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On Survival Data and Statistical Modeling

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بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

وَيَسْأَلُونَكَ عَنِ الرُّوحِ ۖ قُلِ الرُّوحُ مِنْ أَمْرِ

رَبِّي وَمَا أُوتِيتُمْ مِنَ الْعِلْمِ إِلَّا قَلِيلًا

صدق الله العلي العظيم

من سورة الأسراء (الآية 85)

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Dedication

To my family ….. husband and children

To my big family….. parents, sisters and brothers.

To all who teach me a letter or a lesson in my life.

Acknowledgments

First, I wish to thank Allah, my Lord and creator, for his help and inspiration

That make this work more than a dream after all what happened. I would like to express my deep thanks and gratitude to my supervisor ***prof. Dr. Kareema Abdel AL-Kadim*** *for her valuable guidance and encouragement during my work.*

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Abstract

The objective of this dissertation is constricting a mathematical formula based on survival function, this formula has a basic parameter called transmuted parameter. In addition, experimenting this formula within Exponential and Weibull distributions to produce three different lifetime distributions. The process of this production contains three parts: the first part includes the mathematical formula and the proof of its validity statistically. The second part is about applying this formula on two of lifetime distributions. Finally, based on the results of the first and second parts three mixed models are given. Moreover, realizing the mathematical and statistical properties of these models as the r^{th} moment about the origin, moment's generation function, mean, median and other statistical properties. Furthermore, explaining the shapes of the density, cumulative and hazard function for the three models. Using the Akaike information criterion, corrected Akaike information criterion and the Bayesian information criterion to compare the performance of the models. We illustrate the usefulness of the given distributions by applications of three real data sets with curve fitting.

Later, to estimate the parameters of the three models used three methods they are (Maximum Likelihood estimation method, Least Squared method and Moments method). the Mean Squared Error

is measured to compare the methods above for different size of the generated samples (10,30, 50 and 100).

Finally, we illustrate the usefulness of the given distributions by applications of three real data sets with curve fitting.

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List of Symbols

T	Random variable
t	Value of random variable
E	Exponential distribution
W	Weibull distribution
TSE	Transmuted Survival Exponential
TSW	Transmuted survival Weibull
TSEW	Transmuted Survival Exponential Weibull
$F_*(x)$	Baseline cumulative function
λ	Transmuted parameter
$S(x)$	Survival function
$\propto$	parameter
S_0	Survival function
R	Real numbers
R_+	Positive real numbers
p	parameter
$S_{ew}(x)$	Survival function of exponential-Weibull
$S_e(x)$	Survival of exponential function
$S_w(x)$	Survival of Weibull
$F_{ew}(x)$	Cumulative of exponential- Weibull
$f_w(x)$	Density function of Weibull
$w(x)$	Weight function for standard distribution
$E[w(x)]$ TSF	Expectation of the weight function for the standard distribution Transmuted survival formula
P	Probability function
$S_*(t)$	Survival of baseline distribution

$f_*(t)$	Density of baseline distribution
γ	parameter
$S_{TSE}(t)$	Survival function of TSE distribution
$F_{TSE}(t)$	Cumulative function of TSE distribution
$f_{TSE}(t)$	Density of TSE distribution
α	parameter
$F_W(t)$	Cumulative of Weibull
$S_W(t)$	Survival of weibull
$S_{TSW}(t)$	Survival function of TSW distribution
$F_{TSW}(t)$	Cumulative function of TSW distribution
$f_{TSW}(t)$	Density of TSW distribution
$S_{TSEW}(t)$	Survival function of TSEW distribution
$F_{TSEW}(t)$	Cumulative function of TSEW distribution
$f_{TSEW}(t)$	Density of TSEW distribution
$h(t)$	Hazard function
$h_{TSE}(t)$	hazard of TSE distribution
$h_{TSW}(t)$	hazard function of TSW distribution
$h_{TSEW}(t)$	Hazard function of TSEW distribution
Γ	Gamma function
$E(T)$	Mathematical Expectation
$var(t)$	Variance
CV	Coefficient of Variation
MGF	The moment generation function
CS	The Coefficient of Skewness
μ	The mean
CK	The Coefficient of Kurtosis
AIC	Akaiki information criterion

CAIC	Corrected Akaiki information criterion
BIC	Bayesian information criterion
SSE	Sum of squares due to error
SST	Sum of squares about the mean
RMSE	Root mean squared error
MLE	Maximum likelihood estimator
OLS	Ordinary least square estimator
Mom	Moment estimator
mse	Mean square error

Chapter One

Preliminaries

Chapter Two

Methodology and Statistical Properties

Chapter Three

Estimation Methods

and

Information Criterions

Chapter Four

Empirical and Applied Part

المستخلص

الهدف من هذه الاطروحة هو انشاء صيغة رياضية تعتمد في تكوينها على دالة البقاء، وهذه الصيغة تتكون في الأساس من معلمة تسمى معلمة التحويل. كذلك اختبار هذه الصيغة من خلال توزيعي الاسي وويبل لإنتاج ثلاثة توزيعات بقاء مختلفة. عملية الإنتاج هذه تتكون من ثلاث أجزاء: الجزء الأول يتكون من الصيغة الرياضية واثبات صحتها احصائيا. الجزء الثاني يتكون من تطبيق الصيغة الرياضية على توزيعين أساسيين من توزيعات البقاء وأخيرا وبالاعتماد على نتائج الجزء الأول والثاني نحصل على ثلاث صيغ لثلاثة توزيعات مختلفة. ثم بعد ذلك إعطاء الخصائص والصفات الإحصائية للتوزيعات المعطاة مثل الوسيط والعزوم والدالة المولدة للعزوم وغيرها.

أيضا تم دراسة اشكال مختلفة للتوزيعات مع دالة هزار لكل توزيع. تم اختبار أداء وفعالية كل توزيع من خلال حساب بعض المعايير الإحصائية واختبار حسن المطابقة لكل توزيع مع اجراء المقارنات بين التوزيعات واختبارها على بيانات حقيقية موجودة ضمن المصادر العلمية المثبتة.

اعتمدت ثلاث طرق للتقدير(الإمكان الأعظم , المربعات الصغرى و طريقة العزوم) التي قادت في معادلاتها الى طرق عددية لتقدير معالم التوزيعات الثلاث. تم اجراء محاكاة بالاعتماد على الدالة التجميعية لكل توزيع لتوليد عينات مختلفة (حجم 10, 30, 50 و 100) من كل توزيع والاعتماد عليها لتقدير معالم التوزيعات الثلاث ولقيم مختلفة للمعالم وتكرار تجربة المحاكاة عند عدد دورات 500 وال 1000 وتحديد الطريقة الأفضل للتقدير بالاعتماد على حساب متوسط مربعات الخطأ.

Introduction

Statistical modeling is an important concept in statistical analysis studies. As it helps understand and solve many natural phenomena especially nowadays in many fields. Statisticians have tried for decades to work hard to produce new distributions and expand the existing one in order to cover the new problems in life especially fields of lifetime phenomena such as medical or devices. Therefore, we have focused our attention on building a formula that can create a new distribution by using the traditional one in order to use the advantage of such distribution the mix between the most famous with a study of the mathematical and statistical properties of this distribution. In addition, important to compare the validity and performance of the introduced distributions through the use of information criterions and statistics quantities such as Akaike criterion and goodness of fit quantities. We define the new formula and test it with two famous distributions the Exponential and Weibull to produce three different lifetime distributions. This dissertation consists of four chapters, chapter one includes the introduction, aims of the study, literature review and concepts that related to this work. Methodology of the new formula that introduce the new distributions with study to their shapes are given in Chapter two also studies the mathematical and statistical characteristics of the given new distributions. Chapter three deals with the estimations methods such that we use three classical estimation methods: maximum likelihood estimator, least squared estimator and the moment estimator. Some of these estimators lead to the use of numerical approaches like Newton-Raphson methods. Information's criterions and curve fitting are being used also in this chapter. Chapter four utilizes a simulation study to *generate different sizes od samples. We make a Comparison of the methods of estimation above by using mean square error for the estimation of the parameters. Then in same chapter we test the given distributions by two real data set and compare them with the original distributions. Also, it consists of conclusions, and future works and references of the thesis.*

1.1 Introduction

Statistical modeling is an important concept in statistical analysis studies. As it, helps understand and solve many natural phenomena especially nowadays in many fields. Statisticians have tried for decades to work hard to produce new distributions and expand the existing one in order to cover the new problems in life especially fields of lifetime phenomena such as medical or devices. Therefore, we have focused our attention on building a formula that can create a new distribution by using the traditional one in order to use the advantage of such distribution the mix between the most famous with a study of the mathematical and statistical properties of this distribution. In addition, important to compare the validity and performance of the introduced distributions with information criterions and statistics quantities such as Akaike criterion and goodness of fit quantities. We define the new formula and test it with two famous distributions the Exponential and Weibull to produce three different lifetime distributions. This dissertation consists of four chapters, chapter one includes the introduction, aims of the study, literature review and concepts that related to this work. Methodology of the new formula that introduce the new distributions with study to their shapes are given in Chapter two also studies the mathematical and statistical characteristics of the given new distributions. Chapter three deals with the estimations methods such that we use three classical estimation methods: maximum likelihood estimator, least squared estimator and the moment estimator. Some of these estimators lead to the use of numerical approaches like Newton-Raphson methods. Information's criterions and curve fitting are being used also in this chapter. Chapter four utilizes a simulation study to generate different sizes od samples. We make a Comparison of the methods of estimation above by using mean square error for the estimation of the parameters. Then in same chapter we test the given distributions by two real data set and compare them with the original distributions. Also, it consists of conclusions, and future works and references of the thesis.

1.2 The Aim the Study

This work has testified a number of main objects as follows:

1. Constructing a new distributions: Transmuted Survival Exponential (TSE) distribution, Transmuted Survival Weibull distribution (TSW) and Transmuted Survival Exponential Weibull (TSEW) based on new formula that use survival function of major distributions: Exponential and Weibull distribution; we proved that this formula is well defined statistically.
2. Discussing and proving the statistical and mathematical properties of the given distributions such as mean, mood, quantile function and moment generation function. Also we test some estimating methods to find the best methods by using numerical technique. With following good algorithms to estimate the parameters of the proposed distributions and matlab language.
3. Comparing the new disruptions with each other and with the Exponential and Weibull distribution. We clarify the utility of proposed by application of real data.

1.3 Literature Review

Our new formula depends on three concepts: transmuted formula, survival function and mixed distributions. There are many researchers studied these concepts to produce new distributions especially lifetime distributions. We give some as follows:

1. (Warren Gilchrist, 2000) [11] refers to use maps that depends on cumulative functions $F(x)$ and its invers these maps were used by William T. Shaw and Ian R. C. Buckley 2007 and 2009 such that they used the transmutation map especially the quadratic form:

   $$F(x) = (1+\lambda)F_*(x) - \lambda {F_*}^2(x) \quad ; \quad |\lambda| \leq 1$$

   Where $F_*(x)$ is the cdf of base distribution.

And apply it with lifetime distributions like exponential, Weibull, log-logistic and uniform distributions.

In 2015 Abdus Saboor, Mustafa Kamal and Munir Ahmad [25] apply the transmuted technique to Exponential-Weibull (TEW) distribution to make Weibull distribution more flexible, and make it wider within application in lifetime of certain system.

Daniele Cristina Tita Granzotto 2017 [12] propose the transmuted log-logistic model using the quadratic rank transmutation map and in his book he presents transmuted family of model which has the property that the extra parameter λ can take any real value hence without restricted parameter space λ transmuted technique be a convenient way of constricting new distributions.

Kareema A. Al-Kadim and Ashraf A. Mahdi (2018) combine Transmuted map and Exponentiated formula [1] such that they present new distribution called Exponentiated Transmuted Exponential (ETE) distribution of three parameters:

$$F(x;\ \alpha,\beta,\lambda) = ((1-e^{-\lambda x})(1+\beta e^{-\lambda x}))^{\alpha} \qquad ; \quad x > 0, \alpha\ ,\beta,\lambda > 0$$

2. Albert W. Marshall and Ingram Olkin (1997)[16] introduce a new method for adding a parameter to a family of distribution with application to the Exponential and Weibull family starting with survival function $S(x)$ and CDF $F(x)$, the one-parameter family of survival function:

$$S(x;\ \propto) = \frac{\propto S(x)}{F(x)+\propto S(x)} \qquad ; \quad -\infty < x < \infty\ , \ \ 0 <\propto< \infty$$

While Vasileios Pappas, Konstantinous Adamidis and Sotirios Loukas (2012) [18] use a formula starting with a distribution with survival function S_0 , the survival function of the proposed family with the additional parameter p is given by :

$$S(x) = \frac{\ln\{\ 1-(1-p)\, S_0(x)\}}{\ln p} \ ; \ \ x \in R, p \in R_{+} - \{0\}$$

In 2014 Cordeiro, Ortega and Lemont [4], introduce a new method to mix the distributions by using a tail of these distributions, and represente this new method to mix between Exponential of one parameter and Weibull of two parameters

$$S_{ew}(x) = S_e(x)S_w(x) \quad and \;\; F_{ew}(x) = 1 - S_{ew}(x)$$

Where $S_e(x)$ and $S_w(x)$ are the survival functions of Exponential of one parameter and Weibull of two parameters. The $S_{ew}(x)$ is the survival function of the new distribution and $F_{ew}(x)$ is the CDF.

In continuation to the above, and due to the novelty we have constructed a mathematical formula such that it can be considered as quadratic transmuted map that takes the survival function of a distribution and introduce a new one. This new survival function can produce probability distributions. In addition, we combine the new distributions to have another such that the full technique with prove will be presented in the next chapter and analyzing to the new distributions shapes.

1.4 Concepts that Related to the Survival Modeling

Survival Data: Numerous assays outcome in data on event studied and collected over time, and on the study of elements that associated and effect on the appearance for those events. One of such assay observes and gauges the time until appearance of precise or fixed event.

This data oftentimes called as "survival data " or " failure-time data". as example for such events the lifetime of machine components in industrial applications, disease progression, or time to death of cancer patients. In addition, the statistical methods that deal and analysis such data usually known as "survival analysis". And there are three major functions that related to this branch of statistics survival function, probability density function and hazard function.

Statistical Model: is a mathematical exemplification and formulation of spotted dataset. And Statistical modeling is set of procedures of applying statistical analysis to a dataset. Moreover, when the dataset represented by a T a non-negative random variable adapts the waiting time until the appearance of an event then the statistical model called “survival model" or "lifetime model".

Distribution of Survival Time

Let T be non-negative continuous random variable the probability density function (pdf) and cumulative distribution function (cdf) are most commonly used to characterize the distribution of survival data and we shall denote these by $f(\cdot)$ and $F(\cdot)$, respectively:

$$pdf: f(t)$$

$$cdf: F(t)$$

Because T is non-negative and usually denotes the elapsed time until an event, it is commonly characterized by survival function which we will discusses in next section.

2.1 Introduction

In this chapter, we introduce the methodology of the new formula and driving the three new distributions with discussing and analyzing to its shapes. In addition, we drive some of statistical characteristics to these new distributions.

2.2 The Transmuted Survival Formula

The new distribution (or formula) depends firstly, on survival function$S(t)$, which is defined by

$$\begin{aligned} S(t) &= P(T > t) \quad , \ t \geq 0 \qquad \text{(2-1)} \\ &= 1 - P(T \leq t) \\ &= 1 - F(T) \\ &= \int_t^{\infty} f(t)dt \end{aligned}$$

Where T is a continues random variable (failure time), the distribution function *F(t)* describe the probability the time to event (T) is smaller or equal compared to a fixed time (t) and is given as:

$$F(t) = P(T \leq t) \qquad \text{(2-2)}$$

f(t) is the (probability density function) probability that the failure time occurs at exactly time t (out of all possible times) and is given as

$$f(t) = \lim_{\Delta t \to 0} \frac{Prob(t \leq T \leq t + \Delta t)}{\Delta t} \qquad \text{(2-3)}$$

Secondly, on computing a new $S(t)$ from the following transmuted formula:-

$$S(t) = (1+\lambda)S_*(t)^2 - \lambda S_*(t) \qquad (2\text{-}4)$$

Where $S_*(t)$ is the survival function of the baseline distribution and λ is a real number called transmuted parameter . By differentiation law

$$dS(t) = -dF(t) = -f(t)$$

We have

$$-f(t) = -2(1+\lambda)S_*(t)f_*(t) + \lambda f_*(t)$$

$$f(t) = 2(1+\lambda)S_*(t)f_*(t) - \lambda f_*(t)$$

$$f(t) = f_*(t)[2(1+\lambda)S_*(t) - \lambda] \qquad (2\text{-}5)$$

According to [17], a function $f(\cdot)$ that is defined as $f: R \to [0, \infty)$ represents a probability density function if and only if

1. $f(x) \geq o \; for \; all \; x > 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

Therefore, the first property is satisfied for all x > 0 and at:

$$\lambda < \frac{2S_*(x)}{1 - 2S_*(x)}$$

The second property is shown below:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} f(x)dx = 2(1+\lambda)\int_{0}^{\infty} S_*(x) \; f_*(x)dx - \lambda \int_{0}^{\infty} f_*(x)dx$$

Now,

Put $u = F_*(x) \quad . S_*(x) = 1 - u \quad and \quad du = dF_* = f_*(x)dx$

Moreover, since u is uniformly distributed variable, hence:

$$2(\,1+\lambda)\int_0^{\infty} S_*(x)\ f_*(x)dx =\ 2(1+\lambda)\int_0^{1}(1-u)\,du$$

$$= 2(1+\lambda)\left[1-\frac{1}{2}\right]$$

$$= (1+\lambda)$$

And $\lambda\int_0^{\infty} f_*(x)dx\ =\ \lambda$

$$Therefor\ \int_0^{\infty} f(x)dx = (1+\lambda)-\lambda = 1.$$

Therefore, equation (2-5) is a density function

2.2.1 The Transmuted Survival Exponential Distribution (TSE).

The cdf, pdf and survival function of Exponential distribution are respectively

$$f_e(t) = \gamma e^{-\gamma t} \quad \gamma > 0\,, t > 0 \qquad (2\text{-}6)$$

$$F_e(t) = 1 - e^{-\gamma t} \qquad (2\text{-}7)$$

$$S_e(t) = e^{-\gamma t} \qquad (2\text{-}8)$$

We take the tail (survival) function of exponential distribution and by using transmutation (2-4); we get the Transmuted-Survival-Exponential distribution:

$$S_{TSE}(t) = (1+\lambda)e^{-2\gamma t} - \lambda e^{-\gamma t} \qquad (2\text{-}9)$$

The cdf of this distribution is:

$$F_{TSE}(t) = 1 - S_{TE}(t)$$

$$F_{TSE}(t) = 1 - (1+\lambda)e^{-2\gamma t} + \lambda e^{-\gamma t} \qquad (2\text{-}10)$$

Then the pdf of the new distribution from the derivative of the cdf distribution where $\gamma > 0$, is the scale parameter

$$f_{TSE}(t) = 2\gamma(1+\lambda)e^{-2\gamma t} - \lambda\gamma e^{-\gamma t} \qquad (2\text{-}11)$$

Or we can write (2-11) in the following:

$$f_{TSE} = \gamma e^{-2\gamma t}(2 + \lambda(2 - e^{-\gamma t})) \qquad (2\text{-}12)$$

It is easy to show that the new distribution satisfies the following properties:

$$f_{TSE}(t) \geq 0 \quad and \quad \int_0^\infty f_{TSE}(t)\, dt = 1$$

The following graph figure (2.a & 2.b) show some shapes of the pdf of the_TSE distribution for selected values of the parameters λ , γ

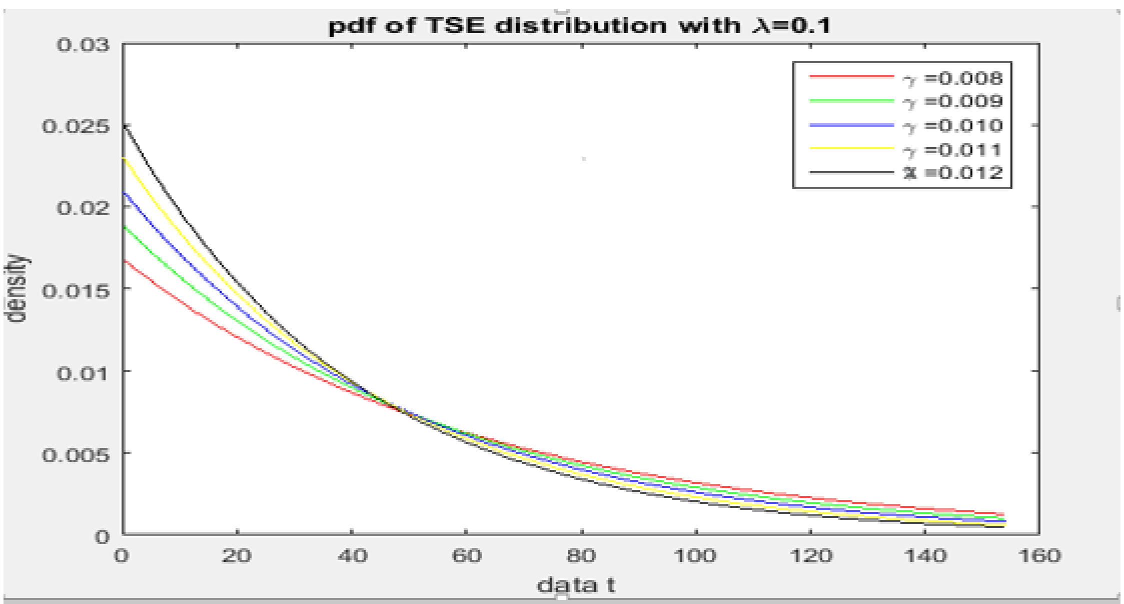


Figure 2.1: The pdf TSE distribution with different values of γ and fixed λ.

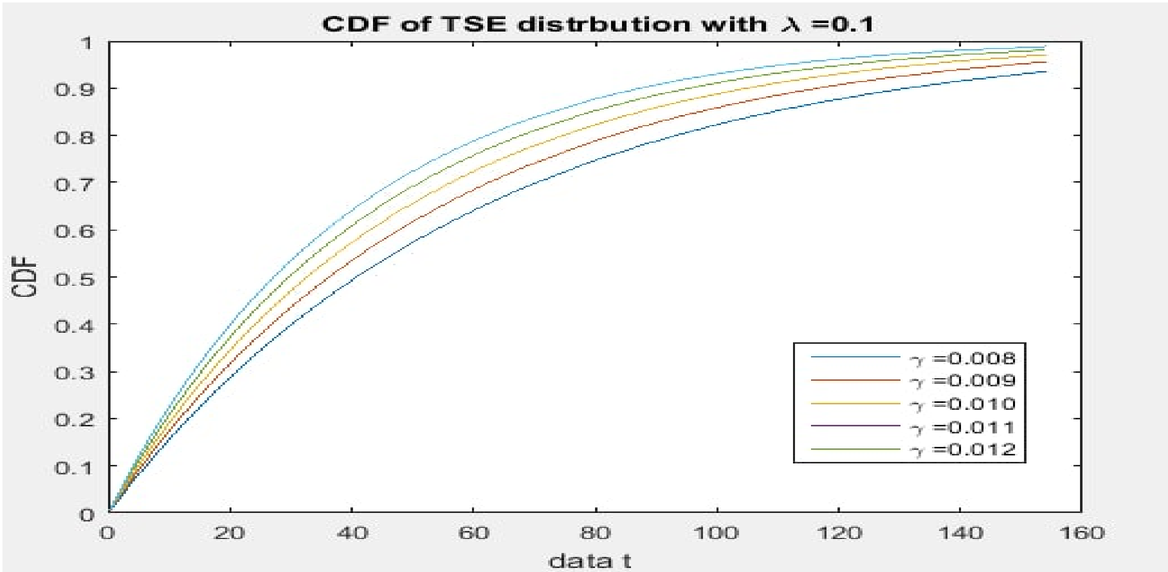


Figure2. 2 : The cdf of TSE distribution with different values of $\gamma$and fixed λ.

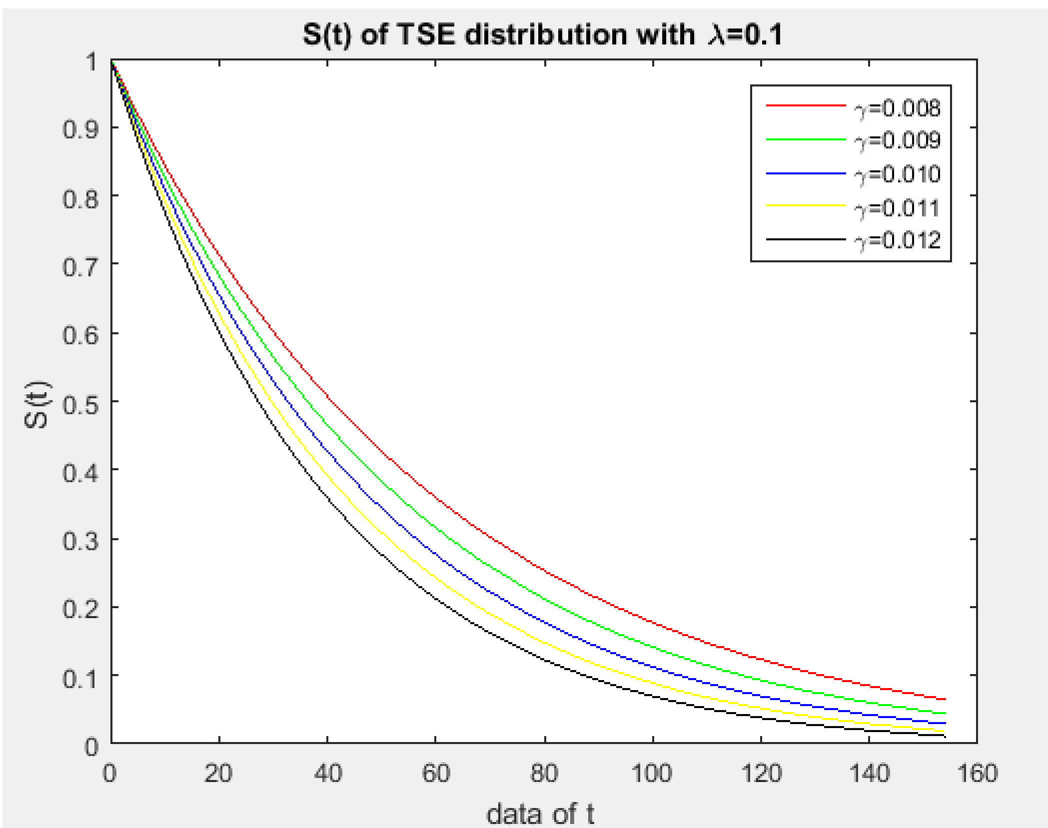


Figure2. 3: The Survival function of TSE distribution with different values of $\gamma$and fixed λ.

The pdf of TSE distribution in figure (2.1) exhibit a decreasing curve with $\lambda = 0.1$ and different small values of γ. The cdf of TSE in figure (2.2) exhibit increasing curve with fixed value of $\lambda = 0.1$ and different values of γ and the cdf's curve have the same shape but different positions. For curve of survival functions in figure (2.3), the probability of surviving decreasing to the right direction at $\lambda = 0.1$ and different small values of γ.

2.2.2 The Transmuted Survival Weibull Distribution (TSW).

The standard Weibull distribution pdf and cdf functions is given by respectively:

$$f_W(t) = \alpha t^{\alpha-1} e^{-t^{\alpha}} \quad \alpha > 0. t > 0 \qquad (2\text{-}13)$$

$$F_W(t) = 1 - e^{-t^{\alpha}} \qquad (2\text{-}14)$$

In addition, survival function that clearly gives the same information as cdf is the right tail distribution function

$$S_W(t) = e^{-t^{\alpha}} \qquad (2\text{-}15)$$

Apply formula (1-4) we get:

$$S_{TSW}(t) = (\lambda + 1) S_W(t)^2 - \lambda S_W(t) \qquad (2\text{-}16)$$

$$= (\lambda + 1) e^{-2t^{\alpha}} - \lambda e^{-t^{\alpha}} \qquad (2\text{-}17)$$

Where the real number λ called the transmuted parameter.

$$F_{TSW}(t) = 1 - S_{TSW}(t)$$

$$F_{TSW}(t) = 1 - (\lambda + 1) e^{-2t^{\alpha}} + \lambda e^{-t^{\alpha}} \qquad (2\text{-}18)$$

$$f_{TSW}(t) = 2\alpha t^{\alpha-1} (\lambda + 1) e^{-2t^{\alpha}} - \alpha t^{\alpha-1} \lambda e^{-t^{\alpha}} \qquad (2\text{-}19)$$

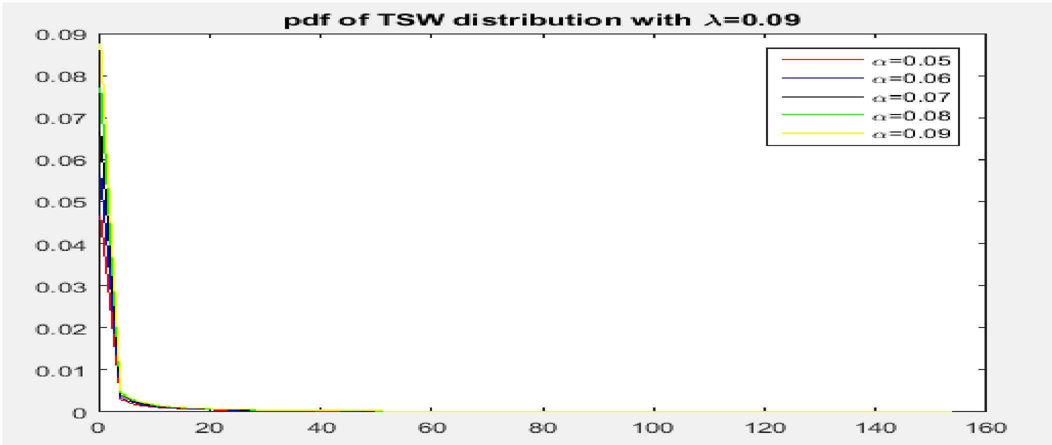


Figure 2.4 :The pdf of TSW distribution with different values of α and fixed λ.

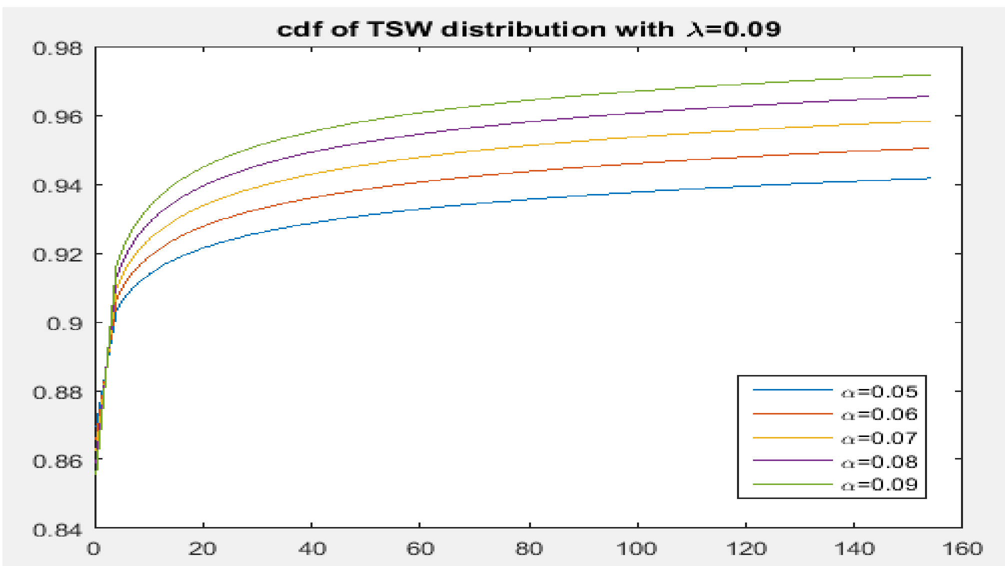


Figure 2.5: The cdf of TSW distribution with different values of α and fixed λ.

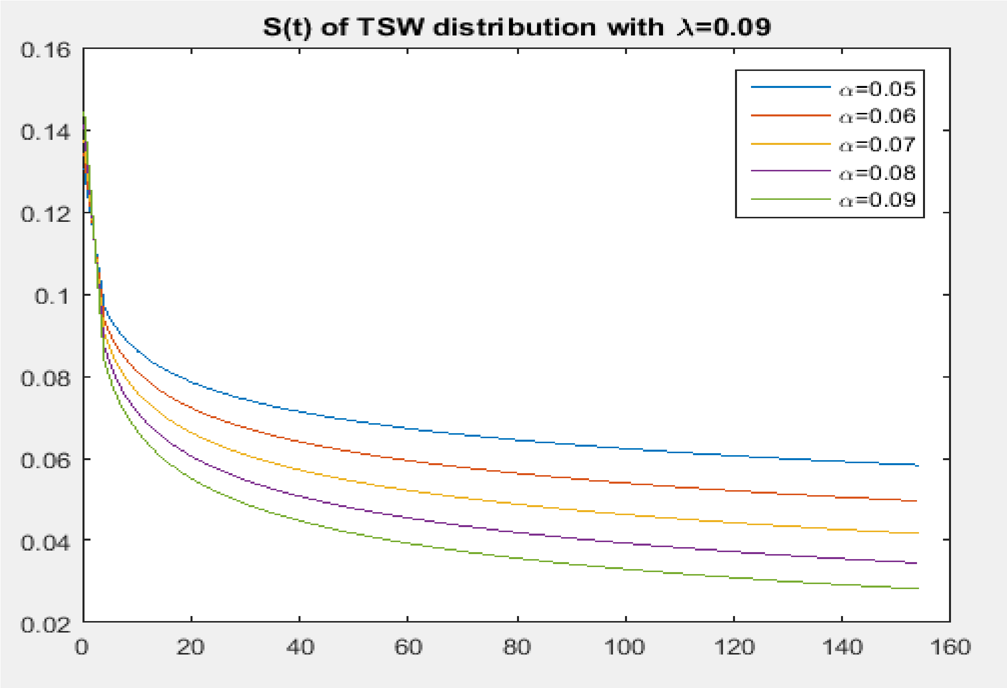


Figure 2.6: The $S(t)$ of TSE distribution with different values of α and fixed λ.

The pdf of TSW distribution in figure (2.4) exhibit a decreasing curve with $\lambda = 0.09$ and different small values of α with very close at positions. The cdf of TSW in figure (2.5) exhibit increasing curve with fixed value of $\lambda = 0.09$ and different values of α and the cdf's curve have the same shape but different positions at t get increase. For survival curve, the probability of survival is decreasing in its value as t get increase in figure (2.6).

2.2.3 The Transmuted Survival Exponential-Weibull Distribution (TSEW).

In this sub section, we introduce the mixture distribution by applying (2-4) on both exponential and Weibull distributions together within same formula as follows:

$$S_{TSEW}(t) = (1+\lambda)e^{-2\gamma t} - \lambda e^{-t^{\alpha}} \qquad (2\text{-}20)$$

Where the $S_*(t)^2$ in (2-4) belongs to the exponential distribution and $S_*(t)$ is the survival of weibull distribution.

The cdf of TSEW is:

$$F_{TSEW}(t) = 1 - (1+\lambda)e^{-2\gamma t} + \lambda e^{-t^{\alpha}} \qquad \text{(2-21)}$$

Then the pdf of the new distribution is:

$$f_{TSEW}(t) = 2\gamma(1+\lambda)e^{-2\gamma t} - \lambda\alpha t^{\alpha-1}e^{-t^{\alpha}} \qquad \text{(2-22)}$$

Now we will proof that the pdf of TSEW is a probability density function by proofing firstly $\int f_{TSEW} = 1$ and secondly that $f_{TSEW} \geq 0$ such that for all $t \geq 0$.

So, to show that $\int f_{TSEW}\,(t) = 1$

Take the integral of (2-22) as:

$\int_0^\infty f_{TSEW}\,(t)dt = (1+\ \lambda)\int_0^\infty 2\gamma\, e^{-2\gamma t}dt -\ \lambda\int_0^\infty \alpha t^{\alpha-1}e^{-t^{\alpha}}\,dt$

Set $I = (1+\ \lambda)\int_0^\infty 2\gamma\, e^{-2\gamma t}dt$ and $II = \lambda\int_0^\infty \alpha t^{\alpha-1}e^{-t^{\alpha}}\,dt$

Since $I = 2(1+\ \lambda)\ \int_0^\infty S_e\,(t)f_e(t)dt$ where $S_e f_e =\ e^{-\gamma t}\,\gamma e^{-\gamma t}$

Use the following substitutions:

$u = F_e(t), \quad S_e = 1-u \quad , \quad du = dF_e(t) =\ f_e(t)\,dt$

So,

$$I =\ 2(1+\lambda)\int_0^1 (1-u)\,du$$
$$= (1+\lambda)$$

And

$$II = \lambda \int_0^\infty f_e(t)dt$$

$$= \lambda$$

Thus $\int_0^\infty f_{TSEW}\ (t)dt = I - II = 1.$

Now we need to prove second condition

We can guarantee that $f_{TSEW} \geq 0$ for all $t \geq 0$ if we catch the following restriction:

We need to insure the inequality

$$2\gamma(1+\lambda)e^{-2\gamma t} - \lambda\alpha t^{\alpha-1}e^{-t^\alpha} \geq 0$$

Which can be when we insure:

$$2\gamma(1+\lambda)e^{-2\gamma t} \geq \lambda\alpha t^{\alpha-1}e^{-t^\alpha}$$

And

$$\frac{1+\lambda}{\lambda} \geq (2^{-1}\gamma^{-1}e^{2\gamma t})\ (\alpha t^{\alpha-1}e^{-t^\alpha})$$

$$\lambda^{-1} + 1 \geq (2^{-1}\gamma^{-1}e^{2\gamma t})\ (\alpha t^{\alpha-1}e^{-t^\alpha})$$

$$\lambda^{-1} \geq (2^{-1}\gamma^{-1}e^{2\gamma t})\left(\alpha t^{\alpha-1}e^{-t^\alpha}\right) - 1$$

Hence

$$\lambda \leq \left(2^{-1}\gamma^{-1}\alpha t^{\alpha-1}e^{2\gamma t - t^\alpha}\right) - 1$$

The last inequality can be achieved numerically.

Thus $f_{TSEW}(t)$ is a probability density function.

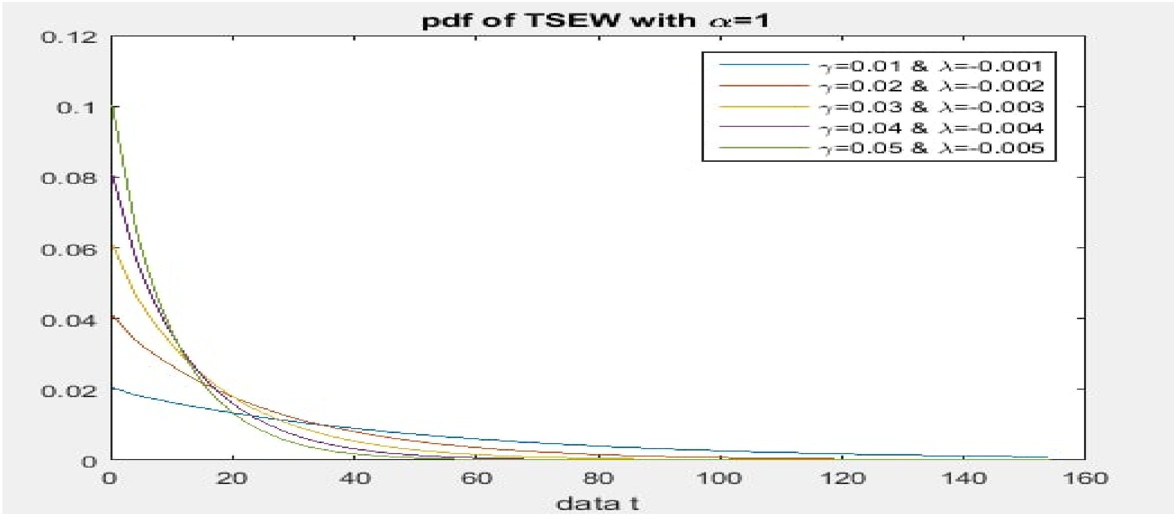


Figure 2. 7.a :The pdf of TSEW distribution with different values of γ, λ and fixed α.

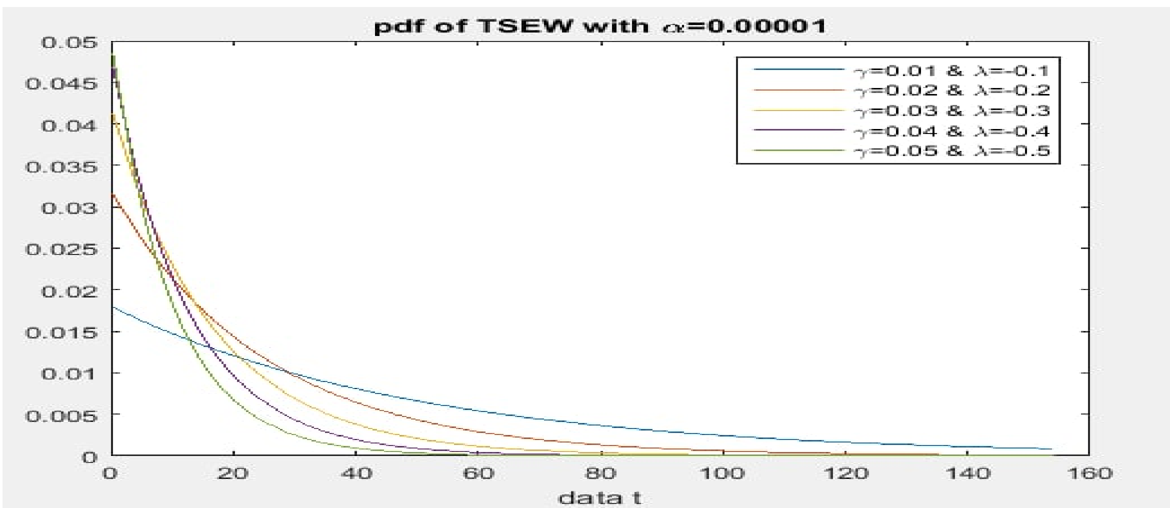


Figure 2.7.b: The pdf of TSEW distribution with different values of γ, λ and fixed α.

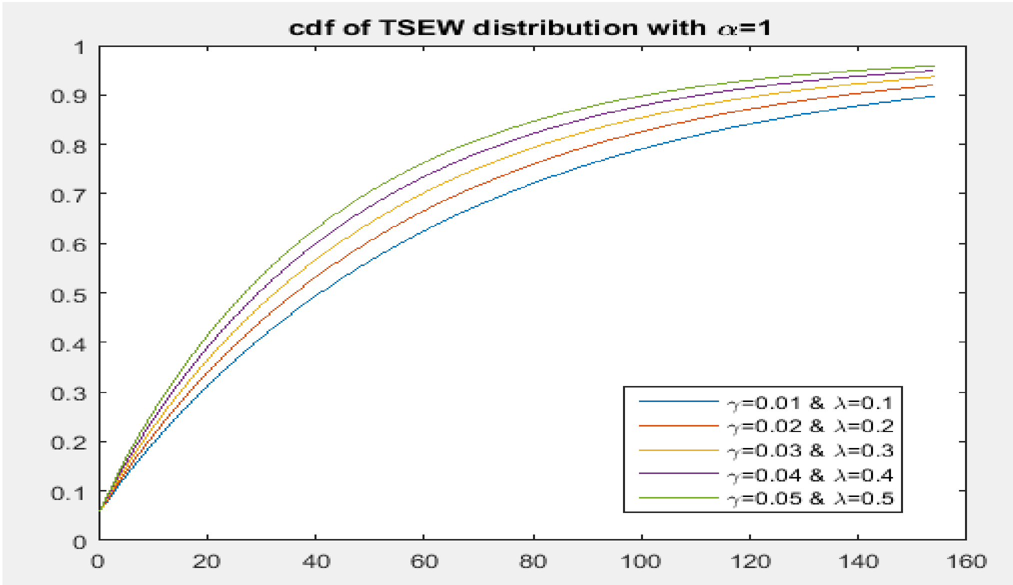


Figure 2.8: The cdf of TSEW distribution with different values of γ, λ and fixed α.

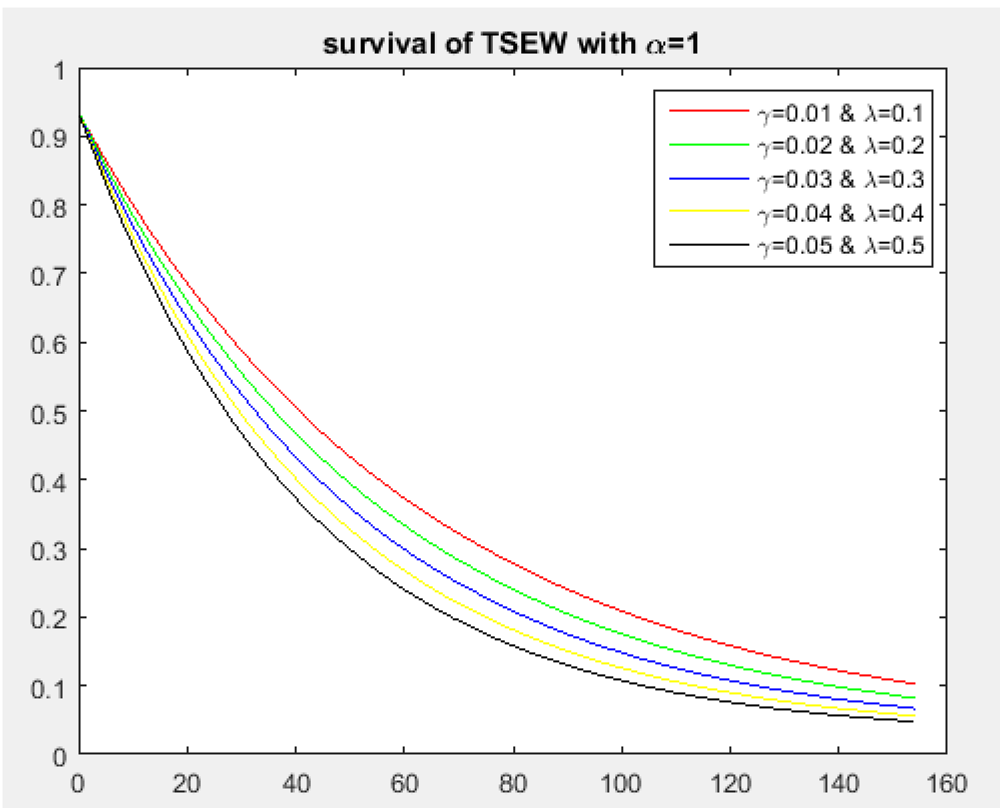


Figure 2.9: The $survival\ function$ of TSEW distribution with different values of γ, λ and fixed α.

The pdf of TSEW distribution in figure (2.7.a, b) exhibit a decreasing curve with fixed value of $\alpha = 1$ and different values of γ, λ which give varies in curves positions. Changing values of λ with very small fixed value of $\alpha = 0.00001$ and fixed value of γ effected on position and separation. In addition, the curve of cdf in figure (2.8) is increasing towards y=1 as t goes to zero. Moreover, the curve of survival function in figure (2.9) is decreasing towards y=zero as variable t goes to zero.

2.3 Shapes of New Distributions

We look at the shapes of the given three new distributions to give a visualization of those distributions by discuss the shapes of the density functions, survival functions and the hazard rate functions of the TSE, TSW and TSEW distributions.

2.3.1 Shape of TSE distribution

Consider equations (2-10) and (2-11)

$$\lim_{t\to 0} f_{TSE}(t,\gamma,\lambda) = \lim_{t\to 0} 2\gamma(1+\lambda)e^{-2\gamma t} - \lambda\gamma e^{-\gamma t}$$
$$= 2\gamma(1+\lambda)\lim_{t\to 0} e^{-2\gamma t} - \lambda\gamma\lim_{t\to 0} e^{-\gamma t}$$
$$= 2\gamma(1+\lambda) - \lambda\gamma$$
$$= \gamma(2+\lambda)$$

While

$$\lim_{t\to\infty} f_{TSE}(t,\gamma,\lambda) = 0$$

The limit of cdf of TSE:

$$\lim_{t\to 0} F_{TSE}(t,\gamma,\lambda) = \lim_{t\to 0} 1-(1+\lambda)e^{-2\gamma t} + \lambda e^{-\gamma t}$$

$$= 1 - 1 - \lambda + \lambda$$

$$= 0\,.$$

In addition, it is cl$ear\ that$ $\lim_{t \to \infty} F_{TSE}(t, \gamma, \lambda) = 1$

To have more clarify about distribution's curve, we need to find first and second derivative of $f_{TSE}(t)$ in order to decide the stationary point:

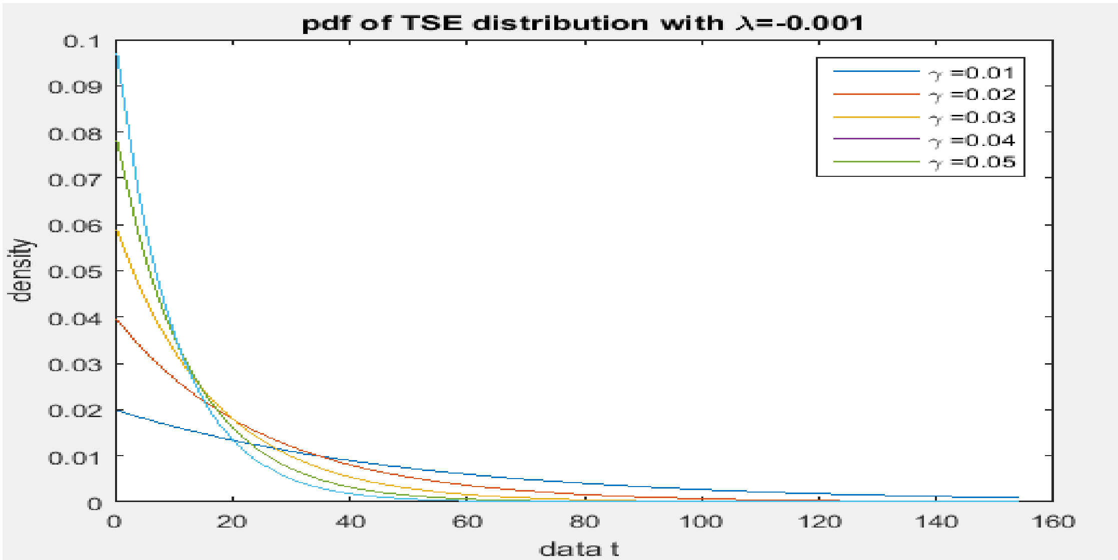
So,

$$\frac{\partial f_{TSE}}{\partial t} = -4\gamma^2(1+\lambda)e^{-2\gamma t} + \lambda\gamma^2 e^{-\gamma t} \qquad \text{(2-23)}$$

$$= \lambda\gamma^2 e^{-\gamma t} - 4\gamma^2(1+\lambda)e^{-2\gamma t}$$

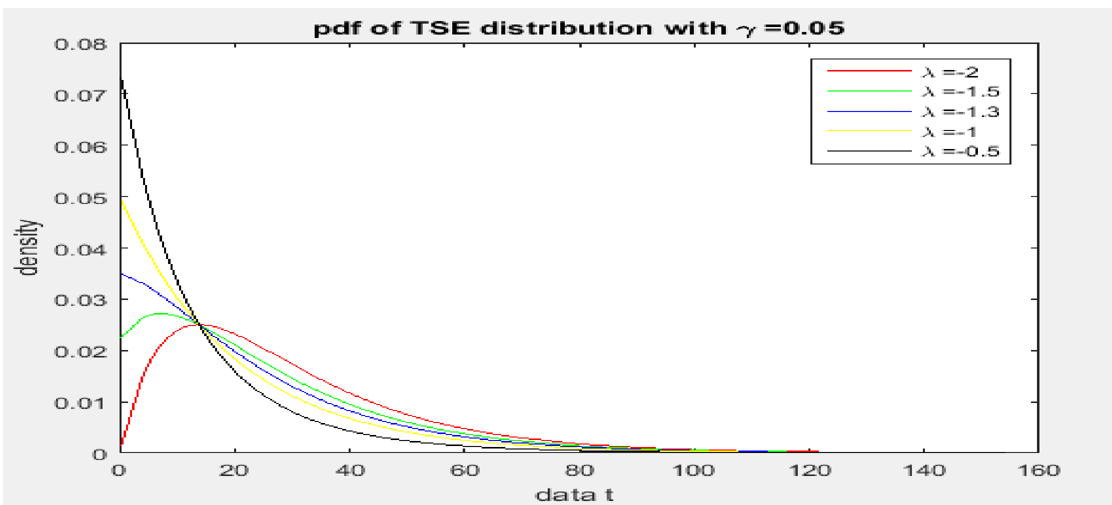
There is more than one root of this equation that can be founded numerically. So, suppose $t = t_1$ Then, it depending to local maximum, minimum or a point of inflection which depending on the $\frac{\partial^2 f(t_1)}{\partial t^2} < 0$, $\frac{\partial^2 f(t_1)}{\partial t^2} > 0\ or\ \frac{\partial^2 f(t_1)}{\partial t^2} = 0$, where

$$\frac{\partial^2 f(t)}{\partial t^2} = 8\gamma^3(1+\lambda)e^{-2\gamma t} - \lambda\gamma^3 e^{-\gamma t}$$



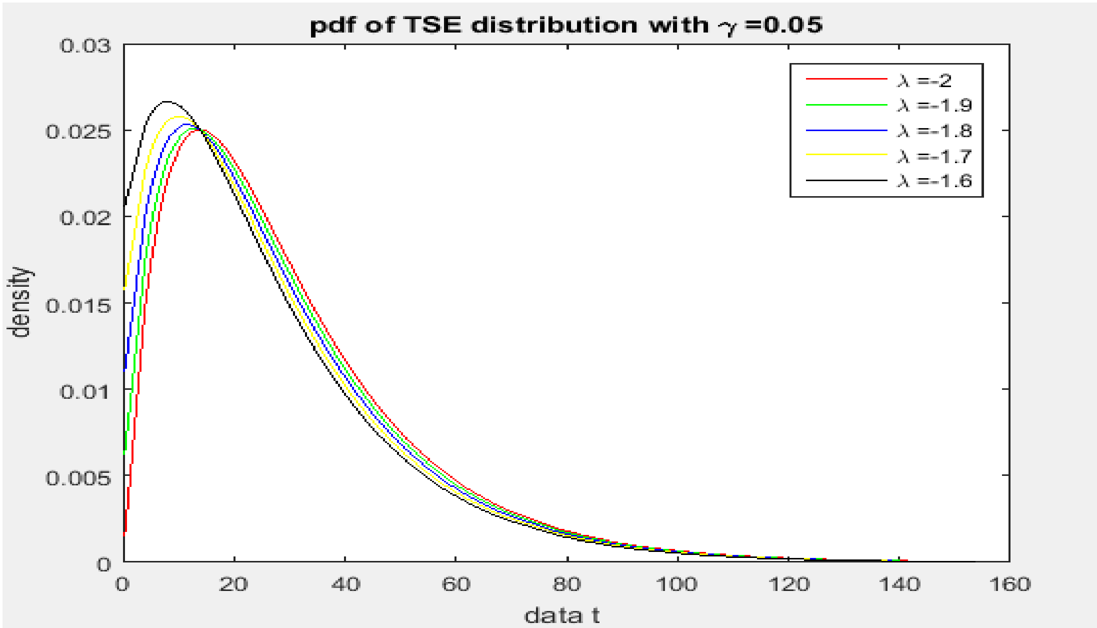
pdf of TSE distribution with λ=-0.001
density
data t
γ =0.01
γ =0.02
γ =0.03
γ =0.04
γ =0.05

-a-

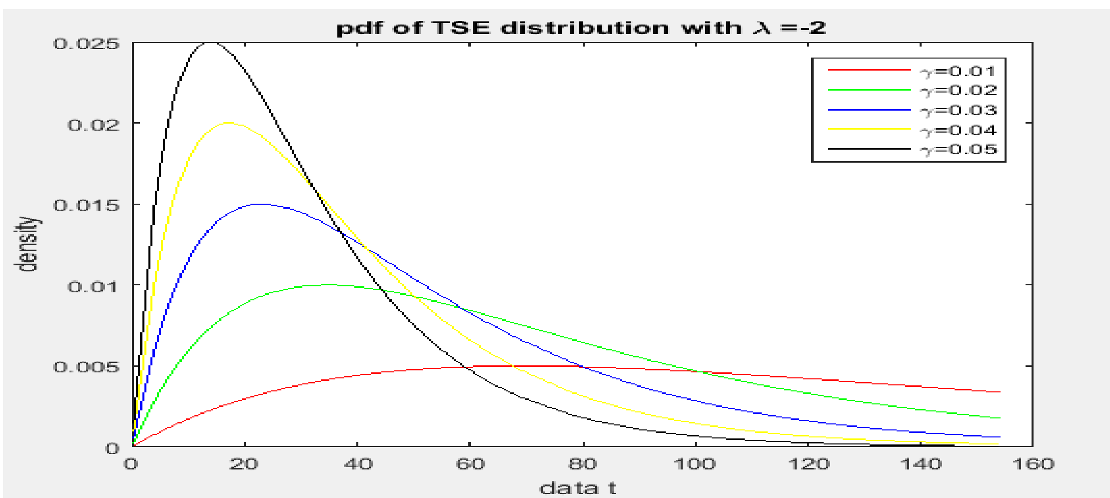


pdf of TSE distribution with γ =0.05
density
data t
λ =-2
λ =-1.5
λ =-1.3
λ =-1
λ =-0.5

-b-



-c-



-d-

Figure 2.10: Shapes of density function with different scale (γ) parameter value and different shape (λ) transmuted parameter.

Figure (2.10) shows different faces, in (2.10,a) the density f_{TSE} becomes decreasing function and without general change at shape at specific transmuted parameter $\lambda = -0.001$ and different values of γ in (2.10,b) the effect of changing values of λ is obvious such that at specific point the curve of f_{TSE} distribution turn from increasing to decreasing at fixed γ. This effect becomes clearer at (2.10, c) and (2.10,d). In addition, the shape transmuted parameter λ who causes the bending in the curve of distribution with one peak while the scale parameter γ is response upon the stretching out in the curve as we can note in (2.10, d). Beside when one set $\lambda = -1$ at the f_{TSE} in (2-11), then the TSE distribution is just the Exponential distribution:

$$f_{TSE}(t) = \gamma e^{-\gamma t}$$

In addition, when $\lambda = 0$, $f_{TSE}(t)$ is another formula of Exponential distribution with multiple values of data:

$$f_{TSE}(t) = 2\gamma e^{-2\gamma t}$$

However, when $\lambda = 1$, $f_{TSE}(t)$ = $2\gamma e^{-2\gamma t} - \gamma e^{-\gamma t}$is not probability density function because $\int f_{TSE}(t)\, dt = 0$ so λ must not be one.

Hazard Rate Function

The hazard rate function also known as the "force of mortality" is defined by

$$h(t) = P(\, T < t + dt | T \geq t)$$

$$= -\frac{S(t)'}{S(t)}$$

$$= -\frac{d \ln(S(t))}{dt}$$

$$= \frac{2\gamma(1+\lambda)e^{-2\gamma t} - \gamma\lambda e^{-\gamma t}}{(1+\lambda)e^{-2\gamma t} - \lambda e^{-\gamma t}}$$

Hence the hazard transmuted-survival-exponential distribution is:-

$$h_{TSE}(t) = \frac{2\gamma(1+\lambda) - \gamma\lambda e^{\gamma t}}{(1+\lambda) - \lambda e^{\gamma t}}$$

The limits of this function are:

$$\lim_{t\to 0} h_{TSE}(t,\gamma,\lambda) = \lim_{t\to 0} \frac{2\gamma(1+\lambda) - \gamma\lambda e^{\gamma t}}{(1+\lambda) - \lambda e^{\gamma t}}$$

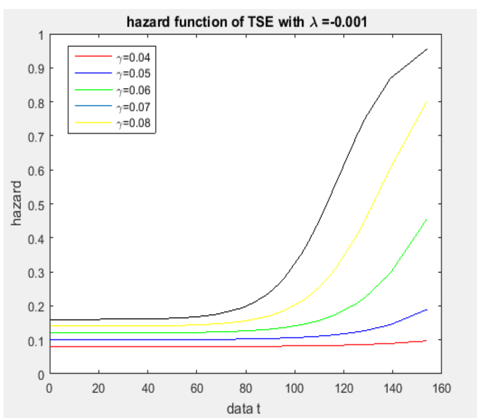
$$= \gamma\ (2+\lambda\ )qQ$$

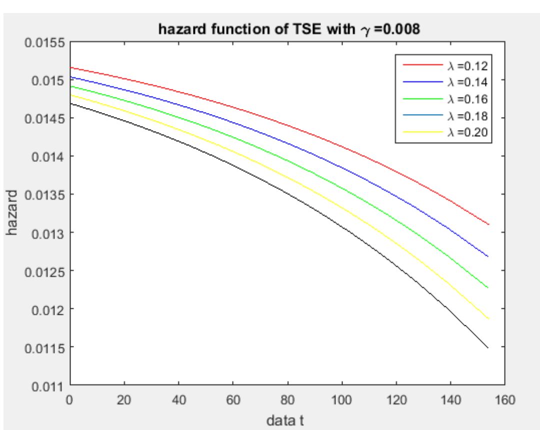
And

$$\lim_{t\to \infty} h_{TSE}(t,\gamma,\lambda) = \lim_{t\to \infty} \frac{2\gamma(1+\lambda) - \gamma\lambda e^{\gamma t}}{(1+\lambda) - \lambda e^{\gamma t}}$$

Can be computed by L'hopetal rule:

$$\lim_{t\to \infty} h_{TSE}(t,\gamma,\lambda) = \gamma$$





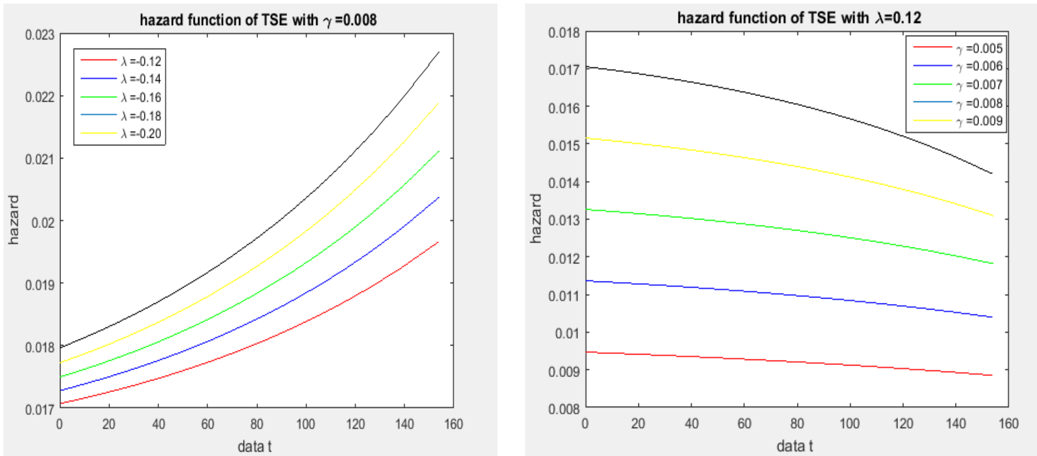


Figure 2.11 Shapes of hazard function with different values of γ *and* λ .

The above sketches give different cases of hazard function as one can note that $h_{TSE}(t)$ depends on the values of *and* λ . Therefore, it turns between increasing, decreasing and almost constancy.

2.3.2 Shape of TSW distribution

Consider equations (2-18) and (2-19)

$$\lim_{t\to 0} f_{TSW}(t) = \lim_{t\to 0}(2\alpha t^{\alpha-1}(\lambda+1)e^{-2t^{\alpha}} - \alpha t^{\alpha-1}\lambda e^{-t^{\alpha}})$$

It is clear that:

$$\lim_{t\to 0} f_{TSW}(t) = 0$$

And

$$\lim_{t\to \infty} f_{TSW}(t) = \lim_{t\to \infty} (2\alpha t^{\alpha-1}(\lambda+1)e^{-2t^{\alpha}} - \alpha t^{\alpha-1}\lambda e^{-t^{\alpha}})$$

Also

$$\lim_{t\to \infty} f_{TSW}(t) = 0$$

While $\lim_{t\to 0} F_{TSW}(t) = 1-(\lambda+1)e^{-2t^{\alpha}} + \lambda e^{-t^{\alpha}}$

$$= 1-(\lambda+1)+\lambda = 0$$

And

$$\lim_{t\to\infty} F_{TSW}(t) = 1-(\lambda+1)e^{-2t^{\alpha}} + \lambda e^{-t^{\alpha}} = 1$$

In addition, we can have more details about the curve of f_{TSW} by first and second derivative like the one we do in last section such that:

$$\frac{\partial f_{TSW}}{\partial t} = 2\alpha(\lambda+1)(\alpha-2\alpha t^{\alpha}-1)t^{\alpha-2}\, e^{-2t^{\alpha}} - \alpha\lambda(\alpha-\alpha t^{\alpha} - 1)t^{\alpha-2}\, e^{-t^{\alpha}} \qquad (2\text{-}24)$$

In addition, the second derivative is:

$$\frac{\partial^2 f_{TSW}}{\partial t^2} = 2\alpha(\alpha-2)(\lambda+1)(1-2\alpha t^{\alpha}) - 4\alpha^2\,(\lambda+1)(1-2\alpha t^{\alpha})t^{2\alpha-3}e^{-2t^{\alpha}} - 4\alpha^3\,(\lambda+1)t^{2\alpha-3}e^{-2t^{\alpha}} - \alpha^2\lambda t^{2\alpha-3}e^{-t^{\alpha}} + \alpha(\alpha-2)\lambda t^{\alpha-3}\, e^{-t^{\alpha}}$$

The first and second derivative can be solved numerically for t to find the stationary, maximum and minimum points to have clear image about the f_{TSW}. Next, we give visualization for different shape's cases of different values of parameters (λ *and* α) .

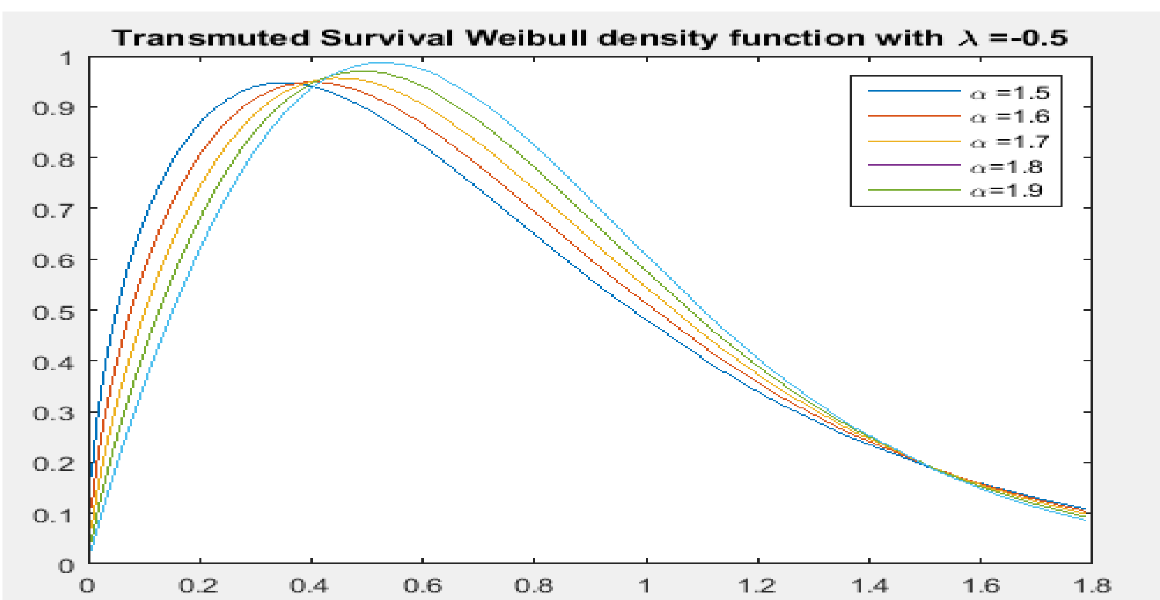


Figure 2.12, a

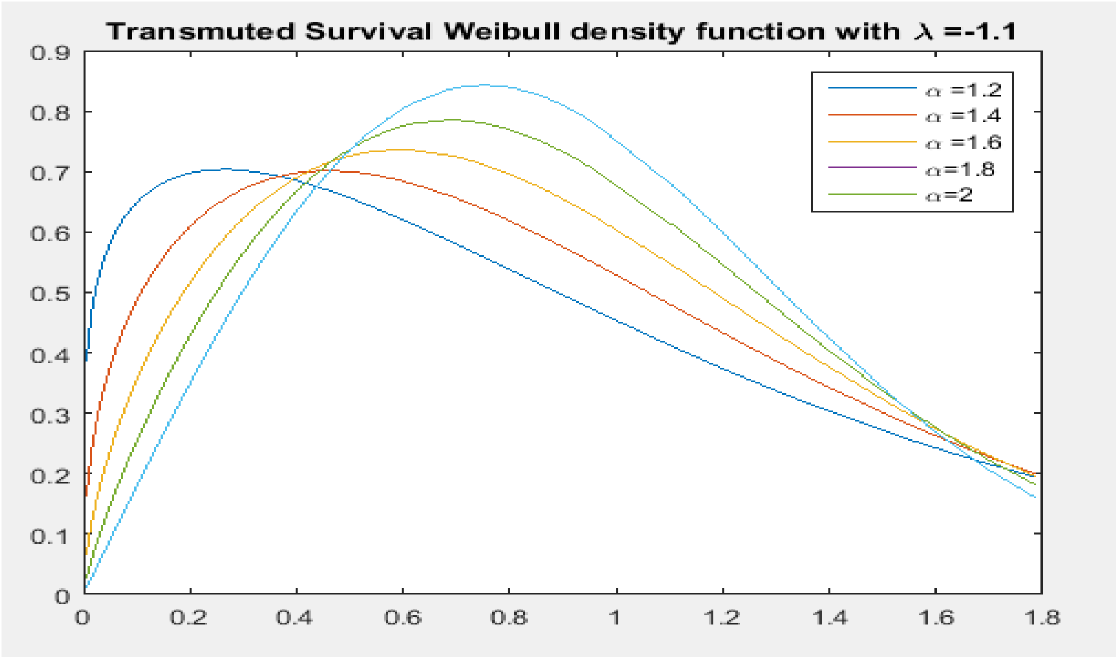


Figure 2.12, b

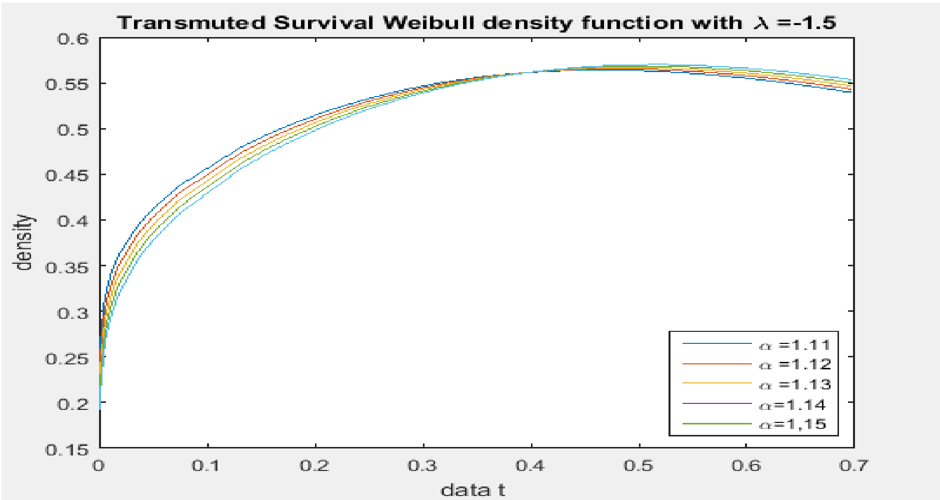


Figure 2.12, c

Figure (2.12, a,b,c) shows that when we fix the value of transmuted parameter λ and change the values of scale's parameter α the behavior of the curve of TSW distribution changes from increment to decrement, with one peak. Moreover, the increase in α values has the effect of stretching out the pdf to the left once and to the right another once depending on the vary of λ values as figure 2.13, d shows.

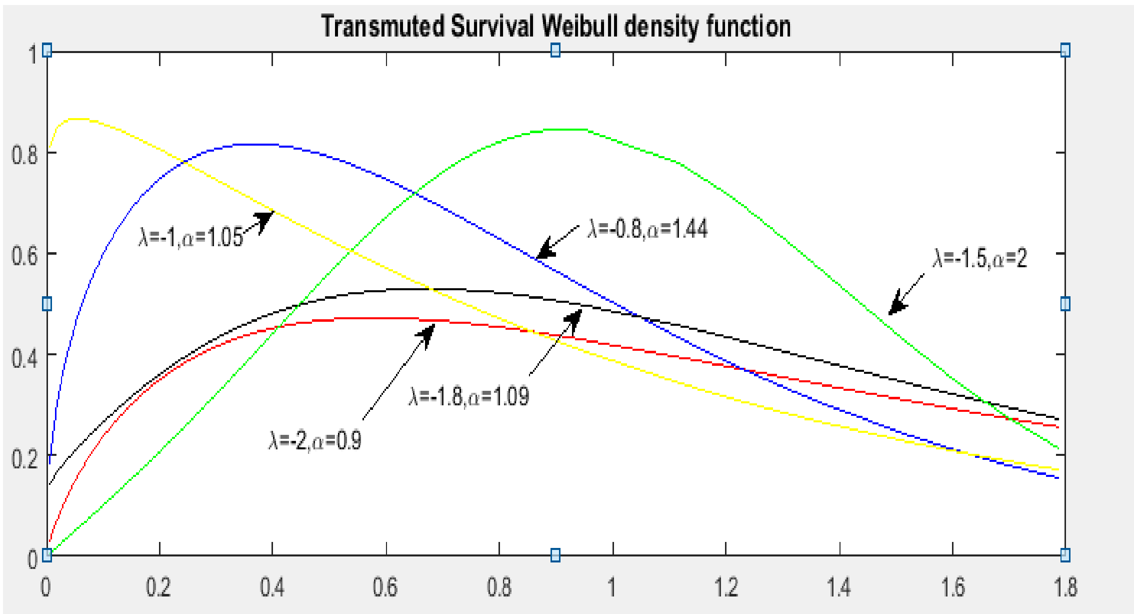


Figure 2.13, d

Figure 2.13, d shows that shape transmuted parameter λ is what gives the TSW its flexibility. By changing, the value of λ the TSW can model a wide variety of data:

When $\lambda = -1$ the probability density function (2-19) is just the standard Weibull distribution $f_{TSW}(t) = \alpha t^{\alpha-1} e^{-t^{\alpha}}$, when $\lambda = 0$ *the is* $f_{TSW}(t) = 2\alpha t^{\alpha-1} e^{-2t^{\alpha}}$ which is a special case of tow parameters Weibull distribution with shape parameter =2. In addition, when $\lambda = 1$ we get

$$f_{TSW}(t) = 4\alpha t^{\alpha-1} e^{-2t^{\alpha}} - \alpha t^{\alpha-1} e^{-t^{\alpha}}$$

Which is also lead to $\int f_{TSW}(t)\, dt = 0$.

Moreover, we give the formula of $h_{TSW}(t)$ such =that:

$$h_{TSW}(t) = -\frac{S_{TSW}(t)'}{S_{TSW}(t)}$$

$$h_{TSW}(t) = -\frac{d\ (\ln S_{TSW}(t))}{dt}$$

Thus

$$h_{TSW}(t) = -\frac{d}{dt}\ \ln((\lambda+1)e^{-2t^{\alpha}} +\ \lambda e^{-t^{\alpha}}$$

$$h_{TSW}(t) = -\frac{d}{dt}\ \ln[e^{-t^{\alpha}}((\lambda+1)e^{-t^{\alpha}} +\ \lambda)]$$

$$h_{TSW}(t) = -\frac{d}{dt}\ [-t^{\alpha} + \ln[((\lambda+1)e^{-t^{\alpha}} +\ \lambda)]$$

$$h_{TSW}(t) = \alpha t^{\alpha-1} - \frac{-\alpha t^{\alpha-1}(\lambda+1)e^{-t^{\alpha}}}{(\lambda+1)e^{-t^{\alpha}} +\ \lambda}$$

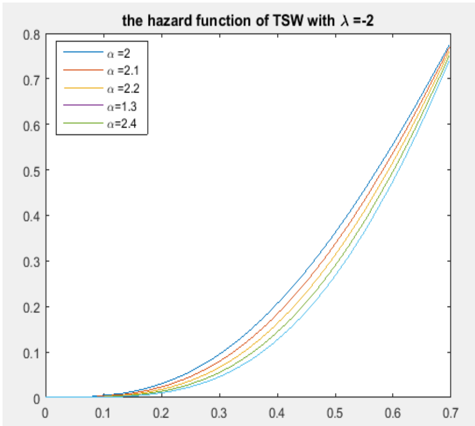
Now we can find the limit:

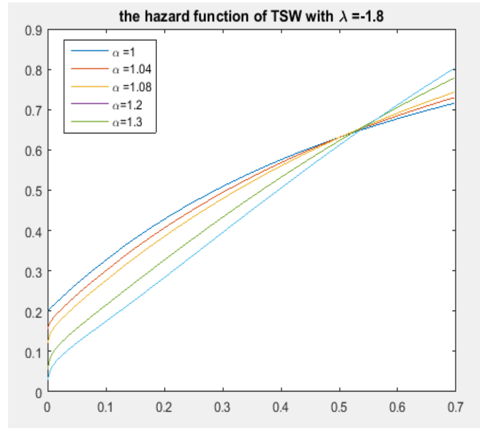
$$\lim_{t\to 0} h_{TSW}(t,\alpha,\lambda) = \lim_{t\to 0} \alpha t^{\alpha-1} - \lim_{t\to 0} \frac{-\alpha t^{\alpha-1}(\lambda+1)e^{-t^{\alpha}}}{(\lambda+1)e^{-t^{\alpha}} +\ \lambda}$$

$$\lim_{t\to 0} h_{TSW}(t,\alpha,\lambda) = 0$$

And

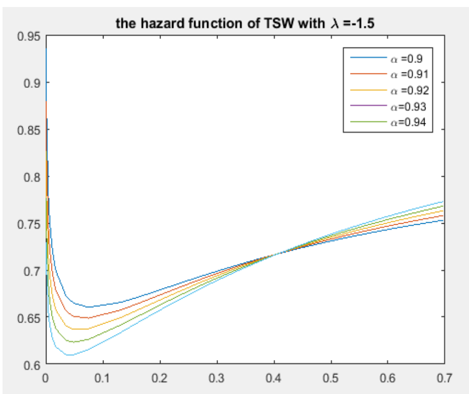
$$\lim_{t\to\infty} h_{TSW}(t,\alpha,\lambda) = \infty$$

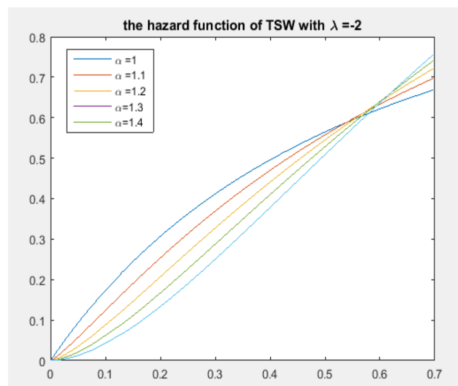




-a-

-b-





-c-

-d-

Figure 2.14 Shape of hazard function with different values of $and\ \alpha$.

As figure 2.14 the plot for the curve of h_{TSW} function indicates that when the time data tend toward the initial time $t = 0$ the curve tend to the zero point of y-axis and when time t increases , the curve of h_{TSW} increases with increment of y-axis .

2.3.3 Shape of TSEW distribution

As same procedure in the last two sections consider (2-21) and (2-22)

$$\lim_{t\to 0} f_{TSEW}(t,\gamma,\alpha,\lambda) = \lim_{t\to 0} 2\gamma(1+\lambda)e^{-2\gamma t} - \lim_{t\to 0} \alpha t^{\alpha}\lambda e^{-t^{\alpha}}$$

$$= 2\gamma(1+\lambda)$$

And

$$\lim_{t\to \infty} f_{TSEW}(t,\gamma,\alpha,\lambda) = \lim_{t\to \infty} 2\gamma(1+\lambda)e^{-2\gamma t} - \lim_{t\to \infty} \alpha t^{\alpha}\lambda e^{-t^{\alpha}}$$

$$= 0$$

The limits of the F_{TSEW} are:

$$\lim_{t\to 0} F_{TSEW}(t,\gamma,\alpha,\lambda) = \lim_{t\to 0} 1 - \lim_{t\to 0}(1+\lambda)e^{-2\gamma t} + \lim_{t\to 0} \lambda e^{-t^{\alpha}}$$

$$= 0$$

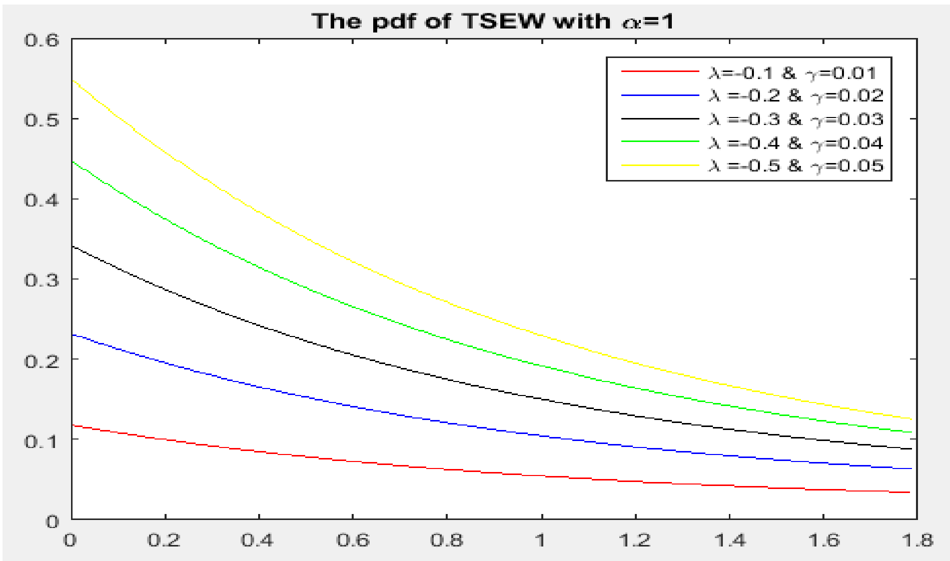
And

$$\lim_{t\to \infty} F_{TSEW}(t,\gamma,\alpha,\lambda) = 1$$

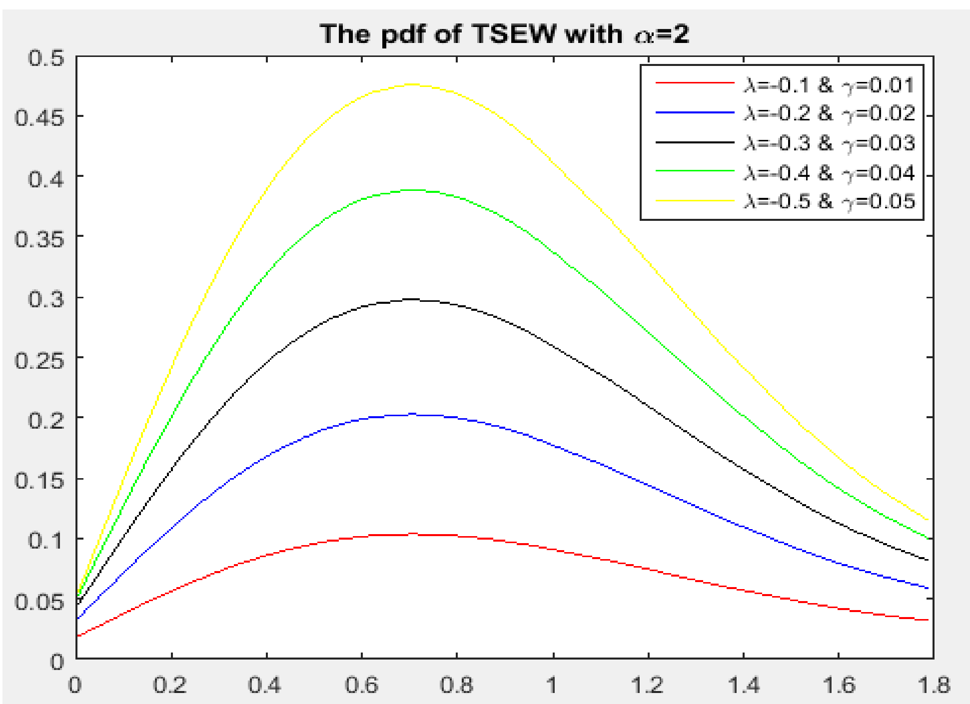
The first and second derivative are:

$$\frac{\partial f_{TSEW}}{\partial t} = -4\gamma^2(1+\lambda)e^{-2\gamma t} - \frac{\alpha^2 t^{\alpha}\lambda\, e^{-t^{\alpha}}}{t} + \frac{\alpha^2 (t^{\alpha})^2 \lambda\, e^{-t^{\alpha}}}{t} \qquad \text{(2-25)}$$

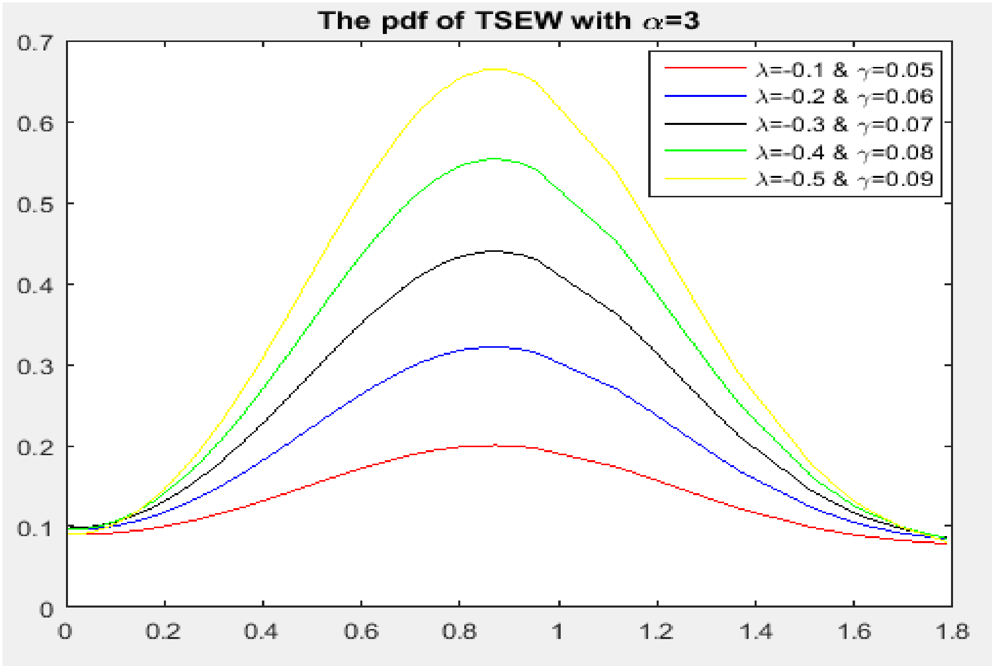
$$\frac{\partial^2 f_{TSEW}}{\partial t^2} = 8\gamma^3(1+\lambda)e^{-2\gamma t} - \alpha^3\lambda e^{-t^{\alpha}}(t^{\alpha-2} - 3t^{2\alpha-2} + t^{3\alpha-2}) + \alpha^2\lambda e^{-t^{\alpha}}(t^{\alpha-2} - t^{2\alpha-2})$$


The pdf of TSEW with α=1
λ=-0.1 & γ=0.01
λ =-0.2 & γ=0.02
λ =-0.3 & γ=0.03
λ =-0.4 & γ=0.04
λ =-0.5 & γ=0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8

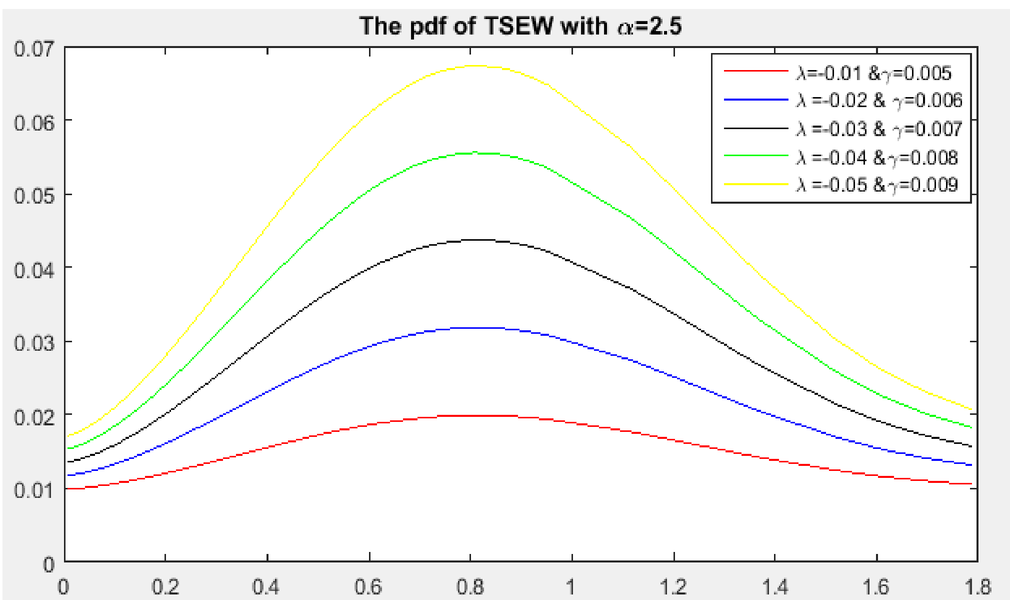
-a-


The pdf of TSEW with α=2
λ=-0.1 & γ=0.01
λ=-0.2 & γ=0.02
λ=-0.3 & γ=0.03
λ=-0.4 & γ=0.04
λ=-0.5 & γ=0.05
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8

-b-



-c-



-d-

Figure 2.15 Shapes of probability density function of TSEW distribution with different values of parameters $(\alpha, \lambda \; and \; \gamma)$.

This new mix of distributions has distinguished oneself, which is the transmuted-shape parameter λ, the scale parameter γ and the location parameter α as figure (2.15) emphasize. Where changing the values of λ caused the presence of the peaks in the TSEW distribution's curves and using different values of γ caused the widening. Also variation in the value of α has effect of changing the location of the distribution curves as figure (2.15,a,b,c and d) say.

Also the TSEW distribution has another feature, when $\lambda = -1$ the probability density function (2-22) reduces to the standard Weibull distribution. And, when $\lambda = 0$, we have the Exponential distribution. While if $\lambda = 1$, the mix distribution (2-22) becomes

$$f_{TSEW}(t) = 4\gamma e^{-2\gamma t} - \alpha t^{\alpha-1} e^{-t^{\alpha}} \qquad \text{(2-26)}$$

And $\int f_{TSEW}(t)\,dt = 0$.

Now we derive the hazard rate function:

$$f_{TSEW}(t) = 2\gamma(1+\lambda)e^{-2\gamma t} - \lambda \alpha t^{\alpha-1} e^{-t^{\alpha}}$$

$$h_{TSEW}(t) = -\frac{\acute{S}_{TSEW}(t)}{S_{TSEW}(t)}$$

$$h_{TSEW}(t) = \frac{2\gamma(1+\lambda)e^{-2\gamma t} - \lambda \alpha t^{\alpha-1} e^{-t^{\alpha}}}{(1+\lambda)e^{-2\gamma t} - \lambda e^{-t^{\alpha}}}$$

$$h_{TSEW}(t) = \frac{2\gamma(1+\lambda)e^{-2\gamma t}}{(1+\lambda)e^{-2\gamma t} - \lambda e^{-t^{\alpha}}} - \frac{\lambda \alpha t^{\alpha-1} e^{-t^{\alpha}}}{(1+\lambda)e^{-2\gamma t} - \lambda e^{-t^{\alpha}}}$$

$$h_{TSEW}(t) = \frac{(1+\lambda)e^{-2\gamma t}}{(1+\lambda)e^{-2\gamma t}}\left(\frac{2\gamma}{1-\frac{\lambda}{1+\lambda}e^{-t^{\alpha}+2\gamma t}}\right) - \frac{\lambda e^{-t^{\alpha}}}{\lambda e^{-t^{\alpha}}}\left(\frac{\alpha t^{\alpha-1}}{\frac{(1+\lambda)}{\lambda}e^{-2\gamma t+t^{\alpha}}-1}\right)$$

Now we can find the limit to image the general behavior of the failure arte of TSEW distribution when t close to zero and at t increase to the infinity:

$$\lim_{t\to 0} h_{TSEW}(t,\gamma,\alpha,\lambda) = \lim_{t\to 0} \frac{2\gamma}{1-\frac{\lambda}{1+\lambda}e^{-t^{\alpha}+2\gamma t}} - \lim_{t\to 0} \frac{\alpha t^{\alpha-1}}{\frac{(1+\lambda)}{\lambda}e^{-2\gamma t+t^{\alpha}}-1}$$

$$\lim_{t\to 0} h_{TSEW}(t,\gamma,\alpha,\lambda) = \frac{2\gamma}{1-\frac{\lambda}{1+\lambda}}$$

$$\lim_{t\to 0} h_{TSEW}(t,\gamma,\alpha,\lambda) = 2\gamma(1+\lambda)$$

So, when t close to the initial time the hazard rate function will depend on the tow parameter $\gamma \; and \; \lambda$ where both the parameter of the TSE distribution. In addition:

$$\lim_{t\to \infty} h_{TSEW}(t,\gamma,\alpha,\lambda) = \lim_{t\to \infty} \frac{2\gamma}{1-\frac{\lambda}{1+\lambda}e^{-t^{\alpha}}e^{2\gamma t}} - \lim_{t\to \infty} \frac{\alpha t^{\alpha-1}}{\frac{(1+\lambda)}{\lambda}e^{-2\gamma t}e^{t^{\alpha}}-1}$$

$$\lim_{t\to \infty} h_{TSEW}(t,\gamma,\alpha,\lambda) = \frac{2\gamma}{1} + \infty$$

Hence

$$\lim_{t\to \infty} h_{TSEW}(t,\gamma,\alpha,\lambda) = \infty$$

That means when t increasing its value, the curve of the hazard rate function will increase too asymptotic to the x-axis, as we will see in the following figures:

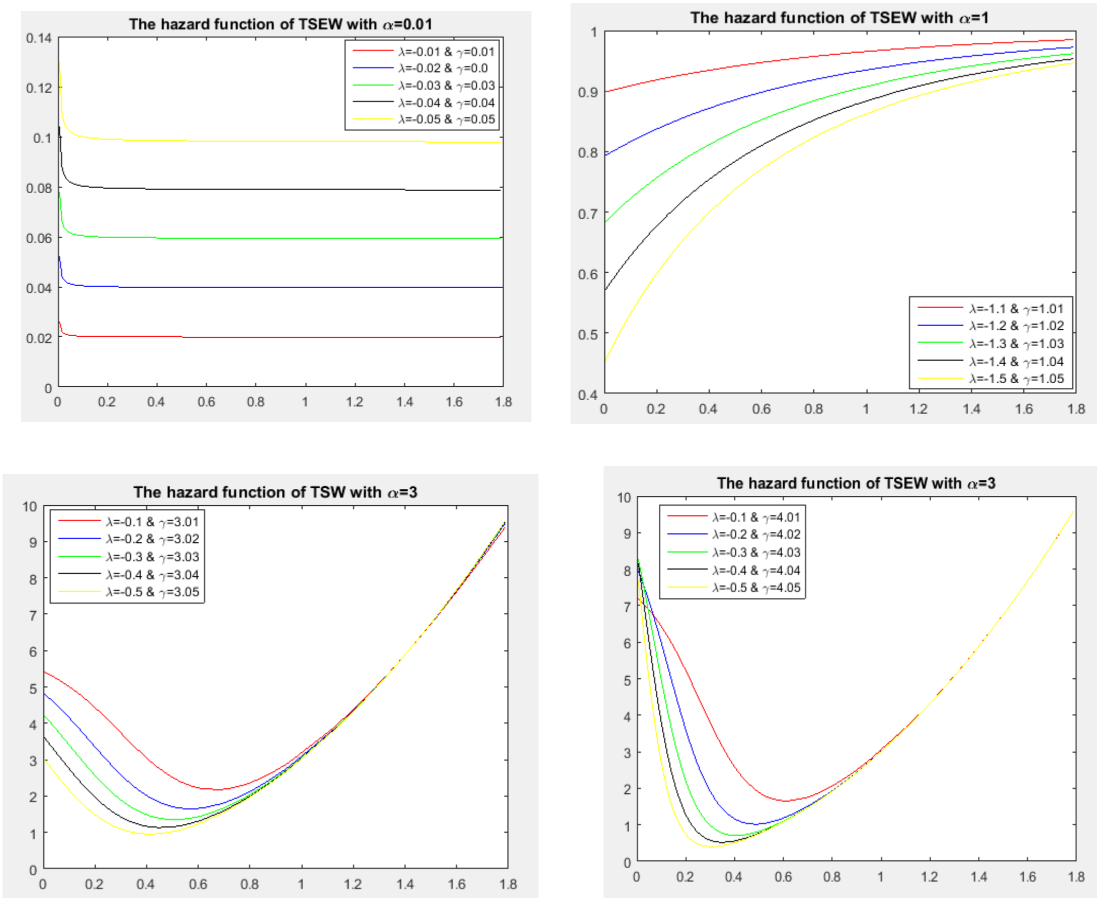


Figure 2.16 Shape of hazard rate function of TSEW with different values of parameters λ, γ and α.

Next, we derive the statistical characteristics of the introduced distributions (TSE, TSW and TSEW).

2.4 Statistical Characteristics of TSE Distribution

2.4.1 Mode:

The Mode is most useful as a measure of central tendency and the Mood is an approach of wording distinguish value. This value is no need to be unique. In addition, if the pdf of distribution has more than one maximum point then we can tail that every maximum point is mode of the distribution.

Consider the pdf of TSE distribution (2-11) and the first derivative (2-23) with:

$$\frac{\partial f_{TSE}}{\partial t} = 0$$

Hence:

$$\lambda\gamma^2 e^{-\gamma t} - 4\gamma^2(1+\lambda)e^{-2\gamma t} = 0$$

And

$$4(1+\lambda) = \lambda e^{\gamma t}$$

$$e^{\gamma t} = \frac{4(1+\lambda)}{\lambda}$$

Then

$$t = \frac{\ln(4(1+\lambda)/\lambda)}{\gamma} \qquad (2\text{-}27)$$

2.4.2 Median: The common definition of the median t_{med} is the value of random variable t where half the probability comes before it and half comes after it:

$$P(T \le t) \ge \frac{1}{2} \quad and \quad P(T \le t) \le \frac{1}{2}$$

Hence, we can find it by solve the equation:

$$F(t) = \frac{1}{2} \;, \;\; for\ t$$

Consider (1-10) and let:

$$F_{TSE}(t) = \frac{1}{2}$$

$$1-(1+\lambda)e^{-2\gamma t}+\lambda e^{-\gamma t}=\frac{1}{2}$$

$$(1+\lambda)e^{-2\gamma t}+\lambda e^{-\gamma t}=\frac{1}{2}$$

$$e^{-2\gamma t}-\frac{\lambda}{(\lambda+1)}e^{-\gamma t}-\frac{1}{2(\lambda+1)}=0$$

$$\text{Let } x=e^{-\gamma t},x^2=e^{-2\gamma t}$$

then the last equation become

$$x^2-\frac{\lambda}{(\lambda+1)}x-\frac{1}{2(\lambda+1)}=0$$

And let $b=\frac{\lambda}{(\lambda+1)}$ $and\ c=\frac{1}{2(\lambda+1)}$ the quadratic equation can be solved algebraically for x as:

$$x=\frac{-b\ \mp\ \sqrt{b^2-4ac}}{2a}$$

Then we can find t_{med} after solving the equation $x=e^{-\gamma t}$such that

$$\ln x=\ln e^{-\gamma t}$$

Hence

$$t=-\ln x/\gamma \qquad (2\text{-}28)$$

2.4.3 The Moments: For a distribution, moments are important to understand the various characteristics of a frequency distribution and the shape of it can be depicted by its diverse 'moments'. Finding a general formula of these moments, central tendency, dispersion, skewness and kurtosis of a distribution can be computed so easily.

Theorem 1: The rth moment M'_r of TSE distribution about the origin is

$$M'_r = \Gamma(r+1)\,[\,\frac{\lambda+1}{(2\gamma)^{r-1}} - \frac{\lambda}{(\gamma)^{r-1}}\,] \qquad (2\text{-}29)$$

And

Therefore the rth moment M_r of TSE distribution about the means

$$M_r = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}M'_i\,(M'_1)^{r-i} \qquad (2\text{-}30)$$

Where $M'_i = E\big(\,T^i\big)$.

Proof:

$$E(t^r) = \int_0^\infty t^r\, f_{TSE}(x)\,dx = \int_0^\infty t^r\ \ [2(\lambda+1)\,\gamma e^{-2\gamma t} - \lambda\gamma e^{-\gamma t}]dt$$

$$= \int_0^\infty t^r\, 2(\lambda+1)\,\gamma e^{-2\gamma t}\,dt - \int_0^\infty t^r\ \lambda\gamma e^{-\gamma t}\,dt$$

$$= \frac{\lambda+1}{(2\gamma)^{r-1}}\int_0^\infty (2\,\gamma t)^{(r+1)-1}\,e^{-2\gamma t}\,dt - \frac{\lambda}{\gamma^{r-1}}\int_0^\infty (t\gamma)^{(r+1)-1}\ e^{-\gamma t}\,dt$$

$$= \frac{\lambda+1}{(2\gamma)^{r-1}}\ \Gamma(r+1) - \frac{\lambda}{(\gamma)^{r-1}}\ \Gamma(r+1)$$

$$= \Gamma(r+1)\,[\,\frac{\lambda+1}{(2\gamma)^{r-1}} - \frac{\lambda}{(\gamma)^{r-1}}\,]$$

Now

The general equation for converting the rth moment about the origin to the moment about the mean is

$$M_r = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}M'_i\,(M)^{r-i}\ .\ \textit{where } M \textit{ is the mean of the dsitribution}$$

Therefore the rth moment M_r' of TSE distribution about the mean μ is

$$E(T-\mu)^r = \ M_r = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}M_i'(M_1')^{r-i}$$

Where M_1' is the mean of TSE distribution calculated from (2-29).

The mean of the TSE distribution is (when r = 1):

$$E(T) = \ \Gamma(2)$$

And, the second moment of the TSE distribution is (when r = 2):

$$E(T^2) = \frac{\Gamma(3)}{2\gamma}(\ 1-\lambda)$$

Then, the variance of the TSE distribution is:

$$var(t) = \ E(T^2) - \left(E(T)\right)^2$$

$$var\ (T) = \frac{\Gamma(3)}{2\gamma}(\ 1-\lambda) - \ \Gamma(2)^2$$

The Coefficient of Variation CV is given by:

$$CV = \frac{\sqrt{var\ (T)}}{E(t)}$$

$$CV = \frac{\sqrt{\frac{\Gamma(3)}{2\gamma}(\ 1-\lambda) - \ \Gamma(2)^2}}{\Gamma(2)}$$

2.4.4 The Moment Generation Function

Theorem 2: The (mgf) of T for the TSE distribution is:

$$M_t(z) = \frac{2\gamma(\lambda+1)}{2\gamma-z} - \frac{\gamma\lambda}{\gamma-z} \quad . \; z \in R \qquad (2\text{-}30)$$

Proof:

$$M_t(z) = E(e^{zt}) = \int_0^\infty e^{zt} f_{TSE}(t)dt$$

$$= \int_0^\infty e^{zt}[2\gamma(\lambda+1)\; e^{-2\gamma t} - \lambda\gamma\, e^{-\gamma t}]dt$$

$$= \int_0^\infty e^{zt} 2\gamma(\lambda+1)\; e^{-2\gamma t}\, dt - \int_0^\infty e^{zt}\, \gamma\lambda\; e^{-\gamma t}\, dt$$

$$= 2\gamma(\lambda+1)\int_0^\infty e^{zt}\; e^{-2\gamma t}\, dt - \gamma\lambda \int_0^\infty e^{zt}\; e^{-\gamma t}\, dt$$

$$= \frac{2\gamma(\lambda+1)}{2\gamma-z}\int_0^\infty (2\gamma-z)e^{-(2\gamma-z)t}\; dt - \frac{\gamma\lambda}{\gamma-z}\int_0^\infty (\gamma-z)\,)e^{-(\gamma-z)t}\; dt$$

Since $\int_0^\infty (2\gamma-z)\,)e^{-(2\gamma-z)t}\; dt = \int_0^\infty (\gamma-z)\,)e^{-(\gamma-z)t}\; dt = 1$

Hence

$$M_t(z) = \frac{2\gamma(\lambda+1)}{2\gamma-z} - \frac{\gamma\lambda}{\gamma-z}$$

2.4.5 The Factorail Moments Generation Function

Factorial moments are beneficial for studying non-negative integer-valued random variables. Factorial moments serve as analytic implement in mathematical field of combinatorial.

Theorem 3: The Factorial Moments Generating function of t denoted by:

$$M_t(z) = \frac{2\,\gamma(\lambda+1)}{2\gamma - lnz} - \frac{\gamma\lambda}{\gamma - lnz} \qquad (2\text{-}31)$$

Proof:

$$M_t(z) = E(z^t) = \int_0^\infty z^t\, f_{TSE}(t)dt$$

$$= \int_0^\infty e^{\ln z^t}[2\,\gamma(\lambda+1)\ e^{-2\gamma t} - \lambda\gamma\, e^{-\gamma t}]dt$$

$$= \int_0^\infty e^{\ln z^t} 2\,\gamma(\lambda+1)\ e^{-2\gamma t}\,dt - \int_0^\infty e^{\ln z^t}\gamma\lambda\ e^{-\gamma t}\,dt$$

$$= 2\,\gamma(\lambda+1)\int_0^\infty e^{t\ln z}\ e^{-2\gamma t}\,dt - \gamma\lambda\int_0^\infty e^{t\ln z}\ e^{-\gamma t}\,dt$$

$$= \frac{2\,\gamma(\lambda+1)}{2\gamma - \ln z}\int_0^\infty (2\gamma - \ln z\ )e^{-(2\gamma - lnz)t}\ dt - \frac{\gamma\lambda}{\gamma - \ln z}\int_0^\infty (\gamma - \ln z)\ )e^{-(\gamma - lnz)t}\ dt$$

Since $\int_0^\infty (2\gamma - lnz)\ )e^{-(2\gamma - lnz)t}\ dt = \int_0^\infty (\gamma - lnz)\ e^{-(\gamma - lnz)t}\ dt = 1$

Hence

$$M_t(z) = \frac{2\,\gamma(\lambda+1)}{2\gamma - lnz} - \frac{\gamma\lambda}{\gamma - lnz} \qquad ,\gamma > lnz$$

2.4.6 The Quantile Function

The quintile function or inverse cumulative distribution function returns the value t such that

$$t = Q(p) = F^{-1}(p) \qquad where \quad 0 < p < 1$$

So this function is a very important tool in statistical modeling precisely in simulate data.

Consider the cdf in (2-10) the quantile function of TSE distribution may be expressed within implicit form as:

$$p = 1 - (\mathbf{1} + \lambda)\boldsymbol{e}^{-2\gamma t} + \lambda \boldsymbol{e}^{-\gamma t}$$

$$1 - p = (\mathbf{1} + \lambda)\boldsymbol{e}^{-2\gamma t} - \lambda \boldsymbol{e}^{-\gamma t} \qquad \text{(2-32)}$$

$$\ln(1 - p) = -\boldsymbol{\gamma t} + \ln[(\mathbf{1} + \lambda)\boldsymbol{e}^{-\gamma t} - \lambda] \qquad (2-33)$$

Which is nonlinear equation can be solved for t numerically.

2.4.7 The Coefficient of Skewness

Skewness is a tool that measures the symmetry, or more precisely, the deficiency of symmetry. A distribution is symmetric if it appears the same to the left and right of the center point within its curve

The coefficient of skewness of a random variable T CS is given by:

$$CS = \frac{E(t-\mu)^3}{\sigma^3} = \frac{\sum_{i=0}^{3}(-1)^i\binom{3}{i}M'_{3-i}\left(M'_1\right)^i}{\sigma^3} \qquad \text{(2-34)}$$

Or

$$CS = \frac{E(T^3) - 3E(T)\sigma^2 - \mu^3}{\sigma^3} \qquad \text{(2-35)}$$

Where $\sigma^3 = (\sigma^2)^{3/2}$

μ is the mean ($E(t)$). So by using Theorem 1:

$$CS = \frac{\frac{\Gamma(4)}{(2\gamma)^2}(1-3\lambda) - 3\Gamma(2)\left(\frac{\Gamma(3)}{2\gamma}(1-\lambda) - \Gamma(2)^2\right) - \Gamma(2)^3}{(\frac{\Gamma(3)}{2\gamma}(1-\lambda) - \Gamma(2)^2)^{3/2}}$$

2.4.8 The Coefficient of Kurtosis

The coefficient of kurtosis is a tool that identify if the tails of given distribution have extreme values or not.

The Coefficient of kurtosis CK is given by

$$CK = \frac{E(T-\mu)^4}{\sigma^4}$$

Where

$$E(t-\mu)^4 = E(T^4) - 4E(T)E(T^3) + 6E(T^2)E(T)^2 - 3E(T)^4$$

And $\sigma^4 = var\ (T)^2$.

According to Theorem (1) the kurtosis of TSE distribution is:

$$CK = \frac{\Gamma(5)(\frac{(1+\lambda)}{(2\gamma)^3} - \frac{\lambda}{\gamma^3}) - 4\Gamma(2)\frac{\Gamma(4)}{(2\gamma)^2}(1-3\lambda) + 6\,\Gamma(2)^2\frac{\Gamma(3)}{2\gamma}(1-\lambda) - 3\Gamma(2)^4}{(\frac{\Gamma(3)}{2\gamma}(\,1-\lambda) - \,\Gamma(2)^2)^2}$$

2.5 Statistical Characteristics of TSW Distribution

2.5.1 Mode

Consider (2-19) and (2-24) with

$$\frac{\partial f_{TSW}}{\partial t} = 0$$

And

$$2\alpha(\lambda+1)(\alpha-2\alpha t^{\alpha}-1)t^{\alpha-2}\, e^{-2t^{\alpha}} - \alpha\lambda(\alpha-\alpha t^{\alpha}-1)t^{\alpha-2}\, e^{-t^{\alpha}} = 0$$

$$2\alpha(\lambda+1)(\alpha-2\alpha t^{\alpha}-1)t^{\alpha-2}\, e^{-2t^{\alpha}} = \alpha\lambda(\alpha-\alpha t^{\alpha}-1)t^{\alpha-2}\, e^{-t^{\alpha}}$$

$$2(\lambda+1)(\alpha-2\alpha t^{\alpha}-1) = \lambda(\alpha-\alpha t^{\alpha}-1)\, e^{t^{\alpha}} \qquad \text{(2-36)}$$

We can solve (2-36) numerically and it has many root. Subsequently if we suppose $t = t_0$ as root for (2-36) then the value of the mode will depend on the second derivative at t_0 whether

$$\frac{\partial^2 f_{TSW}(t_0)}{\partial t^2} < 0, \quad \frac{\partial^2 f_{TSW}(t_0)}{\partial t^2} > 0 \; or \; \frac{\partial^2 f_{TSW}(t_0)}{\partial t^2} = 0$$

2.5.2 Median

Consider (2-18) and let

$$F_{TSW}(t) = \frac{1}{2}$$

$$1-(\lambda+1)e^{-2t^{\alpha}} + \lambda e^{-t^{\alpha}} = \frac{1}{2}$$

$$(\lambda+1)e^{-2t^{\alpha}} - \lambda e^{-t^{\alpha}} = \frac{1}{2}$$

$$e^{-2t^{\alpha}} - \frac{\lambda}{(\lambda+1)} e^{-t^{\alpha}} - \frac{1}{2(\lambda+1)} = 0$$

Let $p = e^{-t^{\alpha}}$. $p^2 = e^{-2t^{\alpha}}$ then the last equation become

$$p^2 - \frac{\lambda}{(\lambda+1)} p - \frac{1}{2(\lambda+1)} = 0$$

And let $b = \frac{\lambda}{(\lambda+1)}$ $and\ c = \frac{1}{2(\lambda+1)}$ $with\ a = 1$ the quadratic equation can be solved algebraically for p as:

$$p = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

Then we can find t_{med} after solving the equation $p = e^{-t^{\alpha}}$ such that

$$\ln p = \ln e^{-t^{\alpha}}$$

Hence

$$t = -(\ln p)^{\frac{1}{\alpha}} \qquad \text{(2-37)}$$

2.5.3 The Moments:

Theorem 4: the rth moment of TSW distribution about the origin is:

$$M_r' = (\, 2^{1-\frac{r}{\alpha}}(1+\lambda) - \lambda)\Gamma \frac{r+\alpha}{\alpha} \qquad \text{(2-38)}$$

Therefore the rth moment $E(T-\mu)^r$ of TSW distribution about the means

$$M_r = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} M_i' (M_1')^{r-i} \qquad \text{(2-39)}$$

Proof:

$$E(T^r) = \int_0^{\infty} t^r \, f_{TSW}(t)dt$$

$$= \int_0^{\infty} t^r \left(2\alpha t^{\alpha-1}(1+\lambda)e^{-2t^{\alpha}} - \alpha\lambda t^{\alpha-1}e^{-t^{\alpha}}\right)dt$$

$$= \int_0^{\infty} t^r \left(2\alpha t^{\alpha-1}(1+\lambda)e^{-2t^{\alpha}}\right)dt - \int_0^{\infty} t^r \left(\alpha\lambda t^{\alpha-1}e^{-t^{\alpha}}\right)dt$$

$$= 2\alpha(1+\lambda)\int_0^{\infty} t^{r+\alpha-1}e^{-2t^{\alpha}} \, dt - \alpha\lambda\int_0^{\infty} t^{r+\alpha-1}e^{-t^{\alpha}} \, dt$$

Now to find the last two integrals we will use the following transformation:

Put

$$I = 2\alpha(1+\lambda) \int_0^{\infty} t^{r+\alpha-1}e^{-2t^{\alpha}} \, dt$$

And

$$II = \alpha\lambda\int_0^{\infty} t^{r+\alpha-1}e^{-t^{\alpha}} \, dt$$

Let

$$y = t^{\alpha}. \;\; 2t^{\alpha} = 2y \;\; . t = (y)^{\frac{1}{\alpha}} \;, dt = \frac{1}{\alpha} \, (y)^{\frac{1}{\alpha}-1} \, dy \, . \; t^r = y^{\frac{r}{\alpha}} \;\; and$$

$$t^{-1} = (y)^{-\frac{1}{\alpha}} \; .$$

Hence,

$$I = 2\alpha(1+\lambda)\int_0^{\infty} t^{r+\alpha-1}e^{-2t^{\alpha}} \, dt$$

$$= \frac{1}{2^{\frac{r}{\alpha}}}\ 2(1+\lambda)\ \int_0^\infty (2y)^{\frac{r}{\alpha}+1-1} e^{-2y}\ dy$$

$$=\ 2^{-\frac{r}{\alpha}+1}(1+\lambda)\ \Gamma\frac{r+\alpha}{\alpha}$$

For integral II:

$$II =\ \alpha\lambda \int_0^\infty t^{r+\alpha-1} e^{-t^\alpha}\ dt =\ \lambda \int_0^\infty y^{\frac{r}{\alpha}+1-1} e^{-y}\ dy$$

Hence,

$$II = \lambda\Gamma\frac{r+\alpha}{\alpha}$$

Finally,

$$E(T^r) = 2^{1-\frac{r}{\alpha}}(1+\lambda)\ \Gamma\frac{r+\alpha}{\alpha} -\ \lambda\Gamma\frac{r+\alpha}{\alpha}$$

$$= \Gamma\frac{r+\alpha}{\alpha}[\ 2^{1-\frac{r}{\alpha}}(1+\lambda) - \lambda]$$

Therefore the rth moment $E(T-\mu)^r$ of TSW distribution about the mean μ is

$$E(T-\mu)^r =\ \ M_r = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} M_i'(M_1')^{r-i}$$

Where M_1' is the mean of TSE distribution calculated from (2-38) and $M_i' = E\left(\ t^i\right)$.

The mean of the TSW distribution is (when r = 1):

$$E(T)\ = \Gamma\frac{1+\alpha}{\alpha}[\ 2^{\frac{\alpha-1}{\alpha}}(1+\lambda) - \lambda]$$

And, the second moment of the TSW distribution is (when r = 2) :

$$E(T^2) = \Gamma\frac{2+\alpha}{\alpha}\left[\, 2^{\frac{\alpha-2}{\alpha}}(1+\lambda) - \lambda\right]$$

Then, the variance of the TSW distribution is:

$$var(T) = E(T^2) - \left(E(T)\right)^2$$

$$var\,(T) = \Gamma\frac{2+\alpha}{\alpha}\left[\, 2^{\frac{\alpha-2}{\alpha}}(1+\lambda) - \lambda\right] - (\Gamma\frac{1+\alpha}{\alpha})^2[\, 2^{\frac{\alpha-1}{\alpha}}(1+\lambda) - \lambda]^2$$

And the Coefficient of Variation CV is given by:

$$CV = \frac{\sqrt{var\,(T)}}{E(T)}$$

Hence

$$CV = \frac{\sqrt{\Gamma\frac{2+\alpha}{\alpha}\left[\, 2^{\frac{\alpha-2}{\alpha}}(1+\lambda) - \lambda\right] - (\Gamma\frac{1+\alpha}{\alpha})^2[\, 2^{\frac{\alpha-1}{\alpha}}(1+\lambda) - \lambda]^2}}{\Gamma\frac{1+\alpha}{\alpha}[\, 2^{\frac{\alpha-1}{\alpha}}(1+\lambda) - \lambda]}$$

2.5.4 The Moment Generation Function

Theorem 5: The (mgf) of T for TSW, distribution is given by:

$$M_t(z) = \sum_{i=0}^{\infty}\frac{(z)^i}{i!}\Gamma\frac{i+\alpha}{\alpha}[\, 2^{1-\frac{i}{\alpha}}(1+\lambda) - \lambda\,] \quad (2\text{-}40)$$

Proof:

$$M_t(z) = E(e^{zt}) = \int_0^\infty e^{zt} f_{TSW}(t)dt$$

$$= 2\alpha(1+\lambda)\int_0^\infty t^{\alpha-1}e^{zt}\, e^{-2t^\alpha}\, dt - \alpha\lambda\int_0^\infty t^{\alpha-1}e^{zt}e^{-t^\alpha}\, dt$$

Use the expansion

$$e^{zt} = \sum_{i=0}^{\infty} \frac{(zt)^i}{i!}$$

$$M_t(z) = 2\alpha(1+\lambda)\sum_{i=0}^{\infty} \frac{(z)^i}{i!} \int_0^{\infty} t^{i+\alpha-1}\, e^{-2t^{\alpha}}\, dt$$

$$-\ \alpha\lambda\sum_{i=0}^{\infty} \frac{(z)^i}{i!} \int_0^{\infty} t^{i+\alpha-1} e^{-t^{\alpha}}\, dt$$

And with the following transmutation:

$$y = t^{\alpha},\ 2t^{\alpha} = 2y\ ,\ t = (y)^{\frac{1}{\alpha}}, dt = \frac{1}{\alpha}\,(y)^{\frac{1}{\alpha}-1}\, dy\, ,\ t^r = y^{\frac{r}{\alpha}}\ \ and\ \ t^{-1} = (y)^{-\frac{1}{\alpha}},\ t^i = (y)^{\frac{i}{\alpha}}\ .$$

Hence,

$$M_t(z) = 2(1+\lambda)\sum_{i=0}^{\infty} \frac{(z)^i}{i!} 2^{-\frac{i}{\alpha}}\, \Gamma\frac{i+\alpha}{\alpha} - \lambda \sum_{i=0}^{\infty} \frac{(z)^i}{i!}\ \Gamma\frac{i+\alpha}{\alpha}$$

$$M_t(z) = \sum_{i=0}^{\infty} \frac{(z)^i}{i!} \Gamma\frac{i+\alpha}{\alpha} [\ 2^{1-\frac{i}{\alpha}}(1+\lambda) - \lambda\ ]$$

According to the last result TSW has heavy right tail since the moment generating function of t is infinite for all $z > 0$.

2.5.5 The Factorail Moments Generation Function

Theorem 6: The Factorial Moments Generating function of t given by:

$$M_t(z) = \sum_{i=0}^{\infty} \frac{(\ln z)^i}{i!} \Gamma \frac{i+\alpha}{\alpha} [\, 2^{1-\frac{i}{\alpha}}(1+\lambda) - \lambda \,] \qquad (2\text{-}41)$$

Proof:

$$M_t(z) = E(z^t) = \int_0^\infty e^{\ln z^t} f_{TSW}(t) dt$$

$$= 2\alpha(1+\lambda) \int_0^\infty t^{\alpha-1} e^{\ln z^t} e^{-2t^\alpha} dt - \alpha\lambda \int_0^\infty t^{\alpha-1} e^{\ln z^t} e^{-t^\alpha} dt$$

Use the expansion

$$e^{t \ln z} = \sum_{i=0}^{\infty} \frac{(t \ln z)^i}{i!}$$

$$M_t(z) = 2\alpha(1+\lambda) \sum_{i=0}^{\infty} \frac{(\ln z)^i}{i!} \int_0^\infty t^{i+\alpha-1} e^{-2t^\alpha} dt$$

$$- \alpha\lambda \sum_{i=0}^{\infty} \frac{(\ln z)^i}{i!} \int_0^\infty t^{i+\alpha-1} e^{-t^\alpha} dt$$

And with the following transmutation:

$$y = t^\alpha, \ 2t^\alpha = 2y \ , \ t = (y)^{\frac{1}{\alpha}}, dt = \frac{1}{\alpha} (y)^{\frac{1}{\alpha}-1} dy \, , \ t^r = y^{\frac{r}{\alpha}} \ and \ \ t^{-1} = (y)^{-\frac{1}{\alpha}}, \ t^i = (y)^{\frac{i}{\alpha}} \ .$$

Hence,

$$M_t(z) = 2(1+\lambda) \sum_{i=0}^{\infty} \frac{(\ln z)^i}{i!} 2^{-\frac{i}{\alpha}} \Gamma \frac{i+\alpha}{\alpha} - \lambda \sum_{i=0}^{\infty} \frac{(\ln z)^i}{i!} \Gamma \frac{i+\alpha}{\alpha}$$

$$M_t(z) = \sum_{i=0}^{\infty} \frac{(\ln z)^i}{i!} \Gamma \frac{i+\alpha}{\alpha} [\, 2^{1-\frac{i}{\alpha}}(1+\lambda) - \lambda \,]$$

2.5.6 The Quantile Function

Consider the cdf in (2-18) and let:

$$F_{TSW}(t) = v$$

$$v = 1 - (\lambda + 1)e^{-2t^{\alpha}} + \lambda e^{-t^{\alpha}}$$

$$1 - v = (\lambda + 1)e^{-2t^{\alpha}} - \lambda e^{-t^{\alpha}} \qquad (2\text{-}42)$$

By using Taylor series representation, we get:

$$1 - v = (\lambda + 1)\sum_{n=0}^{\infty} \frac{(-2t^{\alpha})^n}{n!} - \lambda \sum_{n=0}^{\infty} \frac{(-t^{\alpha})^n}{n!}$$

$$1 - v = (\lambda + 1)\sum_{n=0}^{\infty} \frac{(-2)^n t^{\alpha n}}{n!} - \lambda \sum_{n=0}^{\infty} \frac{(-1)^n t^{\alpha n}}{n!}$$

$$1 - v = (\lambda + 1)\sum_{n=0}^{\infty} t^{\alpha n} \frac{(-2)^n}{n!} - \lambda \sum_{n=0}^{\infty} t^{\alpha n} \frac{(-1)^n}{n!}$$

$$1 - v = \sum_{n=0}^{\infty} t^{\alpha n} \frac{(-2)^n}{n!} + \lambda \sum_{n=0}^{\infty} t^{\alpha n} \left(\frac{(-2)^n}{n!} - \frac{(-1)^n}{n!}\right)$$

$$1 - v = \sum_{n=0}^{\infty} t^{\alpha n} \frac{(-2)^n}{n!} + \lambda \sum_{n=0}^{\infty} t^{\alpha n} \frac{(-2)^n - (-1)^n}{n!}$$

$$1 - v = \sum_{n=0}^{\infty} t^{\alpha n} \frac{(-2)^n + \lambda((-2)^n - (-1)^n)}{n!}$$

Finally,

$$1 - v = \sum_{n=0}^{\infty} t^{\alpha n}\, k(n.\lambda) \qquad (2\text{-}43)$$

Where $k(n.\lambda) = \frac{(-2)^n + \lambda((-2)^n - (-1)^n)}{n!} \qquad .n = 0.1.2. \ldots$

The equation (2-43) can solve numerically for t.

2.5.7 The Coefficient of Skewness

From (2-39) and Theorem 4 we can compute the skewness for TSW distribution:

$$CS = \frac{E(T^3)-3E(T)\sigma^2-\mu^3}{\sigma^3}$$

$$CS = \frac{\Gamma\frac{3+\alpha}{\alpha}\left[2^{1-\frac{3}{\alpha}}(1+\lambda)-\lambda\right]-3\left(\Gamma\frac{1+\alpha}{\alpha}\left[2^{1-\frac{1}{\alpha}}(1+\lambda)-\lambda\right]\right)\Gamma\frac{2+\alpha}{\alpha}\left[2^{\frac{\alpha-2}{\alpha}}(1+\lambda)-\lambda\right]}{(\Gamma\frac{2+\alpha}{\alpha}\left[2^{\frac{\alpha-2}{\alpha}}(1+\lambda)-\lambda\right]-(\Gamma\frac{1+\alpha}{\alpha})^2[2^{\frac{\alpha-1}{\alpha}}(1+\lambda)-\lambda])^{3/2}}$$

$$-\frac{-(\Gamma\frac{1+\alpha}{\alpha})^2[2^{\frac{\alpha-1}{\alpha}}(1+\lambda)-\lambda]-(\Gamma\frac{1+\alpha}{\alpha}[2^{1-\frac{1}{\alpha}}(1+\lambda)-\lambda])^3}{(\Gamma\frac{2+\alpha}{\alpha}\left[2^{\frac{\alpha-2}{\alpha}}(1+\lambda)-\lambda\right]-(\Gamma\frac{1+\alpha}{\alpha})^2[2^{\frac{\alpha-1}{\alpha}}(1+\lambda)-\lambda])^{3/2}}$$

2.5.8 The Coefficient of Kurtosis

According to Theorem (4) the kurtosis of TSW distribution is:

$$CK = \frac{\Gamma\frac{4+\alpha}{\alpha}\left(2^{\frac{\alpha-4}{\alpha}}(1+\lambda)-\lambda\right)-4\Gamma\frac{4+\alpha}{\alpha}(2^{\frac{\alpha-1}{\alpha}}(1+\lambda)-\lambda)\Gamma\frac{3+\alpha}{\alpha}(2^{\frac{\alpha-3}{\alpha}}(1+\lambda)-\lambda)}{(\Gamma\frac{2+\alpha}{\alpha}\left[2^{\frac{\alpha-2}{\alpha}}(1+\lambda)-\lambda\right]-\left(\Gamma\frac{1+\alpha}{\alpha}\right)^2\left[2^{\frac{\alpha-1}{\alpha}}(1+\lambda)-\lambda\right]^2)^2}$$

$$+$$

$$\frac{6\Gamma\frac{2+\alpha}{\alpha}(2^{\frac{\alpha-2}{\alpha}}(1+\lambda)-\lambda))\left(\Gamma\frac{1+\alpha}{\alpha}\right)^2\left[2^{\frac{\alpha-1}{\alpha}}(1+\lambda)-\lambda\right]^2-3\left(\Gamma\frac{1+\alpha}{\alpha}\right)^4\left[2^{\frac{\alpha-1}{\alpha}}(1+\lambda)-\lambda\right]^4}{(\Gamma\frac{2+\alpha}{\alpha}\left[2^{\frac{\alpha-2}{\alpha}}(1+\lambda)-\lambda\right]-\left(\Gamma\frac{1+\alpha}{\alpha}\right)^2\left[2^{\frac{\alpha-1}{\alpha}}(1+\lambda)-\lambda\right]^2)^2}$$

2.6 Statistical Characteristics of TSEW Distribution

2.6.1 Mode

Consider (2-22) and (2-25)

With

$$\frac{\partial f_{TSEW}}{\partial t} = 0$$

And

$$-4\gamma^2(1+\lambda)e^{-2\gamma t} - \alpha\lambda(\alpha - \alpha t^{\alpha} - 1)t^{\alpha-2}\, e^{-t^{\alpha}} = 0$$

$$-4\gamma^2(1+\lambda)e^{-2\gamma t} = \alpha\lambda(\alpha - \alpha t^{\alpha} - 1)t^{\alpha-2}\, e^{-t^{\alpha}} \qquad \text{(2-44)}$$

Again, equation (2-44) is not explicit formula for t. so we can find its root numerically and find the value of mode according to the inequalities

$$\frac{\partial^2 f_{TSEW}(t_0)}{\partial t^2} < 0, \quad \frac{\partial^2 f_{TSEW}(t_0)}{\partial t^2} > 0 \; or \; \frac{\partial^2 f_{TSEW}(t_0)}{\partial t^2} = 0$$

Where t_0 is the root of (2-44).

2.6.2 Median

Consider (2-21) and let

$$F_{TSEW}(t) = \frac{1}{2}$$

$$1 - (1+\lambda)e^{-2\gamma t} + \lambda e^{-t^{\alpha}} = \frac{1}{2}$$

$$(1+\lambda)e^{-2\gamma t} + \lambda e^{-t^{\alpha}} - \frac{1}{2} = 0$$

$$e^{-2\gamma t} - \frac{\lambda}{(\lambda+1)} e^{-t^{\alpha}} - \frac{1}{2(\lambda+1)} = 0$$

Now assume $\alpha = 1 \; and \; \gamma = 1$ as especial case. Then,

$$e^{-2t} - \frac{\lambda}{(\lambda+1)} e^{-t} - \frac{1}{2(\lambda+1)} = 0$$

Hence put $y = e^{-t}$ then

$$y^2 - \frac{\lambda}{(\lambda+1)} y - \frac{1}{2(\lambda+1)} = 0$$

And

$$y = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

Where $b = \frac{\lambda}{(\lambda+1)}$ $and\ c = \frac{1}{2(\lambda+1)}$. Subsequently:

$$t = -\ln y \qquad (2\text{-}45)$$

In addition, we can assume $\gamma = 1/2$ then

$$e^{-t} - \frac{\lambda}{(\lambda+1)} e^{-t} - \frac{1}{2(\lambda+1)} = 0$$

And by use same technique we can have the median value such that

$$t = -\ln(\frac{1}{2})$$

As especial case.

2.6.3 The Moments

Theorem 7: the rth moment of TSEW distribution about the origin is:

$$E(T^r) = \frac{\lambda+1}{(2\gamma)^{r-1}} \Gamma(r+1) - \lambda\Gamma\frac{r+\alpha}{\alpha} \qquad (2\text{-}46)$$

With rth moment $E(T-\mu)^r$ of TSEW distribution about the means

$$M_r = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} M_i' (M_1')^{r-i} \qquad (2\text{-}47)$$

Proof:

$$E(T^r) = \int_0^\infty t^r\, f_{TSEW}(t)dt = \int_0^\infty t^r \left(2(\lambda+1)\,\gamma e^{-2\gamma t} - \alpha\lambda t^{\alpha-1}e^{-t^\alpha}\right)dt$$

By combine Theorem (1) and Theorem (4)

$$E(T^r) = \frac{\lambda+1}{(2\gamma)^{r-1}}\,\Gamma(r+1) - \lambda\Gamma\frac{r+\alpha}{\alpha}$$

In addition, the rth moment $E(T-\mu)^r$ of TSEW distribution about the mean μ is

$$E(T-\mu)^r = M_r = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}M_i'(M_1')^{r-i}$$

Where M_1' is the mean of TSEW distribution calculated from (2-46) and $M_i' = E\left(T^i\right)$.

The mean of the TSEW distribution is (when r = 1):

$$E(T) = \Gamma(2)(\lambda+1) - \Gamma\frac{1+\alpha}{\alpha}\lambda$$

And, the second moment of the TSEW distribution is (when r = 2):

$$E(T^2) = \frac{\Gamma(3)}{2\gamma}(\lambda+1) - \Gamma\frac{2+\alpha}{\alpha}\lambda$$

Then, the variance of the TSEW distribution is:

$$var(T) = E(T^2) - \left(E(T)\right)^2$$

$$var\,(T) = (\frac{\Gamma(3)}{2\gamma}(\lambda+1) - \Gamma\frac{2+\alpha}{\alpha}\lambda) - (\Gamma(2)(\lambda+1) - \Gamma\frac{1+\alpha}{\alpha}\lambda)^2$$

And the Coefficient of Variation CV is given by:

$$CV = \frac{\sqrt{var\ (T)}}{E(T)}$$

Hence

$$CV = \frac{\sqrt{(\frac{\Gamma(3)}{2\gamma}\ (\lambda+1) - \ \Gamma\frac{2+\alpha}{\alpha}\lambda) - (\Gamma(2)(\lambda+1\ ) - \ \Gamma\frac{1+\alpha}{\alpha}\lambda)^2}}{\Gamma(2)(\lambda+1\ ) - \ \Gamma\frac{1+\alpha}{\alpha}\lambda}$$

2.6.4 The Moment Generation Function

Theorem 8: The (mgf) of T for TSEW distribution is given by:

$$M_t(z) = \frac{2\,\gamma(\lambda+1)}{2\gamma-z} - \lambda\ \sum_{i=0}^{\infty}\frac{(z)^i}{i!}\ \Gamma\frac{i+\alpha}{\alpha} \qquad (2\text{-}48)$$

Proof:

$$M_t(z) = E(e^{zt})\ =\ \int_0^\infty e^{zt}\left(2(\lambda+1)\,\gamma e^{-2\gamma t} - \alpha\lambda t^{\alpha-1}e^{-t^\alpha}\right)dt$$

By combine Theorem (2) and Theorem (5):

$\int_0^\infty e^{zt}\ (2(\lambda+1)\,\gamma e^{-2\gamma t})dt = \frac{2\,\gamma(\lambda+1)}{2\gamma-z}$

And $\int_0^\infty e^{zt}\left(\alpha\lambda t^{\alpha-1}e^{-t^\alpha}\right)dt = \lambda\ \sum_{i=0}^{\infty}\frac{(z)^i}{i!}\ \Gamma\frac{i+\alpha}{\alpha}$

Therefor

$$M_t(z) = \frac{2\,\gamma(\lambda+1)}{2\gamma-z} - \lambda\ \sum_{i=0}^{\infty}\frac{(z)^i}{i!}\ \Gamma\frac{i+\alpha}{\alpha}$$

2.6.5 The Factorail Moments Generation Function

Theorem 9: The Factorial Moments Generating function of t given by:

$$M_t(z) = \frac{2\gamma(\lambda+1)}{2\gamma - lnz} - \lambda \sum_{i=0}^{\infty} \frac{(\ln z)^i}{i!} \Gamma \frac{i+\alpha}{\alpha} \qquad (2\text{-}49)$$

Proof: By using theorem 3 and theorem 6, we get the result.

2.6.6 The Quantile Function

Consider the cdf in (2-21) and let:

$$F_{TSEW}(t) = u$$

$$1 - (1+\lambda)e^{-2\gamma t} + \lambda e^{-t^{\alpha}} = u$$

$$1 - u = (1+\lambda)e^{-2\gamma t} - \lambda e^{-t^{\alpha}} \qquad (2\text{-}50)$$

Also, in special case if $\alpha = \gamma = 1$ then (2-24) can be as

$$1 - u = (1+\lambda)e^{-2t} - \lambda e^{-t}$$

Now let $x = e^{-t}, a = 1+\lambda, b = -\lambda \; and \; c = 1-u$

Thus

$$ax^2 + bx - c = 0$$

And $\qquad x = \frac{\lambda \pm \sqrt{\lambda^2 - 4(1-u)}}{2}$

Therefore

$$x = e^{-t}$$

Hence

$$t = -\ln(x) \qquad (2\text{-}51)$$

2.6.7 The Coefficient of Skewness

From (2-47) and theorem 7 the coefficient of skewness for TSEW distribution is:

$$E(T) = \Gamma(2)(\lambda + 1) - \Gamma\frac{1+\alpha}{\alpha}$$

And, the second moment of the TSEW distribution is (when r = 2):

$$E(T^2) = \frac{\Gamma(3)}{2\gamma}(\lambda + 1) - \Gamma\frac{2+\alpha}{\alpha}\lambda$$

Then, the variance of the TSEW distribution is:

$$var\,(T) = (\frac{\Gamma(3)}{2\gamma}(\lambda + 1) - \Gamma\frac{2+\alpha}{\alpha}\lambda) - (\Gamma(2)(\lambda + 1) - \Gamma\frac{1+\alpha}{\alpha}\lambda)^2$$

Thus

$$CS = \frac{(\frac{\Gamma(4)}{2\gamma^2}(\lambda + 1) - \Gamma\frac{3+\alpha}{\alpha}\lambda)}{(\frac{\Gamma(3)}{2\gamma}(\lambda + 1) - \Gamma\frac{2+\alpha}{\alpha}\lambda) - (\Gamma(2)(\lambda + 1) - \Gamma\frac{1+\alpha}{\alpha}\lambda)^{\frac{3}{2}}}$$

$$- \frac{3(\Gamma(2)(\lambda + 1) - \Gamma\frac{1+\alpha}{\alpha}\lambda)(\frac{\Gamma(3)}{2\gamma}(\lambda + 1) - \Gamma\frac{2+\alpha}{\alpha}\lambda) - (\Gamma(2)(\lambda + 1) - \Gamma\frac{1+\alpha}{\alpha}\lambda)^2}{(\frac{\Gamma(3)}{2\gamma}(\lambda + 1) - \Gamma\frac{2+\alpha}{\alpha}\lambda) - (\Gamma(2)(\lambda + 1) - \Gamma\frac{1+\alpha}{\alpha}\lambda)^{\frac{3}{2}}}$$

$$- \frac{(\Gamma(2)(\lambda + 1) - \Gamma\frac{1+\alpha}{\alpha}\lambda)^2}{(\frac{\Gamma(3)}{2\gamma}(\lambda + 1) - \Gamma\frac{2+\alpha}{\alpha}\lambda) - (\Gamma(2)(\lambda + 1) - \Gamma\frac{1+\alpha}{\alpha}\lambda)^{\frac{3}{2}}}$$

2.6.8 The Coefficient of Kurtosis

According to Theorem (7) the kurtosis of TSEW distribution is:

$$CK = \frac{\left(\frac{\Gamma(5)}{(2\gamma)^3}\ (\lambda+1) - \Gamma\frac{4+\alpha}{\alpha}\lambda\right) - 4(\Gamma(2)(\lambda+1\,) - \Gamma\frac{1+\alpha}{\alpha}\lambda)\left(\frac{\Gamma(4)}{(2\gamma)^2}\ (\lambda+1) - \Gamma\frac{3+\alpha}{\alpha}\lambda\right)}{(\frac{\Gamma(3)}{2\gamma}\ (\lambda+1) - \Gamma\frac{2+\alpha}{\alpha}\lambda) - (\Gamma(2)(\lambda+1\,) - \Gamma\frac{1+\alpha}{\alpha}\lambda)^2)^2}$$

$$+ \frac{6(\frac{\Gamma(3)}{2\gamma}\ (\lambda+1) - \Gamma\frac{2+\alpha}{\alpha}\lambda)(\Gamma(2)(\lambda+1\,) - \Gamma\frac{1+\alpha}{\alpha}\lambda)^2 - 3(\frac{\Gamma(3)}{2\gamma}\ (\lambda+1) - \Gamma\frac{2+\alpha}{\alpha}\lambda)^4}{(\frac{\Gamma(3)}{2\gamma}\ (\lambda+1) - \Gamma\frac{2+\alpha}{\alpha}\lambda) - (\Gamma(2)(\lambda+1\,) - \Gamma\frac{1+\alpha}{\alpha}\lambda)^2)^2}$$

3.1 Introduction

In this chapter, the estimation methods of the parameters of TSE, TSW and TSEW distributions respectively are introduced and discussed. These methods are: the maximum likelihood estimator (MLE), the least squared estimation method and the moments estimation method. Statistical criterions are used to compare the given distributions with each other and with base line distributions. Throughout real data sets for consolidating the results.

3.2 Estimation Methods of TSE

We estimate the two parameters $\gamma\ and\ \lambda$ of TSE distributions by the following estimations methods:

3.2.1 Maximum Likelihood Estimation of TSE [31[[20] [9]

The MLE method is most famous classical methods one can use to estimate the distributions' s parameters. It copes with different type of samples (uncensored or censored) (white, 1982) [31]. Moreover, it is often used to estimate parameters because it has good characteristics as efficient capabilities and less contrast property with invariant property too.

Now we will apply this method in order to estimate the parameters of the three given distributions as follows:

Let $\Omega = (\lambda, \gamma)^T$ be the parameter vector of that we want to estimate and ($t_1, t_2, \cdots, t_n$) be a random variables of size n with a known probability density function $f_{TSE}\ (\lambda, \gamma, t)$, the likelihood function defined as the common joint probability density function distribution of the data and can be written as follows:

$$l(\lambda, \gamma, t_1, t_2, \cdots, t_n) = \prod_{i=1}^{n} f_{TSE}(t_i, \lambda, \gamma) \quad (3\text{-}1)$$

$$= \prod_{i=1}^{n} (\boldsymbol{2\gamma}(\mathbf{1}+\lambda)\boldsymbol{e^{-2\gamma t_i}} - \lambda\boldsymbol{\gamma e^{-\gamma t_i}})$$

The log-likelihood function for the vector of parameters $(\lambda.\gamma)^T$ can be written as,

$$L = lnl(\lambda, \gamma, t_i) = \ln \prod_{i=1}^{n} (2\gamma(1+\lambda)e^{-2\gamma t_i} - \lambda\gamma e^{-\gamma t_i})$$

$$\text{L} = \sum_{i=1}^{n} \ln(2\gamma(1+\lambda)e^{-2\gamma t_i} - \lambda\gamma e^{-\gamma t_i}) \qquad (3\text{-}2)$$

Then, by take the partial derivatives of L with respect to unknown parameters (λ, γ) as follows:

Assume that,

$$g(\lambda) = \frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} \frac{2\gamma e^{-2\gamma t_i} - \gamma e^{-\gamma t_i}}{2\gamma(1+\lambda)e^{-2\gamma t_i} - \lambda\gamma e^{-\gamma t_i}}$$

(3-3)

And

$$w(\gamma) = \frac{\partial L}{\partial \gamma} = \sum_{i=1}^{n} \frac{2(1+\lambda)e^{-2\gamma t_i} - 4\gamma\,(1+\lambda)t_i e^{-2\gamma t_i} - \lambda e^{-\gamma t_i} + \lambda\gamma t_i e^{-\gamma t_i}}{2\gamma(1+\lambda)e^{-2\gamma t_i} - \lambda\gamma e^{-\gamma t_i}} \qquad (3\text{-}4)$$

When $\frac{\partial L}{\partial \lambda} = 0 \; and \; \frac{\partial L}{\partial \gamma} = 0$, there is no closed solution of the equations (3-3) and (3-4). Therefore, numerical technique (Newton-Raphson method) should be apply to solve these equations as follows:

$$\begin{bmatrix} \hat{\lambda} \\ \hat{\gamma} \end{bmatrix} = \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} - J^{-1} \begin{bmatrix} g(\lambda) \\ w(\gamma) \end{bmatrix}$$

where $$J = \begin{bmatrix} \frac{\partial g(\lambda)}{\partial \lambda} & \frac{\partial g(\lambda)}{\partial \gamma} \\ \frac{\partial w(\gamma)}{\partial \lambda} & \frac{\partial w(\gamma)}{\partial \gamma} \end{bmatrix}$$

It is the Jacobean matrix.

And

$$\frac{\partial g(\lambda)}{\partial \lambda} = -\sum_{i=1}^{n} \frac{\left(2\gamma e^{-2\gamma t_i} - \gamma e^{-\gamma t_i}\right)^2}{(2\gamma(1+\lambda)e^{-2\gamma t_i} - \lambda\gamma e^{-\gamma t_i})^2} \qquad (3\text{-}5)$$

$$\frac{\partial g(\lambda)}{\partial \gamma} = \sum_{i=1}^{n} \frac{2e^{-2\gamma t_i} - 4\gamma t_i e^{-2\gamma t_i} - e^{-\gamma t_i} + \gamma t_i e^{-\gamma t_i}}{2\gamma(1+\lambda)e^{-2\gamma t_i} - \lambda\gamma e^{-\gamma t_i}} - \sum_{i=1}^{n} \frac{(2\gamma e^{-2\gamma t_i} - \gamma e^{-\gamma t_i})(2(1+\lambda)e^{-2\gamma t_i} - 4\gamma(1+\lambda)t_i e^{-2\gamma t_i} - \lambda e^{-\gamma t_i} + \gamma\lambda t_i e^{-\gamma t_i}}{(2\gamma(1+\lambda)e^{-2\gamma t_i} - \lambda\gamma e^{-\gamma t_i})^2} \tag{3-6}$$

$$\frac{\partial w(\gamma)}{\partial \lambda} = \sum_{i=1}^{n} \frac{2e^{-2\gamma t_i} - 4\gamma t_i e^{-2\gamma t_i} - e^{-\gamma t_i} + \gamma t_i e^{-\gamma t_i}}{2\gamma(1+\lambda)e^{-2\gamma t_i} - \lambda\gamma e^{-\gamma t_i}} = - \sum_{i=1}^{n} \frac{(2\gamma e^{-2\gamma t_i} - \gamma e^{-\gamma t_i})(2(1+\lambda)e^{-2\gamma t_i} - 4\gamma(1+\lambda)t_i e^{-2\gamma t_i} - \lambda e^{-\gamma t_i} + \lambda\gamma t_i e^{-\gamma t_i}}{(2\gamma(1+\lambda)e^{-2\gamma t_i} - \lambda\gamma e^{-\gamma t_i})^2} \tag{3-7}$$

$$\frac{\partial w(\gamma)}{\partial \gamma} = \sum_{i=1}^{n} \frac{-8+8\gamma(1+\lambda)t_i{}^2 e^{-2\gamma t_i} + 2(1+\lambda)t_i e^{-2\gamma t_i} + 2\lambda t_i e^{-\gamma t_i} - \lambda\gamma t_i{}^2 e^{-\gamma t_i}}{2\gamma(1+\lambda)e^{-2\gamma t_i} - \lambda\gamma e^{-\gamma t_i}} - \sum_{i=1}^{n} \frac{(2(1+\lambda)e^{-2\gamma t_i} - 4\gamma(1+\lambda)t_i e^{-2\gamma t_i} - \lambda e^{-\gamma t_i} + \lambda\gamma t_i e^{-\gamma t_i})^2}{(2\gamma(1+\lambda)e^{-2\gamma t_i} - \lambda\gamma e^{-\gamma t_i})^2} \tag{3-8}$$

To insure that the Jacobean matrix is non-singular symmetric matrix, we can take $\frac{\partial w(\gamma)}{\partial \lambda}$ to be the same as $\frac{\partial g(\lambda)}{\partial \gamma}$.

3.2.2 Least Square Estimation of TSE [6] [27].

The basic idea of least squared approach is to minimize the sum of the squares of the residuals that came from subtracting the exact form of the model and the approximate one. This approach has many applications in data fitting, functions approximation, regression analysis, and estimating parameters of the statistical models. In estimating, the least squared method have many properties such as the estimator has linearity property, unbiased property, consistency, and others. Therefore, in this subsection, our goal is to estimate the parameters of the three given models by minimizing the summation of squared differences between the exact form of the cumulative distributions functions and the approximated forms.

For two parameter λ, γ and sample data of size n ($t_1, t_2, \cdots, t_n$) the cdf of TSE distribution is:

$$F(\lambda, \gamma, t_i) = 1 - (1 + \lambda)e^{-2\gamma t_i} + \lambda e^{-\gamma t_i} \qquad (3\text{-}9)$$

The least square estimators $\hat{\lambda}.\hat{\gamma}$ can be obtained by minimizing:

$$Les(\lambda, \gamma) = \sum_{i=1}^{n}[F(\lambda, \gamma, t_i) - \frac{i}{n+1}]^2 \qquad (3\text{-}10)$$

With respect to λ *and* γ. Similarly, they can also be obtained by solving the following non-linear equations:

$$\frac{\partial Les(\lambda, \gamma)}{\partial \lambda} = 0 \quad and \quad \frac{\partial Les(\lambda, \gamma)}{\partial \gamma} = 0$$

Such that:

$$\frac{\partial Les(\lambda, \gamma)}{\partial \lambda} = 2[F(\lambda, \gamma, t_i) - \frac{i}{n+1}](e^{-\gamma t_i} - e^{-2\gamma t_i}) \qquad (3\text{-}11)$$

$$\frac{\partial Les(\lambda, \gamma)}{\partial \gamma} = -2[F(\lambda, \gamma, t_i) - \frac{i}{n+1}](2(1 + \lambda)t_i e^{-2\gamma t_i} + \lambda t_i e^{-\gamma t_i}) \quad (3\text{-}12)$$

By numerical iterations method (Newton-Raphson).

3.2.3 Moments Estimation of TSE [7] [20].

The method of moments is one of the oldest methods used to estimate parameters of statistical models; Karl Pearson (1894, 1895) introduces it. This approach is much quickly, easily and consistent. Its steps depend on equating, as many population moments to sample moments as there are parameters to estimate. Mathematical support for this procedure comes from the principle of moments as discussed in details in Kendall and Stuart (1969) this principle says that two distributions that have a finite number of lower moments in common will be approximations of one another.

For the two parameter TSE distribution, the first three moments about the origin are:

$$M_1 = E(T) = \Gamma(2)$$

$M_2 = E(T^2) = \frac{\Gamma(3)}{2\gamma}(1-\lambda)$ (3-13)

$M_3 = E(T^4) = \frac{\Gamma(4)}{4\gamma^2}(1-3\lambda)$ (3-14)

In addition, the corresponding sample moments are:

$$m_r = \sum_{i=1}^{n} \frac{t^r}{n}$$

Where n is the number of observations. Now apply the equation $m_r = M_r$

We get the following equations:

$m_2 = \sum_{i=1}^{n} \frac{{t_i}^2}{n} = \frac{\Gamma(3)}{2\gamma}(1-\lambda)$ (3-15)

$m_3 = \sum_{i=1}^{n} \frac{{t_i}^3}{n} = \frac{\Gamma(4)}{4\gamma^2}(1-3\lambda)$ (3-16)

Such that $\Gamma(3) = 2 \ \ and \ \ \Gamma(4) = 6$.

Equation (3-15) and (3-16) can be solve algebraically such that:

$\hat{\lambda} = 1 - \hat{\gamma}\, m_2 \quad and \quad \hat{\gamma} = \frac{3\, m_2 \mp \sqrt{9 - \frac{12}{m_3}}}{2}$.

3.3 Estimation Methods of TSW [14]

In this sub section, the two parameter $\lambda\, and\, \alpha$ of the TSW distribution using the MLE, LSE and Moments methods are estimated.

3.3.1 Maximum Likelihood Estimation of TSW

Let $\mathcal{L}(\lambda, \alpha)$ denotes likelihood function of the TSW distribution, respectively. Then the maximum likelihood estimator $\hat{\lambda}\ and\ \hat{\alpha}$ can be formulated as follows:

$$\mathcal{L}(\lambda, \alpha, t_1, t_2, \cdots, t_n) = \prod_{i=1}^{n} f_{TSW}(\lambda, \alpha, t_i)$$

We re-write equation (2- 19) for simplification as

$$f_{TSW} = \alpha \boldsymbol{t^{\alpha-1}} \boldsymbol{e^{-2t^{\alpha}}} \ (2(1+\lambda) - \lambda \boldsymbol{e^{t^{\alpha}}})$$

Hence

$$\mathcal{L}(\lambda, \alpha, t_1, t_2, \cdots, t_n) = \prod_{i=1}^{n} [\ \alpha \boldsymbol{t_i}^{\boldsymbol{\alpha-1}} \boldsymbol{e}^{\boldsymbol{-2t_i^{\alpha}}}][2(1+\lambda) - \lambda \boldsymbol{e}^{\boldsymbol{t_i^{\alpha}}}]$$

Now take the log to the likelihood function:

$$\ln \mathcal{L}(\lambda, \alpha, t_1, t_2, \cdots, t_n) = \ln \prod_{i=1}^{n} f_{TSW}(\lambda, \alpha, t_i)$$

$$= \ln \prod_{i=1}^{n} [\ \alpha \boldsymbol{t_i}^{\boldsymbol{\alpha-1}} \boldsymbol{e}^{\boldsymbol{-2t_i^{\alpha}}}][2(1+\lambda) - \lambda \boldsymbol{e}^{\boldsymbol{t_i^{\alpha}}}]$$

$$= \sum_{i=1}^{n} \ln(\ [\ \alpha \boldsymbol{t_i}^{\boldsymbol{\alpha-1}} \boldsymbol{e}^{\boldsymbol{-2t_i^{\alpha}}}]\ [2(1+\lambda) - \lambda \boldsymbol{e}^{\boldsymbol{t_i^{\alpha}}}])$$

$$\ln \mathcal{L} = \sum_{i=1}^{n} \ln [\ \alpha \boldsymbol{t_i}^{\boldsymbol{\alpha-1}} \boldsymbol{e}^{\boldsymbol{-2t_i^{\alpha}}}] + \sum_{i=1}^{n} \ln[2(1+\lambda) - \lambda \boldsymbol{e}^{\boldsymbol{t_i^{\alpha}}}]$$

To maximize the log-likelihood function we need to solve the following nonlinear equations:

$$\left.\frac{\partial \ln \mathcal{L}}{\partial \lambda}\right|_{\lambda=\hat{\lambda}} = 0 \ \ and \ \ \left.\frac{\partial \ln \mathcal{L}}{\partial \alpha}\right|_{\alpha=\hat{\alpha}} = 0$$

Where

$$\frac{\partial \ln \mathcal{L}}{\partial \lambda} = \sum_{i=1}^{n} \frac{2 - \boldsymbol{e}^{\boldsymbol{t_i}^{\boldsymbol{\alpha}}}}{2(1+\lambda) - \lambda \boldsymbol{e}^{\boldsymbol{t_i}^{\boldsymbol{\alpha}}}}$$

And

$$\frac{\partial \ln \mathcal{L}}{\partial \alpha} = \sum_{i=1}^{n} (\,\alpha^{-1} + \ln t_i - 2{t_i}^{\alpha} \ln t_i) - \frac{\lambda {t_i}^{\alpha} \boldsymbol{e}^{\boldsymbol{t_i}^{\boldsymbol{\alpha}}} \ln \mathrm{t_i}}{2(1+\lambda) - \lambda \boldsymbol{e}^{\boldsymbol{t_i}^{\boldsymbol{\alpha}}}}$$

We can solve these nonlinear equations by Newton-Raphson method as follows:

$$\text{Let } i = 1, \cdots, n$$

$$\begin{bmatrix} \hat{\lambda}_{i+1} \\ \hat{\alpha}_{i+1} \end{bmatrix} = \begin{bmatrix} \hat{\lambda}_{i} \\ \hat{\alpha}_{i} \end{bmatrix} + J^{-1} \begin{bmatrix} \dfrac{\partial \ln \mathcal{L}}{\partial \lambda} \\ \dfrac{\partial \ln \mathcal{L}}{\partial \alpha} \end{bmatrix}$$

Where

$$J = \begin{bmatrix} \dfrac{\partial^2 \ln \mathcal{L}}{\partial \lambda^2} & \dfrac{\partial^2 \ln \mathcal{L}}{\partial \lambda \partial \alpha} \\ \dfrac{\partial^2 \ln \mathcal{L}}{\partial \alpha \partial \lambda} & \dfrac{\partial^2 \ln \mathcal{L}}{\partial \alpha^2} \end{bmatrix}$$

$Where$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \lambda^2} = -\sum_{i=1}^{n} \left(2 - e^{{t_i}^{\alpha}}\right)^2 (2(1+\lambda) - \lambda \boldsymbol{e}^{\boldsymbol{t_i}^{\boldsymbol{\alpha}}})^{-2}$$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \lambda \partial \alpha} = \sum_{i=1}^{n} \frac{\left(2 - e^{{t_i}^{\alpha}}\right)(\lambda {t_i}^{\alpha} \boldsymbol{e}^{\boldsymbol{t_i}^{\boldsymbol{\alpha}}} \ln \mathrm{t_i})}{(2(1+\lambda) - \lambda \boldsymbol{e}^{\boldsymbol{t_i}^{\boldsymbol{\alpha}}})^2} - \sum_{i=1}^{n} \frac{{t_i}^{\alpha} \boldsymbol{e}^{\boldsymbol{t_i}^{\boldsymbol{\alpha}}} \ln \mathrm{t_i}}{2(1+\lambda) - \lambda \boldsymbol{e}^{\boldsymbol{t_i}^{\boldsymbol{\alpha}}}}$$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \alpha^2} = -\sum_{i=1}^{n} \alpha^{-1} + 2{t_i}^{\alpha} \ln t_i + \frac{\lambda {t_i}^{\alpha} \boldsymbol{e}^{\boldsymbol{t_i}^{\boldsymbol{\alpha}}} \ln \mathrm{t_i}}{2(1+\lambda) - \lambda \boldsymbol{e}^{\boldsymbol{t_i}^{\boldsymbol{\alpha}}}}$$

The Jacobean matrix is non-singular, that means $\frac{\partial^2 \ln \mathcal{L}}{\partial \alpha \partial \lambda} = \frac{\partial^2 \ln \mathcal{L}}{\partial \lambda \partial \alpha}$

3.3.2 Least Square Estimation of TSW

Consider the cdf of TSW distribution

$$F_{TSW}(t) = 1 - (1+\lambda)e^{-2t^{\alpha}} + \lambda e^{-t^{\alpha}}$$

The least square estimators $\hat{\lambda}.\hat{\alpha}$ can be obtained by minimizing:

$$Les(\lambda, \alpha\,) = \sum_{i=1}^{n} [F(\lambda, \alpha, t_i) - \frac{i}{n+1}]^2$$

With respect to λ and α. Similarly, they can also be obtained by solving the following non-linear equations numerically:

$$\frac{\partial Les(\lambda, \alpha\,)}{\partial \lambda} = 0 \quad and \quad \frac{\partial Les(\lambda, \alpha\,)}{\partial \alpha} = 0$$

Such that:

$$\frac{\partial Les(\lambda,\alpha\,)}{\partial \lambda} = 2[F(\lambda, \gamma, t_i) - \frac{i}{n+1}]\,(e^{-2t_i{}^{\alpha}} + e^{-t_i{}^{\alpha}})$$

$$\frac{\partial Les(\lambda, \alpha\,)}{\partial \alpha} = 2\left[\,F(\lambda, \gamma, t_i) - \frac{i}{n+1}\right](\,-2(1+\lambda)t_i{}^{\alpha} \ln t_i \quad e^{-2t_i{}^{\alpha}}$$
$$-\ \lambda t_i{}^{\alpha}\ \ln t_i\, e^{-2t^{\alpha}}\,)$$

3.3.3 Moments Estimation of TSW

For the tow parameter TSW distribution, the first two moments about the origin are:

$$M'_r = \Gamma \frac{r+\alpha}{\alpha} [\,2^{1-\frac{r}{\alpha}}(1+\lambda) - \lambda]$$

$$M_1 = E(t) = \Gamma \frac{1+\alpha}{\alpha} [2^{\frac{\alpha-1}{\alpha}} (1+\lambda) - \lambda]$$

$$M_2 = \Gamma \frac{2+\alpha}{\alpha} [2^{\frac{\alpha-2}{\alpha}} (1+\lambda) - \lambda]$$

after equating with the corresponding sample moments m_r we get the following equations:

$$\sum_{i=1}^{n} \frac{t^1}{n} = \Gamma \frac{1+\alpha}{\alpha} [2^{\frac{\alpha-1}{\alpha}} (1+\lambda) - \lambda]$$

$$\sum_{i=1}^{n} \frac{t^2}{n} = \Gamma \frac{2+\alpha}{\alpha} [2^{\frac{\alpha-2}{\alpha}} (1+\lambda) - \lambda]$$

That can be solved numerically

3.4 Estimation Methods of TSEW

To estimate the three parameters $(\gamma , \lambda \ and \ \alpha)$ of the TSEW distribution we applied the three previous methods as follows:

3.4.1 Maximum Likelihood Estimation of TSEW

Specified the data $t_1, t_2, \cdots, t_n$ the log likelihood function is given by the set functions:

$$\mathcal{L}(\lambda, \alpha, t_1, t_2, \cdots, t_n) = \prod_{i=1}^{n} f_{TSEW}(\gamma, \lambda, \alpha, t_i)$$

Rewrite equation (2-22) as

$$f_{TSEW}(\gamma, \lambda, \alpha, t_i) = e^{-2\gamma t_i}(2\gamma(1+\lambda) - \alpha \ \lambda {t_i}^{\alpha-1} \ e^{2\gamma t_i - {t_i}^{\alpha}}$$

$$\mathcal{L}(\gamma, \lambda, \alpha, t_1, t_2, \cdots, t_n) = \prod_{i=1}^{n} [\boldsymbol{e}^{-\mathbf{2}\gamma t_i}(2\gamma(1+\lambda) - \alpha \ \lambda {t_i}^{\alpha-1} \ e^{2\gamma t_i - {t_i}^{\alpha}}]$$

$$\ln \mathcal{L}(\gamma, \lambda, \alpha, t_1, t_2, \cdots, t_n) = \ln \prod_{i=1}^{n} f_{TSEW}(\gamma, \lambda, \alpha, t_1, t_2, \cdots, t_n)$$

$$= \sum_{i=1}^{n} \ln[\, \boldsymbol{e}^{-2\gamma t_i}(2\gamma(1+\lambda) - \alpha\, \lambda {t_i}^{\alpha-1}\, e^{2\gamma t_i - {t_i}^{\alpha}}]$$

$$= \sum_{i=1}^{n} [-\mathbf{2}\gamma t_i + \ln(2\gamma(1+\lambda) - \alpha\, \lambda {t_i}^{\alpha-1}\, e^{2\gamma t_i - {t_i}^{\alpha}})$$

To find the M.L. E's we need to solve the following nonlinear equations:

$$\left.\frac{\partial \ln \mathcal{L}}{\partial \gamma}\right|_{\gamma=\hat{\gamma}} = 0 \ , \quad \left.\frac{\partial \ln \mathcal{L}}{\partial \lambda}\right|_{\lambda=\hat{\lambda}} = 0 \ \ and \quad \left.\frac{\partial \ln \mathcal{L}}{\partial \alpha}\right|_{\alpha=\hat{\alpha}} = 0$$

Where

$$\frac{\partial \ln \mathcal{L}}{\partial \gamma} = \sum_{i=1}^{n} [\, \mathbf{2} t_i + \frac{(2(1+\lambda) - \alpha\, \lambda {t_i}^{\alpha-1} \mathbf{2} t_i e^{2\gamma t_i - {t_i}^{\alpha}})}{(2\gamma(1+\lambda) - \alpha\, \lambda {t_i}^{\alpha-1}\, e^{2\gamma t_i - {t_i}^{\alpha}})}$$

$$\frac{\partial \ln \mathcal{L}}{\partial \lambda} = \sum_{i=1}^{n} [\frac{2\gamma - \alpha\, {t_i}^{\alpha-1} e^{2\gamma t_i - {t_i}^{\alpha}}}{2\gamma(1+\lambda) - \alpha\, \lambda {t_i}^{\alpha-1}\, e^{2\gamma t_i - {t_i}^{\alpha}}}]$$

And

$$\frac{\partial \ln \mathcal{L}}{\partial \alpha} = \sum_{i=1}^{n} [\frac{-\lambda {t_i}^{\alpha-1} e^{2\gamma t_i - {t_i}^{\alpha}} - \alpha\, \lambda {t_i}^{\alpha-1} \ln(t)\, e^{2\gamma t_i - {t_i}^{\alpha}} + \alpha \lambda {t_i}^{\alpha-1}\, {t_i}^{\alpha} \ln(t)\, e^{2\gamma t_i - {t_i}^{\alpha}}}{(2\gamma(1+\lambda) - \alpha\, \lambda {t_i}^{\alpha-1}\, e^{2\gamma t_i - {t_i}^{\alpha}}}]$$

So, to estimate the un known parameters we can use the numerical iterative technique Newton-Raphson as follows:

$$\text{Let } i = 1, \cdots, n$$

$$\begin{bmatrix}\hat{\gamma}_{i+1}\\ \hat{\lambda}_{i+1}\\ \hat{\alpha}_{i+1}\end{bmatrix} = \begin{bmatrix}\hat{\gamma}_{i}\\ \hat{\lambda}_{i}\\ \hat{\alpha}_{i}\end{bmatrix} + J^{-1}\begin{bmatrix}\frac{\partial \ln \mathcal{L}}{\partial \gamma}\\ \frac{\partial \ln \mathcal{L}}{\partial \lambda}\\ \frac{\partial \ln \mathcal{L}}{\partial \alpha}\\ \\ \end{bmatrix}$$

Where

$$J = \begin{bmatrix}\frac{\partial^2 \ln \mathcal{L}}{\partial \gamma^2} & \frac{\partial^2 \ln \mathcal{L}}{\partial \gamma \partial \lambda} & \frac{\partial^2 \ln \mathcal{L}}{\partial \gamma \partial \alpha}\\ \frac{\partial^2 \ln \mathcal{L}}{\partial \lambda \partial \gamma} & \frac{\partial^2 \ln \mathcal{L}}{\partial \lambda^2} & \frac{\partial^2 \ln \mathcal{L}}{\partial \lambda \partial \alpha}\\ \frac{\partial^2 \ln \mathcal{L}}{\partial \alpha \partial \gamma} & \frac{\partial^2 \ln \mathcal{L}}{\partial \alpha \partial \lambda} & \frac{\partial^2 \ln \mathcal{L}}{\partial \alpha^2}\end{bmatrix}$$

The Jacobean matrix is non-singular, that means $\frac{\partial^2 \ln \mathcal{L}}{\partial \gamma \partial \lambda} = \frac{\partial^2 \ln \mathcal{L}}{\partial \lambda \partial \gamma}, \frac{\partial^2 \ln \mathcal{L}}{\partial \gamma \partial \alpha} = \frac{\partial^2 \ln \mathcal{L}}{\partial \alpha \partial \gamma}$ and $\frac{\partial^2 \ln \mathcal{L}}{\partial \lambda \partial \alpha} = \frac{\partial^2 \ln \mathcal{L}}{\partial \alpha \partial \lambda}$

And

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \gamma^2} = -\sum_{i=1}^{n} \frac{4\left(\alpha\lambda {t_i}^{\alpha-1}\left({t_i}^2 e^{2\gamma t_i - {t_i}^{\alpha}}\right)\right)}{(2\gamma(1+\lambda) - \alpha\, \lambda {t_i}^{\alpha-1}\, e^{2\gamma t_i - {t_i}^{\alpha}}} + \frac{(2(1+\lambda) - 2(\,\alpha\lambda {t_i}^{\alpha-1}\left(t_i e^{2\gamma t_i - {t_i}^{\alpha}}\right)))^2}{(2\gamma(1+\lambda) - \alpha\, \lambda {t_i}^{\alpha-1}\, e^{2\gamma t_i - {t_i}^{\alpha}})^2}$$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \gamma \partial \lambda} = \sum_{i=1}^{n} \frac{2 - 2(\alpha {t_i}^{\alpha-1}\left(t_i e^{2\gamma t_i - {t_i}^{\alpha}}\right))}{(2\gamma(1+\lambda) - \alpha\, \lambda {t_i}^{\alpha-1}\, e^{2\gamma t_i - {t_i}^{\alpha}}} - \frac{(2(1+\lambda) - 2(\alpha\lambda {t_i}^{\alpha-1}\left(t_i e^{2\gamma t_i - {t_i}^{\alpha}}\right)))(2\gamma - \alpha {t_i}^{\alpha-1} e^{2\gamma t_i - {t_i}^{\alpha}})}{(2\gamma(1+\lambda) - \alpha\, \lambda {t_i}^{\alpha-1}\, e^{2\gamma t_i - {t_i}^{\alpha}})^2}$$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial\gamma\partial\alpha} =$$

$$\sum_{i=1}^{n} \frac{-2\left(\lambda\, t_i{}^{\alpha-1}\left(t_i e^{2\gamma t_i - t_i{}^{\alpha}}\right)\right) - 2\alpha\lambda \ln(t_i) t_i\, e^{2\gamma t_i - t_i{}^{\alpha}} + 2(\alpha\lambda\, t_i{}^{\alpha-1}(t_i t_i{}^{\alpha} \ln t_i\, e^{2\gamma t_i - t_i{}^{\alpha}})))}{2\gamma(1+\lambda) - \alpha\, \lambda t_i{}^{\alpha-1}\, e^{2\gamma t_i - t_i{}^{\alpha}}}$$

$$-\sum_{i=1}^{n} \frac{((\,2(1+\lambda) - 2\left(\alpha\lambda\, t_i{}^{\alpha-1}\left(t_i\, e^{2\gamma t_i - t_i{}^{\alpha}}\right)\right))(-\lambda t_i{}^{\alpha-1} e^{2\gamma t_i - t_i{}^{\alpha}} - \alpha\lambda t_i{}^{\alpha-1} \ln t_i e^{2\gamma t_i - t_i{}^{\alpha}}}{(2\gamma(1+\lambda) - \alpha\, \lambda t_i{}^{\alpha-1}\, e^{2\gamma t_i - t_i{}^{\alpha}})^2}$$

$$+ \frac{\alpha\lambda t_i{}^{\alpha-1}(t_i{}^{\alpha} \ln t_i\, e^{2\gamma t_i - t_i{}^{\alpha}})}{\left(2\gamma(1+\lambda) - \alpha\, \lambda t_i{}^{\alpha-1}\, e^{2\gamma t_i - t_i{}^{\alpha}}\right)^2}$$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial\lambda^2} = -\sum_{i=1}^{n} \frac{(2\gamma - \alpha\, t_i{}^{\alpha-1}\, e^{2\gamma t_i - t_i{}^{\alpha}})^2}{(2\gamma(1+\lambda) - \alpha\, \lambda t_i{}^{\alpha-1}\, e^{2\gamma t_i - t_i{}^{\alpha}})^2}$$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial\lambda\partial\alpha}$$

$$= \sum_{i=1}^{n} \frac{-t_i{}^{\alpha-1}\, e^{2\gamma t_i - t_i{}^{\alpha}} - \alpha t_i{}^{\alpha-1}\, \ln t_i\, e^{2\gamma t_i - t_i{}^{\alpha}} + \alpha t_i{}^{\alpha-1} t_i{}^{\alpha} \ln t_i\, e^{2\gamma t_i - t_i{}^{\alpha}}}{2\gamma(1+\lambda) - \alpha\, \lambda t_i{}^{\alpha-1}\, e^{2\gamma t_i - t_i{}^{\alpha}}}$$

$$-\sum_{i=1}^{n} \frac{1}{\left(2\gamma(1+\lambda) - \alpha\, \lambda t_i{}^{\alpha-1}\, e^{2\gamma t_i - t_i{}^{\alpha}}\right)^2}\, (\,2\gamma - \alpha t_i{}^{\alpha-1}\, e^{2\gamma t_i - t_i{}^{\alpha}})$$

$$(-\lambda t_i{}^{\alpha-1}\, e^{2\gamma t_i - t_i{}^{\alpha}} - \alpha\, \lambda t_i{}^{\alpha-1} \ln t_i\;\, e^{2\gamma t_i - t_i{}^{\alpha}}$$

$$+\; \alpha\, \lambda t_i{}^{\alpha-1}\, t_i{}^{\alpha} \ln t_i\, e^{2\gamma t_i - t_i{}^{\alpha}}.$$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \alpha^2}$$

$$= \sum_{i=1}^{n} \frac{1}{2\gamma(1+\lambda) - \alpha\, \lambda t_i^{\alpha-1}\, e^{2\gamma t_i - t_i^{\alpha}}} \left(-2\left(\lambda t_i^{\alpha-1} \ln t_i\, e^{2\gamma t_i - t_i^{\alpha}}\right)\right.$$

$$+2\left(\lambda t_i^{\alpha-1} t_i^{\alpha} \ln t_i\, e^{2\gamma t_i - t_i^{\alpha}}\right) - \alpha\, \lambda t_i^{\alpha-1} \ln t_i^{2}\, e^{2\gamma t_i - t_i^{\alpha}} +$$

$$2(\alpha\, \lambda t_i^{\alpha-1} \ln t_i)\left(t_i^{\alpha} \ln t_i\, e^{2\gamma t_i - t_i^{\alpha}}\right) + \alpha\, \lambda t_i^{\alpha-1}\, t_i^{\alpha} \ln t_i^{2}\, e^{2\gamma t_i - t_i^{\alpha}} -$$

$$t_i^{\alpha 2} \ln t_i^{2}\, e^{2\gamma t_i - t_i^{\alpha}} -$$

$$\sum_{i=1}^{n} \frac{-\lambda\, t_i^{\alpha-1}\, e^{2\gamma t_i - t_i^{\alpha}} - \alpha\, \lambda t_i^{\alpha-1} \ln t_i\, e^{2\gamma t_i - t_i^{\alpha}} + \alpha\, \lambda t_i^{\alpha-1}\left(t_i^{\alpha} \ln t_i\, e^{2\gamma t_i - t_i^{\alpha}}\right)^2}{\left(2\gamma(1+\lambda) - \alpha\, \lambda t_i^{\alpha-1}\, e^{2\gamma t_i - t_i^{\alpha}}\right)^2}$$

3.4.2 Least Square Estimation of TSEW

We proceed with the next method of estimation to estimate the three parameter of TSEW distribution (γ , λ *and* α) :

The cdf of TSEW distribution is given by

$$F_{TSEW}(t) = 1 - (1+\lambda)e^{-2\gamma t} + \lambda e^{-t^{\alpha}}$$

And the least square estimators $\hat{\gamma}.\hat{\lambda}$ *and* $\hat{\alpha}$ can be obtained by minimizing:

$$Les(\gamma, \lambda, \alpha\ ) = \sum_{i=1}^{n} [F(\gamma, \lambda, \alpha, t_i) - \frac{i}{n+1}]^2$$

With respect to γ λ *and* α. Similarly, they can also be obtained by solving the following non-linear equations numerically:

$$\frac{\partial Les(\gamma, \lambda, \alpha\ )}{\partial \gamma} = 0\,, \qquad \frac{\partial Les(\gamma, \lambda, \alpha\ )}{\partial \lambda} = 0 \quad and \quad \frac{\partial Les(\gamma, \lambda, \alpha\ )}{\partial \alpha} = 0$$

Such that:

$$\frac{\partial Les(\gamma,\lambda,\alpha\,)}{\partial\gamma} = 4\sum_{i=1}^{n}\left[F(\gamma,\lambda,\alpha,t_i) - \frac{i}{n+1}\right]\ (1+\lambda)t_i e^{-2\gamma t_i}$$

$$\frac{\partial Les(\gamma,\lambda,\alpha\,)}{\partial\lambda} = 2\sum_{i=1}^{n}\left[F(\gamma,\lambda,\alpha,t_i) - \frac{i}{n+1}\right]\ (-\ e^{-2\gamma t_i} + e^{-t_i^{\alpha}})$$

$$\frac{\partial Les(\gamma,\lambda,\alpha\,)}{\partial\alpha} = -2\sum_{i=1}^{n}\left[F(\gamma,\lambda,\alpha,t_i) - \frac{i}{n+1}\right]\lambda\, t_i^{\alpha}\ln t_i e^{-t_i^{\alpha}}$$

3.4.3 Moments Estimation of TSEW

For the three parameter TSEW distribution, the first three moments about the origin are:

$$M_1 = E(t)\ = \Gamma(2)(\lambda+1\,) -\ \Gamma\frac{1+\alpha}{\alpha}\lambda$$

$$M_2 = E(t^2) =\ \frac{\lambda+1}{2\gamma}\ \Gamma(3) -\ \lambda\Gamma\frac{2+\alpha}{\alpha}$$

$$M_3 = E(t^3) =\ \frac{\lambda+1}{2\gamma^2}\ \Gamma(4) -\ \lambda\Gamma\frac{3+\alpha}{\alpha}$$

And by equating these moments to the corresponding sample moments:

$$m_r =\ \sum_{i=1}^{n}\frac{t^r}{n}$$

We get three equations we can solve them numerically

$$\sum_{i=1}^{n}\frac{t_i}{n} = \Gamma(2)(\lambda+1\,) -\ \Gamma\frac{1+\alpha}{\alpha}\lambda$$

$$\sum_{i=1}^{n} \frac{t_i^{\,2}}{n} = \frac{\lambda + 1}{2\gamma}\ \Gamma(3) - \ \lambda\Gamma\frac{2+\alpha}{\alpha}$$

$$\sum_{i=1}^{n} \frac{t_i^{\,2}}{n} = \frac{\lambda + 1}{2\gamma^2}\ \Gamma(4) - \ \lambda\Gamma\frac{3+\alpha}{\alpha}$$

3.5 Information Criterion and Curve Fitting

Choosing convenient and suitable model among other models that describe the phenomena under study, is a major concern of modern science so many statistical tools have been proposed over the years for dealing with this issue such as information criteria and goodness of fit. The aim of computing such information criteria is to find out how the good model is at explaining the relationship between the variables and identifying the “best model” among a set of elects models. While the purpose of curve fitting is to construct a curve, that has the best fit to a series of original data point possibly subject to constraints with evaluation statistical quantities that measure the goodness of fit.

In this work, we use three different types of information criterions: the Akaiki Information Criterion (AIC), Corrected Akaiki Information Criterion (CAIC) [10] and Bayesian Information Criterion (BIC) [28]. Also we test four types of curve fitting quantities to identify the best fitted curve among the tested models they are: the sum of Squared due to Error, R-square, Adjusted R-square and root Mean Square Error (RMSE).

3.5.1 Akaike Information Criterion and Corrected Akaike Information

The AIC was introduced firstly in 1971 by Hirotugu Akaike [28]. [19]

It is a criterion for model selection defined for each model by

AIC= -2 (maximum log-likelihood of the model)

+2(number of free parameters of the model).

The model that indicates lowest value of AIC among a group of models will be the most appropriate one. In some cases, the AIC could have a significant negative bias [9] for such a case Hurvich and Tsai introduces the corrected Akaike Information criterion CAIC where the vantage of the CAIC on the AIC is that the expected difference is estimated with a tendency lower than the AIC. The formula of the AIC can be written as follows:

$$CAIC = AIC + \frac{2m(m+1)}{n-m-1}$$

Where

n : is the sample size of the data application.

m: is the number of parameters in the statistical distribution.

3.5.2 Bayesian Information Criterion (BIC)

The BIC can be evaluated by the following:

$$BIC = m\ln(n) - 2\ (\text{maximum}\ \log_{\text{likelihood}}\ \text{of the model})$$

n : is the sample size of the data application.

m: is the number of parameters in the statistical distribution.

This criterion was introduced by Schwarz in (1978) [19] and it is one of the most widely known tools in choosing best statistical models that is according to simplicity in computational and effective performance.

We used ACI, CAIC and BIC to indicate the better fit among the three introduced distributions such that a lower value of used criterions will indicate a best fit among the tested models.

3.5.3 Curve Fitting [24]

In order to decide which model exhibited better curve fit among our three tested models: TSE, TSW and TSEW distributions we examine four quantities:

1- The sum of squares due to error (SSE): this statistic measures the total deviation of the response values from the fit to the response values. It is also called the summed square of residuals.

$$SSE = \sum_{i=1}^{n} w_i (f_i - \widehat{f_i})^2$$

Where $\widehat{f_i}$the response is computed by smoothing spline method and w_i is the smoothing parameter.

2- R-squared: This statistic measures how successful the fit is in explaining the variation of the data. R-square is defined as the ratio of the sum of squares of the regression (*SSR*) and the total sum of squares (*SST*). *SSR* is defined as:

$$SSR = \sum_{i=1}^{n} w_i (\hat{f}_i - \bar{f}\ )^2$$

SST is also called the sum of squares about the mean, and is defined as

$$SST = \sum_{i=1}^{n} w_i (f_i - \bar{f}\,)^2$$

R-square is expressed as:

$$R - square = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

R-square can take any value between 0 and 1, with a value closer to 1 indicating that a greater proportion of variance is accounted for by the model.

3- Adjusted R-squared: is generally the best indicator of the fit quality when one compare two models. The adjusted R-square statistic can take on any value less than or equal to 1, with a value closer to 1 indicating a better fit.
4- Root mean squared error (RMSE): mean squared error is the measure of the closeness of a regression line to a set of points. It measures the distance called errors and squares to remove any negative signs. Lower values of RMSE mean that the regression line is close to the data points, indicating a better fit.

Those quantities evaluated by Curve Fitting Application offered by matlab environment. This curve fitting application is using numerical smoothing spline method to perform the fitting and the goodness of fit statistics.

4.1 Introduction

In this chapter, we used the three estimation methods which are discussed in Chapter three to estimate the parameters $(\gamma.\lambda and\ \alpha)$ of the given distributions. That means, finding a numerical result by using appropriate algorithm. We companied between the analytic solution to the estimated methods that we derived in Chapter 3 and the numerical technique to compute the estimators and examine these estimators with simulation.

4.2 Simulation Experiment

In order to study the behavior of the three models TSE, TSW and TSEW and through generating different sizes of sample we simulate an experiment. This simulation depends on the cumulative function of the models. The Mean Square Error used to compare between the parameter estimator results that founded by MLE, OLS and Moment estimation methods.

4.2.1 Algorithm of Experiments

In this subsection we use the following algorithm of simulation to generate random data samples. This algorithm depends on the cumulative function of the TSE, TSW and TSEW distributions and it has the following steps:

<u>Step (1)</u>

We adopted two ways in choosing initial parameters such that with TSE we make the value of λ fixed number with two changeable values for γ and vice versa. For TSW and TSEW we take two and one set of parameters respectively.

<u>Step (2)</u>

Choose small, median and large sizes of samples: $n = 10.30.50.100$.

Step (3)

Use the command U=Rand to generate uniform random variables $(t_i, i = 1, \cdots, n$.

Step (4)

Transform the generated uniform variables $(t_i, i = 1, \cdots, n)$ to the tested distributions by substituting t_i in the inverse of cumulative distribution function that we derived in Chapter 2.

Step (5)

Stabilize the initial values for the New-Raphson method to solve the non-linear systems that comes out from estimations methods.

Step (6)

Replicate the previous steps 500 and 1000 times.

Step (7)

Use the three estimated methods in Chapter 3 to estimate the three parameters of TSE, TSW and TSEW distributions.

Step (8)

Comparer the results of estimation methods by computing the mean square error (MSE) which is defined as a distance between the value of density function at estimated parameters and the actual value of it.

$$MSE = \frac{1}{R}\sum_{i=1}^{R}(f_i - \widehat{f_i})^2$$

Where, R is number of repeating of each experiment and equal to R=500, 1000.

4.3 Numerical Results

This section contains the numerical results of the three estimator methods: MLE, OLS and the Moment that mentioned in Chapter three for the TSE, TSW and TSEW distributions. Moreover, the following tables present the numerical results of estimators of $\gamma\,,\lambda\ and\ \alpha$ at n=10, 30, 50,100 with MSE of them.

MSE of $\gamma\,,\lambda\ and\ \alpha$ for repeating R=500, 1000 respectively.

4.3.1 Numerical Results of TSE Parameters (R=500)

In this subsection we estimate parameters of TSE distribution ($\gamma = 1\,,0.001\,,0.005\ and\ \lambda = -0.9\,,0.005,-0.001$)

At n=10,30,50 and 100 with r=500 listed in the following tables: -

- Numerical values of estimator $\hat{\gamma}$ at n=10, 30 ,50 100 for initial values ,λ=-0.9,$\gamma = 1$ are given in Table (4-1) , the values of estimator $\hat{\gamma}$ are oscillation near the initial value and the best estimator is ($\hat{\gamma} =$ 0.920416044) (at sample size n=30 with (MSE= 8.91887638006e-05) computed by OLS method.
- Table (4-1) indicates that OLS has lowest values of error comparison to other estimator methods so the best method to estimate the value of $\gamma = 1$ at R=500 with different value of sample n is the least squared method then the MLE method.
- At n= 10, MLE give better estimator $\hat{\gamma}$ =0.9154825641 with MSE=(1.7537077100e-04).

Table (4-1): Estimator values with $\gamma = 1$, λ=-0.9 and R=500 and MSE values given between parentheses.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\gamma}$	0.9154825641 (1.7537077100e-04)	0.769571462358529 (0.0017269297071)	0.277025387582362 (0.36335513430093)	MLE
30	$\hat{\gamma}$	0.7981678503 (7.3283141346e-04)	0.920416044 (8.9188763800e-05)	0.4147506501 (0.32984558675)	OLS
50	$\hat{\gamma}$	0.8846387748 (0.003866386836999)	0.677972145759177 (0.003220936032183)	0.372500518358321 (0.31293021256796)	OLS
100	$\hat{\gamma}$	1.1529380136 (0.01641423989)	1.02499483045 (0.010746999051)	0.28073300112 (0.27334551142)	OLS

- Numerical values of Estimator $\hat{\gamma}$ (R=500) at n=10, 30 ,50 100 and for initial values . λ=-0.9, $\gamma = 0.001$ are given in Table (4-2) .the values of estimator $\hat{\gamma}$ are oscillation near the initial value and the best estimator is ($\hat{\gamma}$ =0.001061851) (at sample size n=100 with (MSE = 7.190240481011e-10)
- Table (4-2) indicates that OLS has lowest values of error comparison to other estimator methods so the best method to estimate the value of $\gamma = 0.001$ at R=500 with different value of sample n is the least squared method then the mle method and mom method.

- initial value $\gamma = 0.001$ give good result in estimating than $\gamma = 1$ according to the values of MSE listed in Tables (4-1), (4-2).
- Table (4-2) indicates decreasing values of mean square error as sample values n increasing.

Table (4-2) Estimator value for,$\gamma = 0.001.\lambda$=-0.9 and R=500 and MSE values given between parentheses.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\gamma}$	0.0012705 (1.739327083e-07)	0.0015440921 (6.5686476937e-08)	0.106982402 (4.440544426e-07)	OLS
30	$\hat{\gamma}$	0.0009346 (6.840957052e-09)	0.0008015 (7.05734359339e-09)	0.266878152 (2.965071312e-07)	OLS
50	$\hat{\gamma}$	0.0012502 (3.665183172e-09)	0.001133986 (3.6051169554e-09)	0.22635094 (3.659531228e-07)	OLS
100	$\hat{\gamma}$	0.00124695 (9.432999022e-10)	0.001061851 (7.190240481011e-10)	0.23507235 (3.453317223e-07)	OLS

- Numerical values of estimator $\hat{\gamma}$ (R=500) at n=10, 30 ,50 100 and for initial values $.\lambda$=-0.9,$\gamma = 0.05$ are given in Table (4-3) the estimator values are oscillation near the initial value and the best estimator is ($\hat{\gamma}$ =0.0533482938) (at sample size n=50 with (MSE= 3.7627923895e-06) as shown in Table (4-3)

- Table (4-3) indicates that OLS has lowest values of error comparison to other estimator methods so the best method to estimate the value of $\gamma = 0.05$ at R=500 with different value of sample n is the least squared method then the MLE method and MOM method.
- initial value $\gamma = 0.001$ is better estimator than $\gamma = 1$ *and* 0.05 according to the values of MSE listed Tables (4-1), (4-2), (4-3)
- Table (4-3) indicates oscillating values of mean square error as sample values n increasing.

Table (4-3) Estimator value for,$\gamma = 0.05$. λ=-0.9 and R=500 and MSE values given between parentheses.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\gamma}$	0.034412158 (9.38707658618e-07)	0.04127267 (1.1353899476e-05)	0.03809384 (5.3699884245e-02)	MLE
30	$\hat{\gamma}$	0.070379963 (9.90152314073e-06)	0.0570320517 (7.81148097432e-06)	0.0513546535 (0.074084468315244)	OLS
50	$\hat{\gamma}$	0.0401004719 (4.7654596087e-06)	0.0533482938 (3.7627923895e-06)	0.0713833325 (0.18376574855)	OLS
100	$\hat{\gamma}$	0.06422070976 (3.8419186323e-06)	0.05553579924 (5.5006860293e-06)	0.052681429433 (0.107538108820)	MLE

- Numerical values of Estimator $\hat{\lambda}$ (R=500) at n=10, 30 ,50 100 and for initial values . λ=0.005,$\gamma = 1$ are given in table (4-4). the estimator values are oscillation near the initial value and the best estimator is ($\hat{\lambda}$ =0.08167205295) (at sample size n=100 with (MSE= 0.0018576890831) as shown in Table (4-4) .

- Table (4-4) indicates that MLE has lowest values of error comparison to other estimator methods so the best method to estimate the value of λ=0.005 at R=500 with different value of sample n is the MLE then the OLS method and MOM method.
- Table (4-4) indicates oscillating values of mean square error as symbol values n increasing.
- At n=10 the moment method give better estimator than MLE and OLS estimations methods such that $\hat{\lambda} = 0.1710230118998$ according to MSE in Table (4-4) such that (MSE=0.0088131476482) then when n increase the OLS indicate better values of MSE.

Table (4-4) Estimator value for λ=0.005 ,$\gamma = 1$ and R=500 and MSE values given between parentheses.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\lambda}$	-0.3041039899 (0.011900994707)	-0.084249144 (0.014393851809)	0.1710230118998 (0.0088131476482)	MOM
30	$\hat{\lambda}$	-0.4687701183 (0.06902327393)	-0.3057055881 (0.20542589597)	-0.5132039508 (0.81491910267)	MLE
50	$\hat{\lambda}$	-0.0482453308 (0.0472486001)	-0.1404324297 (0.018645022003)	-0.048245330875 (0.552686016706)	OLS
100	$\hat{\lambda}$	0.08167205295	0.4010593885	-0.996540469020	MLE

		(0.0018576890831)	(0.0235818290722)	(1.046822117532)	

- Numerical values of Estimator $\hat{\lambda}$ (R=500) at n=10, 30 ,50 100 and for initial values , λ=-0.9,$\gamma = 1$ are given in table (4-5). the estimator values are oscillation near the initial value and the best estimator is ($\hat{\lambda}$ =-1.00283463313) (at sample size n=50 with (MSE = 2.7766713161e-05) as shown in Table (4-5) .
- Table (4-5) indicates that MLE has lowest values of error comparison to other estimator methods so the best method to estimate the value of λ=-0.9 at R=500 with sample n =50 and 100 is MLE method
- Table (4-5) indicates oscillating values of mean square error as sample values n increasing.

n	parameter	MLE	OLS	MOM	
10	$\hat{\lambda}$	-0.58181722160 (0.243735770233664)	-0.488622024459 (0.2874022665881)	-5.120032411645 (0.107541440620)	MOM
30	$\hat{\lambda}$	-0.47479257134 (0.009946865469091)	-1.460069176215 (0.0052868800802)	-4.33224963288 (0.3232796288850)	OLS

50	$\hat{\lambda}$	-1.00283463313 (2.7766713161e-05)	-1.614695159385 (0.018380116934)	-3.90169412080 (0.31795092216)	MLE
100	$\hat{\lambda}$	-1.17084471779 (0.0087141699663)	-1.0644410074290 (0.010692405506)	-3.08697157485 (0.37552776125)	MLE

At n=10 the moment method give better estimator than MLE and OLS estimations methods such that $\hat{\lambda}$ =-5.120032411645 and MSE =0.107541440620.

Table (4-5) Estimator value for , λ=-0.9 ,$\gamma = 1$ and R=500 and MSE values given between parentheses.

- Numerical values of Estimator $\hat{\lambda}$ (R=500) at n=10, 30 ,50 100 and for initial values , λ=-0.001,$\gamma = 1$ are given in table (4-6) the estimator values are oscillation near the initial value and the best estimator is ($\hat{\lambda}$ =-0.04187396663) (at sample size n=100 with (MSE= 1.0423081347e-04) as shown in Table (4-6).
- Table (4-6) indicates that MLE has lowest values of error comparison to other estimator methods so the best method to estimate the value of λ=-0.001 at R=500 with different value of sample n is the maximum likelihood method then the OLS method and MOM method.

Table (4-6) indicates decreasing values of mean square error as sample values n increasing.

n	parameter	MLE	OLS	MOM	The best

10	$\hat{\lambda}$	-0.50332932763 (0.03465895158327)	0.04097074850367 (0.065390488783189)	0.238627319694 (0.2442594931804)	MLE
30	$\hat{\lambda}$	-0.43266450561 (9.139277952778e-04)	0.052008474472 (0.006547411687099)	-1.79994330295 (1.098995578970231)	MLE
50	$\hat{\lambda}$	-0.30758530393 (6.0384623985e-04)	-0.5523865804839 (0.0020089396079)	0.126560987213 (0.083176818312)	MLE
100	$\hat{\lambda}$	-0.04187396663 (1.0423081347e-04)	-0.1895654184595 (0.0014953600475)	0.052527355580 (0.140772711325)	MLE

Table (4-6) estimator value for ,λ=-0.001 ,$\gamma = 1$ and R=500 and MSE values given between parenthesis

4.3.2 Numerical results of TSE parameters (R=1000)

- Numerical values of Estimator $\hat{\lambda}$ (R=1000) at n=10, 30 ,50 ,100 and at initial values .λ=-0.9 ,$\gamma = 1$ are given in table (4-7). the estimator values are oscillation near the initial value and the best estimator is ($\hat{\gamma} =$ 0.989332217452) (at sample size n=50 with (MSE= 2.7766713161e-05) as shown in table (4-7).
- table (4-7) indicates that at n=50 and 100 MLE has lowest values of error comparison to other estimator methods so the best method to estimate the value of γ=1 at R=500 with different value of sample n is the maximum likelihood method then the OLS method and mom method.

- Table (4-7) indicates oscillating values of mean square error as sample values n increasing.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\gamma}$	0.7644483970428 (0.107541440620934)	1.431878523902 (0.287402266588141)	1.4556425207 (0.24373577023)	MLE
30	$\hat{\gamma}$	0.8426803680 (0.009946865469091)	1.34230506258 (0.005286880080236)	0.190229459 (0.323279628885)	OLS
50	$\hat{\gamma}$	0.98933217452 (2.7766713161e-05)	1.45429624159 (0.0183801169349)	0.445020288228 (0.317950922169)	MLE
100	$\hat{\gamma}$	1.38207622841 (0.00871416996)	1.30059778076 (0.0106924055)	0.291438236443 (0.37552776125)	MLE

Table (4-7) estimator values of λ =-0.9 ,$\gamma = 1$, R=1000 and MSE .

- Numerical values of Estimator $\hat{\lambda}$ (R=1000) at n=10, 30 ,50 100 and for initial values .λ=-0.9,$\gamma = 0.001$ are given in table (4-8). the estimator values are oscillation near the initial value and the best estimator is ($\hat{\gamma} =$ 0.0009685978782) (at sample size n=100 with (MSE= 1.7031719273e-10) as shown in table (4-8)
- table (4-8) indicates that at n=30 and 50 OLS has lowest values of error comparison to other estimator methods while at n=100 the MLE has indicate better result in MSE values to estimate the value of . γ=0.001 at R=1000.

- Table (4-8) indicates decreasing values of MSE values as sample values n increasing.
- According to tables (4-8) and (4-4) there were an improve in MSE values when R=500 and R=1000 especially the performance of MOM method.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\gamma}$	0.000548317909 (3.1763938780647e-09)	0.000689795851 (8.330428794649e-09)	0.2853188325 (2.06124951651e-07)	MLE
30	$\hat{\gamma}$	0.001082947547 (2.881012266271e-08)	0.001328657429 (2.416255970696e-08)	0.25499091767 (4.08133767398e-07)	OLS
50	$\hat{\gamma}$	0.00113262032 (1.6291693013e-09)	0.0010629298181 (8.3823298415e-10)	0.30990068023 (3.583390017e-07)	OLS

100	$\hat{\gamma}$	0.001229807028 (1.5070000953e-10)	0.0009685978782 (1.7031719273e-10)	0.30751581004 (3.194195025e-07)	MLE

Table (4-8) estimator values of λ =-0.9 ,$\gamma = 0.001$ and R=1000 and MSE values given between parenthesis

- Numerical values of Estimator $\hat{\gamma}$ (R=1000) at n=10, 30 ,50 100 and for initial values .λ=-0.9 ,$\gamma = 0.05$ given in Table (4-9). the estimator values are oscillation near the initial value and the best estimator is ($\hat{\gamma}$ =0.0459953279204) (at sample size n=100 with (MSE= 2.5421914135e-06) as shown in table (4-9).
- Table (4-9) indicates that OLS has lowest values of error comparison to other estimator methods so the best method to estimate the value of $\gamma = 0.05$ at R=1000 with different value of sample n is the least squared method then the MLE method and mom method.
- Table (4-9) indicates decreasing values of mean square error as sample values n increasing.
- According to Tables (4-6) and (4-9) there were improve in values of MSE in estimating $\gamma = 0.05$ at n=100.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\gamma}$	0.053699176377 (1.8297566421398e-04)	0.072837756193 (1.276410407024e-04)	2.98757201711 (9.72810054820e-04)	OLS
30	$\hat{\gamma}$	0.03865736496	0.042869084588	0.074553097563	MLE

		(4.21119281310650e-06)	(9.340336661708e-06)	(0.1969267136776)	
50	$\hat{\gamma}$	0.0379072991 (3.3401989419e-06)	0.05459732464 (3.5818400897e-06)	0.07284027811 (0.023300649421)	MLE
100	$\hat{\gamma}$	0.0525838875405 (8.5156960192e-06)	0.0459953279204 (2.5421914135e-06)	0.061776236847 (0.004008718957)	OLS

Table (4-9) estimator values of λ =-0.9 ,$\gamma = 0.05$ and R=1000 and MSE values given between parenthesis

- Numerical values of Estimator $\hat{\lambda}$ (R=1000) at n=10, 30 ,50 100 and at initial values .λ=0.005 ,$\gamma = 1$ are given the estimator values are oscillation near the initial value and the best estimator is ($\hat{\gamma} =$ -0.301837621) (at sample size n=30 with (MSE= 5.504030665832e-04) as shown in table (4-10).
- table (4-10) indicates that at n=30 and 50 OLS has lowest values of error comparison to other estimator methods so the best method to estimate the value of . γ=0.001 at R=1000 with different value of sample n is the Ordinary least squared method then the MLE method and MOM method.
- Table (4-10) indicates decreasing values of mean square error as sample values n increasing.
- According to Tables (4-10) and (4-4) there were an improve in MSE values when R=500 and R=1000.

n	parameter	MLE	OLS	MOM	The best

10	$\hat{\lambda}$	-0.3710810097 (0.34825885234409)	-0.4589377934 (0.191809205406108)	-0.333338050568 (1.2294838399304)	MLE
30	$\hat{\lambda}$	-0.3539797759 (0.004139591582072)	-0.3018376210 (5.504030665832e-04)	0.0736596310161 (0.20201575014400)	OLS
50	$\hat{\lambda}$	-0.2972295663 (0.006683904579)	-0.2090545147 (0.004491334808)	-1.213291259001 (1.05201265472)	OLS
100	$\hat{\lambda}$	-0.4382962339 (0.0090200874885)	-0.3049261904 (0.004876300109)	-0.396500962387 (0.003128559262)	MOM

Table (4-10) estimator values of $.\lambda$=0.005 ,$\gamma = 1$ and R=1000 and MSE values given between parenthesis

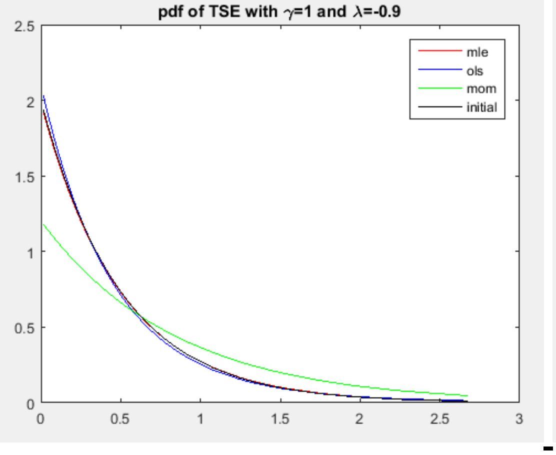
- Numerical values of Estimator $\hat{\lambda}$ (R=1000) at n=10, 30 ,50 100 and at initial values $.\lambda$=- 0.005 ,$\gamma = 1$ are given in Table (4-11) the estimator values are oscillation near the initial value and the best estimator is ($\hat{\gamma}$ = -0.45040695410) (at sample size n=100 with (MSE= 0.00133144936109) as shown in table (4-11).
- Table (4-11) indicates that at n=100 MSE has lowest values of error comparison to other estimator methods so the best method to estimate Table (4-11) indicates decreasing values of mean square error as sample values n increasing.

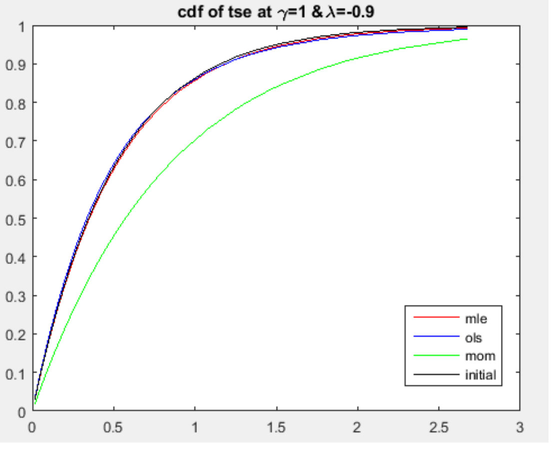
n	parameter	MLE	OLS	MOM	The best

10	$\hat{\lambda}$	-0.42576121010 (0.03431294771951)	-0.22336584938 (0.03431294771951)	-0.96418818790 (1.053776668684)	MLE
30	$\hat{\lambda}$	-0.43581184270 (0.012614728917187)	0.053095648188 (0.039380586344685)	0.1769659460195 (1.010137540324936)	MLE
50	$\hat{\lambda}$	-0.46817016141 (0.1504184165320)	0.706757075163 (0.0283248147761)	-0.333333386422 (1.009096578975)	OLS
100	$\hat{\lambda}$	-0.45040695410 (0.00133144936109)	0.351364498833 (0.025315411672)	0.134499626306 (0.112100788495)	MLE

Table (4-11) estimator values of $\lambda = -0.001$ and $\gamma = 1$ at $R = 1000$ and MSE values given between parenthesis

The following are the shapes of pdf and cdf for TSE distribution with estimated parameters at R=1000.





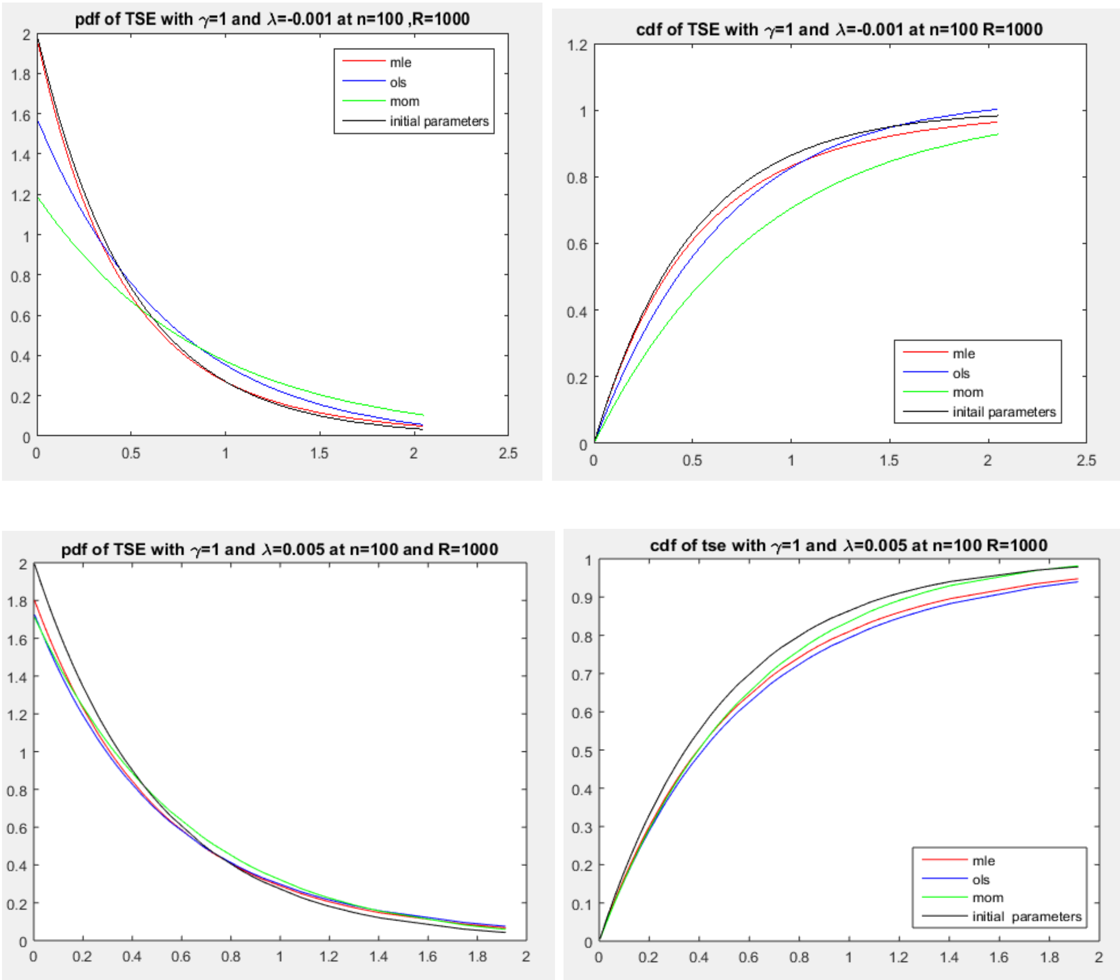


Figure 4.1 Shapes of TSE (pdf and cdf) at $\gamma = 1$ and different cases of $\lambda = (\ 0.005\ and - 0.001)$

Shapes of pdf and cdf of TSE distribution in figure (4.1) are sketched by using the estimated parameters γ and λ . The shapes of these functions are compatible with numerical results in previous tables such that the curves of functions computed by MLE and OLS estimators $\hat{\gamma}\ and\ \hat{\lambda}$ are close to the curve of initial parameters while the MOM estimators did not exhibit that closeness except the shapes with $\gamma = 1$ and $\lambda = 0.005$ and according to table (4-10) the estimators of three methods are close in its values.

4.3 .3 Numerical results of TSW parameters (R=500)

In this subsection we estimate parameters of TSW distribution ($\alpha = 1.1\ ,1.5\ and\ \lambda = -1\ ,0.005$)

At n=10,30,50 and 100 with R=500 listed in the following tables: -

- numerical result of estimator $\alpha = 1.1$ in tables (4-12) indicate that the best estimator is ($\hat{\alpha} = 1.16036777697161$) at n=100 with (mse=5.7815617739e-04) .
- The MLE method has indicate lowest values of MSE as listed in table (4-12) than OLS and MOM estimation methods.
- Table (4-12) has recorded decreasing values of MSE when n is increasing.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\alpha}$	1.000960377079 (0.184238399998)	0.9282736014272 (0.1847878683483)	1.062310436393 (0.1782236580902)	MOM
30	$\hat{\alpha}$	1.2691461517725	1.041849025049	1.0387474433136	MOM

		(0.213090393539)	(0.1909287319548)	(0.185471547431)	
50	$\hat{\alpha}$	1.4450887958646 (0.0143836669871)	1.56719251397742 (0.0239315508342)	1.04492041152604 (0.026154445035)	MLE
100	$\hat{\alpha}$	1.16036777697161 (5.781561773e-04)	1.175494421268041 (8.7247147364e-04)	1.039453416535810 (0.033056596561)	MLE

Table (4-12) estimator values of $\alpha = 1.1 \; and \; \lambda = -1 \; with \; R = 500$ and MSE values given between parenthesis

- Numerical result of estimator $\alpha = 1.5$ in table (4-13) indicates that the best estimator is ($\hat{\alpha} = 1.4161387898083$) at n=100 with (MSE=0.00265873465317) .
- The MLE method has indicate lowest values of mse as listed in table (4-13) than OLS and MOM estimation methods.
- Table (4-13) has recorded decreasing values of MSE when n is increasing.
- At n=50 in table (4-13) the OLS estimation method give best estimator than other methods.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\alpha}$	2.302494520639	2.0302768572903	1.0578371326274	MOM

		(0.175711527005)	(0.0540890693423)	(0.0401610952654)	
30	$\hat{\alpha}$	1.250619602967 (0.0360826144928)	1.14091906952 7 (0.0616489080168)	1.0663896013573 (0.1915916714722)	MLE
50	$\hat{\alpha}$	1.8016717759539 (0.0234558774295)	1.722569679847 (0.01421510615)	1.0585518759786 (0.089366764485)	OLS
100	$\hat{\alpha}$	1.4161387898083 (0.00265873465317)	1.4097243167687 (0.00487931640)	1.06515579152196 (0.147519311778)	MLE

Table (4-13) estimator values of $\alpha = 1.5$ and $\lambda = 0.005$ $with$ $R = 500$ and MSE values given between parenthesis

- Numerical result of estimator $\lambda = -0.9$ in tables (4-14) indicate that the best estimator is ($\hat{\lambda} = -0.95473487105$) at n=100 with (MSE =5.7815617739e-04) .

- The MLE method has indicate lowest values of MSE as listed in table (4-14) than OLS and MOM estimation methods.
- Table (4-14) has recorded decreasing values of MSE when n is increasing.
- At n=10 ,30 in table (4-14) the OLS estimation method give best estimator than other methods.

Table (4-14) estimator values of $\alpha = 1.1$ and $\lambda = -0.9$ $with$ $R = 500$ and MSE values given between parenthesis

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\lambda}$	-0.57284449677 (0.0200474939615)	-0.62541159796 (0.0135377153021)	0.025377895177 (0.1000814476511)	MOM
30	$\hat{\lambda}$	-0.76245609704 (0.0111555558256456)	-0.85467309825 (0.002915123031269)	-0.06789431241 (0.100700497669068)	OLS
50	$\hat{\lambda}$	-0.98051415569 (0.0143836669871)	-0.93001241796 (0.02393155083428)	-0.39247069465 (0.02615444503543)	MLE
100	$\hat{\lambda}$	-0.95473487105 (5.781561773e-04)	-0.93044995035 (8.7247147364e-04)	-0.49936594835 (0.033056596561)	MLE

- Numerical result of estimator $\lambda = 0.005$ in tables (4-15) indicates that the best estimator is ($\hat{\lambda} = -0.1476647983593$) at n=100 with (MSE = 0.00230662978581) .
- The MLE method has indicate lowest values of MSE as listed in table (4-15) than OLS and MOM estimation methods.
- Table (4-15) has recorded decreasing values of MSE when n is increasing.
- At n=50 in table (4-15) the OLS estimation method give best estimator than other methods

Table (4-15) estimator values of $\alpha = 1.5$ *and* $\lambda = 0.005$ *with* $R = 500$

and MSE values given between parenthesis

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\lambda}$	0.54130477145564 (0.175711527005470)	0.095025893775865 (0.054089069342342)	0.179433116449149 (0.040161095265469)	MOM
30	$\hat{\lambda}$	0.27056643878175 (0.008733251739169)	0.389250974608801 (0.022047596792641)	0.183596326739269 (0.109360840929939)	MLE
50	$\hat{\lambda}$	0.10466796777152 (0.00457966816876)	0.22842584161430 (0.0044189274183)	0.138659669366535 (0.108457016543)	OLS
100	$\hat{\lambda}$	-0.1476647983593 (0.00230662978581)	-0.1914442448268 (0.0052078494962)	0.154487225316063 (0.125008051191)	MLE

The following are the shapes of pdf and cdf for TSW distribution with estimated parameters at R=1000.

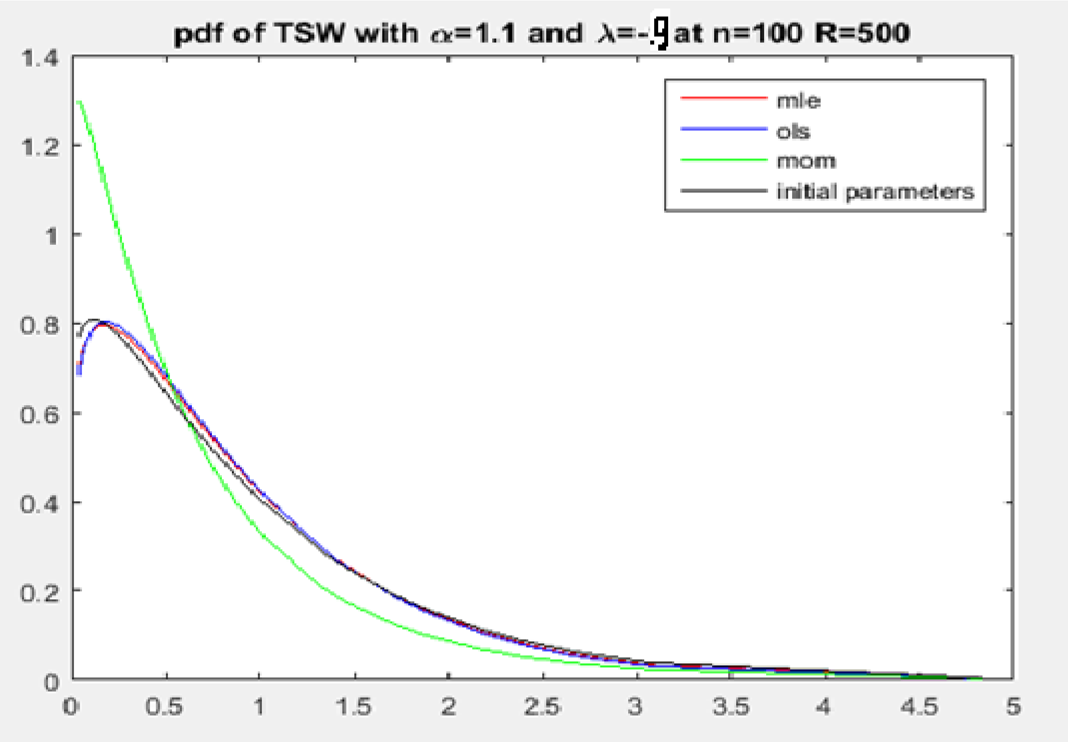


Figure 4.2 shape of density function of TSW with $\alpha = 1.1 \; and \; \lambda = -0.9 \; at \; n = 100 \; R = 500$

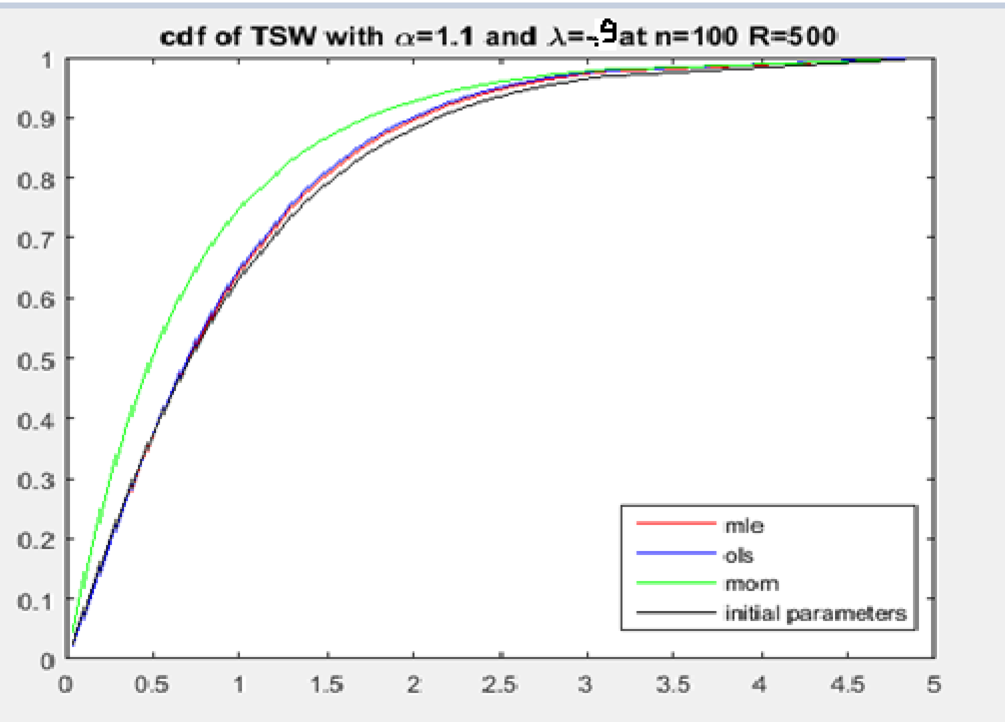


Figure 4.3 shape of cdf of TSW distribution with $1.1 \; and \; \lambda = -0.9 \; at \; n = 100 \; R = 500$

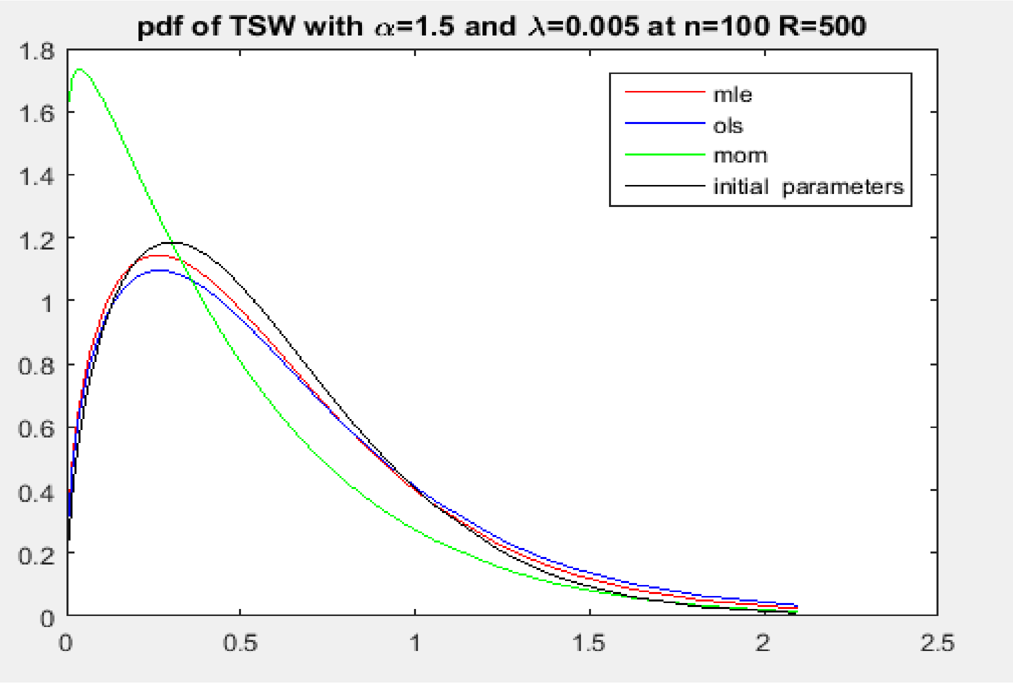


Figure 4.4 shape of pdf of TSW distribution with $\alpha = 1.5\ and\ \lambda = 0.005\ at\ n = 100\ R = 500$

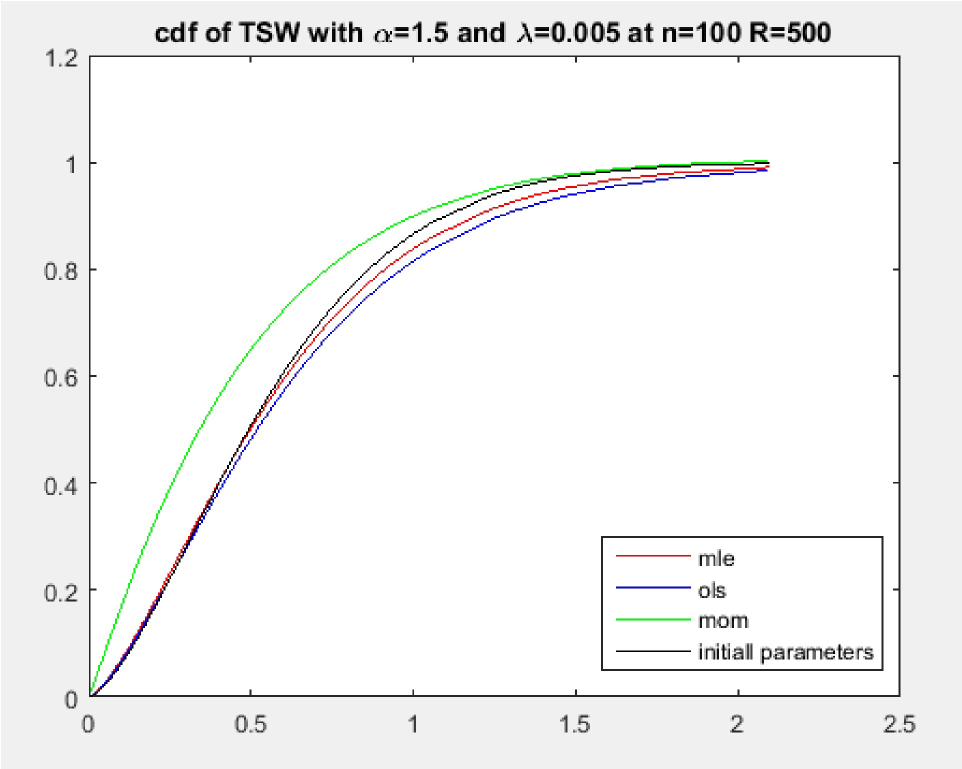


Figure 4.5 shape of cdf of TSW distribution with $\alpha = 1.5\ and\ \lambda = 0.005\ at\ n = 100\ R = 500$

4.3.4 Numerical results of TSW parameters (R=1000)

In this subsection we listed the numerical results of estimating parameters of TSW with R=1000 as in the following tables: -

- Numerical result of estimator $\alpha = 1.1$ in tables (4-16) indicates that the best estimator is ($\hat{\alpha} = 1.107202767088$) at n=50 with (MSE = 2.0987758412e-05) .
- The OLS method has indicate lowest values of MSE as listed in table (4-16) than MLE and MOM estimation methods.
- Table (4-16) has recorded oscillated values of MSE when n is increasing.
- At n=30 in table (4-16) the MLE estimation method give best estimator than other methods.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\alpha}$	2.1019881632387 (0.0767049556698)	1.56526689138146 (0.028288297709316)	1.05275093047256 (0.054366149777887)	OLS
30	$\hat{\alpha}$	0.83578688673413 (0.044435568829109)	0.77353771019535 (0.065245426917461)	1.0485509057709 (0.069400398916692)	MLE
50	$\hat{\alpha}$	0.9571301073481 (0.0025388463458)	1.107202767088 (2.0987758412e-05)	1.055524763758 (0.045170543771)	OLS

100	$\hat{\alpha}$	0.94984097828222 (0.00891173362624)	0.9575797703361 (0.0077902998529)	1.04045696493951 (0.023316764331)	OLS

Table (4-16) estimator values of $\alpha = 1.1$ *and* $\lambda = -0.9$ *with* $R = 1000$ and MSE values given between parenthesis.

- Numerical result of estimator $\alpha = 1.5$ in tables (4-17) indicates that the best estimator is ($\hat{\alpha} = 1.50016406649156$) at n=50 with (MSE = 2.6176417975e-04) .
- The MLE method has indicate lowest values of MSE as listed in table (4-17) than OLS and MOM estimation methods.
- Table (4-17) has recorded oscillated values of MSE when n is increasing.
- At n=50 in table (4-17) the OLS estimation method give best estimator than other methods.

Table (4-17) estimator values of $\alpha = 1.5$ *and* $\lambda = 0.005$ *with* $R = 1000$

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\alpha}$	2.224033403801 (0.121213668641590)	1.789899354784 (0.064306125393843)	1.051059926592 (0.03174752906377)	MOM
30	$\hat{\alpha}$	1.528691252305 (4.2173893687756e-04)	1.564536229202 (0.002767681642355)	1.065703830908 (0.11866578925612)	MLE

50	$\hat{\alpha}$	1.646589361098 (0.00655013691078)	1.634060125373 (0.0061439539682)	1.060250513243 (0.103278017289)	OLS
100	$\hat{\alpha}$	1.50016406649156 (2.6176417975e-04)	1.442069093016 (0.0019351947577)	1.06077642920 (0.116835025809)	MLE

and MSE values given between parenthesis.

- Numerical result of estimator $\lambda = -0.9$ n tables (4-18) indicates that the best estimator is ($\hat{\lambda} = -0.963601915130$) at n=100 with (MSE = 6.0715442814e-04) .
- The OLS method has indicate lowest values of MSE as listed in table (4-18) than MLE and MOM estimation methods.
- Table (4-18) has recorded decreasing values of MSE when n is increasing.
- At n=100 in table (4-18) the MLE estimation method give close estimator $\hat{\lambda} = -0.99440475098546$ to the initial $\hat{\lambda} = -0.9$.

Table (4-18) estimator values of $\alpha = 1.1\ and\ \lambda = -0.9\ with\ R = 1000$

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\lambda}$	0.072033084032536 (0.093521230747537)	-0.528291512240 (0.009039149153863)	0.20455546522351 (0.255227866172623)	ols
30	$\hat{\lambda}$	-0.72541662880355	-0.749163454539	-0.1601025556976	ols

		(0.010424537981162)	(0.004843651086102)	(0.067285687012547)	
50	$\hat{\lambda}$	-0.73224214417366 (0.00564698228702)	-0.652381845784 (0.0096945725915)	-0.2496024833162 (0.063986500821)	mle
100	$\hat{\lambda}$	-0.99440475098546 (0.00387864933700)	-0.963601915130 (6.0715442814e-04)	41.983056115571 (2.300179280002)	ols

and MSE values given between parenthesis.

- Numerical result of estimator $\lambda = 0.005\ in$ tables (4-19) indicate that the best estimator is ($\hat{\lambda} = 0.0013861431724 10$) at n=100 with (MSE = 0.00948926822703) .
- The MLE method has indicate lowest values of MSE as listed in table (4-19) than OLS and MOM estimation methods.
- Table (4-19) has recorded decreasing values of MSE when n is increasing.

n	parameter	MLE	OLS	MOM	
10	$\hat{\lambda}$	-0.1740757948019 (0.011122865278488)	-0.373981157831 (0.061753391108497)	0.140225872287294 (0.121222454027630)	ols

30	$\hat{\lambda}$	0.422995242903286 (0.036261288886868)	-0.059105633919 (0.029186206970612)	0.422995242903286 (0.139326868453565)	ols
50	$\hat{\lambda}$	-0.29921230390294 (0.01582246226845)	-0.354158877574 (0.0170228139340)	0.100927864249939 (0.111413901224)	mle
100	$\hat{\lambda}$	0.0013861431724l0 (0.00948926822703)	-0.060679349876 (0.0099617928072)	0.146001988633387 (0.155874422322)	mle

Table (4-19) estimator values of $\alpha = 1.5$ *and* $\lambda = 0.005$ *with* $R = 1000$ and MSE values given between parenthesis

Next, we give shapes for the pdf and cdf at estimated parameters of TSW distribution:

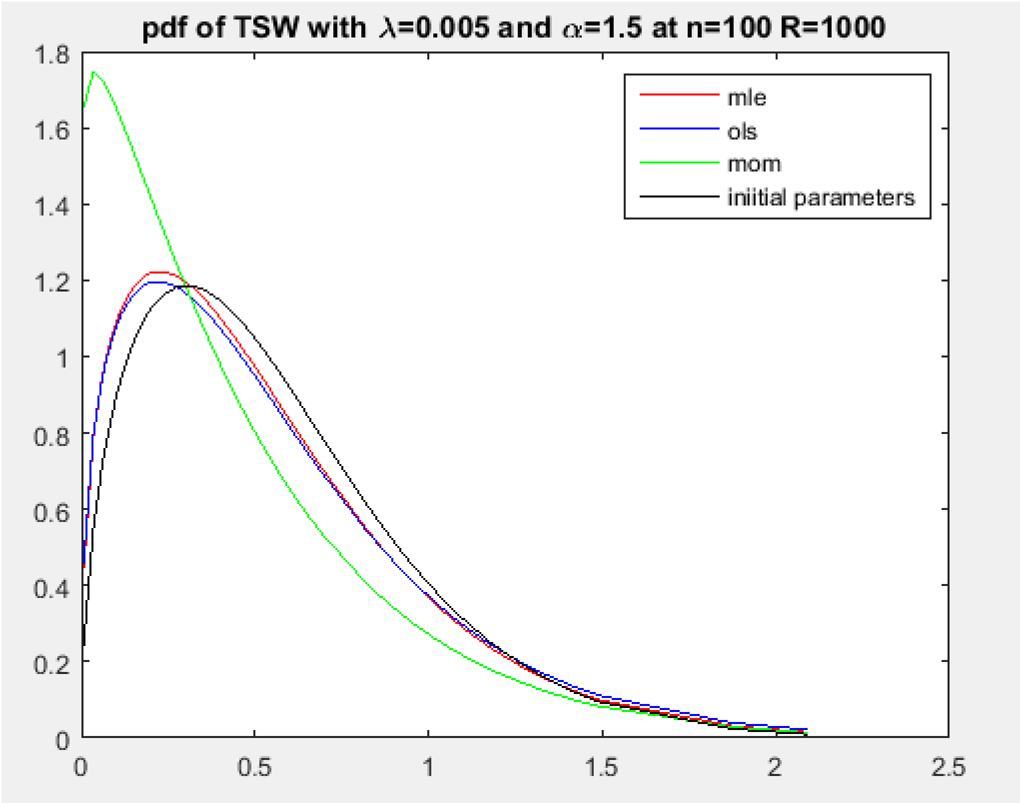


Figure 4.6 shape of pdf of TSW distribution with $\alpha = 1.5\ and\ \lambda = 0.005\ at\ n = 100\ R = 1000$

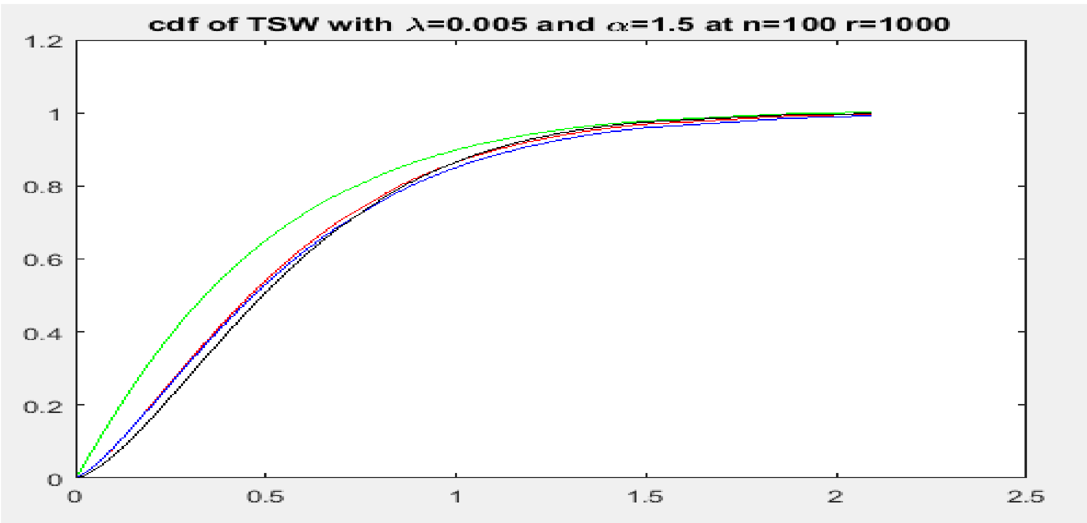


Figure 4.7 shape of shape of cdf of TSW distribution with $\alpha = 1.5\ and\ \lambda = 0.005\ at\ n = 100\ R = 1000$

4.3.5 Numerical Results of TSEW parameters (R=500 and R=1000)

we listed in this subsection the numerical result of estimating TSEW distribution's parameters $\gamma, \alpha\ and\ , \lambda$ with two set of initial parameters ($\gamma = 0.001, \lambda = 0.05\ , \alpha = 2$) and ($\gamma = 0.05,\ \alpha = 0.05\ , \lambda = -0.9\ for$ R=500 and R= 1000 listed in the following tables: -

- Numerical result of estimator $\gamma = 0.001$in table (4-20) indicates that the best estimator is $\hat{\gamma} = 0.00316627653049$at n=100 with $MSE = 1.0849943301\text{e} - 05.$
- The MLE method has indicate lowest value of MSE as listed in table (4- 20) than OLS and MOM estimation method.
- Table (4-20) has indicate oscillator values of MSE when n increasing.

Table (4-20) estimator values of $\gamma = 0.001, \lambda = 0.05\ , \alpha = 2\ and\ R = 500$ and MSE values given between parenthesis.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\gamma}$	0.0123366361476 (2.77646135865e-05)	0.0026794975921 (3.95503845413e-05)	6.564691123612 (0.2208774890105)	MLE
30	$\hat{\gamma}$	0.013929148392 (4.32586917752e-05)	0.0015209320149 (3.25699585510e-04)	4.118232118504 (.912422752313306)	OLS
50	$\hat{\gamma}$	0.0105831473395 (0.0070027352082)	0.0023937303453 (8.233352328e-04)	4.088793134946 (4.135252344e-02)	OLS
100	$\hat{\gamma}$	0.00316627653049 (1.084994330e-05)	0.0035159247507 (2.949839714e-04)	3.901524494641 (6.836247479e-04)	MLE

- Numerical result of estimator $\gamma = 0.001$in table (4-21) indicate that the best estimator is $\hat{\gamma} = 0035159247507$ n=100 with $MSE = 1.166357504e - 07$.
- The MLE method has indicate lowest value of MSE as listed in table (4- 21) than OLS and MOM estimation method.
- Table (4-21) has indicate decreasing values of MSE when n increasing.
- At n=30 MOM estimator method give close value to the initial parameter $\gamma = 0.001$.

Table (4-21) estimator values of $\gamma = 0.001, \lambda = 0.05\ , \alpha = 2\ and\ R = 1000$ and MSE values given between parenthesis.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\gamma}$	0.0123366361476 (4.69590674480e-06)	0.0026794975921 (1.46211809289e-06)	0.06564691123612 (7.426825115e-03)	OLS
30	$\hat{\gamma}$	0.013929148392 (9.23641436626e-05)	0.0015209320149 (8.66156166094e-06)	0.00118232118504 (4.3477233618e-06)	MOM
50	$\hat{\gamma}$	0.0105831473395 (4.14110982e-05)	0.0023937303453 (1.02231380e-08)	4.088793134946 (1.1226658e-06)	OLS
100	$\hat{\gamma}$	0.0035159247507 (1.16635750e-07)	0.0386627653049 4.1952844416e-05	3.901524494641 9.424319772e-01	MLE

- Numerical results of estimator $\alpha = 2$ in table (4-22) indicate that the best estimator is $\hat{\alpha} = 1.26799392891160$ n=100 with $MSE = 0.01344427394900$.
- The MLE method has indicate lowest value of MSE as listed in table (4- 22) than OLS and MOM estimation method.
- Table (4-22) has indicate decreasing values of MSE when n increasing.
- At n=30 MLE estimator method give close value to the initial parameter $\alpha = 2$ than other sample value.

Table (4-22) estimator values of $\alpha = 2, \gamma = 0.001, \lambda = 0.05$ and $R = 500$ with MSE values given between parenthesis.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\alpha}$	1.8119881037229 (.242204997473944)	2.1486980718387 (0.968308095055189)	2.413031241748 (0.133218677194724)	MOM
30	$\hat{\alpha}$	1.014493580857924 (0.014393932825125)	2.074047277451204 (0.195329073095225)	3.49954120015842 (0.652688697798415)	MLE
50	$\hat{\alpha}$	2.64946308227815 (0.0258338003186)	1.9403522776704 (0.2783711952803)	3.672790690365 (0.032335928741)	MLE
100	$\hat{\alpha}$	1.26799392891160	2.2926839679575	3.637932722617	MLE

		(0.0134442739490)	(0.1992000970411)	(1.879720817722)	

- Numerical result of estimator $\alpha = 2$ in table (4-23) indicate that the best estimator is $\hat{\alpha} = 1.57410277312597$n=100 with $mse = 0.01373850857983.$
- The MLE method has indicate lowest value of MSE as listed in table (4- 23) than OLS and MOM estimation method.
- Table (4-23) has indicate oscillating values of MSE when n increasing.
- At n=30 OLS estimator method give close value 2.030449171719541 to the initial parameter $\alpha = 2$ than other sample value.

n	parameter	MLE	OLS	MOM	
10	$\hat{\alpha}$	1.00004716909027 (0.093694869831708)	2.3445386173239 (0.470416168607473)	2.639301487903 (0.201919189466503)	MLE
30	$\hat{\alpha}$	0.963258552601377 (0.26078949925576)	2.030449171719541 (0.242029277224155)	3.36369644839237 (0.96826779479158)	OLS

50	$\hat{\alpha}$	1.05939531927665 (0.03687794791764)	2.1524691018949 (0.4330039092268)	3.197552002026 (0.518196160066)	MLE
100	$\hat{\alpha}$	1.57410277312597 (0.01373850857983)	2.0515074564458 (0.2286731432670)	3.569014045545 (1.363783256609)	MLE

Table (4-23) estimator values of $\alpha = 2, \gamma = 0.001, \lambda = 0.05$ and $R1000$ with MSE given between parenthesis.

- Numerical result of estimator $\lambda = 0.05$ in table (4-24) indicate that the best estimator is $\hat{\lambda} = 0.0575701166993$ n=50 with $MSE = 2.73866328842e-05$.
- The OLS method has indicate lowest value of MSE as listed in table (4- 24) than MLE and MOM estimation method.
- Table (4-24) has indicate oscillating values of MSE when n increasing.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\lambda}$	0.00000000007144 (4.391351596503e-04)	1.8150234336737 (0.212066700622896)	0.425938531159 (0.0022868602203)	MLE
30	$\hat{\lambda}$	0.071157149989808 (0.007148574350593)	1.536997294937292 (0.138051292052250)	0.40184302932056 (0.0705424082866)	MLE

50	$\hat{\lambda}$	0.00000000000109 (3.226392152e-05)	0.0575701166993 (2.738663288e-05)	0.572745168779 (0.62544834950)	OLS
100	$\hat{\lambda}$	.0272361051904 (6.1650742215e-04)	0.4354485567609 (0.1453999484297)	0.687099642121 (0.83469309111)	MLE

Table (4-24) estimator values of $\lambda = 0.05. \alpha = 2, \gamma = 0.001, \ and\ R = 500$ with MSE given between parentheses

- Numerical result of estimator $\lambda = 0.05$in table (4-25) indicate that the best estimator is $\hat{\lambda} = 0.00423213959222$ n=50 with $MSE = 4.82167684278\text{e} - 04.$
- The MLE method has indicate lowest value of MSE as listed in table (4- 25) than OLS and MOM estimation method.
- Table (4-25) has indicate oscillating values of MSE when n increasing.

Table (4-25) estimator values of $\lambda = 0.05. \alpha = 2. \gamma = 0.001. \ and\ R = 1000$ with MSE values given between parentheses.

n	Parameter	MLE	OLS	MOM	The best
10	$\hat{\lambda}$	0.6344104075269 (0.009487159259787)	1.6314059214854 (0.104621913728345)	0.605830529759 (0.6398120957837)	MLE
30	$\hat{\lambda}$	0.64157295605269	1.8181000846338	0.38823682915732	MOM

		(0.0409360246750)	(0.2500790514036)	(0.0371363735842)	
50	$\hat{\lambda}$	0.00423213959222 (4.8216768427e-04)	1.8832765629625 (0.1763513653279)	0.616291309100 (0.68349571037)	MLE
100	$\hat{\lambda}$	0.81978515752773 (0.1199213065233)	1.6608684208985 (0.0950627123634)	0.442287457737 (0.00295911549)	MOM

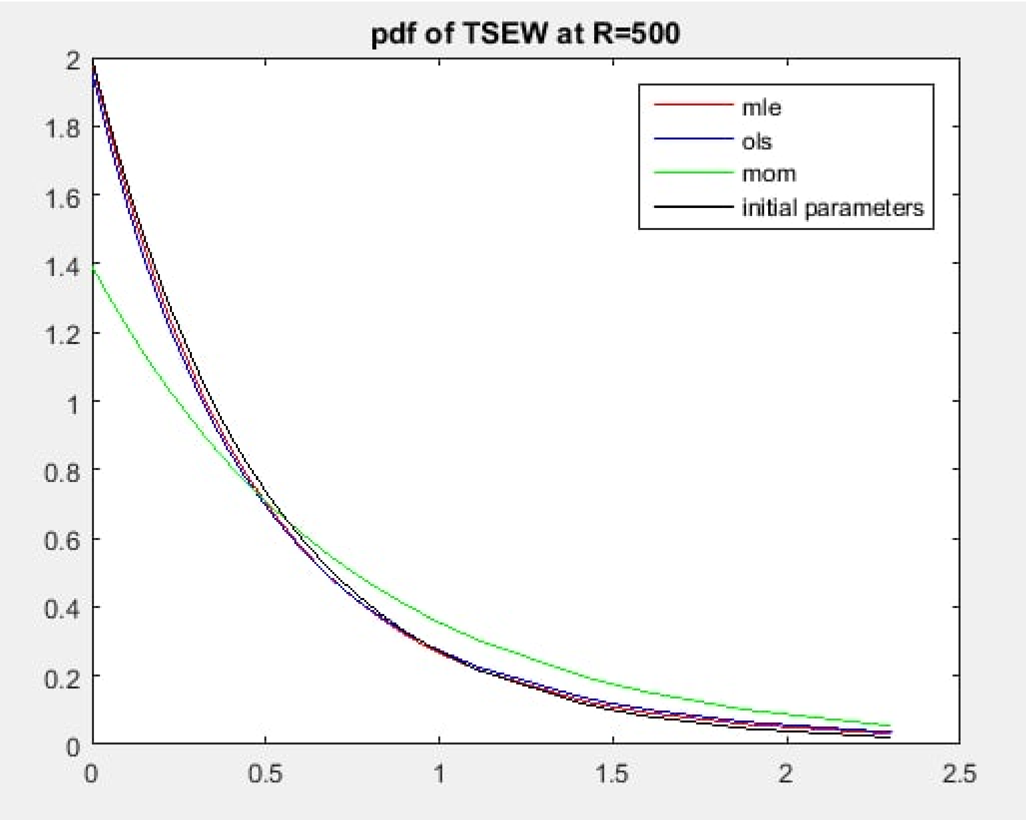


Figure 4.8 shape of pdf of TSEW with R=500

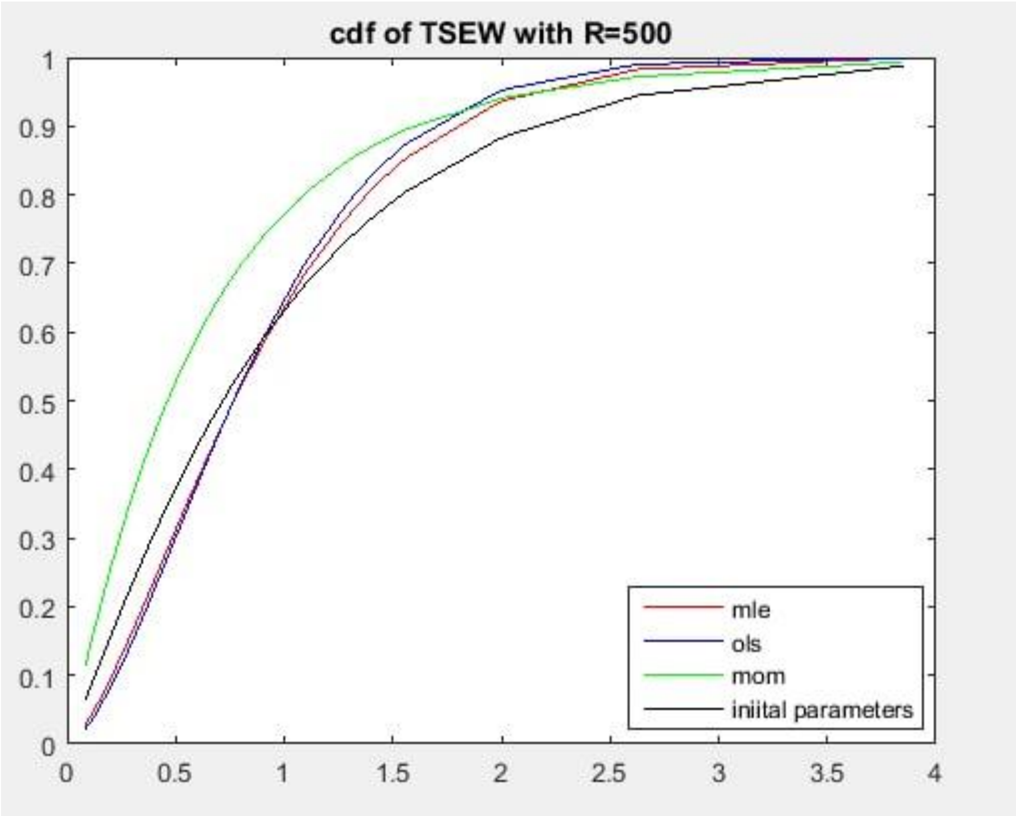


Figure 4.9 shape of cdf of TSEW with R=500

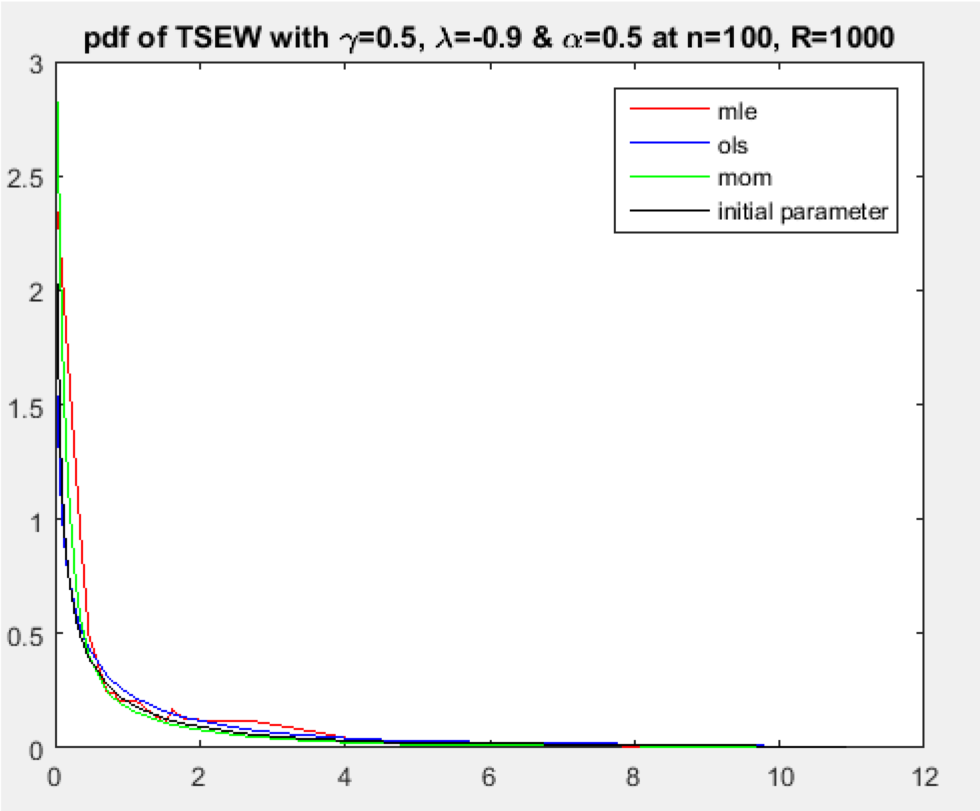


Figure 4.10 shape of cdf of TSEW with R=1000

- Numerical result of estimator $\lambda = 0.05$in table (4-26) indicate that the best estimator is $\hat{\gamma} = 0.53941801175575$ at n=50 with $MSE = (0.0094656199219)$.
- The OLS method has indicate lowest value of MSE as listed in table (4- 26) than MLE and MOM estimation method.
- At n=10 and 100 the MLE distinct with good estimation to γ .
- The results of estimators by MOM method are far-reaching from $\gamma = 0.05$.
- Table (4-26) has indicate oscillating values of MSE when n increasing.

Table (4-26) estimator values of $\gamma = 0.05$, $\alpha = 0.05$ $\lambda = -0.9$ $and\ R = 500$ with MSE values given between parentheses.

n	Parameter	MLE	OLS	MOM	The Best
10	$\hat{\gamma}$	0.5001180612200 (0.043134243643)	0.24833291996645 (0.3603960912404)	11.31732420418348 (0.5713295795587)	MLE
30	$\hat{\gamma}$	0.50168863678210 (2.1190435206921)	0.57189356223723 (0.0515207951542)	5.359448438321067 (0.61664382969998)	OLS
50	$\hat{\gamma}$	0.55514967363922 (0.8089041209502)	0.53941801175575 (0.0094656199219)	4.526815618724223 (0.4549442046398)	OLS
100	$\hat{\gamma}$	0.4998857208064 (0.0184075000265)	0.01502525142808 (0.155061482366)	8.021299171384046 (0.545636257570)	MLE

- Numerical result of estimator $\alpha = 0.05$in table (4-27) indicate that the best estimator is $\hat{\alpha} = 0.5254464805356443$at n=100 with $MSE = (0.017821661783909)$.
- The MLE method has indicate lowest value of MSE as listed in table (4- 27) than OLS and MOM estimation method.
- At n=30 the OLS distinct with good estimation to γ than MOM method.
- Table (4-27) has indicate oscillating values of MSE when n increasing.

Table (4-27) estimator values of $\gamma = 0.05$, $\alpha = 0.05$ $\lambda = -0.9$ $and\ R = 500$ with MSE values given between parentheses.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\alpha}$	0.34838261833158 (0.461211350037)	0.23337882688384 (0.548073263550)	0.3989982743618 (0.26084476631)	MOM
30	$\hat{\alpha}$	0.58230848468392 (0.076909161919)	0.49986076902482 (0.048872781415)	0.6226227533723 (0.14103231040)	OLS
50	$\hat{\alpha}$	0.55665907356345 (0.026785656680)	0.28536206605535 (0.678805133733)	0.6095051800656 (0.94026785656)	MLE
100	$\hat{\alpha}$	0.52544648053564 (0.017821661783)	0.61196818324465 (0.283396593230)	0.6005256435292 (0.13687521395)	MLE

- Numerical result of estimator $\lambda = -0.9$in table (4-28) indicate that the best estimator is $\hat{\lambda} = -0.999806467356401$ at n=100 with $MSE = (5.27895587996\text{e} - 06)$.
- The MLE method has indicate lowest value of MSE as listed in table (4- 28) than OLS and MOM estimation method.
- At n=10 the OLS distinct with good estimation to λ than MOM method.
- The results of estimators by MOM method are far-reaching from $\lambda = -0.9$.

- Table (4-28) has indicate oscillating values of MSE when n increasing.

Table (4-28) estimator values of $\gamma = 0.05$, $\alpha = 0.05$ $\lambda = -0.9$ $and\ R = 500$ with MSE values given between parentheses.

n	parameter	MLE	OLS	MOM	The Best
10	$\hat{\lambda}$	-0.93824036885129 (0.0388062114376)	-0.888250371811355 (0.35815329892836)	-0.22469630573469 (0.29955384810718)	MLE
30	$\hat{\lambda}$	-0.89592488004604 (0.0809339819943)	-1.424151062721465 (0.15744475879429)	-0.099574854675640 (0.32348917928730)	MLE
50	$\hat{\lambda}$	-0.94978751384473 (0.0280851822394)	-1.343271722380049 (0.14612062544017)	-3.254164649760135 (0.26022997720213)	MLE
100	$\hat{\lambda}$	-0.99980646735640 (5.2789558799e-06)	-2.053584260991757 (0.00868481753774)	- 0.00000000000170 (0.32630409681928)	MLE

Next the result of estimations with R=1000 for $\gamma = 0.05$, $\alpha = 0.05$ $\lambda = -0.9$.

- Numerical result of estimator $\gamma = 0.05$in table (4-29) indicate that the best estimator is $\hat{\gamma} = 0.033036079910794$at n=100 with $MSE = (0.08866517000482)$.
- The MLE method has indicate lowest value of MSE as listed in table (4- 29) than OLS and MOM estimation method.

- The results of estimators by MOM method are far-reaching from $\gamma = 0.05$.
- At n=100 the OLS give the best estimator with R=500 while with R=1000 the MLE give that result.
- The MSE value of estimating γ when R=500 is less than MSE with R=1000 at n=100.

- Table (4-29) has indicate oscillating values of MSE when n increasing.

Table (4-29) estimator values of $\gamma = 0.05$, $\alpha = 0.05$ $\lambda = -0.9$ $and\ R = 1000$ with MSE values given between parentheses.

n	parameter	MLE	OLS	MOM	The Best
10	$\hat{\gamma}$	0.4383053384728 (0.169814403411)	0.20429602697733 (0.7712492584972)	0.998136148967209 (0.57540565307752)	MLE
30	$\hat{\gamma}$	0.59233170701765 (0.7656764478042)	0.34462466689673 (0.1014920855580)	2.7062481194567 (0.14736295980517)	OLS
50	$\hat{\gamma}$	0.5388571192786 (0.3657259527837)	0.29915858165400 (0.1864820018008)	3.243812999056264 (0.95854408399799)	MLE
100	$\hat{\gamma}$	0.03303607991079 (0.088665170004)	0.48472025553976 (	0.856408211550036 (	MLE

			0.2987879271502)	0.16621970298136)	

- Numerical result of estimator $\alpha = 0.05$in table (4-30) indicate that the best estimator is $\hat{\alpha} = 0.459358801836391$n=100 with $MSE = (0.119368375904886)$.
- The OLS method has indicate lowest value of MSE as listed in table (4- 30) than MLE and MOM estimation method.
- Table (4-30) has indicate oscillating values of MSE when n increasing.
- At n=100 the MLE give the best estimator with R=500 while with R=1000 the OLS give that result.

Table (4-30) estimator values of $\gamma = 0.05\,, \alpha = 0.05\ \lambda = -0.9\quad and\ R = 1000$ with MSE values given between parentheses.

n	parameter	MLE	OLS	MOM	The best
10	$\hat{\alpha}$	0.991795901640555 (0.322245311689018)	0.425555203995880 (0.028060561611959)	0.682959309315327 (0.354236788757919)	OLS
30	$\hat{\alpha}$	0.978950957406207 (0.242612761786729)	0.129390467293814 (0.516100225908277)	0.805086460649760 (0.324996026317281)	MOM
50	$\hat{\alpha}$	0.980667331699470 (0.119852437111949)	0.620301734695967 (0.128081977002242)	1.069255108040444 (1.460625024604022)	OLS
100	$\hat{\alpha}$	0.749824278496497 (0.218288603687679)	0.459358801836391 (0.119368375904886)	0.617066387605972 (0.263487073416755)	OLS

- Numerical result of estimator $\lambda = -0.9$ in table (4-31) indicate that the best estimator is $\hat{\lambda} = -0.994402849410370$=100 with $MSE = (0.03046078919464)$.
- The MLE method has indicate lowest value of MSE as listed in table (4- 31) than OLS and MOM estimation method.
- Table (4-31) has indicate oscillating values of MSE when n increasing.
- At n=100 the MLE give the best estimator with R=500 while with R=1000 the MLE give that result at n=50.
- The results of estimators by MOM method are far-reaching from $\lambda = -0.9$.

n	parameter	MLE	OLS	MOM	The Best
10	$\hat{\lambda}$	-1.163775069847096 (0.09103223375137)	-0.720275695256362 (0.08171085523662)	-0.590205453019547 (0.108718488947004)	OLS
30	$\hat{\lambda}$	-0.998847373460307 (0.05227047367061)	-0.811666014004037 (0.03182539440820)	-2.328007915019533 (0.902696081142395)	MLE
50	$\hat{\lambda}$	-0.994402849410370 (0.03046078919464)	-0.947185580955048 (0.07395186765506	-6.070310352144946 (0.402863453247218)	MLE
100	$\hat{\lambda}$	-0.997164264442364	-1.096645243747452	0.868263875016195	MLE

		(0.05597469224831)	(0.07178920482088)	0.170989201480659	

Table (4-31) estimator values of $\gamma = 0.05$, $\alpha = 0.05$ $\lambda = -0.9$ $and\ R = 1000$ with MSE values given between parentheses.

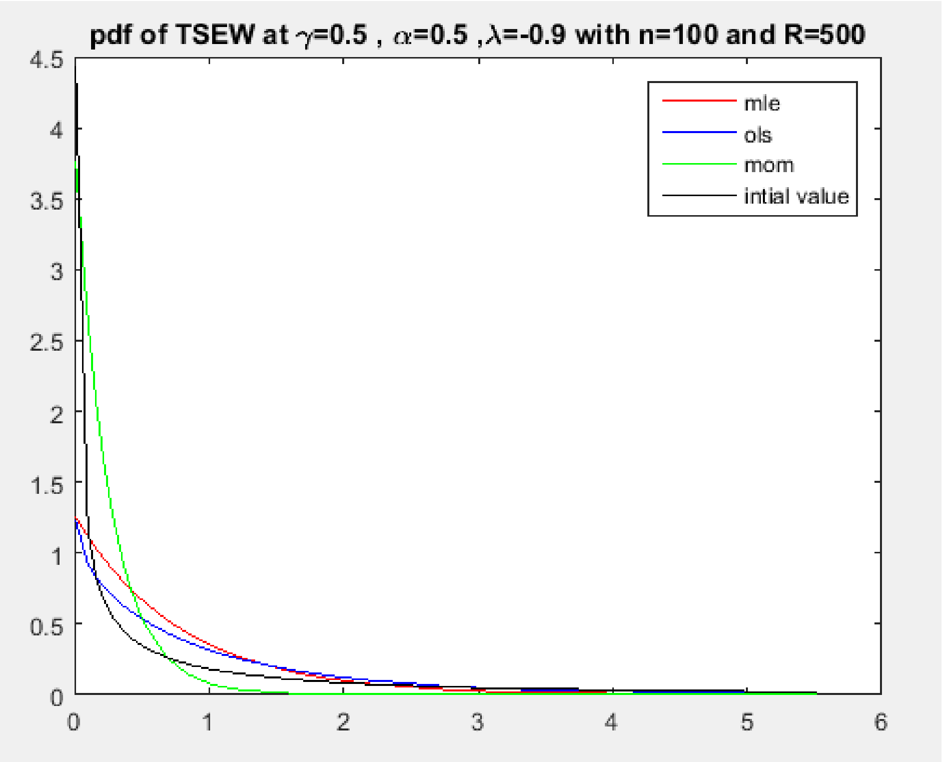


Figure 4.11 shape of pdf of TSEW with R=500

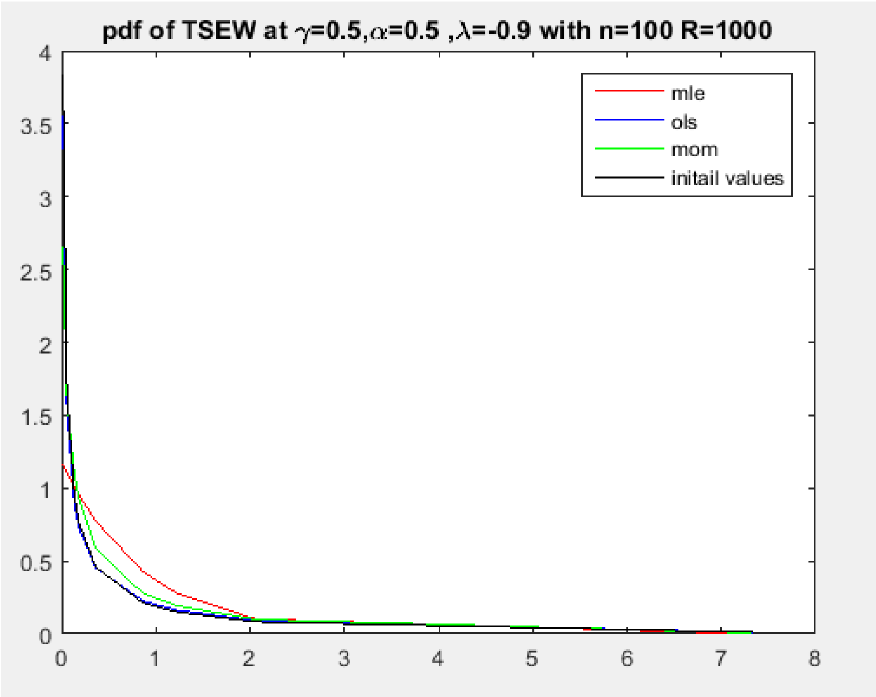


Figure 4.12 shape of pdf of TSEW with R=1000

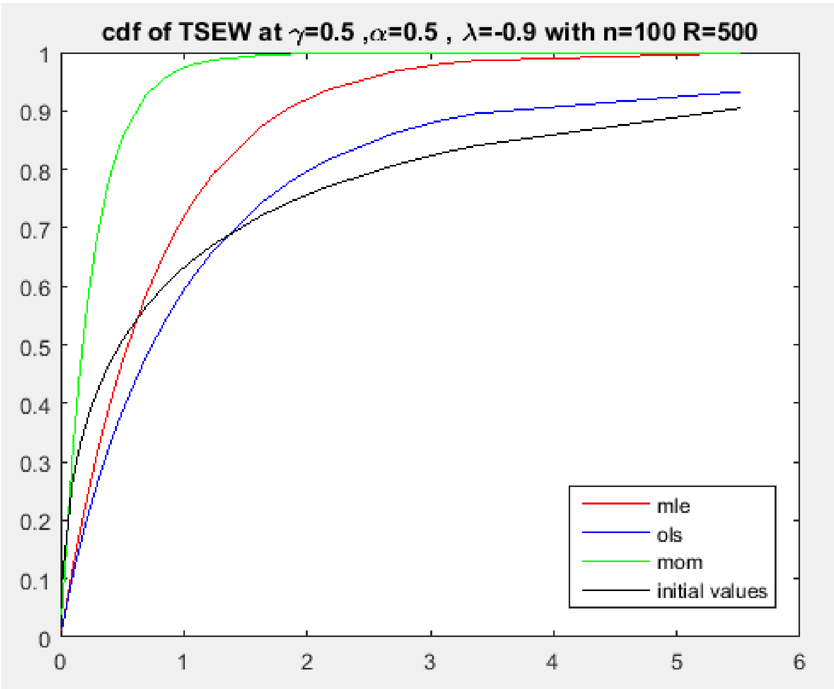


Figure 4.13 shape of cdf of TSEW with R=500

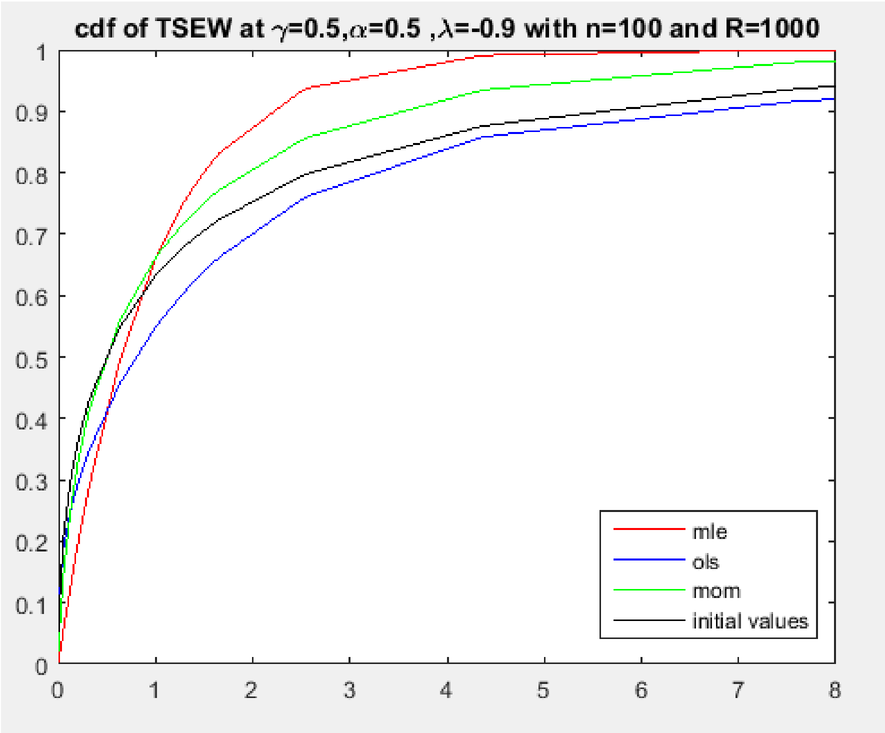


Figure 4.14 shape of cdf of TSEW with R=1000

Table (4-32) the number of preferences for the estimation methods

Distribution R		Sample size 10	30	50	100	Number of preferences
TSE 500	MLE	3	3	2	4	12
	OLS	2	3	4	2	11
	MOM	1	1	0	0	2
The preference Method		MLE	MLE & OLS	OLS	MLE	MLE
TSE 1000	MLE	4	1	2	3	10
	OLS	2	3	3	1	10

	MOM	0	1	0	1	2
The preference Method		MLE	OLS	OLS	MLE	MLE & OLS
Distribution R		Sample size				
		10	30	50	100	
TSW 500	MLE	0	2	2	4	8
	OLS	1	1	2	0	4
	MOM	3	1	0	0	4
The best		MOM	MLE	MLE & OLS	MLE	MLE
TSW 1000	MLE	0	2	2	2	6
	OLS	3	2	1	2	8
	MOM	1	0	0	0	1
The best		OLS	MLE &OLS	MLE	MLE	MLE
TSEW 500	MLE	4	3	3	6	16
	OLS	0	3	3	0	6
	MOM	2	0	0	0	2
The best		MLE	MLE & OLS	MLE & OLS	MLE	MLE
TSEW 1000	MLE	3	1	4	4	12
	OLS	3	2	2	1	8
	MOM	0	3	0	1	4

In table (4-32) we set the number of preferences of the three estimation methods according to the values of MSE listed in the previous tables. Table (4-32) indicates that the number of preferences of MLE is better than OLS and MOM and the number of preferences of OLS is better than MOM method. According to this result we suggest the MLE method to estimate the parameters of our distributions in next section.

4.4 Applications

In this section, we test the three models TSE, TSW and TSEW with the following real data sets and compare the results using information criterions and curve fitting tools. And according to the numerical results of previous section, the MLE method gives good results more than OLS and MOM methods that's why we adopted the MLE method to estimate the parameters of our distributions.

Data set (1): real data set represent the survival times of 121 cases with breast cancer taken from a large hospital censored from 1929 to 1938 This data set has recently been studied by [1]**:**

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3,11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5,17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0,31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0,41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0,54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0,78.0, 80.0,83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0,126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

Data set (2)

The following data set are the remission times (in months) of a random sample of 128 bladder cancer patients [1]. The data are

[0.08 , 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34,5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69].

Data set (3)

The following data set are the total cases of Covid-19 collected from Aruba

at time (24-3 until 19-7 in 2020).

[55;55;60;62;64;64;71;74;77;82;86;92;92;92;92;93;95;96;96;97;97;97;100;100;100;100;100;100;100;100;100;100;100;100;100;100;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;101;103;103;103

;104;104;105;105;105;105;105;105;105;105;105;105;1
06;106;108;108;108];

4.4.1 The Result and Discussion

<u>**First:**</u> comparing the three distribution and other related distributions by test data set (1) and the results tabulated in the table (4-33)

Table (4-33) : Data of breast cancer of 121 cases.

Distribution	Estimation parameters	Standard Error	L	AIC	CAIC	BIC
Exponential	$\hat{\gamma}$= 0.0216	0.0684	-585.1277	1172.26	1172.29	1175.05
Weibull	$\hat{\alpha}$=0.2202	0.0062	-968.7623	1955.5	1941.6	1961.1
TSE	$\hat{\gamma}$=0.0092 $\hat{\lambda}$= 0.2235	0.0015 0.0135	-581.8901	1167.78	1164.78	1173.37
TSW	$\hat{\alpha}$= 0.2174 $\hat{\lambda}$= 0.7320	0.0074 0.0020	-934.75	1873.5	1859.6	1879.1
TSEW	$\hat{\gamma}$=0.3724 $\hat{\lambda}$=-1.9993 $\hat{\alpha}$= 0.28	0.0276 0.0006 0.0060	-714.3966	1432.8	1418.9	1438.4

Notice that in general the TSE distribution have the smallest AIC, CAIC and BIC criterions comparing with other listed distributions. Also TSW distribution indicate smallest vales of criterions comparing to the Weibull distribution. Besides that, the effect of Weibull distribution is clear on the

TSEW distribution such that its indicate larger criterions values than TSE distribution. This is according to the type of tested data.

<u>**Second:**</u> the result of testing and comparing the three distribution and other related distributions by using data set (2) and the results tabulated in the table (4-34).

Table (4-34) : Data of the remission times (in months) of a random sample of 128 bladder cancer patients.

Distribution	Estimation parameters	Standard Error	L	AIC	CAIC	BIC
Exponential	$\hat{\gamma}$=0.0041	0.0059	-708.5014	1419.0	1417.0	1421.9
Weibull	$\hat{\alpha}$=0.2499	0.00008	-531.9165	1065.8	1063.9	1068.7
TSE	$\hat{\gamma}$=0.0959 $\hat{\lambda}$= -0.8073	0.0059 0.3927	-414.4087	832.8175	818.9112	838.5215
TSW	$\hat{\alpha}$=0.2499 $\hat{\lambda}$= -6.6094	0.017 0.0001	-376.7525	757.505	743.5988	763.2091
TSEW	$\hat{\gamma}$=0.2498 $\hat{\lambda}$=0.8211 $\hat{\alpha}$= 1.3587	0.0498 0.0911 0.0213	-632.0297	1268.1	1254.2	1273.8

When we comparing the distributions according to the criterions giving in Table (4-34) we find that:

1- TSE distribution has lower criterion values than other distribution and Weibull.
2- TEW distribution record lowest criterion values than other distribution.

Third The following table contains the result of comparing the listed distributions using data set (3):

Table (4-35) : Data of covid-19 .

Distribution	Estimation parameters	Standard Error	L	AIC	CAIC	BIC
E Exponential	$\hat{\gamma}$= 0.0102	0.0268	-613.9989	1230.0	1228.0	1232.7
Weibull	$\hat{\alpha}$= 0.1762	0.5636	-851.8033	1705.6	1703.6	1708.3

TSE	$\hat{\gamma}$=0.0188 $\hat{\lambda}$=-1.9	0.0002 0.9130	-587.2901	1178.6	1164.7	1184.0
TSW	$\hat{\alpha}$= 0.2040 $\hat{\lambda}$= -2.7323	0.0540 0.3023	-756.1485	1516.3	1502.4	1521.7
TSEW	$\hat{\gamma}$=0.0157 $\hat{\lambda}$=-0.7166 $\hat{\alpha}$= 0.0672	0.0137 0.0634 0.0382	-817.7930	1639.6	1625.7	1645.0

Criterions information AIC, CAIC and BIC listed above indicate that:

1- Exponential distribution is fitted the data better than Weibull distribution.
2- When we compare between TSE and Exponential distribution the new distribution fitted the data better than Exponential distribution.
3- AIC, CAIC and BIC of TSW distribution have lowest values than Weibull distribution that is to say the TSW give better fit to the data comparing with Weibull distribution.
4- The mixture TSEW distribution improve the performance of Weibull distribution such that it exhibits smallest AIC,CAIC and BIC than Weibull distribution.

The results in Tables (4-33, 4-34 and 4-35) have been computed using maximum likelihood estimated method to estimate the parameters of the listed distributions and utilizing for calculated L, AIC, CAIC and BIC criterions as listed in above tables.

4.4.2 Results of Curve Fitting

This subsection contains results of curve fitting statistics (SSE , R-squared , Adjusted R-squared and RMSE) in order to compare the performances of the TSE, TSW and TSEW in fitting the forth sets of given data listed in the next table:

Table (4-36) : statistics to the goodness of fit

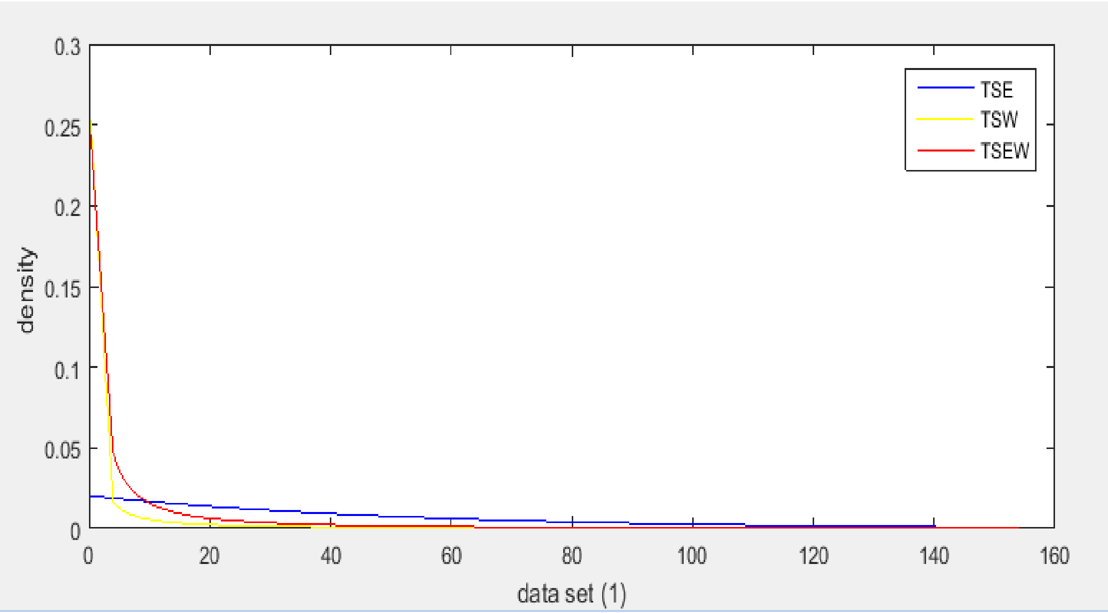
Distribution		**SSE**	**R-squared**	**Adj-R squared**	**RMSE**
Set (1)	TSE	1.960778E-7	0.99999388	0.99993496375	4.1670E-5
	TSW	3.358736E-5	0.99973113	0.99951554989	7.1015E-4
	TSEW	4.97360E-5	0.99958375	0.99933144989	8.1590E-4
Set (2)	TSE	1.28076E-5	0.99988476	0.99987476033	3.3106E-4
	TSW	1.02136E-6	0.99999836	0.99998657886	2.5726E-4
	TSEW	6.6658E-6	0.999833999288	0.99968849528	9.9243E-4
Set (3)	TSE	3.973339E-12	0.9999	.9999	2.02580E-7
	TSW	1.330724E-11	0.9998	0.9998	3.57150E-7
	TSEW	7.920043E-11	0.9992	0.9991	8.63868E-7

Statistics (SEE, R-squared, Adj R-squared and RMSE) listed in table (3-36) with figure (4.1) indicate that :

1- For data set (1): TSE distribution with (SSE= 1.960778E-7 RMSE=4.1670E-5) which is the smallest comparing to TSW and TSEW distributions is fitted the data better than others distrebutions and this result compatible with results of Table (3-33). And SSE with SSR values closer to 0 indicate that the fit is more useful for prediction.

2- For data set (2): TSW distribtuion have (SSE=1.0214e-06 and RMSE=2.5726-04) such that its cose to zero and have lowest values than TSE and TSEW. Also TSE with (SSE=1.28076E-5 and RMSE=2.5726E-4) make better fit to the data than TSEW distribution.

3- For data set (3) and compatible with Table (3-34) , TSE achive better goodness of fit statistics than other listed distribuion while TSW with (SSE=1.330724E-11 and RMSE=3.57150E-7) indicate better fit than TSEW distribution.



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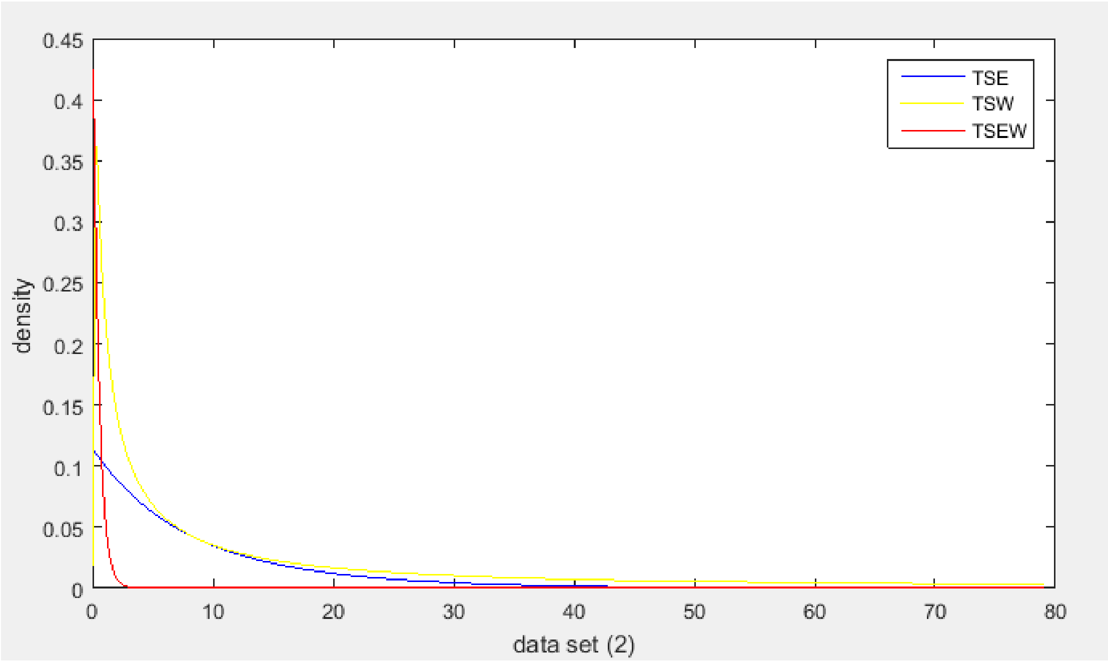

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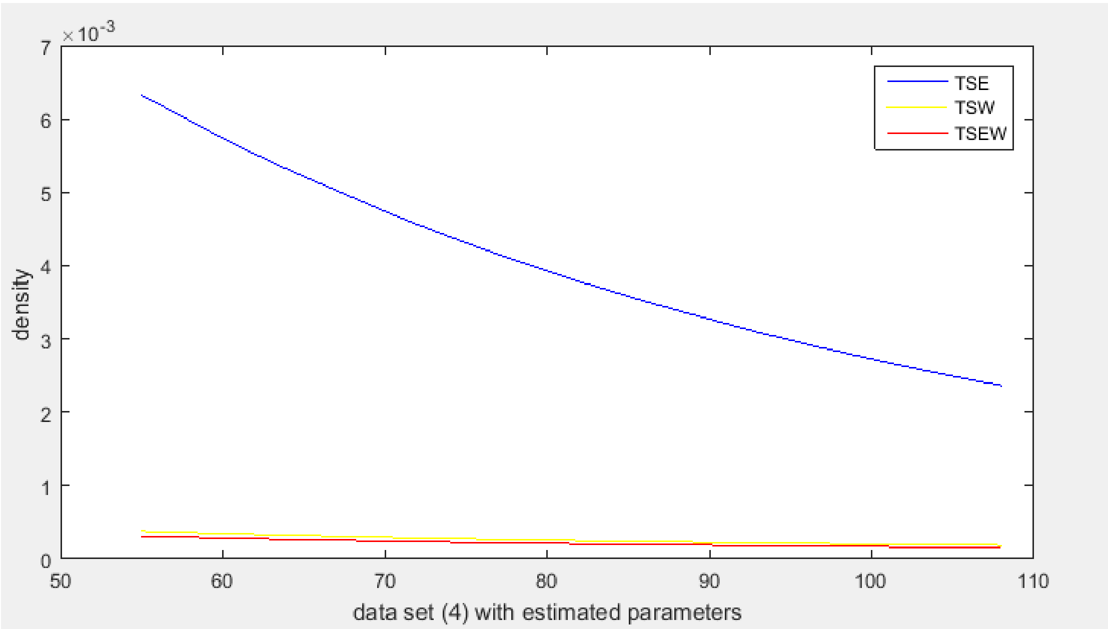

-c-

Figure 4.15 probability density function of TSE, TSW and TSEW distribution with estimated parameters ($\hat{\gamma}$, $\hat{\lambda}$ and $\hat{\alpha}$) for the data sets(1,2 and 3).

The results in Tables (4-33, 4-34,4-35 and 4-36) have been computed using maximum likelihood estimated method to estimate the parameters of the listed distributions and utilizing for calculated L, AIC, CAIC and BIC criterions with goodness of fit statistics listed in above tables and figure (4.15 a, b and c).

Conclusion and Future Work

4.5 Conclusion:

There are many significant conclusions of this study which are as following

1. Within simulation study, the MLE estimated method exhibits better estimator than OLS and MOM methods in estimating the parameter γ of TSE distribution at R=500 and oscillating between MLE and OLS at R=1000.
2. Estimating the transmuted parameter λ of TSE at R=500 and R=1000 is computed and has better estimator from MLE and OLS depending on sample size n.
3. The Mean Squared Error values are minimized when sample size n increasing and R=1000 for TSE distribution.

4. For TSW distribution, we observing that, MLE method is better than OLS and MOM in estimating at R=500 and R=1000.

5. Observing that, MSE values are decreasing when n get increasing within TSW distribution.

6. Observing that, The MOM estimation method give close result to the initial parameters of TSW distribution when R=500 but not very effective when R=1000 most of time.

7. At R=1000, estimating parameters of TSW distribution are close to the initial values through MLE and OLS methods.

8. The MLE and OLS exhibit good results in estimating parameters of TSEW distribution.

9. The MOM estimators of TSEW mark improve at n=10 when R=500 and at n=30, 100 when R=1000.

10. We apply two ways in choosing the initial values of parameters the first one with TSE distribution in order to show the effect of changing parameters on shapes of pdf and cdf and there was no big changing in shapes while the second way allowed to MOM method to give good result in estimating with TSW and TSEW distribution such that the first way enclose the good estimation by OLS then MLE.

11. According the table of preferences, we choose the MLE method to estimate the parameters of our distributions in testing three real data sets.

4.6 Future Work

There are many important thoughts and notes which can be used and applied as future study such as:

1. Study the transmuted formula through apply it on other distributions.
2. Study the transmuted formula through different type of distributions not just lifetime one.
3. Find other estimation methods to estimate the three parameters of the three distributions
4. Modify algorithms to initialized the parameters of the given distributions
5. Analyze real sample data with the best estimated parameters of each distribution and apply it on real experiment.
6. Generalized the transmuted formula through expand the parameter of it.

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