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On i -topological Spaces Generated by Proximity Relation

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿اللَّهُ وَلِيُّ الَّذِينَ آمَنُوا يُخْرِجُهُم مِّنَ الظُّلُمَاتِ إِلَى النُّورِ وَالَّذِينَ كَفَرُوا أَوْلِيَاؤُهُمُ
الطَّاغُوتُ يُخْرِجُونَهُم مِّنَ النُّورِ إِلَى الظُّلُمَاتِ أُولَٰئِكَ أَصْحَابُ النَّارِ هُمْ فِيهَا
خَالِدُونَ﴾

صدق الله العلي العظيم

البقرة (257)

List of Symbols

| | |
|---------------------------|--|
| $I_{\phi}(x)$ | The set of all local set of the point x |
| $\phi(A)$ | The occlusion set of A |
| $Fd(A)$ | The derived set of A via the focal set |
| $Fcl(A)$ | The closure set of A via the focal set |
| $Fadh(A)$ | the set of adherent point of the set A via the focal set |
| $i - d(A)$ | The derived set of A via the i – open set. |
| $i - cl(A)$ | The closure set via i – open set |
| $i - adh(A)$ | The set of adherent points via i – open set. |
| $i - dense$ | The dense set via i – open set. |
| $FOdense$ | The dense set via focal set. |
| $FDdense$ | The focal derived dense set |
| ϕ dense | The focal occlusion dense set |
| ϕ_T dense | The occlusion dense set via i – open set. |
| \aleph_T dense | Nested dense via i – open set. |
| $\aleph_{I_{\phi}}$ dense | Focal nested dense via focal set. |
| $Fadh$ dense | Focal adherent dense. |
| $\psi_T(A)$ | ψ – Operator via i – open set. |
| $\psi_{\delta}(A)$ | ψ – Operator via focal set. |
| P_f -congested set | The congested set via focal set |
| P -congested set | The congested set via i-open set |
| $\delta_p(I, T)$ | The set of all proximity congested sets. |
| $\phi_I(x)$ | All the focal sets containig x. |
| $\delta_n(I, T)$ | The set of all proximity not congested sets. |
| $\delta_n(I, T)$ | The set of all P_n –congested sets. |

| | |
|----------------|--------------------------------------|
| $F_{fn}(T, I)$ | The set of all Pf_n –crowded sets. |
| $F_n(T, I)$ | The set of all P_n –crowded sets. |

Publications

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Abstract

This study aims at determining how we can build a new space by using the i -topological space via the proximity theory, and study of all the topological concepts that can be studied in it. Also Studying the effect of the axioms of proximity theory on the topological concepts with respect to i -topological space . And in this context first we introduced the meaning of the concept of i – topological space (X, T, I) and the proximity spaces (X, δ) . Some of topological concepts have been studied here including : i – closure, i – dense, i – interior, i – continuous, i – open function, i – closed function and i – homeomorphism .Some of properties related to these concepts used in this study were proved , the relationships between current results and past results were given .

After that a new space was built which is the i – topological proximity space which represents the first stage of this study.

The second stage of our study included defining a new set which we call the focal set and study some of the characteristics and theories related to it.

And via the focal set we defined the following concepts: focal function, focal closure and focal nested sets. Some of the characteristics and theories related to these concepts have been studied. Also a new type of function was defined by using the focal set ,it is the formatting function

In third stage we introduced the concept of density in i -topological proximity spaces , where we studied four types of them.

The fourth stage of this study included the study of how to collect points in the i -topological proximity space, and it was discussed in two different ways ,the first involves the use of classical topological conditions and we called it the proximity crowded set, which is determined by using the i – closure set and the nested set. The other form, which has been studied using properties and axioms about the proximity relationship, is called the proximity congested set . Some properties related to these two concepts have been studied.

Finally, in the fifth stage of this study, the ψ – operator was studied in its two forms, the first using the i – open set and the other form using the focal set, with the most important

properties related to it. Two types of ψ –sets are introduced and study and they are , ψ_δ – set and $F\psi$ -set .

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Introduction

There are mathematical concepts in-depth in the non-contemporary history of mathematics and it has acquired a wide horizon and resonance at the present time as a result of the development of electronics and the large number of scientific, engineering and technical problems that needed unconventional quick solutions, which stimulated researchers to search and excavate in old studies in order to invest and link them with the mathematical concepts, and among these concepts is the concept of ideal , where several problems are introduced via the ideal. In the recently years, there has been a rapidly growing interest in applying the concept of ideals in study different spaces ,and one of these spaces is the topological space. Ideal study on topological space is not a new concept today, it has been studied in twentieth century by Kuratowski ,K [20] in 1958 and Vidyanathaswamy ,R. [35] in 1960 ,after that the researchers Jankovic, Dragan, and T. R.Hamlett [18] in 1990 defined the concept of I-open sets,and then various mathematicians have applied theses concepts to topological spaces and linked it with different spaces ,some of which we mention, in 2002 ,Kozae ,Abd El-Monem [23] study the relation between some topological concepts with respect to τ and τ^* . In 2004 ,Ma,Zhihao,Junde Wu, and Shijie Lu. [27]give some algebraic application in ideal topological spaces.In 2005 ,Hatir , E. and Takashi Noiri [15] study some of topological concept with respect to semi –I-open sets in ideal topological spaces .In 2009 ,Ekici , E. and T. Noiri [11] introduced and study the notion of \star -extremally disconnected ideal topological spaces and obtained Many characterizations of this space .In 2011, Khan , M. and T. Noiri [19] introduced the notion of s^* -g-closed sets and they find that the family of all s^* -g-open sets is a topology for X which is finer than T ,also they obtained some characterizations of s^* -normality and some preservation theorems in ideal topological spaces . In 2012, Modak ,shyampada [28] introduce and study a new topologies in ideal topological space. In 2013 Al-Omari ,ahmed, and Takashi Noiri [2] introduce a new definition of local function by using the operator $\Gamma (A)(I, \tau)$ called the local closure function of A with respect to I and τ in ideal topological spaces .In 2014 , Al-Omari,wadei and Mohd Salmi Md Noorani and ahmad Al omari[7] introduced and study new classes of contra continuous function in ideal topological spaces.In 2015, Ekici ,Erdal and ozlem Elmal [10] study the notions of \mathcal{RPI} -sets,

\mathcal{RPCI} -sets and \mathcal{RCI} -sets, various decompositions in ideal spaces are established via generalized closeness .In 2015 , Janakia C. and A. Jayalalkshmb [17] studied the generalized closed set via the regular local function in ideal topological spaces .In 2016 ,Pavlović, Aleksandar [34] introduced an investigate properties of the local closure function, as a generalization of the θ -closure and the local function in ideal topological spaces.In 2017 ,Maragthavalli ,S. and D. Vinodhini [31] applied the notion of α Ig-closed sets in ideal topological spaces to present and study a new class of functions called contra α Ig-continuous function and contra α Ig-irresolute functions. In 2018, Ahmad Abdullah Al-omari Modak, Shyamapada, [1] introduce the concept of binary ideal topological space and discuss about generalized binary closed sets and generalized kernel in the same topological space. In 2019 Selim, S., Noiri, T., Modak, S., and Kumomoto-ken, Y. S.[36] defined the filter with respect to an ideal I defined on the topological space (X,T) .

One of the important concepts in ideal topological spaces is the concept of local function ,where it is indicated by A^* and defined by the set $A^*(I, \tau) = \{x \in X | U \cap A \notin I, \text{ for every } U \in \tau(x)\}$,which was invested by Natkaniec T. [32] in order to define ψ –operator as follow : $\psi: P(X) \rightarrow T$ by the complement factor for this operator and it was defined as follows $\psi(A) = X / (X / A)^*$.After that Modak Shyamapada [28] introduced the definition of ψ - set in (X, τ, I) as the set satisfy $A \subseteq \text{Int}(\text{Cl}((A)))$, and the collection of all ψ -set in (X, τ, I) is indicated by τ^ψ ,also another definition of ψ - set introduced by Modak,shyamapada and Md Monirul Islam [29] ,which is indicated by $a^{*\psi}$ -set and this type of sets satisfy that $A \subseteq (\psi(A^*))$. Some of the researches that included studies related to that operator we can see : Al-omari Ahmad, and Takashi Noiri introduce and study $\tilde{\Psi}_{\mathcal{G}}$ -sets and utilize the $\Psi_{\mathcal{G}}$ -operator and define interesting generalized open sets and study their properties in [2] ,while in [3] A. Vizary and Lazarow E. define the Ψ_I -density point and Ψ_I -density topology, and generated by it analogously to the classical I -density topology on the real line. Islam ,md Monirul and Shyamapada modak[16] study the sets in ideal topological spaces ,which defined by the operators $*$ and Ψ -operator which is called $*$ -set in [30] .

The second pillar of the study presented here is the use of the new old theory of proximity , this type of relationship was defined by Riesz,Frigeys [37] ,and rediscovered by

V.A.Efremovich[12] in the 1952 , who is sketched the concepts of proximity spaces in his "theory of enchainment", which received no further development at that time. Proximity theory has achieved of applications for various fields, directly or indirectly such as[5,6, 13,24,38].

The development of proximity spaces is growing rapidly ,where lots of researchers are interested to generalize this theory in many fields of applications like, Künzi, H-PA in 1986 [21] He studies the class of topological spaces that admit a coarsest quasi-proximity, while Cheon, Mee-Kyoung in his paper [8] in the same year shows that the quasi-proximity spaces induced by a quasi-uniformity have the same property , and a quasi-proximity is a nearness structure with the same topological closure operate . In 1994 ,Ferrer, Jesús[14] proved that every T_1 topological space with a unique compatible quasi-proximity should be hereditarily compact, and showd that it is true for product spaces as well as for locally hereditarily Lindelöf spaces. Latecki, Longin, and Frank Prokop in 1995[25] use semi-proximity spaces to establish a formal relationship between the “topological” concepts of digital image processing and their continuous counterparts in \mathbb{R}^n . In 2017 Kandil, A., S. A. El-Sheikh, and E. Said [22] introduce a new approach of proximity structure based on the grill notion . Mukherjee, M. N., D. Mandal, and Dipankar Dey in 2019 [26] introduce a kind of proximity structure, termed μ -proximity, on a set X, which ultimately gives rise to a generalized topology on the ambient set X.

This study included the construction of a new space which is called , the i-topological proximity space,by the connected between the proximty spaces and a special type of ideal topological spaces, which is the i-topological space defined by Irina [39] in 2006, which includes the fulfillment of a set of conditions on the family T defined on the space, and through a relationship defined by the ideal I .

The rest of this dissertation is organized as follows:

Chapter 1 contains three sections , in the first we provide a background on i-topological spaces ,and in the second section we study the following concepts: i-interior , i- derived ,i-adherent , i- closure sets , i-continuous , i-open , i-closed and i-homeomorphism functions .

in section three we introduce some of basic definitions and properties of the proximity spaces, with some theorems that we are needed in our study. finally in this chapter we are constructed the i -topological proximity space.

Chapter 2 involves three sections, in the first we presented an integrated study related to the topic of the focal set with its characteristics, theories and relationships. in section two we define a new set by using the focal set which is the focal nested set also as another type of this set, it is the nested set which is defined by using the i -open set. in third section of this chapter we define the concepts: focal closure, focal derived, focal adherent, set.

Chapter 3 includes a studying on a new operator which is the focal function, and for that we have two sections, in the first we define the focal function via the focal set and in section two we define it by using the i -open set.

In chapter 4 some types of density are discussed in i -topological proximity spaces, in section one we study the focal nested density and nested density sets, while in section two we define the focal density, focal derived density, focal adherent density, occlusion density set.

Chapter 5 includes the study of the effect of the proximity relationship on the nature of the crowding of points in space X , which presented as a detailed study in three sections. The first included a study on proximity focal uncongested set and proximity focal congested set. The second section included a study on proximity non congested and proximity congested set. Section three includes a study on proximity crowded set, proximity focal crowded set, proximity not always crowded and proximity focal not always crowded set.

In chapter 6 a new definition of ψ -operator introduced by using the focal set and proximity relation as follow: in section one, and by using the i -open set and proximity relation in section two. in section three we define two types of ψ -set in i -topological proximity space in the first we used the focal function concept and denoted by ψ_δ -set, and in the other type we used the focal closure concept and denoted by $F\psi$ -set.

CHAPTER ONE
REVIEW OF NECESSARY
BACKGROUND MATERIALS

1.1 Study on the i – Topological Space

In this section we will review some concepts and definitions in the i -topological space ,with some of theorems and properties related to it .

Definition (1-1-1)[20] :

Anon empty family I of subsets of a set X is called an ideal on X if and only if its satisfy the following conditions

1. If $A \in I$ and $B \subseteq A$ then $B \in I$.
2. If $A, B \in I$ then $A \cup B \in I$.
3. $\emptyset \in I$,but $X \notin I$ in general .

In our study Λ –ideal means that for any index Λ , $A_\lambda \in I$, then $\bigcup_{\lambda \in \Lambda} A_\lambda \in I$

Definition (1-1-2) [41]:

Let I be an ideal defined on a set X , and let A, B are subsets of X ,the relation α defined on X as follows:

$A \alpha B$ if and only if $A - B \in I$, also a relation \approx defined on X by the formula : $A \approx B$ if and only if $A - B \cup B - A \in I$.

Some properties of the relation α and \approx are showed in the propositions below.

Proposition (1-1-3):

Let I be any ideal defined on X and A, B, C are subsets of X , then:

1. $A \alpha X$ for each subset A of X .
2. $A \alpha \emptyset$ if and only if $A \in I$.
3. If $A \in I$ then $A \alpha B$ for each subset B of X .
4. If $C \subseteq A$ such that $A \alpha B$ then $C \alpha B$.
5. If $B \subseteq D$ such that $A \alpha B$ then $A \alpha D$.
6. If $A \alpha B_1$, and $A \alpha B_2$ then
 - i. $A \alpha B_1 \cap B_2$
 - ii. $A \alpha B_1 \cup B_2$.
7. If $A \alpha B_\lambda$ for each $\lambda \in \Lambda$ where Λ is any index , then

- i. $A \alpha \cup_{\lambda \in \Lambda} B_\lambda$ ii. $A \alpha \cap B_\lambda$, for I is Λ –ideal
8. If $A_\lambda \alpha B$ for each $\lambda \in \Lambda$, where Λ is any index then
- i. $\cap_{\lambda \in \Lambda} A_\lambda \alpha B$ ii. $\cup A_\lambda \alpha B$, I is Λ –ideal
9. $A \alpha A$ for each subset A of X .
10. If $A \alpha B$ and $B \alpha C$ then $A \alpha C$.
11. If $A_\lambda \alpha B_{\ell}$ for each λ and $\ell \in \Lambda$ then the following statements are true for I is Λ –ideal
- .
- i. $\cup A_\lambda \alpha \cap B_{\ell}$ ii. $\cap A_\lambda \alpha \cup B_{\ell}$.

Proof:

1. By definition (1-1-2) the result exist
2. Since $A \in I$ and by definition (1-1-2) we get the result
3. Since $A \cap B^c \subseteq A$ and by definition (1-1-2) $A \alpha B$
4. Since $C \subseteq A$, so $C \cap B^c \subseteq A \cap B^c$ and by defintion (1-1-2) the result exist
5. Since $B \subseteq D$, we have that $A \cap D^c \subseteq A \cap B^c$ and by definition (1-1-2) we get that $A \alpha D$
6. i. Since $A \cap B_1^c \in I$ and $A \cap B_2^c \in I$, so $A \cap (B_1 \cap B_2)^c \in I$ and by definition (1-1-1) , $A \alpha B_1 \cap B_2$
- ii. Since $B_1 \subseteq B_1 \cup B_2$, and by (5) we get that $A \alpha B_1 \cup B_2$
7. i. By (5) and since $A \alpha B_\lambda$ for each λ and because I is Λ –ideal,so by definition (1-1-2) we get that $A \alpha \cup_{\lambda \in \Lambda} B_\lambda$
- ii. Since $A \cap B_\lambda^c \in I$, for each $\lambda \in \Lambda$ so $A \cap (\cap B_\lambda)^c \in I$ and then $A \alpha \cap B_\lambda$
8. i. By (4) and since $\cap_{\lambda \in \Lambda} A_\lambda \subseteq A_\lambda$ for each λ and by definition (1-1-2) we get that $\cap_{\lambda \in \Lambda} A_\lambda \alpha B$
- ii. since $A_\lambda \cap B^c \in I$, for each $\lambda \in \Lambda$ then $(\cup A_\lambda) \cap B^c \in I$,and then $\cup A_\lambda \alpha B$, for I is Λ –ideal .
9. By definition (1-1-2) we get that result
10. Since $A \cap C^c \subseteq A \cap B^c \cup B \cap A^c$, hence $A \alpha C$

11. Since $A_\lambda \alpha B_{\ell}$ for each $\lambda \in \Lambda$ and $\ell \in \Lambda$ then $A_\lambda - B_{\ell} \in I$, so
 $\cup A_\lambda \alpha \cap B_{\ell}$.

Corollary (1-1-4):

Let I be an ideal defined on X such that $A_i \alpha B_i$, $i = 1, \dots, n$ then:

1. $\cup_{i=1}^n A_i \alpha \cup_{i=1}^n B_i$
2. $\cap_{i=1}^n A_i \alpha \cap_{i=1}^n B_i$
3. $\cap_{i=1}^n A_i \alpha \cap_{i=1}^n B_i$

Proposition (1-1-5):

Let I be an ideal defined on the space X, and let A,B,C are subset of X then:

1. $A \approx A$ for each subset A of X .
2. $A \approx \emptyset$ for each $A \in I$.
3. $A \approx X$ for each $A \subseteq X$ such that $A^c \in I$.
4. If $A \approx B$ and $B \approx C$ then $A \approx C$.
5. $A \approx B$ then $B \approx A$.
6. If $A \approx C$ and $B \approx C$ then $A \cup B \approx C$.
7. If $A_j \approx B_j$, $j \in \Lambda$ then
 - i. $\cup A_j \approx \cap B_j$ ii. $\cap A_j \approx \cup B_j$, for Λ is finite set .
8. If $A_j \approx B_\lambda$, $\lambda, j \in \Lambda$ then $\cap A_j \approx \cup B_j$, for Λ is finite set .
9. If $A_j \approx B$, for each $j \in \Lambda$ then
 - i. $\cap A_j \approx B$ ii. $\cup A_j \approx B$, for I is Λ –ideal
10. If $A \approx B_j$ for each $j \in \Lambda$ then
 - i. $A \approx \cap B_j$ ii. $A \approx \cup B_j$, for I is Λ –ideal

Proof:

2. By definition (1-1-2) and since $A \in I$ we get that $A \approx \emptyset$
3. Since $A^c \in I$ and by definition (1-1-2) we get that $A \approx X$
6. By proposition (1-1-3) (6) and by definition (1-1-2) we get that $A \cup B \approx C$.

7.

i. Since $A_j \alpha B_j$ and $B_j \alpha A_j$ so by corollary (1-1-4) we have $\cup A_j \alpha \cup B_j$ and $\cup B_j \alpha \cup A_j$, hence $\cup A_j \approx \cup B_j$.

ii. Since $A_j \alpha B_j$ and $B_j \alpha A_j$, so by corollary (1-1-4) (3) we have $\cap A_j \alpha B_j$ and $\cap B_j \alpha A_j$ from that we get $\cap A_j \approx \cap B_j$.

8. Since $A_j \alpha B_j$ and $B_j \alpha A_j$, so by corollary (1-1-4) (2) and by proposition (1-1-3) (11) we have $\cap A_j \alpha \cup B_\lambda$ and $\cup B_\lambda \alpha \cap A_j$ hence $\cap A_j \approx \cup B_\lambda$

9. i. Since $A_j \alpha B$ and $B \alpha A_j$ and by proposition (1-1-3) (8) we get that

$$\cap A_j \approx B.$$

ii. Since $A_j \alpha B$ and $B \alpha A_j$, so by proposition (1-1-3) (8) we get that $\cup A_j \approx B$

10. By proposition (1-1-3) (11) we get the result.

Definition (1-1-6) [41] :

Let I be an ideal on X , an i – topological space on X is a family T of subsets of X that check satisfies conditions :

1. $X, \emptyset \in T$.
2. For any $U \subseteq T$, there exist $W \in T$ such that $\cup U \approx W$.
3. For any $U, W \in T$, there exist $H \in T$ such that $U \cap W \approx H$.
4. $I \cap T = \{\emptyset\}$.

Then (X, T, I) is called i – topological space, and an item of T is called i – open set .

Example (1-1-7):

$(X, T_D, \{\emptyset\})$ is a discrete i – topological space, while (X, T_D, I) when we take $I = \{\emptyset, \{x\}\}, x \in X$, is not i – topological space for any space X .

Remark (1-1-8) :

1. If (X, T) is a topological space then there exist an ideal I defined on X such that (X, T, I) is i – topological space.

2. $(X, T_D, \{\emptyset\})$ is called the discrete i – topological space where T_D is the discrete topology

3. (X, T_i, I) is called the indiscrete i – topological space where T_i is the indiscrete topological and I is any ideal defined on X .

Noted that if (X, T_j, I_j) , $j = 1, 2$ are i – topological space, such that T_1, T_2 are topological spaces, then $(X, T_1 \cap T_2, I_1 \cap I_2)$ is i – topological space but $(X, T_1 \cup T_2, I_1 \cap I_2)$ is not i – topological space as in the following example .

Example (1-1-9) :

Let $X = \{a, b, c\}$ and $T_1 = \{X, \emptyset, \{a\}, \{b\}\}$, $I_1 = \{\emptyset, \{c\}\}$, $T_2 = \{X, \emptyset, \{b\}, \{c\}\}$ and $I_2 = \{\emptyset, \{a\}\}$, then clearly that $T_1 \cup T_2 = \{X, \emptyset, \{a\}, \{b\}, \{c\}\}$, $I_1 \cap I_2 = \{\emptyset\}$ so $(X, T_1 \cup T_2, I_1 \cap I_2)$ is not i – topological space because there exist a family $U = \{a, b\}$, but $\cup U \neq W$, for each i -open set W in X .

Proposition (1-1-10) :

i – topological space (X, T, I) , may be not a topological space.

Example (1-1-11) :

Let $X = \{a, b, c\}$, $T = \{\emptyset, X, \{a\}, \{b\}\}$ and $I = \{\emptyset, \{c\}\}$, then (X, T, I) is i - topological space, but not topological space , where (X, T) is not a topological space .

1.2 Some Topological concepts in i-Topological space

In this section we will study the following concepts: open set, interior set, derived set, closure set, and dense set in i – topological space.

Definition (1-2-1):

Let (X, T, I) be an i – topological space .A point $x \in A$ is called i – interior point of $A \subseteq X$ if and only if there exist i – open set H such that $x \in H \subseteq A$ and the set of all i – interior point of A is denoted by $i - \text{int} (A)$.

Proposition (1-2-2):

Let (X, T, I) be an i – topological space and let A, B are subset of X then :

1. $i - \text{int} (A) = \bigcup \{H \in T: H \subseteq A\}$.
2. If $A \subseteq B$ then $i - \text{int}(A) \subseteq i - \text{int}(B)$.
3. $i - \text{int}(A \cap B) \subseteq i - \text{int}(A) \cap i - \text{int}(B)$.
4. $i - \text{int}(A) \cup i - \text{int}(B) \subseteq i - \text{int}(A \cup B)$.
5. if $A \in T$ then $A = i - \text{int}(A)$.
6. $i - \text{int}(A) \subseteq A$.

The following example shows that $i - \text{int} (A)$ is not necessary i – open set and also if $A = i - \text{int}(A)$ then it is not necessary that $A \in T$ as in the example below .Also the converse of cases (3) and (4) are explained .

Example (1-2-3) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}, \{b\}\}$, and $I = \{\emptyset, \{c\}\}$, then if $A = \{a, b\}$ we get that $i - \text{int}(A) \notin T$, also we have $A = i - \text{int}(A)$ but $A \notin T$,and when we take $A = \{a, c\}$, $B = \{b, c\}$,then $i - \text{int}(A \cup B) = X \not\subseteq i - \text{int}(A) \cup i - \text{int}(B) = \{a, b\}$.

Example (1-2-4):

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, and $I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ then if $A = \{a, b\}$, $B = \{a, c\}$,so $i - \text{int}(A) \cap i - \text{int}(B) = \{a\} \not\subseteq i - \text{int}(A \cap B) = \emptyset$.

The following proposition shows that ideal has no effect when the case of inclusion is studied with respect to i -interior set .

Proposition (1-2-5):

Let $(X, T_j, I), j = 1,2$ be an i – topological spaces such that $T_1 \subset T_2$ then, $i - \text{int}_{T_1}(A) \subseteq i - \text{int}_{T_2}(A)$.

Proposition (1-2-6):

Let (X, T, I) be an i – topological space, then $i - \text{int}(A) = \emptyset$,for each $A \in I$.

Proof:

1. If possible that $i - \text{int}(A) \neq \emptyset$, so there exist $x \in i - \text{int}(A)$, then there exist i – open set H ,such that $x \in H \subseteq A$,and by definition (1-1-1) we get that $H \in I$ and this contradiction .

Clearly that for indiscrete i -topological space with the ideal $I = \{\emptyset, \{x\}\}$, if we take $A = \{x_i\}$, x_i belong to X ,then $i - \text{int}(A) = \emptyset$,but A doesn't belong to I .

Definition (1-2-7):

let (X, T, I) be an i – topological space and let $A \subseteq X$,then x is called i – limit point of A if and only if for each i – open set U of x , $U_x \cap A/x \neq \emptyset$.The set of all limit points of A is called the i – derived set and denoted by $i - d(A)$.

The following example shows that it is not necessary that $i - d(A) = \emptyset$, when $A \in I$.

Example (1-2-8):

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, and $I = \{\emptyset, \{b, c\}, \{b\}, \{c\}\}$,then if $A = \{b, c\}$,so $i - d(\{b, c\}) = \{a\}$ which is not empty set.

Some properties of i -derived set are discussed below.

Proposition (1-2-9):

Let (X, T, I) be an i – topological space and let A, B are subset of X then each of the following are achieved:

1. $A \subseteq B$ then $i - d(A) \subseteq i - d(B)$.
2. $i - d(A \cup B) = i - d(A) \cup i - d(B)$.
3. $i - d(A \cap B) \subseteq i - d(A) \cap i - d(B)$.
4. $i - d(A) \subseteq A$, for each i -closed set A of X .

Proof:

1. The prove exist by definition (1-2-7) and by hypothesis that $A \subseteq B$.

The prove of (2) and (3) exist by definition (1-2-7) and by (1).

4. Let $x \in i - d(A)$, so $U \cap A/x \neq \emptyset$, for each $U \in T(x)$, where $T(x) = \{U \in T : x \in U\}$, If $x \notin A$ then $x \in A^c$, and A^c is i -open set, so $A^c \cap A/x \neq \emptyset$ and this contradiction.

The converse of case (3) and (4) is not true as in the following example:

Example (1-2-10) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}, \{b\}\}$ and $I = \{\emptyset, \{c\}\}$ and if $A = \{c\}$, so $i d(A) \subseteq A$ but A is not i -closed set.

We are discussed the state of inclusion with respect to the i - derivative concept in this proposition.

Proposition (1-2-11):

Let (X, T_j, I) , $j = 1, 2$ be an i -topological spaces such that $T_1 \subset T_2$ then $i - d_{T_2}(A) \subseteq i - d_{T_1}(A)$.

Definition (1-2-12):

Let (X, T, I) be an i -topological space and let A is a subset of X , the i -closure of A is the intersection of all i -closed sets consist of A , and is denoted by $i - cl(A)$, i.e, $i - cl(A) = \bigcap \{H : H \text{ is } i\text{-closed set}, A \subseteq H\}$

Remark (1-2-13) :

1. $i - cl(A)$ is not necessary is i -closed set.
2. $A \subseteq i - cl(A)$, for each subset A of X .
3. $i - cl(i - cl(A)) = i - cl(A)$, for each subset A of X .

$$4. i - \text{int}(A) \neq X - (i - \text{cl}(A)).$$

The following example explains the case (1) above

Example (1-2-14) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}, \{b\}\}$ and $I = \{\emptyset, \{c\}\}$ then If $A = \{c\}$ we get that $i - \text{cl}(A) = \{c\}$ which is not $i - \text{closed}$ set. And if $A = \{a\}$ we get that $i - \text{int}(A) = \{a\} \neq X - (i - \text{cl}(A)) = X - \{a, c\}$.

Now below some propositions related to i -closure are discussed.

Proposition (1-2-15):

Let (X, T, I) be an $i - \text{topological}$ space then:

1. If A is $i - \text{closed}$ set then $A = i - \text{cl}(A)$ but not conversely.
2. $A \cup i - d(A) = i - \text{cl}(A)$.
3. $X - (i - \text{cl}(A)) \subseteq i - \text{cl}(X - A)$

Proof:

1. Let A be an $i - \text{closed}$ set , then by definition (1-2-12) we get that $A = i - \text{cl}(A)$.
2. Let $x \in A \cup i - d(A)$ so $x \in A$ or $x \in d(A)$;if $x \in A$ and by definition (1-2-12), then $x \in i - \text{cl}(A)$ and if $x \notin A$ and $x \in i - d(A)$, so for each $i - \text{open}$ set U of x , $U \cap A/x \neq \emptyset$ hence $U \cap A \neq \emptyset$,and by proposition (1-2-15) we get that $x \in i - \text{cl}(A)$,from this conversation we get that $A \cup id(A) \subseteq i - \text{cl}(A)$. Now let $x \in i - \text{cl}(A)$, if possible that $x \notin A \cup id(A)$ so $x \notin A$ and $x \notin id(A)$ hence $U \cap A/x = \emptyset$,for some $i - \text{open}$ set U of x and since $x \notin A$ so by proposition (1-2-15), $x \notin i - \text{cl}(A)$ and this contradiction .
3. Let $x \in X - (i - \text{cl}(A))$, then $x \notin i - \text{cl}(A)$ so by proposition (1-2-15) there exist i -open set H of x , satisfy that $H \cap A = \emptyset$,hence $H \subseteq (X - A)$, and then $x \in (X - A)$, from that and by remark (1-2-13)(2) we get that $x \in i - \text{cl}(X - A)$.

Proposition (1-2-16):

Let (X, T, I) be an $i - \text{topological}$ space ,then $a \in i - \text{cl}(A)$ if and only if $U \cap A \neq \emptyset$ for each $i - \text{open}$ set U of a .

Proof:

Let $a \in i - \text{cl}(A)$ and suppose that $U \cap A = \emptyset$, for some $U \in \mathcal{T}(a)$, hence $A \subseteq X - U$, and since $i - \text{cl}(A)$ is the intersection of all i - closed set containing A , so $i - \text{cl}(A) \subseteq X - U$, and then $a \in X - U$, and this contradiction. Conversely if possible that $a \notin i - \text{cl}(A)$, then by proposition (1-2-15)(2) $a \notin A \cup i - d(A)$, but $U \cap A \neq \emptyset$, for each i - open set U of a , and this is contradiction.

Proposition (1-2-17) :

Let (X, \mathcal{T}, I) be an i - topological space, and let A, B are subset of X , then each of the following are exist:

1. $A \subseteq B$ so $i - \text{cl}(A) \subseteq i - \text{cl}(B)$.
2. $i - \text{cl}(A \cup B) = i - \text{cl}(A) \cup i - \text{cl}(B)$.
3. $i - \text{cl}(A \cap B) \subseteq i - \text{cl}(A) \cap i - \text{cl}(B)$.

Proof:

1. The proof exist by proposition (1-2-9) and by hypothesis.
2. The proof exist by (1) and by proposition (1-2-9).
3. The proof exist by (1) and by proposition (1-2-9).

The following examples explain the converse of case (3)

Example (1-2-18):

Let $X = \{a, b, c\}$, $\mathcal{T} = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, and $I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ so If $A = \{a, c\}$, $B = \{b, c\}$ then $i - \text{cl}(A) \cap i - \text{cl}(B) = X \not\subseteq i - \text{cl}(A \cap B) = \{c\}$.

Proposition (1-2-19):

Let (X, \mathcal{T}_j, I) , $j = 1, 2$ be an i - topological spaces such that $\mathcal{T}_1 \subset \mathcal{T}_2$, then $i - \text{cl}_{\mathcal{T}_2}(A) \subseteq i - \text{cl}_{\mathcal{T}_1}(A)$

Proof:

By proposition (1-2-11) we get the result.

Proposition (1-2-20):

Let (X, T, I) be an i – topological space, then $U \cap i - cl(A) \subseteq i - cl(U \cap A)$ for each i -open set A of X .

Proof:

Let $x \in U \cap i - cl(A)$, then $x \in U$ and $x \in i - cl(A)$, if possible that $x \notin i - cl(U \cap A)$, then there exist $w \in T(x)$ such that $w \cap U \cap A = \emptyset$ and this contradiction.

Example (1-2-21):

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$ and $I = \{\emptyset, \{c\}\}$ if $U = \{a, c\}$, $A = \{b, c\}$ then $U \cap i - cl(A) = \{a, c\} \cap \{X\} = \{a, c\} \not\subseteq i - cl(U \cap A) = \{c\}$

Remark (1-2-22):

$U \cap i - cl(A) = U \cap A$, for each i -closed set A of X .

Definition (1-2-23):

Let (X, T, I) be an i – topological space and let $A \subseteq X$ then A is called i – dense set if $i - cl(A) = X$.

Example (1-2-24) :

In the indiscrete i – topological space $(X, T_i, \{\emptyset\})$, every subset of X is i – dense set .

Proposition (1-2-25) :

Let (X, T, I) be an i – topological space and let A, B are subset of X , then :

1. $A \subseteq B$ and A is i – dense set then B is i – dense set .
2. $A \cap B$ is i – dense set then A and B are i – dense set .
3. If A or B are i – dense set then $A \cup B$ is i – dense set .
4. If A is i -dense set and $U \in T$, then $U \subseteq i - cl(U \cap A)$.

We can see in example (1-2-21) if $A=\{a,b\}$, $B=\{b,c\}$, then clearly that A, B are i -dense sets but $A \cap B = \{b\}$ is not i -dense set. Also if $A=\{b\}, B=\{c\}$, then $A \cup B$ is i -dense set but not A niether B is i -dense set.

Proposition (1-2-26) :

Let (X, T, I) be an i – topological space then A is i – dense if and only if $U \cap A \neq \emptyset$, for each i – open set U .

proof :

If possible that $U \cap A = \emptyset$ for some $U \in T$, so for some $x \in X$, $x \notin i - \text{cl}(A)$ and this contradiction . Conversely, since $U \cap A \neq \emptyset$ for each $U \in T$, so for each $x \in X$ and each $U \in T(x)$, $U \cap A \neq \emptyset$ hence $i - \text{cl}(A) = X$.

In the proposition below we will discuss the inclusion with respect to the i -density concept.

Proposition (1-2-27) :

Let (X, T_j, I) , $j = 1, 2$ be an i – topological space such that $T_1 \subset T_2$ then every i_{T_2} – dense set is i_{T_1} – dense set.

Proof :

By proposition (1-2-19) and definition (1-2-23) we get the result .

Now the concept of resolvable spaces discusses below.

Definition (1-2-28) :

Let (X, T, I) be an i – topological space then a subset A of X is called i – nowhere dense set if $i - \text{int}(i - \text{cl}(A)) = \emptyset$,and if $i - \text{int}(i - \text{cl}(A)) \neq \emptyset$, we say that A is i – somewhere dense set .

The set of all i – nowhere dense set of (X, T, I) is denoted by $I_N(T, I)$.

Clearly that if A is i -dense set then A is i -somewhere dense set.

Proposition (1-2-29) :

Let (X, T, I) be an i – topological space and A, B are subset of X ,then each of the following are holds.

1. $A \subseteq B$ and B is i – nowhere dense ,then A is i – nowhere dense .
2. If A or B is i – nowhere dense, then $A \cap B$ is i – nowhere dense set.
3. If $A \cup B$ is i – nowhere dense set, then A and B are i – nowhere dense set.

Proposition (1-2-30) :

Let (X, T, I) be an i – topological space, then if A is i – nowhere dense set ,then $i - int(A) = \emptyset$.

Proof :

Since $i - int(A) \subseteq i - int(i - cl(A))$,so from that we get $i - int(A) = \emptyset$.

The following example shows that the union of two i – nowhere dense sets is not necessary i – nowhere dense set.

Example (1-2-31) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, and $I = \{\emptyset, \{c\}, \{b\}, \{b, c\}\}$.If $A = \{b\}$, $B = \{c\}$ then A and B are i – nowhere dense sets, but $A \cup B$ is not i – nowhere dense set

Proposition (1-2-32) :

If (X, T, I) is i – topological space , then $T \cap I_N(X, T) = \emptyset$.

Proof :

If possible that $T \cap I_N(X, T) \neq \emptyset$,so there exist $\emptyset \neq A \in T$,and A is i – nowhere dense set and hence $i - int(A) = \emptyset$.and by proposition (1-2-2) (5) this contradiction.

Remark (1-2-33) :

Since the union of two i -nowhere dense sets is not i -nowhere dense set then $I_N(X, T)$ is not an ideal.

Some of definitions and theorems deals with the concept of functions that we are needed in this study we are discussed below :

Definition (1-2-34) :

A mapping $f: (X, T_X, I_X) \rightarrow (Y, T_Y, I_Y)$ is called i – continuous function if and only if the inverse image of each i – open set in Y is i -open set in X .

Definition (1-2-35) :

A mapping $f: (X, T_X, I_X) \rightarrow (Y, T_Y, I_Y)$ is called i – open function if and only if the image of each i – open set in X is i – open set in Y .

Definition (1-2-36) :

A mapping $f: (X, T_X, I_X) \rightarrow (Y, T_Y, I_Y)$ is called i – closed function if and only if the image of each i – closed set in X is i – closed set in Y .

Definition (1-2-37) :

A function $f : (X, T_X, I_X) \rightarrow (Y, T_Y, I_Y)$ is called i – homeomorphism function if and only if f is bijective , i -continuous and f^{-1} is i -continuous function .

Theorem (1-2-38) :

Let $f: (X, T_X, I_X) \rightarrow (Y, T_Y, I_Y)$ is i – closed function then $f(i - cl(A)) = i - cl(f(A))$ for each i –closed set A of X .

Proof :

Let A is i – closed set then by (1-2-16)(1) $i - cl(A) = A$,and then $i - cl(A)$ is i – closed set, because f is i – closed function we get that $f(i - cl(A))$ is i – closed set, so $f(A) = f(i - cl(A))$ hence $i - cl(f(A)) = f(i - cl(A))$.

Theorem (1-2-39) :

Let $f: (X, T_X, I_X) \rightarrow (Y, T_Y, I_Y)$ be a bijective function then each of the following are equivalent:

1. f is i – homeomorphism function.
2. f is i –open and i – continuous function.
3. f is i – closed and i – continuous function.

Definition (1-2-40) [36]

A non empty subset M of $P(X)$ is said to be filter if each of the following conditions are exist:

1. if $A, B \in M$, then so is their intersection.

2. if $A \in M$, and $A \subseteq B$, then $B \in M$.
3. $\emptyset \in M$, $X \in M$.

1.3 The Proximity spaces

The definition of proximity spaces and some of properties are introduced in this section.

Definition (1-3-1) [33] :

A binary relation δ defined on the power set of X is called proximity on X if and only if it satisfies the following axioms :

1. $A\delta B$ implies $B\delta A$
2. $A \cup B\delta C$ if and only if $A\delta C$ or $B\delta C$
3. $A\delta B$ implies $A \neq \emptyset$ and $B \neq \emptyset$
4. $A \cap B \neq \emptyset$ implies $A\delta B$
5. $A\bar{\delta}B$ implies there exists a subset E of X such that $A\bar{\delta}E$ and $X - E\bar{\delta}B$.

The pair (X, δ) is called a proximity space .

Definition (1-3-2) [33]:

A proximity space δ on X is separated if it satisfies that $x\delta y$ implies $x = y$ and (X, δ) is called separated proximity space . Note that $x\delta y$ means that $\{x\}\delta\{y\}$.

Example (1-3-3)[33] :

For any space X if we defined the relation δ by $A\delta B$ if and only if $A \cap B \neq \emptyset$ for any subsets A and B of X then (X, δ) is called a discrete proximity space also (X, δ) is called separated discrete proximity space .

Example (1-3-4) :

Let X be a non empty set and δ defined as follow , $A\delta B$ if and only if $A \neq \emptyset , B \neq \emptyset$ then δ is called the trivial proximity space (X, δ) which is not separated .

Some of axioms of proximity spaces listed in the following propositions below .

Proposition (1-3-5) [33] :

Let (X, δ) be an a proximity space then each of the following are exist:

1. If $A\bar{\delta}B$,then $A\bar{\delta}C$ for each $C \subset B$

2. If $A\bar{\delta}B$, then $C\bar{\delta}B$ for each $C \subset A$
3. If $A\bar{\delta}B$, then $A \cap B = \emptyset$
4. If $U\bar{\delta}V$, then $V\bar{\delta}U$
5. If $A\bar{\delta}B$, then $\{x\}\bar{\delta}B$ for each $x \in A$
6. If $A\bar{\delta}C$ and $B\bar{\delta}C$, then $A \cup B\bar{\delta}C$

Proposition (1-3-6)[33] :

Let (X, δ) be an a proximity space then :

1. If $B\delta C$, then $A\delta C$ where $B \subseteq A$
2. If $B\delta C$, then $B\delta D$ where $C \subseteq D$
3. If $A\delta D^c$ or $D\delta B$, then $A\delta B$ for each $D \subseteq X$
4. If $A\delta B$, then $\{x\}\delta B$ for some $x \in A$ and also $A\delta B$ then $A\delta\{y\}$ for some $y \in B$.

A definition of proximity neighborhoods with some properties on it are discuss below

Definition (1-3-7)[10] :

A subset B of a proximity space (X, δ) is called δ – neighborhood of A if $A\bar{\delta}X - B$ and this is denoted by $A \ll B$.

Theorem (1-3-8)[30] :

In a proximity space (X, δ) the relation \ll has following properties :

1. $X \ll X$.
2. $\emptyset \ll A$ for any subset A of X .
3. $A \ll B$ implies $A \subset B$.
4. $A \subset B$, $B \ll C$ and $C \subseteq D$ imply $A \ll D$
5. $A \ll B_i, i = 1, \dots, n$ if and only if $A \ll \cap \{B_i : i = 1, \dots, n\}$
6. $A \ll B$ implies $(X - B) \ll (X - A)$
7. If $A \ll B$ implies, there is $C \subset X$ such that $A \ll C \ll B$
8. If δ is separated then $x \ll (X - y)$ if and only if $x \neq y$
9. If $A_i \ll B_i, i = 1, \dots, n$ then $\cup \{A_i : i = 1, \dots, n\} \ll \cap \{B_i : i = 1, \dots, n\}$, and $\cup \{A_i : i = 1, \dots, n\} \ll \cup \{B_i : i = 1, \dots, n\}$

10. $A \ll X$, for each subset A of X .

11. If $\{x\} \ll A$ then $x \in A$

12. $A \ll B$ then $\{x\} \ll B$ for all $x \in A$

Definition (1-3-9)[9]:

The proximity δ defined on a non – empty universal X is called σ – proximity if for any arbitrary family $\{\mu_\lambda ; \lambda \in \beta\}$ of subsets of X , it has the following feature $B \delta \cup_{\lambda \in \beta} \mu_\lambda$ iff $B \delta \mu_{\lambda_0}$ for some $\lambda_0 \in \beta$.

Definition (1-3-10)[9] :

A mapping $f : (X, \delta) \rightarrow (Y, \delta_Y)$ is said to be proximity or δ – continuous if $A \delta_X B$ then $f(A) \delta_Y f(B)$ for each $A, B \subseteq X$.

Definition (1-3-11)[9] :

A proximity space (X, δ) is called δ – Connected space if every δ – continuous function on X to a discrete space is constant. A subset A of X is δ – connected if it is δ – connected as a proximity subspace, and otherwise is called δ – disconnected.

Proposition (1-3-12)[9] :

In any proximity space (X, δ) the following statement are equivalent :

1. (X, δ) is δ – connected
2. $A \delta_X - A$ for each $\emptyset \neq A \neq X$
3. If $X = A \cup B$ and $A \bar{\delta} B$ then one of sets A , B is empty .

Proposition (1-3-13)[9] :

A mapping f from a proximity space (X, δ) into a proximity space (Y, δ_Y) is δ – continuous if and only if for each $P, Q \subset Y$ $P \bar{\delta}_Y Q$ implies $f^{-1}(P) \bar{\delta}_X f^{-1}(Q)$.

Corollary (1-3-14)[9] :

A mapping $f : X \rightarrow Y$ of a proximity space (X, δ) into a proximity space (Y, δ_Y) is δ – continuous if and only if $P \ll_Y Q$ implies that $f^{-1}(P) \ll_X f^{-1}(Q)$ for each , $Q \subset Y$.

Corollary (1-3-15)[9] :

Let f be a mapping from a proximity space X into a proximity space (Y, δ_Y) then $\delta_X = f^{-1}(\delta_Y)$ is the coarsest proximity on X for which f is δ – continuous mapping .

Definition (1-3-16)[9] :

If $f : (X, \delta) \rightarrow (Y, \delta_Y)$ is bijective δ – continuous mapping and $f^{-1} : (Y, \delta_Y) \rightarrow (X, \delta)$ is δ – continuous mapping then f is said to be is proximally isomorphic or δ – homeomorphism from X onto Y .

Definition (1-3-17) :

The quadruple (X, T, I, δ) is called i – topological proximity space where (X, T, I) is i – topological space and (X, δ) is a proximity space.

CHAPTER TWO
THE FOCAL SET AND ITS
APPLICATIONS IN i -
TOPOLOGICAL PROXIMITY
SPACE

2.1 Study on the Focal set

Through the characteristics and properties of ideal we were able to give a new definition to neighborhood of a certain point. We presented the most important results and their properties in this section.

Definition (2-1-1) :

1. Let (X, T, I, δ) be an i – topological proximity space, then a subset A is named a focal set of a point $x \in X$ if we have $U \in T(x)$ such that $U \alpha A$. The system of all focal sets of a point x is denoted by $I_\delta(x) = \{A \subseteq X: \exists U \in T(x), U \alpha A\}$. Noted that X is a focal set for each $x \in X$.

There are several properties to focal sets among them and then is also a relationship between them and ideal on one hand and $T(x)$ on the other hand shown by the following theorem:

Theorem (2-1-2) :

Let (X, T, I, δ) be an i – topological proximity space and let A, B are subsets of X , then each of the following properties are holds :

1. For each $A \in T(x)$, then $A \in I_\delta(x)$ and $\emptyset \notin I_\delta(x)$.
2. If $B \in I_\delta(x)$ and $B \subseteq A$, then A is a focal set of x .
3. $A, B \in I_\delta(x)$ if and only if $A \cap B \in I_\delta(x)$.
4. For each $B \in I_\delta(x)$, then we have $A \subseteq X$, such that $B \alpha A$, and $A \in I_\delta(y)$ for each $y \in A$.
5. For each $A \in I$, then $A \notin I_\delta(x)$.
6. If $A \in I_\delta(x)$, then $A^c \notin I_\delta(x)$.
7. If $A \in I$, then $A^c \in I_\delta(x)$, for each $x \in X$.
8. If $A, B \in I_\delta(x)$, then $A \cup B \in I_\delta(x)$.

Proof:

1. Let A is i -open set then the result exists by the definition, also because $T \cap I = \{\emptyset\}$ and $\emptyset^c = X$ so for each $U \in T, X \cap U = U \notin I$.

2. Exist by proposition (1-1-3) part (5) and by definition (2-1-1)
3. let $x \in X$ and $U_1, U_2 \in T(x)$ such that $U_1 \alpha B$ and $U_2 \alpha A$, since $U_1, U_2 \in T$ then we have $W \in T$ such that $U_1 \cap U_2 \approx W$ which impty that $(U_1 \cap U_2) \alpha W$ and $W \alpha (U_1 \cap U_2)$, since $U_1 \alpha B$ and $U_2 \alpha A$ and by collary (1-1-4)(3) we have $U_1 \cap U_2 \alpha A \cap B$ hence $W \alpha U_1 \cap U_2 \alpha A \cap B$ sol by proposition (1-1-3) (10)we have $W \alpha A \cap B$.

Now to prove that $x \in W$, if bearable that $W \notin T(x)$, so for each $H \in I_{\phi(x)}$ we get $W - H \notin I$, but $W \alpha U_1 \cap U_2 \subseteq U_1$, so by proposition (1-1-3)(5) we get that, $W \alpha U_1$ and $U_1 \in I_{\phi(x)}$ and this contradiction, hence x belong to W .

Conversely, let $A \cap B \in I_{\phi(x)}$ then there exist $W \in T(x)$ such that $W \alpha A \cap B$, hence $W \cap (A \cap B)^c \in I$ so, $(W \cap A^c) \cup (W \cap B^c) \in I$ and hence $W \cap A^c \in I$ and $W \cap B^c \in I$ and from that we get $A \in I_{\phi(x)}$ and $B \in I_{\phi(x)}$.

4. Let $B \in I_{\phi(x)}$ and $A \in T(x)$, such that $A \alpha B$, therefor for each $y \in A$, $A \in T(y)$, but $A \alpha A$ so $A \in I_{\phi(y)}$.
5. Suppose that $A \in I_{\phi(x)}$ so we have $U \in T(x)$ such that $U \cap A^c \in I$, but $A \in I$, then by definition (1-1-6)(4) $(U \cap A^c) \cup A \in I$ from that we get $U \in I$ and this is contradiction.
6. Let $A \in I_{\phi(x)}$, so by (2) (1) $A^c \notin I_{\phi(x)}$.
7. Let $A \in I$, if bearable that $A^c \notin I_{\phi(x)}$, then for every $U \in T(x)$, $U \cap A \notin I$ and this is contradiction because $U \cap A \subseteq A$.
8. The proof is similar to (3).

The converse of case (1),(8), (5) explain in the following examples :

Example (2-1-3) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}\}$, and $I = \{\emptyset\}$, then $I_{\phi(a)} = \{X, \{a, b\}\} = I_{\phi(b)}$,

$I_{\phi(c)} = \{X\}$, then $\{a, b\} \in I_{\phi(a)}$, but $\{a\}, \{b\} \notin I_{\phi(x)}$.

Example (2-1-4) :

In the space (X, T_i, I) , where T_i is the indiscrete topology, and $I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

then $I_{\phi(a)} = I_{\phi(b)} = I_{\phi(c)} = \{X, \{c\}, \{a, c\}, \{b, c\}\}$

so clearly that $\{c\}$ is a focal set of c but it is not i – open set .

Example (2-1-5) :

Let $\tau = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}, \{b\}\}$, and $I = \{\emptyset, \{c\}\}$,then $I_\phi (a) = \{X, \{a, b\}\} = I_\phi (b)$,
 $I_\phi (c) = \{X\}$,then clearly that $\{a, b\} \in I_\phi (a)$, but $\{a\} \subset \{a, b\}$, and $\{a\} \notin I_\phi (x)$.

The relation between the system of i -neighborhoods and the focal set is explained in this proposition .

Proposition (2-1-6) :

Let (X, T, I, δ) be an i – topological proximity space, then $N(x) \subseteq I_\phi (x)$, for each $x \in X$,where $N(x)$ is the i -neighborhood system of the point x .

Proof:

Let $A \in N(x)$ then there exist $G \in T$, such that $x \in G \subseteq A$, so $G - A \in I$ and then $A \in I_\phi (x)$

Remark (2-1-7) :

2. By example (2-1-4)we can see that if $A \in I_\phi (x)$ then it is not necessary that $x \in A$, like $\{c\}$ is a focal set of the point a but $a \notin \{c\}$.

Proposition (2-1-8) :

Let (X, T, I) be a space . Then the system of focal set constructed a filter for each x in X .

Proof :

By theorem (2-1-2)(2,3) and remark (2-1-7)(2).

Remark (2-1-9):

Let (X, T, I) be a space , then $T(x) \subseteq I_\phi (x)$ for each x in X .

The influence of the concept of embedding for ideals and embedding for topological on the focal set is discussed in the following propositions:

Proposition (2-1-10):

Let (X, T, I_j) , $j = 1, 2$ be a spaces such that $I_1 \subseteq I_2$, then $I_{1\phi}(x) \subseteq I_{2\phi}(x)$.

Proof :

Let $A \in I_{1\phi}(x)$ then we have $U \in T(x)$ such that $U \alpha A$, so $U \cap A^c \in I_1$ hence $U \cap A^c \in I_2$ and then $A \in I_{2\phi}(x)$.

Noted that in proposition (2-1-10) if we have two i-topological spaces (X, T_j, I) , $j=1, 2$ such that $T_1 \subseteq T_2$, we can get that $I_{T_1\phi}(x) \subseteq I_{T_2\phi}(x)$.

Proposition (2-1-11):

Let (X, T_j, I_j) , $j = 1, 2$ be ani – topological spaces such that T_2 is finer than T_1 and I_2 is finer than I_1 then :

- 1) $I_{\phi T_1}(x) \subseteq I_{\phi T_2}(x)$
- 2) $I_{1\phi T_1}(x) \subseteq I_{2\phi T_2}(x)$

Proof :

- 1- Let $A \in I_{1\phi T_1}(x)$ then we have $U \in T_1(x)$ such that $U \alpha A$, so $U \cap A^c \in I_1$ hence by assume $U \in T_2(x)$, $U \cap A^c \in I_2$ and then $A \in I_{2\phi}(x)$
- 2- Since $I_1 \subseteq I_2$ and by above proposition the result exists .

Definition (2-1-12) :

A function $f: (X, T, I_x) \rightarrow (Y, T_2, I_y)$ is called formatting mapping (simply F – map) if satisfy that $f(U_x) \in I_{\phi}(f(x))$, for each $x \in X$ and $U \in I_{\phi}(x)$.

From this definition we can deduce the following proposition .

Proposition (2-1-13)

If $f: (X, T, I_x) \rightarrow (Y, T_2, I_y)$ is i-homeomorphism, then $f^{-1}(U_{f(x)}) \in I_{\phi}(x)$, for each $U_{f(x)} \in I_{\phi}(f(x))$, $x \in X$.

Proof:

Let $U_y \in I_{Y\phi}(y)$ so because f is onto there exist $x \in X$ such that $y = f(x)$ also there exist $H \in T_y(y)$ satisfy that $H - U_y \in I_Y$ and because f is homeomorphism $f^{-1}(H) - f^{-1}(U_y) = f^{-1}(H - U_y) \in f^{-1}(I_Y)$, but $f^{-1}(H) \in T(x)$ since f is i -continuous hence $f^{-1}(U_y) \in I_\phi(x)$.

2.2 The Nested set

This section includes presenting and studying different forms of nested set in i -topological proximity spaces in addition to studying some of the characteristics related to them and the relationships between them.

Definition (2-2-1) :

Let (X, T, I, δ) be an i – topological proximity space then a point x in X is called a nested point of a sub set A of X if there exist a proper subset U of X such that $U \in T(x)$ and $U \delta A$ the set of all nested points of A is called a nested set and denoted by $\aleph_T(A) = \{x \in X: \exists U \neq X, U \in T(x), U \delta A\}$.

The relation between i -interior and nested set is discussed below .

Proposition (2-2-2) :

Let (X, T, I, δ) be an i – topological proximity space then $i - \text{int} (A) \subseteq \aleph_T(A)$ for each proper subset A of X

proof :

let $x \in i - \text{int} (A)$, so there exist $U \in T$,such that $x \in U \subset A$ exist $U \in T(x)$,but $x \delta A$ by definition (1-3-1)part (4), and $x \in U$ so by proposition (1-3-6)part (1) $U \delta A$,then $x \in \aleph_T(A)$

Example (2-2-3) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}, \{c\}\}$, $I = \{\emptyset, \{b\}\}$,with a trivial proximity δ defined on X , so if $A = \{b\}$, then clearly that $i - \text{int} (A) \not\subseteq \aleph_T(A) = \{a, c\}$.

Remark (2-2-4) :

1. In the i – topological proximity space (X, T, I, δ) , $i - \text{int} (X) \supseteq \aleph_T(X)$ for each proximity relation δ on X
2. $\aleph_T(X) \neq X$ in general for each i – topological proximity space X .
3. $\aleph_T(A) \neq X - i - \text{cl}(A)$.
4. $\aleph_T(A)$ is not i – open set in general .

5. $\mathfrak{N}_T(A)$ is not necessary a subset of A .

6. It is not necessary that A is i – open set if and only if $\mathfrak{N}_T(A) = A$.

7. $\mathfrak{N}_T(\emptyset) = \emptyset$.

Cases (3),(5),(6) are explained in the examples below

Example (2-2-5) :

Let $X=\{a,b,c\}$, $T = \{X, \emptyset, \{a, b\}\}$, and $I = \{\emptyset\}$,where δ is the discrete proximity , then if $A = \{a\}$,so $\mathfrak{N}_T\{a\} = \{a, b\} \not\subseteq A$.Also $\mathfrak{N}_T(A) \neq X-(i - cl \{a\}) = \emptyset$,and $\mathfrak{N}_T(X) = \{a, b\} \neq X$.

Example (2-2-6) :

Let $X = \{a, b, c\}$, $T = P(X)$, $I = \{\emptyset\}$ and δ is the discrete proximity defined on X , then if $A = \{a\}$,which is i – open set , clearly that $\mathfrak{N}_T\{a\} \neq \{a\}$.

Proposition (2-2-7) :

Let (X, T, I, δ) be an i -topological proximity space ,and let A be a subset of X ,then each of the following are exist:

1. $\mathfrak{N}_T(A) = \cup \{U \in T(x): x \in A\}$
2. $\mathfrak{N}_T(A) \in I_\delta(x)$, for some $x \in A$.

Proof :

1. Let $x \in \mathfrak{N}_T(A)$ such that $x \in A$, so $U \delta A$,for some $U \in T(x)$, hence $x \in \cup \{U \in T(x): x \in A\}$. Conversely , let $x \in \cup \{U \in T(x): x \in A\}$, then there exist $U \in T(x)$, and $x \in A$,so $U \delta A$,imply that $x \in \mathfrak{N}_T(A)$.
2. If possible that $\mathfrak{N}_T(A) \notin I_\delta(x)$, such that $x \in A$, then for each i – open set H of x , $H - \mathfrak{N}_T(A) \notin I$, hence $H - \mathfrak{N}_T(A) \neq \emptyset$, so there exist $y \in H$ and $y \notin \mathfrak{N}_T(A)$ and by definition (2-1-1) for each $w \in T(y)$, $w \bar{\delta} A$, but $H \in T(y)$, so $H \bar{\delta} A$, for each $H \in T(x)$,but by assumption we have $x \in A$,and this contradiction.

Example (2-2-8) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}, \{b\}\}$ and $I = \{\emptyset, \{c\}\}$ and δ is the discrete proximity defined on X , then if $A = \{a, c\}$, $\{a\} = \mathfrak{N}_T(A)$ then $\mathfrak{N}_T(A) \in I_\delta(a)$, but $\mathfrak{N}_T(A) \notin I_\delta(c)$.

The following proposition shows the properties of nested set .

Proposition (2-2-9) :

Let (X, T, I, δ) be an i – topological proximity space and let A, B are subsets of X then the following holds :

1. $A \subseteq B$ then $\mathfrak{N}_T(A) \subseteq \mathfrak{N}_T(B)$.
2. $\mathfrak{N}_T(A \cap B) \subseteq \mathfrak{N}_T(A) \cap \mathfrak{N}_T(B)$.
3. $\mathfrak{N}_T(A \cup B) = \mathfrak{N}_T(A) \cup \mathfrak{N}_T(B)$.
4. If $A \in T$, then $A \subseteq \mathfrak{N}_T(A)$.
5. $\mathfrak{N}_T(\mathfrak{N}_T(A)) = \mathfrak{N}_T(A)$.
6. If A is i -closed set then $(\mathfrak{N}_T(A))^c \subseteq \mathfrak{N}_T(A^c)$

Proof :

1. Let $x \in \mathfrak{N}_T(A)$, then there exist $U \in T(x)$ such that $U \delta A$ but $A \subseteq B$ then by proposition (1-3-6) we have $U \delta B$,hence $x \in \mathfrak{N}_T(B)$.
2. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$ and by (1) the result exist .
3. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$,so by (1)we have that $\mathfrak{N}_T(A) \cup \mathfrak{N}_T(B) \subseteq \mathfrak{N}_T(A \cup B)$. Now let $x \in \mathfrak{N}_T(A \cup B)$ so there exist $U \in T(x)$ such that $U \delta A \cup B$ and by definition (1-3-1) we have $U \delta A$ or $U \delta B$, hence, $x \in \mathfrak{N}_T(A) \cup \mathfrak{N}_T(B)$.
4. Let $x \in A$, and by assumption $A \in T$, so $A \in T(x)$ hence $A \delta A$, and then $x \in \mathfrak{N}_T(A)$.
5. Let $x \in \mathfrak{N}_T(\mathfrak{N}_T(A))$,so by proposition (2-2-7)(1), $x \in \cup \{U \in T(x) : x \in \mathfrak{N}_T(A)\}$, hence $x \in \mathfrak{N}_T(A)$.Now let $x \in \mathfrak{N}_T(A)$,then If possible that $x \notin \mathfrak{N}_T(\mathfrak{N}_T(A))$ hence for each $V \in T(x)$, $V \bar{\delta} \mathfrak{N}_T(A)$ so $V \cap \mathfrak{N}_T(A) = \emptyset$ and this contradiction .
6. Let $x \in (\mathfrak{N}_T(A))^c$,then $x \notin \mathfrak{N}_T(A)$, so for each i -open set H containing x $H \bar{\delta} A$, and then $x \in A^c$, and because A is i -closed set by assume then by (4) $x \in \mathfrak{N}_T(A^c)$.

The converse of cases (1), (2)and (4) are explained in the following example

Example (2-2-10) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$ and $I = \{\emptyset, \{c\}\}$, with the discrete proximity, If $B = \{c\}$, $A = \{a\}$,then $\mathfrak{N}_T(A \cap B) = \emptyset$, but $\mathfrak{N}_T(A) = X$, $\mathfrak{N}_{T(x)}(B) = \{a, c\}$, $\mathfrak{N}_T(A) \cap$

$\mathfrak{N}_T(B) = \{a, c\} \not\subseteq \mathfrak{N}_T(A \cap B)$.Also $\mathfrak{N}_T\{c\} = \{a, c\} \subseteq \mathfrak{N}_T\{a, b\} = X$,but $\{c\} \not\subseteq \{a, b\}$ and $\{b\} \subseteq \mathfrak{N}_T(\{b\}) = \{a, b\}$ but $\{b\} \notin T$

The effect of inclusion of ideals on the nested set discussed in proposition below .

Proposition (2-2-11) :

Let $(X, T_j, I, \delta), j = 1, 2$, be an i – topological proximity space such that T_2 is finer than T_1 , then $\mathfrak{N}_{T_1}(A) \subseteq \mathfrak{N}_{T_2}(A)$.

Proof :

Let $x \in \mathfrak{N}_{T_1}(A)$ so there exist $U \in T_1(x)$ such that $U \delta A$,hence there exist $U \in T_2(x), U \delta A$ then $x \in \mathfrak{N}_{T_2}(A)$.

Now a new definition of nested set via the focal set shown below

Definition (2-2-12) :

Let (X, T, I, δ) be an i – topological space a proximity space then a point $x \in X$ is called a focal nested point of $A \subseteq X$ if and only if there exist $U \in I_\phi(x)$ is a proper subset of X , $x \in U$ such that $U \delta A$.The set of all focal nested points of a set A is called a focal nested set and denoted by $\mathfrak{N}_{I_\phi}(A)$, i.e , $\mathfrak{N}_{I_\phi}(A) = \{x \in X: \exists U \in \phi_I(x), U \delta A\}$, where $\phi_I(x) = \{ U \in I_\phi(x), x \in U\}$

Example (2-2-13) :

Let $(X, T_i, \{\emptyset\})$ where T_i is the indiscrete topology and δ is the discrete proximity defined on X , so $\mathfrak{N}_{I_\phi}(A) = \emptyset$, for each subset A of X .

The relation between the two types of nested set is discussed below

Proposition (2-2-14) :

Let (X, T, I, δ) be an i – topological proximity space , then $\mathfrak{N}_T(A) \subseteq \mathfrak{N}_{I_\phi}(A)$ for each subset A of X .

Proof :

Let $x \in \mathfrak{N}_{T(x)}(A)$ so there exist $U \in T(x)$, such that $U \delta A$ and by theorem (2-1-2) (1) we have $U \in I_{\phi}(x)$, hence $x \in \mathfrak{N}_{I_{\phi}}(A)$.

Some of properties of focal nested set are show below .

Proposition (2-2-15) :

Let (X, T, I, δ) be an i – topological proximity space . Then $\mathfrak{N}_{I_{\phi}}(A) \subseteq \cup \{U \in I_{\phi}(x): U \subset X, x \in A\}$.

Proof :

Let $x \in \mathfrak{N}_{I_{\phi}}(A)$ such that $x \in A$, so there exist $U \in I_{\phi}(x), U \delta A$, hence $x \in \cup \{U \in I_{\phi}(x): U \subset X, U \delta A, x \in A\}$.

Example (2-2-16) :

Let $X = \{a, b, c\}, T = \{X, \emptyset, \{a\}, \{c\}\}, I = \{\emptyset, \{b\}\}$ and δ is the discrete Proximity. Let $A = \{c\}, \mathfrak{N}_{I_{\phi}}\{c\} = \{a, c\}$ and $\cup \{U \in I_{\phi}(x): U \delta A, x \in A\} = X \not\subseteq \mathfrak{N}_{I_{\phi}}$.

Remark (2-2-17) :

Let (X, T, I) be an i – topological proximity space , then we have the following features of focal nested sets :

1. $\mathfrak{N}_{I_{\phi}}(A) \neq X/i - cl(A)$.
2. $\mathfrak{N}_{I_{\phi}}(A)$ is not necessary i – open set ingeneral .
3. It is not necessary that $\mathfrak{N}_{I_{\phi}}(A) \subseteq A$ in general .
4. It is not necessary that A is i – open if and only if $A = \mathfrak{N}_{I_{\phi}}(A)$.

The converse of cases (2),(3),(4) of

Example (2-2-18) :

Let $X = \{a, b, c\}, T = \{X, \emptyset, \{a\}, \{c\}\}, I = \{\emptyset, \{b\}\}$ and δ is the discrete proximity defined on X , then if $A = \{a, c\}$ then $\mathfrak{N}_{I_{\phi}}(A) = \{a, c\}$ which is not i – open set, also $A = \mathfrak{N}_{I_{\phi}}(A)$ but A

is not i – open set. And clearly that when we take $B = \{a\}$ which is i – open set , $\aleph_{I_\phi}(B) = \{a, c\} \neq B$ also $\aleph_{I_\phi}(B) \not\subseteq B$.and $X - (i - \text{cl}(B)) = \{c\} \neq \aleph_{I_\phi}(B)$.

The following proposition concluding all the properties that the focal nested sets exist with respect to $I_\phi(x)$.

Proposition (2-2-19) :

Let (X, T, I, δ) be an i – topological proximity space such that $A, B \subseteq X$ then each of the following are holds :

1. If $A \subseteq B$ then $\aleph_{I_\phi}(A) \subseteq \aleph_{I_\phi}(B)$.
2. $\aleph_{I_\phi}(A \cap B) \subseteq \aleph_{I_\phi}(A) \cap \aleph_{I_\phi}(B)$.
3. $\aleph_{I_\phi}(A \cup B) = \aleph_{I_\phi}(A) \cup \aleph_{I_\phi}(B)$.
4. $\aleph_{I_\phi}(A) \subseteq \aleph_{I_\phi}(\aleph_{I_\phi}(A))$.
5. If $A \in T$, then $A \subseteq \aleph_{I_\phi}(A)$.

Proof :

1. Let $x \in \aleph_{I_\phi}(A)$, so there exist $U \in \phi_I(x)$, $U \delta A$ but $A \subseteq B$, then by proposition (1-3-6) $U \delta B$ hence $x \in \aleph_{I_\phi}(B)$.
2. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$ so by (1) the result exist .
3. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ so by (1) we get that, $\aleph_{I_\phi}(A) \cup \aleph_{I_\phi}(B) \subseteq \aleph_{I_\phi}(A \cup B)$. Now let $x \in \aleph_{I_\phi}(A \cup B)$, then there exist $U \in I_\phi(x)$, $x \in U$ and $U \delta(A \cup B)$ hence by definition (1-3-1) , we have $U \delta A$ or $U \delta B$,and from that we get $x \in \aleph_{I_\phi}(A) \cup \aleph_{I_\phi}(B)$.
4. Let $x \in \aleph_{I_\phi}(A)$, if possible that $x \notin \aleph_{I_\phi}(\aleph_{I_\phi}(A))$,then for each $U \in \phi_I(x), U \bar{\delta} \aleph_{I_\phi}(A)$, but $x \in U$ and $x \in \aleph_{I_\phi}(A)$ and this is contradiction so $x \in \aleph_{I_\phi}(\aleph_{I_\phi}(A))$.

5. Let $A \in T$, so for each $U \in T(x), x \in A, U \delta A$, hence there exist a focal set U of x such that $U \delta A$, from that we get $x \in \aleph_{I_\emptyset}(A)$.

The converse of proposition (2-2-20) part 4 is not true always

Example (2-2-20) :

Let $X = \{a, b, c\}, T = \{X, \emptyset\}, I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, and δ is the discrete proximity defined on X , so if $A = \{a\}$, then $\aleph_{I_\emptyset}(A) = \{a, c\}$, but $\aleph_{I_\emptyset}(\aleph_{I_\emptyset}(A)) = X \not\subseteq \{a, c\}$, and when we take $B = \{a, c\}$, then $\aleph_{I_\emptyset}(B) = X$, but $B \notin T$. Also if $A = \{b\}, B = \{c\}$, then $\{b, c\} = \aleph_{I_\emptyset}(B) \cap \aleph_{I_\emptyset}(A) \not\subseteq \aleph_{I_\emptyset}(B \cap A) = \emptyset$.

The effect of the inclusion of ideals on the focal nested set show below

Proposition (2-2-21) :

Let (X, T, I_j, δ) , $j = 1, 2$, be an i – topological proximity space, if I_2 is finer than I_1 then $\aleph_{I_1}(A) \subseteq \aleph_{I_2}(A)$ for each A of X .

Proof :

By proposition (2-1-10) the result exist .

Proposition (2-2-22)

In the i – topological proximity space $(X, T, \{\emptyset\}, \delta)$, $\aleph_{I_\emptyset}(A) \neq \emptyset$, for each focal proper subset A of x .

Proof:

If possible that $\aleph_{I_\emptyset}(A) = \emptyset$, so for each point x of X and focal set U of x , $U \bar{\delta} A$, and this means that $U \cap A = \emptyset$, from that we get $X - A$ is a focal set which contradiction because A is a focal set .

Proposition (2-2-23) :

Let (X, T, I) be an i – topological space and A is i – dense set, then $N_T(A) \neq \emptyset$ and then $\aleph_{I_\emptyset}(A) \neq \emptyset$.

Proof:

If possible that $\mathfrak{N}_T(A) = \emptyset$, so for each $x \in X, x \notin \mathfrak{N}_T(A)$ and by definition (2-2-1), for each i – open set U of $x, U \bar{\delta} A$, hence there is i – open set U of x such that $U \cap A = \emptyset, x \notin i - \text{cl}(A)$ and by assumption this contradiction also by proposition (2-2-15) $N_{I_\phi}(A) \neq \emptyset$.

Proposition (2-2-24) :

Let (X, T, I, δ) be an i – topological proximity space ,then each of the following are exist :

- 1- $U \cap \mathfrak{N}_T(A) \subseteq \mathfrak{N}_T(A \cap U)$, $U, A \in I_\phi(x)$.
- 2- $U \cap \mathfrak{N}_T(A) \subseteq \mathfrak{N}_{I_\phi}(A \cap U)$, $U, A \in I_\phi(x)$.

Proof :

- 1- Let $x \in U \cap \mathfrak{N}_T(A)$ then there exist $w \in T(x), w \delta A$, so if $x \notin \mathfrak{N}_T(A \cap U)$,hence for each $H \in T(x)$, we have $H \bar{\delta}(U \cap A)$, and this means that $U \cap A \cap H = \emptyset$, but $H \cap U \cap A$ is a focal set ,so by theorem (2-1-2)(1) we get a contradiction .
- 2- By (1) and by proposition (2-2-14).

2.3 On some Topological concept via the Focal set

This section includes a study of the closure concept in i -topological proximity spaces which is called focal closure set and ,also we will introduce the concept focal derived set and focal a adherent set .

Definition (2-3-1) :

Let (X, T, I) be an i – topological space and $A \subseteq X$, $x \in X$ then x is called a focal limit point of A , if and only if for each $U \in \mathcal{F}_I(x)$, $U_x \cap A/\{x\} \neq \emptyset$, and the set of all focal limit points is called the focal derived set and denoted by $Fd(A)$ also the focal clouser of the set A denoted by $Fcl(A)$ and defined by $Fcl(A) = A \cup Fd(A)$.

Example (2-3-2) :

Let $(X, T_d, \{\emptyset\})$ be an i – topological space , then $Fd(A) = \emptyset$ for each subset A of X and $Fcl(A) = A$.

Remark (2-3-3) :

Let (X, T, I) be an i – topological space , then

1. $Fcl(A)$ is not necessary i – closed set .
2. If $Fcl(A) = A$ then A is not necessary i – closed set .
3. $Fcl(A) \neq \bigcap \{H: H \text{ is } i\text{-closed set}, A \subseteq H\}$

The converse of these two cases are explained in this example

Example (2-3-4) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}\}$, $I = \{\emptyset, \{b\}\}$,then $Fcl\{a\} = \{a\}$ and $\{a\}$ is not i – closed set .Also if $A=\{b\}$, then $Fcl(A)=\{b\} \not\subseteq \bigcap \{H: H \text{ is } i\text{-closed set}, A \subseteq H\}$.

We have several properties of the focal derived set in the following proposition:

Proposition (2-3-5) :

Let (X, T, I) be an i – topological space ,and A, B are subsets of X . Then each of the following are holds:

1. If $A \subseteq B$,then $(A) \subseteq Fd(B)$.

2. $Fd(A) \cap Fd(B) \supseteq Fd(A \cap B)$.
3. $Fd(A \cup B) = Fd(A) \cup Fd(B)$.
4. $Fd(\emptyset) = \emptyset$.
5. $Fd(X) = X$.
6. $Fd(A) = \emptyset$,for each $A \in I$.
7. $U \cap Fd(A) \subseteq Fd(U \cap A)$, for each focal set U of X .

Proof :

1. Let $x \in Fd(A)$ then $U \cap A/\{x\} \neq \emptyset$ for each $U \in \mathcal{F}_I(x)$ and since $A \subseteq B$,we get $U \cap B/\{x\} \neq \emptyset$, hence $x \in Fd(B)$.
2. Since $A \cap B \subseteq A$ and B ,then by (1) we get the result .
3. Let $x \in Fd(A \cup B)$, then $U \cap (A \cup B)/\{x\} \neq \emptyset$ for each $U \in \mathcal{F}_I(x)$ and $x \in U$, hence at least $U \cap A/\{x\} \neq \emptyset$,or $U \cap B/\{x\} \neq \emptyset$, there for $x \in Fd(A)$ or $x \in Fd(B)$.
4. If possible that $Fd(\emptyset) \neq \emptyset$, then there exist $x \in Fd(\emptyset)$, hence $U \cap \emptyset/\{x\} \neq \emptyset$,and this contradiction
5. By definition (2-3-1), since we have $U \cap X/\{x\} \neq \emptyset$, for each $x \in X$ and each $U \in I_\phi(x)$, then $Fd(X) = X$.
6. Let $A \in I$ and if possible that $Fd(A) \neq \emptyset$,then there exist $x \in Fd(A)$,so by definition (2-3-1) $U \cap A/\{x\} \neq \emptyset$,for each $U \in I_\phi(x)$ and $x \in U$, now if $x \notin A$ hence $x \in A^c$, so $A^c \cap A/\{x\} \neq \emptyset$,and this contradiction , and if $x \in A$, then $U \cap A \neq \emptyset$, but $U \cap A \in I$ and this means that this intersection may be empty set and also this contradiction .
7. Let $x \in U \cap Fd(A)$, then $x \in U$ and $W \cap A/\{x\} \neq \emptyset$,for each $W \in I_{\phi(x)}$,and $x \in W$,now $x \in U \cap W$, for each $W \in I_{\phi(x)}$, and $U \cap W \in I_{\phi(x)}$,hence $(U \cap W) \cap A/\{x\} \neq \emptyset$, and then $x \in Fd(U \cap A)$.

Corollary (2-3-6) :

Let (X, T, I, δ) be an i – topological proximity space and let A, B are subset of X then each of the following are existed :

1. If $A \subseteq B$ then $Fcl(A) \subseteq Fcl(B)$.

2. $Fcl(A \cap B) \subseteq Fcl(A) \cap Fcl(B)$.
3. $Fcl(A \cup B) = Fcl(A) \cup Fcl(B)$.
4. $Fcl(A) = A$,for each $A \in I$.

Proof :

1. By proposition (2-3-5) (1) we get the result .
2. By proposition (2-3-5)(2) we get the result .
3. By proposition (2-3-5)(3) we get the result .
4. By proposition (2-3-5)(6) we get that $Fcl(A) = A$.

The converse of case (1) ,(2) of proposition (2-3-5) explain in this example

Example (2-3-7) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, $I = \{\emptyset, \{c\}\}$, clearly that $Fd\{b, c\} = \{b\} \subseteq \{b, c\} = Fd\{a\}$, but $\{b, c\} \not\subseteq \{a\}$, also $\{b, c\} \notin I$.

The relation between focal closure and nested set is shown below

Proposition (2-3-8) :

Let (X, T, I) be an i – topological space and let A is a subset of X , then $Fcl(A) = A$, for each i –closed set A of X .

Proof :

Let $x \in Fcl(A)$, so $U \cap A \neq \emptyset$, for each i -open set U of x , If possible that $x \notin A$, then $x \in A^c$, but A^c is a focal set of x then $A^c \cap A \neq \emptyset$,and this contradiction from that we get $Fcl(A) \subseteq A$,and by definition (2-3-1)we have $A \subseteq Fcl(A)$.

We can see by example (2-3-7),if $A=\{b,c\}$,then $Fcl(A)=A$, but A is not i -closed set .

Propositions below show the effect of inclusion of ideals and families with respect to focal derived set .

Proposition (2-3-9) :

Let (X, T, I_j) , $j = 1, 2$ be an i – topological space such that I_2 is finer than I_1 then

1. $Fd_{I_2}(A) \subseteq Fd_{I_1}(A)$.

2. $Fcl_{I_2}(A) \subseteq Fcl_{I_1}(A)$.

Proof :

1. By proposition (2-1-10) we get that $I_{1\phi}(x) \subseteq I_{2\phi}(x)$ hence the result exists.

2. By (1) and by proposition (2-1-10) we get the result .

Corollary (2-3-10) :

Let (X, T_j, I) be an i – topological proximity spaces , such that T_2 is finer than T_1 then

1. $Fd_{T_2}(A) \subseteq Fd_{T_1}(A)$.

2. $Fcl_{T_2}(A) \subseteq Fcl_{T_1}(A)$.

Proof :

By proposition (2-1-11) we get that $I_{\phi T_1}(x) \subseteq I_{\phi T_2}(x)$ and by proposition (2-3-9) we get the result for (1) and (2) .

The converse of proposition (2-3-10) is explained in the following example:

Example (2-3-11) :

Let $X = \{a, b, c\}$, $T_2 = \{X, \emptyset, \{a\}, \{b\}\}$, $T_1 = \{X, \emptyset, \{a\}\}$ and $I = \{\emptyset, \{c\}\}$

If $A = \{a\}$ so $Fd_{T_1}(A) = \{b, c\}$, $Fd_{T_2}(A) = \{c\}$ $Fd_1(A) \not\subseteq Fd_{T_2}(A)$ also

$Fcl_{T_1}(A) \not\subseteq Fcl_{T_2}(A) = \{a, c\}$.

The relation between focal closure and i - closure explained below

Proposition (2-3-12) :

Let (X, T, I) be an i – topological space and A is a subset of X , then

$Fcl(A) \subseteq i - cl(A)$.

Proof :

Let $p \in Fcl(A)$ then $p \in A$ or $p \in Fd(A)$ or both. One case If $p \in A$ so

$p \in i - cl(A)$. Case two if $p \in Fd(A)$ hence for each $U \in I_\phi(x)$, $x \in U$, $U \cap A/\{x\} \neq \emptyset$. Now if $x \notin i - cl(A)$ then by proposition (1-2-15) we have $U \in T(x)$ such that $U \cap A = \emptyset$ and this is contradiction.

The converse is not true as in the following example :

Example (2-3-13) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, $I = \{\emptyset, \{c\}\}$ if $A = \{b, c\}$ then $i - cl \{A\} = X \not\subseteq Fcl\{A\} = \{b, c\}$.

Proposition (2-3-14) :

Let (X, T, I) be an i – topological space .If $x \in Fcl(A)$ then $U \cap A \neq \emptyset$ for each $U \in T(x)$.

Proof :

By proposition (2-3-12) then the result is existed .

The converse of above proposition is not true always as in the example below

Example (2-3-15) :

Let $X = \{a, b, c\}$ $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, and $I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$, then $Fcl\{b, c\} = \{b, c\}$, and for a point $x = a$, then for each $U \in T(a)$, $U \cap A \neq \emptyset$, but $a \notin Fcl(A)$.

Proposition(2-3-16) :

Let (X, T, I) be an i – topological space, then $U \cap Fcl(A) \subseteq Fcl(U \cap A)$, for each $U \in T$.

Proof :

Let $x \in U \cap Fcl(A)$ then $x \in U$ and $x \in Fcl(A)$.if possible that $x \notin Fcl(U \cap A)$, hence there exist $w \in T(x)$, $w \cap (U \cap A) = \emptyset$, but $w \cap U \in \phi_I(x)$, so if $x \in A$, then $w \cap (U \cap A) \neq \emptyset$, and this contradiction, also if $x \in Fd(A)$, then $U \cap A/\{x\} \neq \emptyset$, for each $H \in \phi_I(x)$, but $w \cap U \in \phi_I(x)$, also this contradiction hence $x \in Fcl(U \cap A)$.

Converse of above proposition is not true as in the following example.

Example (2-3-17):

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, $I = \{\emptyset, \{c\}\}$, If $U = \{a, c\}$ and $A = \{b, a\}$, so $U \cap \text{Fcl}(A) = \{a, c\} \not\subseteq \text{Fcl}(U \cap A) = \{X\}$

Corollary(2-3-18)

Let (X, T, I) be an i – topological space , $U \cap \text{Fcl}(A) = \text{Fcl}(U \cap A)$, for each $A \in I$,and $U \subseteq X$.

Proof :

Let $A \in I$, so $U \cap A \subseteq A$, hence by ideal definition (1-1-1) and corollary (2-3-6) , $U \cap A \in I$, $U \cap \text{Fcl}(A) = U \cap A = \text{Fcl}(U \cap A) = U \cap A$

Proposition (2-3-19) :

Let (X, T, I) be an i -topological space ,then $\text{Fcl}(U \cap A) = \text{Fcl}(U) \cap \text{Fcl}(A)$,for each $U, A \in \phi_I(x)$.

Proof :

Let $x \in \text{Fcl}(U) \cap \text{Fcl}(A)$,then by proposition (2-3-14), for each $W \in T(x)$, $W \cap U \neq \emptyset$, $W \cap A \neq \emptyset$,so by assume and by proposition (2-1-2)(3) we get that $W \cap U \in I_{\phi(x)}$, $x \in W \cap U$, hence if possible that $(W \cap U) \cap A = \emptyset$,then $(X-A)$ is a focal set of x and this contradiction , hence $(W \cap U) \cap A \neq \emptyset$,and then $x \in \text{Fcl}(U \cap A)$. Conversely ,by proposition (2-3-6)(2) ,we have $\text{Fcl}(U \cap A) \subseteq \text{Fcl}(U) \cap \text{Fcl}(A)$.

Definition (2-3-20) :

Let (X, T, I) be an i – topological space then a point $x \in X$, is called focal adherent point of a subset A of X , if and only if for each $U \in I_{\phi}(x)$,such that $U \cap A \neq \emptyset$. The set of all focal adherent of A is denoted by $\text{Fadh}(A)$.

Example (2-3-21) :

Let $(X, T_i, \{\emptyset\})$ be an i – topological space ,then $\text{Fadh}(A) = X$, for each subset A of X .

There is relationship between the focal derived set and focal adherent set also between focal closure set as shown in the following proposition :

Proposition (2-3-22) :

Let (X, T, I) be an i – topological space and let A is a subset of X ,then each of the following are holds :

1. $Fadh(A) \subseteq Fcl(A)$.
2. If $A \in I$,then $Fadh(A) = \emptyset$.

Proof :

1. Let $x \in Fadh(A)$,hence for each focal set U of x , $U \cap A \neq \emptyset$, hence $U \cap A \neq \emptyset$ for each $U \in T(x)$, then $x \in Fcl(A)$.
2. Let $A \in I$, if possible that $Fadh(A) \neq \emptyset$,then there exist $x \in Fadh(A)$,so for each $U \in I_\phi(x)$, $U \cap A \neq \emptyset$, but $A^c \in I_\phi(x)$, hence $A^c \cap A \neq \emptyset$,and this is contradiction .

The converse of case (3) explain in the following example

Example (2-3-23) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, and $I = \{\emptyset, \{c\}\}$, if $A = \{b\}$, so $Fadh(A) = \emptyset$ but $A \notin I$.

Some properties of focal adherent sets discussed in this proposition

Proposition (2-3-24) :

Let (X, T, I, δ) be an i – topological proximity space ,and let A, B are subset of X , then each of the following are holds :

1. $A \subseteq B$,then $Fadh(A) \subseteq Fadh(B)$.
2. $Fadh(A \cap B) \subseteq Fadh(A) \cap Fadh(B)$.
3. $Fadh(A \cup B) = Fadh(A) \cup Fadh(B)$.
4. $A = Fadh(A)$, for each focal set $A \in \phi_I(x)$.

Proof :

1. Let $x \in Fadh(A)$,then $U \cap A \neq \emptyset$, for each $U \in I_\phi(x)$, and then $U \cap B \neq \emptyset$, so $x \in Fadh(B)$.
2. Since $A \cap B \subseteq A$ and B ,so by (1) we get the result .

3. Let $x \in \text{Fadh}(A \cup B)$, we get that $U \cap (A \cup B) \neq \emptyset$, for each focal set U of x , so $(U \cap A) \cup (U \cap B) \neq \emptyset$, hence $x \in \text{Fadh}(A)$ or $x \in \text{Fadh}(B)$. Conversely since A and $B \subseteq A \cup B$, and by (1) we get that $\text{Fadh}(A) \cup \text{Fadh}(B) \subseteq \text{Fadh}(A \cup B)$.
4. Let $x \in A$, then if possible $x \notin \text{Fadh}(A)$, we get that there exist a focal set H of x , satisfy that $H \cap A = \emptyset$, so by remark (2-1-7) this is contradiction. Conversely let $x \in \text{Fadh}(A)$, then by assume $x \in A$.

The converse of case (2) is shown in the following example

Example (2-3-25) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}, \{b\}\}$, $I = \{\emptyset, \{c\}\}$, with the discrete proximity. if $A = \{a\}, \{b\} = B$ then $A \cap B = \emptyset$, $\text{Fadh}(A \cap B) = \emptyset$, but $\text{Fadh}(A) = \{a, c\}$, $\text{Fadh}(B) = \{b, c\}$, $\text{Fadh}(A) \cap \text{Fadh}(B) = \{c\} \not\subseteq \emptyset$.

The inclusion of ideals and families is explained below

Proposition (2-3-26):

Let $(X, T, I_j), j = 1, 2$ be an i – topological space such that I_2 is finer than I_1 then $\text{Fadh}_{I_2}(A) \subseteq \text{Fadh}_{I_1}(A)$.

Proof :

Let $x \in \text{Fadh}_{I_2}(A)$, then for each $U \in I_2 \phi(x), U \cap A \neq \emptyset$, but by proposition (2-1-10) we get $U \in I_1 \phi(x)$ and $U \cap A \neq \emptyset$, so $x \in \text{Fadh}_{I_1}(A)$.

Proposition (2-3-27) :

Let $(X, T_j, I), j = 1, 2$ be an i – topological space such that T_2 is finer than T_1 then $\text{Fadh}_{T_1}(A) \subseteq \text{Fadh}_{T_2}(A)$.

Proposition (2-3-28) :

Let (X, T, I) be an i – topological space then $U \cap \text{Fadh}(A) \subseteq \text{Fadh}(U \cap A)$ for each $U \in T(x), x \in X$.

Proof :

Let $x \in U \cap \text{Fadh}(A)$, then $W \cap A \neq \emptyset$, for each focal set W of x , and $U \in I_\phi(x)$, if possible that $x \notin \text{Fadh}(U \cap A)$, then there exist $V \in I_\phi(x)$ satisfy $(U \cap V) \cap A = \emptyset$, but $U \cap V \in I_\phi(x)$, which contradiction and hence $x \in \text{Fadh}(U \cap A)$.

Proposition (2-3-29):

Let $f: (X, T_X, I_X) \rightarrow (Y, T_Y, I_Y)$ be an i – closed function then $f(\text{Fcl}(A)) = \text{Fcl}(f(A))$ for each i – closed set A of X .

Proof :

Let A is i – closed set ,then by proposition (2-3-8) (2) $\text{Fcl}(A) = A$, and then $\text{Fcl}(A)$ is i – closed set, and because f is i – closed function we get that $f(\text{Fcl}(A))$ is i – closed set. Now since A is i -closed ,then $f(A)$ is i -closed set ,so $\text{Fcl}(f(A))=f(A)$, but $f(\text{Fcl}(A))$ is i – closed then $f(\text{Fcl}(A)) = \text{Fcl}(f(A))$.

CHAPTER THREE

THE OCCLUSION SET IN

i – TOPOLOGICAL PROXIMITY

SPACES

3.1 The Focal Function via the Focal set

New definition to the focal function we introduced by using both proximity relation and the idea of the neighborhoods that were indicated in chapter two .

Definition (3-1-1) :

Let (X, T, I, δ) be an i – topological proximity space , then a point $x \in X$ is called occlusion point of a subset B of X if and only if for each focal set $U \in \mathcal{F}_I(x)$, $U \not\subseteq B$, the set of all occlusion point of B denoted by $\phi(B)$, also we will call that occlusion set $\phi(B)$ is a focal set ,and easily see that $\phi(\emptyset) = \emptyset$.

Below is an example illustrating the above definition .

Example (3-1-2):

In the i – topological proximity space $(X, T_i, \{\emptyset\})$ and δ is the discrete proximity defined on X , $\phi(A) = X$, for each subset A of X .

The relationship between the focal function and its set is shown below.

Proposition (3-1-3) :

In any i – topological proximity space (X, T, I, δ) , then for each $A \subseteq X$, $A \subseteq \phi(A)$.

Proof:

Let $x \in A$, if possible that $x \notin \phi(A)$ hence there exist $U \in I_\phi(x)$, $x \in U$, $U \not\subseteq A$, and this contradiction so $x \in \phi(A)$.

Basic characteristics of the focal function are demonstrated in this proposition .

Proposition (3-1-4) :

Let (X, T, I, δ) be an i – topological proximity space ,then for a subset A, B of X we have :

1. If $A \subseteq B$, then $\phi(A) \subseteq \phi(B)$.
2. $\phi(A \cap B) \subseteq \phi(A) \cap \phi(B)$.
3. $\phi(A \cup B) = \phi(A) \cup \phi(B)$.
4. $(\phi(A))^c \subseteq \phi(A^c)$.

5. $U \cap \phi(A) \subseteq \phi(U \cap A)$, for each $A \in I_\phi(x)$.
6. $\phi(A) \in I_\phi(x)$,for each $A \in I_\phi(x)$ and for some $x \in A$.

Proof :

1. Let $x \in \phi(A)$,then for each $U \in I_\phi(x)$ and $x \in U$, $U \delta A$, so by proposition (1-3-6) we have that for each $U \in I_\phi(x)$, $x \in U$, $U \delta B$,hence $x \in \phi(B)$.
2. Since $(A \cap B) \subseteq A$ and $(A \cap B) \subseteq B$, then by (1) $\phi(A \cap B) \subseteq \phi(A) \cap \phi(B)$.
3. $\phi(B \cup A) = \{x \in X: \text{for each } U \in I_\phi(x), x \in U, U \delta A \cup B\}$
 $= \{x \in X: \text{for each } U \in I_\phi(x), x \in U, U \delta A \text{ or } U \delta B\}$
 $= \{x \in X: \text{for each } U \in I_\phi(x), x \in U, U \delta A\}$
or $\{x \in X: \text{for each } U \in I_\phi(x), x \in U, U \delta B\}$
 $= \phi(A) \cup \phi(B)$.
4. Let $x \in (\phi(A))^c$, then $x \notin \phi(A)$, hence there exist a focal set W containing x $W \bar{\delta} A$,and this means that $W \cap A = \emptyset$,hence $W \subseteq A^c$, therefor $x \in A^c$, so for each $H \in I_\phi(x)$, $x \in H$, $H \cap A^c \neq \emptyset$ and then $H \delta A^c$, If possible that $x \notin \phi(A^c)$,then there exist $W \in I_\phi(x)$, $x \in W$, $W \bar{\delta} A^c$,and this contradiction so $x \in \phi(A^c)$.
5. Let $x \in U \cap \phi(A)$, then $x \in U$, and $x \in \phi(A)$,if possible that $x \notin \phi(U \cap A)$,hence by proposition (3-1-3) $x \notin U \cap A$,and this contradiction .
6. By proposition (3-1-3-) and proposition (2-1-2)(2) .

The following proposition includes some properties related to the focal function .

Proposition (3-1-5):

Let (X, T, I, δ) be an i – topological proximity space ,then for a subset A, B of X we have :

1. $\phi(\phi(B)) = \phi(B)$, for each $B \in I_\phi(x)$.
2. $\phi(B) = \text{Fcl}(\phi(B))$, for each $B \in I_\phi(x)$.
3. $\phi(B) \subseteq i - \text{cl}(\phi(B))$ for each subset B of X .

Proof:

1. Let $x \in \phi(B)$, so $U \delta B$, for each $U \in I_\phi(x)$ and $x \in U$, if possible that $x \notin \phi(\phi(B))$, then there exist $W \in \phi_I(x)$, $W \bar{\delta} \phi(B)$, hence $W \cap \phi(B) = \emptyset$, and this contradiction since $x \in W$ and $x \in \phi(B)$.

Now let $x \in \phi(\phi(B))$, if possible that $x \notin \phi(B)$, then there exist $W \in \phi_I(x)$, $W \bar{\delta} B$ and therefore we get $W \cap B = \emptyset$, from that we get $X-B$ is a focal set, but B is a focal set, so by theorem (2-1-2)(1,3) this contradiction and this means that $x \in \phi(B)$ and hence $\phi(\phi(B)) \subseteq \phi(B)$.

2. Let $x \in \text{Fcl}(\phi(B))$, so if $x \notin \phi(B)$, then there exist $U \in \phi_I(x)$, and then $U \bar{\delta} B$, hence $U \cap B = \emptyset$, and by theorem (2-1-2)(1,3) this contradiction. Conversely, exist by definition (2-3-1) we have $\phi(B) \subseteq \text{Fcl}(\phi(B))$, and then we get the result.

3. The result is existed by (3) and by proposition (2-3-12).

Relationship of i – closure and the focal function is explained below

Proposition (3-1-6) :

Let (X, T, I, δ) be an i – topological proximity space, such that $U \cap A \notin I$, for each $U \in I_\phi(x)$ and for each $x \in A$, then $\phi(A) = i - \text{cl}(\phi(A))$.

Proof :

Let $x \in i - \text{cl}(\phi(A))$ so $U \cap \phi(A) \neq \emptyset$ for each $U \in T(x)$, and if possible that $x \notin \phi(A)$, hence there exist $W \in I_\phi(x)$, $x \in W$ and $W \bar{\delta} A$, therefor $W \cap A \in I$, and this contradiction, also $\phi(A) \subseteq i - \text{cl}(\phi(A))$ then $\phi(A) = i - (\phi(A))$.

Noted that $\phi(A \cap B) \subseteq \phi(A) \subseteq \phi(A \cup B)$.

The following proposition explains the conditions necessary to achieve a state of equality.

Proposition (3-1-7) :

Let (X, T, I, δ) be an i – topological proximity space, then $\phi(A) = \phi(A - B) = \phi(A \cup B)$ where $B \in I$ and for each $A \in \phi_I(x)$, for each $x \in X$.

Proof :

First we prove that $\phi(A) = \phi(A - B)$. Let $x \in \phi(A)$ hence $U\delta A$ for each $U \in \phi_I(x)$, if possible that $x \notin \phi(A - B)$ so there exist $W \in \phi_I(x)$ and $W\bar{\delta}A - B$ hence $W \cap A \cap B^c = \emptyset$ and there for $W \cap A \subseteq B$, but $W \cap A \in I_\phi(x)$, so $B \in I_\phi(x)$ and by (2-1-2)(5) this contradiction hence $\phi(A) \subseteq \phi(A - B)$. Also by (3-1-4)(1) we have $\phi(A - B) \subseteq \phi(A)$, and then the result exists.

Now to prove $\phi(A - B) = \phi(A \cup B)$ since $A - B \subseteq A \subseteq A \cup B$, so $\phi(A - B) \subseteq \phi(A \cup B)$. Now let $x \in \phi(A \cup B)$ if possible $x \notin \phi(A - B)$ then there exist $W \in \phi_I(x)$, $W\bar{\delta}(A \cap B^c)$, then $W \cap A \cap B^c = \emptyset$, hence $W \cap A \subseteq B$, so by theorem (2-1-2)(2) $B \in I_\phi(x)$ and by the theorem (2-1-2) (5) this contradiction.

The relation of inclusion between ideal or between i – topological spaces is explained in the proposition below

Proposition (3-1-8) :

Let (X, T, I_j, δ) , $j = 1, 2$, be an i – topological proximity space then :

1. If $I_1 \subset I_2$ then $\phi_{I_2}(A) \subseteq \phi_{I_1}(A)$ for each subset A of X .
2. If $T_1 \subset T_2$ then $\phi(A)_{T_2} \subseteq \phi(A)_{T_1}$.

Proof :

1. Let $x \in \phi_{I_2}(A)$ hence $U\delta A$ for each each $U \in \phi_{I_2}(x)$, so by proposition (2-1-10) for each $U \in \phi_{I_1}(x)$, $U\delta A$ then $x \in \phi_{I_1}(A)$.
2. By proposition (2-1-10) and by (1) we get the result

Proposition (3-1-9) :

Let (X, T_j, I_j, δ) , $j = 1, 2$, be an i – topological proximity space then

1. $\phi(A)_{T_2 \cap T_1} \supseteq \phi(A)_{T_1}$ and $\phi(A)_{T_2}$
2. $\phi_{I_1 \cap I_2}(A) \supseteq \phi_{I_1}(A)$ and $\phi_{I_2}(A)$

Proof :

1. Since $T_1 \cap T_2 \subseteq T_1$ and T_2 then by proposition (3-1-8)(2) $\phi(A)_{T_1} \subseteq \phi(A)_{T_1 \cap T_2}$, and $\phi(A)_{T_2} \subseteq \phi(A)_{T_1 \cap T_2}$.
2. Since $I_1 \cap I_2 \subseteq I_1$ then by proposition (3-1-8)(1) the result exist.

The relationship of the focal function with the derived set and the closure set is discussed below.

Proposition (3-1-10) :

Let (X, T, I, δ) be an i – topological proximity space , then $Fd(A) \subseteq \phi(A)$ for each subset A of X .

Proof :

Let $x \in Fd(A)$ so $U \cap A/x \neq \emptyset$ for each $U \in \phi_I(x)$, and hence $U \delta A$ so $x \in \phi(A)$.

Example (3-1-11) :

Let $X = \{a, b, c\}$ and $T = \{X, \emptyset, \{a\}, \{b\}\}$, $I = \{\emptyset, \{c\}\}$, then if δ is the discrete proximity defined on X and let $A = \{b\}$ then $Fd(A) = \{c\} \not\subseteq \phi(\{b\}) = \{b, c\}$.

Proposition (3-1-12) :

Let (X, T, I, δ) be an i – topological proximity space ,then $Fcl(A) = \phi(A)$ for each $A \in I_\phi(x)$, $x \in X$.

Proof :

Let $x \in Fcl(A)$, so by assumption $x \in A$, if and only if $W \cap A \neq \emptyset$ for each $W \in \phi_I(x)$ if and only if $x \in \phi(A)$.

Corollary (3-1-13):

Let (X, T, I, δ) be an i - topological proximity space ,then for each $A \in I_\phi(x)$, $x \in X$ $Fadh(A) \subseteq \phi(A)$.

Proposition (3-1-14) :

Let (X, T, I, δ) be an i – topological proximity space and $Fcl(A) = X$, then $\phi(A) = X$.

Proof :

Let $Fcl(A)=X$, then if possible that $\phi(A) \neq X$,then there exist $x \in X$ such that $x \notin \phi(A)$ hence there exist $U \in I_\phi(x)$, $x \in U$, $U \bar{\delta} A$, so $U \cap A = \emptyset$ and then $x \notin Fcl(A)$ and this is contradiction so $\phi(A) = X$.

The converse is not true as in the following example.

Example (3-1-15) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}\}$ and $I = \{\emptyset, \{b\}\}$ where δ is the trivial proximity then if $A = \{b\}$ then $\phi\{b\} = X$ but $Fcl\{b\} = \{b\} \neq X$.

The relation between the sets of occlusion points and focal adherent points of a set A shows below

Proposition (3-1-16) :

Let (X, T, I, δ) be an i – topological proximity space then $Fadh(A) \subseteq \phi(A)$ for each subset A of X.

Proof :

Let $x \in Fadh(A)$ hence for each $U \in I_\phi(x)$, $U \cap A \neq \emptyset$ and then for each $U \in I_\phi(x)$, $x \in U$, $U \delta A$, from that we get $x \in \phi(A)$.

The converse is not true as in the following example

Example (3-1-17) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$ and $I = \{\emptyset, \{c\}\}$,so if δ is the discrete proximity then if $A = \{c\}$, so $\phi(\{c\}) \not\subseteq Fadh(A)$.

The following proposition explains the effect of empty focal function on the focal derived set but not conversely

Proposition (3-1-18) :

Let (X, T, I, δ) be an i – topological proximity space if $\phi(A) = \emptyset$, then $Fd(A) = \emptyset$.

Proof :

Suppose that $Fd(A) \neq \emptyset$ hence there exist, $x \in Fd(A)$, so for each $U \in I_{\phi}(x)$, $x \in U$, $U \cap A/\{x\} \neq \emptyset$ and then $U \delta A$, there for $x \in \phi(A)$ and this contradiction .

The converse is explained in the following example .

Example (3-1-19) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}, \{b\}\}$, $I = \{\emptyset, \{c\}\}$ and δ is the discrete proximity defined on X , let $A = \{c\}$ then $Fd(A) = \emptyset$ but $\phi(A) = \{c\} \neq \emptyset$.

3.2 The Focal Function with respect to i-open set

After we studying the focal function by using the concept of focal set , in this section we study the other type of focal function via the i-open set.

Definition (3-2-1) :

Let (X, T, I, δ) be an i – topological proximity space and let A is a sub set of X then the occlusion set of A with respect to $T(x)$ defined by $\phi_T(A) = \{x \in X: \text{for each } U \in T(x), U \delta A\}$.

Example (3-2-2) :

In the i – topological proximity pace $(X, T_i, \{\emptyset\}, \delta)$ where δ is the discrete proximity then $\phi_T(A) = X$, for each subset A of X .

Some of basic properties of ϕ_T are shown in this proposition

Proposition (3-2-3) :

Let (X, T, I, δ) be an i – topological proximity space .

1. $A \subseteq \phi_T(A)$.
2. $\phi(A) \subseteq \phi_T(A)$.
3. $(\phi_T(A))^c \subseteq \phi_T(A^c)$.
4. $A \subseteq B$ then $\phi_T(A) \subseteq \phi_T(B)$.
5. $\phi_T(A) \cap \phi_T(B) \supseteq \phi_T(A \cap B)$.
6. $\phi_T(A) \cup \phi_T(B) = \phi_T(A \cup B)$.
7. $\phi_T(A) = \emptyset$ if and only if $A = \emptyset$.
8. $\phi_T(\phi_T(A)) = \phi_T(A)$, for each $A \in I_\phi(x), x \in X$

Proof :

1. Let $x \in A$, so $x \in U \cap A$ for each i – open set U of x and hence $x \in \phi_T(A)$.

2. Let $x \in \phi(A)$, so for each $U \in I_{\phi(x)}$ and $x \in U, U\delta A$, hence for each $U \in T(x)$, $U\delta A$ and then $x \in \phi_T(A)$.
3. Let $x \in (\phi_T(A))^c$, then $x \notin \phi_T(A)$, hence there exist $w \in T(x), w\bar{\delta}A$ from that we get $x \in A^c$ and then $w\delta A^c$ for each $w \in T(x)$, so $x \in \phi_T(X - A)$.
4. Let $A \subseteq B$ and let $x \in \phi_T(A)$ hence for each $U \in T(x), U\delta A$ and by proposition (1-3-6) we get that $U\delta B$ hence $x \in \phi_T(B)$.
5. Since $A \cap B \subseteq A$ and B , so by (4) we get the result.
6. The proof exist by proposition (1-3-1) and by (4).
7. Let $A = \emptyset$ then $U\bar{\delta}A$, for each $U \in T(x)$ and then $\phi_T(A) = \emptyset$. Conversely, let $\phi(A) = \emptyset$ if possible that $A \neq \emptyset$ then there exist $x \in A$ and by (1) $x \in \phi_T(A)$ and this is contradiction.
8. Let $x \in \phi_T(\phi_T(A))$ then $U\delta \phi_T(A)$, for each $U \in T(x)$, if possible that $x \notin \phi_T(A)$ hence there exist $U \in T(x), U\bar{\delta}A$, so $U \cap A = \emptyset$, and U, A are focal set this is contradiction. Now by (1) since $A \subseteq \phi_T(A)$ then $\phi_T(A) \subseteq \phi_{T(x)}(\phi_{T(x)}(A))$.

Some relations related to ϕ_T are explained below :

Proposition (3-2-4) :

Let (X, T, I, δ) be an i – topological proximity space and let A be a subset of X then $\phi_T(A) \in I_{\phi(x)}$ for each $A \in I_{\phi(x)}$ and for some $x \in A$.

Proof :

Let $A \in I_{\phi(x)}$, so by proposition (3-1-4)(6) $\phi(A) \in I_{\phi(x)}$ and by proposition (3-2-3)(2) we get that $\phi_T(A) \in I_{\phi(x)}$.

The converse of cases (1), (2), (3), (4) and (5) is explain in the following example

Example (3-2-5) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, $I = \{\emptyset, \{c\}\}$, δ defined by the discrete proximity, then clearly that if $A = \{b, c\}$ then $\phi_{I_{\phi(x)}}(A) = \{b, c\}$ and $\phi_T(A) = X$, hence $\phi_T(A) \notin$

$\phi_T(A)$,also if $A = \{b\}$,then $(\phi_T(A))^c = \{a, c\}$,but $\phi_T(A^c) = X \not\subseteq (\phi_T(A))^c$, and if $A = \{a, b\}$, $B = \{b, c\}$,then clearly that $\phi_T(A) \cap \phi_T(B) \not\subseteq \phi_T(A \cap B)$.

Some of relations between i- closure ,focal closure , focal derived ,focal adherent sets discusses below

Proposition (3-2-6) :

Let (X, T, I, δ) be an i – topological proximity space , the following propositions includes some cases related to ϕ_T

1. $i - cl(A) \subseteq \phi_T(A)$ for each A of X .
2. If $\phi_T(A) = \emptyset$ then $Fd(A) = \emptyset$.
3. If $\phi_T(A) = \emptyset$ then $Fcl(A) = A$.
4. $Fd(A) \subseteq Fcl(A) \subseteq \phi_T(A)$.
5. $Fadh(A) \subseteq \phi_T(A)$.
6. If $Fcl(A) = X$, then $\phi_T(A) = X$.
7. $\phi_T(A) = \phi_T(A - B) = \phi_T(A \cup B)$ for each $B \in I$ and $A \in I_\phi(x)$, for each $x \in X$.

Proof :

1. Let $x \in i - cl(A)$,so by proposition (1-2-15) we have that $U \cap A \neq \emptyset$,for each i-open set U of x , from that we get $U\delta A$, for each $U \in T(x)$ and hence $x \in \phi_T(A)$.
2. If possible that $Fd(A) \neq \emptyset$, then there exist $x \in Fd(A)$, so $U \cap A/x \neq \emptyset$ for each $U \in I_\phi(x)$, $x \in U$ and hence $x \in \phi_T(A)$ and this contradiction ,then $Fd(A) = \emptyset$.
3. Exist by (2) and by definition (2-3-1).
4. Let $x \in Fcl(A)$, so $U \cap A \neq \emptyset$ for each $U \in T(x)$, hence $U\delta A$ for each $U \in T(x)$ and then $x \in \phi_T(A)$,and by definition (2-3-1) we get the result .
5. Let $x \in Fadh(A)$, so $U \cap A \neq \emptyset$ for each $U \in I_\phi(x)$, hence $U\delta A$ for each $U \in T(x)$ and then $x \in \phi_T(A)$.
6. By part (4) , we have $X = Fcl(A) \subseteq \phi_T(A)$, thus $\phi_T(A) = X$.

7. First we prove that $\phi_T(A) = \phi_T(A - B)$. Let $x \in \phi_T(A)$, if possible that $x \notin \phi_T(A - B)$, hence there exist $w \in T(x)$, such that $w\bar{\delta}A \cap B^c$ and then $w \cap A \cap B^c = \emptyset$, but $w \cap A \cap B^c \in I_\phi(x)$, but by theorem (2-1-2)(1,3) the empty set is not a focal set and w, A, B^c are focal set, hence we get a contradiction, also since $A - B \subseteq A$ by proposition (3-2-3) (4) we have $\phi_T(A - B) \subseteq \phi_T(A)$ and then $\phi(A) = \phi(A - B)$. Now to prove $\phi(A) = \phi_T(A \cup B)$, hence let $x \in \phi_T(A \cup B)$, if possible that $x \notin \phi_T(A)$ hence there exist $w \in T(x)$, $w\bar{\delta}A$, so $w \cap A = \emptyset$ but $w \cap A \in I_\phi(x)$ and this contradiction.

The following example explains the converse of case (6)

Example (3-2-7) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, and $I = \{\emptyset, \{c\}\}$ with the discrete proximity, if $A = \{b, c\}$ then $Fcl\{b, c\} = \{b, c\} \neq X$ where $\phi_{T(x)}(A) = X$.

Proposition (3-2-8) :

Let (X, T, I, δ) be an i – topological proximity space, then $U \cap \phi(A) \subseteq \phi_T(U \cap A)$ for each $U \in \phi_I(x)$, $x \in X$.

Proof :

Let $x \in U \cap \phi(A)$, so $x \in U$ and $w\delta A$ for each $w \in \phi_I(x)$, if possible that $x \notin \phi_T(U \cap A)$ then there exist $H \in T(x)$, such that $H\bar{\delta}U \cap A$, then $H \cap U \cap A = \emptyset$, but $U \cap H \in \phi_I(x)$, $x \in H \cap U$, hence $x \in H \cap U \cap A$ and this contradiction.

The inclusion relationship with respect to ϕ_T between i – topological space is explained below.

Proposition (3-2-9) :

Let (X, T_j, I, δ) be an i – topological proximity space such that $T_1 \subset T_2$, then $\phi_{T_2}(A) \subseteq \phi_{T_1}(A)$ for each subset A of X .

Proof :

Let $x \in \phi_{T_2}(A)$, so for each $U \in T_2(x)$, $U \delta A$ and then for each $U \in T_1(x)$, $U \delta A$, hence $x \in \phi_{T_1}(A)$.

Proposition (3-2-10) :

In any i-topological space $\phi_T(A) \neq A$, for each $A \in I$.

Example (3-2-11) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, and $I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ with the discrete proximity, if $A = \{b, c\}$ then $\phi_T(A) = X \neq A$.

CHAPTER FOUR

SOME TYPES OF DENSITY IN

***i* – TOPOLOGICAL PROXIMITY**

SPACES

4.1 On Nested density set

In this section we discuss two types of nested density, the first is nested density with respect to i -open set, and the other is the focal nested density via the focal set.

Definition (4-1-1) :

Let (X, T, I, δ) be an i -topological proximity space, we say that a subset A of X is \aleph_T dense set if and only if $\aleph_T(A) = X$.

Example (4-1-2) :

In the i – topological proximity space $(X, P(x), \{\emptyset\})$, where δ is the discrete proximity defined on the space X , we have that A is \aleph_T dense set for each subset A , and for any space X .

Some properties of nested density have been studied in this proposition.

Proposition (4-1-3) :

Let (X, T, I, δ) be an i – topological proximity space, and let A, B are subset of X then of the following are holds :

1. $A \subseteq B$ and A is \aleph_T dense set then B is \aleph_T dense set .
2. If $A \cap B$ is \aleph_T dense set then A and B are \aleph_T dense set
3. If A or B is \aleph_T dense set if and only if $A \cup B$ is \aleph_T dense set .

Proof :

1. Since $X = \aleph_T(A) \subseteq \aleph_T(B)$ then $\aleph_T(B) = X$.
2. By proposition (1-3-6) and by (1) we get the result .
3. Let A is \aleph_T dense set, so by proposition (1-3-6) and by (1) we get that $A \cup B$ is \aleph_T dense set. Conversely, if $A \cup B$ is \aleph_T dense set then for each $x \in X$, there exist $H \in T(x)$ such that $H \delta A \cup B$ and then by definition (1-3-1) $H \delta A$ or $H \delta B$, so A is \aleph_T dense or B is \aleph_T dense set .

Converse of case (2) in the above proposition is shown in this example.

Example (4-1-4) :

Let $X=\{a,b,c\}$,so for the i-topological proximity space $(X, P(x), \{\emptyset\})$,where δ is the discrete proximity defined on the space X ,we have that $\{a\}, \{b\}$ are \aleph_T dense , where $\aleph_T(\{a\} \cap \{b\}) = \emptyset$ hence $\{a\} \cap \{b\}$ is not \aleph_T dense set .

Proposition (4-1-5) :

Let (X, T, I, δ) be an i – topological proximity space . A is \aleph_T dense set if and only if $U \subseteq \aleph_T(U \cap A)$, for each $U \in T(x)$ and $U \cap A \neq \emptyset$.

Proof :

Let A is \aleph_T dense set so by proposition (2-2-24) and since $\aleph_T(A) = X$ we get the result .Conversely , if possible that $\aleph_{T(x)}(A) \neq X$, then there exist a point x of X ,such that , $x \notin \aleph_T(A)$, so for each $H \in T(x)$, $H \bar{\delta} A$ hence $H \cap A = \emptyset$ and then $A \cap U = \emptyset$,and this contradiction by assume .

Definition (4-1-6) :

Let (X, T, I, δ) be an i – topological proximity space and let A be a subset of X ,then we say that A is focal Nested dense set if and only if $\aleph_{I_\phi}(A) = X$,and it is denoted by \aleph_{I_ϕ} dense set .

Example (4-1-7) :

Let $(X, P(x), \{\emptyset\})$,where δ defined on the space X by the discrete proximity , then A is \aleph_{I_ϕ} dense set for each subset A of X .

The following property shows the relationship between the nested density and focal nested density .

Proposition (4-1-8) :

Let (X, T, I, δ) be an i – topological proximity space ,then every \aleph_T dense set is \aleph_{I_ϕ} dense set.

Proof :

By proposition (2-2-14) we have $\aleph_T(A) \subseteq \aleph_{I_\phi}(A)$, and $\aleph_T(A) = X$, then A is \aleph_{I_ϕ} dense set .

Proposition (4-1-9) :

Let (X, T, I, δ) be an i – topological proximity space and let A, B are subset of X then each of the following are satisfy :

1. If A is \aleph_{I_ϕ} dense set and $A \subseteq B$, then B is \aleph_{I_ϕ} dense set .
2. If $A \cap B$ is \aleph_{I_ϕ} dense set then A and B are \aleph_{I_ϕ} dense set .
3. If A or B is \aleph_{I_ϕ} dense set if and only if $A \cup B$ is \aleph_{I_ϕ} dense set .

Proof :

1. If possible that B is not \aleph_{I_ϕ} dense set then there exist $x \in X$ and $x \notin \aleph_{I_\phi}(B)$ and then for each focal set H containing x , $H \bar{\delta} B$ but $A \subseteq B$, so by proposition (1-3-5)(2) we get that $H \bar{\delta} A$,and this is contradiction .
2. Existed by proposition (1-3-5) and by (1).
3. If A or B is \aleph_{I_ϕ} dense set then by proposition (1-3-6) and by (1) $A \cup B$ is \aleph_{I_ϕ} dense set .

4.2 On Focal Closure ,Focal Derived and Focal Adherent density

Another kind of density has been defined in i-topological proximity spaces in this section .

Definition (4-2-1) :

Let (X, T, I, δ) be an i – topological space and let A is a subset of X then we say that A is focal derived dense set if and only if $Fd(A) = X$, and its denoted by FD dense set .

Proposition (4-2-2) :

Let (X, T, I, δ) be an i – topological proximity space .A subset A of X is FDdense set if and only if $U \subseteq Fd(U \cap A)$,for each $U \in I_{\phi(x)}$.

Proof :

Let A is FDdense set so by proposition (2-3-5)(7) and definition (4-2-1) we get the result .Conversely , if possible that A is not FDdense set then $Fd(A) \neq X$,hence there exist a point x in X ,such that $x \notin Fd(A)$, from that we get $x \notin Fd(A \cap U)$, so by assume $x \notin U$, for each focal set U of x , and this contradiction .

Proposition (4-2-3) :

Let (X, T, I, δ) be an i – topological proximity space and let A, B are subset of X then each of the following are satisfied :

1. If A is FD dense set and $A \subseteq B$, then B is FD dense set .
2. If $A \cap B$ is FD dense set then A and B are FD dense set .
3. If A or B is FD dense set if and only if $A \cup B$ is FD dense set .

Proof:

1. Since $A \subseteq B$, so by proposition (2-3-5)(1) and because $Fd(A)=X$, then $Fd(B)=X$.
2. By (1) and proposition (2-3-5)((2) we get the result .
3. Let A is FDdense set , so by proposition (2-3-5)(3) and (1) the result exist. Conversely, let $A \cup B$ is FD dense set then for each x in X ,and focal set H containing x , we have $H \cap (A \cup B) /_x \neq \emptyset$, hence there exist a point z of X such that $z \in H \cap$

$(A \cup B), x \neq z$, from that we get at lest $H \cap A /_x \neq \emptyset$,or $H \cap B /_x \neq \emptyset$, therefor
 $FD(A) = X$ or $FD(B) = X$.

Definition (4-2-4) :

Let (X, T, I, δ) be an i – topological proximity space and let $A \subseteq X$, then we say that A is focal adherent dense set and it is denoted by $Fadh$ dense if and only if $Fadh(A) = X$.

Proposition (4-2-5) :

Let (X, T, I, δ) be an i – topological proximity space and let A is $Fadh$ dense set then $U \subseteq Fadh(U \cap A)$ for each $U \in I_{\phi(x)}$

Proof :

By proposition (2-3-28) and by definition (4-2-5) we get the result

Properties of $Fadh$ density are shown below

Proposition (4-2-6) :

Let (X, T, I, δ) be an i – topological proximity space and let A, B are subset of X then each of the following are exist :

1. $A \subseteq B$ and A is $Fadh$ dense set then B is $Fadh$ dense set
2. If $A \cap B$ is $Fadh$ dense set then A and B are $Fadh$ dense set
3. If A or B are $Fadh$ dense set if and only if $A \cup B$ is $Fadh$ dense set
4. If $A \in I$ and A is not $Fadh$ dense set

Proof :

1. If possible that $Fadh(B) \neq X$, then there exist $x \notin Fadh(B)$, so there exist $U \in I_{\phi(x)}$, $U \cap B = \emptyset$,and then $U \cap A = \emptyset$,and this contradiction.
2. Since $A \cap B \subseteq A$ and B , so by (1) we get the result
3. Let A or B is $Fadh$ dense set ,also we have that A and $B \subseteq A \cup B$, hence by (1)we get the result .Conversely , let $A \cup B$ is $Fadh$ dense set then for each x in X ,and focal set H , we have $H \cap (A \cup B) \neq \emptyset$, hence there exist a point z of X such that

$z \in H \cap (A \cup B), x \neq z$, from that we get at least $H \cap A \neq \emptyset$,or $H \cap B \neq \emptyset$, therefor $\text{Fadh}(A) = X$ or $\text{Fadh}(B) = X$.

4. Since $A \in I$, so by proposition (2-1-2) $A^c \in I_{\phi(x)}$, and then by definition (2-3-20) A is not Fadh dense set .

Definition (4-2-7) :

Let (X, T, I) be an i – topological space ,then a subset A of X is called focal dense if and only if $\text{Fcl}(A) = X$,and it is denoted by FOdense .

Example (4-2-8) :

Let $(X, T_i, \{\emptyset\})$ be an i – topological space then A is FOdense for each subset A of X .

The relationship between Fadh density and FOdense are shown below .

Proposition (4-2-9) :

Let (X, T, I, δ) be an i – topological proximity space ,and let A be a subset of X , such that A is Fadh dense set ,then A is FOdense set .

Proof :

By proposition (2-3-22) and since $\text{Fcl}(A) = X$, so A is FOdense set.

A relationship between FOdense , i – density and nested set shown below:

Proposition (4-2-10) :

Let (X, T, I) be an i – topological space then each of the following are exist :

1. every FOdense subset of X is i – dense set.
2. if A is FOdense set ,then $\aleph_T(A) \neq \emptyset$,and then $\aleph_{I\phi}(A) \neq \emptyset$.

proof :

1. Let A be a FOdense set, then $\text{Fcl}(A) = X$,but by proposition (2-3-12) we have that $\text{Fcl}(A) \subseteq i\text{-cl}(A)$,hence $i\text{-cl}(A) = X$,and by definition (1-2-23) A is i – dense set .

2. By (1) A is i -dense set and by proposition (2-2-23) we get that $\aleph_T(A) \neq \emptyset$ and $\aleph_{I\phi}(A) \neq \emptyset$.

The converse of case (1) of proposition (4-2-10) is shown below.

Example (4-2-11) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$ and $I = \{\emptyset, \{c\}\}$ then if $A = \{b, c\}$ so A is i -dense but not FODense .

Properties of FODensity are shown below .

proposition (4-2-12) :

let (X, T, I) be an i -topological space ,then each of the following holds for a subsets A , B of X :

1. If $A \subseteq B$,such that A is FODense ,then B is FODense .
2. If $A \cap B$ is FODense ,then A and B are FODense .
3. If A or B are FODense if and only if $A \cup B$ is FODense

Proof :

1. Let $A \subseteq B$ and A is Fodense set ,then $Fcl(A) = X$ and by proposition (2-3-6) (1) we have that $Fcl(A) \subseteq Fcl(B)$,so $Fcl(B) = X$,hence B is FODense set .
2. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$ so by (1) we get that A and B are Fodense
3. Let A or B is FODense set ,also we have that $A \subseteq A \cup B$ and $B \subseteq A \cup B$ and by (1) we get that $A \cup B$ is Fodense .Conversely if $A \cup B$ is FODense set ,then $Fcl(A \cup B) = X$, so for each point x of X and each i -open set H of x we have $H \cap (A \cup B) \neq \emptyset$, hence at least we have $H \cap B \neq \emptyset$ or $H \cap A \neq \emptyset$, therefore A or B is FODense set .

The converse of case (2) explains in the following example

Example (4-2-13) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}\}$, $I = \{\emptyset\}$ then if $A = \{a, c\}$ then A , B are FODense but $A \cap B$ is not FODense

A relationship between FODensity , FD density , Fadh and FD density are shown below

Proposition (4-2-14) :

Let (X, T, I, δ) be an i – topological proximity space and A be a subset of X then each of the following are exist :

1. If A is FD dense set subset of X , then A is FODense set .
2. If A is FD dense set ,then A is Fadh dense set .

Proof :

1. By definition (2-3-1) we get the result .
2. By proposition (2-3-22) we get the result .

Proposition (4-2-15) :

Let (X, T, I) be an i – topological space and A is FODense set then $U \cap A \neq \emptyset$ for each $U \in T$.

Proof :

By proposition (2-3-14) and by proposition (4-2-10)(1)we have every FODense is i – dense, so from proposition (1-2-26) we get the result .

The converse is not true as in the following example .

Example (4-2-16) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$ and $I = \{\emptyset, \{c\}\}$. If $A = \{b, c\}$,then $A \cap U \neq \emptyset$, for each $U \in T$ but A is not FODense set .

Proposition (4-2-17) :

Let (X, T, I) be an i – topological space ,then $U \subseteq Fcl(U \cap A)$, for each $U \in T$ if and only if A is FODense set .

Proof :

Let A be a FODense set , then by proposition (2-3-16) we get that $U \subseteq Fcl(U \cap A)$. Conversely , suppose that $Fcl(A) \neq X$ so there exist $x \in X$ and $x \notin Fcl(A)$,hence there exist

$W \in \mathcal{T}(X)$, $W \cap A = \emptyset$, and then $\text{Fcl}(W \cap A) = \emptyset$, and by assume $W = \emptyset$, and this contradiction hence $x \in \text{Fcl}(A)$.

Proposition (4-2-18) :

Let $(X, \mathcal{T}, I_j), j = 1, 2$ be an i – topological space such that $I_1 \subseteq I_2$, then every FO_{I_2} dense is FO_{I_1} dense set .

Proof :

Let A is FO_{I_2} dense set then $\text{Fcl}_{I_2}(A) = X$ and by proposition (2-3-9) we get $\text{Fcl}_{I_1}(A) = X$ and ,hence A is FO_{I_1} dense .

Proposition (4-2-19) :

Let $(X, \mathcal{T}_j, I), j = 1, 2$, be an i – topological space such that $\mathcal{T}_1 \subset \mathcal{T}_2$, then every $\text{Fo}_{\mathcal{T}_2}$ dense set is $\text{Fo}_{\mathcal{T}_1}$ dense set .

Proof :

Let A is $\text{Fo}_{\mathcal{T}_2}$ dense set , then by proposition (2-1-10) and proposition (2-3-10)(2) we get the result .

Proposition (4-2-20) :

Let $(X, \mathcal{T}_j, I), j = 1, 2$, be an i – topological spaces, a subset A of X is $\text{FO}_{\mathcal{T}_1}$ dense set or $\text{FO}_{\mathcal{T}_2}$ dense set ,then A is $\text{FO}_{\mathcal{T}_1 \cap \mathcal{T}_2}$ dense set .

Proof :

Let A is $\text{FO}_{\mathcal{T}_1}$ dense set or $\text{FO}_{\mathcal{T}_2}$ dense set and since $\mathcal{T}_1 \cap \mathcal{T}_2 \subseteq \mathcal{T}_1$ and by (4-2-19) A is $\text{FO}_{\mathcal{T}_1 \cap \mathcal{T}_2}$ dense set .

Definition (4-2-21) :

Let $(X, \mathcal{T}, I, \delta)$ be an i -topological proximity space ,then a subset A of X is called occlusion focal density and denoted by ϕ dense if and only if $\phi(A) = X$.

Example (4-2-22) :

In the i -topological proximity space $(X, T_i, \{\emptyset\})$ every subset A of X is ϕ dense, where δ defined on the space X by the discrete proximity.

The following proposition discusses the relation between focal density and occlusion density.

Proposition (4-2-23) :

Let (X, T, I, δ) be an i -topological proximity space, then every FOdense set is ϕ dense set but not conversely.

Proof :

Let A is FOdense set, then by proposition (3-1-14) we get that A is ϕ dense.

The converse of this proposition is not true as in the following example.

Example (4-2-24) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset\}$ and $I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and δ defined on X by the trivial proximity. Let $A = \{a\}$, then $\phi(A) = X$, so A is ϕ dense but $Fcl(A) = \{a\} \neq X$.

Proposition (4-2-25) :

Let (X, T, I, δ) be an i -topological proximity space, then for a subset A, B of X each of the following are holds :

1. If $A \subseteq B$ such that A is ϕ dense set, then B is ϕ dense set
2. If $A \cap B$ is ϕ dense set then A and B are ϕ dense set
3. If A or B is ϕ dense set then $A \cup B$ is ϕ dense set

Proof :

1. By proposition (3-1-4) (1) we get the result
2. By (1) and proposition (3-1-4)(2) we get the result
3. By (1) and proposition (3-1-4)(3) we get the result

Proposition (4-2-26) :

Let (X, T, I, δ) be an i -topological proximity space ,then A is ϕ dense if and only if $U \subseteq \phi(U \cap A)$ for each $U \in I_{\phi(x)}$

Proof :

By proposition (3-1-4) (5) we have $U \subseteq \phi(U \cap A)$ for each $U \in I_{\phi(x)}$ and since $\phi(A) = X$, so $U \subseteq \phi(U \cap A)$.Conversely , let $U \subseteq \phi(U \cap A)$, for each $U \in I_{\phi(x)}$ and suppose that $\phi(A) \neq X$ then there exist $x \in X$ and $x \notin \phi(A)$, so there exist $W \in I_{\phi(x)}$, $x \in W$, $W \bar{\delta} A$, but $W \subseteq \phi(W \cap A)$, hence $W = \emptyset$, and this contradiction so $x \in \phi(A)$.

Proposition (4-2-27) :

Let $(X, T, I_j, \delta), j = 1,2$, be an i -topological proximity spaces, such that $I_1 \subset I_2$,then every ϕ_{I_2} dense if ϕ_{I_1} dense set .

Proof :

Let A is ϕ_{I_2} dense set , hence $\phi_{I_2}(A) = X$ and by proposition (3-1-8)

$\phi_{I_2}(A) = X$ and then A is ϕ_{I_1} dense set .

Proposition (4-2-28) :

Let $(X, T_j, I, \delta), j = 1,2$, be an i -topological proximity spaces , such that $T_1 \subset T_2$,then every ϕ_{T_2} dense set is ϕ_{T_1} dense set .

Proof :

Let A is ϕ_{T_2} dense set , hence $\phi_{T_2}(A) = X$, and by proposition (3-1-8)(2) we get the result .

Proposition (4-2-29) :

in any i -topological proximity spaces $(X, T_j, I, \delta) , j=1,2$, If A is $\phi_{T_1 \text{ or } T_2}$ dense set ,then A is $\phi_{T_1 \cap T_2}$ dense set .

Proof :

Let A is ϕ_{T_1} dense set or ϕ_{T_2} dense set and since $T_1 \cap T_2 \subseteq T_1$ and T_2 , then by proposition (4-2-28) we get that A is $\phi_{T_1 \cap T_2}$ dense set

Proposition (4-2-30) :

Let (X, T, I_j, δ) , $j = 1, 2$, be an i -topological proximity space, and let A is ϕ_{I_1} dense or ϕ_{I_2} dense set then A is $\phi_{I_1 \cap I_2}$ dense.

Proof :

since $I_1 \cap I_2 \subseteq I_1$ and I_2 then by proposition (4-2-27) we get that result.

Definition (4-2-31) :

Let (X, T, I, δ) be an i -topological proximity space then we say that a subset A of X is ϕ_T dense set if and only if $\phi_T(A) = X$.

Proposition (4-2-32) :

Let (X, T, I, δ) be an i -topological proximity space, then every ϕ dense set is ϕ_T dense set.

Proof :

By proposition (3-2-3) (2) we get the result

Proposition (4-2-33) :

Let (X, T, I, δ) be an i -topological proximity space and A is a ϕ dense subset of X , then $U \subseteq \phi_T(U \cap A)$, for each $U \in T$.

Proof :

By proposition (3-2-8) and by Definition (4-2-21) we get the result.

Example (4-2-34) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}, \{b\}\}$ and $I = \{\emptyset, \{c\}\}$. If $U = \{a\}$ and $A = \{a, c\}$, $\phi(A) = \{a, c\}$, then clearly that $\{a\} = U \subseteq \phi_T(U \cap A) = \{a, c\}$, but A is not ϕ dense set. where δ defined by the discrete proximity on X

Proposition (4-2-35) :

In any i -topological proximity space, if A is FOdense set, then A is ϕ_T dense set.

Proof :

If possible that $x \in X$ and $x \notin \phi_T(A)$ by proposition (3-2-3) $x \notin \phi(A)$ and by proposition (4-2-23) we get a contradiction, hence $x \in \phi_T(A)$.

Proposition (4-2-36) :

In any i -topological proximity space, if A is i -dense set then, A is ϕ_T dense set

Proof :

If possible $x \in X$ and $x \notin \phi_T(A)$, so there exist $W \in T(x)$, $W \bar{\delta} A$ and hence $W \cap A = \emptyset$, so $x \notin i-cl(A)$ and this contradiction.

The following example explains the converse of proposition (4-2-35) and proposition (4-2-32)

Example (4-2-37) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$ and $I = \{\emptyset, \{c\}\}$ then if $A = \{b, c\}$ then A is ϕ_T dense set but not ϕ dense set also A is not FOdense set where δ defined by the discrete proximity on X .

The following example explains the converse of proposition (4-2-36).

Example (4-2-38) :

In above example if $A \delta B$, if and only if $A \neq \emptyset$, $B \neq \emptyset$, and if $A = \{b\}$ then A is ϕ_T dense set but not i -dense set.

CHAPTER FIVE
CONGESTED AND CROWDED
SETS IN i – TOPOLOGICAL
PROXIMITY SPACES

5.1 On the Proximity Congested set

This section includes a study related to the concept of proximity congested sets and proximity not congested sets in i -topological proximity spaces which were defined using two concepts the i -closure concept and nested set and then utilized it in studying a new type of concept namely CPC- congested set, CP- congested set, PC- congested set. Some of the characteristics and theories of these new sets were presented .

Definition (5-1-1):

Let (X, T, I, δ) be an i – topological proximity space then a subset A of X is called

1. Proximity not congested set (simply, P_n – congested) if $\mathfrak{N}_T(A) \bar{\delta}i - cl(A)$ and the collection of all P_n – congested set is denoted by $\delta_n(I, T)$.
2. If $\mathfrak{N}_T(A) \delta i - cl(A)$ we say that A is proximity congested (simply, P – congested) set. The set of all proximity congested sets on X is denoted by $\delta_p(I, T)$.

Remark (5-1-2) :

1. By definition (1-3-1)(3) we get that \emptyset is P_n – congested set but not P – congested.
2. The universal non empty set X is P – congested.(except for the indiscrete i -topological proximity space)
3. By definition (1-2-23) and proposition (2-2-23) every i – dense set is P – congested set.Also by definition (4-1-1) every \mathfrak{N}_T -dense set is P -congested set ,but the converse is not true except for the indiscrete topology as in the following example.

Example (5-1-3):

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}, \{c\}\}$ and $I = \{\emptyset, \{b\}\}$ with δ defined by the discrete proximity , so for $A = \{a\}$ we have that A is P – congested set but not i – dense set.

Remark (5-1-4) :

1. Clearly that by properties of proximity space any subset A of X such that $\aleph_T(A) = \emptyset$ is P_n – congested set.
2. By proposition (1-3-5) we have that the union of two P_n – congested set is not P_n – congested set , but by proposition (1-3-6) the union of p -congested is p -congested .
3. By (2) $\delta_n(I, T)$ is not an ideal on the other hand by remark (5-1-2)(1) $\delta_p(I, T)$ is also not an ideal .

In the following proposition we will list the most important features of the P_n – congested set.

Proposition (5-1-5) :

For any i – topological proximity space (X, T, I, δ) and any subset A, B of X the following properties are achieved:

1. If $A \subseteq B$ and B is P_n – congested set then A is P_n – congested set.
2. $A \cup B$ is P_n – congested set then A, B are P_n – congested set.
3. If A or B is P_n – congested set then $A \cap B$ is P_n – congested set.

Proof :

1. Let B is P_n – congested set then $\aleph_T(B) \bar{\delta}i = cl(B)$ and by proposition (2-2-9) $\aleph_T(A) \subseteq \aleph_T(B)$ and then by proposition (1-3-5) (3) $\aleph_T(A) \bar{\delta}i = cl(B)$, and then $\aleph_T \bar{\delta}i = cl(A)$, so A is P_n – congested.
2. Since A and B are subset of $A \cup B$, then by (1) A and B are P_n – congested set.
3. By (1) and since $A \cap B$ is a subset of A and B we get the result.

The converse of above proposition is explained in the example below.

Example (5-1-6) :

By example (5-1-3) for $A = \{b\}$, so A is P_n – congested set , but $B = \{b, c\}$ or $B = \{a, b\}$ where $A \subseteq B$, B is not P_n – congested.

Remark (5-1-7) :

$\delta_n(I, T)$ and T have a disjoint intersection since if A is a non empty i – open set such that $A \in \delta_n(I, T)$, so by proposition (2-2-9)(4) we have $A \subseteq \aleph_T(A)$, and by definition (5-1-1), $\aleph_T(A) \bar{\delta}i - \text{cl}(A)$ so $A \bar{\delta}i - \text{cl}(A)$ and this contradiction.

In a similar way the following proposition discusses the cases of proposition (5-1-5) via the P -congested set .

Proposition (5-1-8) :

Let (X, T, I, δ) be an i – topological proximity space and A, B are subset of X then

1. If $A \subseteq B$ and A is P – congested set ,then B is P – congested set.
2. If $A \cap B$ is P – congested set ,then A and B are P – congested.
3. If A or B is P – congested set ,then $A \cup B$ is P – congested set.

Proof :

1. Since A is P – congested set so $\aleph_T(A) \bar{\delta}i - \text{cl}(A)$ and because $A \subseteq B$,then by proposition (2-2-9) we there $\aleph_T(A) \subseteq \aleph_T(B)$, hence by proximity axioms $\aleph_T(B) \bar{\delta}i - \text{cl}(A)$ and ,then $\aleph_T(B) \bar{\delta}i - \text{cl}(B)$,therefor B is P – congested set

(2) and (3) exist by (1) and proposition (1-3-6).

Proposition (5-1-9) :

Let (X, T, I, δ) be an i – topological proximity space and $\{H_i : i \in \alpha\}$, α is arbitrary set ,be a collection of subset of T .If $\cup_{i \in \alpha} H_i$ is P_n – congested set then $\aleph_T(\cup_{i \in \alpha} H_i) \bar{\delta}i - \text{cl}(\cup_{i \in \alpha} i - \text{cl}(H_i))$.

Proof :

First we will prove that $i - \text{cl}(\cup_{i \in \alpha} i - \text{cl}(H_i)) = i - \text{cl}(\cup_{i \in \alpha} H_i)$, since we have $\cup_{i \in \alpha} H_i \subseteq \cup_{i \in \alpha} (i - \text{cl}(H_i))$ which imply that for property of $i - \text{cl}(\cup_{i \in \alpha} H_i) \subseteq i - \text{cl}(\cup_{i \in \alpha} i - \text{cl}(H_i))$, conversly, let $y \in i - \text{cl}(\cup_{i \in \alpha} i - \text{cl}(H_i))$, so for each $M_y \in T(y)$ satisfy $M_y \cap (\cup_{i \in \alpha} i - \text{cl}(H_i)) \neq \emptyset$ hence there exist $i_0 \in \alpha$, such that $M_y \cap i - \text{cl}(H_{i_0}) \neq \emptyset$, so there exist at least one point say z belong to this intersection, but by proposition (1-2-20)

$M_y \cap i - \text{cl}(H_{i_0}) \subseteq i - \text{cl}(M_y \cap H_{i_0})$ imply that $z \in i - \text{cl}(M_j \cap H_{i_0})$ and $z \in M_y$ and then for each $V \in T(z)$, $V \cap (M_y \cap H_{i_0}) \neq \emptyset$, for $z \in M_y$ we have $V \cap M_y \cap H_{i_0} = V \cap (M_y \cap H_{i_0}) \neq \emptyset$ and $M_y \cap H_{i_0} \subseteq \bigcup_{i \in \alpha} H_i$ thus $y \in i - \text{cl}(\bigcup_{i \in \alpha} H_i)$. Now we have come to the main step, by using the definition (5-1-1) that is $\aleph_T(\bigcup_{i \in \alpha} H_i) \bar{\delta} i - \text{cl}(\bigcup_{i \in \alpha} H_i) = i - \text{cl}(\bigcup_{i \in \alpha} i - \text{cl}(H_i))$.

The following proposition discusses the image and inverse image of p_n -congested set.

Proposition (5-1-10) :

Let $f: (X, T_1, I, \delta_x) \rightarrow (Y, T_2, I, \delta_y)$ be an i -homeomorphism and δ -homeomorphism function. A is p_n -congested set if and only if $f(A)$ is p_n -congested for each i -closed set A of X .

Proof :

Let A be p_n -congested set, to show that $f(A)$ is p_n -congested set. First we must prove that $f(\aleph_{T_1}(A)) = \aleph_{T_2}(f(A))$. Let $y \in f(\aleph_{T_1}(A))$, then there exist $x \in X$ such that $f(x) = y$ and $x \in \aleph_{T_1}(A)$, so there is at least one i -open \mathcal{U}_x containing x and $\mathcal{U}_x \delta_x A$, by using the definition of proximity continuous function we have $f(\mathcal{U}_x) \delta_y f(A)$, but f is i -open function, so $f(\mathcal{U}_x) \in T_2(y)$ hence y belong to $\aleph_{T_2}(f(A))$. Conversely let $y \in \aleph_{T_2}(f(A))$, then there is $\mathcal{U}_y \in T_2(y)$ with property $\mathcal{U}_y \delta f(A)$, but f is δ -homeomorphisms which means that $f^{-1}(\mathcal{U}_y) \delta_x A$, and f^{-1} is i -continuous function, so $f^{-1}(\mathcal{U}_y) \in T_1(x)$ which imply to $x \in \aleph_{T_1}(A)$, about this we get $y = f(x) \in f(\aleph_{T_1}(A))$. Now by assumption $\aleph_{T_1}(A) \bar{\delta}_x i - \text{cl}(A)$, then we get and by proximity function f we get $\aleph_{T_1}(f(A)) = f(\aleph_{T_1}(A)) \bar{\delta}_y f(i - \text{cl}(A))$, because if possible $f(\aleph_{T_1}(A)) \delta_y f(i - \text{cl}(A))$, since f^{-1} is δ -continuous, so $f^{-1}(f(\aleph_{T_1}(A)) \delta_y f(i - \text{cl}(A)))$, but f is bijective, $\aleph_{T_1}(A) \delta_x i - \text{cl}(A)$ which contradiction. now by using the property of homeomorphism we set $\aleph_{T_2}(f(A)) \bar{\delta}_y i - \text{cl}(f(A))$. therefore $f(A)$ is p_n -congested set. Conversely, let $f(A)$ is p_n -congested in Y first to show that $f^{-1}(\aleph_{T_2}(f(A))) = \aleph_{T_1}(A)$. let $x \in f^{-1}(\aleph_{T_2}(f(A)))$, then

$f(x) \in \mathfrak{N}_{T_2}(f(A))$, so there is $\mathcal{V} \in T_2(f(x_1))$ satisfy that $\mathcal{V} \delta_Y f(A)$, but f^{-1} is δ -continuous and bijective so $f^{-1}(\mathcal{V}) \delta_X f^{-1}(f(A)) = A$, from this fact and by assumption that f is i -continuous we get $x \in \mathfrak{N}_{T_1}(A)$. conversely, for any $x \in \mathfrak{N}_{T_2}(A)$ there is $\mathcal{U} \in T_1(x)$ with $\mathcal{U} \delta_X A$, but f is δ -continuous and i -open we get $f(\mathcal{U}) \delta_Y f(A)$, so $f(x) \in \mathfrak{N}_{T_2}(f(A))$, that is $x \in f^{-1}(\mathfrak{N}_{T_2}(f(A)))$. Now since $f(A)$ is p_n -congested set we get $\mathfrak{N}_{T_2}(f(A)) \delta_Y i\text{-cl}(f(A))$ by assumption and the above fact we get $\mathfrak{N}_{T_1}(A) = f^{-1}(\mathfrak{N}_{T_2}(f(A)) \delta_X f^{-1}(f(i\text{-cl}(A)))) = i\text{-cl}(A)$ therefore A is p_n -congested set in X .

Proposition (5-1-11) :

Let $\{H_i; i \in \alpha\}$ be a collection of subset of the space X . If $\cup_{i \in \alpha} H_i$ is P -conested then $\mathfrak{N}_T(\cup_{i \in \alpha} i\text{-cl}(H_i)) \delta i\text{-cl}(\cup_{i \in \alpha} H_i)$.

Proof :

Since $H_i \subseteq i\text{-cl}(H_i)$ for each i , then $\cup_{i \in \alpha} H_i \subseteq \cup_{i \in \alpha} i\text{-cl}(H_i)$ and by proposition (2-2-9)(1), we have $\mathfrak{N}_T(\cup_{i \in \alpha} H_i) \subseteq \mathfrak{N}_T(\cup_{i \in \alpha} i\text{-cl}(H_i))$. Now by assume $\mathfrak{N}_T(\cup_{i \in \alpha} H_i) \delta i\text{-cl}(\cup_{i \in \alpha} H_i)$, so by proximity axioms we get that $\mathfrak{N}_T(\cup_{i \in \alpha} i\text{-cl}(H_i)) \delta i\text{-cl}(\cup_{i \in \alpha} H_i)$.

The following proposition discusses the image and inverse image of P -congested set .

Proposition (5-1-12) :

Let $f: (X, T_1, I, \delta_x) \rightarrow (Y, T_2, I, \delta_y)$ be an i -homeomorphism, δ -homeomorphism function, then a subset A of X is P -congested if and only if $f(A)$ is P -congested set for each i -closed set A of X .

Proof :

Let A is P -congested set then $\mathfrak{N}_{T_1}(A) \delta_X i\text{-cl}(A)$ since by conversation of proposition (5-1-10) we prove that $f(\mathfrak{N}_{T_1}(A)) = \mathfrak{N}_{T_2}(f(A))$, so because f is δ -continuous function $f(\mathfrak{N}_{T_1}(A)) \delta_Y f(i\text{-cl}(A))$ and then $\mathfrak{N}_{T_2}(f(A)) \delta_Y f(i\text{-cl}(A))$ and since f is i -closed function, so by proposition (1-2-38) we get $\mathfrak{N}_{T_2}(f(A)) \delta_Y i\text{-cl}(f(A))$, hence $f(A)$ is P -conested set.

Conversly let $f(A)$ is P – congested set then $\aleph_{T_2}(f(A))\delta_Y i - \text{cl}(f(A))$. Now by conversation of proposition (5-1-10) we have that $f^{-1}(\aleph_{T_2}(f(A))) = \aleph_{T_1}(A)$, so because f is δ – homeomorphism and $f(A)$ is P – congested set then $f^{-1}(\aleph_{T_2}(f(A)))\delta_X f^{-1}(i - \text{cl}(f(A)))$ and then $\aleph_{T_1}(A)\delta_X f^{-1}(i - \text{cl}(f(A)))$, and since f is 1-1, onto function then by proposition (1-2-38) we get $\aleph_{T_1}(A)\delta_X i - \text{cl}(A)$, hence A is P – congested set.

Now we will define the congested concept via the complement of the set.

Definition (5-1-13) :

Let (X, T, I, δ) be an i – topological proximity space and let A is a subset of X then

1. If the complement of A is P – congested then A is called CPC– congested set.
2. A is called PC – congested set, if and only if $\aleph_T(A)\delta i - \text{cl}(A^c)$
3. A is called CP – congested set , if and only if $\aleph_T(A^c)\delta i - \text{cl}(A)$.

Proposition (5-1-14) :

In any i – topological proximity space (X, T, I, δ) the intersection of any familly of CPC – congested sets is CPC – congested set.

Proof :

Let $\{H_i; i \in \alpha\}$ is a family of CPC – congested sets then H_i^c is P – congested set for each i , and then $\aleph_T(H_i^c)\delta i - \text{cl}(H_i^c)$ and since $\aleph_T(H_i^c) \subseteq \aleph_T(\cap H_i)^c$ also $i - \text{cl}(H_i^c) \subseteq i - \text{cl}(\cap H_i)^c$ hence by a proximity axioms we get that $\aleph_T(\cap H_i)^c\delta i - \text{cl}(\cap H_i)^c$, therefor $\cap H_i$ is CPC – congested set.

Remark (5-1-15) :

1. Union of any family of CPC – congested sets is not CPC – congested set.
2. Union and intersection of any family of CP – congested (resp. PC – congested) sets is not CP – congested (resp. PC – congested) set.

Example (5-1-16) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}\}$, $I = \{\emptyset\}$ and δ defined by the discrete proximity . we have that $\{a\}$, $\{b\}$ are PC-congested sets but the union and intersection of $\{a\}$, $\{b\}$ is not PC-congested set also $\{b\}$, $\{a, c\}$ is CP-congested sets but the union and intersection is not CP-congested set .

Through the following proposition , we can divide any space into four parts via i-topological proximity space.

Proposition (5-1-17) :

Any subset A of i – topological proximity space (X, T, I, δ) is P – congested or CPC – congested or PC – congested or CP – congested set.

Proof :

Since we have that $\mathfrak{N}_T(X) \delta_X X$ but $X = A \cup A^c$ for any subset A of X , also by proposition (2-2-9)(3) we have $\mathfrak{N}_T(A) \cup \mathfrak{N}_T(A^c) = \mathfrak{N}_T(X)$ and $i - cl(A) \cup i - cl(A^c) = i - cl(X)$,so we get that $\mathfrak{N}_T(A) \cup \mathfrak{N}_T(A^c) \delta i - cl(A) \cup i - cl(A^c)$, hence by definition (1-3-1)(2) we have $\mathfrak{N}_T(A) \delta i - cl(A)$ or $\mathfrak{N}_T(A) \delta i - cl(A^c)$ or $\mathfrak{N}_T(A^c) \delta i - cl(A)$ or $\mathfrak{N}_T(A^c) \delta i - cl(A^c)$ from that A is P – congested or PC – congested or CP – congested or CPC – congested set respectively.

Remark (5-1-18) :

Any i – dense set is P – congested set but neither CPC nor CP, PC – congested set.

Proposition (5-1-19) :

Let (X, T, I, δ) be an i – topological proximity space , with δ is σ – proximity and $\{\mu_\lambda^c ; \lambda \in \beta\}$ be locally finite intersection then , for some $\lambda_0 \in \beta$, μ_{λ_0} is CPC – congested if and only if $\bigcap_{\lambda \in \beta} \mu_\lambda$ is CPC -congested set .

Proof :

Suppose that for some $\lambda_0 \in \beta$, μ_{λ_0} is CPC – congested then by definition (5-1-1) , $\mu_{\lambda_0}^c$ is p – congested then $\mathfrak{N}(\mu_{\lambda_0}^c) \delta i - cl(\mu_{\lambda_0}^c)$ and by proposition (1-3-6) , we get $\mathfrak{N}_T((\bigcap_{\lambda \in \beta} \mu_\lambda)^c) = \mathfrak{N}_T(\bigcup_{\lambda \in \beta} \mu_\lambda^c) \delta i - cl(\bigcup_{\lambda \in \beta} \mu_\lambda^c) = i - cl((\bigcap_{\lambda \in \beta} \mu_\lambda)^c) = \bigcup_{\lambda \in \beta} (i - cl(\mu_\lambda^c))$,therefore

$\cup_{\lambda \in \beta} \mu_\lambda$ is CPC – congested set .Conversely, first to show that $\mathfrak{N}_T(\cup_{\lambda \in \beta} \mu_\lambda^c) = \cup_{\lambda \in \beta} \mathfrak{N}_T(\mu_\lambda^c)$, since $\mu_\lambda^c \subseteq \cup_{\lambda \in \beta} \mu_\lambda^c$, hence $\mathfrak{N}_T(\mu_\lambda^c) \subseteq \mathfrak{N}_T(\cup_{\lambda \in \beta} (\mu_\lambda^c))$ imply that $\cup_{\lambda \in \beta} \mathfrak{N}_T(\mu_\lambda^c) \subseteq \mathfrak{N}_T(\cup_{\lambda \in \beta} \mu_\lambda^c)$. Conversely let $x \in \mathfrak{N}_T(\cup_{\lambda \in \beta} \mu_\lambda^c)$, so there is $\mathcal{U} \in T(x)$ such that $\mathcal{U} \delta \cup_{\lambda \in \beta} \mu_\lambda^c$, then by definition (1-3-1) part 2,we get there is $\lambda_0 \in \beta$ with $\mathcal{U} \delta \mu_{\lambda_0}^c$, hence $x \in \mathfrak{N}_T(\mu_{\lambda_0}^c)$, for some $\lambda_0 \in \beta$.Therefore $\mathfrak{N}_T(\cup_{\lambda \in \beta} \mu_\lambda^c) \subseteq \mathfrak{N}_T(\mu_{\lambda_0}^c) \subseteq \cup_{\lambda \in \beta} \mathfrak{N}_T(\mu_\lambda^c)$. Now by assumption that $\mathfrak{N}_T((\cap_{\lambda \in \beta} \mu_\lambda)^c) \delta i - cl((\cap_{\lambda \in \beta} \mu_\lambda)^c)$ imply that $\mathfrak{N}_T(\cup_{\lambda \in \beta} \mu_\lambda^c) \delta i - cl(\cup_{\lambda \in \beta} \mu_\lambda^c)$ by using the properties (locally finite intersection) and σ - proximity , we get $\mathfrak{N}_T(\mu_{\lambda_0}^c) \delta i - cl(\mu_{\lambda_0}^c)$ for some $\lambda_0 \in \beta$,therefore μ_{λ_0} is cpc – congested set for some $\lambda_0 \in \beta$.

The image and inverse image of CPC , CP,PC – congested set are discussed in propositions (5-1-20) and (5-1-21).

Proposition (5-1-20):

Let $f : (X, T_1, I, \delta_x) \rightarrow (Y, T_2, I, \delta_y)$ be an i-homeomorphism and δ – homomorphism function, then A is CPC – congested set if and only if $f(A)$ is CPC – congested set , for each i-open set A of X .

Proof :

Let A be an i-open CPC – congested set hence A^c is i-closed P-congested set then by definition (5-1-13) and proposition (5-1-12) we get the proof .

Proposition (5-1-21) :

Let $f : (X, T_1, I, \delta_x) \rightarrow (Y, T_2, I, \delta_y)$ be i- homeomorphism and δ –homeomorphism function then

- 1- A is PC – congested set if and only if $f(A)$ is PC – congested set , for each i-open set A of X .
- 2- A is CP – congested set if and only if $f(A)$ is CP– congested set for each i-closed set A of X .

Proof :

1. Let A be an i -open PC – congested set , first we must prove that $f(\mathfrak{N}_{T_1}(A)) = \mathfrak{N}_{T_2}(f(A))$,so let $y \in f(\mathfrak{N}_{T_1}(A))$ and because f is bijection then there exist $x \in X$ such that $f(x) = y$ and $x \in \mathfrak{N}_{T_1}(A)$, so there is at least one i – open set \mathcal{U}_x containing x and $\mathcal{U}_x \delta_x A$, by using definition (1-3-10) we have $f(\mathcal{U}_x) \delta_y f(A)$, but f is i – open function , so $f(\mathcal{U}_x) \in T_2(x)$ hence y belong to $\mathfrak{N}_{T_2}(f(A))$, Conversely let $y \in \mathfrak{N}_{T_2}(f(A))$, then there is $\mathcal{U}_y \in T_2(y)$ with property $\mathcal{U}_y \delta_y f(A)$, but f^{-1} is δ –continuous which means that $f^{-1}(\mathcal{U}_y) \delta_x A$ and f is i -continuous function , so $f^{-1}(\mathcal{U}_y) \in T_1(x)$ which imply to $x \in \mathfrak{N}_{T_1}(A)$, about this we get $y = f(x) \in f(\mathfrak{N}_{T_1}(A))$.Now by assumption $\mathfrak{N}_{T_1}(A) \delta_x i - cl(A^c)$ and by definition (1-3-10) , we get $\mathfrak{N}_{T_1}(f(A)) = f(\mathfrak{N}_{T_1}(A)) \delta_y f(i - cl(A^c))$ and by using theorem (1-2-38) we set $\mathfrak{N}_{T_2}(f(A)) \delta_y i - cl(f(A^c))$. therefore $f(A)$ is PC – congested set .Conversely , let $f(A)$ is PC – congested in Y first to show that $f^{-1}(\mathfrak{N}_{T_2}(f(A))) = \mathfrak{N}_{T_1}(A)$. let $x \in f^{-1}(\mathfrak{N}_{T_2}(f(A)))$, then $f(x) \in \mathfrak{N}_{T_2}(f(A))$, so there is $\mathcal{V} \in T_2(f(x_1))$ satisfy that $\mathcal{V} \delta_y f(A)$, but f is δ – homeomorphism so $f^{-1}(\delta_y) = \delta_x$, hence $f^{-1}(\mathcal{V}) \delta_x f^{-1}(f(A)) = A$ from this fact and by assumption that f is i – continuous we get $x \in \mathfrak{N}_{T_1}(A)$. conversely , for any $x \in \mathfrak{N}_{T_2}(A)$ there is $\mathcal{U} \in T_1(x)$ with $\mathcal{U} \delta_x A$, but f is δ – continuous and i – open we get $f(\mathcal{U}) \delta_y f(A^c)$, so $f(x) \in \mathfrak{N}_{T_2}(f(A))$, that is $x \in f^{-1}(\mathfrak{N}_{T_2}(f(A)))$. Now since $f(A)$ is PC – congested set we get $\mathfrak{N}_{T_2}(f(A)) \delta_y i - cl(f(A)^c)$,and by assumption and by the above fact we get $\mathfrak{N}_{T_1}(A) = f^{-1}(\mathfrak{N}_{T_2}(f(A))) \delta_x f^{-1}(f(i - cl(A^c))) = i - cl(A^c)$, therefore A is PC – congested set.
2. Let A is CP – congested set then $\mathfrak{N}_{T_1}(A^c) \delta_x i - cl(A)$. First we will prove that $f(\mathfrak{N}_{T_1}(A^c)) = \mathfrak{N}_{T_2}(f(A^c))$ which is exist by conversation of (1) and because f is δ – homeomorphism, we get that $f(\mathfrak{N}_{T_1}(A^c)) \delta_y f(i - cl(A))$,and since f is i -closed function

then $\mathfrak{N}_{T_2}(f(A)^c)\delta_i - \text{cl}(f(A))$, which means that $f(A)$ is CP – congested. Conversely, let $f(A)$ is CP – congested set then $\mathfrak{N}_{T_2}(f(A)^c)\delta_{Y_i} - \text{cl}(f(A))$ and because f is δ –homeomorphis we get that $f^{-1}(\mathfrak{N}_{T_2}(f(A)^c)\delta_{Y_i}) = \mathfrak{N}_{T_1}(f^{-1}(f(A)^c))\delta_{X_i} - \text{cl}(f^{-1}(f(A)))$. Since f is 1-1, onto function hence $f^{-1}(\mathfrak{N}_{T_2}(f(A)^c)\delta_{Y_i}) = \mathfrak{N}_{T_1}(f^{-1}(f(A)^c))\delta_{X_i} - \text{cl}(A)$ and by conversation of (1) we get that $\mathfrak{N}_{T_1}(A^c)\delta_i - \text{cl}(A)$ therefore A is CP – congested set.

Remark (5-1-22):

For the i -homeomorphism and δ -homeomorphism functions in the above propositions if (X, T_1) and (Y, T_2) are topological spaces then A is CPC, (resp. CP, PC) congested sets if and only if $f(A)$ is CPC (resp. CP, PC) congested sets.

5.2 On a Proximity Focal Congested set

All the concepts introduced in section one ,we will studying in this section via the focal sets .

Definition (5-2-1)

Let (X, T, I, δ) be an i – topological proximity space and A be a subset of X then A is called

1. Focal proximity un congested, simply, (P_f –un congested) set if $\aleph_{I_\emptyset}(A) \bar{\delta}Fcl(A)$.
2. Focal proximity congested (simply, P_f – congested) set if $\aleph_{I_\emptyset}(A) \delta Fcl(A)$.

The set of all P_f –un congested sets denoted by $\delta_{fn}(I, T)$ and the family of all P_f – congested sets denoted by $\delta_f(I, T)$.

Remark (5-2-2)

1. \emptyset is P_f – uncongested set and not P_f – congested set
2. The universal non empty set X is P_f – congested set
3. Except for the indiscrete topology every FODense set is P_f – congested set and this exist by definition (5-2-1) and proposition (4-2-10), also if A is \aleph_{I_\emptyset} –dense set then A is P_f – congested set .The converse is not true as in the example below .

Example (5-2-3)

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}\}$ and $I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$, where δ defined by the discrete proximity if $H = \{b\}$ then H is P_f – congested but H is not FODense set .

Remark (5-2-4)

1. For any i – topological proximity space and for any subset A of X which satisfy $\aleph_{I_\emptyset}(A) = \emptyset$, then A is P_f – uncongested set .
2. Union of tow P_f – uncongested set is not necessary P_f – uncongested set
3. Since $\aleph_{I_\emptyset}(A), \aleph_{I_\emptyset}(B)$ is a subset of $\aleph_{I_\emptyset}(A \cup B)$ also $Fcl(A), Fcl(B)$ is a subset of $Fcl(A \cup B)$, so we get that by using the proximity axioms the union of two P_f –congested is P_f –congested set .

We can see that from example (5-2-3) $\{b\}, \{c\}$ are P_f – uncongested sets then clearly that $\{b,c\}$ is

Now we list the most important features of the P_f – uncongested sets .

Proposition(5-2-5)

For any i – topological proximity space and any subsets A , B of the universal set X the following properties are achieved :

1. If $A \subseteq B$ and B is a P_f – un congested set , then A is P_f – un congested set then A is P_f –un congested set .
2. If $A \cup B$ is a P_f – uncongested set , then A and B are P_f – uncongested set .
3. If A or B is a P_f – uncongested set , then $A \cap B$ is P_f – uncongested set.

Proof :

1. Let B is P_f – uncongested set then $\mathfrak{N}_{I_\phi}(B) \bar{\delta}Fcl(B)$ and since we have that $\mathfrak{N}_{I_\phi}(A) \subseteq \mathfrak{N}_{I_\phi}(B)$, hence by proposition (1-3-5) we get that $\mathfrak{N}_{I_\phi}(A) \bar{\delta}Fcl(B)$, so by assume and by corollary (2-3-6)we get $\mathfrak{N}_{I_\phi}(A) \bar{\delta}Fcl(A)$, so A is P_f – uncongested set.
2. The proof exist by (1) and by proposition (1-3-5)
3. The proof exist by (1) and by proposition (1-3-5)

In a similar way we will discuss the cases of proposition (5-2-5)via the focal proximity congested set in proposition (5-2-6)

Proposition (5-2-6) :

Let (X, T, I) be an i – Topological proximity space and A , B are subset of X then

1. If $A \subseteq B$ and A is P_f – congested ,then B is P_f – congested set .
2. If $A \cap B$ is P_f – congested ,then A and B are P_f – congested set .
3. A or B is P_f – congested if and only if $A \cup B$ is P_f – congested set .

Proof :

1. Since $A \subseteq B$, and A is P_f – congested set , then $\aleph_{I_\phi}(A) \delta Fcl(A)$, so by assume and proposition (2-2-19) $\aleph_{I_\phi}(A) \subseteq \aleph_{I_\phi}(B)$, also by corollary (2-3-6) we have $Fcl(A) \subseteq Fcl(B)$, therefor by proximity axioms we get that B is P_f – congested set .
2. Exist by (1) and proposition (1-3-6).
3. Let A or B is P_f – congested set then by (1)and proposition (2-2-19), corollary (2-3-6) we get that $A \cup B$ is P_f – congested set .Conversely , let $A \cup B$ is P_f – congested set then $\aleph_{I_\phi}(A \cup B) \delta Fcl(A \cup B)$, and by definition (1-3-1) and by proposition (2-2-19)(3),corollary (2-3-6)(3) we get that A or B is P_f – congested set .

Remark (5-2-7) :

1. By remark (5-2-4)(2) $\delta_{f_n}(I, T)$ is not an ideal.
2. $\delta_{f_n}(I, T)$ has a disjoint intersection with T because if there is i – open set $A \in \delta_{f_n}(I, T)$,and by use proposition (2-2-19)(5)we have that $A \subseteq \aleph_{I_\phi}(A)$ hence definition (5-2-1) imply that $A \bar{\delta} Fcl(A)$ and this contradiction .

Proposition (5-2-8):

Let $\{H_i; i \in \alpha\}$ be a collection of subset of i – topological proximity space (X, T, I) . If $\cup_{i \in \alpha} H_i$ is P_f – un congested set , then $\aleph_{I_\phi}(\cup_{i \in \alpha} H_i) \bar{\delta} Fcl(\cup_{i \in \alpha} Fcl(H_i))$

Proof :

First we prove that $Fcl(\cup_{i \in \alpha} Fcl(H_i)) = Fcl(\cup_{i \in \alpha} H_i)$.Since $H_i \subseteq Fcl(H_i)$ for each i , so $\cup H_i \subseteq \cup (Fcl(H_i))$,and then $Fcl(\cup_{i \in \alpha} H_i) \subseteq Fcl(\cup_{i \in \alpha} Fcl(H_i))$, Now let $m \in Fcl(\cup_{i \in \alpha} Fcl(H_i))$,then for each $U_m \in T(m)$, $U_m \cap (\cup_{i \in \alpha} Fcl(H_i)) \neq \emptyset$, hence there is $i_0 \in \alpha$, where $U_m \cap Fcl(H_{i_0}) \neq \emptyset$, so there is at least one point in the intersection say k , but by proposition (2-3-16)we have $U_m \cap Fcl(H_{i_0}) \subseteq Fcl(U_m \cap H_{i_0})$,so $k \in Fcl(U_m \cap H_{i_0})$, and $k \in U_m$ and hence for each i – open set V of k , satisfy $V \cap (U_m \cap H_{i_0}) \neq \emptyset$, for $k \in U_m$ and for each $U_m \in T(m)$ we have $V \cap U_m \cap H_{i_0} \subset (\cup_{i \in \alpha} H_i)$, thus $m \in Fcl(\cup_{i \in \alpha} H_i)$.

Now we will be back to the main step by using definition(5-2-1) that is $\aleph_{I_\phi}(\cup_{i \in \alpha} H_i) \bar{\delta} Fcl(\cup_{i \in \alpha} H_i) = Fcl(\cup_{i \in \alpha} (Fcl(H_i)))$.

Image and inverse image of focal proximity uncongested set are explained in this proposition

Proposition (5-2-9):

Let $f: (X, T, I, \delta_x) \rightarrow (Y, T_Y, I_Y, \delta_y)$ be i -homeomorphism formatting and δ – homeomorphism function then A is P_f –un congested set if and only if $f(A)$ is P_f – un congested set for each i -closed set A of X .

Proof :

Let A be an i -closed P_f – un congested set . First we will prove that $f\left(\aleph_{I_\phi T_1}(A)\right) = \aleph_{I_\phi T_2}(f(A))$. Let $m \in f\left(\aleph_{I_\phi T_1}(A)\right)$, then there exist $x \in X$, such that $m = f(x)$ and $x \in \aleph_{I_\phi T_1}(A)$, so there is at least one focal set H_x , containing x and $H_x \delta_x A$, by using the definition of proximity continuous function we have $f(H_x) \delta_y f(A)$, but f is a formatting function then $f(H_x) \in I_\phi T_2(f(x))$ and hence $m = f(x) \in \aleph_{I_\phi T_2}(f(A))$, conversly , let $m \in \aleph_{I_\phi T_2}(f(A))$, then there is a focal set $H_m \in I_\phi T_2(m)$ with property $H_m \delta_y f(A)$, but f^{-1} is δ –continuous which means that $f^{-1}(H_m) \delta_x A$, and since f is formatting function, so by proposition (2-1-13) $f^{-1}(H_m) \in I_\phi T_1(x)$, which $x \in \aleph_{I_\phi T_1}(A)$ and then $f(x) = m \in f\left(\aleph_{I_\phi T_1}(A)\right)$,now since A is P_f –un congested set $\aleph_{I_\phi T_1}(A) \bar{\delta}_X Fcl(A)$, and because f is δ –continuous then $f(\aleph_{I_\phi T_1}(A)) \bar{\delta}_Y f(Fcl(A))$,so by above conversation and proposition (2-3-29) we get that $\aleph_{I_\phi T_2}(f(A)) \bar{\delta}_Y (Fcl(f(A)))$,hence $f(A)$ is P_f – un congested set .

. Conversely, let $f(A)$ is P_f – un congested set in Y , first to show that $f^{-1}(\aleph_{I_\phi T_2}(f(A))) = \aleph_{I_\phi T_1}(A)$. Let $m \in f^{-1}(\aleph_{I_\phi T_2}(f(A)))$ then $f(m) \in \aleph_{I_\phi T_2}(f(A))$, so there is $H_{f(m)} \in I_\phi T_2(f(A))$ which satisfy $H_{f(m)} \delta_y f(A)$, but f^{-1} is δ – continuous , so $f^{-1}(H_{f(m)}) \delta_x A$, from this fact and by assumption that f is formatting function we get that $m \in \aleph_{I_\phi T_1}(A)$.

Conversly, for any $m \in \aleph_{I_\phi T_1}(A)$ there is a focal set H of m with $H\delta_X A$, but f is δ – continuous , so $f(H)\delta_Y f(A)$ and by definition (2-1-12), $f(H)$ is a focal set of $f(m)$, and hence $f(m) \in \aleph_{I_\phi T_2}(f(A))$,so $m \in f^{-1}\left(\aleph_{I_\phi T_2}(f(A))\right)$.

Now since $f(A)$ is P_f – un congested set we have that $\aleph_{I_\phi T_2}(f(A))\bar{\delta}Fcl(f(A))$ and by assumption and the above conversation we get that $\aleph_{I_\phi T_1}(A) = f^{-1}(\aleph_{I_\phi T_2}(f(A))\bar{\delta}_X f^{-1}(f(Fcl(A))) = Fcl(A)$ therefor A is P_f – un congested .

Proposition (5-2-10) :

Let $\{H_i: i \in \alpha\}$ be a collection of subset of the i – topological proximity space X . if $\cup_{i \in \alpha} H_i$ is P_f – congested set ,then $\aleph_{I_\phi}(\cup Fcl(H_i))\delta Fcl(\cup_{i \in \alpha} H_i)$.

Proof :

Since $H_i \subseteq Fcl(H_i)$ for each i , so $\cup_{i \in \alpha} H_i \subseteq \cup_{i \in \alpha} Fcl(H_i)$ and then $\aleph_{I_\phi}(\cup_{i \in \alpha} H_i) \subseteq \aleph_{I_\phi}(\cup_{i \in \alpha} Fcl(H_i))$, and since $\cup_{i \in \alpha} H_i$ is P_f – congested set so by proposition (1-3-6)we get $\aleph_{I_\phi}(\cup_{i \in \alpha} H_i)\delta Fcl(\cup_{i \in \alpha} H_i)$ so $\aleph_{I_\phi}(\cup Fcl(H_i))\delta Fcl(\cup H_i)$.

Image and inverse image of focal proximity congested set are discussed in this proposition

Proposition (5-2-11) :

Let $f: (X, T_1, I_X, \delta_X) \rightarrow (Y, T_2, I_Y, \delta_Y)$ be a formatting i -homeomorphism and δ – homeomorphism function . then A is P_f – congested if and only if $f(A)$ is P_f – congested for each i -closed set A of X .

Proof :

Let A is P_f – congested then $\aleph_{I_\phi T_1}(A)\delta_X Fcl(A)$ and since f is δ – homeomorphism then $f\left(\aleph_{I_\phi T_1}(A)\right)\delta_Y f(Fcl(A))$ and by the proof of proposition (5-2-9) $f\left(\aleph_{I_\phi T_1}(A)\right) = \aleph_{I_\phi T_2}(f(A))$ we get that $\aleph_{I_\phi T_2}(f(A))\delta_Y f(Fcl(A))$ and by proposition (2-3-29) $\aleph_{I_\phi T_2}(f(A))\delta_Y Fcl(f(A))$ therefor $f(A)$ is P – congested , conversly let $f(A)$ is P_f – congested

then $\aleph_{I_{\phi T_2}}(f(A))\delta_Y Fcl(f(A))$ and since f^{-1} is δ – continuous we get that

$f^{-1}(\aleph_{I_{\phi T_2}}(f(A))\delta_X f^{-1}(Fcl(f(A))))$ by the proof of proposition (5-2-9)

$f^{-1}(\aleph_{I_{\phi T_2}}(f(A))) = \aleph_{I_{\phi T_1}}(A)$ hence $\aleph_{I_{\phi T_1}}(A)\delta_X Fcl(A)$ and therefore A is P_f – congested.

As in definition (5-1-3) we will discuss the focal congested concept of the complement set .

Definition (5-2-12) :

Let (X, T, I, δ) is i – topological proximity space and A is a subset of X then

1. If the complement of A is P_f – congested set, then A is called CP_fC – congested set .
2. A is called P_fC – congested set if $\aleph_{I_{\phi}}(A)\delta Fcl(A^c)$
3. A is called CP_f – congested set if $\aleph_{I_{\phi}}(A^c)\delta Fcl(A)$

Example (5-2-13) :

Every subset of a space X with the discrete topological space and the empty ideal is a CP_fC – congested , CP_f – congested set and P_fC – congested .

Proposition (5-2-14) :

In any i – topological proximity space the intersection of any family of CP_fC – congested set is CP_fC – congested set .

Proof :

Let $\{H_i : i \in \alpha\}$ is any family of CP_fC – congested sets, so H_i^c is P_f – congested set and , then $\aleph_{I_{\phi}}(H_i^c)\delta Fcl(H_i^c)$, for each i , by using the proximity axioms and nested properties we have that $\aleph_{I_{\phi}}(H_i^c) \subseteq \aleph_{I_{\phi}}((\cap H_i)^c)\delta Fcl(H_i^c) \subseteq Fcl((\cap H_i)^c)$ Hence $\cap H_i$ is CP_fC – congested set .

Remark (5-2-15) :

1. The union of any family of CP_fC – congested set is not necessary CP_fC – congested set .
2. The union and intersection of any family of CP_f or P_fC – congested sets is not necessary CP_f or P_fC – congested sets respectively .

Example (5-2-16):

Let $X = \{a, b, c\}$ with the indiscrete topology and $I = \{\emptyset, \{a\}\}$ if we take the family $\{\{b\}, \{c\}\}$ then the union of this family is $\{b, c\}$ which is not CP_fC – congested set. For the same family we have that union is not CP_f – congested set and the intersection of this family $\{\{a, b\}, \{a, c\}\}$ is not CP_f – congested set .

Through the following proposition ,we can divid any space into four parts via i – topological proximity space .

Proposition (5-2-17) :

Any subset A of i – topological proximity space (X, T, I, δ) is P_f – congested or CP_fC – congested or P_fC – congested or CP_f – congested set.

Proof :

Since we have that $\aleph_{I_\phi}(X) \delta X$, also we have $X = A \cup A^c$ for any subset A of X and $\aleph_{I_\phi}(A) \cup \aleph_{I_\phi}(A^c) = \aleph_{I_\phi}(X)$, $Fcl(A) \cup Fcl(A^c) = Fcl(X)$ from all above we get that $\aleph_{I_\phi}(A) \cup \aleph_{I_\phi}(A^c) \delta Fcl(A) \cup Fcl(A^c)$ hence by proximity axioms we get that $\aleph_{I_\phi}(A) \delta Fcl(A^c)$ or $\aleph_{I_\phi}(A^c) \delta Fcl(A)$ or $\aleph_{I_\phi}(A^c) \delta Fcl(A^c)$ or $\aleph_{I_\phi}(A) \delta Fcl(A)$,and this result that we need.

Proposition (5-2-18):

Let (X, T, I, δ) be an i – topological proximity space with δ is σ – proximity and $\{H_i: i \in \alpha\}$ be locally finite intersection then for some $i_0 \in \alpha$, H_{i_0} is CP_fC – congested set if and only if $\bigcap_{i \in \alpha} H_i$ is CP_fC – congested set .

Proof :

Suppose that for some $i_0 \in \alpha$, H_{i_0} is CP_fC – congested set, so by definition (5-2-12) $\aleph_{I_\phi}(H_{i_0}^c) \delta Fcl(H_{i_0}^c) \supseteq Fcl(\bigcap H_i^c)$ from that we get $\aleph_{I_\phi}(\bigcup H_i)^c \delta Fcl(\bigcup H_i)^c$ hence $\bigcup_{i \in \alpha} H_i$ is CP_fC – congested set .

Conversely ,First we prove that $\aleph_{I_\phi}(\cup_{i \in \alpha} H_i^c) = \cup_{i \in \alpha} \aleph_{I_\phi}(H_i^c)$ since $H_i^c \subseteq \cup_{i \in \alpha} H_i^c$ then $\aleph_{I_\phi}(H_i^c) \subseteq \aleph_{I_\phi}(\cup_{i \in \alpha} H_i^c)$ imply that $\cup_{i \in \alpha} \aleph_{I_\phi}(H_i^c) \subseteq \aleph_{I_\phi}(\cup_{i \in \alpha} H_i^c)$. Now let $x \in \aleph_{I_\phi}(\cup_{i \in \alpha} H_i^c)$ then there is afocal set w of x , $x \in w, w \delta \cup_{i \in \alpha} H_i^c$, so there exist $i_0 \in \alpha$ such that $w \delta H_{i_0}^c$ and then $x \in \aleph_{I_\phi}(H_{i_0}^c) \subseteq \cup \aleph_{I_\phi}(H_{i_0}^c)$ by above conversation and assumption that $\aleph_{I_\phi}(\cup H_i^c) \delta Fcl(\cup H_i^c)$ imply $\cup \aleph_{I_\phi}(H_i^c) \delta Fcl(\cup H_i^c)$, so by σ – proximity of the family $\{H_i^c : i \in \alpha\}$ and locally finite intersection we get that for some $i_0 \in \alpha$, $\aleph_{I_\phi}(H_{i_0}^c) \delta Fcl(H_{i_0}^c)$ and then H_{i_0} is CP_fC – congested set .

Image and inverse image of CP_fC , CP_f , $P_f C$ - congested sets are explained in propositions (5-2-20) and (5-2-21).

Proposition (5-2-19) :

Let $f: X \rightarrow Y$ be an i -homeomorphism and δ – homeomorphism function then A is CP_fC – congested set if and only if $f(A)$ is CP_fC – congested set for each i -open set A of X .

Proof :

Let A is i – open CP_fC – congested set then A^c is i – closed P_f – congested set and since f is i -homeomorphism $f(A^c)$ is i – closed and $f(A^c) = (f(A))^c$ then by proposition (5-2-11) $f(A^c)$ is CP_fC – congested set. Now if $f(A)$ is $CP_f C$ -congested ,so $f(A)^c$ is P_f – congested hence $f(A^c)$ is P_f – congested then by proposition (5-2-11) then A^c is P_f – congested ,therefor A is CP_fC – congested set .

Proposition (5-2-20) :

Let $f: (X, T_1, I_X, \delta_X) \rightarrow (Y, T_2, I_Y, \delta_Y)$ is formatting i -homeomorphism and δ – homeomorphism function then :

1. A is i – open P_fC – congested set if and only if $f(A)$ is P_fC – congested set .
2. A is i – closed CP_f – congested set if and only if $f(A)$ is CP_f – congested set.

Proof :

1. Let A is i – open P_fC – congested set, first we have by proposition (5-2-9) that

$f\left(\aleph_{I_{\phi T_1}}(A)\right) = \aleph_{I_{\phi T_2}}(f(A))$. So by assume that $\aleph_{I_{\phi T_1}}(A)\delta_X Fcl(A^c)$ and since f is δ – continuous then $f\left(\aleph_{I_{\phi T_1}}(A)\right)\delta_Y f(Fcl(A^c))$ and then $\aleph_{I_{\phi T_2}}(f(A))\delta_Y f(Fcl(A^c))$, because f is i -homeomorphism formatting function we get that $\aleph_{I_{\phi T_2}}(f(A))\delta_Y Fcl(f(A)^c)$, hence $f(A)$ is P_fC – congested set .

Conversely let $f(A)$ is P_fC – congested set in Y , so $\aleph_{I_{\phi T_2}}(f(A))\delta_X Fcl(f(A)^c)$.

By proposition (5-2-9) we have that $f^{-1}\left(\aleph_{I_{\phi T_2}}(f(A))\right) = \aleph_{I_{\phi T_1}}(A)$

. by assume we have $\aleph_{I_{\phi T_2}}(f(A))\delta_Y Fcl(f(A)^c)$ and because f^{-1} is δ – continuous $f^{-1}(\aleph_{I_{\phi T_2}}(f(A))\delta_X f^{-1}(Fcl(f(A)^c))$, and f is 1-1, into function we get that $\aleph_{I_{\phi T_1}}(A)\delta_X Fcl(A)$, hence A is P_fC – congested .

2 -Let A is CP_f – congested set then $\aleph_{I_{\phi T_1}}(A^c)\delta Fcl(A)$. first we prove that

$f\left(\aleph_{I_{\phi T_1}}(A^c)\right) = \aleph_{I_{\phi T_2}}(f(A^c))$ and for that let $y \in f\left(\aleph_{I_{\phi T_1}}(A^c)\right)$ and because f is onto and

1-1 then there is $m \in \aleph_{I_{\phi T_1}}(A^c)$ such that $y = f(m)$ and by definition (2-2-12) there is

$H_m \in I_{\phi T_1}(m)$ such that $H_m \delta_X A^c$ and since f is δ – homeomorphism then $f(H_m)\delta_Y f(A^c)$,

so $f(m) \in \aleph_{I_{\phi T_2}}(f(A^c))$. conversly let $y \in \aleph_{I_{\phi T_2}}(f(A^c))$, so by definition (2-2-12) there is

$H_y \in I_{\phi T_2}(y)$ such that $H_y \delta_Y f(A^c)$ and since f is δ – homeomorphsim , so

$f^{-1}(H_y)\delta_X f^{-1}(f(A^c))$ and because f is bijective formatting function then $f^{-1}(H_y)$ is a

focal set of m in X , hence $m \in \aleph_{I_{\phi T_1}}(A^c)$ and there for $y = f(m) \in f\left(\aleph_{I_{\phi T_1}}(A^c)\right)$. Now

we will back to assumption above $\aleph_{I_{\phi T_1}}(A^c)\delta_X Fcl(A)$, $f\left(\aleph_{I_{\phi T_1}}(A^c)\right)\delta_Y f(Fcl(A))$ and by

above conversation $\aleph_{I_{\phi T_2}}(f(A^c))\delta_Y f(Fcl(A))$ and because A is i – closed and f is bijective

function we get that $\aleph_{I_{\phi T_2}}(f(A))^c \delta_Y Fcl(f(A))$ so $f(A)$ is CP_f – congested set . Conversely

let $f(A)$ is CP_f – congested set then $\aleph_{I_{\phi T_2}}(f(A))^c \delta_Y Fcl(f(A))$. and by conversiation first

we will prove that $f^{-1}(\mathfrak{N}_{I_{\phi T_2}}(f(A))^c) = \mathfrak{N}_{I_{\phi T_1}}(A^c)$ and for that let $m \in f^{-1}(\mathfrak{N}_{I_{\phi T_2}}(f(A))^c)$ which is equal to $f^{-1}(\mathfrak{N}_{I_{\phi T_2}}(f(A^c)))$ and then $f(m) \in \mathfrak{N}_{I_{\phi T_2}}(f(A^c))$, hence there is afocal set $H_{f(m)}$ containing $f(m)$ such that $H_{f(m)}\delta_Y f(A^c)$ and because f is δ – homeomorphism and 1-1, onto function then $f^{-1}(H_{f(m)})\delta_X A^c$ also $f^{-1}(H_{f(m)})$ is afocal set of m in X , hence $m \in \mathfrak{N}_{I_{\phi T_1}}(A^c)$. conversly, let $m \in \mathfrak{N}_{I_{\phi T_1}}(A^c)$ then there is a focal set H_m containing m such that $H_m\delta_X A^c$ and because f is δ – homeomorphism then $f(H_m)\delta_Y f(A^c)$ which is mean that $f(m) \in \mathfrak{N}_{I_{\phi T_2}}(f(A^c))$ where $f(H_m)$ is a focal set of $f(m)$ and containing it, from that we get $m \in f^{-1}(\mathfrak{N}_{I_{\phi T_2}}(f(A^c)))$. Now we will back to first step $\mathfrak{N}_{I_{\phi T_2}}(f(A)^c)\delta_Y Fcl(f(A))$, since f is δ – homeomorphism and 1-1, onto function then $f^{-1}(\mathfrak{N}_{I_{\phi T_2}}(f(A)^c)\delta_X Fcl(A))$, hence $\mathfrak{N}_{I_{\phi T_1}}(A^c)\delta_X Fcl(A)$ and then A is CP_f – congested set .

5.3 Study on the Proximity Crowded set

In this section we will define the proximity crowded set in i -topological proximity spaces via the nested set, focal nested set, i -closer, focal closure concepts. the relationship with the sets that were defined in the first and second sections of this chapter.

Definition (5-3-1) :

A subset A of i -topological proximity space (X, T, I, δ) is called:

1. Proximity non – crowded (simple p_n – crowdeds , $\aleph_T(i - cl(A)) = \emptyset$.
2. Proximity not always crowded (simply p_{na} – crowded) if $\aleph_T(i - cl(A)) \neq \emptyset$.
3. Proximity focal non – crowded (simply, pf_n – crowded) if $\aleph_{I_\phi}(Fcl(A)) = \emptyset$.
4. Proximity focal not always crowded (simply , pf_{na} – crowded) if $\aleph_{I_\phi}(Fcl(A)) \neq \emptyset$.

Example (5-3-2):

Let $(X, T_i, \{\emptyset\}, \delta)$ be an i -topological proximity space, where δ is the discrete proximity, then A is P_n – crowded and P_{na} – crowded for each subset A of X .

Remark (5-3-3) :

Let (X, T, I, δ) is i -topological proximity space and T is any family except the indiscrete topological space then:

1. If A is F_0 dense, then A is P_{na} – crowded set.
2. If A is ϕ dense, then A is P_{na} – crowded set.
3. The set of all proximity focal non crowded is denoted by $F_{fn}(T, I)$ and the set of all proximity non crowded set is denoted by $F_n(T, I)$.
4. Noted that \emptyset is P_n – crowded and P_{fn} –crowded set .
5. Union of two P_n – crowded (P_{fn} –crowded)set is P_n – crowded (P_{fn} –crowded) set

The following theorem explains the effect of the nested set on the P_n – crowded set.

Theorem (5-3-4) :

Let (X, T, I, δ) be an i – topological proximity space , then $\aleph_T(A) = \emptyset$ if and only if A is P_n – crowded set

Proof:

let $\aleph_T(A) = \emptyset$ if possible that $\aleph_T(i - cl(A)) \neq \emptyset$,then there exist $x \in \aleph_T(i - cl(A))$, so by definition (2-2-1)there exist $u \in T(x)$, $u \delta (i - cl(A))$,hence $i - cl(A) \neq \emptyset$, so there is $z \in i - cl(A)$, from that we get $w \cap A \neq \emptyset$,for each $w \in T(z)$,so there exist $W \in T(z)$, $W \delta A$, then $z \in \aleph_T(A)$ and this contradiction, hence $\aleph_T(i - cl(A)) = \emptyset$.

Conversely, suppose that $\aleph_T(i - cl(A)) = \emptyset$ and since $A \subseteq i - cl(A)$ so $\aleph_T(A) \subseteq \aleph_T(i - cl(A)) = \emptyset$, hence $\aleph_T(A) = \emptyset$.

Corollary (5-3-5) :

Let (X, T, I, δ) be an i – topological proximity space , $\aleph_{I_\phi}(A) = \emptyset$ if and only if A is P_{fn} – crowded set.

The relation between proximity focal non crowded and proximity non crowded are explained in the proposition below

Proposition (5-3-6):

Let (X, T, I, δ) be an i -topological proximity space ,then every P_{fn} – crowded is P_n – crowded .

Proof:

By corollary (5-3-5) if A is P_{na} – crowded , then $\aleph_{I_\phi}(A) = \emptyset$ hence by proposition (2-2-14) , $\aleph_T(A) = \emptyset$, so by theorem (5-3-4)then A is P_n –crowded.

Some properties of the above types of crowded sets in definition (5-3-1)are proven below

Proposition (5-3-7):

Let (X, T, I, δ) be an i – topological proximity space , and let A, B are subsets of X such that $B \subseteq A$, then each of the following are achieved :

1. if B is P_{na} – crowded set ,then A is P_{na} – crowded set .
2. if A is P_n – crowded set ,then B is P_n – crowded set .
3. if A is P_{fn} – crowded set ,then B is P_{fn} – crowded set .
4. if B is P_{fna} – crowded set ,then A is P_{fna} – crowded set .

Proof:

1. Let B is P_{na} – crowded set ,then $\mathfrak{N}_T(i - cl(B)) \neq \emptyset$, so by assumption and propositions (2-2-9), (1-2-17) , $\mathfrak{N}_T(i - cl(B)) \subseteq \mathfrak{N}_T(i - cl(A))$, hence A is P_{na} – crowded set.
2. Let A is P_n – crowded set ,then $\mathfrak{N}_T(i - cl(A)) = \emptyset$,so by assumption and propositions (2-2-9), (1-2-17) , $\mathfrak{N}_T(i - cl(B)) \subseteq \mathfrak{N}_T(i - cl(A)) = \emptyset$, hence B is P_n – crowded set.
3. Let A is P_{fn} – crowded set ,then $\mathfrak{N}_{I\phi}(F - cl(A)) = \emptyset$,so by assumption and propositions (2-2-19), (2-3-6) $\mathfrak{N}_{I\phi}(F - cl(B)) \subseteq \mathfrak{N}_T(F - cl(A)) = \emptyset$, hence A is P_{fn} – crowded set.
4. Let B is P_{fna} – crowded set ,then $\mathfrak{N}_{I\phi}(F - cl(A)) \neq \emptyset$,so by assumption and propositions (2-2-19), (2-3-6) , $\mathfrak{N}_{I\phi}(F - cl(B)) \subseteq \mathfrak{N}_T(F - cl(A)) \neq \emptyset$, hence A is P_{fn} – crowded set.

Some instances of ideals or the family embedding are showed in propositions (5-3-8) , (5-3-9).

Proposition (5-3-8):

Let (X, T, I_j, δ) , $j=1,2$, be an i - topological proximity spaces , such that $I_1 \subseteq I_2$ then if A is P_{fn} – crowded set with respect to I_2 ,then A is P_{fn} – crowded set with respect to I_1 .

Proof:

Let A is P_{f_n} – crowded with respect to I_2 , hence, $\aleph_{I_2\phi}(A) = \emptyset$ and by proposition (2-2-21) $\aleph_{I_1\phi}(A) \subseteq \aleph_{I_2\phi}(A) = \emptyset$, so $\aleph_{I_1\phi}(A) = \emptyset$, and by corollary (5-3-5) A is P_{f_n} – crowded with respect to I_1 .

In a similar way we can discuss the case that if A is $P_{f_{na}}$ – crowded set via I_1 ,then A is $P_{f_{na}}$ – crowded set via I_2

Proposition (5-3-9):

Let (X, T_j, I, δ) $j = 1, 2$, be an i – topological proximity spaces, such that $T_1 \subset T_2$,then if A is P_{naT_1} – crowded set ,then A is P_{naT_2} – crowded set.

Proof:

Let A is P_{naT_1} – crowded set, so $\aleph_{T_1}(i - cl(A)) \neq \emptyset$,so if possible that $\aleph_{T_2}(i - cl(A)) = \emptyset$,then for each $x \in X, x \notin \aleph_{T_2}(i - cl(A))$, hence for each $H \in T_2(x), H\bar{\delta}i - cl(A)$,and by assume for each $H \in T_1$ we get that $H\bar{\delta}i - cl(A)$ and that for each $x \in X$, hence $\aleph_{T_1}(i - cl(A)) = \emptyset$ and this contradiction.

In a similar way we can discuss the case that if A is P_{nT_2} – crowded set , then A is P_{nT_1} – crowded set

Example (5-3-10):

$X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}\}$, $I_1 = \{\emptyset, \{c\}\}$, $I_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and δ defined the discrete proximity space , If $A = \{c\}$ then A is P_{na} – crowded with respect to I_1 But not P_{na} – crowded with respect to I_2 .

Below the relation between the set of every proximity non crowded and proximity focal non crowded are discussed .

Proposition (5-3-11):

Let (X, T_j, I, δ) , $j = 1, 2$, be an i – topological proximity spaces ,such that $T_1 \subset T_2$ then $F_n(I, T_2) \subseteq F_n(I, T_1)$

Proof:

Let $A \in F_n(I, T_2)$, then $\mathfrak{N}_{T_2}(i - \text{cl}_{T_2}(A)) = \emptyset$, so if possible that $\mathfrak{N}_{T_1}(i - \text{cl}(A)) \neq \emptyset$, hence there is $x \in \mathfrak{N}_{T_1}(i - \text{cl}_{T_1}(A))$, and then there exist $H \in T_1(x)$, such that $H\delta i - \text{cl}_{T_1}(A)$, so $i - \text{cl}_{T_1}(A) \neq \emptyset$, and then there is $x \in i - \text{cl}_{T_1}(A)$ so for each $W \in T_1(x)$, $W \cap A \neq \emptyset$, hence there is $W \in T_2(x)$, $H\delta W$, and then $x \in \mathfrak{N}_{T_2}(A)$, and this contradiction.

Example (5-3-12):

Let $X = \{a, b, c\}$, $T_1 = \{X, \emptyset, \{a\}\}$, $I = \{\emptyset\}$, $T_2 = \{X, \emptyset, \{a, b\}\}$ and δ defined by the discrete proximity space, then $f_n(I, T_1) = \{\emptyset\}$, $f_n(I, T_2) = \{\emptyset, \{c\}\}$ But $T_2 \not\subseteq T_1$

Proposition (5-3-13):

Let (X, T, I, δ) be an i -topological proximity space, then $F_{fn}(I, T)$ and $F_n(I, T)$ are ideals.

Proof:

By remark (5-3-3)(4),(5) and proposition (5-3-7) we get the result.

Image and inverse image of all sets in definition (5-3-1) are discussed in propositions (5-3-14), (5-3-15)

Proposition (5-3-14):

Let $f: (X, T_X, I_X) \rightarrow (Y, T_Y, I_Y)$ be δ -continuous, i -homeomorphism function, then following statement exist

1. A is P_n -crowded set if and only if $f(A)$ is P_n -crowded set
2. A is P_{na} -crowded set if and only if $f(A)$ is P_{na} -crowded set.

Proof :

1. Let A is P_n -crowded set, so $\mathfrak{N}_{T_X}(i - \text{cl}_X(A)) = \emptyset$ in order to prove that $f(A)$ is P_n -crowded set and for that first we will prove $\mathfrak{N}_{T_Y}(i - \text{cl}_Y(f(A))) = f(\mathfrak{N}_{T_X}(i - \text{cl}_X(A)))$.

Let $y \in \mathfrak{N}_{T_Y}(i - \text{cl}_Y(f(A)))$ then exist $x \in X$, such that $y = f(x)$ also there exist i -open set H of y proper subset of Y satisfy $H\delta_Y i - \text{cl}_Y(f(A))$ and since f^{-1} is δ -

continuous, then $f^{-1}(H)\delta_X f^{-1}(i - cl_Y(f(A)))$, but $f^{-1}(H)$ is i -open set of x , because f is i -homeomorphism there exist i -open set $f^{-1}(H)$ of the point x such that $f^{-1}(H)\delta_X i - cl_X(A)$, hence $x \in \mathfrak{N}_{T_X}(i - cl_X(A))$ and then $f(x) \in f(\mathfrak{N}_{T_X}(i - cl_X(A)))$. Conversely, let $y \in f(\mathfrak{N}_{T_X}(i - cl_X(A)))$, so $x \in \mathfrak{N}_{T_X}(i - cl_X(A))$, and by definition (2-2-1) we have i -open set G of x , satisfy $G\delta_X i - cl_X(A)$, since f is δ -continuous the $f(G)\delta_Y f(i - cl_X(A))$ also f is i -homeomorphism then $f(G)\delta_Y i - cl_Y(f(A))$ and $f(G)$ is i -open set of y , hence $y \in \mathfrak{N}_{T_Y}(i - cl_Y(f(A)))$. Now will back to the first step that is $\mathfrak{N}_{T_X}(i - cl_X(A)) = \emptyset$, so $f(\mathfrak{N}_{T_X}(i - cl_X(A))) = \emptyset$ and by above conversation $\mathfrak{N}_{T_Y}(i - cl_Y(f(A))) = \emptyset$, hence $f(A)$ is P_n -crowded set. Conversely, let $f(A)$ is P_n -crowded set then $\mathfrak{N}_{T_Y}(i - cl_Y(f(A))) = \emptyset$ and by a bove conversation we have that $\mathfrak{N}_{T_Y}(i - cl_Y(f(A))) = f(\mathfrak{N}_{T_X}(i - cl_X(A)))$, hence $f(\mathfrak{N}_{T_X}(i - cl_X(A))) = \emptyset$ and because f is i -homeomorphism we get that $\mathfrak{N}_{T_X}(i - cl_Y(A)) = \emptyset$ and then A is P_n -crowded set.

2. Let A is P_{n_a} -crowded set then $\mathfrak{N}_{T_X}(i - cl_X(A)) \neq \emptyset$. First we will prove that $f^{-1}(\mathfrak{N}_{T_Y}(i - cl(f(A)))) = \mathfrak{N}_{T_X}(i - cl_X(A))$. Let $x \in f^{-1}(\mathfrak{N}_{T_Y}(i - cl(f(A))))$ then $f(x) \in \mathfrak{N}_{T_Y}(i - cl_Y(f(A)))$, so there exist i -open set H of y satisfy $H\delta_Y i - cl(f(A))$ and because f^{-1} is δ -continuous, then $f^{-1}(H)\delta_X f^{-1}(i - cl(f(A)))$, but f is i -homeomorphism we have $f^{-1}(H)$ is i -open set of x and $f^{-1}(H)\delta_X i - cl_X(A)$ and then $x \in \mathfrak{N}_{T_X}(i - cl_X(A))$. Now let $x \in \mathfrak{N}_{T_X}(i - cl_X(A))$, then there exist i -open set G of x such that $G\delta_X i - cl_X(A)$ and since f is δ -continuous then $f(G)\delta_Y f(i - cl_X(A)) = i - cl_Y(f(A))$, so there exist i -open set $f(G)$ of y satisfy $f(G)\delta_Y i - cl_Y(f(A))$, which means that $y \in \mathfrak{N}_{T_Y}(i - cl_Y(f(A)))$ and then $x = f^{-1}(y) \in f^{-1}(\mathfrak{N}_{T_Y}(i - cl_Y(f(A))))$, therefor $f^{-1}(\mathfrak{N}_{T_Y}(i - cl_Y(f(A)))) = \mathfrak{N}_{T_X}(i - cl_X(A))$. We will back to first step, if possible that $\mathfrak{N}_{T_Y}(i - cl_Y(f(A))) = \emptyset$, so $f^{-1}(\mathfrak{N}_{T_Y}(i - cl_Y(f(A)))) = \emptyset$ and this mean

that $\aleph_{T_Y}(i - cl_X(A)) = \emptyset$ and this contradiction. Conversely, let $f(A)$ is open crowded set then $\aleph_{T_Y}(I - cl_Y(f(A))) \neq \emptyset$, so by conversation of (1) we have that $\aleph_{T_Y}(i - cl_Y(f(A))) = f(N_{T_X}(i - cl_X(A)))$, so if possible that $\aleph_{T_X}(i - cl_X(A)) = \emptyset$ then $f(\aleph_{T_X}(i - cl_X(A))) = \emptyset$ and then $\aleph_{T_Y}(i - cl_Y(f(A))) = \emptyset$ and this contradiction, hence A is P_{n_a} - crowded set.

Proposition (5-3-15):

Let $f: (X, T_X, I_X, \delta_X) \rightarrow (Y, T_Y, I_Y, \delta_Y)$ be formatting, i - homeomorphism, δ - continuous function then :

1. A is P_{f_n} - crowded set if and only if $f(A)$ is P_{f_n} - crowded set.
2. A is $P_{f_{n_a}}$ - crowded set if and only if $f(A)$ is $P_{f_{n_a}}$ - crowded set.

Proof :

1. Let A is $P_{f_{n_a}}$ - crowded set ,then $\aleph_{I_{\phi_X}}(Fcl_X(A)) = \emptyset$. If possible that $\aleph_{I_{\phi_Y}}(Fcl_Y(f(A))) \neq \emptyset$,this means that there exist $y \in Y$ such that $y \in \aleph_{I_{\phi_Y}}(Fcl_Y(f(A)))$ and since f is i - homeomorphism then f is onto so there is $x \in X$, such that $y = f(x)$ also by definition (2-2-12) we have $H \in I_{\phi}(y)$ satisfy $H\delta_Y Fcl_Y(f(A))$. Because f is δ - continuous then $f^{-1}(H)\delta_X f^{-1}(Fcl_Y(f(A)))$, but f is i - homeomorphism $f^{-1}(H)\delta_X Fcl_X(A)$, and by proposition (2-1-13), $f^{-1}(H)$ is a focal set of the point x , from that we get $x \in \aleph_{I_{\phi_X}}(Fcl_X(A))$, and this contradiction, so $\aleph_{I_{\phi_Y}}(Fcl_Y(f(A))) = \emptyset$, and then $f(A)$ is $P_{f_{n_a}}$ - crowded set. Conversely, let $f(A)$ is $P_{f_{n_a}}$ - crowded set , so $\aleph_{I_{\phi_Y}}(Fcl_Y(f(A))) = \emptyset$, if possible that $\aleph_{I_{\phi_X}}(Fcl_X(A)) \neq \emptyset$, then there exist $x \in X$ such that $x \in \aleph_{I_{\phi_X}}(Fcl_X(A))$ so by focal nested definition we have a focal set G of x satisfy $G\delta_X Fcl_X(A)$, because f is δ - continuous then $f(G)\delta_Y(Fcl_Y(f(A)))$, also f is formatting i - homeomorphism function we have $f(G)$ is

focal set of y in Y and then $y \in \mathfrak{N}_{I_\phi_Y}(\text{Fcl}_Y(f(A)))$ and this contradiction hence $\mathfrak{N}_{I_\phi_X}(\text{Fcl}(A)) = \emptyset$.

2. Let A is Pf_{n_a} – crowded set, then $\mathfrak{N}_{I_\phi_X}(\text{Fcl}(A)) \neq \emptyset$. If possible that $\mathfrak{N}_{I_\phi_Y}(\text{Fcl}(f(A))) = \emptyset$ then $f(A)$ is Pf_n – crowded set, so by (1) A is Pf_n – crowded set and this contradiction hence $f(A)$ is Pf_{n_a} –crowded set. Conversely, let $f(A)$ is Pf_{n_a} – crowded set, then $\mathfrak{N}_{I_\phi_Y}(\text{Fcl}_Y(f(A))) \neq \emptyset$. If possible $\mathfrak{N}_{I_\phi_X}(\text{Fcl}_X(A)) = \emptyset$, this means that A is Pf_n – crowded set, so $f(A)$ is Pf_n – crowded set and this contradiction.

The relation between crowded and congested concepts are discussed below .

Proposition(5-3-16) :

Let (α, T, I, δ) be an i – topological proximity space then

1. Every P – congested(P_f – congested)set is P – crowded(P_f – crowded set).
2. Every P_n – crowded (Pf_n – crowded)set is P_n – congested (Pf_n – congested) set.

Proof :

1. Let A is P – congested set, so $\mathfrak{N}_T(A)\delta i - \text{cl}(A)$,and then $i - \text{cl}(A) \neq \emptyset$. If possible that $\mathfrak{N}_T(i - \text{cl}(A)) = \emptyset$, then for each $x \in X$ and $U \in T(x)$, $U\bar{\delta}i - \text{cl}(A)$, hence $U \cap i - \text{cl}(A) = \emptyset$, but $x \in U$, hence $x \notin i - \text{cl}(A)$, for each $x \in X$ and this contradiction, therefor $\mathfrak{N}_T(i - \text{cl}(A)) \neq \emptyset$, imply that A is P – crowded set.
2. Let A is P_n – crowded set, then $\mathfrak{N}_T(i - \text{cl}(A)) = \emptyset$, and since $A \subseteq i - \text{cl}(A)$, we get that $\mathfrak{N}_T(A) = \emptyset$,hence $\mathfrak{N}_T(A)\bar{\delta}i - \text{cl}(A)$, therefor A is P_n – congested set .

CHAPTER SIX
 ψ – OPERATOR
IN i – TOPOLOGICAL PROXIMITY
SPACES

6.1 ψ –operator via Focal set

Another form of ψ -operator is defined in this section by using employing two pillars, they are i-topological spaces and the proximity spaces.

Definition (6-1-1) :

Let (X, T, I, δ) be an i – topological proximity space , we define the operator $\psi_\delta: P(X) \rightarrow P(X)$ as : $\psi_\delta(A) = \{x \in X : \text{there are } u \in I_{\phi(x)} \text{ such that } U \ll A\}$.

Example (6-1-2) :

Let $(X, T_i, \{\emptyset\}, \delta)$,where δ defined by the discrete proximity space be an i – topological proximity space , then $\psi_\delta(A) = \emptyset$ for each subset A of X .

Some of the basic properties of ψ_δ –operator are proven in the following proposition .

Proposition (6-1-3) :

Let (X, T, I, δ) be an i – topological proximity space then each of the following are held :

1. $\psi_\delta(A) \subseteq \cup \{U \in I_{\phi(x)}, x \in A: U - A \in I\}$
2. $\psi_\delta(A) \supseteq \cup \{U \in I_{\phi(x)}, x \in A: U \ll A\}$
3. $\psi_\delta(A) \subseteq \cup \{U \in I_{\phi(x)}, x \in A: U \approx A\}$
4. $\psi_\delta(\emptyset) = \emptyset$ and $\psi_\delta(X) = X$

Proof :

1. Let $x \in \psi_\delta(A)$, so there exist $U \in I_{\phi(x)}$, such that $U \ll A$, hence $U - A = \emptyset \in I$,so If $x \notin \cup \{U \in I_{\phi(x)}: U - A \in I\}$ then $x \notin U$, for each $U \in I_{\phi(x)}$, $U - A \in I$, and this contradiction .
2. Let $x \in \cup \{U \in I_{\phi(x)}, U \ll A\}$ and by definition (6-1-1) , $x \in \psi_\delta(A)$
3. Let $x \in \psi_\delta(A)$ and if possible that $x \notin \cup \{U \in I_{\phi(x)}: x \in A, A \approx U\}$ then for each $U \in I_{\phi(x)}$, $U - A \cup A - U \notin I$ and then $U - A \notin I$,hence for each $U \in I_{\phi(x)}$, $U - A \notin I$

and from that we get $x \notin \cup \{U \in I_{\phi(x)} : U - A \in I\}$ and by (1) $x \notin \psi_{\delta}(A)$ and this contradiction .

4. If there exist $x \in \psi_{\delta}(A)$ then there exist $W \in I_{\phi(x)}, W \ll \emptyset$ and this contiadiction also if there exist $x \notin \psi_{\delta}(X)$ then there exist $W \in I_{\phi(x)}, W \ll X$.

Converse of above cases explain in this example

Example (6-1-4) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}, \{b\}\}$, $I = \{\emptyset, \{c\}\}$ and $A\delta B$ defined by the discrete proximity. If $A = \{a\}$, then $\cup \{U \in I_{\phi(x)} : U - A \in I\} \not\subseteq \psi_{\delta}\{A\} = \{a\}$ also if $B = \{a, b\}$ then $\psi_{\delta}(B) = X \not\subseteq \cup \{U \in I_{\phi(x)} : x \in B, U \ll B\} = \{a, b\}$ and if $H = \{a\}$ then $\cup \{U \in I_{\phi(x)}, x \in A, A \approx U\} = \{a, c\} \not\subseteq \psi_{\delta}(H) = \{a\}$

Proposition (6-1-5) :

Let (X, T, I, δ) be an i – topological proximity space and let A are subsets of X then

1. $\psi_{\delta}(A) = \emptyset$, for each $A^c \in I_{\phi(x)}$.
2. $\psi_{\delta}(A)$ is i – open set for each subset A of X
3. $\psi_{\delta}(A) \neq \emptyset$ if and only if $\psi_{\delta}(A) \in I_{\phi(x)}$, for some $x \in A$
4. If $A \subseteq B$ then $\psi_{\delta}(A) \subseteq \psi_{\delta}(B)$
5. $\psi_{\delta}(A \cap B) = \psi_{\delta}(A) \cap \psi_{\delta}(B)$
6. $\psi_{\delta}(A) \cup \psi_{\delta}(B) \subseteq \psi_{\delta}(A \cup B)$
7. $\psi_{\delta}(\psi_{\delta}(A)) = \psi_{\delta}(A)$ for each subset A of X
8. $X - \phi(X - A) \subseteq \psi_{\delta}(A)$
9. $\psi_{\delta}(A) \subseteq \phi(A)$
10. $\psi_{\delta}(A) = \emptyset$ for each $A \in I$
11. $\psi_{\delta}(A^c) = \emptyset$ for each $A \in I_{\phi(x)}$

Proof:

1. If $\psi_\delta(A) \neq \emptyset$, then there is $x \in \psi_\delta(A)$, so there is $W \in I_{\phi(x)}$, $W \ll A$, from that we get $W \subseteq A$, so from theorem (2-1-2) (2) $A \in I_{\phi(x)}$, and by the same theorem part 6 this contradiction .
2. If possible that $\psi_\delta(A) \notin T$, so for each $H \in T, H \not\approx \psi_\delta(A)$, then $H - \psi_\delta(A) \cup \psi_\delta(A) - H \notin I$, hence $H - \psi_\delta(A) \cup \psi_\delta(A) - H \neq \emptyset$, and this mean that $H - \psi_\delta(A) \neq \emptyset$, or $\psi_\delta(A) - H \neq \emptyset$ or both cases are not empty and all of these cases for each $H \in T$, so for case one there exist a point x belong to $H - \psi_\delta(A)$, which is read as $x \in H$,for each i-open set H , and that is contradiction by the empty set .for the second case if there exist a point x belong to $\psi_\delta(A) - H$, which is read as $x \notin H$,for each i-open set H , and that is contradiction by the universal set X So from this conversation we get that $H - \psi_\delta(A) \cup \psi_\delta(A) - H \in I$ and then $\psi_\delta(A)$ is i – open set .
3. By (2) and by proposition (2-1-2)(1)
4. Let $A \subseteq B$ and let $x \in \psi_\delta(A)$, then there exist $W \in I_{\phi(x)}$, $W \ll A$ but $B^c \subseteq A^c$ hence by proposition (1-3-5)(2) we get $W\bar{\delta}B^c$,and then $x \in \psi_\delta(B)$.
5. First since $A \cap B \subseteq A$ and B , so by (4) $\psi_\delta(A \cap B) \subseteq \psi_\delta(A) \cap \psi_\delta(B)$.
Now let $x \in \psi_\delta(A) \cap \psi_\delta(B)$, then there exist $H, W \in I_{\phi(x)}$ such that $H \ll A$ and $W \ll B$ and by proposition (2-1-2) $H \cap W \in I_{\phi(x)}$, so by proposition (1-3-5) $H \cap W \ll A$ and $H \cap W \ll B$ and then by proposition (1-3-8)(13) $H \cap W \ll A \cap B$.
6. Since A and B are subset of $A \cup B$ and by (5) we get that $\psi_\delta(A) \cup \psi_\delta(B) \subseteq \psi_\delta(A \cup B)$.
7. Let $x \in \psi_\delta(\psi_\delta(A))$,then by proposition (6-1-3)(3) we get that $x \in \cup \{U \in I_{\phi(x)}, x \in \psi_\delta(A): U \approx \psi_\delta(A)\}$,hence $x \in \psi_\delta(A)$.Conversely since $\psi_\delta(A)$ is i – open set then by (4) $\psi_\delta(A) \subseteq \psi_\delta(\psi_\delta(A))$.
8. Let $x \in X - \phi(X - A)$,then $x \notin \phi(X - A)$,hence there exist $W \in I_{\phi(x)}$, $x \in W$ and $W\bar{\delta}X - A$ and then $x \in \psi_\delta(A)$.
9. Let $x \in \psi_\delta(A)$,then there exist $W \in I_{\phi(x)}$, $W \ll A$,hence by theorem (2-1-2)(2) , $A \in I_{\phi(x)}$.If possible that $x \notin \phi(A)$ then there exist $H \in I_{\phi(x)}$, $x \in H$,and $H\bar{\delta}A$ so

$H \subseteq A^c$ and by theorem (2-1-2)(2) $A^c \in I_{\phi(x)}$ and by the same theorem part 6 this is contradiction .

10. Suppose that $\psi_{\delta}(A) \neq \emptyset$, so there exist $x \in \psi_{\delta}(A)$, hence there exist $U \in I_{\phi(x)}$, $U \ll A$ and by theorem (2-1-2)(2) then $A^c \in I_{\phi(x)}$ and by the same theorem part 6 this contradiction therefor $\psi_{\delta}(A) = \emptyset$.

11. If possible that $\psi_{\delta}(A^c) \neq \emptyset$, then there exist a point x of X , such that $x \in \psi_{\delta}(A^c)$, so there exist $U \in I_{\phi(x)}$, $U \ll A^c$ and by theorem (2-1-2)(2) $A^c \in I_{\phi(x)}$, but by assumption $A \in I_{\phi(x)}$ and this contradiction by proposition (2-1-2)(6) .

Example (6-1-6) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$ and $I = \{\emptyset, \{c\}\}$ and δ is the discrete proximity space ,then If $A = \{a\}$, $B = \{b\}$, then $\psi_{\delta}(A) \cup \psi_{\delta}(B) = \{a, c\} \cup \emptyset = \{a, c\}$.But $\psi_{\delta}(A \cup B) = \psi_{\delta}(\{a, b\}) = X \not\subseteq \{a, c\}$.Also $\psi_{\delta}(A) \neq \emptyset$, but $A \notin T$ and $\psi_{\delta}(B) \subseteq \psi_{\delta}(A)$ but $B \not\subseteq A$ and $\psi_{\delta}(A) \not\subseteq X - \phi(X - A) = \{a, c\}$ and $\phi(A) = X \not\subseteq \psi_{\delta}(A) = \{a, c\}$.

Proposition (6-1-7) :

Let (X, T, I, δ) be an i – topological proximity space then

1. $\psi_{\delta}(A^c) \subseteq (\psi_{\delta}(A))^c$.
2. $\psi_{\delta}(A) - A = \emptyset$ for each i – closed set A of X .
3. $(\psi_{\delta}(A))^c \subseteq \phi(A^c)$, for each subset A of X .
4. $(\phi(A))^c \subseteq \psi_{\delta}(A^c)$ for each subset A of X .
5. $\psi_{\delta}(A) \subseteq \text{Fcl}(A)$.
6. A is FODexse if and only if $\psi_{\delta}(A^c) = \emptyset$, for each $A \subseteq X$.
7. $\psi_{\delta}(A) \cap A = \emptyset$, for each $A \in I$.
8. If $A \cap B = \emptyset$,then $\phi(A) \cap \psi_{\delta}(B) = \emptyset$ for each A, B are subsets of X .
9. $\psi_{\delta}(A) \subseteq A^c$ for each $A \in I$.

Proof :

1. Let $x \in \psi_\delta(A^c)$, so there exist $U \in I_{\phi(x)}$, $U \ll A^c$, hence $A^c \in I_{\phi(x)}$. If possible that $x \notin (\psi_\delta(A))^c$,then $x \in \psi_\delta(A)$ and hence there exist $W \in I_{\phi(x)}$, $W \bar{\delta} A^c$ and then $W \subseteq A$,so by theorem (2-1-2)(2) we get that $A \in I_{\phi(x)}$ but $A^c \in I_{\phi(x)}$ and this contradiction , hence $x \in (\psi_\delta(A))^c$.
2. If possible that $\psi_\delta(A) - A \neq \emptyset$, so there exist $x \in \psi_\delta(A) - A$, hence there exist $W \in I_{\phi(x)}$, $W \ll A$, so by theorem (2-1-2)(2) $A \in I_{\phi(x)}$ but by the same theorem $A^c \in I_{\phi(x)}$ and this contradiction .
3. Let $x \in (\psi_\delta(A))^c$, so $x \notin \psi_\delta(A)$ then for each $U \in I_{\phi(x)}$, $U \bar{\delta} A^c$ then $x \in \phi(A^c)$.
4. Let $x \in (\phi(A))^c$, then $x \notin \phi(A)$, so there exist $U \in I_{\phi(x)}$, $x \in U$, $U \bar{\delta} A$, hence $x \in \psi_\delta(A^c)$.
5. Let $x \in \psi_\delta(A)$,then there exist $U \in I_{\phi(x)}$,such that $U \ll A$, from that we get $U \subseteq A$, and by theorem (2-1-2)(2) $A \in I_{\phi(x)}$, if $x \notin \text{Fcl}(A)$,then there exist $H \in T(x)$,such that $H \cap A = \emptyset$, hence $A^c \in I_{\phi(x)}$ and by theorem (2-1-2)part 5 this imply the contradiction so $x \in \text{Fcl}(A)$.
6. If possible that $\psi_\delta(A^c) \neq \emptyset$, so there exist $x \in \psi_\delta(A^c)$, hence there exist $U \in I_{\phi(x)}$, $U \ll A^c$,so by proposition (2-1-2)(2) we have that $A^c \in I_{\phi(x)}$, from that we have $H \in T(x)$ such that $H \alpha A^c$, hence $H \cap A \in I$, now if $H \cap A = \emptyset$,and by assume this is contradiction because $x \in \text{Fcl}(A)$, so for each i-open set H of x we have , $H \cap A \neq \emptyset$, and then by proposition (6-1-5)(10) $\psi_\delta(A \cap H) = \emptyset$,also by the same proposition (6) $\psi_\delta(A) \cap \psi_\delta(H) = \emptyset$, but $\psi_\delta(A)$, $\psi_\delta(H) \in I_{\phi(x)}$ and this contradiction hence $\psi_\delta(A^c) = \emptyset$. Conversely , if possible that $\text{Fcl}(A) \neq X$, so there exist $x \in X$ and $x \notin \text{Fcl}(A)$, hence there exist $U \in T(x)$, $U \cap A = \emptyset$, then by theorem (2-1-2)(2) $A^c \in I_{\phi(x)}$, also by proposition (6-1-5)(4) $\psi_\delta(U) \subseteq \psi_\delta(A^c) = \emptyset$,then $\psi_\delta(U) = \emptyset$,but by theorem (2-1-2)(1) the empty set is not a focal set ,also by proposition (6-1-5) (2) this contradiction.
7. Since $A \in I$ and by proposition (6-1-5) (10) $\psi_\delta(A) = \emptyset$ then $\psi_\delta(A) \cap A = \emptyset$, conversly , if $A \cap \psi_\delta(A) = \emptyset$, so $\psi_\delta(A) \subseteq A^c$ and then by proposition (2-1-2)(2)

$A^c \in I_{\phi(x)}$, if $A \notin I$ then $A \neq \emptyset$, so there exist $x \in A$, hence $x \notin \psi_\delta(A)$ and then for each $H \in I_{\phi(x)}$, $H\delta A$, but $A^c \in I_{\phi(x)}$, then $A^c\delta A$, and this contradiction.

8. If possible that $\phi(A) \cap \psi_\delta(B) \neq \emptyset$ so there exist $x \in \phi(A) \cap \psi_\delta(B)$, hence $x \in \phi(A)$, so for each $U \in I_{\phi(x)}$, $U\delta A$ and $x \in U$, and $x \in \psi_\delta(B)$, we have $H \in I_{\phi(x)}$, $H\bar{\delta}B^c$, But $A \subseteq B^c$ then $H\bar{\delta}B^c$ then by proposition (1-3-5) $H\bar{\delta}A$ and this contradiction.

9. Since $A \in I$, so by proposition (6-1-4)part 10, $\psi_\delta(A) = \emptyset$, and then $\psi_\delta(A) \cap A = \emptyset$ also $\psi_\delta(A) \subseteq A^c$.

Example (6-1-8) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$ $I = \{\emptyset, \{c\}\}$, δ is the discrete proximity, then if $A = \{a\}$ then $Fcl(A) = X \not\subseteq \psi_\delta(A)$ also if $B = \{c\}$ then $\psi_\delta(B^c) = X \not\subseteq (\phi(B))^c = \{a, b\}$ and $\phi(A^c) = \{b, c\} \not\subseteq (\psi_\delta(A))^c = \{b\}$ and also $(\psi_\delta(A))^c = \{b\} \not\subseteq \psi_\delta(A^c) = \emptyset$

Proposition (6-1-9) :

Let (X, T, I, δ) be an i -topological proximity space and let A, B are subset of X then:

1. If $A \approx B$ then $\psi_\delta(A) = \psi_\delta(B)$, for each subset A, B of X .
2. $(\phi(A))^c \subseteq \psi_\delta(A^c)$ for each subset A of X .
3. $\psi_\delta((\phi(A))^c) = \emptyset$, for each $A \in T$
4. $\phi(\psi_\delta(A)) \in I_{\phi(x)}$ and $\phi_{T(x)}(\psi_\delta(A)) \in I_{\phi(x)}$ for each subset A of X and for some $x \in X$.
5. $\psi_\delta(A) \subseteq \psi_\delta(\phi(A))$ for each subset A of X
6. if $\psi_\delta(A^c) = \emptyset$, then $\phi(A) = X$.

Proof :

1. Since $A \approx B$, so $A - B \in I$ and then $\psi_\delta(A - B) = \emptyset$, hence by proposition (6-1-5)(6) $\psi_\delta(A) \cap \psi_\delta(B^c) = \emptyset$, then by proposition (6-1-7)(1) $\psi_\delta(A) \subseteq (\psi_\delta(B^c))^c \subseteq ((\psi_\delta(B))^c)^c = \psi_\delta(B)$ and similarly we prove that $\psi_\delta(B) \subseteq \psi_\delta(A)$ hence $\psi_\delta(A) = \psi_\delta(B)$

2. Let $x \in (\phi(A))^c$, so $x \notin \phi(A)$, hence there exist $U \in I_{\phi(x)}$, $x \in U, U \bar{\delta} A$ and then $x \in \psi_{\delta}(A^c)$.
3. Since $A \in T$ so by proposition (3-1-4)(6) we have $\phi(A) \in I_{\phi(x)}$ and by proposition (6-1-5)(1) $\psi_{\delta}((\phi(A))^c) = \emptyset$.
4. If $\phi(\psi_{\delta}(A)) = \emptyset$, then $\phi(\psi_{\delta}(A)) \notin I_{\phi(x)}$. If $\phi(\psi_{\delta}(A)) \neq \emptyset$, then by definition (3-1-1) $\psi_{\delta}(A) \neq \emptyset$, and by proposition (6-1-5)part 3 , Since $\psi_{\delta}(A) \in I_{\phi(x)}$,so by proposition (3-1-4)(6) $\phi(\psi_{\delta}(A)) \in I_{\phi(x)}$.And by proposition (3-2-3)(2) and theorem (2-1-2)(2) $\phi_T(\psi_{\delta}(A)) \in I_{\phi(x)}$.
5. By proposition (3-1-3) $A \subseteq \phi(A)$ for each subset A of X and by proposition (6-1-5)(4) we get that $\psi_{\delta}(A) \subseteq \psi_{\delta}(\phi(A))$.
6. By (3), $(\phi(A))^c \subseteq \psi_{\delta}(A^c)$, $(\phi(A))^c = \emptyset$,then $\phi(A) = X$

Example (6-1-10) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, $I = \{\emptyset, \{c\}\}$, and δ defined by the discrete proximity ,then if $A = \{a\}$ then $\phi(A^c) = \{b, c\} \not\subseteq (\psi_{\delta}(A))^c$,also if $B = \{c\}$ then $\psi_{\delta}(B^c) = X \not\subseteq (\phi(B))^c$,and also $\psi_{\delta}(\phi(A)) = X \not\subseteq \psi_{\delta}(A) = \{a, c\}$.

Example (6-1-11) :

$X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}\}$, $I = \{\emptyset\}$,and δ defined by the discrete proximity ,so if $A = \{a, b\}$ then $\phi(A) = X$ but $\psi_{\delta}(X - A) \neq \emptyset$.

Proposition (6-1-12) :

Let (X, T, I_j, δ) , $j = 1, 2$ be an i – topology proximty space such that $I_1 \subseteq I_2$ then $\psi_{\delta_{I_1}}(A) \subseteq \psi_{\delta_{I_2}}(A)$

Proof :

Since $I_1 \subseteq I_2$, so by proposition (2-1-10) $I_1 \phi(x) \subseteq I_2 \phi(x)$ and hence $\psi_{\delta_{I_1}}(A) \subseteq \psi_{\delta_{I_2}}(A)$

Proposition (6-1-13) :

Let $(X, T_j, I, \delta), j = 1, 2$ be an i – topological proximity space such that $T_1 \subseteq T_2$ then $\psi_{\delta_{T_1}}(A) \subseteq \psi_{\delta_{T_2}}(A)$

Proof:

by proposition (2-1-10) we get that $I_{T_1\phi(x)} \subseteq I_{T_2\phi(x)}$ and by proposition (6-1-12) we get that $\psi_{\delta_{T_1}}(A) \subseteq \psi_{\delta_{T_2}}(A)$.

6.2 ψ –operator with respect to i-open set

A study related to the i – open set was presented in this section with a study of all the traits and characteristics that can be studied in addition to studying the relationship between the ψ_T –operator and ψ_δ -operator and the results related to them.

Definition (6-2-1) :

let (X, T, I, δ) be an i – topological proximity space an operator $\psi_T: P(X) \rightarrow P(X)$ defined as follow : $\psi_T(A) = \{x \in X: \text{there exist } U \in T(x), U \ll A\}$

Example (6-2-2) :

In the indiscrete i-topological proximity space $(X, T, \{\emptyset\}, \delta)$ Let and δ is the discrete proximity , $\psi_T(A) = X$,for each subset A of X .

Proposition (6-2-3) :

Let (X, T, I, δ) be an i – topological proximity space then each of the following are held for every subsets A,B of X :

1. $\psi_T(A) \subseteq A$,for each $A \subseteq X$.
2. $\psi_T(A) \subseteq \psi_\delta(A)$, for each $A \subseteq X$.
3. $\psi_T(A) = \emptyset$, for each $A \in I$.
4. If $A \subseteq B$,then $\psi_T(A) \subseteq \psi_T(B)$.
5. $\psi_T(A) \cup \psi_T(B) \subseteq \psi_T(A \cup B)$.
6. $\psi_T(A \cap B) \subseteq \psi_T(A) \cap \psi_T(B)$ for each $A \subseteq X$.

Proof :

1. Let $x \in \psi_T(A)$, so there exist $U \in T(x)$ such that $U \ll A$ and hence $U \subseteq A$ and then $x \in A$.
2. Let $x \in \psi_T(A)$, so there exist $U \in T(x)$ such that $U \ll A$ and hence there exist $U \in I_{\delta(x)}$, $U \ll A$ and then $x \in \psi_\delta(A)$.
3. By proposition (6-1-5) (11) and by (1) we get that $\psi_T(A) = \emptyset$.

4. Let $x \in \psi_T(A)$, so there exist $U \in T(x)$ such that $U \bar{\delta} A^c$, so by proposition (1-3-5)(1), we have $U \bar{\delta} B^c$ and then $x \in \psi_T(B)$.
5. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ then by (6) we get that $\psi_T(A) \cup \psi_T(B) \subseteq \psi_T(A \cup B)$.
6. Since $A \cap B \subseteq A$,and $A \cap B \subseteq B$ and by (6) we get that $\psi_T(A \cap B) \subseteq \psi_T(A) \cap \psi_T(B)$.

Example (6-2-4) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, $I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$, and δ defined by the discrete proximity, If $A = \{a\}$ then $\psi_\delta(A) = X \not\subseteq \psi_T(A) = \emptyset$ also $\psi_T\{c\} \subseteq \psi_T\{a, b\}$ but $\{c\} \not\subseteq \{a, b\}$ and if $A = \{c\}$ and $B = \{a, b\}$ then $\psi_T(A \cup B) = X$ but $\psi_T(A \cup B) \not\subseteq \psi_T(A) \cup \psi_T(B) = \{a, b\}$ also if $A = \{a, b\}$, $B = \{a, c\}$ then $\psi_T(A) \cap \psi_T(B) = \{a\} \not\subseteq \psi_T(A \cap B) = \emptyset$.

Remark (6-2-5) :

$\psi_{T(x)}(A)$ is not i – open set in general as in the following example :

Example (6-2-6) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}, \{b\}\}$, $I = \{\emptyset, \{c\}\}$,and δ is the discrete proximity ,then if $A = \{a, b\}$ then $\psi_T(A) = \{a, b\} \notin T$.

Proposition (6-2-7) :

Let (X, T, I, δ) be an i – topological proximty space ,then each of the following are holds:

1. $X - \phi_T(X - A) = \psi_T(A)$,for each subset A of X .
2. $\psi_T(\psi_T(A)) \subseteq \psi_T(A)$,for each subset A of X .
3. $\psi_T(A) \subseteq \text{Fcl}(A)$,for each subset A of X .
4. $\psi_T(A^c) \subseteq (\psi_T(A))^c$,for each subset A of X .
5. $\psi_T(A) \subseteq \phi_T(A)$,for each subset A of X .
6. A is FOdense set then $\psi_T(A^c) = \emptyset$.
7. A is ϕ_T dense set if and only if $\psi_T(A^c) = \emptyset$.

Proof :

1. Let $x \in X - \phi_T(X - A)$ if and only if $x \notin \phi_T(X - A)$ if and only if there exist $W \in T(x)$ such that $W\bar{\delta}(X - A)$ if and only if there exist $W \in T(x)$ such that $W \ll A$ if and only if $x \in \psi_T(A)$
2. By (1) and by proposition (3-2-3)(8) $X - \phi_T(X - A) = \psi_T(A)$, so $\psi_T(\psi_T(A)) = X - \phi_T(\phi_T(X - A)) \subseteq (X - \phi_T(X - A))$ we get that $\psi_T(\psi_T(A)) \subseteq \psi_T(A)$.
3. Let $x \in \psi_T(A)$, so there exist $w \in T(x)$ such that $W\bar{\delta}A^c$, hence $W \cap A^c = \emptyset$ and then $x \in A$, if possible that $x \notin Fcl(A)$ then there exist $H \in T(x)$, $H \cap A = \emptyset$, but $x \in H \cap A$ and this contradiction , hence $x \in Fcl(A)$.
4. By proposition (5-2-3)(2) we have $\psi_T(A) \subseteq \psi_\delta(A)$, so $(\psi_\delta(A))^c \subseteq (\psi_T(A))^c$,hence by proposition (6-1-7)(1) $\psi_T(A^c) \subseteq \psi_\delta(A^c) \subseteq (\psi_T(A))^c$.
5. Let $x \in \psi_T(A)$, so there exist $W \in T(x)$, $W\bar{\delta}A^c$,and hence $x \in A$, so $x \in A \cap W$,for each $W \in T(x)$, then $x \in \phi_T(A)$.
6. By proposition (6-2-3)(2) and by proposition (6-1-7)(7) we have that $\psi_\delta(A^c) = \emptyset$ so $\psi_T(A^c) = \emptyset$.
7. If possible that $\psi_T(A^c) \neq \emptyset$ then there exist $x \in \psi_T(A^c)$, hence there exist $W \in T(x)$ such that $W \ll A^c$ and this mean that $W\bar{\delta}A$ hence $x \notin \phi_T(A)$ and this contradiction .
Conversely , suppose that there exist $x \in X$ and $x \notin \phi_T(A)$, so there exist $W \in T(x)$, $W\bar{\delta}A$, then $x \in \psi_T(A^c)$ and this contradiction .

Example (6-2-8) :

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$ and $I = \{\emptyset, \{c\}\}$, and δ defined by the discrete proximity then if $A = \{a\}$ then $Fcl(A) = X \not\subseteq \psi_T(A)$,also $(\psi_T(A))^c = X \not\subseteq \psi_T(A^c)$ and $\phi_T(A) = X \not\subseteq \psi_T(A)$ and if $A = \{b, c\}$, so $\psi_T(B^c) = \emptyset$, but $Fcl(B) \neq X$.

Proposition (6-2-12) :

Let (X, T, I, δ) be an i – topological proximity space then If A is ϕ dense set then $\psi_{T(x)}(A^c) = \emptyset$.

Proof :

By proposition (4-3-16) A is ϕ_T -dense set and by proposition (6-2-7) $\psi_T(A^c) = \emptyset$.

Proposition (6-2-13) :

Let $(X, T_j, I, \delta), j = 1, 2$ be an i – topological proximity space such that T_2 is finer than T_1 then $\psi_{T_1}(A) \subseteq \psi_{T_2}(A)$.

Proof:

Let $x \in \psi_{T_1}(A)$, so there exist $W \in T_1(x)$ such that $W \ll A^c$, so there exist $W \in T_2(x)$ such that $W \ll A$, and hence $x \in \psi_{T_2}(A)$.

Proposition (6-2-14) :

Let (X, T, I, δ) be an i – topological proximity space then :

1. $\psi_\delta(\phi(A)^c) = \emptyset$ and $\psi_\delta(\phi_T(A)^c) = \emptyset$ for each $A \in I_{\phi(x)}$.
2. $\phi(\psi_\delta(A)) \in I_{\phi(x)}$, $\phi_T(\psi_\delta(A)) \in I_{\phi(x)}$ and $\phi_T(\psi_T(A)) \in I_{\phi(x)}$ for some $x \in A$.

Proof:

1. By proposition (3-1-4)(6) we have that $\phi(A) \in I_{\phi(x)}$ and by proposition (6-1-5)(12) we get that $\psi_\delta(\phi(A)^c) = \emptyset$. Also for another case since $A \subseteq \phi_T(A)$ so by proposition (2-1-2)(2) we get that $\phi_T(A) \in I_{\phi(x)}$ and by proposition (6-1-5)(12) we get that $\psi_\delta(\phi_T(A)^c) = \emptyset$.
2. By proposition (6-1-5)(3) and by proposition (3-1-4)(6) , we can prove each of these cases.

6.3 On ψ_δ -set and $\mathcal{F}\psi$ –set

Various forms of ψ - set corresponding to different cases of spaces were introduced in a previous studies and the relationships among them therefore the focus of the study in this section will be on the nature of the effects that can be obtained by using proximity spaces, when studying this type of set by investigating a new class of sets called ψ_δ -set and $\mathcal{F}\psi$ –set in i -topological proximity spaces.

Definition (6-3-1)

Let (X, T, I, δ) be an i – topological proximity space then a subset A of X is said to be ψ_δ – set if and only if $A \delta \phi(\psi_\delta(A))$ and it is called $\mathcal{F}\psi_\delta$ – set if and only if $A \delta \text{Fcl}(\psi_{T(X)}(A))$, the collection of all ψ_δ – set of X indicated by $\psi_\delta(X, T, I, \delta)$ and $\mathcal{F}\psi(X, T, I, \delta)$ stand for the set of all $\mathcal{F}\psi$ – set .

The relation between ψ_δ – set and $\mathcal{F}\psi$ – set showed below

Proposition (6-3-2)

Let (X, T, I, δ) be an i – topological proximity space then every $\mathcal{F}\psi$ – set is ψ_δ – set but not conversely.

Proof:

By proposition (6-2-3) (2) and (3-1-12)we get that $\text{Fcl}(\psi_T(A)) \subseteq \phi(\psi_\delta(A))$,and then since $A \delta \text{Fcl}(\psi_T(A))$, so $A \delta \phi(\psi_\delta(A))$ therefor A is ψ_δ – set.

Example (6-3-3)

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, $I = \{\emptyset, \{c\}\}$,and δ defined by the proximity space. If $A = \{a\}$ then $\phi(\psi_\delta(A)) = X \not\subseteq \text{Fcl}(\psi_T(A))$ hence A is ψ_δ – set but not $\mathcal{F}\psi$ – set.

Proposition (6-3-4)

Let (X, T, I, δ) be an i – topological proximity space then each of the following statement are exist :

1. A is not ψ_δ – set neither $F\psi$ – set for each $A \in I$.
2. If $A, B \in \psi_\delta(X, T, I, \delta)$ then $A \cup B \in \psi_\delta(X, T, I, \delta)$.
3. $X \in \psi_\delta(X, T, I, \delta)$ and $X \in F\psi(X, T, I, \delta)$.

Proof:

1. Let $A \in I$, so by proposition (6-1-5)(10) $\psi_\delta(A) = \emptyset$ and then $A \bar{\delta} \phi(\psi_\delta(A))$ also by proposition (6-2-3)(4) A is not $F\psi$ – set.
2. since $A \delta \phi(\psi_\delta(A))$ and $B \delta \phi(\psi_\delta(B))$, so by definition (1-3-1) we get that $(A \cup B) \delta \phi(\psi_\delta(A))$,also by proposition (1-3-6) $(A \cup B) \delta \phi(\psi_\delta(B))$, $(A \cup B) \delta \phi(\psi_\delta(A)) \cup \phi(\psi_\delta(B))$,and by proposition (6-1-5) (7) ,(3-1-4)we get that $(A \cup B) \delta \phi(\psi_\delta(A \cup B))$.
4. Since $X \delta \phi(\psi_\delta(A))$ for each $A \subseteq X$, so $X \in \psi_\delta(X, T, I, \delta)$.

Now the following proposition discuss the inclusion condition

Proposition (6-3-5)

Let (X, T, I, δ) be an i – topological proximity space and A, B are subset of X ,such that $A \subseteq B$ and A is ψ_δ – set (resp., $F\psi$ – set) , then B is ψ_δ – set (resp. $F\psi$ – set)

Proof:

Because A is ψ_δ – set, then $A \delta \phi(\psi_\delta(A))$,then by proposition (1-3-6) we get that $B \delta \phi(\psi_\delta(A))$, and by proposition (6-1-5)(7), (3-1-4) we have $B \delta \phi(\psi_\delta(B))$, hence B is ψ_δ – set. In a same way we get that B is $F\psi$ – set.

Proposition (6-3-6)

Let (X, T, I, δ) be an i – topological proximity space and A, B are subset of X such that $A \cap B$ is ψ_δ – set ($F\psi$ – set) then A and B are ψ_δ – set (resp. $F\psi$ – set) .

Proof:

By proposition (6-3-5) we get that A, B are ψ_δ – set.

Proposition (6-3-7)

Let (X, T, I, δ) be an i – topological proximity space then every $\psi_\delta(A)$, is ψ_δ – set for every subset A of X such that $\psi_\delta(A) \neq \emptyset$.

Proof:

Since by proposition (3-1-3) we have that $\psi_\delta(A) \subseteq \phi(\psi_\delta(A))$ hence $\psi_\delta(A) \delta \phi(\psi_\delta(A))$ therefor $\psi_\delta(A)$ is ψ_δ – set.

Proposition (6-3-8)

Let (X, T, I, δ) be an i – topological proximity space then $A \in \psi_\delta(X, T, I, \delta)$ for each $A \in I_{\phi(x)}$.

Proof:

By proposition (6-1-9)(4) $\phi(\psi_\delta(A)) \in I_{\phi(x)}$ for some $x \in A$, and by remark (2-1-7)(2) since A is focal set we get that $A \delta \phi(\psi_\delta(A))$ hence A is ψ_δ – set.

The following example showed that proposition (6-3-8) is not exist with respect to $F\psi$ – set.

Example (6-3-9)

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$, $I = \{\emptyset, \{c\}\}$, δ is the discrete proximity space .If $A = \{a\} \in I_{\phi(x)}$ but A is not $F\psi$ – set

Remark(6-3-10)

1. Since $\emptyset \delta \phi(\psi_\delta(A))$ for each $A \subseteq X$, then $\emptyset \notin \psi_\delta(X, T, I, \delta)$ also $\emptyset \notin F\psi(X, T, I, \delta)$.
2. $\psi_\delta(X, T, I, \delta)$, (resp. $F\psi(X, T, I, \delta)$) , is not an ideal and not a filter .
3. If A, B are ψ_δ – set then $A \cap B$ is not ψ_δ – set .Also if A, B are $F\psi$ – set then $A \cap B$ is not $F\psi$ – set.

All the cases are explain in the example below

Example(6-3-11)

Let $X = \{a, b, c\}$, $T = \{X, \emptyset, \{a\}, \{b\}\}$, $I = \{\emptyset, \{c\}\}$, and δ defined by the discrete proximity .So clearly that $\emptyset \notin \psi_\delta(X, T, I, \delta)$ also $\{a\} \cap \{b\} = \emptyset \notin \psi_\delta(X, T, I, \delta)$.

The image and inverse image of ψ_δ – set and $F\psi$ – set explained in propositions (6-3-12)and (6-3-13).

Theorem (6-3-12)

Let $f: (X, T, I, \delta) \rightarrow (Y, T_Y, I_Y, \delta_Y)$ be an i-homeomorphism, formatting function and δ – homeomorphism function then A is ψ_δ – set if and only if $f(A)$ is ψ_δ – set.

Proof:

Let A is ψ_δ – set. We will prove the first condition that $\psi_\delta(f(A)) = f(\psi_\delta(A))$ and for that let $y \in \psi_\delta(f(A))$ so by the onto condition of the function f there exist $x \in X$, such that $y = f(x)$,also by definition (6-1-1) there exist $U_y \in I_{Y\phi}(y)$, satisfy $U_y \overline{\delta_Y}(f(A))^c$, now since f is δ – continuous and f is i- homeomorphism then $f^{-1}(U_y) \overline{\delta_X} f^{-1}(f(A^c)) = A^c$, because f is i- homeomorphisms so by proposition (2-1-13) we have $f^{-1}(U_y) \in I_\phi(x)$ and then $x \in \psi_\delta(A)$, so $f(x) \in f(\psi_\delta(A))$.Conversely let us $y \in f(\psi_\delta(A))$, so $f^{-1}(y) \in \psi_\delta(A)$ and then there exist $U_{f^{-1}(y)} \in I_\phi(f^{-1}(y))$, $U_{f^{-1}(y)} \overline{\delta_X} A^c$ and then by onto condition of f , $U_x \overline{\delta_X} A^c$, and because f is δ – homeomorphism we get $f(U_x) \overline{\delta_Y} f(A^c)$, but f is formatting function then $f(U_x) \in I_{Y\phi}(f(x))$, so we get $f(U_x) \overline{\delta_Y}(f(A))^c$.Then $y \in \psi_\delta(f(A))$. Now to prove the second relation that $\phi(\psi_\delta(f(A)) = f(\phi(\psi_\delta(A)))$.

Let $y \in \phi(\psi_\delta(f(A))$ then for each $U_y \in I_Y(y)$, $y \in U_y$ we have $U_y \delta_Y \psi_\delta(f(A))$ and by the first condition $U_y \delta_Y f(\psi_\delta(A))$, since f is δ – homeomorphism that is $f^{-1}(U_y) \delta_X f^{-1}(f(\psi_\delta(A))) = \psi_\delta(A)$, but by proposition (2-1-13) $f^{-1}(U_y) \in I_{\phi(x)}$, hence $x \in \phi(\psi_\delta(A))$ and $f(x) \in f(\phi(\psi_\delta(A)))$, Conversely , let $y \in f(\phi(\psi_\delta(A))$, then $f^{-1}(y) = x \in \phi(\psi_\delta(A))$,and for each $U_x \in I_{\phi(x)}$, $U_x \delta_X \psi_\delta(A)$ but f is δ – homeomorphism, then $f(U_x) \delta_Y f(\psi_\delta(A))$,and by first condition $f(U_x) \delta_Y \psi_\delta(\phi(A))$, we get that $y \in$

$\phi(\psi_\delta(f(A)))$. Now we go back to the assumption $A\delta_X\phi(\psi_\delta(A))$,and by the second relation, $(A)\delta_Y\phi(\psi_\delta(f(A)))$, hence $f(A)$ is ψ_δ set .Conversely, let $f(A)$ is ψ_δ – set then $f(A)\delta_Y\phi(\psi_\delta(f(A)))$,and by the first condition $f(A)\delta_Yf(\psi_\delta(A))$ and since f is δ – continuous then $f^{-1}(f(A))\delta_Xf^{-1}(f(\phi(A)))$, hence $A\delta_X\phi(\psi_\delta(A))$ and we get A is ψ_δ – set.

Proposition(6-3-13)

For the i -homeomorphism , δ – homeomorphism and formatting function $f:(X,T,I,\delta) \rightarrow (Y,T_Y,I_Y,\delta_Y)$ and for any i – closed set A of X , A is $\mathcal{F}\psi$ – set if and only if $f(A)$ is $\mathcal{F}\psi$ – set.

Proof:

Let A is $\mathcal{F}\psi$ – set , so $A\delta\text{Fcl}(\psi_T(A))$ since f is δ – homeomorphism then $f(A)\delta\text{f}(\text{Fcl}(\psi_T(A)))$. Now we will prove that $\psi_T(f(A)) = f(\psi_T(A))$ and for that let $y \in \psi_T(f(A))$, hence there exist $U_y \in T(y)$, $U_y\bar{\delta}(f(A))^c$ but f is i -homeomorphism and f is δ – continuous, then $f^{-1}(U_y)\delta A^c$,so because f is i -continuous function, then $f^{-1}(U_y) \in T(x)$ imply that $x \in \psi_T(A)$ and then $f(x) \in f(\psi_T(A))$ so by the onto condition of f $y \in f(\psi_T(A))$.Conversely , let $y \in f(\psi_T(A))$ then $f^{-1}(y) \in \psi_T(A)$, so there exist $w \in T(x)$, $w\bar{\delta}A^c$ but f is δ – continuous hence $f(w)\bar{\delta}f(A^c)$ and since f is i -homeomorphism $f(w)\bar{\delta}(f(A))^c$, but $f(w) \in T(y)$, so $y \in \psi_T(f(A))$.

By above conversation we have that $(A)\delta\text{f}(\text{Fcl}(\psi_T(A))) = \text{Fcl}(f(\psi_T(A))) = \text{Fcl}(\psi_T(f(A)))$, hence $f(A)$ is $\mathcal{F}\psi$ – set .

Conversely , let $f(A)$ is $\mathcal{F}\psi$ – set , so $f(A)\delta_Y\text{Fcl}(\psi_T(f(A)))$ so by proposition (2-3-29) $\text{Fcl}(f(A)) = f(\text{Fcl}(A))$ and $\psi_T(f(A)) = f(\psi_T(A))$, also since f is δ – continuous we get that $(f(A))\delta_Xf^{-1}(\text{Fcl}(\psi_T(f(A))))$ therefor $A\delta_X\text{Fcl}(\psi_T(A))$ imply that A is $\mathcal{F}\psi$ – set .

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Conclusion and future work

Through this study related to the focal function, which was limited to i -open set in the i -topological space with the presence of the proximity space, we found that it contained the previously defined focal function by using the focal set and the effect was evident through the failure to achieve some of the previously achieved results related to this type of functions. In addition, the possibility of applying these results to some of researches submitted by.

In the light of the new definitions presented in this study and the results presented in it, we can develop and build studies on the compactness, separation axioms, connectedness and other topological properties that we can build equivalent models with by these definition. we expect that we will obtain impressive and wide – ranging results in this field. For example the synonymous definitions are as follows.

1. Throughout the new definitions of the nested sets, we can construct separated sets the subsets A, B of X and called $\kappa_T(A)$ – separated of $\kappa_T(A) \cap B = \emptyset$ and $A \cap \kappa_T(B) = \emptyset$. Also we can construct separated sets as follows, the subsets A, B of X and called I_ϕ . separated if $\kappa_{I_\phi}(A) \cap B = \emptyset$ and $A \cap \kappa_{I_\phi}(B) = \emptyset$.
2. we can defin the κ_T – disconnected (I_ϕ - disconnected) as follows : a subset B of x is called κ_T – disconnected (I_ϕ – disconnected) if A is union of two non – empty κ_T – separated (I_ϕ – separated) sets .
 - i. we can construct the new concepts of separation axioms depending on the pf_{na} (rowded the i – Topological space is called $i - p_0$ – space ($p_0 f$ – space) if for each $x \neq y$ in X , then there exists p_{na} – crowded (pf_{na} – crowded) set contains one of them .
 - ii. p_1 – space ($p_1 f$ – space) if for each $x \neq y$ in X then exists two p_{na} – crowded (pf_{na} – crowded) lets each of them contains of one of it's points not the other .
 - iii. p_2 – space ($p_2 f$ – space) if for each $x \neq y$ in X there exists two disjing p_{na} – crowded (pf_{na} – crowded) sets each of them contains one of its points .

And we can study the relationship between them and study their characteristics.

This is considered one of the open topological study for future large – scale studies. For example, it is possible to study the different types of weakly open sets in these spaces and to alter a lot of research in this field.

Also through the study presented in this dissertation, we observed that : First, the apparent effect of proximity relationship on how points are rally in the i - topological proximity space, which is represented by congested sets. Secondly, the fact that the nested set is an empty confirms the state of non-congested between the points in the space. Third, these results can

be investigated and applied to different types of sets, like ω -open by defining the p - ω -congested set in i -topological proximity spaces. Also we can discuss all these results in the soft or fuzzy spaces to get the soft or fuzzy congested sets.

Also we note that the two sets ψ - δ -set and $F\psi$ -set which are defined in i -topological proximity spaces are independent in their definitions and the nature of their properties than ψ -set that is known in ideal topological spaces as this is evident through the set of characteristics that have been proven in this research.

المستخلص

تهدف هذه الدراسة إلى بناء فضاء جديد باستخدام الفضاء التوبولوجي- i باستخدام نظرية القرب ، ودراسة جميع المفاهيم التوبولوجية التي يمكن دراستها فيه. كذلك دراسة تأثير بديهيات نظرية القرب على المفاهيم التوبولوجية فيما يتعلق بالفضاء الطوبولوجي- i . وفي هذا السياق، قدمنا أولاً معنى مفهوم الفضاء التوبولوجي i -(X, T, I) وفضاءات القرب (X, δ) ، تمت دراسة بعض المفاهيم التوبولوجية هنا بما منها: i -clouser ، i -density ، i -interior ، i -continuous ، i -open ، i -homeomorphism , closed,

بعض الخصائص المتعلقة بهذه المفاهيم المستخدمة في هذه الدراسة تم إثباتها بالإضافة الى العلاقة بين النتائج الحالية والسابقة. بعد ذلك تم بناء فضاء جديد وهو فضاء القرب التوبولوجي- i والتي تمثل المرحلة الأولى من هذه الدراسة. تضمنت المرحلة الثانية من هذه الدراسة تعريف مجموعة جديدة هي focal set مع دراسة لبعض الخصائص والنظريات المتعلقة بها.

من خلال الدراسة المتعلقة بمجموعات focal set ، قدمنا دراسة للمفاهيم التالية: focal function ، Focal clouser, nested set ، من الدوال باستخدام focal set ، وهي forming map . تم دراسة بعض الخصائص والنظريات المتعلقة بهذه المفاهيم. كما تم تعريف نوع جديد

في المرحلة الثالثة ، قمنا بدراسة مفهوم الكثافة في فضاءات القرب التوبولوجية- i ، حيث درسنا أربعة أنواع منها. اما المرحلة الرابعة من هذه الدراسة فقد تضمنت دراسة لكيفية تجمع النقاط في فضاءات القرب التوبولوجية- i ، وتمت مناقشته بطريقتين مختلفتين ، الأولى تتضمن استخدام الشروط التوبولوجية الكلاسيكية وأطلقنا عليها اسم crowded set ، والتي تم تعريفها باستخدام i -clouser و nested set. اما الشكل الآخر للمجموعات المزدهمة تقاربياً فقد تم دراستها باستخدام البديهيات حول علاقة القرب و تسمى مجموعة congested set. تمت دراسة بعض الخصائص المتعلقة بهذين المفهومين.

أخيراً ، في المرحلة الخامسة من هذه الدراسة ، تمت دراسة ψ -operator في شكلين ، الأول باستخدام المجموعة open i - ، والشكل الآخر باستخدام focal sets ، مع أهم الخصائص المتعلقة به. بالإضافة الى تقديم نوعين من مجموعات ψ - ودراستها وهما ، ψ_δ -set, $F\psi$ -set.



جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

حول الفضاءات التبولوجية- i المتولدة بواسطة علاقة القرب

أطروحة مقدمة إلى مجلس كلية التربية للعلوم الصرفة في جامعة بابل
كجزء من متطلبات نيل درجة الدكتوراه فلسفة في التربية/ الرياضيات

من قبل

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بإشراف

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