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A study in New Subclasses of Harmonic Univalent and Multivalent Functions

A Thesis

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بسم الله الرحمن الرحيم

﴿فَتَعَالَى اللَّهُ الْمَلِكُ الْحَقُّ وَلَا تَعْجَلْ بِالْقُرْآنِ مِن قَبْلِ أَن يُقْضَى إِلَيْكَ وَحْيُهُ وَقُل رَّبِّ زِدْنِي عِلْمًا﴾

صدق الله العظيم

(سورة طه –114)

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Dedication

To My Parents

My wife, My Brother and Sisters

My Son and Daughters

and Friends

Audy Hatim

Acknowledgments

Firstly, I am thankful to God the Almighty for His help me for all the way.

*I would like to express my sincere appreciation to my supervisor, (**Dr**.**Aqeel Ketab Al-Khafaji**), for giving me the major steps to go on exploring the subject, and sharing with me the ideas related to my work.*

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Thank you for all…

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Abstract

The purpose of this work is to study some geometric properties of several subclass of univalent and multivalent analytic functions, as well as univalent and multivalent harmonic functions in open unit disk

$$(\mathfrak{U} = \{z \in \mathbb{C}: |z| < 1\}).$$

We introduce new subclass for any function by different operators for studying the extreme points, closure, coefficient estimates, the growth, the distortion, starlike, convex, sense preserving, convolution, closed under convex combination, integral operator and neighborhoods, addition we studied the properties of dependency of the best dominant function, subordination of multivalent analytic functions, properties related subordination and best dominant on analytic multivalent functions.

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List of Symbols

The symbol	Description
$\mathbb{E}$	The region
$\mathfrak{U}$	Open unit disk. $\mathfrak{U} = \{z \in \mathbb{C}: \lvert z\rvert < 1\}$
$\overline{\mathfrak{U}}$	$\{z \in \mathbb{C}: \lvert z\rvert \leq 1\}$
$\mathbb{C}$	Complex plane
$\mathbb{N}$	The set of natural numbers
$K(\alpha)$	Class of close-to-convex functions of order α
$S^*(\alpha)$	Class of all starlike functions of order α in $\mathfrak{U}$
S^*	Class of all starlike functions of order 0
C	Class of all convex functions of order 0
Λ	Class of all analytic univalent function
$I_{0,z}^{\lambda,\mu,\eta} f(z)$	Saigo hypergeometric fractional integral operator
$D_z^{-\lambda} f(z)$	The fractional integral of f order λ
$D_z^{\lambda} f(z)$	The fractional integral of f order λ
${}_2F_1(a, b; c; z)$	The Gaussian hypergeometric function
R_1	Radius of starlikeness
R_2	Radius of convex
$\mathrm{F} \prec f$	F subordinate f
$(v)_j$	The pochhammer Symbol
$*$	Hadamard product
$\mathbb{N}_0$	$\mathbb{N} + \{0\}$
$\mathbb{Z}$	The set of integer numbers

Γ	Gamma function
$F_{p,y}f(z)$	Bernardi–Libera–Livingston integral operator of multivalent function
R	The set of real numbers
$C(\alpha)$	Class of all convex functions of order α
$\mathfrak{A}(p)$	Class of all analytic multivalent function
$K(z)$	Koebe function.
$\overline{co}$ E	The closed convex hull of E
$f = u + iv$	Complex valued harmonic function
$f = h + g$	The harmonic function, sense-preserving locally injective
$D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)$	Elhaddad et. al. **[15]**, introduced the following differential operator
$\Lambda(\sigma,\delta,\tau)$	class of analytic and univalent functions
$D_{\mu,\lambda,\sigma}^{m}(\alpha,\beta)$	Differential operator. **[28]**
S_H	represented the class of functions in $\mathfrak{A}$
T_s	the subclass of S_H
$T_{\mu,\lambda,\sigma}^{m}(\alpha,\beta)$	class of harmonic univalent functions
$\bar{T}_{\mu,\lambda,\sigma}^{m}(\alpha,\beta)$	The Subclass of $T_{\mu,\lambda,\sigma}^{m}(\alpha,\beta)$
$T_u(f(z))$	Circular Bernardi-Libera-Livingston integral operator
$N_\delta(f)$	define $m-\delta$-Neighborhood of f
$x_p(a_{1,} \dots . a_r, b_1, \dots . b_s : z)$	generalized hypergeometric function
$\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)$	Elhaddad and Darus **[39]**
w	Schwarz function
$\tilde{\psi}_{\lambda,p}^{m}(v,\varrho,a_{1,}b_1,\Omega;A,B)$	class of analytic and multivalent functions
$\mathfrak{m}$	measure of the unit close interval [0, 1]
$f(z,.)$	$\mathfrak{m}$- integral on the close interval [0,1]

$D_q f(z)$	q-derivative of function
D	any simply connected domain
$T_{\eta,p,q}^{n,m}(v)$	the family of harmonic function
$\bar{T}_{\eta,p,q}^{n,m}(v)$	The subclass $T_{\eta,p,q}^{n,m}(v)$
$\mathfrak{G}_{\eta,p,q}^{n,m} f(z)$	the class of multivalent function

Introduction

Complex analysis is one of the branches of mathematics that investigates functions of complex numbers, also known as complexity. It has wide uses in applied mathematics and in many branches of mathematics. The primary interest in complex analysis is analytic functions with complex variables, also known as analysis functions.

Recently, there have been many studies regarding univalent and multivalent analytic functions, as well as univalent and multivalent harmonic functions. It is studying of the theory of the geometric function of the complex variable. Its relationship to various life sciences in general and mathematics in particular. It included several fields such as theoretical physics (heat equation), ordinary differential equations, partial differential equations (the theory of operators), partial calculus, differential dependencies, digital symmetry and the study of surfaces. As the study of the geometric properties of analytical functions plays an important role and contributes to new ideas and results.

The study and investigation began In the analytic functions in year Koebe (1910), where he proved the existence of a constant k so that the analytic function or univalent, the open disk image with the normalize condition $\{f(0) = f'(0) - 1 = 0\}$,must cover the disk ($\mathfrak{U} = \{z: |z| < k\}$).in (1916) the Bieberbach conjecture in which the famous coefficient conjecture originated. The coefficients of each $f \in \Lambda$ satisfy

$f(z) = z + a_2z^2 + a_3z^3 + \cdots$, then $|a_n| \le n$, $(n = 2,3,4, \dots)$.

The purpose of this work to give primitive motivation and background some basic concepts and fundamental theorem in contains and general theorem $\mathbb{C}$.

Chapter one (1.1) that deals with summary of some of the fundamental principle of complex concepts, definitions, examples and some figures.

(1.2) that deal with basic concepts by some lemma (1.3) that deal with by some theorem.

Chapter two is consist of two sections. The first section (2.1) study some properties of univalent function associated by differential operator see **[15]**.Some results obtained coefficient bound , growth and distortion, extreme points and hadamard product by subclass in this section is $\lambda(\sigma, \delta, \tau)$ where the main function is

$$f(z) = \mathrm{z} + \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0, n \in N, z \in \mathfrak{A}.$$

The second section(2.2) studies some properties of harmonic univalent function by differential operator see **[28].**Some results obtained coefficient bound**,** sense -preserving, convolution, extreme point, convex combinations, integral operator, neighborhood by subclass in this section is $\bar{T}^{m}_{\mu,\lambda,\sigma}(\alpha, \beta)$ where the main function is

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \bar{z}^n \quad |b_1| < 1.$$

Chapter three consist of four sections. The first section deals with definitions of subordination, dominant, best dominant and some lemma. The second section deals with properties related subordination on analytic multivalent functions. The third section deals with applications Related on subordination of multivalent functions with differential operator see **[39].** And the subclass in this section is $\tilde{\psi}^{m}_{\lambda,p}\left(v, \varrho, a_{1,} b_1, \Omega; A, B\right)$, where the main function is

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, z \in \mathfrak{U}, (p \in N = \{1,2, \dots .\}).$$

The forth section, studies some properties of harmonic multivalent function by differential operator see **[59].**Some results obtained coefficient bound**,** sense–preserving, convolution, extreme point, convex combinations, neighborhood and the subclass in this section is $\overline{T}_{\eta,p,q}^{n,m}(v)$ where the main function is.

$$f(z) = z^p - \sum_{k=2}^{\infty} |a_{k+p}| z^{k+p-1} + \sum_{k=1}^{\infty} |b_{k+p}| z^{k+p-1},$$

Publications

1. On the Class of Analytic and Univalent Functions Defined by Differential Operator, Iraqi Academics Syndicate International Conference for Pure and Applied Sciences, Journal of Physics: Conference Series 1818 (2021) 012184 IOP Publishing doi:10.1088/1742-6596/1818/1/012184.
2. Certain Subordination Properties on Analytic Multivalent Functions, sixth national scientific third international conference, University of Karbala, Publication Acceptance Letter.

Chapter One

Fundamental Definitions and Basic Concepts

Introduction

In the first part of this chapter, we have some aforementioned the necessary definitions, lemmas, examples, theorems and some results of univalent function, multivalent function, harmonic univalent function and harmonic multivalent function of analytic functions, which are needed in thesis for research. Some proofs and discussions can be found standard texts reference see **[1], [2], [3]** and **[4]** and other references.

1.1 Fundamental Definitions

Definition (1.1.1) [1]. Let w_0 be any point and f any function in complex plane, we say that f is analytic at a point w_0 if its derivative exists at w_0 and for each neighborhoods of w_0. In addition, we say that f is analytic in region $\mathbb{E}$ if it is analytic at every point in $\mathbb{E}$. If the function f is analytic at every point in complex plane $\mathbb{C}$, f is entire.

Definition (1.1.2)[1]. Let f analytic function in the open unit disk $\mathfrak{U} = \{z \in \mathbb{C}: |z| < 1\}$. We say that f is Univalent, if it does not take the same value more than once. In other words f is injective mapping of $\mathfrak{U}$ of any domain on complex plane.

If f takes the same value more than once, then we say that f is Multivalent in $\mathfrak{U}$.

As examples, the function $f(z) = z^3$ is univalent in $\mathfrak{U}$.but $f(z) = z^2$ is not univalent in $\mathfrak{U}$. Also $f(z) = z + \frac{z^{3n}}{3n}$ is univalent in $\mathfrak{U}$, for every positive integer n.

Example (1.1.1) [5]. Let $f(z) = (z+2)^2$is univalent in $\mathfrak{U}$.

Let $z_1, z_2 \in \mathfrak{U}$ and suppose $f(z_1) = f(z_2)$. Then

$(z_1 + 2)^2 = (z_2 + 2)^2$

$\Rightarrow {z_1}^2 + 4z_1 + 4 = {z_2}^2 + 4z_2 + 4$

$\Longrightarrow \quad {z_1}^2 - {z_2}^2 + 4(z_1 - z_2) = 0$

$\Longrightarrow (z_1 - z_2)(z_1 + z_2 + 4) = 0.$

Since $|z_1|, |z_2| < 1$, we know that $z_1 + z_2 + 4 \neq 0$ hence $z_1 - z_2 = 0$, or

$$z_1 = z_2.$$

But the function $f(z) = (1+z)^8$is not univalent in $\mathfrak{U}$.

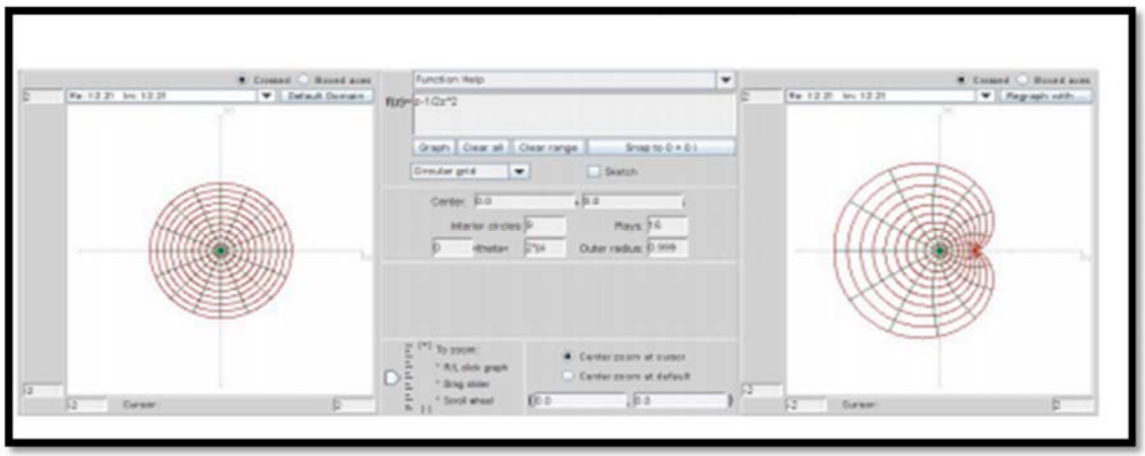

Figure (1.1.1): The Image of The Open Unit Disk Under The $f(z) = (1+z)^2$

Definition (1.1.3) [1]. Let f be function we say that is locally univalent at a point $w_0 \in \mathbb{C}$, when f is univalent in some neighborhood of w_0.For analytic function f the condition $f'(z) \neq 0$ is equivalent to local univalence at w_0.

In this work, we define a class Λ of a function as the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (a_k \geq 0, k \in \mathbb{N}) \qquad (1.1.1)$$

Which is analytic and univalent in $\mathfrak{U}$.

Definition (1.1.4) [1]. Let E any set in $\mathbb{C}$, we say that starlike with respect to $z_0 \in$ E if the line segment joining w_0 to every other point, $z \in E$ lies entirely in E ,while the set E is said to be convex if it is starlike with respect to each of its points.

Definition (1.1.5) [1]. Let $f \in \Lambda$ is any function we say that normalized if it satisfies the conditions $f(0) = 0$ $\& f'(0) = 1$.

Definition (1.1.6) [1]. Let f any function in complex plane we say that is conformal at a point w_0 if it preserves the angle between oriented curves passing through w_0 in magnitude as well as in sense. A function $w = f(z)$ is said to be conformal in the domain D, if it is conformal at each point of the domain. For analytic functions f, the condition $f'(w_0) \neq 0$ is equivalent to local univalent at w_0. An analytic univalent function is called a conformal mapping because of its angle-preserving property.

Definition (1.1.7) [1]. Let $f \in \Lambda$ be function, we say that in the class $s^*(\alpha)$, if satisfy the condition

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha\,, \quad \alpha \in [0,1);\ z \in \mathfrak{U}, f(z) \neq 0) \qquad (1.1.2)$$

Any elements in this class are called starlike function of order α .For example, the function

$$f(z) = \frac{3z}{(1-z)^{4(1-\alpha)}}\,,$$

is starlike of order α.

Definition (1.1.8) [1]. Let $f \in \Lambda$ be function we say that is convex function of order α if satisfies the condition

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha\,, \alpha \in [0,1); z \in \mathfrak{U}, f'(z) \neq 0) \qquad (1.1.3)$$

In this work, we denote the class of all convex functions of order α in $\mathfrak{U}$ by symbol $C(\alpha)$but the symbol C for the convex functions of order 0,$C(0) = C$.

Definition (1.1.9) [1]. Let $f \in \Lambda$ is function we say that is close-to-convex of order α $\{\alpha \in [0,1)\}$ if there is a convex function g satisfies the condition

$$Re\left\{\frac{zf'(z)}{g'(z)}\right\} > \alpha, \ \ (g'(z) \neq 0, z \in \mathfrak{U}) \qquad (1.1.4)$$

In this work we denote by the symbols by $K(\alpha)$, the class of close-to-convex functions of order α, f is normalized by the usual conditions $f(0) = f'(0) - 1 = 0$. By using argument, we can write the condition (1.1.4) as:-

$$\left| arg \frac{f'(z)}{g'(z)} \right| < \frac{\alpha\pi}{2}, \qquad \alpha > 0, z \in \mathfrak{U}. \qquad (1.1.5)$$

Look at $C(\alpha) \subset S^*(\alpha) \subset K(\alpha)$.

Definition (1.1.10) [6]. A function $f: \mathbb{C} \longrightarrow \mathbb{C}$ is Mobius transformation (a bilinear transformation) and it is rational function of the form $f(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$ are constant and $ad - bc \neq 0$.

Example (1.1.2) [1]. Let the function $f(z) = \frac{z}{1-z} \in \Lambda$.

$$\frac{1}{1-z} = \sum_{n=1}^{\infty} z^n,$$

When we multiply by z we get:

$$\frac{z}{1-z} = \sum_{n=0}^{\infty} z^n = z + z^2 + z^3 + \cdots \ .$$

Recall that this is the Mobius transformation that maps $\mathfrak{U}$ onto the right half-plane, whose boundary is the line $-\frac{1}{2} + ic$ where $c \in R$ (see Figure (1.1.2)):

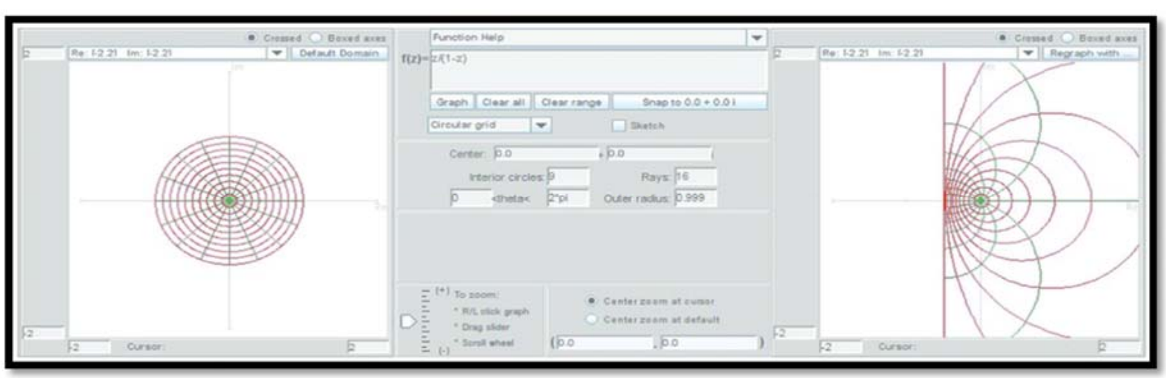

Figure (1.1.2): The Image of the Unit Disk Under The Analytic Right Half-Plane Map In Λ

Example (1.1.3) [1]. The koebe function of Λ is very important in this work and it is given by

$$Ko(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \cdots,$$

the maps in unit disk is to the complement of the ray $\left(-\infty, -\frac{1}{4}\right]$. This can be verified by writing

$$Ko(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4},$$

and noting that $\frac{1+z}{1-z}$ maps are the unit disk conformally onto the right half-plane $\{Re(z) > 0\}$; see Fig. (1.1.3).

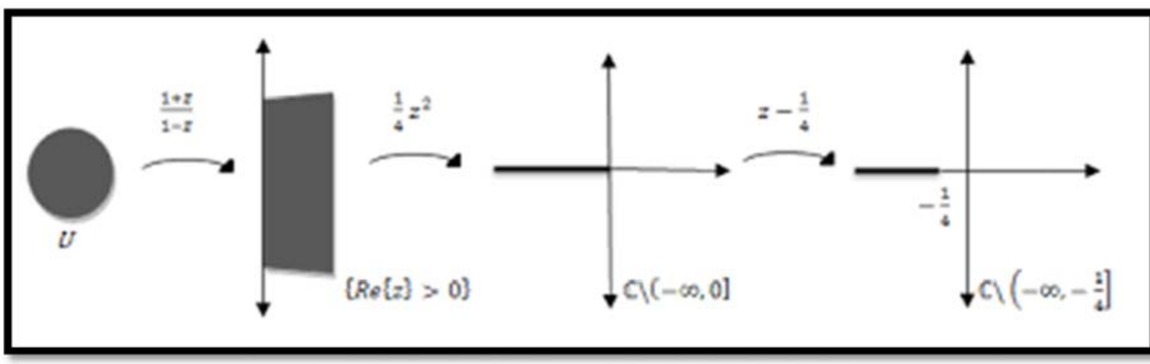

Figure (1.1.3): The Koebe Function Maps 𝔄 Conformally Onto $\mathbb{C}\setminus(-\infty, -\frac{1}{4}]$.

We note that $X_1(z) = \frac{1+z}{1-z}$, $X_2(z) = \frac{1}{4}x_1^2$, $X_3(z) = X_2(z) - \frac{1}{4}$.

Note $x_3 \circ x_2 \circ x_1(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4} = \frac{z}{(1-z)^2}$.

Also X_1 is the Mobius transformation that maps in 𝔄 on to right half-plane whose boundary is the imaginary axis. Also, X_2 is the raised to the power of four function, while X_3 translates the image one space to the left and then multiplies it by a factor of $\frac{1}{4}$.

Look at the Koebe function, which is starlike in this case, but note convex.

In this work we define a class $\mathfrak{A}(p)$ of a multivalent function of the from:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (z \in \mathfrak{A}, p \in N). \tag{1.1.6}$$

We say that f is multivalent starlike of order α, multivalent convex of order α and multivalent close-to-convex of order α, $(\alpha \in [0,p))$, respectively if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha\,, \qquad Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha\,, \qquad Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha\ .$$

Definition (1.1.11)[1]. The Saigo hypergeometric fractional integral operator order real number $\lambda(\lambda > 0)$ is defined for a function f by

$$I_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} f(t)\, {}_2F_1\left(\mu+\lambda, -\eta; \lambda; 1-\frac{t}{z}\right) dt \tag{1.1.7}$$

Where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $Re(z-t) > 0$.

Definition (1.1.12) [7]. The fractional integral of order real number $\lambda(\lambda > 0)$ is defined for a function f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{1.1.8}$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $Re(z-t) > 0$.

Definition (1.1.13) [7]. The fractional derivative of order $\lambda(0 \leq \lambda < 1)$of a function f is defined by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt, \qquad (1.1.9)$$

where f is an analytic function in Definition (1.1.14) and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $\log(z-t)$ to be real, when $Re(z-t) > 0$.

Definition (1.1.14) [8]. The Gaussian hypergeometric function denoted by ${}_2F_1(a, b; c; z)$ is define by

$${}_2F_1(a, b; c; z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} . \frac{z^j}{j!}, \qquad |z| < 1, \qquad (1.1.10)$$

where $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \ldots$, and $(v)_j$ denote the pochhammer symbol define, in terms of the Gamma function Γ, by

$$(v)_j = \frac{\Gamma(v + \mathrm{j})}{\Gamma(v)}$$

$$= \begin{cases} v(v+1) \ldots (\mathrm{v} + j - 1), & j = 1,2,3, \ldots . \\ 1, & j = 0 \end{cases} \qquad (1.1.11)$$

Definition (1.1.15)[1]. Radius of starlikeness of a function f is the largest $R_1, R_1 \in (0,1)$ for which it is starlike in $|z| < R_1$.

Definition (1.1.16) [1]. Radius of convexity of a function f is the largest $R_2, R_2 \in (0,1)$ for which it is convex in $|z| < R_2$

Definition (1.1.17)[7],[9]. The convolution (or Hadamard product) for function f and g denoted by $f * g$ is defined as following for the function in $\mathfrak{A}(p)$ respectively:

If $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$, then

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k, \qquad (1.1.12)$$

Example (1.1.4)[5]. Consider the convolution of the Koebe function $f(z) = \frac{z}{(1-z)^2}$ and the horizontal strip map, $h(z) = \frac{1}{2} log\left(\frac{1+z}{1-z}\right)$. To find the Hadamard $f(z) * h(z)$, we need to compute the Taylor series for h.

$$\log(1-z) = \int \frac{-1}{1-z} dz = -\int \sum_{k=0}^{\infty} z^k dz = \sum_{k=0}^{\infty} \frac{-1}{k+1} z^{k+1}.$$

Likewise, $\log(1+z) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{k+1} z^{k+1}$.

Hence

$$\frac{1}{2} log\left(\frac{1+z}{1-z}\right) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{k+1} z^{k+1} - \sum_{k=0}^{\infty} \frac{-1}{k+1} z^{k+1}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2k+1} z^{2k+1}.$$

Thus,

$$f(z) * h(z) = \frac{z}{(1-z)^2} * \frac{1}{2} log\left(\frac{1+z}{1-z}\right) = \sum_{k=1}^{\infty} kz^k * \sum_{k=0}^{\infty} \frac{1}{2k+1} 2^{2k+1}$$

$$= (z + 2z^2 + 3z^3 + 4z^4 + 5z^5 + \cdots) * \left(z + \frac{1}{3} z^3 + \frac{1}{5} z^5 + \cdots\right)$$

$$= z + z^3 + z^5 + \cdots$$

Since $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$, we have that $\frac{1}{1-z^2} = 1 + z^2 + z^4 + \cdots$

And $\frac{z}{1-z^2} = z + z^3 + z^5 + \cdots$. That is,

$$f(z) * h(z) = \frac{z}{(1-z)^2} * \frac{1}{2} log\left(\frac{1+z}{1-z}\right) = \frac{z}{1-z^2}.$$

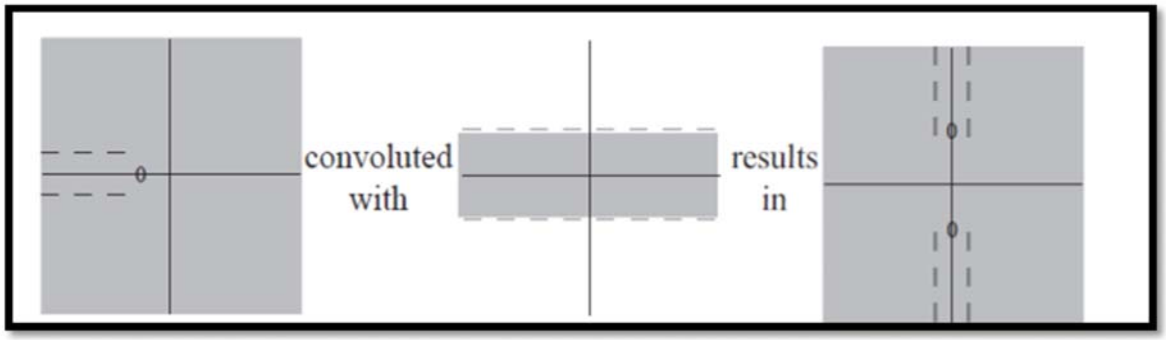

Figure (1.1.4): The Koebe Function Convoluted With A Horizontal Strip Map Yields A Double-Slit Map

Definition (1.1.18)[1]. The weighted mean $w_j$of the functions f and g are defined by

$$w_j(z) = \frac{1}{2}[(1-j)f(z) + (1+j)g(z)], \qquad 0 < j < 1.$$

Also

$$h(z) = \frac{1}{m}\sum_{k=1}^{m} f_k(z)$$

is the arithmetic mean of the functions $f_k(z)$(k=1, 2,… m)

Definition (1.1.19) [3]. Let E $\subset X$. A point $x \in E$ is called an extreme point of E if it has no representation of the form: $x = ty + (1-t)z, t \in (0,1)$ as a proper convex combination of two different points y and z in E.

Definition (1.1.20) [3]. Let X and E be as mentioned in Definition (1.1. 19). then the convex hull of E is the smallest convex set containing E and the closed convex hull of E is the smallest closed convex set containing E, it is the closure of the convex hull of E, we denote the closed convex hull of E by $\overline{co}$ E.

Definition (1.1.21) [10]. A continuous function $f = u + iv$ is a complex valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D.

If $f = u + iv$ be harmonic, then we can find analytic function G and H such that $u = Re\, G$ and $v = Im\, H$, thus $h + \bar{g} = \frac{G+H}{2} + \frac{\bar{G}-\bar{H}}{2}$, where h and g are analytic in D and we say that h is analytic part and g is co-analytic part of f.

Example (1.1.5)[5].The function $f(x, y) = u(x, y) + iv(x, y) =$ $(x^2 - y^2) + i2xy$

Is complex-valued harmonic because

$$u_{xx} + u_{yy} = 2 - 2 = 0 \quad , v_{xx} + v_{yy} = 0 + 0 = 0$$

Definition (1.1.22)[10]. The harmonic function $f = h + \bar{g}$ is sense-preserving and locally injective if $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0, z \in \mathfrak{A}$, where J_f denotes the Jacobian of f.If $f = h + \bar{g}$ is harmonic, sense-preserving and injective, then we say that f is harmonic univalent.

Let $H(\mathfrak{A})$ denotes the class of analytic function in the open unit disk $\mathfrak{A} = \{z \in \mathbb{C}: |z| < 1\}$. For $n \in N = \{1,2,3, \ldots\}$ and $a \in \mathbb{C}$, let $H[a, n] = \{f: f \in H(\mathfrak{A}) and\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$,

With $H_0 \equiv H[0,1]$and $H \equiv H[1,1]$.

And the necessary and sufficient condition for the harmonic function $f = h + \bar{g}$ is locally multivalent and sense-preserving such $|(h(z)')| < |(g(z)')|$ in D (where D is simply connected in $\mathbb{C}$). See [20].

Example (1.1.6) [5]. To show the image of $\mathfrak{A}$ under the harmonic function $f(z) = z + \frac{1}{2}\bar{z}^2$ enter this function in complex tools in the form $z + \frac{1}{2} conj(z^2)$(see Figure (1.1.5))

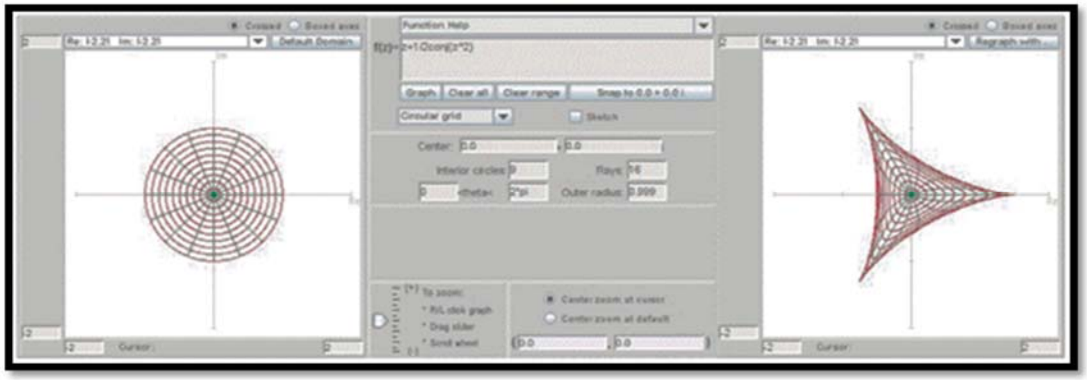

Figure (1.1.5): Image of $\mathfrak{A}$ Under The Harmonic Map.

Note that the harmonic function $f(z) = h(z) + \overline{g(z)}$ can also be written in the form $f(z) = Re\{h(z) + g(z)\} + i\, Im\{h(z) - g(z)\}$. Hence, in the previous example, $f(z) = z + \frac{1}{2}\bar{z}^2$ can also be written as $f(z) = Re\left\{z + \frac{1}{2}z^2\right\} + i\, Im\left\{z - \frac{1}{2}z^2\right\}$. In complex tool you can also enter the harmonic function in this form. To do so, you would type in $Re(z + 1/2z^2) + i\, Im\{z - 1\,/\,2z^2\}$.

Definition (1.1.23) [3]. let f and F be analytic in the unit disk $\mathfrak{A}$. Then F is said to be subordinate to f, written $F \prec f \text{ or } F(z) \prec f(z)$, if there exists a Schwarz function w, which is analytic in $\mathfrak{A}$ with $w(0) = 0$ and $|w(z)| < 1, z \in \mathfrak{A}$, such that $F(z) = f\big(w(z)\big), (z \in \mathfrak{A})$. Furthermore, if the function $F(z)$ is univalent in $\mathfrak{A}$, we have the following equivalence relationship holds true:

$$F(z) \prec f(z), (z \in \mathfrak{A}) \leftrightarrow F(0) = f(0) \text{and } F(\mathfrak{A}) \subset f(\mathfrak{A}).$$

1.2 Basic Results.

Lemma (1.2.1)[11]. Let $\alpha \geq 0, \beta \in [0,1]$. Then, $Re(w) \geq \alpha|w-1| + \beta$ if and only if $Re\left\{w\left(1+\alpha e^{i\theta}\right) - \alpha e^{i\theta}\right\} \geq \beta$, where w be any complex number.

Lemma (1.2.2) [11]. Let $\alpha \geq 0$. Then $Re(w) \geq \alpha$ if and only if $|w-(1+\alpha)| \leq |w+(1-\alpha)|$, where w be any complex number.

Lemma (1.2.3) [3]. Let q be a convex function in $\mathfrak{U}$ and let $h(z) = q(z) + n\alpha z q'(z)$ for $z \in \mathfrak{U}$, where $\alpha > 0$ and n is a positive integer.

If $p(z) = q(0) + p_k z^k + p_{k+1} z^{k+1} + \cdots, for\ z \in \mathfrak{U}$, is analytic in $\mathfrak{U}$ and $p(z) + \alpha z p'(z) \prec h(z)$, for $z \in \mathfrak{U}$, then $p(z) \prec q(z)$ and q is the best dominant.

Lemma (1.2.4) [12]. Let h be a convex function with $h(0) = a$ and $a > 0$. If $p(z) \in H[h(0), n] \cap Q, n \in N$ and if $p(z) + \alpha z p'(z)$ univalent in $\mathfrak{U}$ and $h(z) \prec p(z) + \alpha z p'(z)$, then $q(z) \prec p(z), z \in \mathfrak{U}$, the function q is convex and the best subordinant.

Lemma (1.2.5) [9]. Let q be univalent in the open unit disk $\mathfrak{U}$ and θ and φ be analytic in domain D containing $q(\mathfrak{U})$ with $\varphi(w) \neq 0$ when $w \in q(\mathfrak{U})$. Set $Q(z) = zq'(z)\varphi\left(q(z)\right), h(z) = \theta\left(q(z)\right) + \varphi(z)$. Suppose that

1- $Q(z)$ is starlike univalent in $\mathfrak{U}$ and

2- $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $\in \mathfrak{U}$.

If $\theta\left(p(z)\right) + zp'(z)\varphi\left(p(z)\right) \prec \theta\left(q(z)\right) + zq'(z)\varphi\left(q(z)\right)$, then $p(z) \prec q(z)$ and q is the best dominant.

Lemma (1.2.6) [13]. Let q be univalent in $\mathfrak{U}$. $\xi \in \mathbb{C} \setminus \{0\}$ And suppose that $Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > max\left\{0, -Re\left(\frac{1}{\xi}\right)\right\}$. If $p(z)$ is analytic in $\mathfrak{U}$, with $p(0) = q(0)$ and $p(z) + \xi z p'(z) \prec q(z) + \xi z q'(z)$, then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma (1.2.7) [14]. Let $q(z)$ be convex in $\mathfrak{A}$, $q(0) = a$ and $\xi \in \mathbb{C}, Re(\xi) > 0$. If $p \in H[a, 1]$ and $p(z) + \alpha zp'(z)$ is univalent in $\mathfrak{A}$, then $q(z) + \xi zq'(z) \prec p(z) + \xi zp'(z)$, implies $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

1.3 Basic Theorems

Theorem (1.3.1) [1].(Alexander′s theorem)

Let f be an analytic function in $\mathfrak{A}$ with $f(0) = f'(0) - 1 = 0$. Then, $f \in C$ If and only if $zf' \in S^*$.

Theorem (1.3.2) [1]. (Distortion theorem)

For each $f \in \Lambda$,

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, |z| = r < 1. \quad (1.19)$$

For each $z \in \mathfrak{A}, z \neq 0$ equality occurs if and only if f is a suitable rotation of the koebe function. We say upper and lower bounds for $|f'(z)|$ as distortion bounds.

Theorem (1.3. 3) [1]. (Growth theorem)

For each $f \in \Lambda$

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, |z| = r < 1. \quad (1.20)$$

For each $z \in \mathfrak{A}, z \neq 0$ equality occurs if and only if f is a suitable rotation of the Koebe function.

Theorem (1.3.4) [1]. (Bieberbach Conjecture)

The coefficients of each $f \in \Lambda$ satisfy $|a_n| \leq n$ for n=2, 3… the strict inequality holds for all n unless f is the Koebe function or one of its rotation. In other words

$f(z) = z + a_2z^2 + a_3z^3 + \cdots$ Then $|a_n| \leq n$, $(n = 2,3,4, \dots)$.

Theorem (1.3.5) [1]. (Littlewood′s theorem)

For the constant e, the coefficients of each $f \in \Lambda$ satisfy $|a_n| \leq en$ (for n=2, 3 …)

Theorem (1.3.6) [1]. (Maximum Modulus theorem)

Suppose that a function f is continuous on boundary of $\mathbb{E}$($\mathbb{E}$ and $\mathfrak{A}$ any disk or region). Then, the maximum value of $|f(z)|$, which is always reached, occurs somewhere on the boundary of $\mathbb{E}$ and never in the interior.

Chapter Two

Univalent Function and Harmonic Univalent Function

2.1 On the Class of Analytic and Univalent Functions Defined by Differential Operator

The main object of this section is to introduce a subclass $\boldsymbol{A}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{\tau})$ consisting of analytic and univalent functions in the open unit disk $\mathfrak{A}$. Denoted by Λ the class of analytic functions f is defined on the unit disk $\mathfrak{A}$ with normalization $f(0) = 0$ and $f(0)' = 1$. Such a function has the Taylor series expansion about the origin in the form.

$$f(z) = \mathrm{z} + \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0, n \in N, z \in \mathfrak{A}. \qquad (2.1.1)$$

Let S denote the subclass of Λ consisting of functions that are univalent in $\mathfrak{A}$. For $f \in \Lambda$ given by (2.1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, b_n \geq 0, n \in \mathbb{N}, z \in \mathfrak{A}. \qquad (2.1.2)$$

The convolution (or Hadamard product), denoted by$(f * g)$, is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in \mathfrak{A}) \quad (2.1.3)$$

Note that $f * g \in \Lambda$.

For,$f \in \Lambda$, Elhaddad et. al. **[15]**, introduced the following differential operator:

$$D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z) = z + \sum_{n=2}^{\infty} [1 + ([n]_q - 1)\lambda]^m \frac{\Gamma_q(\delta)(q^{\Upsilon};q)_{n-1}}{\Gamma_q(\sigma(n-1)+\delta)(q;q)_{n-1}} a_n z^n$$

$where, 0 < q < 1, n, m \in \mathbb{N}, \sigma, \delta, \gamma > 0, 0 \leq \lambda \leq 1$ and $z \in \mathfrak{A}$.

Note that:

- If. $q \to 1$ and $\Upsilon = 1,$ we obtained the operator defined in **[16].**
- If, $q \to 1, \sigma = 0, \Upsilon = 0$ *and* $\delta = 1$,we obtained Al-Oboudi operator, see Ref. **[17].**

- If $q \to 1, \sigma = 0, \Upsilon = 1, \delta = 1 and\ \lambda = 1$, we obtained Salagean operator, see Ref. **[18].**
- If $q \to 1, m = 0\ and\ \Upsilon = 1$, we obtained $\mathbb{E}_{\sigma,\delta}(z)$, see Ref. **[19].**

Let $\psi_{\lambda}^{\Upsilon}(\sigma,\delta) = [1 + ([n]_q - 1)\lambda]^m \frac{\Gamma_q(\delta)(q^{\Upsilon};q)_{n-1}}{\Gamma_q(\sigma(n-1)+\delta)(q;q)_{n-1}}$. Then we have

$$D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z) = z + \sum_{n=2}^{\infty} \psi_{\lambda}^{\Upsilon}(\sigma,\delta)\, a_n z^n.$$

By making use of the differential operator, $D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)$, we introduce the class of analytic and univalent functions as illustrated below:-

Definition (2.1.1). For $\tau > 1$, the function *f* given by (2.1.1), is said to be in the class $\Lambda(\sigma,\delta,\tau)$, if and only if satisfies the following condition:

$$\left| \frac{\left(1 + \frac{z\left(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)\right)''}{\left(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)\right)'}\right) + 1}{2\left(1 + \frac{z\left(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)\right)''}{\left(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)\right)'}\right) + 2} \right| < \tau. \qquad (2.1.4)$$

Several authors have studied various classes of analytic and univalent Functions with coefficients, see Refs. **([20], [21], [22], [23], [24], [18], [25], [26] and [27])**. In this work, we investigate and study the class $\Lambda(\sigma,\delta,\tau)$, of analytic and univalent functions. Also, several properties like, coefficient estimate, growth and distortion theorem, extreme points, convolution property, convex linear combination and inclusive properties for functions in our class are obtained.

In the following theorem, we fined the coefficient bounds for function f in the class $\Lambda(\sigma,\delta,\tau)$.

Coefficient Estimate:-

Theorem (2.1.1). The function f of the form (2.1.1) be in the class $\Lambda(\sigma,\delta,\tau)$, if and only if

$$\sum_{n=2}^{\infty}\psi_\lambda^\Upsilon(\sigma,\delta)\mathrm{n}(\mathrm{n}+1+2\mathrm{n}\tau)\,a_n \le 2\,(\tau-1),\qquad \tau>1. \qquad (2.1.5)$$

Proof. Let f in the class $\Lambda(\sigma,\delta,\tau)$, then f satisfies the inequality (2.1.4), which is equivalent to:

$$\left|\frac{2\left(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)\right)'+z\left(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)\right)''}{2\left(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)\right)'+2z\left(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)\right)''}\right|<\tau$$

$$\left|\frac{2+\sum_{\mathrm{n}=2}^{\infty}\psi_\lambda^\Upsilon(\sigma,\delta)2\,a_n\mathrm{n}\mathrm{z}^{\mathrm{n}-1}+\left[\sum_{n=2}^{\infty}\psi_\lambda^\Upsilon(\sigma,\delta)n(n-1)a_{\mathrm{n}}z^{n-1}\right]}{2+\left[\sum_{n=2}^{\infty}\psi_\lambda^\Upsilon(\sigma,\delta)2na_{\mathrm{n}}z^{n-1}\right]+\left[\sum_{n=2}^{\infty}\psi_\lambda^\Upsilon(\sigma,\delta)2na_{\mathrm{n}}(n-1)z^{n-1}\right]}\right|<\tau$$

$$\left|\frac{2+\sum_{\mathrm{n}=2}^{\infty}\psi_\lambda^\Upsilon(\sigma,\delta)\,a_n\mathrm{z}^{\mathrm{n}-1}\mathrm{n}(\mathrm{n}+1)}{2+\sum_{\mathrm{n}=2}^{\infty}\psi_\lambda^\Upsilon(\sigma,\delta)\,a_n\mathrm{z}^{\mathrm{n}-1}2\,\mathrm{n}^2}\right|<\tau,$$

Since, $|Re(z)|\le|z|$, for all z, we have

$$\mathrm{Re}\left\{\frac{2+\sum_{\mathrm{n}=2}^{\infty}\psi_\lambda^\Upsilon(\sigma,\delta)\,a_n\mathrm{z}^{\mathrm{n}-1}\mathrm{n}(\mathrm{n}+1)}{2+\sum_{\mathrm{n}=2}^{\infty}\psi_\lambda^\Upsilon(\sigma,\delta)\,a_n\mathrm{z}^{\mathrm{n}-1}2\,\mathrm{n}^2}\right\}<\tau.$$

Then by choosing the value of z on the real axis and letting z→1 through values, we get:

$$2+\sum_{\mathrm{n}=2}^{\infty}\psi_\lambda^\Upsilon(\sigma,\delta)\,a_n\mathrm{n}(\mathrm{n}+1)<\,2\tau+\sum_{\mathrm{n}=2}^{\infty}\psi_\lambda^\Upsilon(\sigma,\delta)\,\,a_n\tau[2\mathrm{n}^2]$$

hence,

$$\sum_{\mathrm{n}=2}^{\infty}\,\psi_\lambda^\Upsilon(\sigma,\delta)\mathrm{n}(\mathrm{n}+1+2\mathrm{n}\tau)\,a_n<2(\tau-1).$$

Conversely, we assume that (2.1.5) satisfies and |z|=1, then:

$$\left|2\left(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)\right)'+z(D_\lambda^m(\alpha,\beta,\mu)\mathrm{f}(z))''\right|$$
$$-\tau\left|2\left(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)\right)'+2z\left(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)\right)''\right|$$

$$\left|2\left[1+\sum_{n=2}^{\infty}\psi_{\lambda}^{\Upsilon}(\sigma,\delta)\, na_{n}z^{n-1}\right]+z\left[[\sum_{n=2}^{\infty}\psi_{\lambda}^{\Upsilon}(\sigma,\delta)\, n(n-1)a_{n}z^{n-2}]\right]\right|$$

$$-\tau\left|2\left[1+\sum_{n=2}^{\infty}\psi_{\lambda}^{\Upsilon}(\sigma,\delta)\, na_{n}z^{n-1}\right]+2z\left[\sum_{n=2}^{\infty}\psi_{\lambda}^{\Upsilon}(\sigma,\delta)\, n(n-1)a_{n}z^{n-2}\right]\right|$$

$$\left|2+\left|\sum_{n=2}^{\infty}\psi_{\lambda}^{\Upsilon}(\sigma,\delta)\; a_{n}z^{n-1}[n(n+1)]\right|\right|$$

$$-\tau\left|2+\sum_{n=2}^{\infty}\psi_{\lambda}^{\Upsilon}(\sigma,\delta)\; a_{n}z^{n-1}2n^{2})\right|$$

$$\leq 2+\sum_{n=2}^{\infty}\psi_{\lambda}^{\Upsilon}(\sigma,\delta)\, n(n+1)a_{n}-\tau[2+\sum_{n=2}^{\infty}\psi_{\lambda}^{\Upsilon}(\sigma,\delta)\, 2n^{2}a_{n}]$$

$$=\sum_{n=2}^{\infty}\psi_{\lambda}^{\Upsilon}(\sigma,\delta)\, n(n+1+2n\tau)\, a_{n}-2(\tau-1)$$

$$\leq 0. \text{ (by hypothesis)}$$

Then by Maximum modulus theorem, we have $f\in\Lambda(\sigma,\delta,\tau)$. this complete the proof

The result is a sharp for the function

$$f(z)=z+\frac{2(\tau-1)}{\psi_{\lambda}^{\Upsilon}(\sigma,\delta)n(n+1+2n\tau)}z^{n}.\ (n\geq 2)$$

Corollary (2.1.1). Let the function f be in the class $\Lambda(\sigma,\delta,\tau)$,then

$$a_{n}\leq\frac{2(\tau-1)}{\psi_{\lambda}^{\Upsilon}(\sigma,\delta)n(n+1+2n\tau)}\,, \qquad (n\geq 2)$$

Growth and Distortion theorem:-

Theorem (2.1.2). If f an analytic function given by (2.1.1) is in the class $\Lambda(\sigma,\delta,\tau)$, then we have

$$|z|-\frac{2(\tau-1)}{\psi_{\lambda}^{\Upsilon}(\sigma,\delta)2[3+4\tau]}|z|^{2}\leq|f(z)|\leq|z|+\frac{2(\tau-1)}{\psi_{\lambda}^{\Upsilon}(\sigma,\delta)2[3+4\tau]}|z|^{2}.$$

The bounds are sharp, since the equality are attained by the function

$$f(z) = z + \frac{2(\tau - 1)}{\psi_\lambda^\Upsilon(\sigma,\delta)2[3+4\tau]} z^2$$

Proof. In view of Theorem (2.1.1), we have

$$\sum_{n=2}^{\infty} \left[\psi_\lambda^\Upsilon(\sigma,\delta)n[n+1+2n\tau]\right] a_n < 2(\tau-1),$$

and,

$$\psi_\lambda^\Upsilon(\sigma,\delta)2[3+4\tau] \sum_{n=2}^{\infty} a_n \le \sum_{n=2}^{\infty} [\psi_\lambda^\Upsilon(\sigma,\delta)n(n+1+2n\tau)]a_n \le 2(\tau-1).$$

Therefore, we have

$$\sum_{n=2}^{\infty} a_n \le \frac{2(\tau-1)}{\psi_\lambda^\Upsilon(\sigma,\delta)2[3+4\tau]}$$

Thus, for $f \in \Lambda(\sigma,\delta,\tau)$,we obtain

$$|f(z)| = \left|z + \sum_{n=2}^{\infty} a_n z^n\right|$$

$$\le |z| + \sum_{n=2}^{\infty} a_n |z|^2$$

$$\le |z| + \frac{2(\tau-1)}{\psi_\lambda^\Upsilon(\sigma,\delta)2[3+4\tau]} |z|^2$$

The other assertion can be proved as follows

$$|f(z)| = \left|z + \sum_{n=2}^{\infty} a_n z^n\right|$$

$$\ge |z| - \sum_{n=2}^{\infty} a_n |z|^2$$

$$\ge |z| - \frac{2(\tau-1)}{\psi_\lambda^\Upsilon(\sigma,\delta)2[3+4\tau]} |z|^2$$

This completes the proof.

Similarly, following the same method in Theorem (2.1.2), we can prove the following

Theorem (2.1.3): If f an analytic function given by (2.1.1) is in the class $\Lambda(\sigma, \delta, \tau)$,we have

$$1-\frac{2(\tau-1)}{\psi_\lambda^\Upsilon(\sigma,\delta)2[3+4\tau]}|z| \leq |f'(z)| \leq 1+\frac{2(\tau-1)}{\psi_\lambda^\Upsilon(\sigma,\delta)2[3+4\tau]}|z|$$

The bounds are sharp, since the equality are attained by the function

$$f(z)=z+\frac{2(\tau-1)}{\psi_\lambda^\Upsilon(\sigma,\delta)2[3+4\tau]}z^2 \qquad n \geq 2$$

Proof. For $f \in \Lambda(\sigma, \delta, \tau)$, we have

$$|f'(z)| = \left|1-\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \leq 1+2\sum_{n=2}^{\infty} a_n |z|$$

$$\leq 1+\frac{2(\tau-1)}{\psi_\lambda^\Upsilon(\sigma,\delta)n[n+1+2n\tau]}|z|$$

This completes the proof.

Extreme points:-

Theorem (2.1.4). Let $f_1(z)=z$ and

$$f_n(z)=z+\frac{2(\tau-1)}{\psi_\lambda^\Upsilon(\sigma,\delta)n[n+1+2n\tau]}z^n, \qquad n \geq 2$$

then, $f \in \Lambda(\sigma, \delta, \tau)$ if and only if it can be expressed in the form:-

$$f(z)=\sum_{n=2}^{\infty} v_n f_n(z),$$

where,

$$v_n \geq 0 \quad and \quad \sum_{n=1}^{\infty} v_n = 1.$$

Proof. We have $f(z)=\sum_{n=1}^{\infty} v_n f_n(z)$ where $v_n \geq 0$ and $\sum_{n=1}^{\infty} v_n = 1$

$$f(z)=\sum\nolimits_{n=1}^{\infty} v_n f_n(z)$$

$$= v_1 f_1(z) + \sum_{n=2}^{\infty} v_k f_k(z)$$

$$= v_1 f_1(z) + \sum_{n=2}^{\infty} v_n \left\{ z + \frac{2(\tau - 1)}{\psi_\lambda^\Upsilon(\sigma,\delta) n[n+1+2n\tau]} z^n \right\}$$

$$= v_1 z + \sum_{n=2}^{\infty} v_n z + \sum_{n=2}^{\infty} v_n \frac{2(\tau - 1)}{\psi_\lambda^\Upsilon(\sigma,\delta) \mathrm{n}[n+1+2nt]} z^n$$

$$= z - \sum_{n=1}^{\infty} v_n z + \sum_{n=2}^{\infty} v_n \frac{(\tau - 2)}{\psi_\lambda^\Upsilon(\sigma,\delta) \mathrm{n}[n+1+2nt]} z^n$$

$$= \sum_{n=2}^{\infty} v_n \frac{2(\tau - 1)}{\psi_\lambda^\Upsilon(\sigma,\delta) \mathrm{n}[n+1+2nt]} z^n$$

Then, we have

$$\sum_{n=2}^{\infty} \left[\psi_\lambda^\Upsilon(\sigma,\delta)[n+1+2nt]\right] - 2(\tau - 1) \left\{ v_n \frac{2(\tau - 1)}{\psi_\lambda^\Upsilon(\sigma,\delta) \mathrm{n}[n+1+2n\tau]} \right\}$$

$$= \sum_{n=2}^{\infty} v_n (\tau - 2) = (\tau - 2) \left(\sum_{n=1}^{\infty} v_n - v_1 \right) = (1 - v_1) \leq (\tau - 1).$$

The condition (2.1.5) is satisfied. Thus, $f \in \Lambda(\sigma, \delta, \tau)$.

Conversely, we suppose that $f \in \Lambda(\sigma, \delta, \tau)$, since

$$a_n \leq \frac{2(\tau - 1)}{\phi_{\mathrm{n}}(\alpha,\beta,\mu,\tau)\mathrm{n}[n+1+2n\tau]}, \qquad n \geq 2$$

$$\mathrm{v}_n = \frac{\phi_{\mathrm{n}}(\alpha,\beta,\mu,\tau)\mathrm{n}[n+1+2n\tau]}{2(\tau - 1)} a_n, \qquad \mathrm{v}_1 = 1 - \sum_{n=2}^{\infty} v_n.$$

Then,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

$$= z + \sum_{n=2}^{\infty} v_n \frac{2(\tau - 1)}{\psi_\lambda^\Upsilon(\sigma,\delta) \mathrm{n}[n+1-\tau(2n-1)]} z^n$$

$$= \left(1 - \sum_{n=2}^{\infty} v_n\right) z + \sum_{n=2}^{\infty} v_n f_n(z)$$

$$= v_1 f_1(z) + \sum_{n=2}^{\infty} \mathrm{v}_n \mathrm{f}_n(z)$$

$$= \sum_{n=1}^{\infty} \mathrm{v}_n f_n(z).$$

This completes the assertion of Theorem (2.1.4).

Closure theorems:-

Theorem (2.1.5). Let $v_i \geq 0$ for $i = 1,2 \dots l$ and $\sum_{i=1}^{l} v_i = 1$. If the function f_i defined by

$$f_i(z) = z + \sum_{n=1}^{\infty} a_{n,i} z^n, \qquad (a_{i,n} \geq o, i = 1,2, \dots, l) \tag{2.1.6}$$

are in the class $\Lambda(\sigma, \delta, \tau)$ for every $i = 1,2, \dots, l$ then the function $h(z)$ defined by

$$h(z) = z + \sum_{n=2}^{\infty} \left(\sum_{i=1}^{l} v_i a_{n,i}\right) z^n$$

In the class $\Lambda(\sigma, \delta, \tau)$.

Proof. Because,$f_i \in A(\sigma, \delta, \tau)$, for every $i = 1,2, \dots, l$ we have

$$\sum_{\mathrm{n}=2}^{\infty} \left[\psi_\lambda^\Upsilon(\sigma, \delta)\mathrm{n}[\mathrm{n} + 1 + 2\mathrm{n}\tau]\right] a_{k,i} \leq 2(\tau - 1)$$

Hence, we get

$$\sum_{n=2}^{\infty} \{\psi_\lambda^\Upsilon(\sigma, \delta)\mathrm{n}[\mathrm{n} + 1 + 2\mathrm{n}\tau]\} \left(\sum_{i=1}^{l} v_i a_{k,i}\right)$$

$$= \sum_{i=1}^{l} v_i \left(\sum_{\mathrm{n}=2}^{\infty} \left[\psi_\lambda^\Upsilon(\sigma, \delta)\mathrm{n}[\mathrm{n} + 1 + 2\mathrm{n}\tau]\right] a_{k,i}\right)$$

$$\leq 2(\tau - 1) \sum_{i=1}^{l} v_i \leq 2(\tau - 1).$$

This implies that a function h is in the class $\Lambda(\sigma,\delta,\tau)$thus, the proof of the theorem is complete.

Corollary (2.1.2). The class $\Lambda(\sigma,\delta,\tau)$ is closed under convex linear combination.

Proof. Assume that the functions $f_i(i=1,2)$ given by (2.1.6) are in $\Lambda(\sigma,\delta,\tau)$. It suffices to show that the function h defined by

$$h(z)=cf_1(z)+(1-c)f_2(z), \qquad (0\le c\le 1)$$

is in the class $A(\sigma,\delta,\tau)$. By taking $l=c$, $v_1=c$ and $v_2=1-c$, in Theorem (2.1.7), we obtain the corollary.

Theorem (2.1.6). Let $g(z)$ of the form:-

$$g(z)=z+\sum_{n=2}^{\infty} b_n z^n$$

be analytic in $\mathfrak{A}$. If $f\in A(\sigma,\delta,\tau)$ then, the function $f*g$ is also in the class $\Lambda(\sigma,\delta,\tau)$ here the symbol "*" denoted Hadamard product (or convolution).

Proof. Since$f\in\Lambda(\sigma,\delta,\tau)$, we have

$$\sum_{n=2}^{\infty}\left[\psi_\lambda^\Upsilon(\sigma,\delta)\mathrm{n}[\mathrm{n}+1+2\mathrm{nt}]\right]\mathrm{a_n}<2(\tau-1)$$

By utilizing the last inequality and the fact that

$$f(z)*g(z)=z+\sum_{n=2}^{\infty} a_n b_n z^n,$$

we, obtain

$$\sum_{n=2}^{\infty}\left[\psi_\lambda^\Upsilon(\sigma,\delta)\mathrm{n}[\mathrm{n}+1+2\mathrm{nt}]\right]\mathrm{a_n}b_n\le\sum_{n=2}^{\infty}\left[\psi_\lambda^\Upsilon(\sigma,\delta)\mathrm{n}[\mathrm{n}+1+2\mathrm{nt}]\right]\mathrm{a_n}$$
$$<2(\tau-1)$$

Hence, in view of Theorem (2.1.1), the result follows the above object regarding convolution property.

2.2 Certain Properties of a New Subclass of Harmonic Functions

Let $\mathfrak{A}$ be the open unit disk in the complex plane $\mathbb{C}$.Suppose that S_H represents the class of functions in $\mathfrak{A}$ which is normalized by $f(0) = f(z)' - 1 = 0\, with\ f = h + \bar{g} \in S_H$ If f and sense-preserving within $\mathfrak{A}$, where h and g and the class $\mathcal{A}$ is of all analytic functions in $\mathfrak{A}$ as the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , g(z) = \sum_{n=1}^{\infty} b_n z^n \quad |b_1| < 1. \qquad (2.2.1)$$

We say that h an analytic part and g is a co-analytic part, if the S_H is reduce to S of every normalization function analytic univalent, if the co-analytic part of f is equal to zero. See **[10]**.

A necessary and sufficient condition for the harmonic function $f = h + \bar{g}$ is locally univalent and sense-preserving such $|(g(z)')| < |(f(z)')|$ in D (where D is simply connected in $\mathbb{C}$).

[28]. introduces new generalized differential operator $D^m_{\mu,\lambda,\sigma}(\alpha,\beta)$defined on $\mathfrak{A}$. and $f: \mathcal{A} \to \mathcal{A}$ by

$$D^m_{\mu,\lambda,\sigma}(\alpha,\beta) = z + \sum_{z=2}^{\infty} \left[\frac{\mu + \lambda + (n-1)(\lambda - \alpha)(\beta - \sigma)}{\mu + \lambda}\right]^m a_n\, z^n \qquad (2.2.2)$$

$$m \in \mathbb{N}_0 = \{0,1,2,\dots\}, \alpha, \sigma \geq 0, \beta, \lambda, \mu > 0, \lambda \neq \alpha$$

In this operator is reduced to several interesting and differential operators considered earlier for different choices of $\mu, \lambda, \sigma, \alpha\ and\ \beta$as

1. $D^m_{1-\lambda,\lambda,\sigma}(\alpha,\beta) f(z) = z + \sum_{n=2}^{\infty}[1 + (n-1)(\lambda - \alpha)(\beta - \sigma)]^m a_n\, z^n$ was introduced and studied by Ramadan and Darus **[29]**;
2. $D^m_{1-\lambda,0,}(\alpha,\beta) f(z) = z + \sum_{n=2}^{\infty}[1 + (n-1)(\lambda - \alpha)]^m a_n\, z^n$, was introduced and studied by darus and Ibrahem **[30]**

3. $D^m_{\mu,\lambda,0}(0,1)f(z) = z + \sum_{n=2}^{\infty}\left[\frac{\mu+\lambda n}{\mu+\lambda})\right]^m a_n\, z^n$ was introduced and studied by swany **[31]**

4. $D^m_{1-\lambda,0,}(0,1)f(z) = z + \sum_{n=2}^{\infty}[1+(n-1)\lambda]^m a_n\, z^n$ was introduced and studied by AL-oboudi **[17]**

5. $D^m_{0,1,0}(0,1)f(z) = z + \sum_{n=2}^{\infty} n^m a_n\, z^n$ was introduced and studied by salagean **[25].**

Also denote by T_s the subclass of S_H consisting of all functions $f = h + \bar{g}$ where h and g are given by

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \quad , g(z) = \sum_{n=1}^{\infty} |b_n| z^n \qquad |b_1| < 1. \qquad (2.2.3)$$

In this paper, we use the operator of (2.2.2) of harmonic function $f = h + \bar{g}$

$$D^m_{\mu,\lambda,\sigma}(\alpha,\beta)f(z) = D^m_{\mu,\lambda,\sigma}(\alpha,\beta)h(z) + \overline{D^m_{\mu,\lambda,\sigma}(\alpha,\beta)g(z)}$$

Where

$$D^m_{\mu,\lambda,\sigma}(\alpha,\beta)h(z) = z + \sum_{n=2}^{\infty}\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m a_n\, z^n \quad (2.2.4)$$

$$\overline{D^m_{\mu,\lambda,\sigma}(\alpha,\beta)g(z)} = \sum_{n=1}^{\infty}\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m b_n\, z^n \quad (2.2.5)$$

[32]. we denote $\bar{T}^m_{\mu,\lambda,\sigma}(\alpha,\beta)$ subclass of $T^m_{\mu,\lambda,\sigma}(\alpha,\beta)$ where $T_s \cap T^m_{\mu,\lambda,\sigma}(\alpha,\beta) = \bar{T}^m_{\mu,\lambda,\sigma}(\alpha,\beta)$.

Using the operator, $D^m_{\mu,\lambda,\sigma}(\alpha,\beta)$ we introduce the class of harmonic univalent function as follow.

Definition (2.2.1). For $f = h + \bar{g}$ given by (2.2.1), let $\bar{T}^{m}_{\mu,\lambda,\sigma}(\alpha,\beta)$ denote the family of univalent harmonic functions satisfying the following

$$Re\left\{\frac{\left(D^{m}_{\mu,\lambda,\sigma}(\alpha,\beta)h(z)\right)' + \left(D^{m}_{\mu,\lambda,\sigma}(\alpha,\beta)g(z)\right)'}{z}\right\} \geq \alpha \ \ \forall z \in \mathbb{C} - \{0\} \qquad (2.2.6)$$

$m \in \mathbb{N}_0 = \{0,1,2,\dots\}, \sigma \geq 0, \alpha, \beta, \lambda, \mu > 0, \lambda \neq \alpha$

Coefficient Estimate:-

First, we begin the sufficient coefficient condition for functions f in $T^{m}_{\mu,\lambda,\sigma}(\alpha,\beta)$

Theorem (2.2.1). Let $f = h + \bar{g}$, where $h(z)\ and\ g(z)$ are defined by (2.2.1). If

$$\sum_{n=2}^{\infty}\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^{m}|a_n| + \sum_{n=1}^{\infty}\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^{m}|b_n| \leq \frac{2(\alpha+1)}{n} \qquad (2.2.7)$$

then f is sense -preserving, univalent in, and $f \in T^{m}_{\mu,\lambda,\sigma}(\alpha,\beta)$

Proof. Let $\emptyset(z) = \left(\frac{\left(D^{m}_{\mu,\lambda,\sigma}(\alpha,\beta)h(z)\right)' + \left(\overline{D^{m}_{\mu,\lambda,\sigma}(\alpha,\beta)g(z)}\right)'}{z}\right)$ using the fact that $R(z) \geq \alpha$ if and only if

$$|1-\alpha+\emptyset(z)| \geq |1+\alpha-\emptyset(z)| \qquad (2.2.8)$$

It suffices to show that

$$\left|1-\alpha+\left(\frac{\left(D^m_{\mu,\lambda,\sigma}(\alpha,\beta)h(z)\right)'+\left(\overline{D^m_{\mu,\lambda,\sigma}(\alpha,\beta)g(z)}\right)'}{z}\right)\right| - \left|1+\alpha - \left(\frac{\left(D^m_{\mu,\lambda,\sigma}(\alpha,\beta)h(z)\right)'+\left(\overline{D^m_{\mu,\lambda,\sigma}(\alpha,\beta)g(z)}\right)'}{z}\right)\right| \geq 0 \tag{2.2.9}$$

Substituting for h and g in (2.2.4) and (2.2.5) yields

$$= \left|1-\alpha+\frac{\left(\begin{array}{c}1-\sum_{n=2}^{\infty} n\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^m a_n z^{n-1}+\\ \sum_{n=1}^{\infty} n\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^m b_n \bar{z}^{n-1}\end{array}\right)}{z}\right| - \left|1+\alpha-\frac{\left(\begin{array}{c}1-\sum_{n=2}^{\infty} n\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^m a_n z^{n-1}+\\ \sum_{n=1}^{\infty} n\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^m b_n \bar{z}^{n-1}\end{array}\right)}{z}\right| \geq 0 \tag{2.2.10}$$

$$\geq 2\left\{-\alpha-\frac{1}{|z|}\right.$$

$$\left.-\left[\frac{\left(\sum_{n=2}^{\infty} n\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^{m}|a_n||z|^{n-1}+\sum_{n=1}^{\infty} n\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^{m}|b_n||\bar{z}|^{n-1}\right)}{|z|}\right]\right\} \quad (2.2.11)$$

$$\sum_{n=2}^{\infty}\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^{m}|a_n| + \sum_{n=1}^{\infty}\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^{m}|b_n| \leq \frac{2(\alpha+1)}{n} \quad (2.2.12)$$

The proof of theorem is complete.

Theorem (2.2.2). Let $f(z)=h(z)+\bar{g}(z)\in S_H$, where h and g be of the form (2.2.3) then $f\in \bar{T}^{m}_{\mu,\lambda,\sigma}(\alpha,\beta)$if and only if

$$\sum_{n=2}^{\infty}\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^{m}|a_n| + \sum_{n=1}^{\infty}\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^{m}|b_n| \leq \frac{2(\alpha+1)}{n}$$

Proof. Since $\bar{T}^m_{\mu,\lambda,\sigma}(\alpha,\beta) \subset T^m_{\mu,\lambda,\sigma}(\alpha,\beta)$, we only need to prove the "only if" part of the theorem. To this end for function $f(z)$ of the form (2.2.3), we notice that the condition (2.2.6) is equivalent to

$$= Re\left\{\frac{\left(D^m_{\mu,\lambda,\sigma}(\alpha,\beta)h(z)\right)' + \left(D^m_{\mu,\lambda,\sigma}(\alpha,\beta)h(z)\right)'}{z} - \alpha\right\} \geq 0$$

$$Re\left\{\frac{\left(\begin{array}{c}1-\sum_{n=2}^{\infty} n\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |a_n|z^{n-1} - \\ \sum_{n=1}^{\infty} n\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |b_n|\bar{z}^{n-1} - \alpha z\end{array}\right)}{z}\right\} \geq 0 \qquad (2.2.13)$$

If we choose z to be real and let $z \to 1^-$ we get

$$\frac{\left(\begin{array}{c}2(\alpha+1)-\sum_{n=2}^{\infty} n\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |a_n|z^{n-1} - \\ \sum_{n=1}^{\infty} n\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |b_n|\bar{z}^{n-1}\end{array}\right)}{z} \geq 0$$

The above condition (2.2.13) must hold for all values of z on the positive real axis, where $0 < |z| < 1$ we must have

$$2(\alpha+1) - \sum_{n=2}^{\infty} n\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m a_n - \sum_{n=1}^{\infty} n\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m b_n \geq 0$$

The proof is complete.

Convolution (Hadamard product):-

In this section, we show that the class $\bar{T}^m_{\mu,\lambda,\sigma}(\alpha,\beta)$ is close under convolution. For harmonic function

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \bar{z}^n \quad \text{and} \quad F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| z^n$$

The convolution of $f(z)$ and $F(z)$ is given by

$$(f * F)(z) = f(z) * F(z) = z - \sum_{n=2}^{\infty} |a_n A_n| z^n + \sum_{n=1}^{\infty} |b_n B_n| z^n \qquad (2.2.14)$$

Theorem (2.2.3). For $0 \le \Gamma \le \alpha < 1$ let $f(z) \in \bar{T}_{\mu,\lambda,\sigma}^{m}(\alpha, \beta)$ and $F(z) \in \bar{T}_{\mu,\lambda,\sigma}^{m}(\Gamma, \beta)$.then $f * F \in \bar{T}_{\mu,\lambda,\sigma}^{m}(\alpha, \beta) \subset \bar{T}_{\mu,\lambda,\sigma}^{m}(\Gamma, \beta)$

Proof. We need to show that the coefficient $f * F$ satisfies the required condition given in theorem (2.2.2) for $F(z) \in \bar{T}_{\mu,\lambda,\sigma}^{m}(\Gamma, \beta)$. We note that $|A_n| \le 1$ and $|B_n| \le 1$ now, for the convolution function $f * F$ we obtain

$$\sum_{n=2}^{\infty} \frac{\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |a_n|\,|A_n|}{\frac{2(\alpha+1)}{n}} + \frac{\sum_{n=1}^{\infty}\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |b_n||B_n|}{\frac{2(\alpha+1)}{n}}$$

$$\le \sum_{n=2}^{\infty} \frac{\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |a_n|}{\frac{2(\alpha+1)}{n}} + \frac{\sum_{n=1}^{\infty}\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |b_n|}{\frac{2(\alpha+1)}{n}} \le 1$$

Since $0 \le \Gamma \le \alpha < 1$ and $f(z) \in \bar{T}_{\mu,\lambda,\sigma}^{m}(\alpha, \beta)$ therefore $f * F \in \bar{T}_{\mu,\lambda,\sigma}^{m}(\alpha, \beta) \subset \bar{T}_{\mu,\lambda,\sigma}^{m}(\Gamma, \beta)$

Convex combinations:-

In this section, we show that the class $\bar{T}^{m}_{\mu,\lambda,\sigma}(\alpha,\beta)$ is closed under convex combination of its members. Let the functions $f_{k_i}(z)$ be defined for *i =1,2,….,m* by

$$f_{k_i}(z) = z - \sum_{n=2}^{\infty} |a_{n,i}| z^n \quad + \sum_{n=1}^{\infty} |b_n| z^n \qquad (2.2.15)$$

Theorem (2.2.4). Let the function $f_{k_i}(z)$, defined by (2.2.15) be in the class $\bar{T}^{m}_{\mu,\lambda,\sigma}(\alpha,\beta)$, for every i *=1, 2… m*. Then the functions $t_i(z)$ defined by

$$t_i(z) = \sum_{i=1}^{\infty} C_i\, f_{k_i}(z), 0 \le C_i \le 1$$

Are also in the class $\bar{T}^{m}_{\mu,\lambda,\sigma}(\alpha,\beta)$, where $\sum_{i=1}^{\infty} C_i = 1$

Proof. From the definition of $t_i(z)$, we can write

$$t_i(z) = z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} C_i |a_{n,i}|\right) z^n + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} C_i |b_{n,i}|\right) \bar{z}^n$$

Further, since $f_{k_i}(z)$ is in $\bar{T}^{m}_{\mu,\lambda,\sigma}(\alpha,\beta)$ for every i=1, 2… m, then

$$\sum_{n=2}^{\infty}\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m \left(\sum_{i=1}^{\infty} C_i |a_{n,i}|\right)$$

$$+\sum_{n=1}^{\infty}\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m \sum_{i=1}^{\infty} C_i |b_{n,i}|$$

$$= \sum_{i=1}^{\infty} C_i \left[\sum_{n=2}^{\infty}\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m C_i |a_{n,i}|\right.$$

$$\left.+\sum_{n=1}^{\infty}\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m C_i |b_{n,i}|\right]$$

$$\leq \sum_{i=1}^{\infty} C_i \frac{2(\alpha+1)}{n} \leq \frac{2(\alpha+1)}{n}.$$

Integral Operator:-

We check the closure quality of the class $\bar{H}^m_{\mu,\lambda,\sigma}(\alpha,\beta)$ by circular Bernardi- Libera-Livingston integral $T_u(f)$ **[33, 34].** that is given by

$$T_u(f) = \frac{u+1}{z^u} \int_0^2 t^{u-1} f(t)dt, u > -1 \qquad (2.2.16)$$

Theorem (2.2.5). Suppose that $f \in \bar{T}^m_{\mu,\lambda,\sigma}(\alpha,\beta)$ therefore $\bar{H}^m_{\mu,\lambda,\sigma}(\alpha,\beta) \in \bar{T}^m_{\mu,\lambda,\sigma}(\alpha,\beta)$

Proof. By definition of $T_u(f_k(z))$ is defined by (2.2.16) as follows:

$$T_u(f(z)) = \frac{u+1}{z^u} \int_0^2 t^{u-1} (t - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \bar{z}^n)dt,$$

$$= z - \sum_{n=2}^{\infty} \frac{u+1}{u+n} |a_n| z^n + \sum_{n=1}^{\infty} \frac{u+1}{u+n} |b_n| \bar{z}^n$$

$$z - \sum_{n=2}^{\infty} D_n z^n + \sum_{n=1}^{\infty} L_n \bar{z}^n$$

Wherever

$$D_n = \frac{u+1}{u+n} |a_n| and\ L_n = \frac{u+1}{u+n} |b_n|$$

Therefore

$$\sum_{n=2}^{\infty} \frac{\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m \frac{u+1}{u+n}|a_n|}{\frac{2(\alpha+1)}{n}}$$

$$+ \frac{\sum_{n=1}^{\infty} \left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m \frac{u+1}{u+n}|b_n|}{\frac{2(\alpha+1)}{n}}$$

$$\leq \sum_{n=2}^{\infty} \frac{\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |a_n|}{\frac{2(\alpha+1)}{n}} + \frac{\sum_{n=1}^{\infty}\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |b_n|}{\frac{2(\alpha+1)}{n}} \leq 1$$

From theorem 2.2.4

Hence, we have $\bar{H}^m_{\mu,\lambda,\sigma}(\alpha,\beta) \in \bar{T}^m_{\mu,\lambda,\sigma}(\alpha,\beta)$

Extreme Point:

In this section, we obtain the extreme points for the class $\bar{T}^m_{\mu,\lambda,\sigma}(\alpha,\beta)$

Theorem (2.2.6). Let $f(z)$ given by (2.2.3) then $f(z) \in \bar{T}^m_{\mu,\lambda,\sigma}(\alpha,\beta)$ if and only if

$$f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n(z))$$

$$h_1(z) = z, h_n(z) = z - \left(\frac{\frac{2(\alpha+1)}{n}}{\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m}\right) z^n, \ n = 2,3,\ldots$$

$$g_n(z) = z + \left(\frac{\frac{2(\alpha+1)}{n}}{\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m}\right) \bar{z}^n, n = 1,2,3,\ldots \, and \sum_{n=1}^{\infty}(X_n + Y_n) = 1, X_n \geq 0, Y_n \geq 0$$

Proof.

$$f(z)=\sum_{n=1}^{\infty}\left(X_n h_n+Y_n g_n(z)\right)$$

$$=\sum_{n=2}^{\infty}X_n\left\{z-\left(\frac{\frac{2(\alpha+1)}{n}}{\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^m}\right)z^n\right\}$$

$$+\sum_{n=1}^{\infty}Y_n\left\{z+\left(\frac{\frac{2(\alpha+1)}{n}}{\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^m}\right)\bar{z}^n\right\}$$

$$=z-\sum_{n=2}^{\infty}\left(\frac{\frac{2(\alpha+1)}{n}}{\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^m}\right)X_n\,z^n$$

$$+\sum_{n=1}^{\infty}\left(\frac{\frac{2(\alpha+1)}{n}}{\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^m}\right)Y_n\bar{z}^n$$

Therefore

$$\sum_{n=2}^{\infty}\frac{\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^m|a_n|}{\frac{2(\alpha+1)}{n}}$$

$$+\frac{\sum_{n=1}^{\infty}\left[\frac{\left(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\right)}{\mu+\lambda}\right]^m|b_n|}{\frac{2(\alpha+1)}{n}}$$

$$= \sum_{n=2}^{\infty} X_{\boldsymbol{n}} + \sum_{\boldsymbol{n=1}}^{\infty} Y_n$$

$= \sum_{n=2}^{\infty} X_{\boldsymbol{n}} + \sum_{\boldsymbol{n=1}}^{\infty} Y_n - X_1 = 1 - X_1 \leq 1$ *and also* $f(z) \in \bar{T}_{\mu,\lambda,\sigma}^{m}(\alpha, \beta)$

Conversely, suppose that $f(z) \in \bar{T}_{\mu,\lambda,\sigma}^{m}(\alpha, \beta)$, Setting

$$X_n = \frac{\left[\frac{\left(\mu + \lambda + (n-1)(\lambda - \alpha)(\beta - \sigma)\right)}{\mu + \lambda}\right]^m |a_n|}{\frac{2(\alpha+1)}{n}}, 0 \leq X_n \leq 1, \quad n = 2,3, \ldots.$$

$$Y_n = \frac{\left[\frac{\left(\mu + \lambda + (n-1)(\lambda - \alpha)(\beta - \sigma)\right)}{\mu + \lambda}\right]^m |b_n|}{\frac{2(\alpha+1)}{n}}, 0 \leq X_n \leq 1, \quad n = 1,2,3, \ldots.$$

And

$$X_1 = 1 - \left(\sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n\right).$$

Therefore $f(z)$ can be written as

$$f(z) = z - \sum\nolimits_{n=2}^{\infty} |a_n| z^n + \sum\nolimits_{n=1}^{\infty} |b_n| \bar{z}^n$$

$$= \boldsymbol{z} - \sum_{n=2}^{\infty} \left(\frac{\frac{2(\alpha+1)}{n}}{\left[\frac{\left(\mu + \lambda + (n-1)(\lambda - \alpha)(\beta - \sigma)\right)}{\mu + \lambda}\right]^m} \right) X_n z^n$$

$$+ \sum_{n=1}^{\infty} \left(\frac{\frac{2(\alpha+1)}{n}}{\left[\frac{\left(\mu + \lambda + (n-1)(\lambda - \alpha)(\beta - \sigma)\right)}{\mu + \lambda}\right]^m} \right) Y_n \bar{z}^n$$

$$= h_1(z) X_1 + \sum_{n=2}^{\infty} h_n(z) X_n + \sum\nolimits_{n=1}^{\infty} g_n(z) Y_n$$

$$= \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_n(z)).$$

The proof is complete.

Neighborhood:-

In this section we obtain the Neighborhood for the class $\overline{T}^{m}_{\mu,\lambda,\sigma}(\alpha, \beta)$. **[35** and **36]**, define the δ-Neighborhood of the functions (2.2.1) by

$$N_\delta(f) = \left\{ F = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n z^n} : \sum_{n=2}^{\infty} n(|a_n - A_n| + |b_n - B_n|) + |b_1 - B_1| \le \delta \right\}. \quad (2.2.17)$$

We define the $m - \delta$-Neighborhood of the function f as follows

$$N_\delta^m(f) = \left\{ F = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n z^n} : \sum_{n=2}^{\infty} (|a_n - A_n| + |b_n - B_n|) + |b_1 - B_1| \le \delta \right\} \quad (2.2.18)$$

In this case we define $m - \delta$-Neighborhood of f to be the set

$$N_\delta^m(f) = \left\{F: \sum_{n=2}^{\infty}\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |a_n - A_n| + \left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |b_n - B_n| + (\alpha+1)|b_1 - B_1| \le (\alpha+1)\delta\right\} \qquad (2.2.19)$$

Theorem (2.2.7). Let f be given by (2.2.1) satisfies the conditions

$$\sum_{n=2}^{\infty}\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m + \sum_{n=1}^{\infty}\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m \le (\alpha+1) \qquad (2.2.19)$$

where $0 \le \alpha < 1, \delta \le \frac{2(\alpha+1)}{n}$

Then $N_\delta^m(f) \subset \bar{T}_{\mu,\lambda,\sigma}^m(\alpha,\beta)$

Proof. Let f satisfies the condition (7.4) and let

$$\boldsymbol{F(z) = z + \overline{B_1 z} + \sum_{n=2}^{\infty}(A_n z^n + B_n z^n)} \qquad (2.2.20)$$

Belong to $N_\delta^m(f)$ we have

$$(\alpha+1)|B_1| + \sum_{n=2}^{\infty}\left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |A_n| + \left[\frac{(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma))}{\mu+\lambda}\right]^m |B_n| \le (\alpha+1)|b_1 - B_1| + (\alpha+1)|b_1|$$

$$+\sum_{n=2}^{\infty}\left[\frac{\big(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\big)}{\mu+\lambda}\right]^{m}|A_n-a_n|$$

$$+\left[\frac{\big(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\big)}{\mu+\lambda}\right]^{m}|B_n-b_n|$$

$$+\sum_{n=2}^{\infty}\left[\frac{\big(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\big)}{\mu+\lambda}\right]^{m}|A_n|$$

$$+\left[\frac{\big(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\big)}{\mu+\lambda}\right]^{m}|B_n|$$

$$\leq(\alpha+1)\delta+(\alpha+1)|b_1|$$

$$+\sum_{n=2}^{\infty}\left[\frac{\big(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\big)}{\mu+\lambda}\right]^{m}|a_n|$$

$$+\left[\frac{\big(\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)\big)}{\mu+\lambda}\right]^{m}|b_n|$$

$$\leq(\alpha+1)$$

Hence

$$\delta\leq\frac{2(\alpha+1)}{n}$$

$$N_{\delta}^{m}(f)\subset\bar{T}_{\mu,\lambda,\sigma}^{m}(\alpha,\beta)$$

Chapter Three

Multivalent Function and Harmonic Multivalent Function

3.1 Certain Subordination Properties on Analytic Multivalent Functions

In this section, we introduce a new subclass of multivalent functions based on the generalized deferential operator. Use the notation $\mathfrak{A}(p)$ to represent the class of all analytic multivalent functions which are define in the open unit disc $\mathfrak{A}$, and have the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}\,, z \in \mathfrak{U}, (p \in N = \{1,2, \dots .\}) \qquad (3.1.1)$$

The Hadamard product (convolution) of a function f which is defined in (3.1.1), and function g of the form

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \quad z \in \mathfrak{A} \qquad (3.1.2)$$

is define by

$$f(z) * g(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}. \qquad (3.1.3)$$

In 1999 the authors in **[37].**Defined the p -valent function $x_p(a_{1,} \dots . a_r, b_1, \dots . b_s : z)$, which is defined by generalized hypergeometric function as following:

$$X_p(a_1, \dots, a_r, b_1, \dots, b_s; z)$$

$$= z^p + \sum_{j=p+1}^{\infty} \frac{\prod_{i=1}^{r} (a_i)_{j-p}\, z^j}{\prod_{n=1}^{s} (b_n)_{j-p}\, (j-p)!}\,, \quad p \in \mathbb{N} \qquad (3.1.4)$$

where

$a_i \in \mathbb{C}, b_n \in \mathbb{C} \setminus \{0, -1, -2, \dots\}, \{i = 1, \dots r, n = 1, \dots s), and\ r \le s + 1; r, s \in \mathbb{N}_0$, and the pochammer symbol $(v)_j$, is defined by

$$(v)_j = \frac{\Gamma(v + \mathrm{j})}{\Gamma(v)} = \begin{cases} v(v+1) \dots (\mathrm{v} + j - 1), & j = 1,2,3, \dots . \\ 1, & j = 0 \end{cases} \qquad (3.1.5)$$

In **[38, chapter 4]**. Note that for the Gauss hypergeometric function ${}_2F_1$the next relations holds.

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt = \frac{\Gamma(\mathrm{b})\Gamma(\mathrm{c}-\mathrm{b})}{\Gamma(\mathrm{c})} 2F_1(a,b;c;z), \tag{3.1.6}$$

where ${}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$, (3.1.7)

and

$$ {}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z) \tag{3.1.8}$$

Using the Equation (3.1.4), Elhaddad and Darus **[49]**, introduces the following operator:

$$\widetilde{\mathfrak{D}}^m_{\lambda,p}(v,\varrho,a_1,b_1)f(z) = z^p + \sum_{j=\mathrm{P}+1}^{\infty} \left[\frac{[p+(j-p)\lambda]}{p}\right]^m * \frac{\Upsilon_{(j-p,v,\varrho)}(a_r,b_s)}{(j-p)!} a_j z^j, \tag{3.1.9}$$

where, $m \in N_0 = \mathbb{N} \cup \{0\}, \lambda \geq 0, p \in \mathbb{N}$, and $s,r,\varrho,v \in \mathbb{C}$, where $a_i \in \mathbb{C}, b_n \in \mathbb{C}\,\{0,-1,-2,\dots\}$, $Re(v) > 0$, $Re(\varrho) > 0$, and

$$\Upsilon_{(j-p,v,\varrho)}(a_{r,}b_s) = \frac{\Gamma(\varrho)}{\Gamma(v(j-p)+\varrho)}\left(\frac{\prod_{i=1}^r (a_i)_{j-p}}{\prod_{n=1}^s (b_i)_{j-p}}\right) \tag{3.1.10}$$

It is worthy to note the relationship of the above operator with the rest of the previously found operators

- **i.** For $s = 0, r = 1, a_1 = 1\ and\ p = 1$, we get the operator studied by Elhaddad et al. **[16], [40].**
- **ii.** For $s = 0, r = 1, a_1 = 1, v = 0, \varrho = 1\ and\ p = 1$, we get Al-Oboudi operator **[17].**
- **iii.** For $s = 0, r = 1, a_1 = 1, v = 0, \varrho = 1, \lambda = 1 and\ p = 1$, we get sălăgean operator **[25].**
- **iv.** For $s = 0, r = 1, a_1 = 1, m = 0\ and\ p = 1$, we get $E_{\alpha,\beta}(z)$.**[27]**

v. For m =0, $v = 0$ *and* $\varrho = 1$, we get the operator studied by Dozik and Srivastava **[37].**

vi. For $m = 0, v = 0, \varrho = 1, r = 1, s = 0, a_1 = \delta + 1$ *and* $p = 1$, we get the Ruscheweyh presented operator **[41]**.

vii. For $m = 0, v = 0, \varrho = 1, r = 2, s = 1$ *and* $p = 1$, we get the operator which is given by Hohlov **[42]**.

viii. For $m = 0, v = 0, \varrho = 1, r = 2, s = 1, a_2 = 1$ *and* $p = 1$, we get the operator that is given by Carlson and Shaffer **[43].**

We can see from (3.1.5) and for all $f \in \mathfrak{A}(p)$ and ζ we have

$$\zeta z\left(\widetilde{\mathcal{D}}_{\lambda,p}^{m}(v, \varrho, a_{1,} b_1)f(z)\right)' = p\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v, \varrho, a_1, b_1)f(z) - p(1-\zeta)\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v, \varrho, a_1, b_1)f(z) \quad (3.1.11)$$

$$z\left(\widetilde{\mathcal{D}}_{\lambda,p}^{m}(v, \varrho, a_{1,} b_1)f(z)\right)' = (p + \eta - \lambda)\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v, \varrho, a_1, b_1)f(z) - (\eta - \lambda)\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v, \varrho, a_1, b_1)f(z) \quad (3.1.12)$$

and

$$z\left(\widetilde{\mathcal{D}}_{\lambda,p}^{m}(v, \varrho, a_{1,} b_1)f(z)\right)' = (\delta + p)\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v, \varrho, a_1, b_1)f(z) - \delta\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v, \varrho, a_1, b_1).$$

For two analytic functions f and g we say that f subordinates g, and written as $f \prec g$, if there exists a function w (Schwarz function), which is analytic in $\mathfrak{A}$, with $w(0) = 0$, and $|w(z)| < 1, z \in \mathfrak{A}$, where $f(z) = g\big(w(z)\big)$. About more if g is multivalent functions in $\mathfrak{A}$, then we have the equivalence below (see **[44, 45, 46]**).

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{and } f(\mathfrak{A}) \subset g(\mathfrak{A}).$$

By using the deferential operator $\widetilde{\mathcal{D}}_{\lambda,p}^{m}(v, \varrho, a_{1,} b_1)f(z)$,with $m \in N_{0=}\mathbb{N} \cup \{0\}, \lambda, \geq 0, p \in \mathbb{N}$,

$\delta > -p, s, r, \varrho, v \in \mathbb{C}, where\ a_i \in \mathbb{C}, b_n \in \mathbb{C}\backslash \{0, -1, -2, \ldots\}, Re(v) > 0, Re(\varrho) > 0$

and $\Upsilon_{(j-p,v,e)}(a_{r,}b_s)$, is given by

$$\Upsilon_{(j-p,v,\varrho)}(a_{r,}b_s) = \frac{\Gamma(\varrho)}{\Gamma(v(j-p)+\varrho)}\left(\frac{\prod_{i=1}^{r}(a_i)_{j-p}}{\prod_{n=1}^{s}(b_i)_{j-p}}\right),$$

We introduce the following class of multivalent functions.

Definition (3.1.1). Let A and B be any fixed parameters ,with $-1 \le B < A \le 1$, we say that the function $f \in \mathfrak{A}(p)$ is in the class $\tilde{\psi}_{\lambda,p}^{m}(v,\varrho,a_{1,}b_1,\Omega;A,B)$, if it satisfies the subordination condition

$$\frac{1}{p-\Omega}\left(\frac{z\left(\tilde{\mathcal{D}}_{\lambda,p}^{m}(v,\varrho,a_{1,}b_1)f(z)\right)'}{\tilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)} - \Omega\right)$$

$$\prec \frac{1+Az}{1+Bz} \qquad (0 \le \Omega < p),$$

Which is equivalent to,

$$\left|\frac{\dfrac{z\left(\tilde{\mathcal{D}}_{\lambda,p}^{m}(v,\varrho,a_{1,}b_1)f(z)\right)'}{\left(\tilde{\mathcal{D}}_{\lambda,p}^{m}(v,\varrho,a_{1,}b_1)f(z)\right)} - p}{B\dfrac{z\left(\tilde{\mathcal{D}}_{\lambda,p}^{m}(v,\varrho,a_{1,}b_1)f(z)\right)'}{z\left(\tilde{\mathcal{D}}_{\lambda,p}^{m}(v,\varrho,a_{1,}b_1)f(z)\right)} - [pB+(A-B)(P-\Omega)]}\right| < 1 \text{ and } z \in \mathfrak{A}.$$

Remark (3.1.1). Below are the cases for different parameters:

i- $\tilde{\psi}_{\lambda,p}^{m}(v,\varrho,a_{1,}b_1,\Omega;1,-1) =: \tilde{\psi}_{\lambda,p}^{m}(v,\varrho,a_{1,}b_1,\Omega) =$

$$\left\{f \in \mathfrak{A}(\mathrm{p}): Re\frac{z\left(\tilde{\mathcal{D}}_{\lambda,p}^{m}(v,\varrho,a_{1,}b_1)f(z)\right)'}{\tilde{\mathcal{D}}_{\lambda,p}^{m}(v,\varrho,a_{1,}b_1)f(z)} > \Omega, z \in \mathfrak{A}\right\}$$

ii- $\tilde{\psi}_{p,p}^{0}(v,\varrho,a_{1,}b_1,\Omega;A,B) =: \tilde{\psi}^{*}{}_{p}(\Omega;\mathrm{A},\mathrm{B}) = \Big\{f(z) \in$ $\mathfrak{A}(\mathrm{p}): \frac{1}{p-\Omega}\left(\frac{zf(z)'}{f(z)} - \Omega\right) \prec \frac{1+Az}{1+Bz}\Big\}$

And

$$\tilde{\psi}_{p,p}^{0}(v,\varrho,a_{1,}b_1,\Omega;1,-1) =: \tilde{\psi}_{p}^{*}(\Omega) = \left\{f \in \mathfrak{A}(p): Re\frac{zf'(z)}{f(z)} > \Omega, z \in \mathfrak{A}\right\}$$

iii- $\tilde{\psi}^{0}_{1,1}\left(v,\varrho,a_{1,}b_1,\Omega;A,B\right)=\widetilde{\psi^*}(\Omega,A,B)=\Big\{f\in \mathfrak{A}:\frac{1}{p-\Omega}\left(\frac{zf\prime(z)}{f(z)}-\Omega\right)\prec\frac{1+Az}{1+Bz}\Big\}$

and

$$\tilde{\psi}^{0,}_{1,1}(\Omega;1,-1)=:\widetilde{\Psi^*}(\Omega)=\left\{f\in\ \mathfrak{A}:Re\frac{zf'(z)}{f(z)}>\Omega,z\in\mathfrak{A}\right\}.$$

We begin the first-order differential subordination in

$$\mathbb{H}\big(\mathcal{O}(z),z\mathcal{O}'(z)\big)\prec \hbar(z).$$

Let q be multivalent function and $\mathcal{O}(z)$ is analytic function, we say that q is dominant if $\mathcal{O}(z)\prec q(z)$, and satisfies the differentiation subordination. A dominant g is refers to the best dominant if g (z) $\prec$ $q(z)$for every dominants q and it is an implementation, indicate in the Sources to **[48, 49].**

Lemma (3.1.1). [47]. Let that $\hbar\in\mathfrak{A}$ and $\hbar(0)=1$ be a convex function, and let the function ϖ of the formula

$\varpi(z)=1+c_nz^n+c_{n+1}z^{n+1}+c_{n+2}z^{n+2}+\cdots$ be analytic in $\mathfrak{A}$. Hence,

$$\varpi(z)+\frac{z\varpi'(z)}{y}\prec\hbar(z)\qquad (Re\ y\geq 0\ and\ y\neq 0)\qquad (3.1.13)$$

implies, $\varpi(z)\prec\Theta(z)=\frac{y}{n}z^{\frac{-y}{n}}\int_0^z t^{\frac{y}{n}-1}\,\hbar(t)dt\prec\hbar(z)$, and the parameter Θ is the best dominant of (3.1.13).

Lemma (3.1.2) [48]. The measure of the unit close interval [0, 1] is positive and is denoted by $\mathbb{m}$. And suppose that the function $\hbar$ $(z,t)\in\mathbb{C}$ and is defined on $[0,1]\times\mathfrak{A}\ \ni\hbar(.,t)$ belong to $\mathfrak{A}$ $\forall\, t\in[0,1]$ and such that $f(z,.)$ is $\mathbb{m}$- integral on the close interval $[0,1]$, $\forall z\in\mathfrak{A}$. Moreover, let Re $\hbar(z,t)>0,\hbar(-r,t)\in\mathbb{R}$ such that,

$$Re\frac{1}{\hbar(z,t)}\geq\frac{1}{\mathrm{g}(-r,t)},\forall\ |z|\leq r<1,t\in[0,1].$$

If the function $\mathbb{H}$ is defined by

$$\mathbb{H}(z)=\int_0^1\hbar(z,t)d\mathbb{m}(t),$$

then

$Re\,\frac{1}{\mathbb{H}(z)} \geq \frac{1}{\mathbb{H}(-r)}$, $\forall\, |z| \leq r < 1.$

Lemma (3.1.3). [50]. Let $0 \neq \lambda \in \mathbb{R}$, $\frac{a}{\lambda} > 0, \beta \in (0,1)$ be an analytic function in $\mathfrak{A}$, of the form

$$P(z) = 1 + a_n z^{n+1} + \cdots, \quad z \in \mathfrak{A}$$

together,

$$p(z) \prec 1 + Rz \text{ and } R = \frac{aM}{n\lambda + a} ,$$

such that,

$\mathrm{M} = \mathrm{M_n}(\lambda, \Omega, \beta)$

$$= (1 - \beta)|\lambda|\left(1 + \frac{n\lambda}{a}\right).\left[|1 - \lambda + \lambda\beta| + \sqrt{1 + \left(1 + \frac{n\lambda}{a}\right)^2}\right]^{-1}$$

in $\mathfrak{A}$ when p is analytic function of the form $p(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots, z \in \mathfrak{A}$, and satisfies the subordination

$$p(z)\big[1 - \lambda + \lambda\big((1 - \beta)p(z) + \beta\big)\big] \prec 1 + Mz,$$

then $Re\, p(z) > 0, \forall\, z \in \mathfrak{A}$.

It follows lemma is particular case of result **[45, corollary 3.2.].**

Lemma (3.1.4) If $-1 \leq B < A < 1,\ \beta > 0,$ and the y a complex number satisfies:

$$Re\ y\ \geq -\frac{(\beta - \beta A)}{1 - B},$$

and the following differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + y} = \frac{1 + Az}{1 + Bz}, with \quad q(0) = 1,$$

has solution to univalent in $\mathfrak{A}$. Hence,

$$q(z) = \begin{cases} \dfrac{z^{\beta+y}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+y-1}(1+Bt)^{\beta(A-B)/B}dt} - \dfrac{y}{\beta}, & if\ B \neq 0 \\ \dfrac{z^{\beta+y}\exp(\beta Az)}{\beta\int_0^z t^{\beta+y-1}\exp(\beta At)dt} - \dfrac{y}{\beta}, & if\ B = 0. \end{cases}$$

Moreover, the function ϖ a subordination if it is analytic in $\mathfrak{U}$ and satisfies the next

$$\varpi\ (z) + \frac{z\varpi\ '(z)}{\beta\varpi\ (z) + z} \prec \frac{1+Az}{1+Bz} \qquad (3.1.14)$$

then

$$\varpi\ (z) \prec q(z) \prec \frac{1+Az}{1+Bz}$$

and q is the best dominant of (3.1.14).see**[45].**

3.2 Properties Related Subordination on Analytic Multivalent Functions

In thesis we see otherwise mentioned when $-1 \leq B < A \leq 1, p \in \mathbb{N}, m \in N_{0=}\mathbb{N} \cup \{0\}, \lambda, \zeta \geq 0, s, r, \varrho, v \in \mathbb{C}$, where $a_i \in \mathbb{C}, b_n \in \mathbb{C}\backslash \{0, -1, -2, \dots\}$

and

$$\Upsilon_{(j-p,v,e)}(a_{r,}b_s) \text{is given by } \Upsilon_{(j-p,v,\varrho)}(a_{r,}b_s) = \frac{\Gamma(\varrho)}{\Gamma(v(j-p)+\varrho)}\left(\frac{\prod_{i=1}^{r}(a_i)_{j-p}}{\prod_{n=1}^{s}(b_i)_{j-p}}\right),$$

Theorem (3.2.1). If $f \in \mathfrak{A}(p)$ and the subordination condition satisfies:

$$(1-\theta)\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}{z^p} + \theta\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m+1}(v,\varrho,a_1,b_1)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \tag{3.2.1}$$

Then

$$\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}{z^p} \prec \Theta(z) \prec \frac{1+Az}{1+Bz}, \tag{3.2.2}$$

if the function Θ given by

$$\Theta(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)\frac{1}{(1+Bz)}\,{}_2F_1\left(1,1;\frac{p}{\theta\zeta}+1;\frac{Bz}{1+Bz}\right), if\ B \neq 0, \\ 1 + \frac{p}{p+\theta\zeta}Az, \qquad , if\ B = 0 \end{cases}$$

is best dominant of (3.2.1), over and above

$$M < Re\left\{\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(x)}{z^p}\right\}, z \in \mathfrak{A} \tag{3.2.3}$$

where

M

$$= \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)\left[\frac{1}{1+B}\right] {}_2F_1\left(1,1; \frac{p}{\theta\zeta} + 1; \frac{B}{B-1}\right), & if\ B \neq 0, \\ 1 - \frac{p}{p+\theta\zeta} A, & , if\ B = 0. \end{cases}$$

Therefore (3.2.3) has estimate is best possible.

Proof. We have

$$\varpi\ (z) = \frac{\widetilde{\mathfrak{D}}^m_{\lambda,p}(v, \varrho, a_1, b_1) f(z)}{z^p}, z \in \mathfrak{A} \qquad (3.2.4)$$

Since Π in $\mathfrak{A}$ is analytic and differentiating (3.2.4) for to z and put the equation (3.1.11) in the resulting relation, we obtain

$$(1-\theta)\frac{\widetilde{\mathfrak{D}}^m_{\lambda,p}(v, \varrho, a_1, b_1) f(z)}{z^p} + \theta\frac{\widetilde{\mathfrak{D}}^{m+1}_{\lambda,p}(v, \varrho, a_1, b_1) f(z)}{z^p}$$

$$= \varpi\ (z) + \frac{\theta\zeta z \varpi\ '(z)}{p} \prec \frac{1 + Az}{1 + Bz} = \hbar(z).$$

Now, from lemma (3.1.1) and for $y = \frac{p}{\theta\zeta}$, and producing and convert of variables next by the using of the identities (3.1,6), (3.1.7) and (3.1.8), with $a = 1, b = \frac{p}{\theta\zeta}$, and $c - b = 1$, we conclude that

$$\frac{\widetilde{\mathfrak{D}}^m_{\lambda,p}(v, \varrho, a_1, b_1) f(z)}{z^p} \prec \Theta(z) = \frac{p}{\theta\zeta} z^{-\frac{p}{\theta\zeta}} \int_0^z t^{\frac{p}{\theta\zeta}-1}\left(\frac{1 + At}{1 + Bt}\right) dt$$

$$= \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)\frac{1}{(1+Bz)} {}_2F_1\left(1,1; \frac{p}{\theta\zeta} + 1; \frac{Bz}{1+Bz}\right), if\ B \neq 0, \\ 1 + \frac{p}{p+\theta\zeta} Az, \qquad , if\ B = 0, \end{cases}$$

that proving the assertion (3.2.2). Hence, to show the inequality (3.2.3) it is sufficient to prove

$$\inf\{Re\Theta(z): |z| < 1\} = \varpi\ (-1).$$

In fact, we have

$$\frac{1 - Ar}{1 - Br} \leq Re\frac{Az + 1}{Bz + 1}, \quad for \quad |z| \leq r < 1.$$

Setting

$$g(z,s)=\frac{1+A_{sz}}{1+B_{sz}} \qquad and \qquad dv(s)$$

$$=\frac{p}{\varpi\ \zeta}s^{\frac{p}{\theta\zeta}-1}ds\ (0\le s\le 1),$$

That is a positively measure on the interval [0, 1], we obtain

$$\Theta(z)=\int_0^1 g(z,s)dv(s),$$

and

$$Re\ \Theta(z)\ge\int_0^1\frac{1-A_{sr}}{1-B_{sr}}dv(s)=\Theta(-r),|z|\le r<1.$$

Putting $r\to 1$in the last inequality, we get (3.2.3). The best possible like the function Θ is the best dominant of (3.2.1).

In the same way the Theorem 3.2.1, can be proved by using (3.1.12) change in (3.1.11), we can prove the next theorem:

Theorem (3.2.2). A subordination condition satisfies of the function $f\in\mathfrak{A}(p)$ if

$$(1-\theta)\frac{\widetilde{\mathfrak{D}}^{m}_{\lambda,p+1}(v,\varrho,a_1,b_1)f(z)}{z^p}+\theta\frac{\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(v,\varrho,a_1,b_1)f(z)}{z^p} \prec\frac{1+Az}{1+Bz} \qquad (3.2.5)$$

then,

$$\frac{\widetilde{\mathfrak{D}}^{m}_{\lambda,p+1}(v,\varrho,a_1,b_1)f(z)}{z^p}\prec\mathcal{H}(z)\prec\frac{1+Az}{1+Bz},$$

Such that the function $\mathcal{H}$ given by

$$\mathcal{H}(z)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)\frac{1}{(1+Bz)}\,{}_2F_1\left(1,1;\frac{\eta+p-\lambda}{\theta}+1;\frac{Bz}{1+Bz}\right), & if B\neq 0\\ 1+\frac{\eta+p-\lambda}{\eta+p-\lambda+\theta}Az, & if B=0\end{cases}$$

is the best dominant of (3.2.5) in addition to

$$N < Re\frac{\widetilde{\mathfrak{D}}^{m}_{\lambda,p+1}(v,\varrho,a_1,b_1)f(z)}{z^p}, z \in \mathfrak{A} \qquad (3.2.6)$$

, N

$$= \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)\frac{1}{(1-Bz)}\,{}_2F_1\left(1,1;\frac{\eta+p-\lambda}{\theta}+1;\frac{B}{1+Bz}\right), & if B \neq 0, \\ 1 + \frac{\eta+p-\lambda}{\eta+p-\lambda+\theta}A, & if B = 0. \end{cases}$$

Where the inequality (3.2.6) has estimate is best possible.

$\forall$ f $\in \mathfrak{A}$(p) the generalized Bernardi–Libera–Livingston integral operator $F_{p,y} : \mathfrak{A}(p) \to \mathfrak{A}(p)$,with $y > -p$, is defined by **[52].**

$$F_{p,y}f(z) = \frac{y+p}{z^p}\int_0^z t^{y-1}f(t)dt$$

$$= \left(z^p + \sum_{k=1}^{\infty}\frac{y+p}{y+p+k}z^{p+k}\right) * f(z)$$

$$= [z^p.\,{}_2F_1(1,y+p;y+p+1;z)] * f(z) \qquad (3.2.7)$$

It is simply to satisfy that, $\forall f \in \mathfrak{A}(p)$ *we have*

$$z\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(v,\varrho,a_1,b_1)F_{p,y}\,f(z)\right)'$$

$$= (p + y)\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(v,\varrho,a_1,b_1)f(z)$$

$$- y\,\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(v,\varrho,a_1,b_1)F_{p,y}\,f(z) \qquad (3.2.8)$$

Theorem (3.2.3). A subordination condition of the function $f \in \mathfrak{A}(p)$ satisfies if

$$\frac{\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(v,\varrho,a_1,b_1)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \qquad (3.2.9)$$

and the integral operator $F_{p,\lambda}$ is defined by (3.2.7) then

$$\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)F_{p,y}f(z)}{z^p} \prec K(z) \prec \frac{1+Az}{1+Bz},$$

where the function K is given by

$$K(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)\frac{1}{(1+Bz)}\ {}_2F_1\left(1,1;y+p+1;\frac{B}{B-1}\right), & if\ B \neq 0, \\ 1+\frac{y+p}{y+p+1}Az, & if\ B = 0, \end{cases}$$

It has the best dominant for (3.2.9). Moreover

$$Re\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)F_{p,y}f(z)}{z^p} > L, z \in \mathfrak{U} \qquad (3.2.10)$$

where

$$L = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)\frac{1}{(1+B)}\ {}_2F_1\left(1,1;y+p+1;\frac{B}{B-1}\right), & if B \neq 0, \\ 1-\frac{y+p}{y+p+1}A, & if B = 0, \end{cases}$$

the estimation in (3.2.10) is best possible.

Proof. If we allow

$$\mathcal{O}(z) = \frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)F_{p,y}f(z)}{z^p}, \quad z \in \mathfrak{U}, \qquad (3.2.11)$$

then the function $\mathcal{O}$ is analytic in $\mathfrak{U}$. From (3.2.11), differentiating for z and using the identity (3.2.8) in the resulting relation, we obtain

$$\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}{z^p} = \mathcal{O}(z) + \frac{z\mathcal{O}'(z)}{p+y} \prec \frac{1+Az}{1+Bz},$$

and employ the same manner that we need the proof of theorem (3.2.1), the second part of the theorem can be proved in the same way.

Theorem (3.2.4). If $f \in \mathfrak{A}(p)$ satisfies the following condition

$$\left|(1-\theta)\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}{z^p}+\theta\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m+1}(v,\varrho,a_1,b_1)f(z)}{z^p}-1\right|<M_1$$

$$=\frac{p+\theta\zeta}{p}N_{1,}\ \ z\in\mathfrak{U} \qquad (3.2.12)$$

with

$$\frac{P+\theta\zeta}{p}N_1 \le 1 \qquad (3.2.13)$$

where

$$N_1 = min\{x\in(0,1):\Phi(x)=0\}. \qquad (3.2.14)$$

And

$$\Phi(x)=\left[\left(\frac{\theta\zeta(p-\Omega)}{p}\right)^2-2\frac{\theta\zeta(p-\Omega)}{p}-\left(\frac{p+\theta\zeta}{p}\right)^2\right]x^2-2\frac{\theta\zeta(p-\Omega)}{p}\left|1-\frac{\theta\zeta(p-\Omega)}{p}\right|x+\left(\frac{\theta\zeta(p-\Omega}{p}\right)^2, \qquad (3.2.15)$$

then $f\in\tilde{\psi}_{\lambda,p}^{m}(\Omega)$, where $\tilde{\psi}_{\lambda,p}^{m}(\Omega)\coloneqq\tilde{\psi}_{\lambda,p}^{m}(\Omega;1,-1)$ [such as it was in the aforementioned in Remark 3.1.1 (i)].

Proof. If we have

$$\mathcal{O}(z)=\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}{z^p}, z\in\mathfrak{A},$$

From the assumption (3.2.13). Then $\mathcal{O}$ analytic in $\mathfrak{A}$. and according to theorem (3.2.1) for the particular status A=$M_1$and B=0, we get that the assumption (3.2.12) indicate

$$\mathcal{O}(z)\prec 1+\frac{p}{p+\theta\zeta}M_{1z}=1+N_{1z,}$$

which is equivalent to

$$|\mathcal{O}(z)-1|<N_1, z\in\mathfrak{A}. \qquad (3.2.16)$$

Setting

$$p(z)=\frac{1}{p-\Omega}\left(\frac{\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)\right)'}{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}-\Omega\right), \qquad (3.2.17)$$

The assumption (3.2.12) could be written as

$$\left|\left(1-\frac{\theta\zeta(p-\Omega)}{p}\right)\mathcal{O}(z)+\frac{\theta\zeta(p-\Omega)}{p}p(z)\mathcal{O}(z)-1\right|<\frac{p+\theta\zeta}{p}N_1,\ z\in\mathfrak{U}$$

(3.2.18)

Now,

We will show that (3.2.18) implies,

$Rep(z)>0\ ,\forall\ z\in\mathfrak{A},$ that is $f\in\tilde{\psi}^{m}_{\lambda,p}(v,\varrho,a_1,b_1)(\Omega)$.

Supposing that this above inequality is a rong. Therefore $\exists\ z_0\in\mathfrak{A}$, P (0) =1. $and\ x\in\mathbb{R}$, Such that $p(z_0)=ix$. Thus, in order to explain that (3.2.18) indicate Re $p(z)>0$, ∀ z ∈ $\mathfrak{U}$, that is enough to get a contradiction and (3.2.18), its instance

$$ⓔ=\left|\left(1-\frac{\theta\zeta(p-\Omega}{p}\right)\mathcal{O}(z_{0)}+\frac{\theta\zeta(p-\Omega)}{p}p(z_o)\mathcal{O}(z_0)-1\right|$$

$$\geq\frac{p+\theta\zeta}{p}N_1 \qquad (3.2.19)$$

Thus, if we put $\mathcal{O}(z_0)=u+iv$,then

$$ⓔ^2=\left|\left(1-\frac{\theta\zeta(p-\Omega)}{p}\right)\mathcal{O}(z_o)+\frac{\theta\zeta(p-\Omega)}{p}p(z_0)\mathcal{O}(z_0)-1\right|^2 \qquad (3.2.20)$$

$$=(u^2+v^2)\left(\frac{\theta\zeta(p-\Omega)}{p}\right)^2x^2+2xv\frac{\theta\zeta(p-\Omega)}{p}+\left|\left(1-\frac{\theta\zeta(p-\Omega)}{p}\right)\mathcal{O}(z_0)-1\right|^2.$$

By using (3.2.16). We have

$$\left|\left(1-\frac{\theta\zeta(p-\Omega)}{p}\right)\mathcal{O}(z_0)-1\right|$$

$$=\left|\left(1-\frac{\theta\zeta(p-\Omega)}{p}\right)(\mathcal{O}(z_0)-1)-\frac{\theta\zeta(p-\Omega)}{p}\right|$$

$$\geq \frac{\theta\zeta(p-\Omega)}{p} - \left|1-\frac{\theta\zeta(p-\Omega)}{p}\right| N_1 \qquad (3.2.21)$$

Now, we must prove that under our assumption the following inequality is verified:

$$\frac{\theta\zeta(p-\Omega)}{p} - \left|1-\frac{\theta\zeta(p-\Omega)}{p}\right| N_1 \geq 0. \qquad (3.2.22)$$

Thus, if we indicate

$$a = \frac{\theta\zeta(p-\Omega)}{p} > 0, \quad b = \left(\frac{p+\theta\zeta}{p}\right)^2 > 0,$$

Thereafter the function $\mho(z)$, in Equation (3.2.15) becomes apparent

$$\Phi(x) = (a^2 - 2a - b)x^2 - 2a|1-a|x + a^2,$$

and

$$\Phi(0) = a^2 > 0, \ \ \Phi(1) = -2a(|1-a| + 1 - a) - b < 0.$$

If $a = 1$, it is clear that (3.2.22), verified, $\forall\, N_1 > 0$. If $a \neq 1$, therefore,

$$\Phi\left(\frac{a}{|1-a|}\right) = -\frac{a^2(1+b)}{|1-a|^2} < 0,$$

we conclude that if N_1 is given by (3.2.14), then the inequality (3.2.22) is correct. Subsequently from (3.2.20), (3.2.21), and (3.2.22) we get

$$\text{e}^2 - M_1^2 \geq (u^2+v^2)\left(\frac{\theta\zeta(p-\Omega)}{p}\right)^2 x^2 + 2xv\frac{\theta\zeta(p-\Omega)}{p} + \left(\frac{\theta\zeta(p-\Omega}{p} - \left|1-\frac{\theta\zeta(p-\Omega)}{p}\right| N_1\right)^2 - \left(\frac{p+\theta\zeta}{p}\right)^2 N_1^2.$$

Indicating

$$F(x) = (u^2+v^2)\left(\frac{\theta\zeta(p-\Omega}{p}\right)^2 x^2 + 2xv\frac{\theta\zeta(p-\Omega)}{p} + \left(\frac{\theta\zeta(p-\Omega)}{p} - \left|1-\frac{\theta\zeta(p-\Omega)}{p}\right| N_1\right)^2 - \left(\frac{p+\theta\zeta}{p}\right)^2 N_1^2,$$

Since $(u^2+v^2)\left(\frac{\theta\zeta(p-\Omega)}{p}\right)^2>0$, it follows that the inequality $F(x)\geq 0$ holds for all $x\in\mathbb{R}$, if and only if the discriminant $\partial\leq 0$, that is

$$\partial=4\left\{v^2\left(\frac{\theta\zeta(p-\Omega}{p}\right)^2 -(u^2 +u^2)\left(\frac{\theta\zeta(p-\Omega}{p}\right)^2.\left[\left(\frac{\theta\zeta(p-\Omega)}{p} -\left|1-\frac{\theta\zeta(p-\alpha)}{p}\right|N_1\right)^2-\left(\frac{p+\theta\zeta}{p}\right)^2\right]\right\}\leq 0,$$

which is equivalent to

$$v^2\left[1-\left(\frac{\theta\zeta(p-\Omega)}{p}-\left|1-\frac{\theta\zeta(p-\Omega)}{p}\right|N_1\right)^2+\left(\frac{p+\theta\zeta}{p}\right)^2N_1\right] \leq u^2\left[\left(\frac{\theta\zeta(p-\Omega}{p}-\left|1-\frac{\theta\zeta(p-\Omega)}{p}\right|N_1\right)^2-\left(\frac{p+\theta\zeta}{p}\right)^2N_1^2\right] \tag{3.2.23}$$

Putting $\mathcal{O}(z_0)=1+\rho e^{i\varepsilon}$ for some $\varepsilon\in\mathbb{R}$, it is easy to show that

$$\frac{v^2}{u^2}=\frac{\rho^2 sin^2\varepsilon}{(1+\rho cos\varepsilon)^2}\leq\frac{\rho^2}{1-\rho^2},\quad \varepsilon\in\mathbb{R}.$$

From (3.2.16) we have $\rho\leq N_1<1$, and using the above inequality we obtain

$$\frac{v^2}{u^2}\leq\frac{\rho^2}{1-\rho^2}\leq\frac{N_1^2}{1-N_1^2} \tag{3.2.24}$$

Since the function $T(\rho)=\frac{\rho^2}{1-\rho^2},\quad \rho\in[0,1)$,

is a precisely increasing function on the half open interval [0.1), we want to define the largest value of $N_1\in[0,1)$ (the next condition from the above comments). Such that

$$N_1^2\leq\left(\frac{\theta\zeta(p-\Omega)}{p}-\left|1-\frac{\theta\zeta(p-\Omega)}{p}\right|N_1\right)^2-\left(\frac{p+\theta\zeta}{p}\right)^2N_1^2$$

an easy calculation show that the value is obtain from (3.2.13), where Φ has by (3.2.15) according to the above cause, from (3.2.24) it next hat

$$\frac{v^2}{u^2} \le \frac{\rho^2}{1-\rho^2} \le \frac{N_1^2}{1-N_1^2}$$

$$\le \frac{\left(\frac{\theta\zeta(p-\Omega)}{p} - \left|1-\frac{\theta\zeta(p-\Omega)}{p}\right| N_1\right)^2 - \left(\frac{p+\theta\zeta}{p}\right)^2 N_1^2}{1-\left(\frac{\theta\zeta(p-\Omega)}{p} - \left|1-\frac{\theta\zeta(p-\alpha)}{p}\right| N_1\right)^2 - \left(\frac{p+\theta\zeta}{p}\right)^2 N_1^2},$$

$$\frac{v^2}{u^2} \le \frac{\left(\frac{\theta\zeta(p-\Omega)}{p} - \left|1-\frac{\theta\zeta(p-\Omega)}{p}\right| N_1\right)^2 - \left(\frac{p+\theta\zeta}{p}\right)^2 N_1^2}{1-\left(\frac{\theta\zeta(p-\Omega)}{p} - \left|1-\frac{\theta\zeta(p-\Omega)}{p}\right| N_1\right)^2 - \left(\frac{p+\theta\zeta}{p}\right)^2 N_1^2},$$

the last equation is equivalent to (3.2.23), which is $\partial \le 0$. thus (3.2.19) holds, that opposite (3.2.18) and the proof of the theorem is end.

Theorem (3.2.5). If $f \in \mathfrak{A}(p)$, such that $\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}{z^p} \neq 0, \forall z \in \mathfrak{A}$, the subordination condition is verified

$$(1-\theta)\left(\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}{z^p}\right)^{\beta} + \theta\frac{\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m+1}(v,\varrho,a_1,b_1)f(z)\right)'}{pz^{p-1}}\left(\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}{z^p}\right)^{\beta-1} \prec 1+M_2 \qquad (3.2.25)$$

where

$$M_2 = \begin{cases} \dfrac{(p-\Omega)\theta(1+\frac{\theta}{p\beta})}{|p-(p-\Omega)\theta| + \sqrt{p^2+\left(p+\frac{\theta}{\beta}\right)^2}}, & if\ \beta > 0, \\ \dfrac{(p-\Omega)\theta}{p}, & if\ \beta = 0. \end{cases}$$

Then $f \in \tilde{\psi}_{\lambda,p}^{m}(\Omega) \coloneqq \tilde{\psi}_{\lambda,p}^{m}(\Omega;1,-1)$[look at the Remark 3.1.1 (i)].

Proof. Let $\beta = 0$, thus the assumption (3.2.25) is equivalent to

$$\frac{z\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)\right)'}{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}-\Omega \prec (p-\Omega)(z+1),$$

which is implies that $f \in \tilde{\psi}_{\lambda,p}^{m}(\Omega)$.

If we suppose $\beta > 0$, let define the function

$$\mathcal{O}(z)=\left(\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}{z^p}\right)^{\beta}, z\in\mathfrak{A}, \qquad (3.2.26)$$

we pick the main value for the power function. Thus $\mathcal{O}(0)=1$ is analytic in $\mathfrak{A}$, and differentiating (3.2.26) for to z we get

$$(1-\theta)\left(\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}{z^p}\right)^{\beta}$$

$$+\theta\frac{\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)\right)'}{pz^{p-1}}\left(\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}{z^p}\right)^{\beta-1}$$

$$=\mathcal{O}(z)+\frac{\theta}{\beta p}z\mathcal{O}'(z),$$

which, in view of lemma (3.1.1) produce

$$\mathcal{O}(z)\prec 1+\frac{\beta p}{\beta p+\theta}M_2 z.$$

As well, the subordination assumption in (3.2.25), must be written also

$$\left[1-\theta+\theta\left(\left(1-\frac{\Omega}{p}\right)p(z)+\frac{\Omega}{p}\right)\right]\mathcal{O}(z)\prec 1+M_2 z,$$

where p is given by (3.2.17). Hence by Lemma (3.1.3). We conclude that Re $0 < p(z), z\in\mathfrak{A}$,

For the particular case $\lambda=\mu=p, m=0$, Theorem (3.2.5) decrease for the next corollary:

Corollary (3.2.1). If $f\in\mathfrak{A}(\mathrm{p}), \ni \frac{f(z)}{z^p}\neq 0, \forall\, z\in\mathfrak{A}$, and the subordination condition verified

$$(1-\theta)\left(\frac{f(z)}{z^p}\right)^{\beta}+\theta\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\beta}\prec 1+M_2 z,$$

where

$$M_2 = \begin{cases} \dfrac{(p-\Omega)\theta\left(1+\dfrac{\theta}{p\beta}\right)}{|p-(p-\Omega)\theta| + \sqrt{p^2+\left(p+\dfrac{\theta}{\beta}\right)^2}}, & if\beta > 0, \\ \dfrac{(p-\Omega)}{p}\theta & if\ \beta = 0 \end{cases}$$

then $f \in \tilde{\psi}_p^*(\Omega)$ (the powers are the main).See **[46].**

Remark (3.2.1).

- **i.** Look at that the result was obtained by Patel et al. **[53, corollary 4]**;
- **ii.** Choose $p = 1$ in corollary (3.2.1) we get the result of Liu [**45**, Theorem (3.2.2)]; put $n = 1$.

If we choose $\lambda = \mu = p, m = 0\ and\ \theta = 1$ from Theorem (3.2.5), then we get the following corollary:

Corollary (3.2.2). If $f \in \mathfrak{A}(\mathrm{p}) \ni \frac{f(z)}{z^p} \neq 0\ \forall\ z \in \mathfrak{A}$, and the inequality below is verified

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z^p}\right)^\beta - p\right| < \frac{(p-\Omega)(p\beta+1)}{\Omega\beta + \sqrt{p^2\beta^2} + (p\beta+1)^2}, z \in \mathbb{U},$$

Then $f \in \psi_p(\Omega)$ (the powers are the main).See **[46]**

Corollary (3.2.3). If $f \in \mathfrak{A}(\mathrm{p}) \ni \frac{f(z)}{z^p} = 0\ for\ all\ z \in \mathfrak{A}$, and the inequality verified

$$\left|(p-\Omega-1)\left(\frac{f(z)}{z^p}\right)^\beta + \frac{zf'}{pf(z)}\left(\frac{f(z)}{z^p}\right)^\beta - (p-\Omega)\right|$$
$$< \frac{p\beta(p-\Omega)+1}{p\beta\left[(p-1)+\sqrt{p^2+\left(\dfrac{1}{\beta(p-\Omega)}\right)^2}\right]}, z \in \mathfrak{U},$$

then $f \in \tilde{\psi}_p(\Omega)$.

3.3 Applications Related on Subordination of Multivalent Functions

Theorem (3.3.1).

If $f \in \tilde{\psi}_{\lambda,p}^{m+1}(v,\varrho,a_1,b_1,\Omega;A,B) \ni \tilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z) \neq 0, \forall\, z \in \mathfrak{A} \setminus \{0\}$, and

$$\left(\frac{p}{\zeta} - p + \Omega\right)(1-B) + (p-\Omega)(1-A) \geq 0, \tag{3.3.1}$$

then

$$\frac{1}{p-\Omega}\left(\frac{z\left(\tilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)\right)'}{\tilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)} - \Omega\right) \prec q_1(z) \prec \frac{1+Az}{1+Bz} \tag{3.3.2}$$

where

$$q_1(z) = \frac{1}{p-\Omega}\left(\frac{1}{Q_1(z)} - \frac{p}{\zeta} + p - \Omega\right) \tag{3.3.3}$$

and

$$\Theta_1(z) = \begin{cases} \int_0^1 s^{\frac{p}{\zeta}-1}\left(\frac{1+Bzs}{1+Bz}\right)^{\frac{(p-\Omega)(A-B)}{B}} ds, & if\ B \neq 0 \\ \int_0^1 s^{\frac{p}{\zeta}-1}\, exp[(p-\Omega)(s-1)Az]ds, & if\ B = 0, \end{cases}$$

and q_1 is the best dominant of (3.3.2), if moreover (3.3.1),

$$A \leq -\frac{\left(\frac{p}{\zeta} - p + \Omega + 1\right)B}{p-\Omega}, with -1 \leq B < 0 \tag{3.3.4}$$

then,

$$f \in \tilde{\psi}_{\lambda,p}^{m}(\rho_1) \tag{3.3.5}$$

where

$$\rho_1 = \frac{p}{\zeta}\left[{}_2F_1\left(1, \frac{(p-\Omega)(B-A)}{B}; \frac{p}{\zeta}+1; \frac{B}{B-1}\right)\right]^{-1} - \frac{p}{\zeta}$$

$$+ p \text{ and } \tilde{\psi}^m_{\lambda,p}(\rho_1) := \psi^m_{\lambda,p}(\rho_1; 1,, -1)$$

[Look at to Remark 3.1.1 (i)]. The result is the best possible.

Proof. We have

$$\Pi\,(z) = \frac{1}{p-\Omega}\left(\frac{z\left(\widetilde{\mathfrak{D}}^m_{\lambda,p}(v,\varrho,a_1,b_1)f(z)\right)'}{\widetilde{\mathfrak{D}}^m_{\lambda,p}(v,\varrho,a_1,b_1)f(z)} - \Omega\right), \qquad (3.3.6)$$

$$z \in \mathbb{U}$$

then Π is analytic in$\mathfrak{A}$, with $\Pi\,(0) = 1$.Using the identity (3.2.6) in (3.3.6), we obtain

$$\frac{p}{\zeta}\frac{\widetilde{\mathfrak{D}}^{m+1}_{\lambda,p}(v,\varrho,a_1,b_1)f(z)}{\widetilde{\mathfrak{D}}^m_{\lambda,p}(v,\varrho,a_1,b_1)f(z)} = (p-\Omega)\varpi(z) + \frac{p}{\zeta} - p + \Omega \qquad (3.3.7)$$

If differentiating (3.3.7) with respect to z, we get

$$\frac{1}{p-\Omega}\left(\frac{z\left(\widetilde{\mathfrak{D}}^m_{\lambda,p}(v,\varrho,a_1,b_1)f(z)\right)'}{\widetilde{\mathfrak{D}}^m_{\lambda,p}(v,\varrho,a_1,b_1)f(z)} - \Omega\right)$$

$$= \varpi\,(z) + \frac{z\Pi\,'(z)}{(p-\Omega)\varpi\,(z) + \frac{p}{\zeta} - p + \Omega} \prec \frac{1+Az}{1+Bz}.$$

From the assumption (3.3.1), by using Lemma (3.1.4) we obtain,

$$\varpi\,(z) \prec q_1(z) \prec \frac{1+Az}{1+Az},$$

where q_1 is given (3.3.3) is the best dominant of (3.3.8), and this proves (3.3.2).

Hence, we offer that

$$\inf\{Re\; q_1(z): |z| < 1\} = q_1(-1),$$

or, equivalently

$$\inf\left\{\mathrm{Re}\frac{1}{\Theta_1 \mathrm{z}} : |\mathrm{z}| < 1\right\} = \frac{1}{\Theta_1(-1)}. \qquad (3.3.9)$$

Indicating $a := \frac{(p-\Omega)(B-A)}{B}$, $b := \frac{p}{\zeta}$ and $c := \frac{p}{\zeta} + 1$, since $c > b > 0$, from (3.1.6), (3.1.7) and (3.1.8) we conclude that

$$\Theta_1(z) = (1 + Bz)^a \int_0^1 S^{b-1} ds$$

$$= \frac{\Gamma(b)}{\Gamma(c)} \, {}_2\mathrm{F}_1\left(1, a; c; \frac{Bz}{Bz+1}\right) = \frac{\zeta}{\mathrm{p}} \mathbb{H}(\mathrm{z}), \qquad (3.3.10)$$

$$\text{where } \mathbb{H}(\mathrm{z}) = {}_2\mathrm{F}_1\left(1, \mathrm{a}; \mathrm{c}; \frac{\mathrm{Bz}}{\mathrm{Bz}+1}\right),$$

where $B \neq 0$. From (3.3.4), apart from the case of equality, we have $c > a > 0$,thus from (3.1.7) we obtain

$$\mathbb{H}(z) = \int_0^1 f(z,s) dv(s),$$

Where,

$\hbar(z,s) = \frac{1+Bz}{1+(1-s)Bz}$ & $dv(s) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} s^{a-1}(1-s)^{c-a-1} ds, (0 \leq s \leq 1)$.

Which is a positively measure on the interval [0, 1], using the actuality $-1 \leq B \leq 0$, it is simply to test that

$$Re\frac{1}{\mathbb{H}(z)} > 0, z \in \mathfrak{U},$$

$$\hbar(-r,s) \in \mathbb{R}, r \in [0,1), s \in [0,1],$$

$$Re\frac{1}{\hbar(z,s)} \geq \frac{1-(1-s)Br}{1-Br} = \frac{1}{\hbar(-r,s)}, \quad |z| \leq r < 1, s \in [0,1].$$

It follows by Lemma (3.1.2) we conclusion that

$$Re\frac{1}{\mathbb{H}(z)} \geq \frac{1}{\mathbb{H}(-r)}, \qquad |z| \leq r < 1,$$

and by letting r→ 1, taking into the account the relation (3.3.10), we get the inequality (3.3.9).

Moreover, by taking $A \uparrow -\frac{\left(\frac{p}{\zeta}-p+\Omega+1\right)B}{p-\Omega}$ for the case A = $-\frac{\left(\frac{p}{\zeta}-p+\Omega+1\right)B}{p-\Omega}$

We deduce that (3.3.9) holds when the inequality (3.3.4) is verified, that prove (3.3.5). The result is the best possible and the function $q_1$is the best dominant of (3.3.8).

For $A = 1$ and B = -1, the second part of the Theorem 3.3.1, reduces to the following corollary:

Corollary (3.3.1).

If

$\hbar \in \tilde{\psi}_{\lambda,p}^{m+1}\left(v, \varrho, a_1, b_1, \Omega; A, B\right) \ni \tilde{\mathfrak{D}}_{\lambda,p}^{m}(v, \varrho, a_1, b_1)f(z) \neq 0,\ \forall z \in \mathfrak{U} \setminus \{0\}$, and

$$\text{Max}\left\{p - \frac{p}{\zeta}; p - \frac{1}{2}\left(\frac{p}{\zeta} + 1\right)\right\} \leq \Omega < p,$$

then

$$f \in \tilde{\psi}_{\lambda,p}^{m}(v, \varrho, a_1, b_1)(\mathfrak{L}_1),$$

Where $\mathfrak{L}_1$ given by

$$\mathfrak{L}_1 = \frac{p}{\zeta}\left[{}_2F_1\left(1, 2(p-\Omega); \frac{p}{\zeta} + 1; \frac{1}{2}\right)\right]^{-1} - \frac{p}{\zeta} + p.$$

The result is best possible. See **[46]**

Using same pretext to the proof of Theorem (3.3.1), we get the next result:

Theorem (3.3.2).

If

$f \in \tilde{\psi}_{\lambda,p}^{m}\left(v, \varrho, a_1, b_1, \Omega; A, B\right) \ni \tilde{\mathfrak{D}}_{\lambda,p+1}^{m}(v, \varrho, a_1, b_1)f(z) \neq 0\ \forall z \in \mathfrak{U} \setminus \{0\}$,

and

$$(\Omega - \lambda)(1 - B) + (p - \Omega)(1 - A) \geq 0 \tag{3.3.11}$$

then,

$$\frac{1}{p-\Omega}\left(\frac{z\left(\tilde{\mathfrak{D}}_{\lambda,p+1}^{m}(v, \varrho, a_1, b_1)f(z)\right)'}{\tilde{\mathfrak{D}}_{\lambda,p+1}^{m}(v, \varrho, a_1, b_1)f(z)} - \Omega\right) \prec q_2(z) \prec \frac{1+Az}{1+Bz}, \tag{3.3.12}$$

$$\text{Where, } q_2(z) = \frac{1}{p-\Omega}\left(\frac{1}{Q_2(z)} + \lambda - \Omega\right)$$

and

$$\Theta_2(z)\begin{cases}\int_0^1 S^{p-\lambda-1}\left(\frac{1+Bzs}{1+Bz}\right)^{\frac{(p-\Omega)(A-B)}{B}} ds, & if B \neq 0,\\ \int_0^1 S^{p-\lambda-1}\, exp[(p-\Omega)(s-1)Az], & if\ B = 0,\end{cases}$$

and q_2 is the best dominant of *(3.3.12)*. If, moreover *(3.3.11)*

$$A \leq -\frac{(\Omega-\lambda+1)B}{p-\Omega}, \qquad -1 \leq B < 0,$$

Then $f \in \tilde{\psi}^m_{\lambda,p+1}(\rho_2)$,

where,

$$\rho_2 = (p-\lambda)\left[{}_2F_1\left(1, \frac{(p-\Omega)(B-A}{B}; p-\lambda+1; \frac{B}{B-1}\right)\right]^{-1} + \lambda,$$

And $\tilde{\psi}^m_{\lambda,p+1}(v, \varrho, a_1, b_1, \Omega; A, B)(\rho_2) :=$

$\psi^m_{\lambda,p+1}(v, \varrho, a_1, b_1)(\rho_2; 1, -1)$[look at to Remark 3.3.1(i)].*The result is best possible.*

Choose $A = 1$ and $B = -1$ in the other part of theorem (3.3.2) we get next corollary.

Corollary (3.3.2).

If $f \in \psi^m_{\lambda,p}(v, \varrho, a_1, b_1)(\Omega) \ni \psi^m_{\lambda,p}(v, \varrho, a_1, b_1)f(\Omega) \neq 0, \forall z \in \mathfrak{A} \setminus \{0\}$ and

$$max\left\{\lambda - \eta; \frac{p+\lambda-1}{2}\right\} \leq \Omega < p,$$

then

$$f \in \tilde{\psi}^m_{\lambda,p+1}(v, \varrho, a_1, b_1)(\mathfrak{L}_2),$$

where, $\mathfrak{L}_2$ given by

$$\mathfrak{L}_2 = (p-\lambda)\left[\left({}_2F_1(1,2(p-\Omega); p-\lambda+1; \frac{1}{2}\right)\right]^{-1} + \lambda.$$

The result above is the best possible See **[46].**

Theorem (3.3.3).

If $f \in \tilde{\psi}^{m}_{\lambda.p}(v, \varrho, a_1, b_1)(\Omega)$, s.t., $\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(v, \varrho, a_1, b_1)\, F_{p,\gamma} f(z) \neq 0, \forall\, z \in \mathfrak{U} \setminus \{0\}$, and

$$(\Omega + y)(1 - B) + (p - \Omega)(1 - A) \geq 0 \qquad (3.3.13)$$

Then

$F_{p,y} f \in \tilde{\psi}^{m}_{\lambda,p}(v, \varrho, a_1, b_1, \Omega; A, B)$.

where the operator $F_{p,y}$ *is* defined by (3.2.7). Further, if $f \in \tilde{\psi}^{m}_{\lambda,p}(v, \varrho, a_1, b_1)(\Omega; A, B)$, then

$$\frac{1}{p - \Omega}\left(\frac{z\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(v, \varrho, a_1, b_1) f(z)\right)'}{\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(v, \varrho, a_1, b_1) f(z)} - \Omega\right) \prec q_3(z) \prec \frac{1 + Az}{1 + Bz}, \qquad (3.3.14)$$

where,

$$q_3(z) = \frac{1}{p - \Omega}\left(\frac{1}{\Theta_3(z)} - \Omega - y\right),$$

and

$$\Theta_3(z) = \begin{cases} \int_0^1 S^{p+y-1}\left(\frac{1 + Bzs}{1 + Bz}\right)^{\frac{(p-\Omega)(A-B)}{B}} ds, if\ B \neq 0 \\ \int_0^1 S^{p+y-1}\, exp[(p - \Omega)(s - 1)Az]\, ds, if B = 0 \end{cases}$$

then the q_3 is the best dominant of (3.3.14) If, moreover to (3.3.13)

$$A \leq -\frac{(\Omega + y + 1)B}{p - \Omega}, with -1 \leq B < 0,$$

Then

$$F_{p,y} f \in \tilde{\psi}^{m}_{\lambda,p}(v, \varrho, a_1, b_1)(\rho_3),$$

where

$$\rho_3 = (p + y)\left[{}_2F_1\left(1, \frac{(p - \Omega)(B - A)}{B}; p + y + 1; \frac{B}{B - 1}\right)\right]^{-1} - y,$$

and

$$\tilde{\psi}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)(\rho_3):$$

$$= \tilde{\psi}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)(\rho_3;1,-1)[\text{look at to Remark 3.1.1 (i)}],$$

the result is best possible.

Proof. If we let

$$\varpi(z) = \frac{1}{p-\Omega}\left(\frac{z\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)F_{p,y}f(z)\right)'}{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)F_{p,y}f(z)} - \Omega\right), \qquad (3.3.15)$$

$$z \in \mathfrak{A}$$

Then Π is analytic in $\mathfrak{A}$ with $\Pi(0) = 1$. Using the identity (3.2.8) in (3.3.15), we get

$$(p+y)\frac{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)}{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)F_{p,y}f(z)} \qquad (3.3.16)$$

$$= (p-\Omega)\Pi(z) + \Omega + y$$

Differentiating (3.3.16) with respect to z, we get

$$\frac{1}{p-\Omega}\left(\frac{z\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)\right)'}{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)f(z)} - \Omega\right)$$

$$= \Pi(z) + \frac{z\Pi'(z)}{(p-\Omega)\Pi(z) + \Omega + y} \prec \frac{1+Az}{1+Bz},$$

and utilizing the same method that used in the proof of Theorem 3.3.1 the remaining part of the theorem can be proved in a same way.

Putting $A = 1$ and $B = -1$ in theorem 3.3.3, we get the following corollary:

Corollary (3.3.3).

If $f \in \tilde{\psi}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)(\Omega) \ni \widetilde{\mathfrak{D}}_{\lambda,p}^{m}(v,\varrho,a_1,b_1)\mathbb{H}_{p,y}f(z) \neq 0$ for all $z \in \mathfrak{A} \setminus \{0\}$, and

$$\max\left\{-y\,;\frac{p-y-1}{2}\right\} \leq \Omega < p,$$

Then

$$F_{p,y} f \in \tilde{\psi}^{m}_{\lambda,p}(v, \varrho, a_1, b_1)(\mathfrak{L}_3),$$

where

$$\mathfrak{L}_3 = (p - y)\left[{}_2F_1\left(1, 2(p - \Omega); p + y + 1; \frac{1}{2}\right)\right]^{-1} - y\,.$$

The result is the best possible see **[46]**

3.4 Some Properties on Subclass of Harmonic Multivalent Function

Let $\mathfrak{A}$(p) be the represent the class of functions of the form

$$f(z) = z^p + \sum_{k=2}^{\infty} a_{k+p}\, z^{k+p} \quad , z \in \mathfrak{A}, p \in \ \mathbb{N} \ \coloneqq \{\ 1,2, \cdots \} \tag{3.4.1}$$

That are multivalent in the open unit disc $\mathfrak{A}$.

Recently see **[10]** define on harmonic univalent mappings, led to the birth of theory of harmonic univalent mappings. This theory has attracted the function theorists. to look at harmonic analogues of the theory of analytic univalent or multivalent functions, and harmonic multivalent functions have famous to have an increase of applications, in the apparently diverse fields of medicine, engineering, electronics, physics, aerodynamics, operation research and other branches of applied mathematics but, introduce and study of minimal surfaces and also play important roles in a variety of problems in applied mathematics. Where Harmonic function has fruitful applications not only in applied mathematics, but also in physics, engineering, it appears in differential equations and for more details see **[54]**, **[55]**, **[56]**, **[57],[15]** and**[58]**

Our study starts with introduce of the main terms used of q-calculus used.

See **[59].** For $q \in (0,1)$, we define q-derivative operator D_q of a function f by

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0) \end{cases} \tag{3.4.2}$$

From (3.4.2) it follows that if $f \in \mathfrak{A}$(p) has the (3.4.1), then

Jackson **[60, 61]** define q-derivative operator

$\partial_{p,q}^{n} f(z) := zD_q\left(\partial_{p,q}^{n-1} f(z)\right), n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,by

$$\partial_{p,q}^{0} f(z) := f(z), \quad \partial_{p,q}^{n} f(z) := z\left(\partial_{p,q}^{n-1} f(z)\right), n \in \mathbb{N}.$$

Therefore, if $f \in \mathcal{A}(\mathrm{p})$ has from (1.1) it follows that

Where$[k]_q := \frac{1-q^k}{1-q}$, and thus $\lim\limits_{q\to 1}[k]_q = k$.

See **[59].** Define q-derivative operator $\partial_{p,q}^{n} f(z) := zD_q\left(\partial_{p,q}^{n-1} f(z)\right), n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,by

$$\partial_{p,q}^{0} f(z) := f(z), \quad \partial_{p,q}^{n} f(z) := z\left(\partial_{p,q}^{n-1} f(z)\right), n \in \mathbb{N}.$$

Therefore, if $f \in \mathfrak{A}(\mathrm{p})$ has from (3.4.1) it follows that

$$\partial_{p,q}^{n} f(z) = \left(f * G_{p,q}^{n}\right)(z), z \in \mathfrak{A}, p \in \mathbb{N}_0,$$

Where

$$G_{p,q}^{n}(z) := z^p + \sum_{k=1}^{\infty}\left([k+p]_q\right)^n z^{k+p}, z \in \mathfrak{A}, p \in \mathbb{N}, n \in \mathbb{N}_0.$$

Moreover,

$$\partial_{p,q}^{n} f(z) = z^p + \sum_{k=1}^{\infty}\left([k+p]_q\right)^n a_{k+p} z^{k+p}, z \in \mathfrak{A}$$

and

$$\lim_{q\to 1^-} \partial_{p,q}^{n} f(z) = z^p + \sum_{k=1}^{\infty}(k+p)^n a_{k+p} z^{k+p}, z \in \mathfrak{A}.$$

For $\eta \geq 0$, by using the equation above introduce the following q-derivative operator $\mathfrak{G}_{\eta,p,q}^{n,m} : \mathfrak{A}(\mathrm{p}) \to \mathfrak{A}(\mathrm{p})$ by

$$\mathfrak{G}_{\eta,p,q}^{n,0} f(z) := \partial_{p,q}^{n} f(z),$$

$$\mathfrak{G}_{\eta,p,q}^{n,m} f(z) := (1-\eta)\, \mathfrak{G}_{\eta,p,q}^{n,m-1} f(z) + \eta \frac{z}{p}\left(\mathfrak{G}_{\eta,p,q}^{n,m-1} f(z)\right)' \ , m \in \mathbb{N}.$$

From the above definition it follows easily, that if $f \in \mathfrak{A}(\mathrm{p})$ is of the form (3.4.1) then

$$\mathfrak{G}_{\eta,p,q}^{n,m} f(z) = z^p + \sum_{k=2}^{\infty} \left([k+p]_q\right)^n \left(\frac{p+\eta k}{p}\right)^m a_{k+p} z^{k+p} z \qquad (3.4.3)$$
$$\in \mathfrak{A}, m \in \mathbb{N}_0$$

The necessary and sufficient condition for the harmonic function f is locally multivalent and sense-preserving such $|(h(z)')| < |(g(z)')|$ in D (where D is simply connected in $\mathbb{C}$). See **[10].** A continuous function $f = u + iv$ is complex-valued harmonic function in a domain D ⊆ C if both u and v are real harmonic functions in D. In any simply connected domain, denoted by SH the class of all functions of the form $f = h + \bar{g}$ that are harmonic multivalent, normalized and sense preserving within the open unit disc $\mathfrak{A}$ where h and g are analytic in $\mathfrak{A}$. We call h the analytic part and g the co-analytic part of f. Clunie and Sheil-Small noted that the A necessary and sufficient condition for the harmonic function f is locally univalent and sense-preserving such $|(g(z)')| < |(f(z)')|$ in D (where D is simply connected in $\mathbb{C}$.

Let S_H represented the class of harmonic functions $f = h + \bar{g}$ which univalent with $f(0) = f(z)' - 1 = 0 \; with \; f = h + \bar{g} \in S_H$ If f normalized and sense-preserving within $\mathfrak{A}$, where h and g and the class $\mathcal{A}$ of all analytic functions in $\mathfrak{A}$ as the form

$$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p} z^{k+p-1} \quad , g(z) = \sum_{k=1}^{\infty} b_{k+p} z^{k+p-1} \qquad (3.4.4)$$

We say that h is analytic part and g co-analytic part, if the S_H reduce to S of every normalization function analytic univalent, if the co-analytic part of f is equal to zero. See **[10].**

Now we introduce harmonic multivalent functions in $\mathfrak{A}$ it follows by

$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p} z^{k+p-1}, \quad g(z) = \sum_{k=1}^{\infty} b_{k+p} z^{k+p-1},$

In this paper, we modified the operator $\mathfrak{G}_{\eta,p,q}^{n,m} f(z)$ in Equation (3.4.3) of harmonic functions $f = h + g$ as

$$\mathfrak{G}_{\eta,p,q}^{n,m} f(z) = \mathfrak{G}_{\eta,p,q}^{n,m} h(z) + \overline{\mathfrak{G}_{\eta,p,q}^{n,m} g(z)} \tag{3.4.5}$$

Where

$$\mathfrak{G}_{\eta,p,q}^{n,m} h(z) = z^p + \sum_{k=2}^{\infty} \left([k+p]_q\right)^n \left(\frac{p+\eta k}{p}\right)^m a_{k+p} z^{k+p-1} \ . \tag{3.4.6}$$

And

$$\overline{\mathfrak{G}_{\eta,p,q}^{n,m} g(z)} = \sum_{k=1}^{\infty} \left([k+p]_q\right)^n \left(\frac{p+\eta k}{p}\right)^m b_{k+p} z^{k+p-1} \tag{3.4.7}$$

Let $T_{\eta,p,q}^{n,m}(v)$ denote the family of harmonic function f of the form (3.4.1) and let $\overline{T}_{\eta,p,q}^{n,m}(v)$,

Be subclass of $T_{\eta,p,q}^{n,m}(v)$ consisting of harmonic functions $f = h + \bar{g}$, where h and g are of the form

$$h(z) = z^p - \sum_{k=1}^{\infty} |a_{k+p}| z^{k+p-1}, \quad g(z) = \sum_{k=1}^{\infty} |b_{k+p}| z^{k+p-1},$$

Definition (3.4.1). For $0 \le v \le 1$, the function $f = h + \bar{g}$ in the class $T^{n,m}_{\eta,p,q}(v)$ if satisfy the inequality

$$Re\left\{\frac{zD_q\left(\mathfrak{G}^{n,m}_{\eta,p,q} f(z)\right)}{\left(\mathfrak{G}^{n,m}_{\eta,p,q} f(z)\right)}\right\} \ge v \qquad |z| = r < 1. \qquad (3.4.8)$$

First, we begin the sufficient coefficient condition for functions f in $T^{n,m}_{\eta,p,q}(v)$

Theorem (3.4.1). Let $f = h + \bar{g}$, where $h(z)$ and $g(z)$ are defined by (3.4.1). If

$$\sum_{k=2}^{\infty}\left([k+p]_q - 2v\right)|a_{k+p}|\emptyset^{n,m}_{\eta,p,q} + \sum_{k=1}^{\infty}\left([k+p]_q + 2v\right)|b_{k+p}|\,\emptyset^{n,m}_{\eta,p,q} \le 2\left([p]_q - v\right) \qquad (3.4.9)$$

Where $a_1 = 1$, $v \in [0,1)$ and $\emptyset^{n,m}_{\eta,p,q}$ given by

$$\emptyset^{n,m}_{\eta,p,q} = \left|\left([k+p]_q\right)^n\left(\frac{p+\eta k}{p}\right)^m\right|, \qquad (3.4.10)$$

then f is sense –preserving, harmonic, univalent in $\mathfrak{A}$, and $f \in T^{n,m}_{\eta,p,q}(v)$

Proof. If $|z_1| \neq |z_2| < q, then$

$$\left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| \ge 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right|$$

$$= 1 - \left|\frac{\sum_{n=1}^{\infty} b_{k+p}\left(z_1^{k+p-1} - z_2^{k+p-1}\right)}{\left(z_1^p - z_2^p\right) + \sum_{n=2}^{\infty} a_{k+p}\left(z_1^{k+p-1} - z_2^{k+p-1}\right)}\right|$$

$$> 1 - \frac{\sum_{k=1}^{\infty}[k+p]_q|b_{k+p}|}{1 - \sum_{k=2}^{\infty}[k+p]_q|a_{k+p}|} \ge 1$$

$$-\frac{\sum_{n=1}^{\infty}\left[\left([k+p]_q + 2v\right)\emptyset^{n,m}_{\eta,p,q}/\left([p]_q - v\right)\right]|b_{k+p}|}{1 - \sum_{k=2}^{\infty}\left[\left([k+p]_q + 2v\right)\emptyset^{n,m}_{\eta,p,q}/\left([p]_q - v\right)\right]|a_{k+p}|} \ge 0$$

Which proves the multivalent. Observe that, f is sense-preserving in$\mathfrak{A}$, because

$$\left|D_q h(z)\right| \geq \left(1-\sum_{k=2}^{\infty}[k+p]_q\left|a_{k+p}\right||z|^{k+p-1}\right)$$

$$> \left(1-\sum_{k=2}^{\infty}\frac{\left([k+p]_q-2v\right)\emptyset_{\eta,p,q}^{n,m}}{[p]_q-v}\left|a_{k+p}\right|\right)$$

$$\geq \left(\sum_{k=1}^{\infty}\frac{\left([k+p]_q+2v\right)\emptyset_{\eta,p,q}^{n,m}}{[p]_q-v}\left|b_{k+p}\right|\right)$$

$$> \left(\sum_{k=1}^{\infty}\frac{\left([k+p]_q+2v\right)\emptyset_{\eta,p,q}^{n,m}}{[p]_q-v}\left|b_{k+p}\right||z|^{k+p-1}\right)$$

$$\geq \sum_{n=1}^{\infty}[k+p]_q\left|b_{k+p}\right||z|^{k+p-1} \geq \left|D_q g(z)\right|.$$

Then we have $\lim_{q\to 1}\left[\left|D_q h(z)\right| \geq \left|D_q g(z)\right|\right] = \left[|h(z)'| \geq |g(z)'|\right]$.

Now, we show that $f \in T_{\eta,p,q}^{n,m}(v)$.from (3.4.8), we can write

$$\text{Re}\left\{\frac{zD_q\left(\mathfrak{G}_{\eta,p,q}^{n,m} f(z)\right)}{\left(\mathfrak{G}_{\eta,p,q}^{n,m} f(z)\right)}\right\} = Re\left\{\frac{A(z)}{B(z)}\right\},$$

From the equation (3.4.5) we get

$$A(z) = z\, D_q\left(\mathfrak{G}_{\eta,p,q}^{n,m} f(z)\right)$$

$$= [p]_q z^p + \sum_{k=2}^{\infty}[k+p]_q\, \emptyset_{\eta,p,q}^{n,m} a_{k+p} z^{k+p-1}$$

$$- \sum_{k=1}^{\infty}[k+p]_q\, \emptyset_{\eta,p,q}^{n,m}\overline{b_{k+p}}\overline{z^{k+p-1}},$$

and

$$B(z)=\left(\mathfrak{G}_{\eta,p,q}^{n,m}f(z)\right)$$

$$=z^p+\sum_{k=2}^{\infty}\emptyset_{\eta,p,q}^{n,m}a_{k+p}z^{k+p-1}+\sum_{k=1}^{\infty}\emptyset_{\eta,p,q}^{n,m}\overline{b_{k+p}z^{k+p-1}}$$

Using the fact that $Re(w)\geq v$ if and only if $|1-v+w|\geq|1+v-w|$, it suffices to show that

$$|A(z)+(1-v)B(z)|-|A(z)-(1+v)B(z)|\geq 0 \qquad (3.4.11)$$

Replacing for $A(z)$ and $B(z)$ in (3.4.11), we get

$$|A(z)+(1-v)B(z)|-|A(z)-(1+v)B(z)|$$

$$=\left|([p]_q+1-v)z^p+\sum_{k=2}^{\infty}\left([k+p]_q+1-v\right)\emptyset_{\eta,p,q}^{n,m}a_{k+p}z^{k+p-1}\right.$$

$$\left.-\sum_{k=1}^{\infty}\left([k+p]_q-1+v\right)\emptyset_{\eta,p,q}^{n,m}\overline{b_{k+p}z^{k+p-1}}\right|$$

$$-\left|([p]_q-1-v)z^p\right.$$

$$+\sum_{k=2}^{\infty}\left([k+p]_q-1-v\right)\emptyset_{\eta,p,q}^{n,m}a_{k+p}z^{k+p-1}$$

$$\left.-\sum_{n=1}^{\infty}\left([k+p]_q+1+v\right)\emptyset_{\eta,p,q}^{n,m}\overline{b_{k+p}z^{k+p-1}}\right|$$

$$\geq\ 2\left([p]_q-v\right)|z^p|$$

$$-\sum_{k=2}^{\infty}\left([k+p]_q+1-v\right)\emptyset_{\eta,p,q}^{n,m}\left|a_{k+p}\right|\left|z^{k+p-1}\right|$$

$$-\sum_{k=1}^{\infty}\left([k+p]_q-1+v\right)\emptyset_{\eta,p,q}^{n,m}\left|b_{k+p}\right|z^{k+p-1}$$

$$-\left([p]_q-1-v\right)|z^p|$$
$$-\sum_{k=2}^{\infty}\left([k+p]_q-1-v\right)\emptyset_{\eta,p,q}^{n,m}\left|a_{k+p}\right|\left|z^{k+p-1}\right|$$
$$-\sum_{k=1}^{\infty}\left([k+p]_q-1+v\right)\emptyset_{\eta,p,q}^{n,m}\left|b_{k+p}\right|\left|z^{k+p-1}\right|$$

$$\geq 2\left([p]_q-v\right)|z^p|+\sum_{k=2}^{\infty}\left([k+p]_q-2v\right)\emptyset_{\eta,p,q}^{n,m}\left|a_{k+p}\right||z|^{k+p-1}$$
$$+\sum_{k=1}^{\infty}\left([k+p]_q+2v\right)\emptyset_{\eta,p,q}^{n,m}\left|b_{k+p}\right||z|^{k+p-1}$$

$$-2\left([p]_q-v\right)|z^p|$$
$$\geq\sum_{k=2}^{\infty}\left([k+p]_q-2v\right)\emptyset_{\eta,p,q}^{n,m}\left|a_{k+p}\right||z|^{k+p-1}$$
$$+\sum_{k=1}^{\infty}\left([k+p]_q+2v\right)\emptyset_{\eta,p,q}^{n,m}\left|b_{k+p}\right||z|^{k+p-1}$$

$$\sum_{k=2}^{\infty}\left([k+p]_q-2v\right)\left|a_{k+p}\right|\emptyset_{\eta,p,q}^{n,m}+\sum_{k=1}^{\infty}\left([k+p]_q+2v\right)\left|b_{k+p}\right|\emptyset_{\eta,p,q}^{n,m}$$
$$\leq 2\left([p]_q-v\right)$$

By using the enquiringly (3.4.9), we see that the last expression is non negative this implies that $f\in T_{\eta,p,q}^{n,m}(v)$

Now, the necessary and sufficient condition for a function belongs to the class $\overline{T}_{\eta,p,q}^{n,m}(v)$is get.

Theorem (3.4.2). Let $f = h + \bar{g}$ .then $f \in \bar{T}^{n,m}_{\eta,p,q}(v))$ if and only if

$$\sum_{k=2}^{\infty}\left([k+p]_q - 2v\right)|a_{k+p}||z|^{k+p-1}\emptyset^{n,m}_{\eta,p,q} + \sum_{k=1}^{\infty}\left([k+p]_q + 2v\right)|b_{k+p}||z|^{k+p-1}\,\emptyset^{n,m}_{\eta,p,q} \le 2\left([p]_q - v\right) \quad (3.4.12)$$

Where $a_1 = 1, 0 \le v \le 1$ and $\emptyset^{m}_{\lambda}(\alpha,\beta)$ given by (3.4.10)

Proof. Since$\bar{T}^{n,m}_{\eta,p,q}(v) \subseteq T^{n,m}_{\eta,p,q}(v)$ we only need to prove the only if part of the theorem. To this purpose, for functions $f \in \bar{T}^{n,m}_{\eta,p,q}(v)$,we notice that (3.4.8) is equivalent to

$$Re\left\{\frac{zD_q\left(\mathfrak{G}^{n,m}_{\eta,p,q}\,f(z)\right)}{\left(\mathfrak{G}^{n,m}_{\eta,p,q}\,f(z)\right)} - v\right\} \ge 0$$

and

$$\left\{\frac{\begin{pmatrix}([p]_q - v)z^p + \sum_{k=2}^{\infty}([k+p]_q - v)\,\emptyset^{n,m}_{\eta,p,q}a_{k+p}z^{k+p-1} - \\ \sum_{k=1}^{\infty}([k+p]_q - v)\,\emptyset^{n,m}_{\eta,p,q}\overline{b_{k+p}z^{k+p-1}},\end{pmatrix}}{z^p + \sum_{k=2}^{\infty}\emptyset^{n,m}_{\eta,p,q}a_{k+p}z^{k+p-1} + \sum_{k=1}^{\infty}\emptyset^{n,m}_{\eta,p,q}\overline{b_{k+p}z^{k+p-1}}}\right\} \ge 0 \quad (3.4.13)$$

The above condition must hold for all values of z in $\mathfrak{A}$ upon selecting the values of z on the positive real axis where $0 \le z = r < 1$, we must

$$\frac{([p]_q - v) + \sum_{k=2}^{\infty}[k+p]_q(1-v)\emptyset^{n,m}_{\eta,p,q}a_{k+p} - \sum_{k=1}^{\infty}[k+p]_q(1+v)\emptyset^{n,m}_{\eta,p,q}|b_{k+p}|}{1 + \sum_{k=2}^{\infty}\emptyset^{n,m}_{\eta,p,q}a_{k+p} + \sum_{k=1}^{\infty}\emptyset^{n,m}_{\eta,p,q}\overline{b_{k+p}}}$$

$$\ge 0 \quad (3.3.14)$$

If the condition (3.4.12) does not hold, then the numerator in (3.4.14) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in (0,1) for which the quotient of (3.4.14) is negative. Then, $f \in \bar{T}^{n,m}_{\eta,p,q}(v)$ and the proof is end.

Now, we want to show that the class $\bar{T}^{n,m}_{\eta,p,q}(v)$ is closed under the convex combination. Suppose that the function $\mathfrak{G}^{n,m}_{\eta,p,q} f_{k,i}(z)$ is given, where i=1, 2, 3…, m, by

$$\mathfrak{G}^{n,m}_{\eta,p,q} f_{k,i}(z) = z^p - \sum_{k=2}^{\infty} |a_{k+p,i}| z^{k+p-1} + \sum_{k=1}^{\infty} |b_{k+p,i}| \overline{z^{k+p-1}}, \qquad (3.4.15)$$

Extreme points:-

In this section, we provide extreme points for the class $\bar{T}^{n,m}_{\eta,p,q}(v)$.

Theorem (3.4.3). $f \in \bar{T}^{n,m}_{\eta,p,q}(v)$ If and only if f can be expressed as

$$f(z) = \sum_{k=1}^{\infty} (Y_k h_k + M_k g_k)$$

Where $z \in \mathfrak{A}$

$$h_p(z) = z^p, h_k(z) = z^p - \frac{2([p]_q - v)}{([k+p]_q - 2v)\emptyset^{n,m}_{\eta,p,q}} z^{k+p-1},$$

$$g_k(z) = z^p - \frac{2([p]_q - v)}{([k+p]_q + 2v)\emptyset^{n,m}_{\eta,p,q}} \bar{z}^{k+p-1},$$

$$\sum_{k=1}^{\infty} (Y_k + M_k) = 1, \qquad Y_k \geq 0, \text{and } M_k \geq 0$$

In particular, the extreme points of $\overline{T}_{\eta,p,q}^{n,m}(v)$ are $\{h_k\} and \{g_k\}$.

Proof. The function f we can write

$$f(z)=\sum_{k=1}^{\infty}(Y_k h_k+M_k g_k)=$$

$$=\sum_{k=1}^{\infty}(Y_k h_k+M_k g_k)z^p$$

$$=\sum_{k=2}^{\infty}\frac{2([p]_q-v)}{([k+p]_q-2v)\emptyset_{\eta,p,q}^{n,m}}Y_k z^{k+p-1}$$

$$-\sum_{k=1}^{\infty}\frac{2([p]_q-v)}{([k+p]_q+2v)\emptyset_{\eta,p,q}^{n,m}}M_k\bar{z}^{k+p-1}$$

$$=z^p$$

$$-\sum_{k=2}^{\infty}\frac{2([p]_q-v)}{([k+p]_q-2v)\emptyset_{\eta,p,q}^{n,m}}Y_k z^{k+p-1}$$

$$-\sum_{k=1}^{\infty}\frac{2([p]_q-v)}{([k+p]_q+2v)\emptyset_{\eta,p,q}^{n,m}}M_k\bar{z}^{k+p-1}$$

Then

$$\sum_{k=2}^{\infty}\left[\left([k+p]_q-2v\right)\emptyset_{\eta,p,q}^{n,m}\right)\right]\frac{2([p]_q-v)}{([k+p]_q-2v)\emptyset_{\eta,p,q}^{n,m}}Y_k$$

$$-\sum_{k=1}^{\infty}\left[\left([k+p]_q\right.\right.$$

$$\left.\left.+2v\right)\emptyset_{\eta,p,q}^{n,m}\right)\right]\frac{2([p]_q-v)}{([k+p]_q+2v)\emptyset_{\eta,p,q}^{n,m}}M_k$$

$$=2([p]_q-v)\left(\sum_{k=1}^{\infty}(Y_k+M_k)\right)=2([p]_q-v)$$

$$\leq 2([p]_q-v)$$

and so,

$$f \in \bar{T}^{n,m}_{\eta,p,q}(v)$$

Conversely, suppose that

$f \in \bar{T}^{n,m}_{\eta,p,q}(v)$ .setting

$$Y_k = \frac{\left([k+p]_q - 2v\right)\left|a_{k+p}\right|\emptyset^{n,m}_{\eta,p,q}}{2\left([p]_q - v\right)}$$

$$M_k = \frac{\left([k+p]_q + 2v\right)\left|b_{k+p}\right|\emptyset^{n,m}_{\eta,p,q}}{2\left([p]_q - v\right)}$$

We obtain

$f(z) = \sum_{k=1}^{\infty}(Y_k h_k + M_k g_k)$ As required.

In the next three theorems, we prove that the class $\bar{T}^{n,m}_{\eta,p,q}(v)$ is constant under convolution, convex combinations and neighborhood of its members.

And

$$F(z) = z^p - \sum_{k=1}^{\infty}\left|A_{k+p}\right|z^{k+p-1} \quad - \sum_{k=1}^{\infty}\left|B_{k+p}\right|\bar{z}^{k+p-1},$$

The convolution of two harmonic functions $f(z)$and $F(z)$ as

$$(f * F)(z) = f(z) * F(z)$$

$$= z^p - \sum_{k=1}^{\infty}\left|a_{k+p}\right|\left|A_{k+p}\right|z^{k+p-1} - \sum_{k=1}^{\infty}\left|b_{k+p}\right|\left|B_{k+p}\right|\bar{z}^{k+p-1}, \tag{3.4.16}$$

Using the definition above, we show that the class $\overline{T}_{\eta,p,q}^{n,m}(v)$ is closed under convolution.

Theorem (3.4.4) for $0 \leq v \leq w < 1, let\ f \in \overline{T}_{\eta,p,q}^{n,m}(v)\ and\ F \in \overline{T}_{\eta,p,q}^{n,m}(w)$ then

$$(f * F) \in \overline{T}_{\eta,p,q}^{n,m}(w) \subset \overline{T}_{\eta,p,q}^{n,m}(v)$$

Proof. We wish to show that the coefficients of $(f * F)$ satisfy the required condition given in theorem 2.1 for $F \in \overline{T}_{\eta,p,q}^{n,m}(w)$ we note that $0 \leq A_{k+p} \leq 1$ and $0 \leq B_{k+p} \leq 1$. Now, for the convolution $(f * F)$

We obtain

$$\sum_{k=2}^{\infty}\left([k+p]_q - 2w\right)|a_{k+p}||A_{k+p}||z|^{k+p-1}\emptyset_{\eta,p,q}^{n,m} + \sum_{k=1}^{\infty}\left([k+p]_q + 2w\right)|b_{k+p}||B_{k+p}||z|^{k+p-1}\emptyset_{\eta,p,q}^{n,m} \leq 2\left([p]_q - w\right)$$

$$\sum_{k=2}^{\infty}\frac{\left([k+p]_q - 2w\right)|a_{k+p}||A_{k+p}||z|^{k+p-1}\emptyset_{\eta,p,q}^{n,m}}{2\left([p]_q - w\right)} + \sum_{k=1}^{\infty}\frac{\left([k+p]_q + 2w\right)|b_{k+p}||B_{k+p}||z|^{k+p-1}\emptyset_{\eta,p,q}^{n,m}}{2\left([p]_q - w\right)} \leq 1$$

$$\leq \sum_{k=2}^{\infty}\frac{\left([k+p]_q - 2w\right)|a_{k+p}||z|^{k+p-1}\emptyset_{\eta,p,q}^{n,m}}{2\left([p]_q - w\right)} + \sum_{k=1}^{\infty}\frac{\left([k+p]_q + 2w\right)|b_{k+p}||z|^{k+p-1}\emptyset_{\eta,p,q}^{n,m}}{2\left([p]_q - w\right)} \leq 1$$

$$\leq \sum_{k=2}^{\infty} \frac{\left([k+p]_q - 2v\right)|a_{k+p}||z|^{k+p-1} \emptyset_{\eta,p,q}^{n,m}}{2\left([p]_q - v\right)}$$

$$+ \sum_{k=1}^{\infty} \frac{\left([k+p]_q + 2v\right)|b_{k+p}||z|^{k+p-1} \emptyset_{\eta,p,q}^{n,m}}{2\left([p]_q - v\right)} \leq 1$$

Since $0 \leq v \leq w < 1, let\ f \in \bar{T}_{\eta,p,q}^{n,m}(v).$ Therefore

$$(f * F) \in \bar{T}_{\eta,p,q}^{n,m}(w) \subset \bar{T}_{\eta,p,q}^{n,m}(v)$$

Theorem (3.4.5) Let $f, F \in \bar{T}_{\eta,p,q}^{n,m}(v)$ then $f * F \in \bar{T}_{\eta,p,q}^{n,m}(w)$ for

$$f(z) = z^p - \sum_{k=1}^{\infty} |a_{k+p}| z^{k+p-1} - \sum_{k=1}^{\infty} |b_{k+p}| \bar{z}^{k+p-1},$$

$$F(z) = z^p - \sum_{k=1}^{\infty} |A_{k+p}| z^{k+p-1} - \sum_{k=1}^{\infty} |B_{k+p}| \bar{z}^{k+p-1},$$

and

$$f * F = z^p - \sum_{k=1}^{\infty} |a_{k+p}||A_{k+p}| z^{k+p-1} - \sum_{k=1}^{\infty} |b_{k+p}||B_{k+p}| \bar{z}^{k+p-1},$$

where

$$L \geq \frac{4\left([p]_q - v\right)(2v)}{\left([k+p]_q - 2v\right)^2 - \left([p]_q - v\right)(2v)}$$

Proof. $f, F \in \bar{T}_{\eta,p,q}^{n,m}(v)$ and $0 < L < 1$

and so

$$\sum_{k=2}^{\infty} \frac{\left([k+p]_q - 2v\right)}{2\left([p]_q - v\right)} a_{k+p} \leq 1 \qquad (3.4.17)$$

and

$$\sum_{k=1}^{\infty}\frac{\left([k+p]_q+2v\right)}{2\left([p]_q-v\right)}b_{k+p}\leq 1 \tag{3.4.18}$$

We have to find the smallest number w such that

$$\sum_{k=2}^{\infty}\frac{\left([k+p]_q-L\right)}{\left([p]_q-L\right)}a_{k+p}b_{k+p}\leq 1 \tag{3.4.19}$$

By Cauchy-Schwarz inequality

$$\sum_{k=2}^{\infty}\frac{\left([k+p]_q-2v\right)}{2\left([p]_q-v\right)}\sqrt{a_{k+p}b_{k+p}}\leq 1 \tag{3.4.20}$$

Therefore, it's enough to show that

$$\frac{\left([k+p]_q-L\right)}{2\left([p]_q-L\right)}a_{k+p}b_{k+p}\leq\frac{\left([k+p]_q-2v\right)}{2\left([p]_q-v\right)}\sqrt{a_{k+p}b_{k+p}}$$

That is

$$\sqrt{a_{k+p}b_{k+p}}\frac{\left([k+p]_q-2v\right)L}{\left([k+p]_q-L\right)v}$$

From (3.4.20)

$$\sqrt{a_{k+p}b_{k+p}}\leq\frac{2\left([p]_q-v\right)}{\left([k+p]_q-2v\right)}$$

Thus it is enough to show that

$$\frac{2\left([p]_q-v\right)}{\left([k+p]_q-2v\right)}\leq\frac{\left([k+p]_q-2v\right)L}{\left([k+p]_q-L\right)v}$$

Which simplifies to

$$L \geq \frac{\left([p]_q - v\right)(2v)}{\left([k+p]_q - 2v\right)^2 - \left([p]_q - v\right)(2v)}$$

Theorem (3.4.6) the class $\overline{T}_{\eta,p,q}^{n,m}(v)$ is closed under convex combination.

For i =1, 2, 3… suppose $f_i \in \overline{T}_{\eta,p,q}^{n,m}(v)$ where f_i is given by

$$f_i(z) = z^p - \sum_{k=2}^{\infty} a_{i,k+p} z^{k+p-1} - \sum_{k=1}^{\infty} b_{i,k+p} \bar{z}^{k+p-1},$$

Then by

$$\sum_{k=2}^{\infty} \emptyset_{\eta,p,q}^{n,m} a_{i,k+p} + \sum_{k=1}^{\infty} \emptyset_{\eta,p,q}^{n,m} b_{i,k+p} \leq 1 \qquad (3.4.21)$$

for,

$$\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1,$$

The convex combination of $f_i$may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z^p - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{i,k+p}\right) z^{k+p-1}$$
$$+ \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i b_{i,k+p}\right) \bar{z}^{k+p-1}$$

Using the inequality (3.4.21), we obtain

$$\sum_{k=2}^{\infty} \emptyset_{\eta,p,q}^{n,m}\left(\sum_{i=1}^{\infty} t_i a_{i,k+p}\right) + \sum_{k=1}^{\infty} \emptyset_{\eta,p,q}^{n,m}\left(\sum_{i=1}^{\infty} t_i b_{i,k+p}\right)$$

$$\sum_{i=1}^{\infty} t_i \left(\sum_{k=2}^{\infty} \emptyset_{\eta,p,q}^{n,m}\, a_{i,k+p} + \sum_{k=1}^{\infty} \emptyset_{\eta,p,q}^{n,m}\, a_{i,k+p}\right) \leq \sum_{i=1}^{\infty} t_i = 1$$

The proof is complete.

Neighborhood:

See **[36], [62].**Let $s \in \overline{T}_{\eta,p,q}^{n,m}(v)$ is of the form

$$s(z) = z^p - \sum_{k=2}^{\infty} |A_{k+p}| z^{k+p-1} + \sum_{k=1}^{\infty} |B_{k+p}| \bar{z}^{k+p-1}; \quad |B_p| < 1; \quad z \in \mathfrak{U} \tag{3.4.22}$$

For $f = h(z) + \bar{g}(z)$ define by (1.4) belonging to $\overline{T}_{\eta,p,q}^{n,m}(v)$ the neighborhood of in $\overline{T}_{\eta,p,q}^{n,m}(v)$ is defined by

$$N_\delta(f,s) := \left\{ s \in \overline{T}_{\eta,p,q}^{n,m}(v): \sum_{k=2}^{\infty} k(|(|a_k|-|A_k|)| + |(|b_k| - |B_k|)|) + p|(|b_p| - |B_p|)| \le \delta \right\}.$$

In particular for $e(z) = z^p$ we have

$$N_\delta(e,s) := \left\{ s \in \overline{T}_{\eta,p,q}^{n,m}(v) : \sum_{k=2}^{\infty} k(|A_k| + |(|B_k| - |B_k|) + p|B_p| \le \delta \right\}.$$

We have the following inclusion result:

Theorem (3.4.7). Let $s(z)$ given by (3.4.22) be in the class $\overline{T}_{\eta,p,q}^{n,m}(v)$ then $\overline{T}_{\eta,p,q}^{n,m}(v) \subset N_\delta(e,s)$

Where $\delta = 2\big([p]_q - v\big) + p|B_p|$

Proof. Let $s(z) \in \overline{T}_{\eta,p,q}^{n,m}(v)$ from theorem (2.2) we ge

$$\sum_{k=2}^{\infty}\left([k+p]_q-2v\right)\left|A_{k+p}\right|\left(\frac{p+\eta k}{p}\right)^m$$
$$+\sum_{k=1}^{\infty}\left([k+p]_q+2v\right)\left|B_{k+p}\right|\left(\frac{p+\eta k}{p}\right)^m \leq 2\left([p]_q-v\right)$$

Upon simplification, the above inequality reduced to

$$\left(\frac{p+\eta k}{p}\right)^m\sum_{k=2}^{\infty}k(|A_k|+|B_k|)+p\left|B_p\right|-\left(\frac{p+\eta k}{p}\right)^m\sum_{k=2}^{\infty}p\left|A_{k+p}\right|$$
$$+\ p|B_p| \leq 2\left([p]_q-v\right)$$

Note that

$\left(\frac{p+\eta k}{p}\right)^m \geq 1$ For all $v \in [0, 1)$ the above inequality becomes

$$\sum_{k=2}^{\infty}k(|A_k|+|B_k|)+p\left|B_p\right|$$
$$\leq 2\left([p]_q-v\right)$$
$$+\left(\frac{p+\eta k}{p}\right)^m\sum_{k=2}^{\infty}p(|A_k|+p|B_k|) \qquad (3.4.23)$$

Further, applications of theorem 2.2 we also have

$$\left(\frac{p+\eta k}{p}\right)^m\left\{\sum_{k=2}^{\infty}p(|A_k|+p|B_k|)\right\}$$
$$\leq 2\left([p]_q-v\right)+p\left|B_p\right| \qquad (3.4.24)$$

From inequalities (3.4.23) and (34.24) yield

$$\sum_{k=2}^{\infty}k(|A_k|+|B_k|)+p\left|B_p\right| \leq 2\left([p]_q-v\right)+p\left|B_p\right|$$

Hence

$$s(z) \in N_\delta(e,s)$$

Chapter Four

Conclusion and Future Work

Conclusions

At the end of the current study it can be concluded that

- From section (2.1) that the differential operator dependents on the parameters σ and δ and any change in its values impels large change in the class as defined in definition (2.1.1).
- From section (2.2) the inequality (2.2.2) is dependent on the parameters α and β and any change in its values impels large change in the class as defined in definition (2.2.6).
- From section (3.1), that the differential operator depends on the parameters $v, \varrho, a_1$and b_1 and any change in its values leads to large change in the class as define in definition (3.1.1).
- That the subclass $\bar{T}_{\eta,p,q}^{n,m}(v)$ in section (3.4) depends on the parameter v and any change in its values impels large change in the defined in definition (3.4.1).

Future Works

At the end of the study, the researcher suggests that other researches can conduct studies on:

1. Some subclass of meromorphic univalent functions defined by integral operator.
2. Majorization properties for subclass of analytic and univalent functions associated with differential operator including mittag-leffler function.
3. The class of meromorphic p-valent functions defined by differential operator.
4. A new strong differential subordination and superdination defined by the generalized differential operator.
5. Some properties of subclass of meromorfic multivalent harmonic function.
6. A partial sums of subclass of univalent normalized functions based on Mittag-Leffler functions.

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الخلاصة

الغرض من هذا العمل هو دراسة بعض الخصائص الهندسية لفئات فرعية جديدة من الدوال التحليلية الاحادية التكافؤ ومتعددة التكافؤ بالإضافة إلى الدوال التوافقية أحادية التكافؤ ومتعددة التكافؤ في قرص الوحدة المفتوح

$$(\mathfrak{U} = \{z \in \mathbb{C}: |z| < 1\})$$

حيث تم تقديم فئات فرعية جديدة لكل دالة بواسطة مؤثرات مختلفة لدراسة كل من النقاط المتطرفة، والانغلاق، وتقديرات المعلمات، ونظرية النمو والتشويه، والنجمية، والمحدبة، sense preserving، والالتفاف والإغلاق تحت مجموعة محدبة، عامل التكامل والجوارات، بالإضافة إلى أننا درسنا خصائص التبعية وأفضل دالة مهيمنة وتبعية الدوال التحليلية المتعددة التكافؤ، والخصائص المتعلقة بالتبعية لدوال التحليلية متعددة التكافؤ.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

دراسة في فئات فرعية جديدة للدوال التوافقية الاحادية التكافؤ ومتعددة التكافؤ

رسالة

مقدمة الى مجلس كلية التربية للعلوم الصرفة / جامعة بابل كجزء من متطلبات نيل درجة الماجستير في التربية / الرياضيات

من قبل

عدي حاتم صاحب علي

باشراف

أ.م.د. عقيل كتاب الخفاجي

1443هـ 2022