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University of Babylon
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Department of Mathematics

On Primary Subsemimodules

A Research

Submitted to the Faculty of the Department of Mathematics, College of Education for Pure Science, University of Babylon as a Partial Fulfillment of the Requirements for the Degree of High Diploma Education / Mathematics

by

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بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

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لَأَجْرًا غَيْرَ مَمْنُونٍ ﴿٣﴾ وَإِنَّكَ لَعَلَىٰ خُلُقٍ عَظِيمٍ ﴿٤﴾

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Dedication

I dedicate my research to

my family and many friends.

Special gratitude to

my loving parents, whose words of encouragement

and push for tenacity ring in my ears.

My sisters and brothers have never left

my side and are incredibly special.

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I would like to acknowledge everyone who played a role in my academic accomplishments. First of all, my parents, my wife, and children supported me with love and understanding. Without you, I could never have reached this current level of success.

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List of symbols

symbols	Definitions
∩	Intersection
∪	Union
∀	For all
=	Equal
≠	Not equal
⇒	Implies
∈	Belong to
∉	Doesn’t belong
∋	Such that
■	End of the proof
∅	Empty set
√	Nil radical
⊆	Subset
⊊	Proper subset
∃	There exists
ℕ	The set of non-negative integers
∴	Therefore
∞	Infinity
[K:M]	Residual of K
$\boldsymbol{K_2 K_1 \oplus}$	Direct Sum
rad	Jacobson radical

Abstract

This research introduces and investigates the concept of primary subsemimodule over a semiring. It aims to reflect results and related notions that were investigated about these notions in the category of submodules to the category of subsemimodules. The main results of the study are given as follows by assuming K be a primary subsemimodule of an S-semimodule M. If $[K: M]$ is a semi-prime ideal of S, then K is a prime submodule or subsemimodule of M. Then $\sqrt{[K: M]} = \sqrt{[K: c\,]}$, for each $c \notin K$, where $c \in M$. Next, $\sqrt{[K: M]} = \sqrt{[\, K: L]}$ For each subsemimodule $L \ni K \subset L$. After that, [K: I] is a primary subsemimodule of M, for every ideal I of S. Later, if L is a subsemimodule of M such that L is not contained in K, then $K \cap L$ is a primary subsemimodule in K. Finally, if M_1 and M_2 be two S-semimodule and $M = M_1 \oplus M_2$ and $K = K_1 \oplus K_2$ is primary subsemimodule of M, then K_1 and K_2 are primary subsemimodules of M_1 and M_2 respectively and not conversely.

Introduction

The concept of semiring was introduced by Vandaier in 1934 [17]. A semiring is a nonempty set together with two associative binary operations called addition and multiplication (denoted by + and · respectively) that satisfies some condition.

Semirings and semimodules have many applications in computer science, physics, algebra, geometry, algebraic topology, cryptography and many other branches of modern science [7,10]. Much significant construction in pure and applied mathematics can be understood as semimodules over appropriates semiring [11].

Many researchers studied Semimodule over semiring. The book "the theory of semiring with applications in mathematics and theoretical computer science " by J.S.Golan [18] is the first book on semiring written with an algebraic point of view having computer applications. It is beneficial to work on semiring and semimodules and has several applications [3].

The theory of semimodule over semiring with identity generalises the idea of modules over rings with identity [7,10]. As the modules over the ring are essential tools in characterizing the ring properties, we should look at the corresponding structure over semirings. Indeed many of the constructions from ring theory can be transferred, at least partially, to this more general setting [15]

For studying semiring, one needs to study the semimodule over them. The concept of semimodules is one of the essential branches of mathematic [12]. The semirings are commutative if their multiplication is commutative [16].

Given a semiring S, a left S-semimodule M is a commutative monad (M,+) for which we have a function S×M→ M defined by (s, m) → sm (scalar multiplication), which satisfies some conditions. If the condition 1m=m, $\forall m \in M$ hold, then the semimodule M is said to be unitary [2].

The concept of the primary submodules was introduced by Deore [6]. A proper subsemimodule K of S-semimodule M is called primary subsemimodule of M when are sx∈ K, s∈ S, and x∈ M has either x∈ K or

$$\mathrm{s}^{\mathrm{n}} \in [\mathrm{K}\colon \mathrm{M}] = \{\mathrm{s} \in \mathrm{S} : \mathrm{s}^{\mathrm{n}}\ \mathrm{M} \subseteq \mathrm{K}, \text{for some } \mathrm{n} \in \mathrm{Z}_{+}\}.$$

The primary subsimemodule is concerned with generalizing some results in ring and module theory throughout, S is a commutative semiring with identity, M is a unitary left S- semimodule.

The proposition research is divided into two chapters. Chapter one displays the basic concepts of related study, consisting of the definitions, remarks, examples, and relationships between them. This chapter includes three sections. Section one presents the concepts of semiring, ideals and other concepts related to this effort. Section two consists of some definitions, remarks and examples that are needed. Section three adducer the definitions of residual, prime subsememimodule, same remarks and examples that are required. Chapter two consists of the notions resulting from the primary subsimemodules concerned with generalizing some studies in module theory.

Chapter One

Preliminaries

Chapter One

Preliminaries

1.1. Introduction

This chapter will introduce some concepts we need and prove some results. This chapter consists of three sections. The first section states the definitions of a semiring, ideals, and their relation. The second section presents the definitions of semimodule and subsemimodule. In the third section, state the definitions of residual and prime subsemimodule.

1.2. Semiring and Ideals

This section will introduce some of the definitions and remarks needed in the main results. We begin with the definitions of semiring and ideals.

Definition 1.2.1 [10]

A semiring is a nonempty set S on which operations of addition and multiplication have been defined such that the following conditions are satisfied.

1. (S, +) is a commutative monoid with identity element 0.
2. (S,.) is a monoid with identity element $1 \neq 0$ ($1 = 1_S$).

3. Multiplication distributes over addition, i.e.a (b + c) = ab + ac and (a + b) c = ac + bc for all a, b, c $\in$ S.
4. r. 0 = 0. r = 0 for all r $\in$ S.

The semiring S is said to be commutative if its multiplication is commutative.

Example 1.2.2

(Z, +,.) is commutative semiring.

Definition 1.2.3 [10]

If S is a semiring and $0 \neq a \in S$, then a is called a left (right) zero divisor in S if there exists some b ≠ 0 in S such that ab = 0 (ba = 0). A zero divisor is any element of S that is either left or right zero divisor.

Remark 1.2.4

Every ring is semiring, but the converse is not true as in the following example.

Example 1.2.5

If $\mathbb{N}$ = {0, 1, 2, …}, then ($\mathbb{N}$, +,.) is a commutative semiring, but not ring for $-n \notin \mathbb{N}$ for all $n \in \mathbb{N}$.where -n is the additive inverse.

Definition 1.2.6 [10]

A subset S of a semiring S is a subsemiring of S if it contains 0 and is closed under the operations of multiplication and addition in S.

Example 1.2.7

(ℕ, +,.) is a semiring. let ⟨2⟩ = {0, 2, 4, 6, …}, then (⟨2⟩, +,.) is a subsemiring of ℕ.

In general ⟨n⟩ is a subsemiring of ℕ, where $n \in \mathbb{N}$.

Definition 1.2.8 [10]

An element a of a semiring S is said to be nilpotent if $a^n = 0$ for some $n \in Z_+$.

Definition 1.2.9 [10]

A nonempty subset I of a semiring S will be called an ideal of S if ab ∈ I and $r \in S$ imply a + b ∈ I, ra, and ar ∈ I.

Example 1.2.10

(⟨3⟩, +,.) is ideal of (ℕ, +,.) for let a = 3n∈ ⟨3⟩ b=3m for nm ∈ ℕ, then a + b = 3n + 3m = 3(n + m) ∈<3> for any t ∈ ℕ ta = t(3n) = 3tn ∈ ⟨3⟩ and (3n) t=3nt ∈<3>.

Proposition 1.2.11

If I is a proper ideal of a semiring S with identity, then no element of I possesses a multiplicative inverse.

Proof

Let I be a proper ideal of S and suppose that 0≠ a∈I such that a^{-1} exist in S. Since I is closed under multiplication by arbitrary semiring elements, then 1=a

a^{-1}, then 1∈ I and s=s.1 for s ∈ S. So, S ⊆ I ⇒ S =I, which is a contradiction, then no element of I possess a multiplication inverse ■

Corollary 1.2.12

In a semiring with identity, no proper ideal contains the identity elements of multiplication.

Definition 1.2.13 [10]

An ideal I of the semiring S is a prime ideal if for all a,b in S such that ab ∈ I implies either a ∈ I or b∈ I.

Example 1.2.14

< p> is prime ideal of ℕ where p is prime, but <6> is not prime ideal for (2.3) = 6 ∈ <6> where 2 ∉ <6> and 3∉ <6>.

Definition 1.2.15 [10]

An ideal I of the semiring S is said to be a maximal ideal provided that I≠ S and whenever J is an ideal of S with I ⊂ J ⊆ S, then J = S.

Definition 1.2.16 [10]

The Jacobson radical of a commutative semiring S, denoted by rad (S), is the set rad S = ∩{M|M is a maximal ideal of S}. If rad S = {0}, then S is said to be a semiring without Jacobson radical.

For example $(Z_6, +_6, ._6)$, $I_1 = \{\bar{0}, \bar{2}, \bar{4}\}$, $I_2 = \{\bar{0}, \bar{3}\}$ are maximal ideal rad $(Z_6) = \{\bar{0}\}$.

Theorem 1.2.17 [10]

Every maximal ideal is a prime ideal in a commutative semiring with identity.

Definition 1.2.18 [10]

An ideal I of the commutative semiring S is called a primary if the conditions ab$\in I$ and a$\notin$I imply $b^n \in I$ for some positive integer n.

Definition 1.2.19 [10]

Let I be an ideal of the semiring S. The nil radical of I, denoted by $\sqrt{I}$, is the set $\sqrt{I}$ = {r ∈ S | r^n ∈ I for some n ∈ Z_+ }.

Establish the following facts concerning primary ideals:

a. Every prime ideal is a primary ideal.

b. If I is a primary ideal of S, then its nil radical $\sqrt{I}$ is the smallest prime ideal of S containing I.

c. An ideal I of the semiring S is a primary if and only if every zero divisor of the quotient semiring S/I is nilpotent.

The primary ideals of the semiring Z are precisely the ideals (p^n), where p is a prime and n$\in Z_+$.

Definition 1.2.20 [10]

An ideal I of a semiring S is called the semiprime ideal of S if $\sqrt{I}$ =I.

1.3. Semimodule

In this section, we will introduce some definitions we will need. We begin with the definition of a semimodule and subsemimodule.

Definition 1.3.1 [11]

Let S be a semiring. A left S-semimodule is a commutative monoid $(M,+)$ with additive identity 0_M for which we have a function $S \times M \longrightarrow M$ defined by $(s,m) \longrightarrow s\,m$ (scalar multiplication), which satisfies the following conditions for all elements $s, \bar{s} \in$ S and all elements $m, \bar{m} \in$ M.

1. $(s\bar{s})$ m =s $(\bar{s}\,m)$.
2. s (m +$\bar{m}$) = sm + s$\bar{m}$.
3. (s + $\bar{s}$) m =sm + $\bar{s}$m.
4. $s0_M = 0_M = 0_S m$.

Suppose condition $1m = m$ for all $m \in M$ hold, then the semimodule M is said to be unitary. By semimodule, we mean left S-semimodule.

Example 1.3.2

1) Every semiring is a semimodule over itself.
2) If $(\mathbb{N}_6 , +_6 , \cdot_6)$ is commutative monoid, the $(\mathbb{N},+)$ is semimodule over $\mathbb{N}_6$ = [0],[1],[2],[3],[4],[5].

Remark 1.3.3

Every module is a semimodule, but the converse is not true.

Definition 1.3.4 [11]

A nonempty subset K of a S-semimodule M is called subsemimodule of M if K is closed under addition and scalar multiplication that is K is an S-semimodule itself.

Example 1.3.5

1) (<k>, +) is subsemimodule of the ℕ-semimodule ℕ, where k∈ ℕ.
2) If S is a semiring, then the ideal of S are exactly the subsemimodules of the S -semimodule S.
3) Let B = (Z_4 $+_4$) be the monoid of integers 4, then B as an ℕ semimodule has $\{\bar{0}\}$,$\{\bar{0},\bar{2}\}$ proper subsemimodules.

Definition 1.3.6 [10]

A subsemimodule U of a S-semimodule B is called subtractive subsemimodule, if for each b, b+ b^{-} ∈ U, then b^{-} ∈ U.

Definition 1.3.7 [11]

Let M be an S-semimodule and $m \in M$. The set annihilator of m is dented by ann(m)= $\{t \in S | tm = 0\}$.

Remark 1.3.8

ann(m) is a left ideal of S.

Definition 1.3.9 [11]

Let K be a subsimemodule of S-simemodule M. The annihilator of K is defined by ann(K)= $\{t \in S|\ t\ k = 0, \forall\ k \in K\}$.

Example 1.3.10

Let $< \bar{2} >$ be subsemimodule of $\mathbb{N}$-semimodule Z_6 then ann $(< \bar{2} >) = 3\mathbb{N}$.

Definition 1.3.11 [10]

An ideal I of the semiring S is said to be a semiprime ideal if and only if I = $\sqrt{I}$

Definition 1.3.12 [2]

Let M be a left S-semimodule

1. A subset X of M is called a generating set of M if M =< X >.
2. A semimodule (or a subsemimodule) is called finitely generated (briefly f.g) if there exists a finite generating set.
3. A semimodule (or subsemimodule) is called cyclic, if ∃ m∈ M such that <{m}>=M (simply < m > = M).[2]

1.4. Prime semimodule

In this section, we will introduce some definitions and remarks. Such as, we begin with the definition of residual a prim semimodule and the relation between them.

Definition 1.4.1 [9]

Let K be a subsimemodule of an S-simemodule M the residual of K is denoted by [K: M] is defined as [K: M] = {s ∈ S: sM ⊆ K}

Example 1.4.2

Let K = <$\bar{4}$ > be subsimemodule of ℕ a simemodule N_8 .

$N_8 = \{ \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

$0\, N_8 = \bar{0} \subseteq K$

$1N_8 = N_8 \nsubseteq K$, $2N_8$ =<$\bar{2}$>⊈ K

$4N_8 = \{\bar{0}, \bar{4}\} \subseteq K$. *Hence* [<4>:N_8]=<4>

Proposition 1.4.3

The residual of a subsemimodul is an ideal.

Proof

Let K be a subsemimodule of an S-semimodule M. Let $x_1, x_2 \in [K: M]$ $\longrightarrow x_1M \subseteq K \wedge x_2M \subseteq K \longrightarrow (x_1M + x_2M) \subseteq K \longrightarrow (x_1 + x_2)M \subseteq K \longrightarrow x_1 + x_2 \in$ $[K:M]$, $x_1 \in [K:M] \wedge s \in S \longrightarrow x_1M \subseteq K \longrightarrow s(x_1M) \subseteq K \longrightarrow (s\, x_1)M \subseteq K \longrightarrow$ $sx_1 \in [K:M]$.

$sM \subseteq M \longrightarrow x_1(sM) \subseteq x_1M \subseteq K \longrightarrow (sx_1)M \subseteq K \longrightarrow x_1s \in [K:M]$.

So [K: M] is closed under addition and closed under multiplication by elements of S from left and right; therefore, [K: M] is an ideal of S ■

Definition 1.4.4 [9]

A subsemimodule K of an S-semimodule M is called prime subsimemodule if for all sx ∈ K implies either x∈ K or s∈ [K: M] where s∈S and x∈M.

Example 1.4.5

Let L=$<\bar{2} >$ be subsemimodule of $\mathbb{N}_8$. L= $\{\bar{0}, \bar{2}, \overline{4, \bar{6}}\}$ is prime subsemimodule of $\mathbb{N}_8$. But K in example (1.3.2) is not prime subsemimodule for $\bar{4}$ ∈ K and $2.\bar{2} = 4, \bar{4}$ ∈ K and $2\mathbb{N}_8 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \nsubseteq$ K.

Remark 1.4.6

1. Not every semimodule has a prime subsemimodule, as Z_{p^∞} is Z semimodule, which has no prime a subsemimodule.
2. K is prime subsimemodule of M iff $[L: K] = [K: M]$ for all subsimemodules of M ∈ $L \subset K$.

Definition 1.4.7 [4]

An S-semimodule M is said to be a prime semimodule if ann(K) = ann(M) For every non-zero subsemimodule K of M.

Example 1.4.8

Let Z be Z-semimodule ann(Z) ={0} , ∀ n ∈ Z ,<0> is a subsemimodule ann<n> = {0} , ∴ Z is a prime semimodule .

Proposition 1.4.9 [8]

If N is a prime subsimemodul of an S-semimodule M whose residual is P, then P is a prime ideal of S.

The converse of Proposition 1.4.9 is not true in general, as we see in the following example.

Example 1.4.19

Let M = $\mathbb{N} \oplus \mathbb{N}$ be an $\mathbb{N}$-semimodule and K be the subsimemodule generated by (2,0), then [K:M] = {0} which is a prime ideal in $\mathbb{N}$ while K is not prime subsimemodule of M.

Remark 1.4.11

Let A, B be ideals in semiring S, and I prime ideal in S such that $A\cap B \subseteq I$, then $A \subseteq I$ or $B \subseteq I$.

This note is not true in the semimodule as in the example.

Proposition 1.4.12

Let L, N be subsemimodules of an S-semimodule M and K be a prime subsemimodule of M such that $L\cap N \subseteq K$, then either $L \subseteq K$ or $[N:M] \subseteq [K:M]$.

Proof

Suppose that $[N:M] \nsubseteq [K:M]$, then $\exists$ $s\in [N:M]$ where $s \notin [K:M]$, let $t \in L$, the $st \in N\cap L$, then $st\in K$, But K is prime subsemimodule and $s\notin [K:M]$, then $t \in K$, that is mean $L\subseteq K$ ■

Example 1.4.13

In $\mathbb{R}^3$ be over R, any proper subsemimodule of R^3 is prime. If L_1= <(*1,0,0*)> *and* L_2=<(0,1,0)> *and* L_3 = <(0,0,1)> *then* $L_1 \cap L_2$ =0 ,the $L_1 \cap L_2 \subseteq L_3$ But $L_1 \subsetneq L_3$ and $L_2 \subsetneq L_3$.

Chapter Two

Primary Subsemimodule

Chapter Two

Primary Subsemimodule

This chapter aims to study primary subsemimodules and obtain some properties and characterizations of this class of semimodule. Some remarks, examples, and propositions related to this concept will be investigated in this chapter. We talk about multiplicatively closed sets, P-maximal subsemimodule, and multiplication semimodule.

Definition 2.2 [6]

A proper subsemimodule K of a S-semimodule M is called a primary subsemimodule of M if whenever sx∈K, s∈ S and x∈ M then either x ∈K or $s^n \in$ [K:M] for some n∈ Z_+.

Example 2.3

Z_8 is N-semimodule and $\langle\bar{4}\rangle$ is the primary subsemimodule of Z_8. For $<\bar{4}>=\{\bar{0},\bar{4}\}$ $\bar{0} = 2.\bar{4} = 4.\bar{2} = 4.\bar{4} = 4.\bar{6}$,$\bar{4} = 1.\bar{4} = 3.\bar{4} = 4.\bar{3}$ if x $\notin < \bar{4} >$ then $s^n \in [< \bar{4} >: Z_8]$.

Remark 2.4

Every prime subsemimodule is a primary subemimodule. But, the converse is not necessarily true in general, as an example.

Example 2.5

From example (2.3)

2. $\bar{2} = \bar{4} \in \langle \bar{4} \rangle = \{ \bar{0} , 4\}$

$\bar{2}^{2} = \bar{4} \in \langle \bar{4} \rangle \Rightarrow$ Primary

But $\bar{2} \notin \langle \bar{4} \rangle$

$[\langle \bar{4} \rangle : Z_8] = \langle \bar{2} \rangle \nsubseteq \langle \bar{4} \rangle$

$\therefore \langle \bar{4} \rangle$ is Primary but not prime.

Proposition 2.6

If K is a primary subsemimodule of an S -semimodule M and [K:M] is a semi-prime ideal of S, then K is a prime subsemimodule of M.

Proof:

Let s $\in$ S and x$\in$ $M \ni$ sx $\in$ K and x$\in$ K. Since K is a primary subsemimodule, then $s^n \in$ [K:M] for some n $\in Z_+$, hence s $\in \sqrt{[K:M]}$. But [K:M] is semi-prime, then s $\in$ [K:M]. So K is a prime subsemimodule of M■

Proposition 2.7

Let K be a subsemimodule of an S-semimodule M. If K is a primary subsemimodule of M, then $\sqrt{[K : M]}=\sqrt{[K:c]}$ for each c$\notin$ K.

Proof

Let a $\in \sqrt{[K{:}c]}$ and c $\notin$ K, Hence, $a^n c \in$ K for some n $\in Z_+$. But K is a primary subsemimodule of M and c $\notin$ K, then a $\in \sqrt{[K : M]}$. So

$$\sqrt{[K{:}c]} \subseteq \sqrt{[K : M]}. \ . \ . \ . \text{(i)}$$

Let s $\in \sqrt{[K : M]}$, hence $s^n \in$ [K:M] for some n $\in Z_+$. then s^n M ⊆ K. But s^n c ⊈ s^n M, so s^n c ⊈ K. Hence, $s^n \in$ [K:c] for some n $\in Z_+$Therefore, $s^n \in \sqrt{[K{:}c]}$ and

$$\sqrt{[K : M]} \subseteq \sqrt{[K{:}c]} \ . \ . \ . \ . \text{(ii)}$$

From (i) and (ii) implies $\sqrt{[K : M]} = \sqrt{[K{:}c]}$.For each c $\notin$ K ■

Proposition 2.8

Let K be a primary subsemimodule of an S-semimodule M, then $\sqrt{[K{:}M]}$ = $\sqrt{[K{:}L]}$ for each subsemimodule L of M such that K$\subset L$.

Proof:

Since$K \subset L\, , then\ [K : M] \subseteq [K{:}L]$, then

$$\sqrt{[K{:}M]} \subseteq \sqrt{[K{:}L]} \ \ldots\ldots \text{(i)}$$

Since K is the primary subsemimodule, then by Proposition (2.7)

$\sqrt{[K{:}M]}$= $\sqrt{[K{:}c]}$ for each c ∉ K.

Let s $\in \sqrt{[K{:}L]}$, hence s^n L ⊆ K for some n $\in Z_+$ .But K ⊊ L, then there exists x $\in$ L and x $\notin$K. Hence s^n x $\in$ K, then, $s^n \in$ [K: x] $\subseteq \sqrt{[K{:}L]} = \sqrt{[K{:}M]}$ (by

Theorem (2,1,5) which implies that $(s^n)^m \in [K:M]$ for some $m \in Z_+$, and then $s \in \sqrt{[K:M]}$, so that

$$\sqrt{[K:L]} \subseteq \sqrt{[K:M]} \; . \; . \; . \; . \quad \text{(ii)}$$

From (i) and (ii) $\sqrt{[K:M]} = \sqrt{[K:L]}$ for each K $\subsetneq$L. From theorem (1.1.3), we get the following corollary■

Corollary 2.9

Let M be an S-semimodule. If < 0 > is a primary subsemimodule of M, then $\sqrt{\text{ann M}} = \sqrt{\text{ann K}}$ for each non-zero subsemimodule K of M.

We shall give another characterization of primary subsemimodules but first recall the following.

M. D. Larsan and P. J. Mccarlthy in [13] define a multiplicatively closed subset of the ring, and they prove that.

1- Every proper ideal P in a ring is prime if and only if R -P is a multiplicatively closed subset of R.
2- If K is a submodule of an S- module M and S be multiplicatively closed subset of S, then K(s) ={x∈ M | ∃ t∈S, such that tx∈ K} be a submodule of M and K⊆ K(S).

Definition 2.10

We call a subset S of a commutative semiring R multiplicatively closed if $0 \notin S$, $1 \in$ S and ab $\in$ S, whenever a $\in$ S and $b \in$ S.

Proposition 2.11

Every proper ideal I in a semiring S is prime if and only if S-I is a multiplicatively closed subset of S.

Proof

Suppose that I is a prime ideal to prove that S-I is a multiplicatively closed subset of S. If $1 \notin S - I \implies 1 \in I$ which is a contradiction, so 1 $\in$ S-I.

Let a,b $\in$ S-I. If abSI, the$- \notin$ n abI. Since I is prime, then a or b $\in$ I, which is a contradiction, then ab $\in S -$ I. Hence S-I is a multiplicatively closed set of S. Conversely. Suppose that S-I is a multiplicatively closed subset of S to prove that I is a prime ideal. Let ab $\in$ I, if a $\notin$ I and b $\notin$ I, then a $\in$ S $-$ I and b $\in$ S $-$ I. Since S-I is a multiplicatively closed set, then ab$\in S -$ I, which is a contradiction. Hence either a $\in$ I or b $\in$ I.So I is prime ideal ■

Proposition 2.12

Let K be a proper subsemimodule of an R-semimodule M. Then K is a primary subsemimodule if and only if K(S) = K for each S (a multiplicatively closed subset of R) such that.

$$S \cap \sqrt{[K : M]} = \emptyset$$

Proof

If K is a primary subsemimodule, prove K = K(S). It is clear that K ⊆ K(S) so it is enough to show that K(S) ⊆ K. Let x ∈ K(S), so there exists t ∈ S such that t x ∈ K. But K is a primary subsemimodule, so either x∈K or t ∈ $\sqrt{[K:M]}$. But t ∈ $\sqrt{[K:M]}$ implies that t ∈ $S \cap \sqrt{[K:M]} = \emptyset$ which is a contradiction. Thus, x ∈ K and hence K (S) ⊆ K. Thus k(S) = K.

Conversely, if K(S) = K for each multiplicatively closed subset S of R such that $S \cap \sqrt{[N:M]} = \emptyset$. To prove K is a primary subsemimudule. Let rx ∈ K and suppose that r∉$\sqrt{[K:M]}$.

Let S = {1, r, r^2, …….}, then S is a multiplicatively closed subset. $S \cap \sqrt{[K:M]} = \emptyset$, Thus k(S) = K. On the other hand, r x ∈ K and r ∈S imply that x ∈ k(S). Therefore x ∈ K ■

Note

If K is a subsemimodule of an S-semimodule M and I is an ideal of S, then [M: I] = {x ∈ M: x I ⊆ K} is a subsemimodule of M and containing K, and [K: S] = K.

Proposition 2.13

If K is a primary subsemimodule of M, then [K: I] is a primary subsemimodule of M for every ideal I of S.

Proof

If K is a primary subsemimodule of M such that sm $\in$[K:I] where s$\in$S,m$\in$ M and I be an ideal of S, then asm$\in$Kfor all a $\in$I so either am $\in$ K for all a $\in$ I or $s^n \in$ [K: M] for some n $\in Z_+$. The first case implies that m $\in$[K:I]. The second case implies s^n M $\subseteq$ K, but K $\subseteq$ [K: I] and hence s^nM $\subseteq$ [K: I]. It follows that $s^n \in$[[K:I] : M]. Therefore, either m $\in$[K:I] or $s^n \in$ [[K: I]: M] and hence [K: I] is a primary subsemimodule of M ■

Proposition 2.14

Let M be an S-semimodule. If K is the primary subsemimodule of M, then [K:L] is a primary ideal for all K$\subsetneq$ L.

Proof

Let a,b $\in$S such that ab $\in$ [K:L]. Assume that b $\notin$ [K:L] ,that is abx $\in$ K and bx $\notin$ K, for some x $\in$ L. But K is a primary subsemimodule of M, so $a^n \in$ [K: M] $\subseteq$ [K: L] for some n $\in Z_+$. Therefore, [K: L] is a primary ideal of S, for each K $\subsetneq$ L ■

Proposition 2.15

If K is a primary subsemimodule of S-semimodule M, then [K:<x>] is a primary ideal, $\forall\, x \notin$ K.

Proof

Since K is a primary, then [K:L]is a primary L Proposition 2.13. Let a,b$\in$ S, x $\notin$ K such that ab $\in$ [K: $<$ x $>$]ideal, $\forall$ K $\subsetneq$

and suppose that ax K . a $\notin$ [K: x >]then, then K $\subsetneq$ K+ < x > then [K: K+< x >] is a primary ideal of S, so a.b $\in$ [K: K+ < x >] .

But [K : K+ < x >] is a primary ideal of S, So either a$\in$ [K:K+ <x>]

b^n $\in$ [K: K+< X >] for some n$\in$ Z_+. Thus either a(K+<x>)$\subseteq$ K or b^n (K+< x >) $\subseteq$ K. If a(K+<x>) $\subseteq$ K, then ax$\in$ K, which contradicts the assumption. Thus b^n(K+<x>) $\subseteq$ K, so b^n $\in$ [K:< x >]and hence [K:<x>] is a primary ideal of S for each x$\notin$ K ■

Note

The converse of Proposition 2.15 is not true always as in the example.

Example 2.16

Let M be Z-semimodule ZP^j. Let K=(0), then (0) is not the primary subsemimodule of M, but for all x$\notin$ K, X =< $1/P^j$ +z > for som i$\in$ Z_+, hence [0 : <x>] = P^jZ which is primary ideal .

Corollary 2.17

If K is a proper primary subsemimodule of an S-semimodule M, then $\sqrt{[K:M]}$ is a prime ideal.

The converse of corollary (2.17) is not generally true as in the example.

Example 2.18

Let M = ℕ⊕ ℕ as a^n ℕ-semimodule and K=3ℕ⊕ < 0 > be a subsemimodule of ℕ⊕ℕ. Then [K: M] = [3ℕ ⊕ <0>: ℕ⊕ℕ] = 0 which is a primary ideal of $\mathbb{N}$. But K is not a primary subsemimodule of M for (6.0) =3(2,0) ∈ K, (2,0) ∉ K and $3^n \notin$ < 0 > .

The question now is, what is the condition which makes the convers of corollary (2.17) is true.

Definition 2.19

Let M be an R-semimodule and P be an ideal of R. A subsemimodule L of M is P-maximal subsemimodule, if

1. [L: M] = P
2. L is a maximal element in the set of subsemimodules K such that P = [K: M].

The following proposition states the conditions that make the converse of the corollary is true.

Proposition 2.20

Let N be a proper subsemimodule of an S-semimodule M and P be an ideal of S such that N is a P- maximal semimodule of M. Then N is a primary subsemimodule of M if and only if the ideal P = [N: M] is a primary ideal of S.

Proof

Suppose P = [N: M] is a primary ideal of S, to prove N is a primary subsemimodule of M. Let s ∈ S, x ∈ M such that sx ∈ N and suppose that x ∉ N. It is clear that N ⊊ N + ⟨x⟩ = K and K is a subsemimodule of M. Since N is P-maximal subsemimodule, so P = [N: M] ⊊ [K: M]. Then there exists t ∈ [K: M] and t ∉ [N:

M] and so for each $y \in M$, $ty \in K$. Thus, there exist $n \in \mathbb{N}$ such that $ty = tx + n$ and hence $sty = stx + sn \in N$, which means $ts \in [N: M] = P$. But P is a primary ideal and $t \notin P$, so $s^n \in [N: M]$ for some $n \in Z_+$.Therefore N is a primary subsemimodule of M ■

Proposition 2.21

Let N be a proper subsemimodule of S-semimodule M such that $[K: M] \nsubseteq [N: M]$ for each subsemimodule K of M and $N \subsetneq K$. Then N is a primary subsemimodule of M if and only if [N: M] is a primary ideal of S.

Proof

Suppose [N: M] is a primary ideal of S, to prove N is a primary subsemimodule of M. Let $r \in S$ and $x \in M$ such that $rx \in N$ and suppose $x \notin N$. It is clear that the subsemimodule $K = N + \langle x \rangle \supsetneq N$, and so $[K: M] \nsubseteq [N: M]$. Then there exists $s \in [K: M]$ and $s \notin [N:M]$ .Thus $sM \subseteq K$ and $sM \nsubseteq N$ .But $sM \subseteq K$ implies $rsM \subseteq rK = r(N + \langle x \rangle) \subseteq N$ and $rs \in [N: M]$. Since [N: M] is a primary ideal and $s \notin [N: M]$, $r^n \in [N: M]$ for some $n \in Z_+$. Therefore N is a primary subsemimodule of M ■

Definition 2.22 [16]

An S-simemodule M is called multiplication semimodule if for every subsemimodule K of M. there exists an ideal I of S, such that K= IM

Remark 2.23

For every subsemimodule K of a multiplication semimodule M, K= [K: M].

Remark 2.24

If M is cyclic S-semimodule, then M is a multiplication semimodule.

Now, we joint the concept of multiplication semimodule with primary.

Corollary 2.25

Let K be a proper subsemimodule of a multiplication S-semimodule M. Then K is primary if and only if [K: M] is a primary ideal of S.

Proof

For each subsemimodule K of M such that K$\subsetneq N$, we have [K: M] $\nsubseteq$ [N: M] and by Proposition (2.21), the corollary is hold ■

As another consequence of (2.21), we have the following result.

Corollary 2.26

Let K be a proper subsemimodule of a cyclic S-module M. Then K is a primary subsemimodule of M if and only if [K:M] is a primary ideal of S.

Proof

Since M is cyclic, then M is a multiplication semimodule. Hence from corollary 2.26, the result hold ■

The intersection of two primary subsemimodules of an S semimodule M need not be a primary subsemimodule of M.

Example 2.27

The $\mathbb{N}$ simemodule $\mathbb{N}_{12}$ has two primary subsemimodules.K_1=<$\bar{4}$> and K_2 = < $\bar{3}$ > but $K_1 \cap K_2$ = < 0 > is not a primary subsemimodule of $\mathbb{N}_{12}$.

proposition 2.28

Let N and K be two subsemimodules of an S-semimodule M such that N is a primary of M and K is not contained in N. Then, K ∩ N is a primary subsemimodule of K.

Proof

Since $K \not\subseteq N$, $K \cap N$ is a proper subsemimodule of K. Let $r \in R, m \in K$ such that $rm \in K \cap N$, if $m \notin N \cap K$, then $m \notin N$. But N is primary subsemimodule, so $r^n \in [N: M]$ for some $n \in Z_+$ and $r^n M \subseteq N$, therefore $r^n K \subseteq N \cap K$. Hence, $r^n \in [N \cap K: K]$ and $K \cap N$ is a primary subsemimodule of K ■

Note

If N is a primary subsemimodule, then sometimes N is called p-primary subsemimodule, where p =$\sqrt{[\, N: M]}$. and hence if (0) is a primary subsemimodule of M, then (0) is P=$\sqrt{[\, 0: M \,]}$ = $\sqrt{ann : M}$ - primary.

Proposition 2.29

Let S be any semiring, let P be a prime ideal of S, let n be a positive integer and let N be a P – primary subsemimodule of M for each 1≤ i ≤ n. Then $\cap_{i=1}^{n} \mathbb{N}_i$ is also a P-primary subsemimodule of M.

Proof

It is clear that $P = \sqrt{[\cap_{i=1}^{n} N_i : M]}$ Let s∈S and x∈M such that sx $\in \cap_{i=1}^{n}$ N_i and suppose x $\notin \cap_{i=1}^{n} N_i$, then there exists an integer j with $1 \leq j \leq n$ such that x$\notin N$. But sx$\in N$, *and* N , *is* P- primary submodule, it follows that s$\in$ P . *Hence* ,$\cap_{i=1}^{n} N_i$ is a P- primary submodule ■

Proposition 2.30

Let N, L be two subsemiImodules of an S- semimodule M and K be a P- primary submodule of M such that $N \cap L \subseteq K$, then either L$\subseteq K$ *or* $[N: M] \subseteq P = \sqrt{[N: M\,]}$.

Proof

Let t ∈ Suppose$[N_R M] \subseteq P$, then there exists $s \in [N: M]$ and $s \notin$ $\sqrt{[K: M]}$. L, so st ∈ L⋂N and hence stK. but K is a p − primary submodule of M and $s \notin \sqrt{K: M}$ so t ∈ K . thus L ⊆ K■

Proposition 2.31

Let M_1 and M_2 be two S-semimodules and let M= $M_1 \oplus M_2$. If N =$N_1 \oplus N_2$ is a primary subsemimodule of M, then N_1 and N_2 are primary S -subsemimodules of M_1 and M_2 respectively.

Proof

To prove N_1 is a primary S-subsemimodule of M_1 . Let s ∈ S and x ∈ M_1 such that sx ∈ N_1 then s(x , 0) ∈ $N_1 \oplus N_2$. But $N_1 \oplus N_2$ is a primary S – subsemimodule, so either (x,0) ∈ $N_1 \oplus N_2$ or s^n ∈ [$N_1 \oplus N_2$, $M_1 \oplus M_2$] for some n ∈ Z_+.Thus, either x ∈ N_1 or $r^n \in [N_1 : M_1] \cap [N_2 : M_2]$ for some n∈ Z_+ and hence either x ∈ N_1 or $s^n \in [N_1 : M_1]$ for some n∈ Z_+ .

Therefore N_1 is a primary S-subsemimodule of M. By a similar proof, N_2 is a primary subsemimodule of M_2.■

The converse of Proposition 2.31 is not generally true as in the example.

Example 2.32

Consider the ℕ -semimodule M= ℕ ⊕ ℕ and let N= 8ℕ ⊕ 9ℕ . It is clear that 8ℕ and 9ℕ are primary subsemimodules of ℕ. But N=8ℕ ⊕ 9ℕ is not a primary subsemimodule of ℕ ⊕ ℕ, since 4(2.9) ∈N, (2.9) ∉ N and 4^n ∉ [8ℕ ⊕ 9ℕ , ℕ ⊕ ℕ]= 72ℕ for each n ∈ ℕ.■

Chapter Three

Conclusions and Future Works

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Conclusions and Future Works

3.1. Conclusions

This research aims to reflect the resulting notions that were inverstigated about these notions in the category of submodules to the category of subsemimodules.

3.2. Future Works

The scholars can be examining primary subsimemodules and small primary semimodules in the future.

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الخلاصة

في هذا البحث مفهوم شبه المقاسات الجزئية الابتدائية على شبه الحلقة سيقدم ويعمم. الهدف من هذا العمل محاولة ان تعكس بعض النتائج المتحققة بالنسبة للمفاهيم من صنف المقاسات الجزئية الى صنف شبه المقاسات الجزئية. النتائج الرئيسية في هذا العمل هي : ليكن K شبه مقاس جزئي ابتدائي في مقاس S فأن [K:M] شبه مثالي اولي في S , فان K شبه مقاس جزئي اولي في M. $\sqrt{[K:M]} = \sqrt{K:c}$ لاي $c \notin K$ و $c \in M$. $\sqrt{[K:M]} = \sqrt{[K:L]}$ لاي شبه مقاس جزئي L بحيث $K \subset L$ لاي مثالي I في S ,[K:I] شبه مقاس جزئي ابتدائي M. اذا كان L شبه مقاس جزئي في M بحيث L غير محتواه في K فأن $K \cap L$ شبه مقاس جزئي ابتدائي. ليكن M_1, M_2 شبه مقاسان -S و $M = M_1 \oplus M_2$ اذا كان $K = K_1 \oplus K_2$ شبه مقاس جزئي ابتدائي في M فان K_1, K_2 شبه مقاسان جزئيان ابتدائيان في M_1, M_2 وحسب الترتيب وليس العكس.

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حول شبه المقاسات الجزئية الابتدائية

بحث

مقدمة إلى مجلس كلية التربية للعلوم الصرفة في جامعة بابل

كجزء من متطلبات نيل درجة الدبلوم العالي في التربية / الرياضيات

مقدمة من قبل الطالبة

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