal{F}(x) := \sum_{i=0}^{\kappa}(-1)^{\kappa-i}\binom{\kappa}{i}\mathcal{F}(x-\frac{\kappa}{2}h+ih)$$

Be the symmetric difference of order κ, defined for all x and $h \geq 0$, such that $x \pm \frac{\kappa}{2}h \in I$.

The Ditzian-totik(D-T) moduli of smoothness[3] are defined by

$w_{\kappa}^{\vartheta}(\mathcal{F},t)_p := \sup_{0\leq h\leq t}\left\|\Delta_{h\vartheta(x)}^{\kappa}\mathcal{F}(x)\right\|_p$, $t \geq 0$,

Where $\vartheta(x) = \sqrt{1-x^2}$, such that $x \pm \frac{\kappa}{2}h\vartheta(x) \in I$. We also deal with the ordinary moduli of smoothness which are given by the above with $\vartheta(x) \equiv 1$ replacing the above ϑ, namely,

$w_{\kappa}(\mathcal{F},t)_p := \sup_{0\leq h\leq t}\|\Delta_h^{\kappa}\mathcal{F}(x)\|_p$, $t > 0$, such that $x \pm \frac{\kappa}{2}h \in I$.

Denote by $\mathbb{Y}_{\mathbb{s}}, \mathbb{s} \in N$,the set of all collections $Y_{\mathbb{s}} := \{y_i\}_{i=1}^{\mathbb{s}}$, such that $-1 < y_{\mathbb{s}} < \cdots < y_1 < 1$, and for $\mathbb{s} = 0$, we write $\mathbb{Y}_0 := \{\emptyset\}$. For later reference set $y_0 := 1$and $y_{\mathbb{s}+1} := -1$. Finally, let $\Delta^{(2)}(Y_{\mathbb{s}})$ denote the collection of all functions $\mathcal{F} \in L_p$ that change convexity at the set $Y_{\mathbb{s}}$, and are convex in $[y_1, 1]$. Given $n \in N, n > 1$, we set $x_j := x_{j,n} := cos\left(\frac{j\pi}{n}\right), j = 0, \dots, n$, the Chebyshev partition of [-1,1], and we denote $I_j := I_{j,n} := \left[x_j, x_{j-1}\right], j = 1, \dots, n$. Let $\sum_{\kappa,n}$ be the collection of all continuous piecewise polynomials of degree $\kappa - 1$, on the Chebyshev partition and let $\sum_{\kappa,n}^1 \subseteq \sum_{\kappa,n}$ be the subset of all continuously differentiable such function. That is, if $\mathbb{S} \in \sum_{\kappa,n}$, then

$\mathbb{S}|I_j = \mathcal{P}_j$, $j = 1, \dots, n$,

where $\mathcal{P}_j \in \Pi_{\kappa-1}$, the collection of polynomials of degree $\leq \kappa - 1$, and

$\mathcal{P}_j(x_j) = \mathcal{P}_{j+1}(x_j)$, $j = 1, \dots, n-1$

And if $\mathbb{S} \in \sum_{\kappa,n}$, then in addition,

$$\mathcal{P}_j'(x_j) = \mathcal{P}_{j+1}'(x_j), \quad j = 1, \dots, n-1.$$

Given $Y_{\mathbb{s}} \in \mathbb{Y}_{\mathbb{s}}$, let

$\mathcal{O}_i := \mathcal{O}_{i,n}(Y_{\mathbb{s}}) := (x_{j+1}, x_{j-2})$, if $y_i \in [x_j, x_{j-1})$, where $x_{n+1} := -1, x_{-1} := 1$, and denote

$$\mathcal{O} = \mathcal{O}(n, Y_{\mathbb{s}}) := \bigcup_{i=1}^{\mathbb{s}} \mathcal{O}_i, \qquad \mathcal{O}(n, \emptyset) := \emptyset.$$

Finally, we write $j \in H = H(n, Y_{\mathbb{s}})$, if $I_j \cap \mathcal{O} = \emptyset$.

Denote by $\sum_{\kappa,n}(Y_{\mathbb{s}}) \subseteq \sum_{\kappa,n}$ and $\sum_{\kappa,n}^1(Y_{\mathbb{s}}) \subseteq \sum_{\kappa,n}^1$, the subset of those piecewise polynomials for which

$\mathcal{P}_j \equiv \mathcal{P}_{j+1}$, whenever both $j, (j+1) \notin H$.

We wish to approximation a general function $\mathcal{F} \in \Delta^{(2)}(Y_{\mathbb{s}})$ by means of polynomials which are coconvex with $\mathcal{F}$, that is, which belong to $\Delta^{(2)}(Y_{\mathbb{s}})$. we denote by

$E_n^{(2)}(\mathcal{F}, Y_{\mathbb{s}})_p := \inf_{\mathcal{P}_n \in \Pi_n \cap \Delta^{(2)}(Y_{\mathbb{s}})} \|\mathcal{F} - \mathcal{P}_n\|_p$. where Π_n is the set of polynomials of degree not exceeding n.

In [27] Leviatan, and Kopotun proved that if a function $\mathcal{F} \in L_p[-1,1]$ changes convexity at $Y_{\mathbb{s}}$, then

$$E_n^{(2)}(\mathcal{F}, Y_{\mathbb{s}})_p \le c\omega_3^{\vartheta}\left(\mathcal{F}, \frac{1}{n}\right)_p \le c\omega_3\left(\mathcal{F}, \frac{1}{n}\right)_p, \quad n \ge N \tag{3.1.1}$$

where $c = c(\mathbb{s})$ is a constant which depends only on $\mathbb{s}$, and $N = N(Y_{\mathbb{s}})$ is a constant which depends on the location of the points $Y_{\mathbb{s}}$. on the other hand, Wu and Zhou [40] proved that for $\kappa \ge 4$, estimate(3.1.1) cannot be with ω_3 replaced by ω_κ, and Pleshakov and Shatalina [37] have just prove that (3.1.1) is not valid with $N = N(\mathbb{s})$ replacing $N = N(Y_{\mathbb{s}})$. in this chapter we will prove if $\mathbb{s} > 1$, then even

$$E_n^{(2)}(\mathcal{F}, Y_{\mathbb{s}}) \le c\omega\left(\mathcal{F}, \frac{1}{n}\right), \quad n \ge N, \tag{3.1.2}$$

Is not valid with $N = N(\mathbb{s})$ replacing $N = N(Y_{\mathbb{s}})$. In fact we prove more, namely,

3.2 Some Auxiliary Lemmas:

In this section we introduce the results that we need to prove our theorems.

<u>Lemma</u> 3.2.1 [27]

If $\mathcal{X}_j(x) := \mathcal{X}_{(x_j,I]}(x)$ is the characteristic function of $(x_j, I]$, then for $j \in H$, $|\mathcal{X}_j(x) - T_j(x)| \le c\mathfrak{T}_j^{2b-s-1}(x), x \in I$, where T_j is the polynomial of degree$\le cn$,

$$T_j(x) = T_{j,n}(x; b; Y_s) := \frac{1}{d_i}\int_{-1}^{x} t_j^b(u)\Pi(u)du$$

Where $d_i = \int_{-1}^{1} t_j^b(u)\Pi(u)du$. And $t_j(x) \le \frac{c}{(|x-x_j|+h_j)^2} \le ct_j(x), x \in I$, denoting $\mathfrak{T}_j(x) := \frac{h_j}{|x-x_j|+h_j}$.

<u>Lemma</u> 3.2.2 [27]

{If $0 \le j \le i \le n$, then $\frac{1}{2}(\mathcal{J} - j) \le \frac{x_j - x_\mathcal{J}}{x_i - x_{i+1}} \le (\mathcal{J} - j)^2$.

Furthermore , if either $\mathcal{J} \le 3j$ or $n - j \le 3(n - \mathcal{J})$. Then $\frac{1}{2}(\mathcal{J} - j) \le \frac{x_j - x_\mathcal{J}}{x_i - x_{i+1}} \le 2(\mathcal{J} - j)$ }

<u>Lemma</u> 3.2.3 [27]

$\tau_j'(x)\Pi(x)\Pi(x_j) \ge 0,\ x \in I$, where τ_j is polynomial, and $\Pi(x) := \prod_{i=1}^{s}(x - y_i)$.

<u>Lemma</u> 3.2.4 [23]

$$|\tau_j'(x)| \le \frac{c(p)}{h_j}\mathfrak{T}_j^{2b}(x)\frac{|\Pi(x)|}{|\Pi(x_j)|} \le c(p)|\tau_j'(x)|, x \in I,$$

<u>Lemma</u> 3.2.5 [23]

{$\frac{1}{2}(\mathcal{J} - j) \le \frac{x_j - x_\mathcal{J}}{x_i - x_{i+1}} \le 2(\mathcal{J} - j)$ }

<u>Lemma</u> 3.2.6 [23]

{let $\mathcal{P}_\kappa$ be a polynomial of degree not exceeding κ and let $a < \mathfrak{b}$. if

$meas\{x \in [a, \mathfrak{b}]: \mathcal{P}_\kappa'' \leq 0\} < \frac{\mathfrak{b}-a}{16\kappa^3}$

Then for every $x_0 \in (a, \mathfrak{b})$,

$[\mathcal{P}_\kappa;\ a, x_0, \mathfrak{b}] \geq 0.$

<u>Lemma</u> 3.2.7 [23]

measure of $\left\{x \in G_{\vartheta}: \Pi\left(x_{j_\vartheta}\right) \mathbb{S}''(x) < 0\right\} = \frac{1}{16(\kappa-1)^3} |G_\vartheta|$, where $G_\vartheta = (x_{J_\vartheta}, x_{j_\vartheta})$ the connected components of $\mathcal{O} = \mathcal{O}(\mathfrak{n}, Y_{\mathbb{s}})$.

<u>Lemma</u> 3.2.8 [26]

$\mathfrak{b}_{i,j}(\mathbb{s}) := \left\|\mathcal{P}_i - \mathcal{P}_j\right\|_{L_{p(I_i)}} (\frac{h_j}{h_{i,j}})^k, i, j = 1, \dots, \mathfrak{n}$,where $\mathcal{P}_i$, $\mathcal{P}_j$ are polynomials.

<u>Lemma</u> 3.2.9 [26]

$$\frac{|x-y| + \Omega}{2} < |x-y| + \Omega_{\mathfrak{n}}(y) < 2(|x-y| + \Omega), \qquad x, y \in I$$

<u>Lemma</u> 3.2.10 [27]

{If $0 \leq j \leq i \leq \mathfrak{n}$, then $\frac{1}{2}(J - j) \leq \frac{x_j - x_J}{x_i - x_{i+1}} \leq (J - j)^2$}.

<u>Lemma</u> 3.2.11 [27]

$$\frac{\left|x - x_j\right| + h_j}{10} < \left|x - x_j\right| + \Omega\ < 2\left(\left|x - x_j\right| + h_j\right), \qquad x \in I, j = 0, \dots, \mathfrak{n}.$$

<u>Lemma</u> 3.2.12 [27]

$\sum_{j=1}^{\mathfrak{n}} \widetilde{T}_{j,\mathfrak{n}_1}(x) \equiv 1$, where $\widetilde{T}_{j,\mathfrak{n}_1}$ is polynomial.

<u>Lemma</u> 3.2.13 [24]

Let E be an interval which is the union of $\iota \geq 12\mathbb{s}$ of the intervals I_j , and let a set $\mathcal{J} \subseteq E$ be the union of $\iota \leq \mu \leq \iota/4$ of these intervals, then there exists a polynomial $\mathcal{Q}_n(x) = \mathcal{Q}_n(x, E, \mathcal{J})$ of degree$\leq cn$, satisfying $Q''_n(x)\delta(x) \geq c_1 \frac{\iota}{\mu}\left(\frac{\Omega}{max\{\Omega, dist(x,E)\}}\right)^{25(s+1)} \frac{\pi(x)}{\Omega^2}$, $x \in \mathcal{J} \cup [-1,1]\backslash E$),

(we may take $c_1 \leq 1$)

$Q''_n(x)\delta(x) \geq -\frac{\pi(x)}{\Omega^2}, x \in E\backslash\mathcal{J}$,

And

$$\left|Q_n(x)\right| \leq c(p)\iota^6 \sum_{I_j \subseteq E} \frac{h_j}{(|x-x_j|+\Omega)^2}, \ x \in [-1,1].$$

<u>Lemma</u> 3.2.14 [24]

$w_2^{\vartheta}(\mathbb{S}, \frac{1}{n}) \leq c\|\Omega^2 \mathbb{S}''\|_p$, where $\mathbb{S}$ is piecewise polynomial.

<u>Lemma</u> 3.2.15 [24]

Let $\mathcal{A} := \{j_0, \dots, j_0 + \iota_0\}$ and let $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ be such that $\#\mathcal{A}_1 = 2\iota_1$ and$\#\mathcal{A}_2 = 2\iota_2 \in \mathcal{A}_1$. If $\delta_j \in \{-1,1\}, j \in \mathcal{A}_2$, then there exist $2\iota_1$ constants $a_i, i \in \mathcal{A}_1$, such that $|a_i| \leq \left(\frac{\iota_0}{\iota_1}\right)^2$, $i \in \mathcal{A}_1$,

And $\frac{1}{\iota_2}\sum_{j\in\mathcal{A}_2} \delta_j(x - x_j) + \frac{1}{\iota_1}\sum_{i\in\mathcal{A}_1} a_i(x - x_j) \equiv 0$

<u>Lemma</u> 3.2.16

Let $\kappa \geq 3$. Then for each $\mathbb{S} \in \sum_{\kappa,n}(Y_s) \cap \Delta^{(2)}(Y_s) \cap L_p[-1,1]$, there is an $\widetilde{\mathbb{S}} \sum^1_{\kappa,n}(Y_s) \cap \Delta^{(2)}(Y_s) \cap L_p[-1,1]$, such that

$$\left\|\mathbb{S} - \widetilde{\mathbb{S}}\right\|_p \leq c\,(p) w_\kappa^{\vartheta}(\mathbb{S}, \frac{1}{n})_p \qquad (3.2.17)$$

In particular

$$w_\kappa^{\vartheta}\left(\widetilde{\mathbb{S}}, \frac{1}{n}\right)_p \leq c\, w_\kappa^{\vartheta}\left(\mathbb{S}, \frac{1}{n}\right)_p.$$

Proof.

For each $2 \leq j \leq n$, set

$a_j(x) := \frac{1}{2}\frac{x_{j-1}-x_{j-2}}{x_{j-1}-x_j}\frac{\mathcal{P}'_{j-1}(x_{j-1})-\mathcal{P}'_j(x_{j-1})}{x_j-x_{j-2}}(x-x_j)^2$, if j , (j − 1) ∈ H,

$a_j(x) := \frac{1}{2}\frac{\mathcal{P}'_{j-1}(x_{j-1})-\mathcal{P}'_j(x_{j-1})}{x_{j-1}-x_j}(x-x_j)^2$, if j, (j − 1) ∉ H,

And $\quad a_j(x) := 0$, if j ∉ H.

Also for each $1 \le j \le n-1$, set

$b_j(x) := \frac{1}{2}\frac{x_j-x_{j+1}}{x_j-x_{j-1}}\frac{\mathcal{P}'_j(x_j)-\mathcal{P}'_{j+1}(x_j)}{x_{j+1}-x_{j-1}}(x-x_{j-1})^2$, if j , (j − 1) ∈ H,

$b_j(x) := \frac{1}{2}\frac{\mathcal{P}'_j(x_j)-\mathcal{P}'_{j+1}(x_j)}{x_{j-1}-x_j}(x-x_{j-1})^2$, if j , (j − 1) ∉ H

And $\quad b_j(x) := 0$, if j ∉ H

Finally , set $a_1(x) := 0$ and $b_n(x) := 0$.Then ,

$$\tilde{\mathbb{S}}(x) = \mathcal{P}_j(x) + a_j(x) + b_j(x) + \mathcal{J}(x), x \in I_j ,$$

Is the required function, where $\mathcal{J}$ is a piecewise constant function with jumps in at most 2$\mathbb{S}$ points x_j near the y_i‘s; explicitly, the jumps at these x_j‘s are

$$\mathcal{J}(x_j+) - \mathcal{J}(x_j-) := \begin{cases} \frac{1}{2}\left[\mathcal{P}'_j(x_j) - \mathcal{P}'_{j+1}(x_j)\right](x_j - x_{j+1}) \text{if } j \notin H, (j-1) \in H. \\ \frac{1}{2}\left[\mathcal{P}'_j(x_j) - \mathcal{P}'_{j+1}(x_j)\right](x_j - x_{j-1}) \text{if } j \in H, (j-1) \notin H. \end{cases}$$

Indeed, straight forward computations show that $\tilde{\mathbb{S}}\sum^1_{\kappa,n}(Y_s) \cap \Delta^{(2)}(Y_s)$, and by Markov‘s inequality

$$\left\|\mathcal{P}'_j(x_j) - \mathcal{P}'_{j+1}(x_j)\right\|_p \le c\,(p)\frac{2\kappa^2}{x_{j-1}-x_j}\left\|\mathcal{P}_j - \mathcal{P}_{j+1}\right\|_{L_p(I_j)}$$

Write $w := w_\kappa^\vartheta\left(\mathbb{S},\frac{1}{n}\right)_p$, we have to prove that for each i and j ,

$$\left\|\mathcal{P}_i - \mathcal{P}_j\right\|_{L_p(I_i)} \le c\,(p)\left(\frac{|I_{i,j}|}{|I_j|}\right)^\kappa w \qquad (3.2.18)$$

By Whitney Theorem , there exists a polynomial $L(x) := L_{\kappa-1}(\mathbb{S}, x; I_{i,j})$

$\|\mathcal{F} - L_{\kappa-1}(\mathcal{F}, .;\ \mathcal{J})\|_{L_{p(\mathcal{J})}} \leq c\,(p)\,w_{\kappa}^{\vartheta}(\mathcal{F}, |\mathcal{J}|)_p$, $\mathcal{J} \subseteq I$, and for $\mathcal{F} \in L_{p_{\vartheta}}^{r}$,

Implies

$$\|\mathcal{P}_i - L\|_{L_{p(I_i)}} = \|\mathbb{S} - L\|_{L_{p(I_i)}} \leq \|\mathbb{S} - L\|_{L_{p(I_{i,j})}} \leq c\,(p)\,w_{\kappa}^{\vartheta}(\mathbb{S}, |I_{i,j}|)_p$$
$$\leq c\,(p)\,w\ ,$$

Where in the right-hand inequality we applied

$$\frac{2}{n} \leq |I_j| = \frac{|I_j|}{\vartheta\left(\frac{x_j - x_{j-1}}{2}\right)} \leq \frac{\Pi}{n}.$$

Observing that $\mathcal{P}_j - L$ is a polynomial of degree $\leq \kappa - 1$, we have

$$\|\mathcal{P}_j - L\|_{L_{p(I_i)}} \leq \|\mathcal{P}_j - L\|_{L_{p(I_{i,j})}} \leq c\,(p)\|\mathcal{P}_j - L\|_{L_{p(I_j)}} \leq c\,(p)\,w\ ,$$

Hence

$$\|\mathcal{P}_i - \mathcal{P}_j\|_{L_{p(I_i)}} \leq c\,(p)\,w\ ,$$

We have $\|\mathbb{S} - \widetilde{\mathbb{S}}\|_p \leq c\,(p)\,w$ ■

<u>Lemma</u> 3.2.19 [27]

Let $\mathfrak{b}_3 = \mathfrak{b}_2 - \mathbb{s} - 2\kappa - 6 > 0$, and let $\mathcal{A}$ be a proper interval for $\mathbb{S} \in \sum_{\kappa,n}(Y_{\mathbb{s}})$,

$$\left|\mathbb{S}^{(q)}(x) - D_{n_1}^{(q)}(x)\right| \leq c\,\frac{\Gamma(\Omega)}{\Omega^{q}}\left(\mathfrak{b}_{\kappa}(\mathbb{S}, \mathcal{A}) + \mathfrak{b}_{\kappa}(\mathbb{S})\,\frac{n}{n_1}\left(\frac{\Omega}{\Omega + dist(x, [-1,1]\backslash\mathcal{A})}\right)^{\mathfrak{b}_3}\right).$$

$x \in \mathcal{A} \cap \overline{O}_{e}$, where $\overline{O}_{e} := \left\{u : [u - \frac{1}{2}\Omega_n(u), u + \frac{1}{2}\Omega_n(u)] \subseteq \overline{O}\right\}. \cup (\overline{O} \cap (I_1 \cup I_n))$, $q = 0, \ldots, \mathbb{s} + 2$.

Furthermore, if $\mathbb{S} \in \sum_{\kappa,n}^{1}(Y_{\mathbb{s}})$, then for $x \neq x_j$, $0 \leq j \leq n$,

$\left|\mathbb{S}''(x) - D''_{n_1}(x)\right| \le c\frac{1}{\Omega^2}\left(b_\kappa(\mathbb{S},\mathcal{A}) + b_\kappa(\mathbb{S})\frac{n}{n_1}\left(\frac{\Omega}{dist(x,[-1,1]\backslash\mathcal{A})}\right)^{b_3}\right), x \in \mathcal{A}$

3.3 Negative Results:

In this section we introduce the negative results that we need to prove our Theorems.

Lemma 3.3.1

Given $n \ge 1$, for each polynomial $\mathcal{P}_n$ of degree $\le n$, and satisfying

$(x^2 - b^2)\mathcal{P}''_n(x) \ge 0, x \in [-\frac{1}{2},\frac{1}{2}]$, with $b = \frac{1}{2}n^{-\frac{4}{3}}$, we have

$$\|g_b - \mathcal{P}_n\|_p > \frac{b}{40}.$$

Proof.

First we observe that $\mathcal{P}''_n(\pm b) = 0$, and that $\mathcal{P}''_n(x) \le 0$, for $-b < x < b$. Assume that for some $-b < x_0 < b, \mathcal{P}''_n(x) < -\frac{1}{4}$. Then

$$\|[\mathcal{P}''_n; -b, x_0, b]\|_p = \frac{\|\mathcal{P}''_n(x_0)\|_p}{(b - x_0)(b + x_0)} > \frac{1}{4b^2}$$

Since $[\mathcal{P}''_n; -b, x_0, b] = \frac{1}{2}\mathcal{P}_n^{(4)}(0)$, for some $-b < \theta < b\left(\le \frac{1}{2}\right)$, it follows by Bernstein‹s inequality that

$$n^4\|\mathcal{P}_n\|_p \ge \frac{c(p)}{2}\left\|\mathcal{P}_n^{(4)}(\theta)\right\|_p \ge \frac{1}{4b^2}$$

Now by $\|g_b\|_p = \frac{8b}{15} - \frac{b^2}{6} \le \frac{2b}{3}$. and the prescribed value of b

$$\|g_b - \mathcal{P}_n\|_p \ge c(p)\left(\|\mathcal{P}_n\|_p - \|g_b\|_p\right) > \frac{1}{4n^4b^2} - \frac{2b}{3} = \frac{4b}{3} \qquad (3.3.2)$$

If on the other hand,

$\mathcal{P}''_n(x) \ge -\frac{1}{4}$, for all $-b < x < b$,then we represent $\mathcal{P}_n$ in the form

$$\mathcal{P}_n(x) = \mathcal{P}_n(0) + x\mathcal{P}_n'(0) + \int_0^x (x - u)\,\mathcal{P}_n''(u)du$$

Since $\mathcal{P}_n'' \geq 0$ for $b \leq |x| \leq \frac{1}{2}$, it follows that

$$\mathcal{P}_n\left(-\frac{1}{2}\right) - 2\mathcal{P}_n(0) + \mathcal{P}_n\left(\frac{1}{2}\right).$$

$$= \int_0^{\frac{1}{2}} (\frac{1}{2} - u)\mathcal{P}_n''(u)du + \int_0^{-\frac{1}{2}} (-\frac{1}{2} - u)\mathcal{P}_n''(u)du$$

$$\geq \int_0^{b} (\frac{1}{2} - u)\mathcal{P}_n''(u)du + \int_0^{b} (\frac{1}{2} - u)\mathcal{P}_n''(-u)du \geq -\frac{b}{4}$$

Similarly ,

$$g_b\left(-\frac{1}{2}\right) - 2g_b(0) + g_b\left(\frac{1}{2}\right) = 2\int_0^{b} (\frac{1}{2} - u)g''(u)du = -\frac{8b}{15} + \frac{b^2}{3}.$$

Therefore

$$4\|g_b - \mathcal{P}_n\|_p \geq \left[\mathcal{P}_n\left(-\tfrac{1}{2}\right) - g_b\left(-\tfrac{1}{2}\right)\right] - 2\big(\mathcal{P}_n(0) - g_b(0)\big) + [\mathcal{P}_n\left(\tfrac{1}{2}\right) - g_b\left(\tfrac{1}{2}\right)]$$

$$4\|g_b - \mathcal{P}_n\|_p \geq \frac{-b}{4} + \frac{8b}{15} - \frac{b^2}{3} \geq \frac{b}{10}.$$

Thus together with 3.3.2, this concludes the proof of Lemma 3.3.1 ∎

<u>Corollary</u> 3.3.3

For every constant $\mathcal{A} > 1$ there exists an $N(\mathcal{A})$ sufficiently large such that if $n > N(\mathcal{A})$, then for any $s \geq 2$, there is a function $g = g_n \in L_p^3[-1,1]$ which changes convexity s time in $[-1,1]$, and such that any polynomial $\mathcal{P}_n$ of degree $\leq n$ which is coconvex with it, satisfies

$$\|g - \mathcal{P}_n\|_p > \frac{\mathcal{A}\|g^{(3)}\|_p}{n^3},$$

$$\|\mathfrak{g}-\mathcal{P}_{\mathfrak{n}}\|_{p} > \frac{\mathcal{A}\|\mathfrak{g}''\|_{p}}{\mathfrak{n}^{2}},$$

$$\|\mathfrak{g}-\mathcal{P}_{\mathfrak{n}}\|_{p} > \frac{\mathcal{A}\|\mathfrak{g}'\|_{p}}{\mathfrak{n}^{1}}.$$

Proof.

let $N(\mathcal{A}) = (80\mathcal{A})^3$ and let $\mathbb{s} \geq 2$. We take $\mathfrak{b} = \mathfrak{b}_{\mathfrak{n}}, \mathfrak{n} > N(\mathcal{A})$, as in lemma 3.3.1, and let $\mathfrak{g} = \mathfrak{g}_{\mathfrak{b}}$. The function $\mathfrak{g}$ changes convexity at $y_2 = -\mathfrak{b}$ and $y_1 = \mathfrak{b}$, it is convex in $[y_1, 1]$, and if $\mathbb{s} > 2$, then we take $\mathbb{s} - 2$ arbitrary points satisfying $-1 < y_{\mathbb{s}} < \cdots < y_3 < -\frac{1}{2}$, and regard $\mathfrak{g}$ as changing convexity at these points too, hence $\mathfrak{g} \in \Delta^2(y_{\mathbb{s}})$, if the polynomial $\mathcal{P}_{\mathfrak{n}}$ is coconvex with $\mathfrak{g}$, then it satisfies the requirements of Lemma 3.3.1. Therefore, by Lemma 3.3.1,

we have

$$\|\mathfrak{g}-\mathcal{P}_{\mathfrak{n}}\|_{L_{p}{}_{[-1,-\frac{1}{2}]}} > \frac{\mathfrak{b}}{40} \geq \frac{\left\|\mathfrak{g}^{(3)}\right\|_{p}\mathfrak{b}^{2}}{80} > \frac{\mathcal{A}\left\|\mathfrak{g}^{(3)}\right\|_{p}}{\mathfrak{n}^{3}},$$

$$\|\mathfrak{g}-\mathcal{P}_{\mathfrak{n}}\|_{L_{p}{}_{[-1,-\frac{1}{2}]}} > \frac{\mathfrak{b}}{40} = \frac{\|\mathfrak{g}''\|_{p}\mathfrak{b}}{40} > \frac{\mathcal{A}\|\mathfrak{g}''\|_{p}}{\mathfrak{n}^{2}},$$

$$\|\mathfrak{g}-\mathcal{P}_{\mathfrak{n}}\|_{L_{p}{}_{[-1,-\frac{1}{2}]}} > \frac{\mathfrak{b}}{40} = \frac{3\mathfrak{n}\|\mathfrak{g}'\|_{p}}{64\mathfrak{n}} > \frac{\mathcal{A}\|\mathfrak{g}'\|_{p}}{\mathfrak{n}} \blacksquare$$

Remark.

It should be noted that the function $\mathfrak{g}_{\mathfrak{b}}$ above is independent of $\mathcal{A}$

We are ready to prove Theorem 3.3.4.

Theorem 3.3.4 .

For no $\kappa \geq 1, \mathfrak{r} = 0,1,2,3$ and $\mathbb{s} \geq 2$, is it possible to have constants $c = c(\kappa, \mathfrak{r}, \mathbb{s})$ and $N = N(\kappa, \mathfrak{r}, \mathbb{s})$, depending only on $\kappa, \mathfrak{r}$ and $\mathbb{s}$, such that the inequality

$$E_{\mathfrak{n}}^{(2)}(\mathcal{F}, Y_{\mathbb{s}})_{p} \leq \frac{c}{\mathfrak{n}^{\mathfrak{r}}} w_{\kappa}\left(\mathcal{F}^{(\mathfrak{r})}, \frac{1}{\mathfrak{n}}\right)_{p}, \qquad (3.3.5)$$

Holds for all , $\mathfrak{n} \geq N$ and for all $\mathcal{F} \in L_p^{\mathfrak{r}} \cap \Delta^2(Y_{\mathbb{s}})$

On the other hand, we show that if $\mathbb{s} = 1$, then $E_{\mathfrak{n}}^{(2)}(\mathcal{F}, Y_{\mathbb{s}})_p \leq c(p)\omega\left(\mathcal{F}, \frac{1}{\mathfrak{n}}\right)_p$, $\mathfrak{n} \in N$ is valid for $N = 1$; in fact we prove

Proof of Theorem 3.3.4.

The proof readily follows from the observation that for all $\kappa \geq 1$

$$\omega_\kappa\left(\mathcal{F}, \frac{1}{\mathfrak{n}}\right)_p \leq 2^{\kappa-1}\omega\left(\mathcal{F}, \frac{1}{\mathfrak{n}}\right)_p \leq 2^{\kappa-1}\mathfrak{t}\|\mathcal{F}'\|_p,$$

Which by Corollary 3.3.3 does not allow the case $\mathfrak{r} = 0$ in (3.3.5) and

$$\omega_\kappa\left(\mathcal{F}, \frac{1}{\mathfrak{n}}\right)_p \leq 2^\kappa\|\mathcal{F}\|_p,$$

$$E_{\mathfrak{n}}^{(2)}(\mathcal{F}, Y_{\mathbb{s}})_p \leq \|\mathcal{F} - \mathcal{P}_{\mathfrak{n}}\|_p \leq c(p)\omega_\kappa\left(\mathcal{F}, \frac{1}{\mathfrak{n}}\right)_p \leq \frac{c(p)}{\mathfrak{n}^{\mathfrak{r}}}\omega_\kappa\left(\mathcal{F}^{(\mathfrak{r})}, \frac{1}{\mathfrak{n}}\right)_p$$

Which takes case of the other cases ■

<u>Lemma</u> 3.3.6

Let $\mathfrak{b} = 6(\mathbb{s} + 1)$. Then for each $\mathfrak{j} \in H$ there exist polynomials τ_j and $\bar{\tau}_j$ of degree $\leq c\mathfrak{n}$, satisfying

$\tau_j''(x)\Pi(x)\Pi(x_j) \geq 0, \ \ x \in I,$

$$\bar{\tau}_j''(x)\Pi(x)\Pi(x_j) \leq 0, \ \ x \in I\backslash I_j, \qquad (3.3.7)$$

$$\left\|\bar{\tau}_j''(x)\right\|_p \leq \frac{c(p)}{h_j}\left\|\mathfrak{T}_j^{2\mathfrak{b}}(x)\right\|_p \frac{\|\Pi(x)\|_p}{\|\Pi(x_j)\|_p} \leq c(p)\left\|\tau_j''(x)\right\|_p, x \in I, \qquad (3.3.8)$$

Where $\mathfrak{T}_j(x) := \frac{h_j}{|x-x_j|+h_j}$.

And

$$\left\|(x - x_j)_+ - \tau_j\right\|_p \leq c(p)h_j\left\|\mathfrak{T}_j^{2\mathfrak{b}-\mathbb{s}-2}(x)\right\|_p,$$

$$\left\| \left(x - x_{\mathrm{j}}\right)_{+} - \bar{\tau}_{\mathrm{j}} \right\|_{\mathrm{p}} \leq c(\mathrm{p})\mathrm{h}_{\mathrm{j}} \left\| \mathfrak{T}_{\mathrm{j}}^{2\mathfrak{b}-\mathfrak{s}-2}(x) \right\|_{\mathrm{p}}, x \in \mathrm{I} \qquad (3.3.9)$$

Proof.

We will prove only the existence of the polynomials τ_{j}, the other case being completely analogous. for every $\mathrm{j} \in \mathrm{H}$ let T_{j} be defined by

$$\mathrm{T}_{\mathrm{j}}(x) = \mathrm{T}_{\mathrm{j},\mathfrak{n}}(x;\mathfrak{b};\mathrm{Y}_{\mathfrak{s}}) := \frac{1}{d_{\mathrm{i}}}\int_{-1}^{x} \mathrm{t}_{\mathrm{j}}^{\mathfrak{b}}(\mathrm{u})\Pi(\mathrm{u})d\mathrm{u}$$

Where $d_{\mathrm{i}} = \int_{-1}^{1} \mathrm{t}_{\mathrm{j}}^{\mathfrak{b}}(\mathrm{u})\Pi(\mathrm{u})d\mathrm{u}$. We use it to construct τ_{j}. By virtue of Lemma 3.2.1 , (for $\mathrm{j} \in \mathrm{H}$ $\left|\mathcal{X}_{\mathrm{j}}(x) - \mathrm{T}_{\mathrm{j}}(x)\right| \leq c\mathfrak{T}_{\mathrm{j}}^{2\mathfrak{b}-\mathfrak{s}-1}(x), x \in \mathrm{I}$).

$$\int_{-1}^{1}\left|\mathcal{X}_{\mathrm{j}}(x) - \mathrm{T}_{\mathrm{j}}(x)\right|dx \leq c\int_{-1}^{1}\mathfrak{T}_{\mathrm{j}}^{2}(x)\,dx \leq: c_0\mathrm{h}_{\mathrm{j}}, \qquad (3.3.10)$$

If for $\mathfrak{r} := \lceil 6c_0 \rceil$ (where $\lceil a \rceil$ denotes the ceiling of a), both $\mathrm{j} - \mathfrak{r} \geq 0$ and $\mathrm{j} + \mathfrak{r} \leq \mathfrak{n}$, and

if for all $\mathrm{j} - \mathfrak{r} \leq \mathrm{i} \leq \mathrm{j} + \mathfrak{r}$, we have $\mathrm{i} \in \mathrm{H}$, then by Lemma 3.2.2

If $0 \leq \mathrm{j} \leq \mathrm{i} \leq \mathfrak{n}$, then $\frac{1}{2}(\mathcal{J} - \mathrm{j}) \leq \frac{x_{\mathrm{j}} - x_{\mathcal{J}}}{x_{\mathrm{i}} - x_{\mathrm{i}+1}} \leq (\mathcal{J} - \mathrm{j})^2$.

Furthermore , if either $\mathcal{J} \leq 3\mathrm{j}$ or $\mathfrak{n} - \mathrm{j} \leq 3(\mathfrak{n} - \mathcal{J})$. Then $\frac{1}{2}(\mathcal{J} - \mathrm{j}) \leq \frac{x_{\mathrm{j}} - x_{\mathcal{J}}}{x_{\mathrm{i}} - x_{\mathrm{i}+1}} \leq 2(\mathcal{J} - \mathrm{j})$ }

We have $x_{\mathrm{j}-\mathfrak{r}} - x_{\mathrm{j}} \geq 3c_0\mathrm{h}_{\mathrm{j}-\mathfrak{r}-1} \geq c_0\mathrm{h}_{\mathrm{j}-\mathfrak{r}}$ and $x_{\mathrm{j}} - x_{\mathrm{j}+\mathfrak{r}} \geq c_0\mathrm{h}_{\mathrm{j}+\mathfrak{r}}$,

So that it follows from 3.3.10 that

$$\int_{-1}^{1}(\mathrm{T}_{\mathrm{j}-\mathfrak{r}}(x) - \mathcal{X}_{\mathrm{j}}(x))\,dx = \int_{-1}^{1}\left(\mathrm{T}_{\mathrm{j}-\mathfrak{r}}(x) - \mathcal{X}_{\mathrm{j}-\mathfrak{r}}(x)\right)dx + (x_{\mathrm{j}-\mathfrak{r}} - x_{\mathrm{j}}) \leq 0$$

And

$$\int_{-1}^{1}(\mathrm{T}_{\mathrm{j}+\mathfrak{r}}(x) - \mathcal{X}_{\mathrm{j}}(x))\,dx = \int_{-1}^{1}\left(\mathrm{T}_{\mathrm{j}+\mathfrak{r}}(x) - \mathcal{X}_{\mathrm{j}+\mathfrak{r}}(x)\right)dx + (x_{\mathrm{j}} - x_{\mathrm{j}+\mathfrak{r}}) \geq 0,$$

Hence for some $0 \leq \mathcal{S} \leq 1$, we have

$$\mathcal{S}\int_{-1}^{1}(T_{j-r}(x)-\mathcal{X}_j(x))\,dx+(1-\mathcal{S})\int_{-1}^{1}\Big(T_{j+r}(x)-\mathcal{X}_j(x)\Big)\,dx=0$$

We set $\tau_{j,n} \coloneqq \tau_j(x) \coloneqq \mathcal{S}\int_{-1}^{x} T_{j-r}(u)du + (1-\mathcal{S})\int_{-1}^{x} T_{j+r}(u)du$,

So that $\tau_j(1) = 1 - x_j$,

Which by Lemma 3.2.1 in turn implies 3.3.9. now 3.3.7 follows from Lemma 3.2.3 in [27]

$\tau_j'(x)\Pi(x)\Pi(x_j) \geq 0,\ \ x \in I$, and (3.3.8) follows from Lemma 3.2.4 .

$$|\tau_j'(x)| \leq \frac{c(p)}{h_j}\mathfrak{T}_j^{2b}(x)\frac{|\Pi(x)|}{|\Pi(x_j)|} \leq c(p)|\tau_j'(x)|, x \in I,$$

Since by our assumption sgn $\Pi(x_j - r) =$ sgn $\Pi(x_j + r) =$ sgn $\Pi(x_j)$,

If $j - r < 0$, then it suffices to take $\tau_j(x) \coloneqq \int_{-1}^{x} T_j(u)du$

And $j + r > n$, then it suffices to take $\tau_j(x) \coloneqq 1 - x_j - \int_{x}^{1} T_j(u)du$.

We are left with the case where there is an $i \notin H$, such that $0 \leq j - r \leq i \leq j + r \leq n$.

In this case we take the Chebyshev partition of order $2nr$, so that we have $x_j = x_{2rj,2rn}$ and $i \in H(Y_s, 2rn)$, for all $2rj - r \leq 2rj + r$. Thus we set

$$\tau_j(x) \coloneqq \tau_{2rj,2rn}(x),$$

And we observe that by the above construction this τ_j satisfies (3.3.7)–(3.3.9), since by virtue of Lemma 3.2.5 ,

$\{\frac{1}{2}(\mathcal{J} - j) \leq \frac{x_j - x_{\mathcal{J}}}{x_i - x_{i+1}} \leq 2(\mathcal{J} - j)\}$

$h_{2rj,2rn} \leq h_j \leq 4r^2 h_{2rj,2rn}$. ■

<u>Theorem</u> 3.3.11

For every $\kappa, \mathfrak{n} \in \mathbb{N}$ and $\mathbb{s} \in \mathbb{N}_0$, if $\mathbb{S} \in \sum_{\kappa,\mathfrak{n}}(Y_{\mathbb{s}}) \cap \Delta^{(2)}(Y_{\mathbb{s}}) \cap L_p[-1,1]$, then there exists a polynomial $\mathcal{P}_{\mathfrak{n}} \in \Delta^{(2)}(Y_{\mathbb{s}}) \cap L_p[-1,1]$, of degree not exceeding $c\mathfrak{n}$, such that

$$\|\mathbb{S} - \mathcal{P}_{\mathfrak{n}}\|_p \leq c(p)\mathcal{w}_2^{\vartheta}\left(\mathbb{S}, \frac{1}{\mathfrak{n}}\right)_p. \tag{3.3.12}$$

Proof.

for $\kappa = 1$ Theorem 3.3.11 is trivial, we have to prove Theorem 3.3.11 only for $\kappa \geq 2$. Given $\mathfrak{n} \geq 1$, denote by $G_{\mathcal{v}} = (x_{\mathcal{J}_{\mathcal{v}}}, x_{\mathfrak{j}_{\mathcal{v}}})$ the connected components of $\mathcal{O} = \mathcal{O}(\mathfrak{n}, Y_{\mathbb{s}})$. for $\mathfrak{j} = 1, \dots, \mathfrak{n}-1$, let $\tilde{\tau}_{\mathfrak{j}}$ be polynomials of degree$\leq c\mathfrak{n}$ defined as follows :

(a) If $\mathfrak{j} \in H = H(\mathfrak{n}, Y_{\mathbb{s}})$, if $I_{\mathfrak{j}} \cap \mathcal{O} = \emptyset$, the $\tilde{\tau}_{\mathfrak{j}}(x) = \tau_{\mathfrak{j}}(x)$,
where $\tau_{\mathfrak{j}}$ are from Lemma 3.3.6 .

(b) If $\mathfrak{j}_{\mathcal{v}} = 0$ and $0 < \mathfrak{j} < \mathcal{J}_{\mathcal{v}}$, then $\tilde{\tau}_{\mathfrak{j}}(x) := 0$.

(c) If $\mathcal{J}_{\mathcal{v}} = \mathfrak{n}$ and $\mathfrak{j}_{\mathcal{v}} < \mathfrak{j} < \mathfrak{n}$, then $\tilde{\tau}_{\mathfrak{j}}(x) := x - x_{\mathfrak{j}}$

Finally, we have the j‹s for which $0 < \mathfrak{j}_{\mathcal{v}} < \mathfrak{j} < \mathcal{J}_{\mathcal{v}} < \mathfrak{n}$ we divide the v‹s in to two groups , let $\mathfrak{n}_1 := 16\mathbb{s}(\kappa-1)^3\mathfrak{n}$. We say that $\mathcal{v} \in$ 0d if there exists an $\iota_{\mathcal{v}} \in H(\mathfrak{n}_1, Y_{\mathbb{s}})$ such that $I_{\iota_{\mathcal{v}},\mathfrak{n}_1} \cap G_{\mathcal{v}} \neq \emptyset$, and the interval $(x_{\iota_{\mathcal{v}},\mathfrak{n}_1}, x_{\mathfrak{j},\mathfrak{n}})$ contain an odd number of points y_i.

Note that if $\mathcal{v} \notin$ 0d, then the set $G_{\mathcal{v}}$ contains an even number, say $2\mathfrak{m}$, of points y_i, the points $y_{i_0} + 2\mathfrak{m} - 1 < \cdots < y_{i_0}$, say. In this case each two consecutive points $y_{i_0} + 2\mathcal{v}$ and $y_{i_0} + 2\mathcal{v} + 1$, $\mathcal{v} = 0, \dots, \mathfrak{m}-1$, must belong to the union of four consecutive intervals, say $[x_{\iota_{\mathcal{v}+2},\mathfrak{n}_1}, x_{\iota_{\mathcal{v}-2},\mathfrak{n}_1})$,

whence

$$\left\{x \in G_{\mathcal{v}} : \Pi(x_{\mathfrak{j}_{\mathcal{v}}})\mathbb{S}''(x) < 0\right\} \subseteq \cup_{\mathcal{v}=0}^{\mathfrak{m}-1}[x_{\iota_{\mathcal{v}+2},\mathfrak{n}_1}, x_{\iota_{\mathcal{v}-2},\mathfrak{n}_1}]$$

It follows by the left-hand side of Lemma 3.2.5 that

$$\left\{\frac{1}{2}(\mathcal{J} - \mathfrak{j}) \leq \frac{x_{\mathfrak{j}} - x_{\mathcal{J}}}{x_i - x_{i+1}} \leq (\mathcal{J} - \mathfrak{j})^2\right\}.$$

measure of $\{x \in G_{\mathscr{v}}: \Pi(x_{j_{\mathscr{v}}})\mathbb{S}''(x) < 0\} \le \frac{\mathbb{s}}{2} 4max_{I_{\iota,n_1} \subset (x_{\mathcal{J}_{\mathscr{v}}}, x_{j_{\mathscr{v}}})} |I_{\iota,n_1}|$

$$\le \frac{4\mathbb{s}}{2} 2 \frac{|G_{\mathscr{v}}|}{(\mathcal{J}_{\mathscr{v}} - j_{\mathscr{v}})^{\frac{n_1}{n}}} \le 4\mathbb{s} \frac{|G_{\mathscr{v}}|}{4^{\frac{n_1}{n}}} = \frac{1}{16(\kappa-1)^3} |G_{\mathscr{v}}|. \tag{3.3.13}$$

We need the polynomials $\tau_{j_{\mathscr{v}}}$ and $\tau_{\mathcal{J}_{\mathscr{v}}}$; however we note that $j_{\mathscr{v}}$ might not be in H. Since $2j_{\mathscr{v}}$ is always in $H(2n, Y_{\mathbb{s}})$, in the case $j_{\mathscr{v}} \notin H$, we define $\tilde{\tau}_{j_{\mathscr{v}}} := \tau_{j_{\mathscr{v}}} := \tau_{2j_{\mathscr{v}},2n}$.

Similarly, we always have $\mathcal{J}_{\mathscr{v}} \notin H$ and $2\mathcal{J}_{\mathscr{v}} \in H(2n, Y_{\mathbb{s}})$ so we define

$\tilde{\tau}_{\mathcal{J}_{\mathscr{v}}} := \tau_{\mathcal{J}_{\mathscr{v}}} := \tau_{2\mathcal{J}_{\mathscr{v}},2n}$.

(d)If $0 < j_{\mathscr{v}} < j < \mathcal{J}_{\mathscr{v}} < n$ and $\mathscr{v} \notin 0d$, then we let
$\tilde{\tau}_j(x) := \tau_{j_{\mathscr{v}}}(x)$

If on the hand,

(e) $0 < j_{\mathscr{v}} < j < \mathcal{J}_{\mathscr{v}} < n$ and $\mathscr{v} \in 0d$, then we let
$\tilde{\tau}_j(x) := \delta_j \tau_{j_{\mathscr{v}}}(x) + (1 - \delta_j)\tau_{\iota_{\mathscr{v}},n_1}(x)$, where $\delta_j = 0$ or 1 is to be prescribed .

We are in apposition to define $\mathcal{P}_n$. recall that the piecewise linear function L that interpolates $\mathbb{S}$, at the x_j's, satisfies

$$\|\mathbb{S} - L\|_p \le c(p)w_2^{\vartheta}\left(\mathbb{S}, \frac{1}{n}\right)_p, \tag{3.3.14}$$

And may be written in the form

$L(x) = \iota(x) + \sum_{j=1}^{n-1}[\mathbb{S}; x_{j+1}, x_j, x_{j-1}](x_{j-1} - x_{j+1})(x - x_j)_+,$

where $\iota(x)$ is a linear function. Thus, denote

$$\mathcal{P}_n(x) := \iota(x) + \sum_{j=1}^{n-1}[\mathbb{S}; x_{j+1}, x_j, x_{j-1}](x_{j-1} - x_{j+1})\tilde{\tau}_j(x),$$

we begin with the proof of 3.3.12. to this end, we show that for each

We show that for each $j = 1, \dots, n-1$, we have

$$\|(x - x_j)_+ - \tilde{\tau}_j(x)\|_p \le c(p) h_j \|\mathfrak{T}_j^2(x)\|_p, x \in I \quad (3.3.15)$$

Indeed, going through the various cases we see that:

(a) 3.3.15 readily follows from Lemma 3.3.6 of 3.3.9

$$\{\|(x - x_j)_+ - \tau_j(x)\|_p \le c(p) h_j \|\mathfrak{T}_j^{2b-s-2}(x)\|_p,$$

$$\|(x - x_j)_+ - \tilde{\tau}_j(x)\|_p \le c(p) h_j \|\mathfrak{T}_j^{2b-s-2}(x)\|_p, x \in I.\}$$

and b,c of (3.3.15) readily follows from the inequalities

$$h_j \le |G_v| < c h_j, \quad j_v < j < \mathcal{J}_v; \quad (3.3.16)$$

(d) by Lemma 3.3.6 of 3.3.9 and 3.3.16,

$$\|(x - x_j)_+ - \tilde{\tau}_j(x)\|_p \le \|(x - x_j)_+ - (x - x_{j_v})_+\|_p + \|(x - x_{j_v})_+ - \tilde{\tau}_j(x)\|_p$$

$$\le c(p) h_j \|\mathfrak{T}_j^2(x)\|_p + c(p) h_{j_v} \|\mathfrak{T}_{j_v}^2(x)\|_p \le c(p) h_j \|\mathfrak{T}_j^2(x)\|_p;$$

And finally,

(e)If $\delta_j = 1$, then we are back to case (d), and if $\delta_j = o$, then similarly we have

$$\|(x - x_j)_+ - \tilde{\tau}_j(x)\|_p \le c(p) h_j \|\mathfrak{T}_j^2(x)\|_p + \|(x - x_{\iota_v, n_1})_+ - \tilde{\tau}_{\iota_v, n_1}(x)\|_p$$

$$\le c(p) h_j \|\mathfrak{T}_j^2(x)\|_p + \left\|\frac{h_{\iota_v, n_1}^3}{(|x - x_{\iota_v, n_1}| + h_{\iota_v, n_1})^2}\right\|_p \le c(p) h_j \|\mathfrak{T}_j^2(x)\|_p,$$

And 3.3.15 is proved. Since it is well known that

$$\|[\mathbb{S}; x_{j+1}, x_j, x_{j-1}]\|_p \le c(p) h_j^{-1} w_2^{\vartheta}\left(\mathbb{S}, \frac{1}{2}\right)_p, j = 1, \dots, n-1,$$

And

$$\left\|\sum_{j=1}^{n} \mathfrak{T}_j^2(x)\right\|_p \leq c(p),$$

We obtain $\|L - \mathcal{P}_n\|_p \leq c(p)\left\|\sum_{j=1}^{n} \mathfrak{T}_j^2(x)\right\|_p w_2^{\vartheta}\left(\mathbb{S}, \frac{1}{2}\right)_p$.

This together with 3.3.14 concludes the proof of 3.3.12 .

In order to prove that $\mathcal{P}_n \in \Delta^2(Y_s)$ we denote

$$L_j(x) := [\mathbb{S}; x_{j+1}, x_j, x_{j-1}](x_{j-1} - x_{j+1})\tilde{\tau}_j(x), j = 1, \dots, n-1,$$

and

$$\mathcal{P}_n(x) =: \iota(x) + \mathcal{A}(x) + \mathcal{B}(x) + \mathcal{C}(x) + \mathcal{D}(x) + E(x),$$

Where $\mathcal{A}(x) = \sum_{j\in H} L_j(x) + \sum_{\mathcal{J}_v < n} L_{\mathcal{J}_v}(x)$,

$\mathcal{B}(x) = \sum_{j=1}^{\mathcal{J}_v - 1} L_j(x)$, if $j_v = 0$,

$\mathcal{C}(x) = \sum_{j=j_v+1}^{n-1} L_j(x)$, if $j_v = n$,

$$\mathcal{D}(x) = \sum_{v\in 0d} \sum_{j=j_v+1}^{\mathcal{J}_v - 1} L_j(x),$$

And $E(x) = \sum_{v\notin 0d} \sum_{j=j_v+1}^{\mathcal{J}_v - 1} L_j(x) =: \sum_{v\notin 0d} E_v(x)$

It is important to emphasize that we either have $j_v \in H$ or $j_v = \mathcal{J}_{v+1}$, so that indeed all $1 \leq j \leq n-1$ are taken care of.

Again we have to investigate each case separately.

(a) If $j \in H$, then by definition of $\Delta^2(Y_s)$ we have,
$\Pi(x_j)[\mathbb{S}; x_{j+1}, x_j, x_{j-1}] \geq 0$. Hence by Lemma 3.3.6 in 3.3.7

$\Pi(x)L_j''(x) = \Pi(x)[\mathbb{S}; x_{j+1}, x_j, x_{j-1}](x_{j-1} - x_{j+1})\tau_j''(x) \geq 0$,

And similarly $\Pi(x)L_{\mathcal{J}_v}''(x) \geq 0, \mathcal{J}_v < n$,

so that $\Pi(x)\mathcal{A}''(x) \geq 0, x \in I$. b,c since $\mathcal{B}$ and $\mathcal{C}$ are linear functions, we have $\mathcal{B}''(x) \equiv 0$ and $\mathcal{C}''^{(x)} = 0$.

(e) If $\vartheta \in \mathrm{Od}$, then by definition , we have an odd number of point $y_i \in (x_{\iota_\vartheta,n_1}, x_{j_\vartheta})$, which in turn implies that.
$\Pi(x_{\iota_\vartheta,n_1})\Pi(x_{j_\vartheta}) < 0$.

Hence, by Lemma 3.3.6 in (3.3.7) implies

$$\tau''_{\iota_\vartheta,n_1}(x)\tau''_{j_\vartheta}(x) \le 0, x \in I .$$

Hence for each $j = j_\vartheta + 1, \dots, \mathcal{J}_\vartheta - 1$, we may prescribe δ_j so that

$$\Pi(x)L''_j(x) \ge 0, x \in I .$$

With this choice

$$\Pi(x)\mathcal{D}''(x) \ge 0, x \in I .$$

Finally we conclude with the proof of case (d).

(d) If $\vartheta \notin \mathrm{Od}$, then

$$E_\vartheta(x) = \sum_{j=j_\vartheta+1}^{\mathcal{J}_\vartheta-1} L_j(x)$$

$$= \tau_{j_\vartheta}(x) \sum_{j=j_\vartheta+1}^{\mathcal{J}_\vartheta-1} [\mathbb{S}; x_{j+1}, x_j, x_{j-1}](x_{j-1} - x_{j+1})$$

$$= \tau_j(x)([\mathbb{S}; x_{j+1}, x_j, x_{j-1}](x_{j+1} - x_{\mathcal{J}_\vartheta}) + [\mathbb{S}; x_{\mathcal{J}_\vartheta}, x_{\mathcal{J}_\vartheta-1}, x_{j_\vartheta}](x_{j_\vartheta} - x_{j_\vartheta-1})$$

$$=: \tau_{j_\vartheta}(x) e\vartheta .$$

By virtue of Lemma 3.2.6 and 3.2.7,

{let $\mathcal{P}_\kappa$ be a polynomial of degree not exceeding κ and let $a < b$. if measure of $\{x \in [a, b]: \mathcal{P}''_\kappa \le 0\} < \frac{b-a}{16\kappa^3}$

Then for every $x_0 \in (a, b)$, $[\mathcal{P}_\kappa;\ a, x_0, b] \ge 0$.

And { $meas\left\{x \in G_\vartheta : \Pi\left(x_{j_\vartheta}\right) \mathbb{S}''(x) < 0\right\} = \frac{1}{16(\kappa-1)^3}|G_\vartheta|$,

It now follows that $\Pi\left(x_{j_\vartheta}\right) e\vartheta \ge 0$.

Therefore, 3.3.7 of Lemma 3.3.6 implies

$$\Pi(x)E''_{v}(x) = \tau''_{j_v}(x)\Pi(x)\Pi\left(x_{j_v}\right)\frac{ev}{\Pi\left(x_{j_v}\right)} \geq 0\,, x \in I$$

Since $\iota''(x) \equiv 0$, we have shown that $\mathcal{P}_n \in \Delta^2(Y_s)$, and concluded the proof of the Theorem ■

Theorem 3.3.17

Let $\mathcal{F} \in L_p \cap \Delta^2(Y_1)$ that is, changes convexity one on [-1,1]. Then

$$E_n^{(2)}(\mathcal{F}, Y_1) \leq c\, w_2^{\vartheta}\left(\mathcal{F}, \frac{1}{n}\right)_p, n \geq 1. \qquad (3.3.18)$$

Proof .

One notes that the only place one needs the assumption that our function is a piecewise polynomial, is in order to apply Lemma 3.2.6 . Thus for general $\mathcal{F} \in \Delta^2(Y_1)$, if one is guaranteed that n is sufficiently big so that each component G_v contains an odd number of points of Y_s , in particular one point, then one may conclude the same. If $\mathcal{F}$ changes convexity just once, then obviously the requirement that each component G_v contains an odd number of points of Y_s , specifically one point holds for all $n \geq 1$ ■

Lemma (3.3.19)

Suppose that $\kappa \geq 3$,and $\mathbb{S} \in \sum_{\kappa,n}^{(1)}$ is such that

$$b_\kappa(\mathbb{s}) \leq 1, \qquad (3.3.20)$$

If an interval $I_{\mu,v}$contains at least $2\kappa - 5$ intervals I_i and points $x_i^* \in I_i^0$ such that

$$\Omega_n^2(x_i^*)\|\mathbb{s}''(x_i^*)\|_p \leq 1\ , \qquad (3.3.21)$$

then for every $1 \leq j \leq n$,we have

$$\|\Omega^2\mathbb{s}''\|_{L_{p(I_j)}} \leq c(p)[(j-\mu)^{4\kappa} + (j-v)^{4\kappa}] \qquad (3.3.22)$$

Proof.

Fix j and $x \in I_j^0$ where $I_j^0 = (x_i, x_{i-1})$. It follows by Lemma 3.2.8 $(b_{i,j}(s) := \|\mathcal{P}_i - \mathcal{P}_j\|_{L_{p(I_i)}} (\frac{h_j}{h_{i,j}})^k, i, j = 1, \dots, n$, and $b_\kappa(s) \le 1,)$

That for every i, $\quad \|\mathcal{P}_i - \mathcal{P}_j\|_{L_{p(I_i)}} \le c(p)(\frac{h_{i,j}}{h_j})^\kappa$.

Since $\mathcal{P}_i$ and $\mathcal{P}_j$ are polynomials of degree $\kappa - 1$, we get

$$\|\mathcal{P}_i'' - \mathcal{P}_j''\|_p \le \frac{1}{h_i^2}\|\mathcal{P}_i - \mathcal{P}_j\|_p \le \frac{c(p)}{h_i^2}(\frac{h_{i,j}}{h_j})^k.$$

In view of Lemma 3.2.5 and 3.2.9 ,

$\{\frac{1}{2}(\mathcal{J} - j) \le \frac{x_j - x_{\mathcal{J}}}{x_i - x_{i+1}} \le 2(\mathcal{J} - j)\}$

And $\quad (\frac{|x-y|+\Omega}{2} < |x-y| + \Omega_n(y) < 2(|x-y| + \Omega),\ x, y \in I)$ we see that 3.3.21 implies

$\|\mathcal{P}_j''(x_i^*)\|_p \le \frac{c(p)}{h_i^2}(\frac{h_{i,j}}{h_j})^k + \frac{c(p)}{h_i^2} \le \frac{c(p)}{h_i^2}(\frac{h_{i,j}}{h_j})^k$

$\|\mathcal{P}_j''(x_i^*)\|_p \le \frac{c(p)}{h_j^2}(|i-j|+1)^{2k}$ (3.3.23)

By assumption there are $k-2$ points $x_{i_m}^* \in I_{\mu,v}, m = 1, \dots, k-2$, each two being separated by an interval $I_i \subseteq I_{\mu,v}$. Recalling that $x \in I_j$, we have for each $1 \le \iota \le k-2$ and $1 \le m \le k-2$, with $\iota \ne m$,

$$\frac{|x - x_{i_m}^*|}{|x_{i_\iota}^* - x_{i_m}^*|} \le c\frac{h_{j,i_m}}{h_{i_m}} \le c(|i - i_m| + 1)^2$$

$\frac{|x - x_{i_m}^*|}{|x_{i_\iota}^* - x_{i_m}^*|} \le c((i-\mu)^2 + (j-v)^2)$ (3.3.24)

Now ,by virtue of the representation

$$\mathcal{P}_j''(x) \equiv \sum_{\iota=1}^{k-2} \mathcal{P}_j''(x_{i_\iota}^*) \prod_{m=1, m\ne\iota}^{\kappa-2} \frac{x - x_{i_m}^*}{x_{i_\iota}^* - x_{i_m}^*}$$

We obtain form(3.3.23) and (3.3.24)

$$\Omega^2\|\mathbb{s}''(x)\|_p = \Omega^2\left\|\mathcal{P}_j''(x)\right\|_p$$

$\leq c(p)((j-\mu)^{4\kappa-6} + (j-v)^{4\kappa-6}), x \in I_j^0,$

So the prove of 3.3.19 is complete ■

3.4 Zero-preserving Approximation

<u>Lemma</u> 3.4.1

The following relations hold:

$\sum_{j=1}^{n} \widetilde{T}_{j,n_1}(x) \equiv 1$ (3.4.2)

$\widetilde{T}_{j,n_1}'(y_i) = \widetilde{T}_{j,n_1}''(y_i) = 0 \ , 1 \leq i \leq \mathbb{s}, \quad 1 \leq j \leq n,$

$\widetilde{T}_{j,n_1}(y_i) = 0 \ , 1 \leq i \leq \mathbb{s}, \quad 1 \leq j \leq n, \ y_i \notin I_j^*,$ (3.4.3)

$\left\|\widetilde{T}_{j,n_1}^{(q)}(x)\right\|_p \leq c(p)\left\|\frac{1}{\Omega_{n_1}{}^{q}(x)}\left(\frac{\Omega_{n_1}(x)}{\Omega_{n_1}(x)+dist(x,I_j)}\right)^{b_2}\right\|_p, x \in I, 1 \leq j \leq$ $n, \ 0 \leq q \leq \mathbb{s}+2,$ (3.4.4)

Where $b_2 = \frac{1}{2}(b_1 - 1)$

Proof.

Relations 3.4.2 and 3.4.3 follow immediately form $\sum_{j=1}^{n} \widetilde{T}_j(x) \equiv 1$ and

$\widetilde{T}_j'(y_i) = \widetilde{T}_j''(y_i) = 0 \ , 1 \leq i \leq \mathbb{s}, \quad 1 \leq j \leq n,$

$\widetilde{T}_j(y_i) = 0 \ , 1 \leq i \leq \mathbb{s}, \quad 1 \leq j \leq n, \ y_i \notin I_j^*$, when we observe that if $I_{v,n_1} \subseteq I_j$, then $I_{v,n_1}^* \subseteq I_j^*$, Thus we just have to prove 3.4.4.

Note that by Lemma 3.2.10 and 3.2.11 yield

$$\left(\frac{h_{v,n_1}}{|x - x_{v,n_1}| + h_{v,n_1}}\right)^2 \leq c\frac{\Omega_{n_1}(x)}{|x - x_{v,n_1}| + \Omega_{n_1}(x)}.$$

Now if $x < x_j$, then it follows by

$\left\| \widetilde{T}_j^{(q)}(x) \right\|_p \le c(p) \left\| \frac{1}{\Omega^{q}(x)} \left(\frac{h_j}{|x-x_j|+h_j} \right)^{b_1} \right\|_p, x \in I, 1 \le j \le n, \ 0 \le q \le s+2$, that

$$\Omega_1^{q} \left\| \widetilde{T}_{j,n_1}^{(q)}(x) \right\|_p \le c(p) \sum\nolimits_{I_{v,n_1} \subseteq I_j} \left(\frac{h_{v,n_1}}{|x - x_{v,n_1}| + h_{v,n_1}} \right)^{b_1}$$

$$\le c(p) \left\| \Omega_{n_1}^{b_2}(x) \sum\nolimits_{I_{v,n_1} \subseteq I_j} \frac{h_{v,n_1}}{(|x - x_{v,n_1}| + \Omega_{n_1})^{b_2+1}} \right\|_p$$

$$\le c(p) \left\| \Omega_{n_1}(x)^{b_2} \int_{x_j - x}^{\infty} \frac{du}{(u + \Omega_{n_1}(x))^{b_2+1}} \right\|_p$$

$$= c(p) \left\| \left(\frac{\Omega_{n_1}(x)}{\Omega_{n_1}(x) + x_j - x} \right)^{b_2} \right\|_p = c(p) \left\| \left(\frac{\Omega_{n_1}(x)}{\Omega_{n_1}(x) + dist(x, I_j)} \right)^{b_2} \right\|_p,$$

Similar proof yield 3.4.4, if $x_{j-1} < x$, and if $x \in I_j$ ■

Lemma 3.4.5

Let $b_3 = b_2 - s - 2\kappa - 6 > 0$, and let $\mathcal{A}$ be a proper interval for $\mathbb{S} \in \sum_{\kappa,n}(Y_s)$,

$$\left\| \mathbb{S}^{(q)}(x) - D_{n_1}^{(q)}(x) \right\|_p \le c(p) \left\| \frac{\Gamma(\Omega)}{\Omega^{q}} \left(b_\kappa(\mathbb{S}, \mathcal{A}) + b_\kappa(\mathbb{S}) \frac{n}{n_1} \left(\frac{\Omega}{\Omega + dist(x, [-1,1] \backslash \mathcal{A})} \right)^{b_3} \right) \right\|_p. \quad (3.4.6)$$

$x \in \mathcal{A} \cap \bar{O}_e$, where $\bar{O}_e := \left\{ u : [u - \frac{1}{2}\Omega_n(u), u + \frac{1}{2}\Omega_n(u)] \subseteq \bar{O} \right\} . \cup (\bar{O} \cap (I_1 \cup I_n))$, $q = 0, \dots, s+2$.

Furthermore, if $\mathbb{S} \in \sum_{\kappa,n}^{1}(Y_s)$, then for $x \ne x_j$, $0 \le j \le n$,

$$\left\| \mathbb{S}''(x) - D_{n_1}''(x) \right\|_p \le c(p) \left\| \frac{1}{\Omega^2} \left(b_\kappa(\mathbb{S}, \mathcal{A}) + b_\kappa(\mathbb{S}) \frac{n}{n_1} \left(\frac{\Omega}{dist(x, [-1,1] \backslash \mathcal{A})} \right)^{b_3} \right) \right\|_p, x \in \mathcal{A} \quad (3.4.7)$$

Proof.

The proof of the two statements is similar and we will proceed simultaneously in both. Fix $I_v \subseteq \mathcal{A} \cap \bar{O}$ (or simply $I_v \subseteq \mathcal{A}$, if we prove 3.4.7, and let $x \in I_v \cap \bar{O}_e$ (or simply $x \in I_v$) be such that, say

$$x - x_v \le x_{v-1} - x \,. \tag{3.4.8}$$

we will write in this proof Ω_1 for $\Omega_{n_1}(x)$, $\tilde{T}_j$ for $\tilde{T}_{j,n_1}$, and $b_{v,j}$ for $b_{v,j}(\mathbb{S})$ By define $b_{i,j}(\mathbb{S}) := \|p_i - p_j\|_{L_p(I_v)} \left(\frac{h_j}{h_{i,j}}\right)^{\kappa}$, $j, i = 1, \dots, n$

$\|\mathcal{P}_v - \mathcal{P}_j\|_{L_p(I_v)} \le b_{v,j}(\frac{h_{v,j}}{h_j})^{\kappa}$, whence, for each $r \in N$.

$$\|\mathcal{P}_v{}^{(r)} - \mathcal{P}_j{}^{(r)}\|_{L_p(I_v)} \le c(p)\frac{b_{v,j}(\mathbb{S})}{h_v^r}(\frac{h_{v,j}}{h_j})^{\kappa}.$$

First let $j \ne v, v+1$. Then by Lemma 3.2.9,

$\{\frac{|x-y|+\Omega}{2} < |x-y| + \Omega_n(y) < 2(|x-y|+\Omega),\ x, y \in I\}$ and (3.4.8) imply $dist(x, I_j) > \frac{1}{2}\Omega$. Hence Lemma 3.4.1 in 3.4.4 combined with Lemma 3.2.9 and 3.2.11 yields

$$\left\|\mathcal{P}_v{}^{(r)} - \mathcal{P}_j{}^{(r)}\tilde{T}_j^{(q-r)}(x)\right\|_{L_p(I_v)}$$
$$\le c(p)\left\|\frac{b_{v,j}}{h_v^r}(\frac{h_{v,j}}{h_j})^{\kappa}\frac{1}{\Omega_1^{q-r}}(\frac{\Omega_1}{\Omega_1 + dist(x, I_j)})^{b_2}\right\|_p$$

$$\le c(p)\|\frac{b_{v,j}}{h_v^r}(\frac{h_{v,j}}{h_j})^{\kappa+1}\frac{h_j}{h_{v,j}}\frac{1}{\Omega_1^{q-r}}(\frac{\Omega_1}{\Omega_1 + dist(x, I_j)})^{b_2}\|_p$$

$$\le c(p)\left\|\frac{b_{v,j}}{h_v^r}(\frac{\Omega + dist(x, I_j)}{\Omega})^{2(\kappa+1)}\frac{h_j}{h_v}\frac{1}{\Omega_1^{q-r}}(\frac{\Omega_1}{\Omega_1 + dist(x, I_j)})^{q-r+1} \times (\frac{\Omega}{\Omega + dist(x, I_j)})^{b_2-q+r-1}\right\|_p$$

$$\le c(p)\left\|\frac{b_{v,j}}{h_v^{r+1}}h_j\frac{\Omega_1}{\Omega}\frac{1}{\Omega^{q-r}}(\frac{\Omega}{\Omega + dist(x, I_j)})^{b_3+1}\right\|_p$$

$$\le c(\mathrm{p}) \left\| \frac{\mathfrak{b}_{\mathcal{v},\mathrm{j}}}{\Omega^{q}} \frac{\mathfrak{n}}{\mathfrak{n}_1} \Omega^{\mathfrak{b}_3} \mathrm{h}_\mathrm{j} \left(\frac{1}{\Omega + dist(x, \mathrm{I}_\mathrm{j})} \right)^{\mathfrak{b}_3 + 1} \right\|_\mathrm{p}, 0 \le \mathfrak{r} \le q, \qquad (3.4.9)$$

Where in the third inequality we applied the third inequality in Lemma 3.2.9 and 3.2.11, in the next one we used the fact that $dist(x, \mathrm{I}_\mathrm{j}) > \frac{1}{2}\Omega$, and in the last we have applied the straightforward inequality $\frac{\Omega_1}{\Omega} \le \frac{\mathfrak{n}}{\mathfrak{n}_1}$, now, by virtue of the Lemma 3.2.12 , $\{\sum_{\mathrm{j}=1}^{\mathfrak{n}} \widetilde{\mathrm{T}}_{\mathrm{j},\mathfrak{n}_1}(x) \equiv 1\}$ we may represent $\mathbb{S}^{(q)}(x) - \mathcal{D}_{\mathfrak{n}_1}^{(q)}(x)$ as

$$\mathbb{S}^{(q)}(x) - \mathcal{D}_{\mathfrak{n}_1}^{(q)}(x) = \left(\left(\mathcal{P}_{\mathcal{v}}(x) - \mathcal{P}_{\mathcal{v}+1}(x)\right)\tilde{\mathcal{T}}_{\mathcal{v}+1}(x)\right)^{(q)}$$

$$+\left(\sum_{\mathrm{I}_\mathrm{j} \subseteq \mathcal{A}, \mathrm{j} \ne \mathcal{v}, \mathcal{v}+1} + \sum_{\mathrm{I}_\mathrm{j} \nsubseteq \mathcal{A}, \mathrm{j} \ne \mathcal{v}, \mathcal{v}+1}\right) \left(\left(\mathcal{P}_{\mathcal{v}}(x) - \mathcal{P}_\mathrm{j}(x)\right) \tilde{\mathcal{T}}_\mathrm{j}(x)\right)^{(q)}$$

$$:= \sigma_1(x) + \sigma_2(x) + \sigma_3(x),$$

Where we write $\mathcal{P}_{\mathfrak{n}+1} := \mathcal{P}_{\mathfrak{n}}$, if $\mathcal{v} = \mathfrak{n}$.

We begin with the estimate of σ_1 . Note that if $\mathcal{v} = \mathfrak{n}$, then $\sigma_1 \equiv 0$, so that we may assume that $\mathcal{v} < \mathfrak{n}$. we need separate arguments for 3.4.6 and 3.4.7.

First we deal with 3.4.7 .

since $\mathbb{S} \in \sum_{\kappa,\mathfrak{n}}^{1}$, $q = 2$ and $\mathrm{I}_{\mathcal{v}} \subseteq \mathcal{A}$, it readily follows that.

$\|\mathcal{P}_{\mathcal{v}}'' - \mathcal{P}_{\mathcal{v}+1}''\|_{\mathrm{L}_{\mathrm{p}(\mathrm{I}_{\mathcal{v}})}} \le c(\mathrm{p}) \left\| \frac{1}{\Omega^2} \mathfrak{b}_{\mathcal{v},\mathcal{v}+1} \right\|_\mathrm{p}$, which in turn implies

$$\|\mathcal{P}_{\mathcal{v}}'(x) - \mathcal{P}_{\mathcal{v}+1}'(x)\|_\mathrm{p} = \left\| \int_{x_{\mathcal{v}}}^{x} (\mathcal{P}_{\mathcal{v}}'' - \mathcal{P}_{\mathcal{v}+1}'') d\mathfrak{u} \right\|_\mathrm{p}$$

$$\le c(\mathrm{p}) \left\| \frac{1}{\Omega^2} \mathfrak{b}_{\mathcal{v},\mathcal{v}+1} (x - x_{\mathcal{v}}) \right\|_\mathrm{p}$$

And $\|\mathcal{P}_{\mathcal{v}}(x) - \mathcal{P}_{\mathcal{v}+1}(x)\|_\mathrm{p} \le c(\mathrm{p}) \left\| \frac{1}{\Omega^2} \mathfrak{b}_{\mathcal{v},\mathcal{v}+1} (x - x_{\mathcal{v}})^2 \right\|_\mathrm{p}$.

Therefore , by Lemma 3.4.1 in 3.4.4

$\{\left\|\widetilde{T}^{(q)}_{j,n_1}(x)\right\|_p \le c(p)\left\|\frac{1}{\Omega_{n_1}(x)}\left(\frac{\Omega_{n_1}(x)}{\Omega_{n_1}(x)+dist(x,I_j)}\right)^{b_2}\right\|_p, x \in I, 1 \le j \le n,\ 0 \le q \le s+2,$ so by Lemma 3.2.19 ,

$$\|\sigma_1(x)\|_p \le c(p)\left\|\frac{1}{\Omega^2} b_{v,v+1}\left(1+\frac{x-x_v}{\Omega_1}+\frac{(x-x_v)^2}{{\Omega_1}^2}\right)\left(\frac{\Omega_1}{\Omega_1+|x-x_v|}\right)^{b_2}\right\|_p$$

$$\le c(p)\left\|\frac{1}{\Omega^2} b_{v,v+1}\left(\frac{\Omega_1}{\Omega_1+|x-x_v|}\right)^{b_2-2}\right\|_p. \tag{3.4.10}$$

Now, if $I_{v+1} \subseteq \mathcal{A}$, then (3.4.10) implies

$$\|\sigma_1(x)\|_p \le c(p)\left\|\frac{1}{\Omega^2} b_\kappa(\mathbb{S},\mathcal{A})\right\|_p \tag{3.4.11}$$

And if $I_{v+1} \nsubseteq \mathcal{A}$, then (3.4.10) yields

$$\|\sigma_1(x)\|_p \le c(p)\left\|\frac{1}{\Omega^2} b_\kappa(\mathbb{S})\frac{\Omega_1}{\Omega}\frac{\Omega}{\Omega_1+|x-x_v|}\left(\frac{\Omega_1}{\Omega_1+|x-x_v|}\right)^{b_2-3}\right\|_p$$

$$\le c(p)\left\|\frac{1}{\Omega^2} b_\kappa(\mathbb{S})\frac{n}{n_1}\frac{\Omega}{|x-x_v|}\left(\frac{\Omega}{\Omega+|x-x_v|}\right)^{b_2-3}\right\|_p$$

$$\le c(p)\left\|\frac{1}{\Omega^2} b_\kappa(\mathbb{S})\frac{n}{n_1}\left(\frac{\Omega}{dist(x,I\backslash\mathcal{A})}\right)^{b_3}\right\|_p \tag{3.4.12}$$

Now we establish Lemma 3.4.6. since $x \in \bar{O}_e, v \in H$.

If also $(v+1) \notin H$, then $\mathbb{S} \in \sum_{\kappa,n}(Y_s)$ implies $\mathcal{P}_v \equiv \mathcal{P}_{v+1}$.

Hence $\sigma_1 = 0$.

Otherwise $(v+1) \in H$, so that $x \in \bar{O}_e$, implies $x - x_v \ge \Omega$ .Therefore 3.4.9 hold for $j = v+1$, and we may absorb σ_1 either in σ_2 or in σ_3, as the case may be, and which we estimate below, what is left is to estimate σ_2 and σ_3, It follows from 3.4.9 that

$$\|\sigma_3(x)\|_p \le c(p)\left\|\frac{b_\kappa(\mathbb{S})}{\Omega^q}\frac{n}{n_1}\Omega^{b_3}\sum_{I_j \nsubseteq \mathcal{A}, j \ne v,v+1}\frac{h_j}{(\Omega+dist(x,I_j))^{b_3+1}}\right\|_p$$

$$\leq c(\mathrm{p}) \left\| \frac{\mathfrak{b}_{\kappa}(\mathbb{S})}{\Omega^{q}} \frac{\mathfrak{n}}{\mathfrak{n}_1} \left(\frac{\Omega}{\Omega + dist(x, \mathrm{I} \backslash \mathcal{A})} \right)^{\mathfrak{b}_3} \right\|_{\mathrm{p}} \qquad (3.4.13)$$

Similarly , if $dist(x, \mathrm{I}_{\mathcal{v}*}) := \min\{dist(x, \mathrm{I}_{\mathcal{v}-1}), dist(x, \mathrm{I}_{\mathcal{v}+2})\}$

Then we obtain

$$\|\sigma_2(x)\|_{\mathrm{p}} \leq c(\mathrm{p}) \left\| \frac{\mathfrak{b}_{\kappa}(\mathbb{S}, \mathcal{A})}{\Omega^{q}} \frac{\mathfrak{n}}{\mathfrak{n}_1} \left(\frac{\Omega}{\Omega + dist(x, \mathrm{I}_{\mathcal{v}*})} \right)^{\mathfrak{b}_3} \right\|_{\mathrm{p}}$$

$$\leq \frac{c(\mathrm{p}) \mathfrak{b}_{\kappa}(\mathbb{S}, \mathcal{A})}{\Omega^{q}} \qquad (3.4.14)$$

From 3.4.13 and 3.4.14 we get 3.4.6 with the above discussion of σ_1, and from 3.4.10) and 3.4.14 , we get 3.4.7 . This complete the proof ■

<u>Lemma</u> 3.4.15

If $\mathbb{S} \in \sum_{\kappa,\mathrm{n}}(\mathrm{Y}_{\mathbb{s}})$, then

$$\left\|\mathbb{S} - \mathcal{D}_{\mathfrak{n}_1}\right\|_{\mathrm{p}} \leq c(\mathrm{p}) \|\mathfrak{b}_{\kappa}(\mathbb{S})\|_{\mathrm{p}}. \qquad (3.4.16)$$

Moreover , if $\mathbb{S} \in \sum_{\kappa,\mathrm{n}}(\mathrm{Y}_{\mathbb{s}})$ and $\mathbb{S}''(\mathrm{y}_{\mathrm{i}}) = 0 \ , \mathrm{i} = 1, \dots, \mathbb{s},$ (3.4.17)

Then $\mathcal{D}''_{\mathfrak{n}_1}(\mathrm{y}_{\mathrm{i}}) = 0 \ , \mathrm{i} = 1, \dots, \mathbb{s},$ (3.4.18)

Proof.

The proof of 3.4.16 is similar to that of 3.4.7 in Lemma 3.4.5, in fact easier, to prove 3.4.18 . Fix $1 \leq \mathrm{i} \leq \mathbb{s}$, and let $\mathcal{v}$ be such that $\mathrm{y}_{\mathrm{i}} \in \mathrm{I}_{\mathcal{v}}$. Since $\mathcal{P}_{\mathrm{j}} \equiv \mathcal{P}_{\mathcal{v}}$, for all $\mathrm{I}_{\mathrm{j}} \subseteq \mathrm{I}^*_{\mathcal{v}}$, then

$$\mathrm{D}''_{\mathfrak{n}_1}(\mathrm{y}_{\mathrm{i}}) = \sum_{\mathrm{j}=1}^{\mathfrak{n}} (\mathcal{P}_{\mathrm{j}}(\mathrm{y}_{\mathrm{i}}) \widetilde{\mathrm{T}}''_{\mathrm{j}}(\mathrm{y}_{\mathrm{i}}) + \mathcal{P}'_{\mathrm{j}}(\mathrm{y}_{\mathrm{i}}) \widetilde{\mathrm{T}}'_{\mathrm{j}}(\mathrm{y}_{\mathrm{i}}) + \sum_{\mathrm{I}_{\mathrm{j}} \not\subseteq \mathrm{I}^*_{\mathcal{v}}} \mathcal{P}''_{\mathrm{j}}(\mathrm{y}_{\mathrm{i}}) \widetilde{\mathrm{T}}_{\mathrm{j}}(\mathrm{y}_{\mathrm{i}})$$

$$+ \mathcal{P}''_{\mathcal{v}}(\mathrm{y}_{\mathrm{i}}) \sum_{\mathrm{I}_{\mathrm{j}} \subseteq \mathrm{I}^*_{\mathcal{v}}} \widetilde{\mathrm{T}}_{\mathrm{j}}(\mathrm{y}_{\mathrm{i}}).$$

Now, by virtue of $\widetilde{\mathrm{T}}'_{\mathrm{j},\mathfrak{n}_1}(\mathrm{y}_{\mathrm{i}}) = \widetilde{\mathrm{T}}''_{\mathrm{j},\mathfrak{n}_1}(\mathrm{y}_{\mathrm{i}}) = 0, 1 \leq \mathrm{i} \leq \mathbb{s}, 1 \leq \mathrm{j} \leq \mathfrak{n},$

$\widetilde{T}_{j,n_1}(y_i) = 0, 1 \le i \le s, 1 \le j \le n, y_i \in I_j^*$, the first and second sums are zero, and since $\mathcal{P}''_{v}(y_i) = \mathbb{S}''(y_i) = 0$, it follows that the 3.4.18 is prove ■

Lemma 3.4.19

If $\mathcal{A}$ is a proper interval sub set of I, $\mathbb{S} \in \sum_{\kappa,n}^{1}(Y_{s})$, and

$\mathbb{S}''(y_i) = 0 \quad, i = 1, \dots, s$, holds, then

$$\left\|\mathbb{S}''(x) - D''_{n_1}(x)\right\|_p \le$$

$$c_0(p)\left\|\frac{\pi(x)}{\Omega^2}\left(b_\kappa(\mathbb{S},\mathcal{A}) + b_\kappa(\mathbb{S})\frac{n}{n_1}\left(\frac{\Omega}{dist(x,[-1,1]\backslash\mathcal{A})}\right)^{b_3}\right)\right\|_p, x \in \mathcal{A} \quad (3.4.20)$$

where $c_0 = c_0(\kappa, s, b)$, and $\pi(x) := \Pi_{i=1}^{s} \frac{|x-y_i|}{|x-y_i|+\Omega}$.

Proof.

Let $x \in \mathcal{A}$, if $x \notin \bar{O}_e$, then $\pi(x) > c$. and , if $x \notin \bar{O}$, then it follows from $\pi(x) > 2^{-s}, x \in (-1,1)\backslash O$. And we only have to check the case where x is in a connected component, say $[x_\mu, x_v]$, of $\bar{O}$ and either $x + \Omega/2 > x_v$ and $v > 0$, or $x - \frac{\Omega}{2} < x_\mu$, and $\mu < n$.

Clearly, y_i's in this component, so let $y_i \in [x_\mu, x_v]$. It is easily seen that $x + \Omega/2$ is increasing in $[-1, x_1]$ and $x - \frac{\Omega}{2}$ is increasing in $[x_{n-1}, 1]$ .we will show that $x_v < x + \Omega/2$ and $x < \frac{x_v + x_{v+1}}{2}$ cannot hold. If $x_v < x + \Omega/2$ and $x_{v+1} \le x \le x_v$, then $x_v < x + \frac{\Omega}{2} \le x + |I_{v+1}|/2$, which yield that $x - x_{v+1} > |I_{v+1}|/2$. Since $x + \Omega/2$ is increasing this in turn implies that if $x < x_{v+1}$, then $x + \frac{\Omega}{2} < x_v$. Hence if $x_v < x + \Omega/2$, then $x - y_i \ge x - x_{v+1} > |I_{v+1}|/2$, so that

$$\frac{x - y_i}{x - y_i + \Omega} \ge \frac{\frac{|I_{v+1}|}{2}}{\frac{|I_{v+1}|}{2} + |I_{v+1}|} \ge \frac{1}{3}.$$

The case $x-\frac{\Omega}{2}<x_\mu$ is symmetric, thus 3.4.16 follows by Lemma 3.4.5 in 3.4.7. if on the other hand, $x\in\bar{\mathcal{O}}_e\subseteq\bar{\mathcal{O}}$, then $x\in I_j^*$, where I_j^* is a connected component of $\bar{\mathcal{O}}$, such that

$$\Omega_n(u)\le |I_j^*|\le c\Omega_n(u), u\in I_j^*, \tag{3.4.21}$$

And we have $\quad \mathbb{S}(u)=\mathcal{P}_j(u),\ \ u\in I_j^*. \qquad (3.4.22)$

This together with 3.4.21 implies that for $\mathcal{A}_1:=\mathcal{A}\cup I_j^*$, which is a proper interval sub set of I , we have

$\mathfrak{b}_\kappa(\mathbb{S},\mathcal{A}_1)\le c\mathfrak{b}_\kappa(\mathbb{S},\mathcal{A})$. set $I_e^*:=I_j^*\cap\bar{\mathcal{O}}_e$, since $x\in I_e^*$, $dist(x, I\backslash I_j^*)\ge\frac{\Omega}{2}$, and by (3.4.21),

$dist(x, I\backslash I_j^*)\le |I_j^*|\le c dist(x, I\backslash I_j^*)$. Hence

$dist(x, I\backslash\mathcal{A}_1)\le\ |I_j^*|+dist(I_j^*,\ I\backslash\mathcal{A}_1)\le\ c dist(x, I\backslash I_j^*)+dist(x, I\backslash\mathcal{A}_1)\le c dist(x, I\backslash\mathcal{A}_1)$, , $x\in I_e^*$.

By virtue of Lemma 3.4.5 in 3.4.6 we thus obtain

$$\left\|\mathbb{S}^{(q)}(x)-D_{n_1}^{(q)}(x)\right\|_{p(I_e^*)}\le c\,(p)\left\|\frac{1}{|I_j^*|}\Omega\right\|_p ,q=0,\dots,s+2, \tag{3.4.23}$$

With $\quad \Omega:=\left(\mathfrak{b}_\kappa(\mathbb{S},\mathcal{A})+\mathfrak{b}_\kappa(\mathbb{S})\frac{n}{n_1}\left(\frac{|I_j^*|}{|I_j^*|+dist(I_j^*,I\backslash\mathcal{A})}\right)^{\mathfrak{b}_3}\right)$, where we used the fact that $\quad dist(I_j^*,\ I\backslash\mathcal{A}_1)\ge dist(I_j^*, I\backslash\mathcal{A})$. it remains to prove that

$$\left\|\mathbb{S}''(x)-D''_{n_1}(x)\right\|_{p(I_e^*)}\le c\,(p)\left\|\frac{\pi(x)}{|I_j^*|^2}\Omega\right\|_p \tag{3.4.24}$$

To this end, let $\pi_1(x):=\prod_{y_i\in I_j^*}\frac{|x-y_i|}{|x-y_i|+\Omega}$, $\quad \pi_2(x):=\prod_{y_i\notin I_j^*}\frac{|x-y_i|}{|x-y_i|+\Omega}$, so that

$\pi(x)=\pi_1(x).\pi_2(x)$. if $\ y_i\notin I_j^*$, then $\ |x-y_i|>\Omega/2$, whence $\pi_2(x)\ge 3^{-s}$.

Therefore we have to prove 3.4.24 with $\pi_1(x)$ in place of $\pi(x)$. Now by 3.4.22 , $\mathbb{S} - D_{n_1}$ is a polynomial in I_j^*, and by Lemma 3.4.15 in 3.4.17 and 3.4.18 imply

$$\mathbb{S}''(y_i) - D''_{n_1}(y_i) = 0\,,\ i = 1, \dots, s,$$

Hence, if $y_{i_\mu}, 1 \le \mu \le q \le s$, are point of Y_s in I_j^*, then there is a $\theta \in I_e^*$, such that

$$\left\|\mathbb{S}''(x) - D''_{n_1}(x)\right\|_p = \left\|\mathbb{S}^{(\iota+2)}(\theta) - D_{n_1}^{(\iota+2)}(\theta)\right\|_p \prod_{\mu=1}^{\iota} \left|x - y_{i_\mu}\right|$$

$$\le c(p)\left\|\frac{\Omega}{\left|I_j^*\right|^2}\prod_{\mu=1}^{\iota}\frac{\left|x-y_{i_\mu}\right|}{\left|I_j^*\right|}\right\|_p \le c(p)\left\|\frac{\pi_1(x)}{\left|I_j^*\right|^2}\Omega\right\|_p ,$$

Where in the first inequality we applied 3.4.23 and for the second we used the inequality $\left|x - y_{i_\mu}\right| + \Omega \le c\left|I_j^*\right|$. so that is completes the proof of 3.4.24 ■

3.5 proof the Theorem 3.5.1

For every $\kappa, n \in \mathbb{N}$ and $s \in \mathbb{N}_0$ there are constants c and c_*, such that if $\mathbb{S} \in \sum_{\kappa,n}^1(Y_s) \cap \Delta^2(Y_s) \cap L_p(I)$ then there is a polynomial $\mathcal{P}_n \cap \Delta^2(Y_s) \cap L_p(I)$ of degree$\le c_* n$, satisfying

$$\|\mathbb{S} - \mathcal{P}_n\|_p \le c(p) w_\kappa^\vartheta\left(\mathbb{S}, \frac{1}{n}\right).$$

Note that by the above, we have to prove Theorem 3.5.1 only for $\kappa \ge 3$, but the cases $\kappa = 1,2$ are anyway trivial in this setting since $\sum_{2,n}^1 \subseteq \Pi_1$.

Proof.

Recall that we may assume that $\kappa \ge 3$.we begin with some notation, given $\mathcal{A} \subseteq I$ denote $\mathcal{A}^e := \cup_{I_j \cap \mathcal{A} \neq \emptyset} I_j$, and $\mathcal{A}^{2e} := (\mathcal{A}^e)^e$ and $\mathcal{A}^{3e} := (\mathcal{A}^{2e})^e$.without loss of generality we may assume that $b_\kappa(\mathbb{S}) \le 1$ (3.5.2)

So that in view of $\mathfrak{b}_{\kappa}(\mathbb{S}) \leq c w_{\kappa}^{\vartheta}\left(\mathbb{S}, \frac{1}{\mathfrak{n}}\right)) \leq c\mathfrak{b}_{\kappa}(\mathbb{S})$,

In order to prove our assertion , we have to find a polynomial $\mathcal{P}_{\mathfrak{n}}$ of degree $\leq c\,\mathfrak{n}$, such that

$$\|\mathbb{S}(x) - \mathcal{P}_{\mathfrak{n}}(x)\|_{\mathrm{p}} \leq c(\mathrm{p}) \qquad (3.5.3)$$

And

$$\mathcal{P}''_{\mathfrak{n}}(x)\sigma(x) \geq 0, x \in [-1,1] \qquad (3.5.4)$$

where $\sigma(x)$ define the set $\sigma(x) \coloneqq \mathrm{sgn}\,\pi(x), x \in [-1,1]$

And $\pi(x) \coloneqq \prod_{i=1}^{\mathbb{s}}(x - \mathrm{y_i})$

We fix $\mathfrak{b}$ so big that $\mathfrak{b}_3 \geq 25(\mathbb{s}+1)$, ($\mathfrak{b}_3$ was defined in 3.4.12 of Lemma 3.4.5. This makes $\mathcal{C}_0(\kappa, \mathbb{s}, \mathfrak{b})$, the constant in 3.4.22 of Lemma 3.4.21

$$\left\|\mathbb{S}''(x) - \mathcal{D}''_{\mathfrak{n}_1}(x)\right\|_{\mathrm{p}} \leq \mathcal{C}_0(\mathrm{p}) \left\|\frac{\Gamma(\Omega\,)\pi(x)}{\Omega^{\,2}}(\mathfrak{b}_{\kappa}(\mathbb{S}, \mathcal{A})\right\|_{\mathrm{p}} + \left\|\mathfrak{b}_{\kappa}(\mathbb{S})\frac{\mathfrak{n}}{\mathfrak{n}_1}(\frac{\Omega}{dist(x, \mathrm{I}\backslash\mathcal{A})})^{\mathfrak{b}_1}\right\|_{\mathrm{p}}$$

Dependent only on κ and $\mathbb{s}$ we dente $c_2 \coloneqq \mathcal{C}_0$. fix on integer c_3 such that

$$c_3 \geq max\{8\kappa/c_1, 12\mathbb{s}\}, \qquad (3.5.5)$$

where c_1 is constant form

$$\mathrm{Q}''_{\mathfrak{n}}(x)\sigma(x) \geq c_1 \frac{1}{\mu}\left(\frac{\Omega}{max\{\Omega, dist(x,\mathrm{E})}\right)^{25(\mathbb{s}+1)+2\kappa} \frac{\pi(x)\Gamma(\Omega)}{\Omega^{\,2}}, x \in \mathcal{J} \cup [-1,1]\backslash\mathrm{E}.$$

(we may take $c_1 \leq 1$)

And without loss of generality we may assume that $\mathfrak{n}$ is divisible by c_3 ,i.e., $\mathfrak{n} = \mathrm{N}c_3$, where this defines N.

We divide I in to N intervals

$E_q := [x_{qc_3}, x_{(q-1)c_3}] = I_{qc_3} \cup \dots \cup I_{(q-1)c_3+1}, q = 1, \dots, N,$

We will write $j \in \mathbb{N}$, if there is an $x \in I_j$, such that

$$\|\mathbb{S}''(x)\|_p \le 5c_2(p) \left\|\frac{1}{\Omega^2}\right\|_p, \tag{3.5.6}$$

And we will say that $q \in G_1$, if E_q contains at least $2\kappa - 5$ intervals I_j with $j \in \mathbb{N}$.

We will say that $q \in G$ if either $q \in G_1$, or there is a $q^* \in G_1$, such that

$$E^e_{q+v} \cap \mathcal{O} \neq \emptyset, \begin{cases} v = 0,1, \dots, q^* - q, & \text{if} q^* \ge q, \\ v = 0, -1, \dots, q^* - q, & \text{if} q^* < q. \end{cases} \tag{3.5.7}$$

Note that if $q \in G \backslash G_1$,then $|q - q^*| \le 2\mathbb{s}$, hence 3.5.2, 3.5.6 and Lemma 3.3.19 imply

$$\left\|\Omega^2 \mathbb{S}''(x)\right\|_{p(E_q)} \le c(p), q \in G \tag{3.5.8}$$

Now set $\qquad E := \bigcup_{q \notin G} E_q$,

Let $\mathbb{S}'' = \mathbb{s}_1 + \mathbb{s}_2$, where

$$\mathbb{s}_1(x) := \begin{cases} \mathbb{S}''(x), & \text{if } x \notin E^e \\ 0, & \text{if } x \in E^e, \end{cases}$$

And $\mathbb{s}_2 := \mathbb{S}'' - \mathbb{s}_1$,

And putting

$$\mathbb{S}_1(x) := \mathbb{S}(-1) + (x+1)\mathbb{S}'(-1) + \int_{-1}^{x} (x - u)\mathbb{s}_1(u)du,$$

$\mathbb{S}_2(x) := \int_{-1}^{x} (x - u)\mathbb{s}_2(u)du.$

(Note that $\mathbb{s}_1$ and $\mathbb{s}_2$ are well defined for $x \neq x_j, 1 \le j \le n$, so that $\mathbb{S}_1$ and $\mathbb{S}_2$ are well defined everywhere and possess a second derivative again, for $x \neq x_j, 1 \le j \le n$, Thus from now on whenever we write $\mathbb{S}''_\iota(x)$ we will mean $x \neq x_j, 1 \le j \le n$.)It follows that

$\mathbb{S}_1,\ \mathbb{S}_2 \in \sum_{\kappa;\mathfrak{n}}^{1}(\mathrm{Y}_{\mathbb{s}})$, then

$\mathbb{S}_1''(x)\sigma(x) \geq 0, x \in \mathrm{I}$, and $\mathbb{S}_2''(x)\sigma(x) \geq 0, x \in [-1,1]$, by Lemma 3.2.13 and 3.5.8 imply $\mathfrak{b}_\kappa(\mathbb{S}_1) \leq c$, Which by virtue of 3.5.2 yields

$$\mathfrak{b}_\kappa(\mathbb{S}_2) \leq c + 1 \leq \lceil c + 2 \rceil =: c_4 \qquad (3.5.9)$$

The set E is a union of disjoint intervals $\mathrm{F_p} = [a_\mathrm{p}, \mathfrak{b}_\mathrm{p}]$, between any two of which there is an interval $\mathrm{E_q}$ are with $\mathrm{q} \in \mathrm{G}$. we may assume that $\mathfrak{n} > c_3 c_4$, and write $\mathrm{p} \in \mathrm{AG}$ (for ‘‘Almost Good‘‘) if $\mathrm{F_p}$ consists of no more than c_4 intervals $\mathrm{E_q}$, in particular , it consists of

no more than $c_3 c_4$ intervals $\mathrm{I_j}$. Set $\mathrm{F} := \bigcup_{\mathrm{p} \notin \mathrm{AG}} \mathrm{F_p}$, and let

$\mathbb{s}_4 := \begin{cases} \mathbb{S}''(x), & \text{if} x \in \mathrm{F}^{e} \\ 0, & \text{otherwise} \end{cases}$ and

$\mathbb{s}_3 := \mathbb{S}'' - \mathbb{s}_4$. Now put

$$\mathbb{S}_3(x) := \mathbb{S}(-1) + (x+1)\mathbb{S}'(-1) + \int_{-1}^{x} (x - \mathrm{u})\mathbb{s}_3(\mathrm{u})d\mathrm{u},$$

$$\mathbb{S}_4(x) := \int_{-1}^{x} (x - \mathrm{u})\mathbb{s}_4(\mathrm{u})d\mathrm{u}.$$

Then , evidently

$$\mathbb{S}_3, \mathbb{S}_4 \in \sum_{\kappa;\mathfrak{n}}^{1}(\mathrm{Y}_{\mathbb{s}}), \qquad (3.5.10)$$

$$\mathbb{S}_3''(x)\sigma(x) \geq 0, x \in [-1,1] \qquad (3.5.11)$$

and $\mathbb{S}_4''(x)\sigma(x) \geq 0, x \in [-1,1]$ (3.5.12)

For $\mathrm{p} \in \mathrm{AG}$, Lemma 3.3.19 and 3.5.9 imply

$$\|\mathbb{S}_3''(x)\|_\mathrm{p} = \|\mathbb{S}_2''(x)\|_\mathrm{p} \leq c\,(\mathrm{p}) \left\| \frac{\Gamma(\Omega)}{\Omega^{2}} \right\|_{L_p(\mathrm{F_p})}, x \in \mathrm{F_p}.$$

Hence

$$\|\mathbb{S}_3''(x)\|_\mathrm{p} \leq c\,(\mathrm{p}) \left\| \frac{1}{\Omega^{2}} \right\|_\mathrm{p}, x \in [-1,1], \qquad (3.5.13)$$

which by virtue of Lemma 3.2.13 yields that $\mathfrak{b}_\kappa(\mathbb{S}_3) \leq c$.

whence by 3.5.2, $\mathfrak{b}_\kappa(\mathbb{S}_4) \leq c + 1 \leq \lceil c + 2 \rceil =: c_5$ (3.5.14)

In view of 3.5.10 and 3.5.11 , combining Theorem 3.3.11 with 3.5.13 and Lemma 3.2.14, we obtain the existence of a polynomial $\mathfrak{r}_n$ which is coconvex with $\mathbb{S}$, and such that

$$\|\mathbb{S}_3(x) - \mathfrak{r}_n(x)\|_p \le c\,(p)\,. \tag{3.5.15}$$

since $\mathbb{S}_4(x) = \mathbb{S}''(x)\,, x \in F^e$, then by 3.5.2 we have for p $\notin$ AG

$$\mathfrak{b}_\kappa\left(\mathbb{S}_4, F_p^e\right) = \mathfrak{b}_\kappa\left(\mathbb{S}\,, F_p^e\right) \le \mathfrak{b}_\kappa(\mathbb{S}\,) \le 1 \tag{3.5.16}$$

Also for such p, $\mathbb{s}_4(x) = \mathbb{S}_2''(x), x \in F_p^{3e}$, Hence from (3.5.9)

$$\mathfrak{b}_\kappa\left(\mathbb{S}_4, F_p^{3e}\right) = \mathfrak{b}_\kappa\left(\mathbb{S}_2, F_p^{3e}\right) \le \mathfrak{b}_\kappa(\mathbb{S}_2) \le c_4. \tag{3.5.17}$$

We still have to approximate $\mathbb{S}_4$. To this end, applying Lemma 3.2.15, we construct three polynomials Q_n and $\mathcal{M}_n$ of degree $< cn$, and we let $\mathcal{D}_{n_1}(.\,,\mathbb{S}_4)$ of degree cn_1, be define by

$$\mathcal{D}_{n_1}(x) \coloneqq \mathcal{D}_{n_1}(x, \mathbb{S}) \coloneqq \sum_{j=1}^{n_1} \mathcal{P}_j(x)\widetilde{T}_{j;n_1}(x, \mathfrak{b}, Y_{\mathbb{S}}),$$

We begin with Q_n . for each q for which $E_q \subseteq F$, let $\mathcal{J}_q$ be the union of all intervals $I_j \subseteq E_q$ with $j \in UC$. Recall that $q \notin G$, therefore by 3.5.5 , the number of such intervals is at most $2\kappa - 6 < c_3/4$, and the total number of intervals in E_q is c_3 . Thus, by Lemma 3.2.15 is applicable for each E_q, and if we set $Q_n \coloneqq \sum_{E_q \subseteq F} Q_n(.\,, E_q, \mathcal{J}_q),)$

Where on the right-hand side are the polynomials guaranteed by Lemma 3.2.15 (we consider $\sum_{E_q \subseteq F} Q_n(.\,, E_q, \mathcal{J}_q) \equiv 0$, if $\mathcal{J}_q = \emptyset$), and denote

$$\mathcal{J} = \bigcup\nolimits_{E_q \subseteq F} \mathcal{J}_q\,,$$

Then we conclude that $\overline{Q}_n$ satisfies

$$Q_n''(x)\sigma(x) \ge 0, x \in [-1,1]\backslash F, \tag{3.5.18}$$

$$Q_n''(x)\sigma(x) \ge -\frac{\pi(x)}{\Omega^{\,2}}, x \in F\backslash\mathcal{J}\,, \tag{3.5.19}$$

$$Q_n''(x)\sigma(x) \ge \frac{4\pi(x)\Gamma(\Omega)}{\Omega^{\,2}}, x \in \mathcal{J}\,. \tag{3.5.20}$$

Note that 3.5.18-3.5.20 follow since for any given x all relevant $Q''_n(x, E_q, \mathcal{J}_q)$, except perhaps one , have the same sign. Finally, it follows from Lemma 3.2.13.

$$(\|Q_n(x)\|_p \leq c\,(p)\iota^6 \left\| \Omega \sum_{I_j \subseteq E} \frac{h_j}{(|x - x_j| + \Omega^2)} \right\|_p)$$

$\|Q_n(x)\|_p \leq c\,(p)\,, x \in [-1,1].$ (3.5.21)

Next we define the polynomial $\mathcal{M}_n$. For each F_p with $p \notin AG$,

let $\mathcal{J}_{p^-}$ denote the union of the two intervals on the left side of $F_p^{2e} \backslash \mathring{F}_p$, and let $\mathcal{J}_{p^+}$ denote the union of the two intervals on the right side of $F_p^{2e} \backslash \mathring{F}_p$.Also, let F_{p^-} and F_{p^+} be closed intervals each consisting of $\iota := c_3 c_4$ intervals I_j and such that

$$\mathcal{J}_{p^-} \subset F_{p^-} \subset F_p^{2e} \text{ and } \mathcal{J}_{p^+} \subset F_{p^+} \subset F_p^{2e}.$$

Now we set

$\mathcal{M}_n := \sum_{p \notin AG} (Q_n(.\,, F_{p^+}, \mathcal{J}_{p^+}) + Q_n(.\,, F_{p^-}, \mathcal{J}_{p^-})$, since $\iota := c_3 c_4$, $\mu = 2$,

It follows from 3.5.5 that $c_1 \frac{\iota}{\mu} \geq 2c_4$. Again by Lemma 3.2.15

$\mathcal{M}''_n(x)\sigma(x) \geq -2\frac{\pi(x)}{\Omega^2}\,, x \in F,$ (3.5.22)

$\mathcal{M}''_n(x)\sigma(x) \geq \frac{2c_4\pi(x)}{\Omega^2}\,, x \in F^{2e} \backslash F\,,$ (3.5.23)

And $\mathcal{M}''_n(x)\sigma(x) \geq \frac{\pi(x)}{\Omega^2}\left(\frac{\Omega}{dist(x, F^e)}\right)^{25(s+1)}, x \in [-1,1] \backslash F^{2e},$ (3.5.24)

Where in the last inequality we used the inequality,

$$max\{\,\Omega, dist(x, F^{2e})\} \leq\; dist(x, F^e), x \in [-1,1] \backslash F^{2e}.$$

Finally , it readily follows from Lemma 3.2.15

$\|\mathcal{M}_n(x)\|_p \leq c\,(p)$ (3.5.25)

The third auxiliary polynomial the properties of which we need to recall is $\mathcal{D}_{\mathfrak{n}_1}(x) := \mathcal{D}_{\mathfrak{n}_1}(.,\mathbb{S}_4)$. by 3.5.14 and the choice of $\mathfrak{b}$ lemma 3.4.17 yields

$$\left\|\mathbb{S}_4(x) - \mathcal{D}_{\mathfrak{n}_1}(x)\right\|_{\mathrm{p}} \le c\,(\mathrm{p})\,, x \in [-1,1] \qquad (3.5.26)$$

And Lemma 3.4.21 combined with 3.5.10 and 3.5.12 implies that for any proper interval $\mathcal{A}$

$$\left\|\mathbb{S}_4''(x) - \mathcal{D}_{\mathfrak{n}_1}''(x)\right\|_{\mathrm{p}} \le c_2 \left\|\frac{\pi(x)}{\Omega^{\,2}} \mathfrak{b}_\kappa(\mathbb{S}_4,\mathcal{A})\right\|_{\mathrm{L_p}(\mathcal{A})} + c_2c_5\left\|\frac{\pi(x)}{\Omega^{\,2}}\frac{\mathfrak{n}}{\mathfrak{n}_1}\left(\frac{\Omega}{dist(x,\mathrm{I}\backslash\mathcal{A})}\right)^{13(\mathbb{s}-1)}\right\|_{\mathrm{L_p}(\mathcal{A})}, x \in \mathcal{A}. \qquad (3.5.27)$$

Put $\mathfrak{n}_1 = c_5\mathfrak{n}$, and write

$$\mathrm{R}_{\mathfrak{n}} := \mathcal{D}_{\mathfrak{n}_1} + c_2\mathrm{Q}_{\mathfrak{n}} + c_2\mathcal{M}_{\mathfrak{n}}. \qquad (3.5.28)$$

By virtue of 3.5.21 , 3.5.25 , and 3.5.26 , we obtain

$$\|\mathbb{S}_4(x) - \mathrm{R}_{\mathfrak{n}}(x)\|_{\mathrm{p}} \le c\,(\mathrm{p})\,,$$

which combined with 3.5.15 , proves 3.5.3 for $\mathcal{P}_{\mathrm{n}} := \mathrm{R}_{\mathfrak{n}} + \mathfrak{r}_{\mathfrak{n}}$.

Thus, in order to conclude the proof of Theorem 3.5.1 , we should prove that 3.5.4 holds for our $\mathcal{P}_{\mathrm{n}}$.

To this end , we recall that $\mathfrak{r}_{\mathfrak{n}}$ is coconvex with $\mathbb{S}$, so that we only have to deal with $\mathrm{R}_{\mathfrak{n}}$.since 3.5.27 holds for any proper interval $\mathcal{A}$, we will prescribe different ones as needed. As long as $x \in \mathrm{F}$, it suffices to take $\mathcal{A} := \mathrm{F}_{\mathrm{p}}^{e}$, where p is such that $x \in \mathrm{F}_{\mathrm{p}}$ Then the quotient inside the big parentheses in 3.5.27 is bounded by 1 , for all $x \in \mathrm{F}$, and 3.5.16 and 3.5.27 yield

$$\left\|\mathbb{S}_4''(x) - \mathcal{D}_{\mathfrak{n}_1}''(x)\right\|_{\mathrm{p}} \le c_2(\mathrm{p})\left\|\frac{\pi(x)}{\Omega^{\,2}}\mathfrak{b}_\kappa(\mathbb{S}_4,\mathrm{F}_{\mathrm{p}}^{e})\right\|_{L_p(\mathrm{F})} + c_2c_5\left\|\frac{\pi(x)}{\Omega^{\,2}}\right\|_{L_p(\mathrm{F})}\frac{\mathfrak{n}}{\mathfrak{n}_1}$$

$$\leq 2c_2(p)\left\|\frac{\pi(x)}{\Omega^2}\right\|_{L_p(\mathrm{F})}, x \in \mathrm{F}. \qquad (3.5.29)$$

If $x \in \mathrm{F}^{2e}\backslash\mathrm{F}$, then it suffices to take $\mathcal{A} := \mathrm{F}_\mathrm{p}^{3e}$ where p is such that $x \in \mathrm{F}_\mathrm{p}^{2e}$ and , similarly, 3.5.17 and 3.5.27 , imply

$$\left\|\mathbb{S}_4''(x) - \mathcal{D}_{\mathfrak{n}_1}''(x)\right\|_\mathrm{p}$$

$$\leq c_2\,(\mathrm{p})\left\|\frac{\pi(x)}{\Omega^2}\mathfrak{b}_\kappa\left(\mathbb{S}_4, \mathrm{F}_\mathrm{p}^{3e}\right)\right\|_{L_p(\mathrm{F}^{2e})} + c_2c_5\left\|\frac{\pi(x)}{\Omega^2}\right\|_{L_p(\mathrm{F}^{2e})}\frac{\mathfrak{n}}{\mathfrak{n}_1}$$

$$\leq 2c_2c_4\,(\mathrm{p})\left\|\frac{\pi(x)}{\Omega^2}\right\|_{L_p(\mathrm{F}^{2e})}, x \in \mathrm{F}^{2e}. \qquad (3.5.30)$$

Finally , if $x \in [-1,1]\backslash\mathrm{F}^{2e}$, then we take $\mathcal{A}$ to be the connected component of $[-1,1]\backslash\mathrm{F}^{e}$, that contains x, Then by 3.5.27

$$\left\|\mathbb{S}_4''(x) - \mathcal{D}_{\mathfrak{n}_1}''(x)\right\|_\mathrm{p}$$

$$\leq c_2\left\|\frac{\pi(x)}{\Omega^2}\mathfrak{b}_\kappa(\mathbb{S}_4, \mathcal{A})\right\|_\mathrm{p}$$

$$+ c_2c_5\left\|\frac{\pi(x)}{\Omega^2}\frac{\mathfrak{n}}{\mathfrak{n}_1}\left(\frac{\Omega}{dist(x, \mathrm{I}\backslash\mathcal{A})}\right)^{25(\mathbb{s}+1)}\right\|_\mathrm{p}$$

$$= c_2(\mathrm{p})\left\|\frac{\pi(x)}{\Omega^2}\left(\frac{\Omega}{dist(x,\mathrm{I}\backslash\mathrm{F}^e)}\right)^{\mathfrak{b}_1}\right\|_\mathrm{p}, x \in [-1,1]\backslash\mathrm{F}^{2e}, \qquad (3.5.31)$$

. Since by 3.5.28 ,

$$\mathrm{R}_\mathfrak{n}''(x)\sigma(x) \geq c_2\mathrm{Q}_\mathfrak{n}''(x)\sigma(x) + c_2\mathcal{M}_\mathfrak{n}''(x)\sigma(x) + \mathbb{S}_4''(x)\sigma(x)$$
$$- \left\|\mathbb{S}_4''(x) - \mathcal{D}_{\mathfrak{n}_1}''(x)\right\|_\mathrm{p}, x \in [-1,1],$$

It follows by 3.5.20 , 3.5.22 , 3.5.12 and 3.5.29 , that

$$\mathrm{R}_\mathfrak{n}''(x)\sigma(x) \geq \frac{c_2\pi(x)}{\Omega^2}(4 - 2 + 0 - 2) = 0\,, x \in \mathcal{J}\,.$$

If $x \in \mathrm{F}\,\backslash\mathcal{J}$, then 3.5.6 is violated so that

$$\mathbb{S}_4''(x)\sigma(x) \geq \frac{5c_2}{\Omega^2} \geq \frac{5c_2}{\Omega^2}\pi(x)$$

Hence, by virtue of 3.5.19 , 3.5.22 and 3.5.29 , we get

$$R''_{n}(x)\sigma(x) \geq \frac{c_2\pi(x)\Gamma(\Omega)}{\Omega^{2}}(-1-2+5-2) = 0, x \in F\backslash\mathcal{J}.$$

Next , if $x \in F^{2e}\backslash F$, then by inequalities 3.5.18 , 3.5.23 , 3.5.12 and 3.5.30 ,

we obtain

$$R''_{n}(x)\sigma(x) \geq 0 , \qquad (3.5.32)$$

And finally, 3.5.12 , 3.5.18 , 3.5.24 and 3.5.31 imply 3.5.32

for $x \in [-1,1]\backslash F^{2e}$. Thus, 3.5.32 holds for all $x \in [-1,1]$, and so we have constructed a polynomial $\mathcal{P}_{n}$, satisfying 3.5.3 and 3.5.4 , for each $n > c$, divisible by c_3. For all other n‘s Theorem 3.5.1 follows by the inclusion $\sum_{\kappa;n}^{1}(Y_{s}) \subseteq \sum_{\kappa;c_3 n}^{1}(Y_{s})$ .This completes the proof ■

Future Work

We intend to prove direct and inverse theorem, saturation problems, negative theorems, Marched inequality for positive, piecewise positive, monotone, piecewise monotone and k-monotone, approximation using piecewise algebraic polynomials that are copositive, comonotone and co k-monotone $\mathrm{k} \geq 3$ with $\mathcal{F}$ in $\mathrm{L_p}$-quasi normed spaces for $0 < p < 1$. We intend to relax the co k-monotonicity $k = 0,1,2,3, \ldots$. In order to have degree of approximation in terms of moduli of higher order.

Conclusion

In spite of the shape preserving constraints we can approximate a piecewise convex function by a pointwise polynomial in terms of higher orders moduli of smoothness.

In chapter one we introduce some approximating properties for this approximation.

In chapter two we study the piecewise convex approximation or coconvex approximation on the quasi normed space in terms of the ordinary modulus of smoothness.

In chapter three we improve our results in chapter two, by proving Direct theorems for piecewise convex functions in quasi normed spaces using piecewise convex piecewise polynomials in terms of Ditizian Totik modulus of smoothness.

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الملخـــــــــص

في عملنا درسنا التقريب الحاصل للتحدب للدوال في الفضاءات L_p عندما $0 < p < 1$.

درس العديد من الباحثين تقريب الدوال المستمرة المحدبة باستخدام متعددات الحدود الجبرية. في عملنا قمنا بتعميم اعمالهم واستخدمنا متعددات الحدود القطعية لدراسة التقريب في الفضاءات L_p عندما $0 < p < 1$. بعدها قدمنا بعض الخواص التقريبية لمتعددات الحدود التي استخدمناها في التقريب.

قدمت مؤخرا بعض البحوث تضمنت تقريب الدوال متغيرة التحدب والمستمرة. قمنا بتطوير هذا التقريب ودرسنا تقريب الدوال متغيرة التحدب في الفضاءات L_p عندما $0 < p < 1$ باستخدام متعددات الحدود القطعية وكانت درجة التقريب الافضل بدلالة مقياس النعومة الاعتيادي.

كذلك قدمنا نوعا من نظريات جاكسون لدوال متغيرة التحدب التي تنتمي الى الفضاءات L_p عندما $0 < p < 1$ ، باستخدام متعددات الحدود القطعية متغيرة التحدب.

جمهورية العراق
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قسم الرياضيات

التقريب النقطي باستخدام متعددات الحدود

أطروحة

مقدمــــة الى مجلس كلية التربيـــة للعـــلوم الصرفة / جامعة بابـــل كجزء من متطلبات نيـــل درجــــــة الدكتوراه فلسفة في التربيــــــة / الرياضيــــات

من قبــل الطالبــــــة

اكتفاء ضياء جليل كريم

أشـــــــــــراف

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