Constructing Statistical Distribution Models with an Application

A Dissertation

Submitted to The Council of The College of Education for Pure Sciences in University of Babylon in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Education / Mathematics.

By

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بسم الله الرحمن الرحيم

وَأَن لَّيْسَ لِلْإِنسَٰنِ إِلَّا مَا سَعَىٰ (٣٩) وَأَنَّ سَعْيَهُۥ سَوْفَ يُرَىٰ (٤٠) ثُمَّ يُجْزَىٰهُ ٱلْجَزَآءَ ٱلْأَوْفَىٰ (٤١)

صدق الله العلي العظيم

سورة النجم آية
(41-39)

Approval of the Scientific Supervisor

I certify that this dissertation *"**Constructing Statistical Distribution Models with an Application**"* written student "**Sarmad Rahman Hussein Abdullah Al-Hilali**" was prepared under my supervision at the University of Babylon, Collage of Education for Pure Sciences, in a partial fulfillment of the requirements for the degree of Doctor of Philosophy in Education / Mathematics.

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Dedication

To whom God has entrusted with prestige and dignity, to whom taught me to give without waiting, to whom I carry his name with pride "to the soul of my dear father"

To my angel in life, to the meaning of life and devotion, to whom her supplication was the secret of my success, "my beloved mother".

Contents:

Acknowledgments

Abstract

We have introduced three new three-parameter distributions which are Generalized Uniform Distribution Using the Exponential Quantile Function , Generalizing Uniform Distribution Using The Quantile Function and Exponential- Generalizing Uniform Distribution. Therefore discussion some of their mathematical and statistical properties as the probability density function (pdf) , cumulative distribution function (cdf), some reliability functions, , mode, moments, variance, coefficient of (variation, skewness, kurtosis), moment generation function, quantile, order statistics, and entropy. Parameters were estimated using estimation methods(Maximum Likelihood Method, Method of moments ,Least Squares of Method).

The usefulness of these distributions is explained by applying real data, and then compare the proposed distributions (GUEQ, GURQ,E-GURQ) among themselves using scales such as **AIC**, BIC and HQIC.

Matlab(2013b) program was used to compare and estimate the parameters given in a table in addition to plotting the special functions of each distribution.

List of Abbreviations:

Abbreviation	Description
$\mathrm{var}(.)$	The variance
$(C.V.)$	Coefficient of Variation
K_a	The coefficient of skewness
L	The coefficient of kurtosis
$L.s$	Least Squares of Method
AIC	Akaike Information Criterion.
HQIC	Hannan-Quinn Information Criterion
BIC	Bayesian Information Criterion
KS	Kolmogorov-Smirnov test.

Figures Catalog

Publications:

1. Hussein, S. R., & Al-Kadim, K. A. (2021, May). Generalized Uniform Distribution Using the Exponential Quantile Function. In Journal of Physics: Conference Series (Vol. 1897, No. 1, p. 012002). IOP Publishing.

2. Hussein, S. R., & Al-Kadim, K. A. (2021, March). Generalizing Uniform Distribution Using the Quantile Function. In Journal of Physics: Conference Series (Vol. 1818, No. 1, p. 012067). IOP Publishing.

3. Hussein, S. R., & Al-Kadim, K. A .Exponential- Generalizing Uniform Distribution (Publication Acceptance).

Introduction

A statistical distribution is a mathematical function that determines how the results of a randomized trial attempt will occur in a probable manner , it is important for parametric inferences and applications to fit real world phenomena.

Many methods have been developed to generate statistical distributions in the literature. Some well-known methods in the early days for generating univariate continuous distributions include methods based on differential equations developed by Pearson (1895), methods of translation developed by Johnson (1949), and the methods based on quantile functions developed by Tukey (1960).

Due to the different applications of distributions in daily life, there has been a growing interest in research and studies in this aspect, and there was an urgent need to develop classical distributions into more flexible distributions and to benefit from them by finding new statistical models to study the variables that occur in society or in the economic, engineering, medical and other fields.

In 2002 [10] , Eugene and et al . introduce a general class of distributions generated from the logit of the beta random variable. A special case of this family is the beta-normal distribution.

In 2004 [1] Arellano-Valle and et al., studied a new family of asymmetric normal distributions that contains Azzalini's skew-normal (SN) distribution as a special case.

In the same year G. and R. [12] , derived a generalization of the basic Azzalini model proposed by Balakrishnan, as a discussant of Arnold and Beaver (2002).

In 2005[28], Vicari and Kotz , trace developments in the theory of skewed continuous distributions (univariate and multivariate) which commenced in the late 19th century and after some dormant period during

the most of the 20th century were invigorated in the middle 80's of the 20th century and has become in the last 20 years an area of rapid advances.

In 2008 [29] Yadegari and et al. introduced a new generalization of the Balakrishnan skew-normal distribution by explaining some important properties of this distribution. Also, they described three methods for constructing this distribution. Finally, its multivariate extension has been presented.

In 2011[7], Choudhury and Matin., presented a more flexible distribution, extended skew generalized normal distribution, is developed to represent the skewness as well as the kurtosis. This distribution is potentially useful for the data that has more skewness and kurtosis.

In 2013 [9] Domma and Condino defined a five-parameter Beta-Dagum distribution from which moments, hazard and entropy, and reliability measures are then derived. These properties show the high flexibility of the said distribution.

In the same year Oluyede and et al. [21] fined a new class of distributions called Mc-Dagum distribution . This class of distributions contains several distributions such as beta-Dagum, beta-Burr III, beta-Fisk, Dagum, Burr III and Fisk distributions as special cases.

Also Oluyede and et al. in 2014 [22] a new class of generalized Dagum distribution called gamma-Dagum distribution is presented. The gamma-Dagum (GD) distribution which includes the gamma-Burr III (GB III), gamma-Fisk or gamma-log logistic (GF of GLLog), Zografos and Balakrishnan-Dagum (ZB-D), ZB-Burr III (ZB-B III), ZB-Fisk of ZB-Log logistic (ZB-F or ZB-LLog), Burr III (B III), and Fisk or Log logistic (F or LLog) as special cases was proposed and studied.

Also in 2014,Huang and Oluyede [13] provided a new family of distributions called exponentiated Kumaraswamy-Dagum (EKD)

distribution is proposed and studied. This family includes several well-known sub-models, such as Dagum (D), Burr III (BIII), Fisk or Log-logistic (F or LLog).

Aljarrah and et al. ,2014 [2] state some of general properties of this T-X system of distributions are studied and defined the T-X{Y} family as

$$G(x) = \int_a^{Q_Y(F(x))} r(t)dt = R\left(Q_Y\left(F(x)\right)\right) \qquad (1)$$

It is shown that several existing methods of generating univariate continuous distributions can be derived using this T-X system.

In 2014[3] Alzaatreh and et al. proposes four families of generalized normal distributions using the T-X framework. defined as

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = P[T \leq Q_Y\left(F_R(x)\right)] = F_T\left(Q_Y\left(F_R(x)\right)\right) \quad (2)$$

These four families of distributions are named as T-normal families arising from the quantile functions of (i) standard exponential, (ii) standard log-logistic, (iii) standard logistic and (iv) standard extreme value distributions. Some general properties including moments, mean deviations and Shannon entropy of the T-normal family are studied. Four new generalized normal distributions are developed using the T-normal method.

In addition Nasiru and et al., 2017 [20] showed in this study, the exponentiated generalized exponential Dagum distribution has been proposed and studied.

In 2018[30], Zubair and et al introduced new generalized classes of exponential distribution, called T-exponential {Y} class using the quantile functions of well-known distributions.

Shahzad and Asghar,2019, [27] found a transmuted Dagum distribution which can be used for income distribution, actuarial, survival and reliability analysis. The main motivation for generalizing the standard distribution is to provide more flexible distribution to model a variety of data. The

extended distribution has been expressed using quadratic rank transmutation map and its tractable properties like moments, moment generating, quantile, reliability and hazard functions are derived.

In 2019 [16] , Jamal, F., and Nasir, M. define some new members of T-X family which also be viewed as new families of probability distributions.

In 2020[11], Ekum and et, al. Recently, different distributions have been generalized using the T-R {Y} framework but the possibility of using Dagum distribution has not been assessed. the T-R {Y} combines three distributions, with one as a baseline distribution, with the strength of each distribution combined to produce greater effect on the new generated distribution. the e new generated distributions would have more parameters but would have high flexibility in handling bimodality in datasets and it is a weighted hazard function of the baseline distribution. this paper therefore generalized the Dagum distribution using the quantile function of Lomax distribution. A member of T-Dagum class of distribution called exponentiated-exponential-Dagum {Lomax} (EEDL) distribution was proposed.

This study generalizes the work in methods based on quantile functions, where the first distribution is generalized uniform distribution using the exponential quantile function, GUEQD using the exponential quantile function and the substitution by the T-X{Y} family to find the distribution function cdf for the GUEQ distribution. As for generalizing uniform distribution using the quantile function, GURQD , the quantile function reliability function was used to find its GURQ distribution function by substituting the T-X{Y} family.

And then the distribution function was found for the two distributions, Exponential- Generalizing Uniform Distribution (E-GURQD) and the

exponential - the generalization of the uniform distribution (E-GUEQD) using the T-normal families arising from the quantum functions, which the distribution function has knowledge of Equation 2, but has been changed It has to become more complex for my agencies.

$$F_X(x) = \int_a^{F_R(Q_Y(x))} f_T(t)\,dt = F_T\left(F_R\left(Q_Y(x)\right)\right) \qquad (3)$$

Through which cdf was found for each of E-GURQD .

In order to clarify the content of this dissertation and to diagnose its purpose, we note that it was divided into three chapters.

Chapter 1 dealt with some important definitions and concepts.

Chapter 2 included two distributions, namely the generalized uniform distribution using the exponential quantile function, GUEQD and exponential- generalizing uniform distribution using the quantile function (E- GUEQD) which is considered an anomaly, where it was clarified how to build each distribution in addition to the pdf and cdf for each distribution as well as some reliability analysis functions, mode, moments, coefficient of skewness , kurtosis and variation, moment generating function, quantile and the median, order statistics, entropy.

Chapter 3 included two distributions generalizing uniform distribution using the quantile function, GURQD and exponential- generalizing uniform distribution (E- GURQD)**,** where it was clarified how to build each distribution in addition to the pdf and cdf for each distribution as well as some reliability analysis functions, mode, moments, coefficient of skewness , kurtosis and variation, moment generating function, quantile and the median, order statistics, entropy. Conclusions and future work are also presented in this chapter**.**

The maximum likelihood estimation approach is used to estimate the three parameters of the (GUEQ, GURQ and E- GURQ) distributions. Applying them to real-world data sets to demonstrate the significance of the new distributions(GUEQ, GURQ and E- GURQ).

Chapter One

Definitions and Some

Important Concepts

Definitions and Some Important Concepts

Introduction

The first chapter includes some of the basic concepts and mathematical formulas used during our research work, as well as the definitions adopted in the research. Two new definitions have been presented that can be used in a new model resulting from the composition of two distributions.

1.1. Statistical Model [15]:

The statistical model is a classification of mathematical model, which personifies a set of statistical hypotheses related to the formation of a statistical sample, and similar data from a larger statistical community. The statistical model represents the process of creating data. The included assumptions in the statistical model describe a set of probability distributions, some of which are assumed to approximate the distribution from which a particular sample of data was taken. we can say that the model is a statistical description of a basic system, which aims to match the real situation as much as possible. The population model is fitted to a sample by estimating the parameters in the model. It is then possible to conduct hypothesis testing, build confidence intervals and draw conclusions about the population.

1.2. Statistical Parameter[15]:

A statistical parameter is a numerical quantity that identifies and "summarizes" the probabilistic distribution of a population. The statistical parameter can also be defined as a numerical characteristic of a statistical community. For instance, if we impose a population with a normal probability distribution, the two statistical parameters that can

characterize this distribution are the mean and the standard deviation. Likewise, it is a

statistical measure that computes all the component of society without exception, and to describe the community in studying and research.

1.3. Continuous Uniform Distribution [24]:

Let the random variable X denote the outcome when a point is selected at random from an interval [a, b]. We want to find the probability of the event X ≤ x, that is we would like to determine the probability that the point selected from [a, b] would be less than or equal to x . To compute this probability, we need a probability measure μ that satisfies the three axioms of Kolmogorov (namely nonnegativity, normalization and countable additivity).For continuous variables, the events are interval or union of intervals. The length of the interval when normalized satisfies all the three axioms and thus it can be used as a probability measure for one-dimensional random variables. Hence

$$P(X \leq x) = \frac{length\,of\,[a,x]}{length\,of\,[a,b]}$$

Thus, the cumulative distribution function F is

$$F(x) = P(X \leq x) = \frac{x-a}{b-a}, \quad a \leq x \leq b$$

where a and b are any two real constants with a < b. To determine the probability density function from

$$f(x) = \frac{d}{dx}F(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

1.4. Exponential Distribution [18]:

The Exponential distribution is commonly used to model waiting times between occurrence of rare events and lifetime of electrical or mechanical devices. A continuous random variable X is said to have an Exponential

Distribution (ED) with parameter ($\lambda > 0$), if it has the following probability density function:

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad (1.1)$$

and the cdf is defined by

$$F(x;\lambda) = 1 - e^{-\lambda x} \qquad (1.2)$$

1.5. Some Survival Function, [25, 4]:

1.5.1. Survival function:

Survival function is defined as the probability that a system will operate satisfactorily for a given period of time and denoted by $S(x)$

$$S(x) = \Pr(X > x) = 1 - F(x) = 1 - \int_0^x f(x)dx \qquad (1.3)$$

1.5.2. Hazard Function:

The probability that an item will survive to a specified point in time based on its survival until a previous time (t). In other words, the possibility is that if something remains alive for one moment, it will remain alive until the next stage. The hazard rate only applies to items that cannot be repaired. The hazard rate can be determined at any time using the following equation:

$$h(x) = \frac{f(x)}{s(x)} \quad or \quad h(x) = \frac{f(x)}{1 - F(x)} \qquad (1.4)$$

1.5.3. Cumulative Hazard Rate Function (CHF):

The cumulative hazard function can be obtained, denoted by H(x), which is the sum of the failure rate and is expressed in the following form:

$$H(x;\theta) = \int_0^x h(t;\theta)dt = \int_0^x \frac{f(t;\theta)}{s(t;\theta)}dt = -\ln(s(t;\theta)) \qquad (1.5)$$

1.5.4. Reverse Hazard Rate Function :

The reverse hazard function is expressed in the following formula.

$$r(x) = \frac{f(x)}{F(x)} \qquad (1.6)$$

Where $f(x)$ is a pdf and $F(x)$ is a cdf.

1.6. Definition of Coefficients, [14,5,8,23]:

1.6.1. Coefficient Of Skewedness:

In the case of identical distributions, the mean, median, and mode are equal, and the more after the curve is the similarity. These values are separated from each other, and therefore, the difference between these values can be used as a measure of skewedness, but this difference does not measure the skewedness completely. The difference may be large and the skewedness is small, because the data values are dispersed large. And the difference may be small and the skewedness is large because the group is dispersed small, and therefore this difference must be attributed to the corresponding dispersion scale (of the same type as the average value scale used). The resulting scale is called the skewedness coefficient. It is denoted by the symbol

$$CS = \frac{E(X-\mu)^3}{\sigma^3} \qquad (1.7)$$

1.6.2. Coefficient of Kurtosis :

When the iterative distribution is represented as an iterative curve, this curve may be flat, or pointed. So when more values are concentrated near the middle of the curve, and lower at both ends, the curve is pointed, and when a larger number is concentrated on both ends of the curve, and decreases near the middle the curve is flat. It is denoted by the symbol

$$CK = \frac{E(X - \mu)^4}{\sigma^4} \tag{1.8}$$

1.6.3. Coefficient of Variation:

The coefficient of variation is a measure of the relative dispersion. It is a unitless scale. It is used to compare the relative dispersion or homogeneity of different data sets. The group of data with the largest coefficient of variation is more dispersed relative, that is, it is less homogeneous and vice versa. The variation coefficient of the sample whose mean μ and its standard deviation σ is defined as the follows formula:

$$CV. = \frac{\sigma}{\mu} \tag{1.9}$$

1.7. Maximum Likelihood Function [19]

The likelihood function of n random variables $x_1, x_2, ..., x_n$ is defined to be the joint density function of the n random variables say: $f_{x_1, x_2, ..., x_n}(x_1, x_2, ..., x_n; \theta)$, which is considered to be a function of θ. Notation of the likelihood function as $L(\theta; x_1, x_2, ..., x_n)$. Thus the likelihood function as function of θ.

$$L\left(\theta_1,\theta_2,...,\theta_n\right)= \prod_{i=1}^{n} f\left(x_i;\theta_1,\theta_2,...,\theta_n\right) \qquad (1.10)$$

then $\dfrac{\partial L\left(\theta_1,\theta_2,...,\theta_n\right)}{\partial\theta_1}=0,\ \dfrac{\partial L\left(\theta_1,\theta_2,...,\theta_n\right)}{\partial\theta_2}=0,...,\dfrac{\partial L\left(\theta_1,\theta_2,...,\theta_n\right)}{\partial\theta_n}=0$

In this case it may also be easier to work with the logarithm of likelihood.

1.8. Method of Moments [19]

Let $f\left(x;\theta_1,\theta_2,...,\theta_n\right)$ be a density of a random variable x which has n

parameters $\theta_1,\theta_2,...,\theta_n$. As before let μ'_r denote the rth moment about

zero, that is $\mu'_r = E\left(X_r\right)$. In general μ'_r will be a known function of the

n parameters $\theta_1,\theta_2,...,\theta_n$, denote this by writing:

$\mu'_r = \mu'_r\left(\theta_1,\theta_2,...,\theta_n\right)$. Let $X_1,X_2,...,X_n$ be a random sample from the

density $f\left(x;\theta_1,\theta_2,...,\theta_n\right)$ and $\mu'_r = \sum_{j=1}^{n}\dfrac{1}{n}X_r^j$ such that $j=1,2,...,n$ in

the n variables $\theta_1,\theta_2,...,\theta_n$ and let $\hat\theta_1,\hat\theta_2,...,\hat\theta_n$ be their solution.

1.9. Entropy[26]:

Entropy plays a vital role in science, engineering and probability theory, and has been used in various situations as a measure of variation of uncertainty of a random variable (Claude Shannon) is defined by:

Entropy $= E\left(-\log w\left(X\right)\right)$ \qquad (1.11)

1.10. T-Normal Families of Distributions,[20]:

Let *T, R* and *Y* be random variables with CDF $F_T(x) = P(T \leq x)$, $F_R(x) = P(R \leq x)$ and $F_Y(x) = P(Y \leq x)$. The corresponding quantile functions are $Q_T(p), Q_R(p)$ and $Q_Y(p)$ where the quantile function is defined as $Q_Z(z) = \inf\{z : F_Z(z) \geq p\}, 0 < p < 1$. If densities exist, we denote them by $f_T(x), f_R(x)$ and $f_Y(x)$. Now assume the random variable $T \in (a, b)$ and $Y \in (c, d)$, for $-\infty \leq a < b \leq \infty$ and $-\infty \leq c < d \leq \infty$. The CDF of the random variable *X* is defined as

$$F_X(x) = \int_a^{Q_Y\left(F_R(x)\right)} f_T(t)\,dt = P\left[T \leq Q_Y\left(F_R(x)\right)\right] = F_T\left(Q_Y\left(F_R(x)\right)\right) \quad (1.12)$$

$F_X(x)$ will be called the *T-R{Y}* family of distributions.

1.11. Purpose of the study:

The dissertation objectives to present three new distributions with a study of the properties of each of them, statistical and mathematical, with application. These distributions are as follows

- The Generalized Uniform Distribution Using the Exponential Quantile Function, GUEQD.

- Generalizing Uniform Distribution Using the Quantile Function, GURQD.

- Exponential- Generalizing Uniform Distribution (E- GURQD).

Chapter Two

The Generalized Uniform Distribution Using the Exponential Quantile Function

The Generalized Uniform Distribution Using the Exponential Quantile Function.

In this section , we will introduce the generalized uniform distribution using the exponential quantile function and then finding p.d.f and c.d.f and some properties (mathematics and statistical). We also estimate parameters using some estimation methods.

2.1. Construction of the Generalized Uniform Distribution Using the Exponential Quantile Function, GUEQD

Let R (X) represent a reliability function for any distribution function given by the following formula**:**

$$R(x)=1-F(x) \text{ then } R'(x)=-f(x)$$

if $X \sim Exp(\lambda)$ then $F(x)=\left(1-e^{-\lambda x}\right)I_{(0,\infty)}(x)$

As we know that $F(x)\sim U(0,1)$ then

$$x = \ln(1-u)^{\frac{-1}{\lambda}} \text{ and there is } x = Q(F(x)), \ 0<u<1$$

Now let $Y \sim U(a,b)$, $-\infty<a<b<\infty$ then $W(y)=\dfrac{y-a}{b-a} I_{(a,b)}(y)$

Thus $W(x)=\int_a^{Q(F(x))} \dfrac{dy}{b-a}$ then $W(x)=\dfrac{Q(F(x))-a}{b-a}$

such $W(x)$ is the cumulative distribution function of the random variable U.

$$W\left[Q\left(F\left(x\right)\right)\right]=W\left[\ln(1-u)^{\frac{-1}{\lambda}}\right]$$

$$=\dfrac{\dfrac{-\ln(1-u)-\lambda a}{\lambda}}{b-a}$$

$$= \frac{-\left[\ln(1-u)+\lambda a\right]}{\lambda(b-a)}$$

Let $w(u)=W'\left[Q(F(x))\right]$

$$=\left(\frac{-1}{\lambda(b-a)}\right)\left(\frac{-1}{(1-u)}\right)$$

$$w(u)=\frac{1}{\left[\lambda(b-a)\right](1-u)}I_{\left(1-e^{-\lambda a},1-e^{-\lambda b}\right)}(u)$$

Now let $W\left(Q\left(F(x)\right)\right)=1$ then $1=\dfrac{\frac{-1}{\lambda}\ln(1-u)-a}{b-a}$

$$b-a=\frac{-1}{\lambda}\ln(1-u)-a$$

$$e^{-\lambda b}=1-u$$

$$u=1-e^{-\lambda b}$$

Let $W\left(Q\left(F(x)\right)\right)=0$ then $0=\dfrac{\frac{-1}{\lambda}\ln(1-u)-a}{b-a}$

$$0=\frac{-1}{\lambda}\ln(1-u)-a$$

$$e^{-\lambda a}=1-u$$

$$u=1-e^{-\lambda a}$$

Then $1-e^{-\lambda a}<u<1-e^{-\lambda b}$

2.2. Probability Density Function (pdf) and Cumulative Distribution Function (cdf) .

2.2.1. The PDF Of GUEQD

$$w(u)=\frac{1}{\lambda(b-a)(1-u)} \quad , \quad 1-e^{-\lambda a}<u<1-e^{-\lambda b} \quad ,a,b,\lambda>0,a<b \qquad (2.1)$$

Where $1-e^{-a\lambda}$ and $1-e^{-b\lambda}$ are constants.

the function $w(u)$, a probability density function?

   1) It is easy to check $w(u) \geq 0$

Since $\lambda, b, a > 0$ then $\dfrac{1}{\lambda(b-a)} > 0$ and $\dfrac{1}{(1-u)} > 0$ for all

$1 - e^{-\lambda a} < u < 1 - e^{-\lambda b}$ then $w(u) \geq 0$

  2) $\int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} w(u)\,du = 1$

Prove :

$$\int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \frac{1}{\lambda(b-a)(1-u)}\,du = \frac{1}{\lambda(b-a)} \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \frac{1}{(1-u)}\,du$$

$$= \frac{-1}{\lambda(b-a)} \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \frac{-1}{(1-u)}\,du = \frac{-1}{\lambda(b-a)} \ln(1-u)\Big]_{1-e^{-\lambda a}}^{1-e^{-\lambda b}}$$

$$= \frac{-1}{\lambda(b-a)} \left[\ln\left(1 - \left[1 - e^{-\lambda b}\right]\right) - \ln\left(1 - \left[1 - e^{-\lambda a}\right]\right) \right]$$

$$= \frac{-1}{\lambda(b-a)} \left[\ln e^{-\lambda b} - \ln e^{-\lambda a} \right]$$

$$= \frac{-1}{\lambda(b-a)} \left[-\lambda b + \lambda a \right]$$

$$= \frac{-1}{\lambda(b-a)} \left(-\lambda[b-a] \right) = 1$$
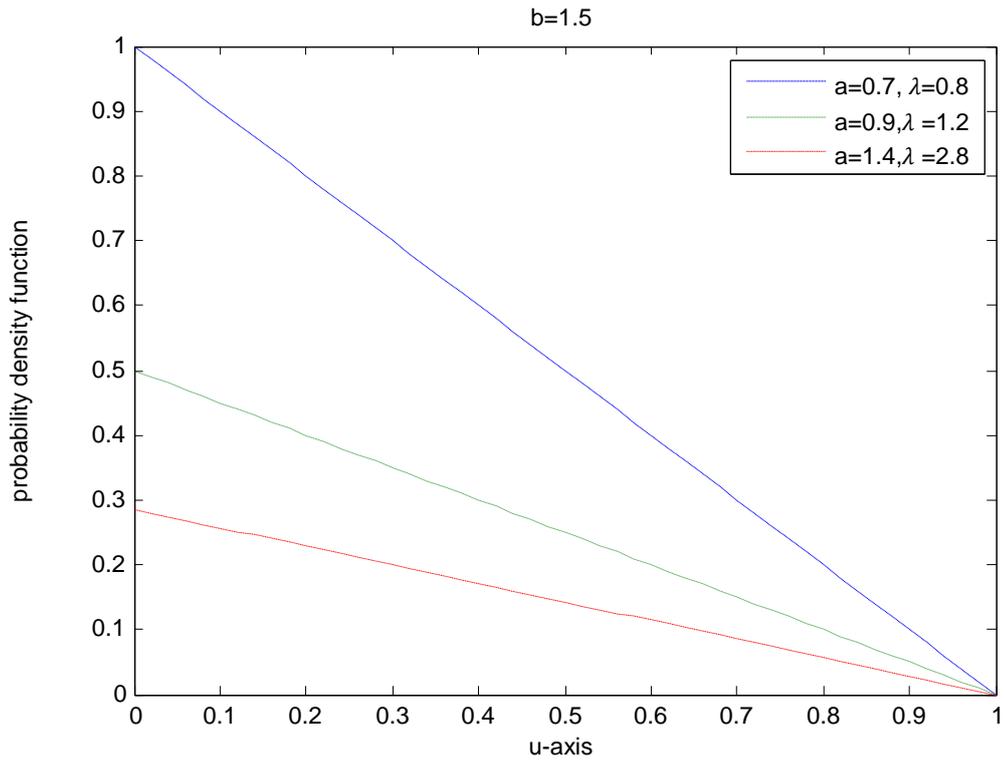
Figure. 1: This graph showed pdf of GUEQD, the parameter $b = 1.5$, $a = 0.7, 0.9, 1.4$, $\lambda = 0.8, 1.2, 2.8$
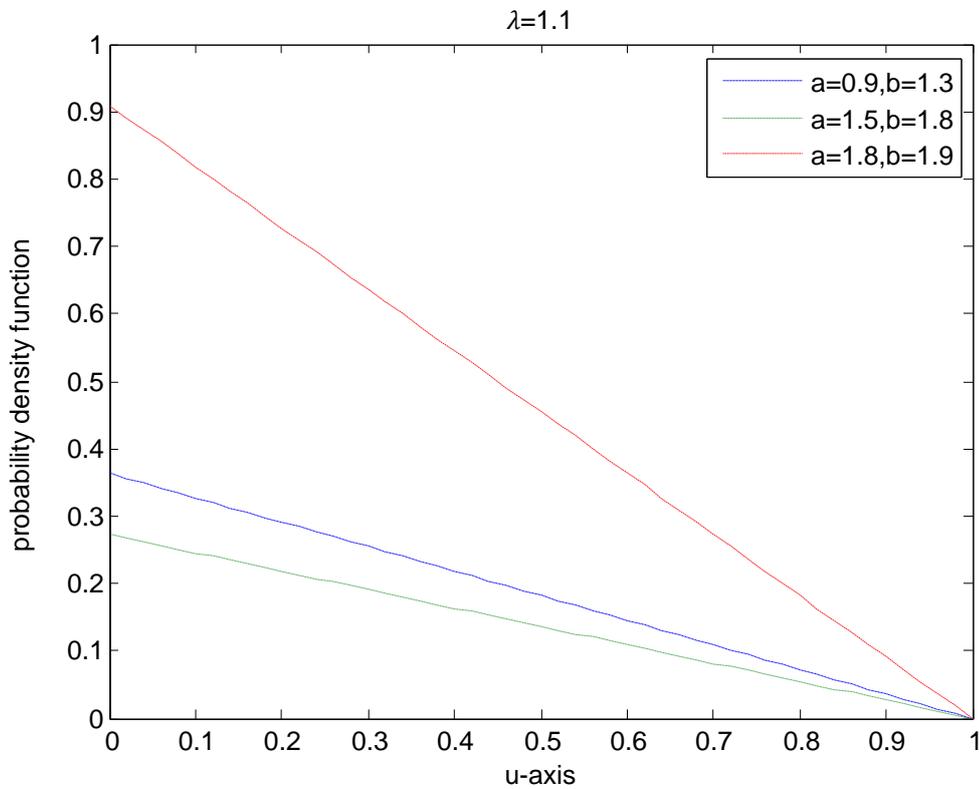
Figure.2: This graph showed pdf of GUEQD, the parameter $\lambda = 1.1$ $a = 0.9, 1.5, 1.8$, $b = 1.3, 1.8, 1.9$
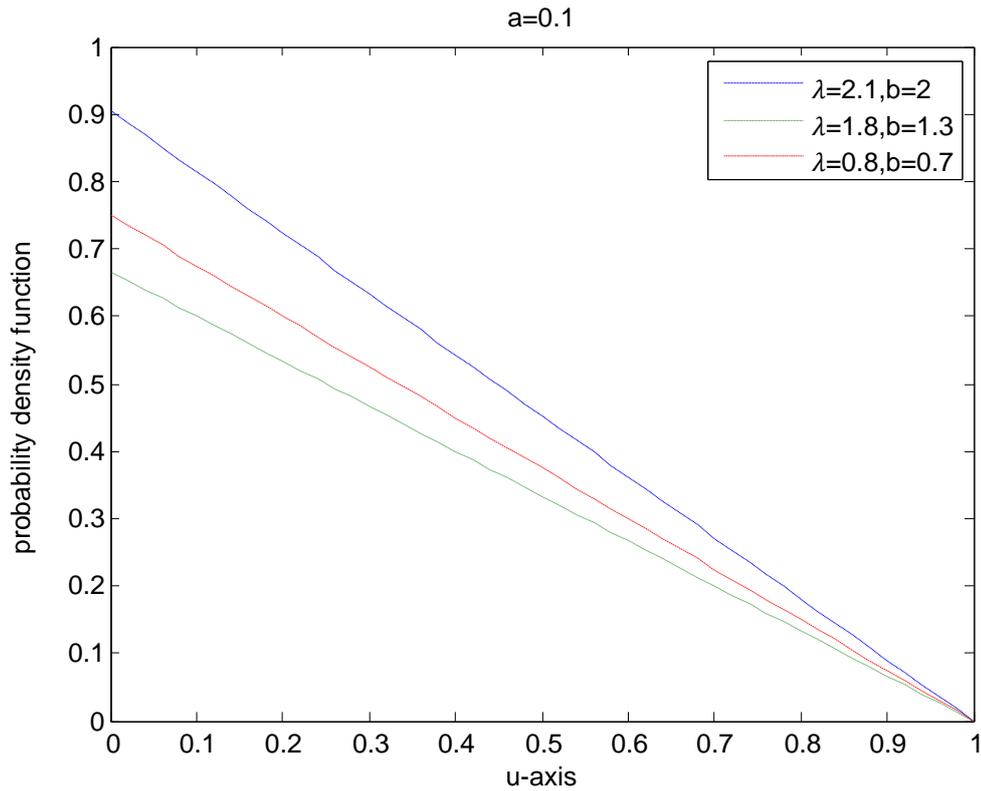
Figure. 3: This graph showed pdf of GUEQD, the parameter a=0.1

$b = 2, 1.3, 0.7, \lambda = 2.1, 1.8, 0.8$

Figures (1), (2), and (3) show that the GUEQ distribution creates a variety of forms, including symmetrical and right skewed geometries.

2.3. The CDF Of GUEQD

The cdf of GUEQD is in

$$W(u) = \frac{-1}{\lambda(b-a)}\left(\ln(1-u) + \lambda a\right), 1 - e^{-\lambda a} < u < 1 - e^{-\lambda b}, a, b, \lambda > 0, a < b \quad (2.2)$$

a=0.1
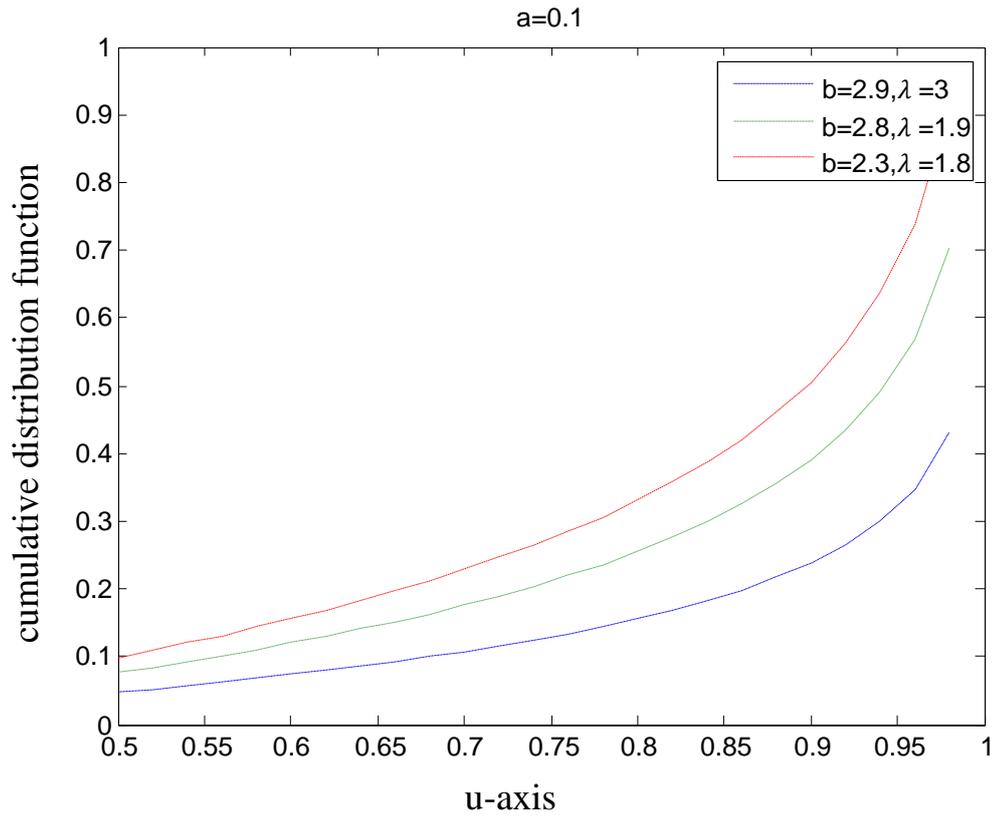
Figure. 4: This graph showed cdf of GUEQD, the parameter a=0.1

$$b = 2.9, 2.8, 2.3, \lambda = 3, 1.9, 1.8$$

b=7
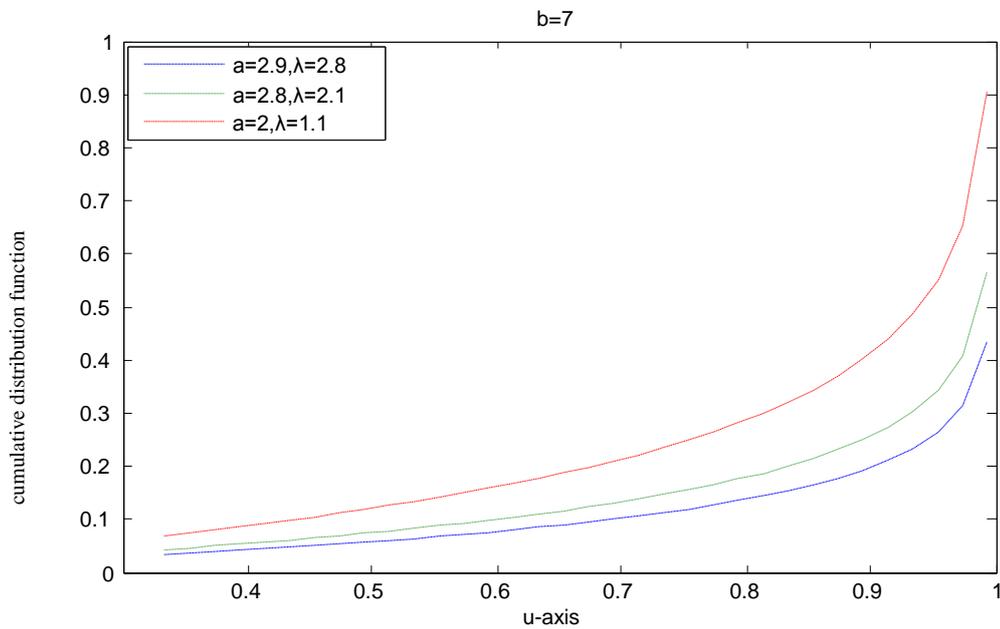
Figure. 5: This graph showed cdf of GUEQD, the parameter b=7

$$a = 2.9, 2.8, 2, \lambda = 2.8, 2.1, 1.1$$

Figures (4) and (5) show that the cdf of the GUEQ distribution is increasing u, as are the parameters a, b and λ .

2.4. Limit of cdf

The limit of the GUEQD is given by the form:

$1.\lim_{u \to 1-e^{-\lambda a}} W_{GUEQD}(u) = 0$

$= \lim_{u \to 1-e^{-\lambda a}} \frac{-1}{\lambda(b-a)} \left(\ln(1-u) + \lambda a \right)$

$= \frac{-1}{\lambda(b-a)} \left(\ln\left(1 - \left[1 - e^{-\lambda a}\right]\right) + \lambda a \right)$

$= \frac{-1}{\lambda(b-a)} (-\lambda a + \lambda a) = 0$

$2.\lim_{u \to 1-e^{-\lambda b}} W_{GUEQD}(u) = 1$

$= \lim_{u \to 1-e^{-\lambda b}} \frac{-1}{\lambda(b-a)} \left(\ln(1-u) + \lambda a \right)$

$= \frac{-1}{\lambda(b-a)} \left(\ln\left(1 - \left[1 - e^{-\lambda b}\right]\right) + \lambda a \right)$

$= \frac{-1}{\lambda(b-a)} (-\lambda b + \lambda a) = 1$

That assertion that $0 \le W(u) \le 1$

2.5. Some Reliability Analysis Functions :

In this subsection, some reliability function are discussed such as (hazard, reverse hazard) rate function and cumulative hazard function (CHF) for the GUEQD.

Subsequently, the probability for a system survives beyond a specified time is known as reliability function or survivor or survival function and is given by the following function.

$R(u)=1-W(u)$

$$=1-\left[\frac{-1}{\lambda(b-a)}\right](\ln(1-u)+\lambda a)=1+\frac{1}{\lambda(b-a)}(\ln(1-u)+\lambda a) \qquad (2.3)$$

Note that $\quad R(u)+W(u)=1$

Therefore

$$1.\lim_{u\to 1-e^{-\lambda a}} R_{GUEQD}(u)=\lim_{u\to 1-e^{-\lambda a}}\left\{1+\frac{1}{\lambda(b-a)}(\ln(1-u)+\lambda a)\right\}$$

$$=1+\frac{1}{\lambda(b-a)}\left(\ln\left(1-\left[1-e^{-\lambda a}\right]\right)+\lambda a\right)$$

$$=1+\frac{1}{\lambda(b-a)}(-\lambda a+\lambda a)=1$$

$$2.\lim_{u\to 1-e^{-\lambda b}} R_{GUEQD}(u)=\lim_{u\to 1-e^{-\lambda b}}\left\{1+\frac{1}{\lambda(b-a)}(\ln(1-u)+\lambda a)\right\}$$

$$=1+\frac{1}{\lambda(b-a)}\left(\ln\left(1-\left[1-e^{-\lambda b}\right]\right)+\lambda a\right)$$

$$=1+\frac{1}{\lambda(b-a)}(-\lambda b+\lambda a)=0$$

The hazard function is described as(hazard , failure) rate and it is given by

$$h_S(u)=\frac{w(u)}{1-W(u)}$$

$$h_S(u)=\frac{1}{(1-u)\left[\lambda(b-a)+(\ln(1-u)+\lambda a)\right]} \qquad (2.4)$$

and the reverse hazard function for the GUEQD is provided by

$$h_r(u)=\frac{w(u)}{W(u)}$$

$$h_r(u)=\frac{-1}{(1-u)(\ln(1-u)+\lambda a)} \qquad (2.5)$$

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and the cumulative hazard function for the GUEQD is provided by

$$\bar{H}_U(u) = -\ln\left(\frac{\lambda(b-a)+(\ln(1-u)+\lambda a)}{\lambda(b-a)}\right) \qquad (2.6)$$

2.6. Mode:

To find the mode of the GUEQD , it must be

$$\frac{dw_{GURQD}(u)}{du} = 0$$

$$\frac{d}{du}\left[\frac{1}{\lambda(b-a)(1-u_{\text{mode}})}\right] = 0$$

$$\frac{1}{\lambda(b-a)}\left[\frac{1}{(1-u_{\text{mode}})^2}\right] = 0$$

$$\frac{1}{(1-u_{\text{mode}})^2} = 0 \qquad\qquad (2.7)$$

However, it is not possible to get an explicit solution of (2.7) in the general case. But from the curve of this distribution , it is easy to find the mode with assumption that $u_{\text{mode}} = 0$, then $w(u)$ is maximum

2.6. Moments about the origin, Moment about Mean

Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution.

2.6.1. Moments about the origin :

The rth order moment $E\left(U^{r}\right)$ about the origin of GUEQD is given by:

$$\mu'_r = E\left(U^{r}\right) = \frac{-1}{\lambda(b-a)} \sum_{k=0}^{r} (-1)^{k} \binom{r}{k} \left(\frac{e^{-k\,\lambda b}}{k} - \frac{e^{-k\,\lambda a}}{k} \right) \tag{2.8}$$

The rth moment of U is obtained by:

$$\mu'_r = E\left(U^{r}\right) = \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} u^{r} w(u)\,du , \ r = 1,2,3,... \tag{2.9}$$

Substituting equation (2.1) into equation (2.9), we obtain:

$$= \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \frac{u^{r}}{\lambda(b-a)(1-u)}\,du$$

$$= \frac{1}{\lambda(b-a)} \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \frac{u^{r}}{(1-u)}\,du$$

Let

$$1-u = e^{x} \implies -du = e^{x}\,dx$$

$$u = 1-e^{x} , \ -\lambda a < x < -\lambda b$$

$$E\left(U^{r}\right) = \frac{-1}{\lambda(b-a)} \int_{-\lambda a}^{-\lambda b} \frac{\left(1-e^{x}\right)^{r}}{e^{x}}\,e^{x}\,dx$$

$$= \frac{-1}{\lambda(b-a)} \int_{-\lambda a}^{-\lambda b} \left(1-e^{x}\right)^{r}\,dx \tag{2.10}$$

Using the binomial series expansion of $\left(1-e^{x}\right)^{r}$ yields:

$$\left(1-e^{x}\right)^{r} = \sum_{k=0}^{r} (-1)^{k} \binom{r}{k} (1)^{r-k} \left(e^{x}\right)^{k}$$

Substitution by equation No. (2.10) yields:

$$= \frac{-1}{\lambda(b-a)} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \int_{-\lambda a}^{-\lambda b} e^{k\,x}\, dx$$

Then

$$\mu_r' = E(U^r) = \frac{-1}{\lambda(b-a)} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \left(\frac{e^{-k\,\lambda b}}{k} - \frac{e^{-k\,\lambda a}}{k} \right)$$

2.6.2. The mean (μ) :

The mean (μ) of U~ GUEQD is obtained by putting r =1 in Eq. (2.8).

Then

$$\mu = \frac{-1}{\lambda(b-a)} \sum_{k=0}^{1} (-1)^k \binom{r}{k} \left(\frac{e^{-k\,\lambda b}}{k} - \frac{e^{-k\,\lambda a}}{k} \right)$$

$$\mu = \frac{-1}{\lambda(b-a)} \left\{ \left[(-1)^0 \binom{1}{0} \lim_{k\to 0} \left(\frac{e^{-k\,\lambda b} - e^{-k\,\lambda a}}{k} \right) \right] \right.$$

$$\left. + \left[(-1)^1 \binom{1}{1} \left(e^{-\lambda b} - e^{-\lambda a} \right) \right] \right\}$$

Using the L'Hôpital's rule of magnitude $\lim_{k\to 0} \left(\frac{e^{-k\,\lambda b} - e^{-k\,\lambda a}}{k} \right)$

$$= \frac{-1}{\lambda(b-a)} \left\{ \left[(-1)^0 \binom{1}{0} \lim_{k\to 0} \left(\frac{-\lambda b\,e^{-k\,\lambda b} + \lambda a\,e^{-k\,\lambda a}}{1} \right) \right] \right.$$

$$\left. - \left[\binom{1}{1} \left(e^{-\lambda b} - e^{-\lambda a} \right) \right] \right\}$$

$$= \frac{-1}{\lambda(b-a)} \left\{ \left[-\lambda b + \lambda a \right] - \left[e^{-\lambda b} - e^{-\lambda a} \right] \right\}$$

$$\mu = 1 + \frac{\left[e^{-\lambda b} - e^{-\lambda a} \right]}{\lambda(b-a)} \tag{2.11}$$

Remark: r =2,3,4 we will obtain the (second , third and fourth) moments.

$$\mu_2 = \frac{-1}{\lambda(b-a)} \sum_{k=0}^{2} (-1)^k \binom{2}{k} \left(\frac{e^{-k\lambda b}}{k} - \frac{e^{-k\lambda a}}{k} \right)$$

$$\mu_3 = \frac{-1}{\lambda(b-a)} \sum_{k=0}^{3} (-1)^k \binom{3}{k} \left(\frac{e^{-k\lambda b}}{k} - \frac{e^{-k\lambda a}}{k} \right)$$

$$\mu_4 = \frac{-1}{\lambda(b-a)} \sum_{k=0}^{4} (-1)^k \binom{4}{k} \left(\frac{e^{-k\lambda b}}{k} - \frac{e^{-k\lambda a}}{k} \right)$$

2.6.3. moment about the mean :

The rth moment about the mean μ_r of a random variable U with pdf $w(u)$ is called Central moments ,stated by

$$\mu_r = E\left[U - \mu_u \right]^r = \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} (u - \mu_u)^r \, w(u) du \quad \text{for } r = 0, 1, 2, \dots$$

Where $\mu_u = \mu$ is the mean of GUEQD and $w(u)$ is the pdf .

$$= \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} (u - \mu_u)^r \frac{1}{\lambda(b-a)(1-u)} du$$

$$= \frac{1}{\lambda(b-a)} \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} (u - \mu_u)^r \frac{1}{(1-u)} du$$

Let $\varphi = \mu_u$

$$= \frac{1}{\lambda(b-a)} \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} (u - \varphi)^r \frac{1}{(1-u)} du$$

Using the binomial series expansion of $(u - \varphi)^r$ and $\frac{1}{1-u}$ yields:

$$= \frac{1}{\lambda(b-a)} \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \sum_{i=0}^{r} (-1)^i \binom{r}{i} u^{r-i} \varphi^i \sum_{j=0}^{\infty} u^j \, du$$

$$= \frac{1}{\lambda(b-a)} \sum_{i=0}^{r} \sum_{j=0}^{\infty} (-1)^i \binom{r}{i} \varphi^i \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} u^{r-i+j} \, du$$

$$\mu_r = \frac{1}{\lambda(b-a)} \sum_{i=0}^{r} \sum_{j=0}^{\infty} \frac{(-1)^i \, \varphi^i \binom{r}{i}}{r-i+j+1} \left(\left[1-e^{-\lambda b}\right]^{r-i+j+1} - \left[1-e^{-\lambda a}\right]^{r-i+j+1} \right) \quad (2.12)$$

2.7. The variance:

The variance for a random variable U~ GUEQD is given by:

$$\text{var}(U) = \frac{1}{\lambda(b-a)} \sum_{i=0}^{2} \sum_{j=0}^{\infty} \frac{(-1)^i \, \varphi^i \binom{2}{i}}{j-i+3} \left(\left[1-e^{-\lambda b}\right]^{j-i+3} - \left[1-e^{-\lambda a}\right]^{j-i+3} \right) \quad (2.13)$$

We know the variance of a random variable U is defined by

$$\text{var}(U) = E\left[(U-\mu)^2\right]$$

As a result, we can calculate the variance of a random variable U ~GUEQD by plugging r=2 into equation (2.12).

Remark: Standard Deviation $(\sigma) = \sqrt{\text{var}(U)}$

2.8. Coefficient of Skewness , Kurtosis and Variation

In this subsection, we use the moment about the mean to drive the skewness, kurtosis, and variation of the **GUEQ** distribution, as shown in the following:

Coefficient of Variation of the **GUEQD** is given by:

$$(C.V.) = \frac{\sigma}{\mu} \qquad (2.14)$$

compensate for σ and Equation No. (2.11) into Equation No. (2.14) we can get (C.V.)

The coefficient of skewness K_a of the **GUEQD** is given by

$$K_a = \frac{E\left[(U-\mu)^3\right]}{E\left[(U-\mu)^2\right]^{\frac{3}{2}}} \tag{2.15}$$

The expression $\dfrac{E\left[(U-\mu)^3\right]}{E\left[(U-\mu)^2\right]^{\frac{3}{2}}}$ can be found by substituting r = 3 for the

numerator and r = 2 for the denominator in equation No. (2.12), and we get

$$K_a = \frac{\dfrac{1}{\lambda(b-a)}\sum_{i=0}^{3}\sum_{j=0}^{\infty}\dfrac{(-1)^i\,\varphi^i\binom{3}{i}}{j-i+4}\left(\left[1-e^{-\lambda b}\right]^{j-i+4}-\left[1-e^{-\lambda a}\right]^{j-i+4}\right)}{\left[\dfrac{1}{\lambda(b-a)}\sum_{i=0}^{2}\sum_{j=0}^{\infty}\dfrac{(-1)^i\,\varphi^i\binom{2}{i}}{j-i+3}\left(\left[1-e^{-\lambda b}\right]^{j-i+3}-\left[1-e^{-\lambda a}\right]^{j-i+3}\right)\right]^{\frac{3}{2}}} \tag{2.16}$$

The coefficient of kurtosis L of GUEQD is given by

$$L = \frac{E\left[(U-\mu)^4\right]}{E\left((U-\mu)^2\right)^2} \tag{2.17}$$

The expression $\dfrac{E\left[(U-\mu)^4\right]}{E\left((U-\mu)^2\right)^2}$ can be found by substituting r = 4 for the

numerator and r = 2 for the denominator in equation No. (2.17), and we get

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$$L = \cfrac{\dfrac{1}{\lambda(b-a)} \sum\limits_{i=0}^{4} \sum\limits_{j=0}^{\infty} \dfrac{(-1)^i \, \varphi^i \binom{4}{i}}{j-i+5} \left(\left[1-e^{-\lambda b}\right]^{j-i+5} - \left[1-e^{-\lambda a}\right]^{j-i+5} \right)}{\left[\dfrac{1}{\lambda(b-a)} \sum\limits_{i=0}^{2} \sum\limits_{j=0}^{\infty} \dfrac{(-1)^i \, \varphi^i \binom{2}{i}}{j-i+3} \left(\left[1-e^{-\lambda b}\right]^{j-i+3} - \left[1-e^{-\lambda a}\right]^{j-i+3} \right) \right]^2}$$

2.9. Moment Generating Function:

The moment generating function $M_U(t)$ of the random variable U is given by:

$$M_U(t) = E\left(e^{tu}\right) = \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} e^{tu} \, w(u) du$$

$$= \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} e^{tu} \frac{1}{\left[\lambda(b-a)\right](1-u)} du$$

$$= \left[\frac{1}{\lambda(b-a)}\right] \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \frac{e^{tu}}{(1-u)} du$$

Use Binomial Series of $\dfrac{1}{1-u}$ we get on

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \ldots = \sum_{k=0}^{\infty} u^k$$

Therefore the Taylor series of e^{tu} get on $e^{tu} = \sum\limits_{m=0}^{\infty} \dfrac{(tu)^m}{m!}$ then

$$M_U(t) = \frac{1}{\lambda(b-a)} \int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \sum_{m=0}^{\infty} \frac{(tu)^m}{m!} \sum_{k=0}^{\infty} u^k \, du$$

$$M_U(t) = \frac{1}{\lambda(b-a)} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^m}{m!} \left(\frac{\left(1-e^{-\lambda b}\right)^{m+k+1}}{m+k+1} - \frac{\left(1-e^{-\lambda a}\right)^{m+k+1}}{m+k+1} \right) \quad (2.18)$$

2.10. Quantile

The quantile u_q of the GUEQD where $\Theta = (a\ ;b\ ;\lambda)$ random variable U

is given by

$$Q(q) = 1 - e^{-\lambda\left(q(b-a)+a\right)}$$

$$q = p\left(X \le x_q\right) = F\left(x_q\right), x_q > 0$$

Starting with the well-known definition of the 100 q^{th} quantile, which is

simply the solution of the following equation, with respect to u_q , $0 < q < 1$

$$q = p\left(U \le u_q\right) = W\left(u_q\right)$$

Using the distribution function of GUEQD we have From the formula and

let

$$W(u) = q \quad \text{then} \quad q = \frac{-1}{\lambda(b-a)}\left(\ln\left(1-u_q\right) + \lambda a\right)$$

$$-q\,\lambda(b-a) = \ln\left(1-u_q\right) + \lambda a$$

$$-q\,\lambda(b-a) - \lambda a = \ln\left(1-u_q\right)$$

$$-\lambda\left(q(b-a)+a\right) = \ln\left(1-u_q\right)$$

$$e^{-\lambda\left(q(b-a)+a\right)} = 1 - u_q$$

$$u_q = 1 - e^{-\lambda\left(q(b-a)+a\right)} \quad\quad\quad (2.19)$$

The Median of the distribution is obtained when q=0.5 in (2.19)

$$Median = u_{0.5} = 1 - e^{-\lambda\left(0.5\,(b-a)+a\right)}$$
(2.20)

2.11. Order Statistics :

Let $u_1, u_2, ..., u_S$ is a simple random sample with distribution Function $W(u)$ and $w(u)$ density function of **GUEQ** Distribution. Let $U_{(1:s)} \leq U_{(2:s)} \leq .... \leq U_{(s:s)}$ indicate the order statistics obtained from this sample. The density function of $U_{(h:s)}$, $1 \leq h \leq s$ is known by:

$$w_{U_{(h:s)}}(u) = \frac{s!}{(h-1)!(s-h)!}[W(u)]^{h-1}(1-W(u))^{s-h} w(u) ,$$
$$1 - e^{-\lambda a} < u < 1 - e^{-\lambda b}$$

$$= \frac{s!}{(h-1)!(s-h)!}\left[\frac{-1}{\lambda(b-a)}\left(\ln(1-u_n)+\lambda a\right)\right]^{h-1}\left(1-\left[\frac{-1}{\lambda(b-a)}\left(\ln(1-u_n)+\lambda a\right)\right]\right)^{s-h}$$
$$\left(\frac{1}{\lambda(b-a)(1-u_n)}\right) , 1 - e^{-\lambda a} < u < 1 - e^{-\lambda b}$$
(2.21)

Use the binomial series expansion of $\left(1-\left[\frac{-1}{\lambda(b-a)}\left(\ln(1-u_n)+\lambda a\right)\right]\right)^{s-h}$ as

follows :

$$\left(1-\left[\frac{-1}{\lambda(b-a)}\left(\ln(1-u_n)+\lambda a\right)\right]\right)^{s-h} = \sum_{t=0}^{s-h}\binom{s-h}{t}(-1)^t\left(\frac{-1}{\lambda(b-a)}\left(\ln(1-u_n)+\lambda a\right)\right)^t$$
(2.22)

From substituting the equation (2.22) into equation (2.21), we get:

$$= \frac{s!}{(h-1)!(s-h)!} \left[\frac{-1}{\lambda(b-a)} \left(\ln(1-u_n) + \lambda a \right) \right]^{h-1} \sum_{t=0}^{s-h} \binom{s-h}{t} (-1)^t$$

$$\left(\frac{-1}{\lambda(b-a)} \left(\ln(1-u_n) + \lambda a \right) \right)^t \left(\frac{1}{\lambda(b-a)(1-u_n)} \right) \quad , 1-e^{-\lambda a} < u_n < 1-e^{-\lambda b}$$

$$= \frac{s!}{(h-1)!(s-h)!} \sum_{t=0}^{s-h} \binom{s-h}{t} \left[\frac{(-1)^{h-1}}{\left[\lambda(b-a) \right]^{h+1}(1-u_n)} \left(\ln(1-u_n) + \lambda a \right)^{t+h-1} \right],$$

$$1-e^{-\lambda a} < u_n < 1-e^{-\lambda b}$$

The pdf of maximum order statistics U_s :

$$w_{U_{(s)}}(u) = s \left[W(u) \right]^{s-1} w(u) \quad , 1-e^{-\lambda a} < u < 1-e^{-\lambda b}$$

$$= s \left[\frac{-1}{\lambda(b-a)} \left(\ln(1-u_s) + \lambda a \right) \right]^{s-1} \left(\frac{1}{\lambda(b-a)(1-u_s)} \right) \quad , 1-e^{-\lambda a} < u_s < 1-e^{-\lambda b}$$

$$= s \left[\left(\frac{(-1)^{s-1}}{\left[\lambda(b-a) \right]^s} \right) \frac{\left(\ln(1-u_s) + \lambda a \right)^{s-1}}{(1-u_s)} \right] \quad , 1-e^{-\lambda a} < u_s < 1-e^{-\lambda b} \qquad (2.23)$$

and the pdf of the smallest order statistics U_s is given by:

$$w_{U_{(1)}}(u) = s \left(1-W(u) \right)^{s-1} w(u) \quad , 1-e^{-\lambda a} < u_{(1)} < 1-e^{-\lambda b}$$

$$= s \left(1 - \left[\frac{-1}{\lambda(b-a)} \left(\ln(1-u_1) + \lambda a \right) \right] \right)^{s-1} \left(\frac{1}{\lambda(b-a)(1-u_1)} \right), 1-e^{-\lambda a} < u_1 < 1-e^{-\lambda b}$$

$$= s \sum_{t=0}^{s-1} \binom{s-1}{t} \left(\frac{1}{\left[\lambda(b-a) \right]^{t+1}(1-u_1)} \left(\ln(1-u_1) + \lambda a \right)^t \right),$$

$$1-e^{-\lambda a} < u_1 < 1-e^{-\lambda b} \qquad (2.24)$$

2.12. Entropy:

If U a random variable has a **GUEQ** distribution ,then the Shannon entropy of U is given by:

Entropy $= E\left(-\log w\left(u\right)\right)$

$$=\int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} w\left(u\right)\left[-\log w\left(u\right)\right]du$$

When w(u) is pdf of **GUEQD** explained by the equation (2.1)

$$=\int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} w\left(u\right)\left[-\log\left(\frac{1}{\lambda(b-a)(1-u)}\right)\right]du$$

$$=\int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \left[\frac{1}{\lambda(b-a)(1-u)}\right]\left[\log\left(\lambda(b-a)(1-u)\right)\right]du$$

$$=\left[\frac{1}{\lambda(b-a)}\right]\left(\int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \left[\frac{1}{(1-u)}\right]\left[\log\lambda(b-a)+\log(1-u)\right]du\right)$$

$$=\frac{1}{\lambda(b-a)}\left(\log\lambda(b-a)\int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \left[\frac{1}{(1-u)}du\right]+\int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \left[\frac{1}{(1-u)}\right]\log(1-u)du\right)$$

$$=\frac{1}{\lambda(b-a)}\left(-\log\lambda(b-a)\int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \left[\frac{-1}{(1-u)}du\right]-\int_{1-e^{-\lambda a}}^{1-e^{-\lambda b}} \left[\frac{-1}{(1-u)}\right]\left[\log(1-u)\right]du\right)$$

$$=\frac{1}{\lambda(b-a)}\left(-\log\lambda(b-a)\ln(1-u)\Big]_{1-e^{-\lambda a}}^{1-e^{-\lambda b}}-\frac{\left[\log(1-u)\right]^2}{2}\Bigg]_{1-e^{-\lambda a}}^{1-e^{-\lambda b}}\right)$$

$$= \frac{1}{\lambda(b-a)} \left(-\left[\log \lambda(b-a) \right] \left[\ln e^{-\lambda b} - \ln e^{-\lambda a} \right] - \left[\frac{\left[\log\left(e^{-\lambda b}\right) \right]^2}{2} - \frac{\left[\log\left(e^{-\lambda a}\right) \right]^2}{2} \right] \right)$$

$$= \frac{1}{\lambda(b-a)} \left(-\left[\log \lambda(b-a) \right] \left[-\lambda b + \lambda a \right] - \left[\frac{\left[-\lambda b \log e \right]^2}{2} - \frac{\left[-\lambda a \log e \right]^2}{2} \right] \right)$$

$$= \frac{1}{\lambda(b-a)} \left(\lambda(b-a) \left[\log \lambda(b-a) \right] - \left[\frac{(\lambda b)^2 \left[\log e \right]^2}{2} - \frac{(\lambda a)^2 \left[\log e \right]^2}{2} \right] \right)$$

$$= \frac{1}{\lambda(b-a)} \left(\lambda(b-a) \left[\log \lambda(b-a) \right] - \left[\lambda^2 \left(b^2 - a^2 \right) \frac{\left[\log e \right]^2}{2} \right] \right)$$

$$Entropy = \log \lambda(b-a) - \lambda(b+a) \frac{\left[\log e \right]^2}{2}$$

$$Entropy = \log \lambda(b-a) - \frac{1}{2} \lambda(b+a) \tag{2.25}$$

2.12. Estimation Methods

We will address in this section several methods for estimating unknown parameters **GUEQD** .

2.12.1. Maximum Likelihood Method.

We will use maximum likelihood method for estimating the parameters of GUEQ. Let $U_1, U_2, ..., U_n$ indicate the sample size n at random from the GUEQD. Then the likelihood function is given by

$$L\left(u_i; \lambda, b, a\right) = \prod_{i=1}^{n} w\left(u_i; \lambda, b, a\right)$$

$$= \prod_{i=1}^{n} \frac{1}{\lambda(b-a)\left(1-u_i\right)}$$

$$\ln L\left(\lambda,b,a\right) = \sum_{i=1}^{n} \ln\left(\frac{1}{\lambda(b-a)\left(1-u_i\right)}\right)$$

$$= -\sum_{i=1}^{n}\left(\ln\lambda + \ln(b-a) + \ln\left(1-u_i\right)\right)$$

$$\ln L\left(\lambda,b,a\right) = -n\ln\lambda - n\ln(b-a) - \sum_{i=1}^{n}\ln\left(1-u_i\right)$$

$$\frac{\partial}{\partial\lambda}\ln L\left(\lambda,b,a\right) = \frac{-n}{\hat{\lambda}} = 0$$

$$-n = \hat{\lambda}*0 \Rightarrow n = 0 \ \left(Absurd\ result\right))$$

$$\frac{\partial}{\partial b}\ln L\left(\lambda,b,a\right) = -\frac{n}{\left(\hat{b}-a\right)} = 0$$

$$-n = \left(\hat{b}-a\right)*0 \Rightarrow n = 0 \ \left(Absurd\ result\right))$$

$$\frac{\partial}{\partial a}\ln L\left(\lambda,b,a\right) = \frac{n}{\left(b-\hat{a}\right)} = 0$$

$$n = \left(b-\hat{a}\right)*0 \Rightarrow n = 0 \ \left(Absurd\ result\right))$$

$$L\left(u_i;\lambda,b,a\right) = \prod_{i=1}^{n} \frac{1}{\lambda(b-a)\left(1-u_i\right)} = \left(\frac{1}{\lambda(b-a)}\right)^{n} \sum_{i=1}^{n}\left(1-u_i\right)^{-1}$$

L is maximum if $\lambda(b-a)$ is minimum

$$\lambda a \leq U_i \leq \lambda b \Rightarrow \lambda a \leq U_{(1)}, U_{(2)}, \ldots, U_{(n)} \leq \lambda b$$

$$\lambda a = U_{(1)} \ and \ \lambda b = U_{(n)} \ then \ a = \frac{U_{(1)}}{\lambda} \ , \ b = \frac{U_{(n)}}{\lambda} \ and \ \lambda = \frac{U_{(1)}}{a} \ , \ \lambda = \frac{U_{(n)}}{b}$$

$$\hat{a} = \frac{\min\left\{U_{(1)}, U_{(2)}, \ldots, U_{(n)}\right\}}{\hat{\lambda}} \ , \ \hat{b} = \frac{\max\left\{U_{(1)}, U_{(2)}, \ldots, U_{(n)}\right\}}{\hat{\lambda}} \tag{2.26}$$

Thus

$$\hat{\lambda}=\frac{\min\left\{U_{(1)},U_{(2)},...,U_{(n)}\right\}}{\hat{a}} \quad , \quad \hat{\lambda}=\frac{\max\left\{U_{(1)},U_{(2)},...,U_{(n)}\right\}}{\hat{b}} \qquad (2.27)$$

2.12.2. Method of moments .

This technique may be discovered by:

$$E\left(U^{r}\right)=\frac{1}{n}\sum_{k=1}^{n}U_{k}^{r} \qquad (2.28)$$

Such that $E\left(U^{r}\right)$ the rth moment about origin.

$$\frac{-1}{\lambda(b-a)}\sum_{k=0}^{r}(-1)^{k}\,C_{k}^{r}\left(\frac{e^{-k\,\lambda b}}{k}-\frac{e^{-k\,\lambda a}}{k}\right)=\frac{1}{n}\sum_{k=1}^{n}U_{k}^{r} \qquad (2.29)$$

The first population moment is the same as the first sample moment, we get

$$1+\frac{\left[e^{-\hat{\lambda}\hat{b}}-e^{-\hat{\lambda}\hat{a}}\right]}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)}=\frac{\sum_{i=1}^{m}u_{i}}{m} \qquad (2.30)$$

Equating the second population and sample moments

$$1+\frac{2\left(e^{-\hat{\lambda}\hat{b}}-e^{-\hat{\lambda}\hat{a}}\right)}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)}-\frac{e^{-2\hat{\lambda}\hat{b}}-e^{-2\hat{\lambda}\hat{a}}}{2\hat{\lambda}\left(\hat{b}-\hat{a}\right)}=\frac{\sum_{i=1}^{m}u_{i}^{2}}{m} \qquad (2.31)$$

Now equating the three population and sample moments

$$1+\frac{3\left(e^{-\hat{\lambda}\hat{b}}-e^{-\hat{\lambda}\hat{a}}\right)}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)}-3\left(\frac{e^{-2\hat{\lambda}\hat{b}}-e^{-2\hat{\lambda}\hat{a}}}{2\hat{\lambda}\left(\hat{b}-\hat{a}\right)}\right)+\left(\frac{e^{-3\hat{\lambda}\hat{b}}-e^{-3\hat{\lambda}\hat{a}}}{3\hat{\lambda}\left(\hat{b}-\hat{a}\right)}\right)$$

$$=\frac{\sum_{i=1}^{m}u_{i}^{3}}{m} \qquad (2.32)$$

We may get estimates for the parameters a, b and λ by solving the equations (2.29), (2.31) ,(2,32) for a, b and λ using numerical methods .

2.12.3. Least Squares of Method

This method is based on minimizing the total squares of errors and can be formulated as follows :

$$Ls = \sum_{\tau=1}^{m} \left[W_{GURQD}(u) - \left(\frac{\tau}{m+1}\right) \right]^2 \quad ...(2.33)$$

Where $W(u)$ is the distribution function of GUEQD .

Such that

$$W(u) = \frac{-1}{\lambda(b-a)}(\ln(1-u) + \lambda a), 1 - e^{-\lambda a} < u < 1 - e^{-\lambda b}, \lambda, b, a > 0, a < b$$

Note that $\dfrac{\tau}{m+1}$ It is an estimator, not a parameter, and is an estimator of the cumulative function of the model GUEQD

$$Ls = \sum_{\tau=1}^{m} \left[\frac{-(\ln(1-u) + \lambda a)}{\lambda(b-a)} - \left(\frac{\tau}{m+1}\right) \right]^2$$

with respect to a , b and λ

$$\frac{\partial Ls}{\partial \lambda} = 2 \sum_{\tau=1}^{m} \left[\frac{-(\ln(1-u) + \lambda a)}{\lambda(b-a)} - \left(\frac{\tau}{m+1}\right) \right] \left[\frac{\ln(1-u)}{\lambda^2(b-a)} \right]$$

$$\frac{\partial Ls}{\partial \lambda} = 2 \sum_{\tau=1}^{m} \left[\frac{-(\ln(1-u) + \lambda a)}{\lambda(b-a)} \left[\frac{\ln(1-u)}{\lambda^2(b-a)} \right] - \left(\frac{\tau}{m+1}\right) \left[\frac{\ln(1-u)}{\lambda^2(b-a)} \right] \right]$$

$$\frac{\partial L.s}{\partial \lambda} = 0$$

$$2 \sum_{\tau=1}^{m} \left[\left(\frac{-(\ln(1-u) + \hat{\lambda}\hat{a})}{\hat{\lambda}(\hat{b}-\hat{a})} \right) \left[\frac{\ln(1-u)}{\hat{\lambda}^2(\hat{b}-\hat{a})} \right] - \left(\frac{\tau}{m+1}\right) \left[\frac{\ln(1-u)}{\hat{\lambda}^2(\hat{b}-\hat{a})} \right] \right] = 0$$

$$\sum_{\tau=1}^{m}\left(\frac{-\left(\ln(1-u)+\hat{\lambda}\hat{a}\right)}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)}\right)\left[\frac{\ln(1-u)}{\hat{\lambda}^2\left(\hat{b}-\hat{a}\right)}\right]=\sum_{\tau=1}^{m}\left(\frac{\tau}{m+1}\right)\left[\frac{\ln(1-u)}{\hat{\lambda}^2\left(\hat{b}-\hat{a}\right)}\right] \qquad (2.34)$$

$$\frac{\partial L.s}{\partial b}=2\sum_{\tau=1}^{m}\left[\frac{-\left(\ln(1-u)+\lambda a\right)}{\lambda(b-a)}-\left(\frac{\tau}{m+1}\right)\right]\left[\frac{\ln(1-u)+\lambda a}{\lambda(b-a)^2}\right]$$

$$\frac{\partial L.s}{\partial b}=0$$

$$2\sum_{\tau=1}^{m}\left[\frac{-\left(\ln(1-u)+\hat{\lambda}\hat{a}\right)}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)}-\left(\frac{\tau}{m+1}\right)\right]\left[\frac{\ln(1-u)+\hat{\lambda}\hat{a}}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)^2}\right]=0$$

$$\sum_{\tau=1}^{m}\left[\left(\frac{-\left(\ln(1-u)+\hat{\lambda}\hat{a}\right)}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)}\right)\left[\frac{\ln(1-u)+\hat{\lambda}\hat{a}}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)^2}\right]-\left(\frac{\tau}{m+1}\right)\left[\frac{\ln(1-u)+\hat{\lambda}\hat{a}}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)^2}\right]\right]=0$$

$$\sum_{\tau=1}^{m}\left(\frac{-\left(\ln(1-u)+\hat{\lambda}\hat{a}\right)}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)}\right)\left[\frac{\ln(1-u)+\hat{\lambda}\hat{a}}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)^2}\right]=\sum_{\tau=1}^{m}\left(\frac{\tau}{m+1}\right)\left[\frac{\ln(1-u)+\hat{\lambda}\hat{a}}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)^2}\right] \qquad (2.35)$$

$$\frac{\partial L.s}{\partial a}=2\sum_{\tau=1}^{m}\left[\frac{-\left(\ln(1-u)+\lambda a\right)}{\lambda(b-a)}-\left(\frac{\tau}{m+1}\right)\right]\left[\frac{-\ln(1-u)+a}{\lambda(b-a)^2}+\frac{1}{(b-a)}\right]$$

$$\frac{\partial L.s}{\partial a}=0 \qquad 2\sum_{\tau=1}^{m}\left[\frac{-\left(\ln(1-u)+\lambda\hat{a}\right)}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)}-\left(\frac{\tau}{m+1}\right)\right]\left[\frac{-\ln(1-u)+\hat{a}}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)^2}+\frac{1}{\left(\hat{b}-\hat{a}\right)}\right]=0$$

$$\sum_{\tau=1}^{m}\left(\frac{-\left(\ln(1-u)+\lambda\hat{a}\right)}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)}\right)\left[\frac{-\ln(1-u)+\hat{a}}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)^2}+\frac{1}{\left(\hat{b}-\hat{a}\right)}\right]=\sum_{\tau=1}^{m}\left(\frac{\tau}{m+1}\right)\left[\frac{-\ln(1-u)+\hat{a}}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)^2}\right.$$

$$\left.+\frac{1}{\left(\hat{b}-\hat{a}\right)}\right] \qquad (2.36)$$

these non-liner questions   are difficult to solve , but   they can be numerically solved

Chapter Three

Generalizing Uniform Distribution Using the Quantile Function

and

Exponential- Generalizing Uniform Distribution

3.1. Generalizing Uniform Distribution Using the Quantile Function

In this section , we will present the generalized uniform distribution using the quantile function, GURQD and then find p.d.f, c.d.f and some properties (mathematical and statistical) of the distribution. We also show some estimation methods to estimate parameters.

3.1.1. Construction of the Generalizing Uniform Distribution Using the Quantile Function, GURQD

Since $R_Y(y) = 1 - F(y)$ as head of $F(x)$, $R(x)$ has $U(0,1)$ distribution then $x = R^{-1}(1 - F(x))$ is quantile function reliability function .if $G(x) = G(R(x))$ where $X \sim U(a,b)$ then $F(x) = \dfrac{x - a}{b - a}$

$x = Q\left(R_y(x)\right)$ where $R_y(x)$ is reliability of exponential distribution such that $R_y(x) = e^{-\lambda x} I_{(0,\infty)}(x)$ then $x = \ln R^{\frac{-1}{\lambda}}$ where $R = R_y$ random variable.

$G(x) = G\left(Q\left(R_y(x)\right)\right) = G(R)$ such that $Q\left(R_y(x)\right)$ is quantile function reliability function .

$G(x) = \dfrac{\ln R^{\frac{-1}{\lambda}} - a}{b - a}$ then $g(x) = \dfrac{-1}{\lambda(b-a)R}$

so you find it is impossible to use this substation unless using the following:

$R(x) = P(X > x) = \int\limits_{x}^{\infty} f(x)dx$

$$R_y(x) = \int_{L_R}^{U_R} \frac{dx}{b-a} \, , \quad L_R < R_y < U_R$$

When $R = 0 = L_R$, $R = 1 = U_R$

$$R_y(x) = \int_{\ln R^{\frac{-1}{\lambda}}}^{b} \frac{dx}{b-a} = \frac{b - l\,n\,R^{\frac{-1}{\lambda}}}{(b-a)} = \frac{b + \frac{1}{\lambda} l\,n\,R}{(b-a)} = \frac{\lambda b + l\,n\,R}{\lambda(b-a)}$$

Let $W_{GURQD}(R) = R_y(x)$ then

$$W_{GURQD}(R) = \frac{1}{\lambda(b-a)}(\ln R + \lambda b) \,, e^{-\lambda b} < R < e^{-\lambda a} \,, a,b,\lambda > 0, a < b \qquad (3.1)$$

$W_{GURQD}(R)$ is a cdf of GURQD.

A random variable R is indicated to be generalizing uniform distribution using the quantile function, GURQD if its pdf is in the form of

Which is a pdf as it can be proved

$$w_{GURQD}(R) = \frac{1}{\lambda(b-a)R} \,, \quad e^{-\lambda b} < R < e^{-\lambda a} \,, a,b,\lambda > 0, a < b \qquad (3.2)$$

 a probability density function ?

   1) It is easy to check $w_{GURQD}(R) \geq 0$

   2) The integral of the function in (3.1) must equal to 1

$$\int_{e^{-\lambda b}}^{e^{-\lambda a}} w_{GURQD}(R)dR = \int_{e^{-\lambda b}}^{e^{-\lambda a}} \frac{1}{\lambda(b-a)R}dR$$

$$= \frac{1}{\lambda(b-a)}\int_{e^{-\lambda b}}^{e^{-\lambda a}} \frac{1}{R}dR$$

$$= \frac{1}{\lambda(b-a)}\ln R \Big]_{e^{-\lambda b}}^{e^{-\lambda a}}$$

$$= \frac{1}{\lambda(b-a)}\left(\ln e^{-\lambda a} - \ln e^{-\lambda b}\right)$$

$$= \frac{1}{\lambda(b-a)}(-\lambda a + \lambda b)$$

$$= \frac{1}{\lambda(b-a)}\left[\lambda(b-a)\right]=1$$

3.1.2. The limit for the pdf as follows :

1. $\lim\limits_{R \to e^{-\lambda b}} W_{GURQD}(R) = \lim\limits_{R \to e^{-\lambda b}} \frac{1}{\lambda(b-a)R}$

$$= \frac{1}{\lambda(b-a)} \lim\limits_{R \to e^{-\lambda b}} \frac{1}{R}$$

$$= \frac{1}{\lambda(b-a)} \frac{1}{e^{-\lambda b}} = \frac{e^{\lambda b}}{\lambda(b-a)}$$

2. $\lim\limits_{R \to e^{-\lambda a}} W_{GURQD}(R) = \lim\limits_{R \to e^{-\lambda a}} \frac{1}{\lambda(b-a)R}$

$$= \frac{1}{\lambda(b-a)} \lim\limits_{R \to e^{-\lambda a}} \frac{1}{R}$$

$$= \frac{1}{\lambda(b-a)} \frac{1}{e^{-\lambda a}} = \frac{e^{\lambda a}}{\lambda(b-a)}$$

a=1.5
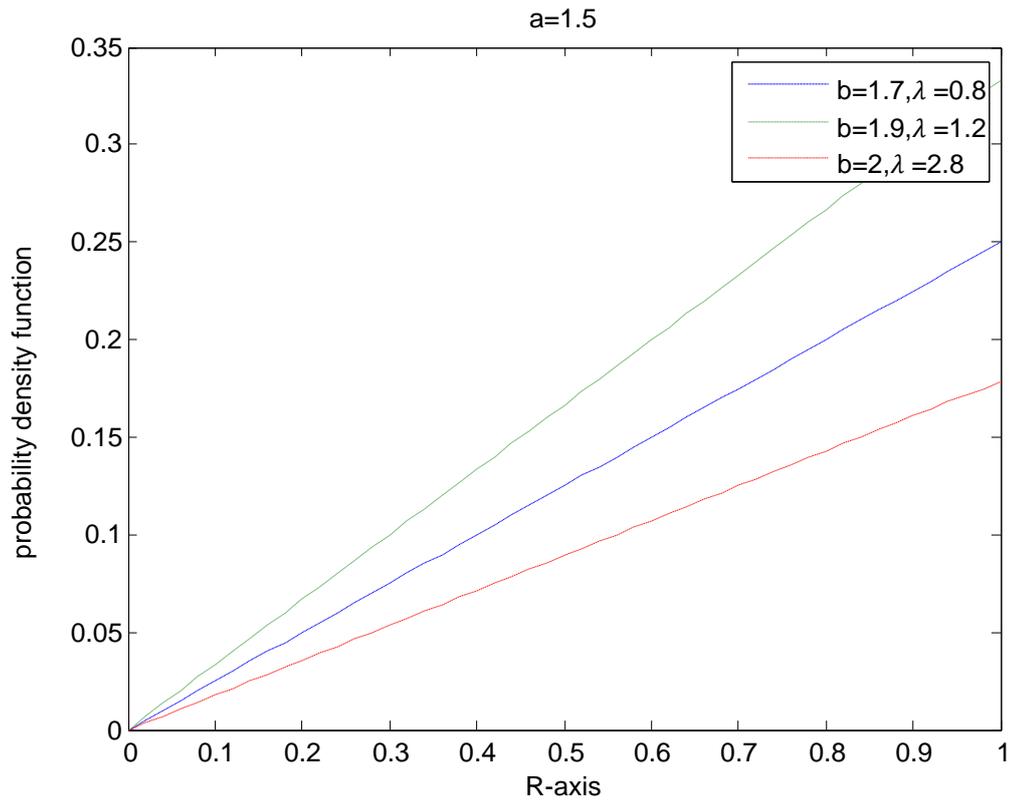
Figure.6:This graph showed pdf of GURQD with the parameters

$a = 1.5$; $b = (1.7, 1.9, 2)$, $\lambda = (0.8, 1.2, 2.8)$.

Figure (6) shows a PDF of the GURQ distribution that generates shapes starting at the origin and skewed to the right and increasingly.
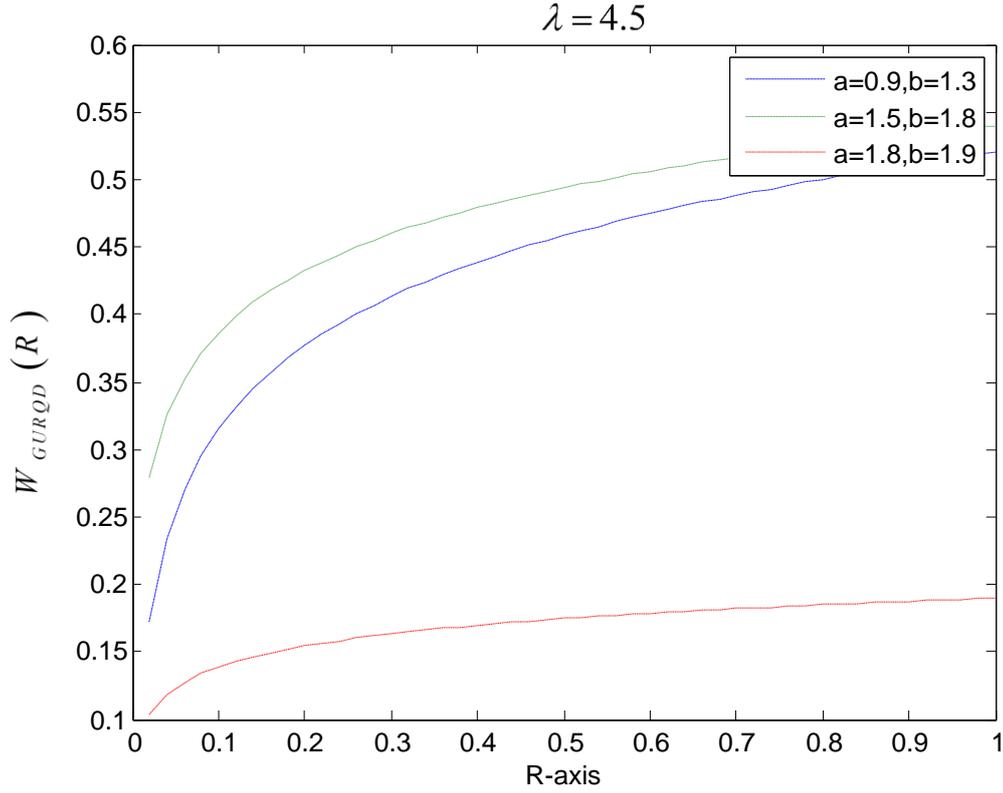
Figure.7:This graph showed cdf of GURQD with the parameters
$\lambda = 4.5$, $b = (1.3, 1.8, 1.9)$, $a = (0.9, 1.5, 1.8)$.

We can see from Figure (7) that cdf for GURQD does not decrease with increasing R *and the parameters λ, a, b.*

3.1.3. Limit of c.d.f

The limit of the GURQD is given by the form:

$$\lim_{R \to e^{-\lambda b}} W_{GURQD}(R) = 0$$

Also,

$$\lim_{R \to e^{-\lambda b}} \frac{1}{\lambda(b-a)}(\ln R + \lambda b)$$

$$\frac{1}{\lambda(b-a)}\left(\ln e^{-\lambda b} + \lambda b\right)$$

$$\frac{1}{\lambda(b-a)}(-\lambda b + \lambda b) = 0$$

And

$$\lim_{R \to e^{-\lambda a}} W_{GURQD}(R) = 1$$

Also,

$$\lim_{R \to e^{-\lambda a}} \frac{1}{\lambda(b-a)}(\ln R + \lambda b)$$

$$= \frac{1}{\lambda(b-a)}\left(\ln e^{-\lambda a} + \lambda b\right)$$

$$= \frac{1}{\lambda(b-a)}(-\lambda a + \lambda b)$$

$$= \frac{1}{\lambda(b-a)}\left[\lambda(b-a)\right] = 1$$

Then $0 \le W_{GURQD}(R) \le 1$

3.1.4. Some Reliabilities Analysis Functions:

The reliability is discussed such as (hazard, reverse hazard) rate function and cumulative hazard function for the GURQD,

Subsequently, the probability for a system survives beyond a specified time is known as reliability function or (survivor , survival) function and It set out in the following function.

$$R_{GURQD}(R) = 1 - W_{GURQD}(R)$$

$$= 1 - \frac{1}{\lambda(b-a)}(\ln R + \lambda b)$$

$$= 1 - \frac{(\ln R + \lambda b)}{\lambda(b-a)} \tag{3.3}$$

Note that $R(R)_{GURQD} + W_{GURQD}(R) = 1$

And

1) $\lim\limits_{R \to e^{-\lambda b}} R_{GURQD}(R) = \lim\limits_{R \to e^{-\lambda b}} \left[1 - \frac{(\ln R + \lambda b)}{\lambda(b-a)} \right] = 1$

2) $\lim\limits_{R \to e^{-\lambda a}} R_{GURQD}(R) = \lim\limits_{R \to e^{-\lambda a}} \left[1 - \frac{(\ln R + \lambda b)}{\lambda(b-a)} \right] = 0$

The hazard function is also defined as hazard rate or failure rate and is given by

$$h(R) = \frac{w_{GURQD}(R)}{1 - W_{GURQD}(R)}$$

$$= \frac{1}{R\left[\lambda(b-a) + (\ln R + \lambda b)\right]} \tag{3.4}$$

and the reverse hazard function for the GURQD, is given by

$$h_r(R) = \frac{w_{GURQD}(R)}{W_{GURQD}(R)}$$

$$= \frac{1}{R(\ln R + \lambda b)} \tag{3.5}$$

and the cumulative hazard function for the GURQD, is given by

$$H_R(R) = -\log\left(1 + \frac{1}{\lambda(b-a)}(\ln R + \lambda R)\right) \tag{3.6}$$

3.1.5. Mode:

To find the mode of the GURQD distribution, it must be

$$\frac{dw_{GURQD}(R)}{dR} = 0$$

$$\frac{d}{dR}\left[\frac{1}{\lambda(b-a)R_{\text{mode}}}\right] = 0$$

$$\frac{1}{\lambda(b-a)}\left[\frac{-1}{R_{\text{mode}}^2}\right] = 0 \tag{3.7}$$

However, it is not possible to get an explicit solution of (3.7) in the general case. Determent that from the curve of the pdf.

3.1.6. Moments

Theorem 3.1.6.1 : Let R denotes the random variable of GURQD, then the rth order moment $E(R^r)$ about the origin of GURQD can be obtained as

$$\mu_r' = E(R^r) = \frac{1}{\lambda r (b-a)} \left[e^{-\lambda r a} - e^{-\lambda r b} \right] \tag{3.8}$$

Proof. The rth moment of R is obtained by:

$$E(R^r) = \int_{e^{-\lambda b}}^{e^{-\lambda a}} R^r w_{GURQD}(R) dR$$

$$= \int_{e^{-\lambda b}}^{e^{-\lambda a}} \frac{R^r}{\lambda (b-a) R} dR$$

$$= \frac{1}{\lambda (b-a)} \int_{e^{-\lambda b}}^{e^{-\lambda a}} R^{r-1} dR$$

$$= \frac{1}{\lambda (b-a)} \frac{R^r}{r} \bigg]_{e^{-\lambda b}}^{e^{-\lambda a}}$$

$$= \frac{1}{\lambda (b-a)} \left[\frac{\left(e^{-\lambda a}\right)^r}{r} - \frac{\left(e^{-\lambda b}\right)^r}{r} \right]$$

Thus, the rth order moment of GURQD, is obtained

$$\mu_r' = \frac{1}{\lambda r (b-a)} \left[e^{-\lambda r a} - e^{-\lambda r b} \right]$$

Remark: If r = 1,2,3,4, we will get the (first , second , third, and fourth) moments.

$$mean = \mu_1 = \frac{1}{\lambda(b-a)}\left[e^{-\lambda a} - e^{-\lambda b}\right] \tag{3.9}$$

$$\mu_2 = \frac{1}{2\lambda(b-a)}\left[e^{-2\lambda a} - e^{-2\lambda b}\right]$$

$$\mu_3 = \frac{1}{3\lambda(b-a)}\left[e^{-3\lambda a} - e^{-3\lambda b}\right]$$

$$\mu_4 = \frac{1}{4\lambda(b-a)}\left[e^{-4\lambda a} - e^{-4\lambda b}\right]$$

Theorem 3.2.6.2: The rth moment about the mean for a random variable X~ GURQD is given by:

$$\mu_r = \frac{1}{\lambda(b-a)}\sum_{\ell=0}^{r}(-1)^{\ell}\binom{r}{\ell}\upsilon^{\ell}\left(\frac{\left(e^{-\lambda a}\right)^{r-\ell}}{r-\ell} - \frac{\left(e^{-\lambda b}\right)^{r-\ell}}{r-\ell}\right) \tag{3.10}$$

Proof: The rth moment about the mean of a random variable R is called Central moments ,stated by

μ_r , is the expected value of $\left(R - \mu_R\right)^r$ Simbolized

$$\mu_r = E\left[\left(R - \mu_R\right)^r\right]$$

$$= \int_{e^{-\lambda b}}^{e^{-\lambda a}}\left(R - \mu_R\right)^r w_{GURQD}(R)dR \text{ for r} = 0, 1, 2, \ldots$$

Let $\upsilon = \mu_R$

$$= \int_{e^{-\lambda b}}^{e^{-\lambda a}} (R - \upsilon)^r \, w_{GURQD}(R) \, dR$$

Using the binomial series expansion of $(R - \upsilon)^r$ yields:

$$(R - \upsilon)^r = \sum_{\ell=0}^{r} (-1)^{\ell} \binom{r}{\ell} R^{r-\ell} \upsilon^{\ell}$$

Then

$$= \frac{1}{\lambda(b-a)} \int_{e^{-\lambda b}}^{e^{-\lambda a}} \sum_{\ell=0}^{r} (-1)^{\ell} \binom{r}{\ell} R^{r-\ell} \upsilon^{\ell} R^{-1} \, dR$$

$$= \frac{1}{\lambda(b-a)} \sum_{\ell=0}^{r} (-1)^{\ell} \binom{r}{\ell} \upsilon^{\ell} \int_{e^{-\lambda b}}^{e^{-\lambda a}} R^{r-\ell-1} \, dR$$

$$\mu_r = \frac{1}{\lambda(b-a)} \sum_{\ell=0}^{r} (-1)^{\ell} \binom{r}{\ell} \upsilon^{\ell} \left(\frac{\left(e^{-\lambda a}\right)^{r-\ell}}{r-\ell} - \frac{\left(e^{-\lambda b}\right)^{r-\ell}}{r-\ell} \right)$$

Then we have the r$^{\text{th}}$ central moment of R about μ_R

Remark (1): The variance for a random variable R~ GURQD is given by:

$$\text{var}(R) = \frac{1}{\lambda(b-a)} \sum_{\ell=0}^{2} (-1)^{\ell} \binom{2}{\ell} \upsilon^{\ell} \left(\frac{\left(e^{-\lambda a}\right)^{2-\ell}}{2-\ell} - \frac{\left(e^{-\lambda b}\right)^{2-\ell}}{2-\ell} \right) \quad ...(3.11)$$

We know the variance of a random variable R is defined by

$$\text{var}(R) = E\left[(R - \mu)^2 \right]$$

As a result, we can calculate the variance of a random variable R GURQD by plugging r=2 into equation (2.10). This is the end of the proof.

Remark (2) : Standard Deviation $(\sigma)=\sqrt{\text{var}(R)}$ $\qquad$ (3.12)

3.1.7. Coefficient of Skewness , Kurtosis and Variation

In this subsection, we use the moment about the mean to drive the skewness, kurtosis, and variation of the GURQ distribution, as shown in the following:

Coefficient of Variation of the **GURQD** is given by:

$$(C.V.)=\frac{\sigma}{\mu} \qquad (2.13)$$

By substituting Equation No. (3.12) and Equation No. (3.9) into Equation No. (2.13) we can get (CV.)

The coefficient of skewness K_a of the **GURQD** is given by

$$K_a = \frac{n}{m^{\frac{3}{2}}} \qquad (2.14)$$

Where

$$n = \frac{1}{\lambda(b-a)}\sum_{\ell=0}^{3}(-1)^{\ell}\binom{3}{\ell}\upsilon^{\ell}\left[\frac{\left(e^{-\lambda a}\right)^{3-\ell}}{3-\ell}-\frac{\left(e^{-\lambda b}\right)^{3-\ell}}{3-\ell}\right]$$

and $m = \frac{1}{\lambda(b-a)}\sum_{\ell=0}^{2}(-1)^{\ell}\binom{2}{\ell}\upsilon^{\ell}\left[\frac{\left(e^{-\lambda a}\right)^{2-\ell}}{2-\ell}-\frac{\left(e^{-\lambda b}\right)^{2-\ell}}{2-\ell}\right]$

The coefficient of kurtosis L of GURQD is given by

$$L = \frac{C}{D^2} \qquad (3.15)$$

Where $\quad C = \dfrac{1}{\lambda(b-a)} \sum\limits_{\ell=0}^{4} (-1)^{\ell}\binom{4}{\ell} v^{\ell} \left[\dfrac{\left(e^{-\lambda a}\right)^{4-\ell}}{4-\ell} - \dfrac{\left(e^{-\lambda b}\right)^{4-\ell}}{4-\ell} \right]$ and

$$D = \dfrac{1}{\lambda(b-a)} \sum\limits_{\ell=0}^{2} (-1)^{\ell}\binom{2}{\ell} v^{\ell} \left[\dfrac{\left(e^{-\lambda a}\right)^{2-\ell}}{2-\ell} - \dfrac{\left(e^{-\lambda b}\right)^{2-\ell}}{2-\ell} \right]$$

3.1.8. Moment Generating Function

Theorem 3.1.8.3.: The moment generating function $M_X(t)$ of GURQD is given by:

$$M(t) = \dfrac{\sum\limits_{i=0}^{\infty} \dfrac{t^{i}}{i!} \left[\dfrac{\left(e^{-\lambda a}\right)^{i}}{i} - \dfrac{\left(e^{-\lambda b}\right)^{i}}{i} \right]}{\lambda(b-a)} \tag{3.16}$$

Proof. The moment generating function of GURQN, was detected by

$$M(t) = E\left(e^{t R}\right) = \int_{e^{-\lambda b}}^{e^{-\lambda a}} e^{t R} \, w_{GURQD}(R) \, dR$$

$$= \int_{e^{-\lambda b}}^{e^{-\lambda a}} e^{t R} \, \dfrac{1}{\lambda(b-a)R} \, dR$$

$$= \dfrac{1}{\lambda(b-a)} \int_{e^{-\lambda b}}^{e^{-\lambda a}} e^{t R} \, \dfrac{1}{R} \, dR$$

From Taylor series of $e^{t R}$ get on $e^{t R} = \sum\limits_{i=0}^{\infty} \dfrac{(t R)^{i}}{i!}$ then

$$= \dfrac{1}{\lambda(b-a)} \int_{e^{-\lambda b}}^{e^{-\lambda a}} \sum\limits_{i=0}^{\infty} \dfrac{(t R)^{i}}{i!} R^{-1} \, dR$$

$$M(t) = \frac{\sum\limits_{i=0}^{\infty} \frac{t^i}{i!} \left[\frac{\left(e^{-\lambda a}\right)^i}{i} - \frac{\left(e^{-\lambda b}\right)^i}{i} \right]}{\lambda(b-a)}$$

3.1.9. Quantile

Theorem 3.1.9.4. : The quantile $Q(q)$ of the GURQD where Θ= (a ;b ;λ) random variable R is given by

$$Q(q) = \exp\left(q\left(\lambda(b-a)\right) - \lambda b\right) \qquad (3.17)$$

Proof:

$$W_{GURQD}(R) = \frac{1}{\lambda(b-a)}(\ln R + \lambda b)$$

By solving $W_{GURQD}(R) = q$ for given value q ,the quantile $Q(q)$ is obtained

$$q = \frac{1}{\lambda(b-a)}(\ln Q + \lambda b)$$

$$Q(q) = \exp\left(q\left(\lambda(b-a)\right) - \lambda b\right)$$

Setting $q = 0.5$ yields the median of GURQD .

Median of GURQD is solution of (3.17) when q=0.5

3.1.10. Order Statistics:

Let $R_1, R_2, ..., R_n$ is a simple random sample with distribution function $W_{GURQD}(R)$ and density function $w_{GURQD}(R)$. Let the order statistics denote as $R_{(1:n)} \le R_{(2:n)} \le .... \le R_{(n:n)}$ obtained from this sample. The density function of $R_{(i:n)}$, $1 \le i \le n$ is provided by:

$$w_{i:n}\left(R_{(i)}\right) = \frac{n!}{(i-1)!(n-i)!}\left[W_{GURQD}\left(R_{(i)}\right)\right]^{i-1}\left(1-W_{GURQD}\left(R_{(i)}\right)\right)^{n-i}$$

$$w_{GURQD}\left(R_{(i)}\right) , e^{-\lambda b} < R_{(i)} < e^{-\lambda a}$$

$$= \frac{n!}{(i-1)!(n-i)!}\left[\frac{1}{\lambda(b-a)}\left(\ln R_{(i)} + \lambda b\right)\right]^{i-1}\left(1-\left[\frac{1}{\lambda(b-a)}\left(\ln R_{(i)} + \lambda b\right)\right]\right)^{n-i}$$

$$\left(\frac{1}{\lambda(b-a)R_{(i)}}\right) , e^{-\lambda b} < R_{(i)} < e^{-\lambda a}$$

The pdf of the maximum order statistics R_n is specified by:

$$w_{R_{(n)}}(R) = n\left[W_{GURQD}\left(R_{(n)}\right)\right]^{n-1} w_{GURQD}\left(R_{(n)}\right) , e^{-\lambda b} < R_{(n)} < e^{-\lambda a}$$

$$= n\left[\frac{1}{\lambda(b-a)}\left(\ln R_{(n)} + \lambda b\right)\right]^{n-1}\left(\frac{1}{\lambda(b-a)R_{(n)}}\right) , e^{-\lambda b} < R_{(n)} < e^{-\lambda a} \quad (3.18)$$

The pdf of smallest order statistics R_n :

$$w_{R_{(1)}}(R) = n\left(1-W_{GURQD}\left(R_{(1)}\right)\right)^{n-1} w_{GURQD}\left(R_{(1)}\right) , e^{-\lambda b} < R_{(1)} < e^{-\lambda a}$$

$$= n\left(1-\left[\frac{1}{\lambda(b-a)}\left(\ln R_{(1)} + \lambda b\right)\right]\right)^{n-1}\left(\frac{1}{\lambda(b-a)R_{(1)}}\right) , e^{-\lambda b} < R_{(1)} < e^{-\lambda a} \quad (3.19)$$

3.1.11. Entropy:

Entropy is a measure of a random variable R. variance or uncertainty   The Renyi and Shannon entropies Renyi (1961), Shannon (2001) are two comm on entropy measures.

Theorem 3.1.11.5. : If R a random variable has a GURQ distribution ,then the Shannon entropy of R is given by:

$$Entropy = \log \lambda(b-a) - \frac{1}{2}\lambda(b+a) \tag{3.20}$$

Proof.

The entropy of a Shannon variable R with pdf $w_{GURQD}(R)$ is defined as

Entropy $= E(-\log w(R))$

$$= \int_{e^{-\lambda b}}^{e^{-\lambda a}} w_{GURQD}(R)\left[-\log w(R)\right]dR$$

$$= \int_{e^{-\lambda b}}^{e^{-\lambda a}} w_{GURQD}(R)\left[-\log\left(\frac{1}{\lambda(b-a)R}\right)\right]dR$$

$$= \int_{e^{-\lambda b}}^{e^{-\lambda a}} \left[\frac{1}{\lambda(b-a)R}\right]\left[\log(\lambda(b-a)R)\right]dR$$

$$= \frac{1}{\lambda(b-a)}\left(\int_{e^{-\lambda b}}^{e^{-\lambda a}}\left[\frac{1}{R}\right]\left[\log\lambda(b-a)+\log R\right]dR\right)$$

$$= \frac{1}{\lambda(b-a)}\left(\frac{\left[\log\lambda(b-a)+\log R\right]^2}{2}\Bigg|_{e^{-\lambda b}}^{e^{-\lambda a}}\right)$$

$$= \frac{1}{\lambda(b-a)}\left(\frac{\left[\log\lambda(b-a)+\log e^{-\lambda a}\right]^2}{2}-\frac{\left[\log\lambda(b-a)+\log e^{-\lambda b}\right]^2}{2}\right)$$

$$= \frac{1}{\lambda(b-a)}\left(\frac{\left[\log\lambda(b-a)-\lambda a\log e\right]^2}{2}-\frac{\left[\log\lambda(b-a)-\lambda b\log e\right]^2}{2}\right)$$

$$= \frac{1}{\lambda(b-a)}\left(\frac{\left[\log\lambda(b-a)-\lambda a\right]^2}{2}-\frac{\left[\log\lambda(b-a)-\lambda b\right]^2}{2}\right)$$

$$= \frac{1}{2\lambda(b-a)}\left(\left[\log\lambda(b-a)\right]^2 - 2\lambda a\log\lambda(b-a) + \left[\lambda a\right]^2 - \left\{\left[\log\lambda(b-a)\right]^2 - 2\lambda b\log\lambda(b-a) + \left[\lambda b\right]^2\right\}\right)$$

$$= \frac{1}{2\lambda(b-a)}\left(-2\lambda a\log\lambda(b-a) + \left[\lambda a\right]^2 + 2\lambda b\log\lambda(b-a) - \left[\lambda b\right]^2\right)$$

$$= \frac{1}{2\lambda(b-a)}\left(2\lambda(b-a)\log\lambda(b-a) - \lambda^2\left(b^2 - a^2\right)\right)$$

$$= \log\lambda(b-a) - \frac{1}{2}\lambda(b+a)$$

3.1.12. Estimation Methods

We will address in this subsection several methods for estimating unknown parameters GURQD.

3.1.12.1. Maximum Likelihood Method

Using the maximum likelihood method ,the parameters of GURQD are estimated . Let $R_1, R_2, ..., R_n$ indicate the sample size n at random from the. Then the likelihood function is given by

$$L\left(R_i; \lambda, b, a\right) = \prod_{i=1}^{n} w\left(R_i; \lambda, b, a\right)$$

$$= \prod_{i=1}^{n} \frac{1}{\lambda(b-a)R_i}$$

$$\ln L\left(\lambda, b, a\right) = \sum_{i=1}^{n} \ln\left(\frac{1}{\lambda(b-a)R_i}\right)$$

$$= -n \ln \lambda - n \ln(b-a) - \sum_{i=1}^{n} \ln R_i$$

But it is noted that then is absurd result when

$$\frac{\partial \ln L}{\partial \lambda} = 0, \ \frac{\partial \ln L}{\partial b} = 0, \ \frac{\partial \ln L}{\partial a} = 0$$

$$L(R_i; \lambda, b, a) = \prod_{i=1}^{n} \frac{1}{\lambda(b-a)R_i} = \left(\frac{1}{\lambda(b-a)}\right)^{n} \sum_{i=1}^{n} R_i^{-1}$$

L is maximum if $\lambda(b-a)$ is minimum

$$e^{-\lambda b} \le R_i \le e^{-\lambda a} \Rightarrow e^{-\lambda b} \le R_{(1)}, R_{(2)}, \dots, R_{(n)} \le e^{-\lambda a}$$

$$e^{-\lambda b} = R_{(1)} \ \text{and} \ e^{-\lambda a} = R_{(n)} \ \text{then} \ b = \frac{\ln R_{(1)}}{-\lambda}, \ a = \frac{\ln R_{(n)}}{-\lambda} \ \text{and} \ \lambda = \frac{\ln R_{(1)}}{-b},$$

$$\lambda = \frac{\ln R_{(n)}}{-a}$$

$$\hat{b} = \frac{\min\left\{\ln R_{(1)}, \ln R_{(2)}, \dots, \ln R_{(n)}\right\}}{-\hat{\lambda}}, \ \hat{a} = \frac{\max\left\{\ln R_{(1)}, \ln R_{(2)}, \dots, \ln R_{(n)}\right\}}{-\hat{\lambda}} \ \text{and}$$

$$\hat{\lambda} = \frac{\min\left\{\ln R_{(1)}, \ln R_{(2)}, \dots, \ln R_{(n)}\right\}}{-\hat{b}}, \ \hat{\lambda} = \frac{\max\left\{\ln R_{(1)}, \ln R_{(2)}, \dots, \ln R_{(n)}\right\}}{-\hat{a}} \quad (3.21)$$

3.1.12.2. Method of Moments

This technique may be discovered by:

$$E(R^r) = \frac{1}{n} \sum_{k=1}^{n} R_k^r \qquad \dots(3.22)$$

Such that $E(R^r)$ the rth moment about origin.

$$\frac{1}{\lambda r (b-a)}\left[e^{-\lambda r a} - e^{-\lambda r b} \right] = \frac{1}{n}\sum_{k=1}^{n} R_k^{\,r} \qquad \text{...(3.23)}$$

If r=1,2,3, then the equation (3.23) becomes as follows:

$$\frac{1}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)}\left[e^{-\hat{\lambda}\hat{a}} - e^{-\hat{\lambda}\hat{b}} \right] = \frac{\sum_{i=1}^{m} R_i}{m} \qquad \text{...(3.24)}$$

$$\frac{1}{2\hat{\lambda}\left(\hat{b}-\hat{a}\right)}\left[e^{-2\hat{\lambda}\hat{a}} - e^{-2\hat{\lambda}\hat{b}} \right] = \frac{\sum_{i=1}^{m} R_i^{2}}{m} \qquad \text{...(3.25)}$$

$$\frac{1}{2\hat{\lambda}\left(\hat{b}-\hat{a}\right)}\left[e^{-2\hat{\lambda}\hat{a}} - e^{-3\hat{\lambda}\hat{b}} \right] = \frac{\sum_{i=1}^{m} R_i^{3}}{m} \qquad \text{...(3.26)}$$

We may get estimates for the parameters a, b and λ by solving the equations (3,24),(3.25),(3.26) for a, b, and λ using numerical methods.

3.1.12.3. Least Squares of Method

This technique, which is based on minimizing total squares of mistakes, may be expressed as follows:

$$L.s = \sum_{\tau=1}^{m}\left[W_{GURQD}(R) - \left(\frac{\tau}{m+1} \right) \right]^{2} \qquad (3.27)$$

Where $W_{GURQD}(R)$ is the GURQ distribution .

Such that

$$L.s = \sum_{\tau=1}^{m}\left[\frac{1}{\lambda(b-a)}(\ln R + \lambda b) - \left(\frac{\tau}{m+1} \right) \right]^{2}$$

$$\frac{\partial L.s}{\partial \lambda} = 0$$

$$\sum_{\tau=1}^{m} \frac{\left(\ln R + \hat{\lambda}\hat{b}\right)}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)} \left(\frac{\ln R}{\hat{\lambda}^2\left(\hat{b}-\hat{a}\right)}\right) = \sum_{\tau=1}^{m} \left(\frac{\ln R}{\hat{\lambda}^2\left(\hat{b}-\hat{a}\right)}\right)\left(\frac{\tau}{m+1}\right)$$ (3.28)

$$\frac{\partial L.s}{\partial b}=0$$

$$\sum_{\tau=1}^{m} \frac{\left(\ln R + \hat{\lambda}\hat{b}\right)}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)} \left(\frac{\hat{\lambda}\hat{a}+\ln R}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)^2}\right) = \sum_{\tau=1}^{m} \left(\frac{\tau}{m+1}\right)\left(\frac{\hat{\lambda}\hat{a}+\ln R}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)^2}\right)$$ (3.29)

$$\frac{\partial L.s}{\partial a}= 2\sum_{\tau=1}^{m}\left[\frac{\left(\ln R + \lambda b\right)}{\lambda\left(b-a\right)} - \left(\frac{\tau}{m+1}\right)\right]\left[\frac{\ln R + \lambda b}{\lambda\left(b-a\right)^2}\right]$$

$$\frac{\partial L.s}{\partial a}=0$$

$$\sum_{\tau=1}^{m} \frac{\left(\ln R + \hat{\lambda}\hat{b}\right)}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)} \left(\frac{\ln R + \hat{\lambda}\hat{b}}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)^2}\right) = \sum_{\tau=1}^{m} \left(\frac{\tau}{m+1}\right)\left(\frac{\ln R + \hat{\lambda}\hat{b}}{\hat{\lambda}\left(\hat{b}-\hat{a}\right)^2}\right)$$ (3.30)

It is clear that the equations (3.28), (3.29), (3.30) are non-linear related, so it can be solved using numerical methods .

3.2. Exponential- Generalizing Uniform Distribution (E- GURQD).

In this section , we will present the exponential - generalized uniform distribution E-GURQD and then find p.d.f, c.d.f and some properties (mathematical and statistical) of the distribution. We also show some estimation methods to estimate parameters.

3.2.1.Cumulative Distribution Function (cdf) of Exponential- Generalizing Uniform Distribution.

This defined the cdf of E- GURQD given by

$$F_X(x)=\int_a^{x=F_R\left[Q_Y(x)\right]} f_T(t)dt =F_T[F_R(Q_Y(x)]\qquad(3.31)$$

Where $f_T(x)$ and $F_T(x)$ are pdf and cdf of Exponential Distribution

such that $F_T(x)=1-e^{-\lambda x}$ and $f_T(x)=\lambda e^{-\lambda x}$, $F_R(x)$ cdf of

uniform distribution such that $F_R(x)=\dfrac{x-a}{b-a}$, $Q_Y(x)$ the quantile

function of GURQD such that the pdf , cdf and quantile function of it are

given as $f_Y(x)=\dfrac{1}{\lambda(b-a)x}$, $e^{-\lambda b}<x<e^{-\lambda a}$, $a,b>0,\lambda>0,a<b$,

$$F_Y(x)=\frac{(\ln x+\lambda b)}{\lambda(b-a)}\text{ and }Q_Y(x)=e^{\left(\left[\lambda(b-a)\right]x-\lambda b\right)}$$

Using (3.32) we get

$$F_X(x)=1-e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)x-\lambda b}-a\right)}\quad,\quad\frac{\ln a+\lambda b}{\lambda(b-a)}\le x<\infty,a<b\qquad(3.32)$$

Equation No. (3.32) represents cdf of E- GURQD.

Limit of cdf

1) $\lim\limits_{x \to \frac{\ln a + \lambda b}{\lambda(b-a)}} F_X(x) = \lim\limits_{x \to \frac{\ln a + \lambda b}{\lambda(b-a)}} \left\{ 1 - e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a \right]} \right\}$

$= 1 - e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)\left[\frac{\ln a + \lambda b}{\lambda(b-a)}\right] - \lambda b} - a \right]}$

$= 1 - e^{\frac{-\lambda}{(b-a)}\left[e^{\ln a + \lambda b - \lambda b} - a \right]}$

$= 1 - e^{\frac{-\lambda}{(b-a)}\left[e^{\ln a} - a \right]}$

$= 1 - e^{\frac{-\lambda}{(b-a)}(a - a)}$

$= 1 - e^0 = 0$

2) $\lim\limits_{x \to \infty} F_X(x) = \lim\limits_{x \to \infty} \left\{ 1 - e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a \right]} \right\}$

$= 1 - e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)\infty - \lambda b} - a \right]}$

$= 1 - e^{-\infty} = 1 - \frac{1}{e^{\infty}} = 1 - 0 = 1$
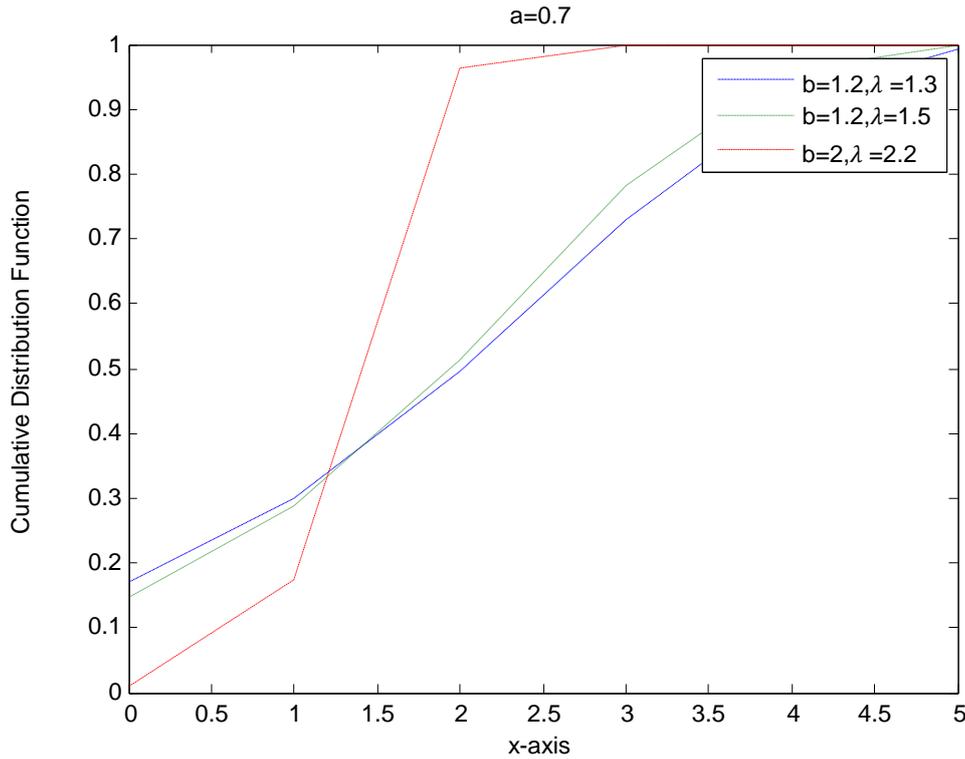
Figure.8:This graph showed cdf of E-GURQD with the parameters

$a = 0.7$ $b = (1.2, 1.2, 2)$ $\lambda = (1.3, 1.5, 2.2)$

We can see from Figure (8) that cdf for E-GURQD does not decrease with increasing x *and the parameters* λ, a, b.

3.2.2. Probability Density Function

The pdf of the new probability distribution, that is,

$$f_X(x) = f_T[F_R(Q_Y(x))][F_R'(Q_Y(x))] q_Y(x) \tag{3.33}$$

Where

$$F_R'(Q_Y(x)) = \frac{d}{dQ_Y} F_R(Q_Y(x))$$

Let $x = F_R\big(Q_Y(x)\big)$ and $F_R'\big(Q_Y(x)\big) q_Y(x) = 1 \Rightarrow F_R'\big(Q_Y(x)\big) = \frac{1}{q_Y(x)}$

Using the fact that $F_R(Q_R(x)) = x$ it follows that

$F'_R(Q_R(x)) \, q_R(x) = 1$ and $\qquad F'_R(Q_R(x)) = \dfrac{1}{q_R(x)}$ so that

$$F'_R(Q_R(x)) = \frac{1}{q_R\left(F_R(Q_R(x))\right)}$$

$$F'_R(p) = \frac{1}{q_R\left(F_R(p)\right)} \quad \text{by taking } p = Q_Y(x)$$

$$f_X(x) = q_Y(x) \frac{f_T[F_R(Q_Y(x))]}{q_R\left(F_R(Q_Y(x))\right)}$$

$$f_X(x) = \lambda e^{\lambda(b-a)x - \lambda b} f_T\left[\frac{e^{\lambda(b-a)x - \lambda b} - a}{b-a}\right]$$

$$f_X(x) = \lambda^2 e^{\lambda(b-a)x - \lambda b} \, e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]} \quad , \quad \frac{\ln a + \lambda b}{\lambda(b-a)} \le x < \infty$$

$$, \ a,b,\lambda > 0 \ , a < b \tag{3.34}$$

It is a **E- GURQD** probability density function which can proved as follows:

It is easy to check $f_X(x) \ge 0$

Therefore

$$\int_{\frac{\ln a + \lambda b}{\lambda(b-a)}}^{\infty} f_X(x) = \int_{\frac{\ln a + \lambda b}{\lambda(b-a)}}^{\infty} \lambda^2 e^{\lambda(b-a)x - \lambda b} \, e^{-\lambda\left[\frac{e^{\lambda(b-a)x - \lambda b} - a}{b-a}\right]} dx$$

$$= - e^{-\lambda\left[\frac{e^{\lambda(b-a)x - \lambda b} - a}{b-a}\right]} \Bigg|_{\frac{\ln a + \lambda b}{\lambda(b-a)}}^{\infty}$$

$$= - \left(e^{-\lambda \left[\dfrac{e^{\lambda (b-a)(\infty)-\lambda b}}{b-a} -a \right]} - e^{-\lambda \left[\dfrac{e^{\lambda (b-a)\left(\frac{\ln a + \lambda b}{\lambda (b-a)} \right)-\lambda b}}{b-a} -a \right]} \right)$$

$$= - \left(e^{-\infty} - e^{0} \right) = 1$$

The limit for the probability density function given as follows :

1) $\lim\limits_{x \to \left(\frac{\ln a + \lambda b}{\lambda (b-a)} \right)} f_X (x) = \lim\limits_{x \to \left(\frac{\ln a + \lambda b}{\lambda (b-a)} \right)} \left(\lambda^2 e^{\lambda (b-a)x - \lambda b} \right.$

$\left. e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)x - \lambda b} -a \right]} \right)$

$= \lambda^2 e^{\lambda (b-a)\left(\frac{\ln a + \lambda b}{\lambda (b-a)} \right)-\lambda b} e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)\left[\frac{\ln a + \lambda b}{\lambda (b-a)} \right]-\lambda b} -a \right]}$

$= \lambda^2 e^{0} e^{0}$

$= \lambda^2$

2) $\lim\limits_{x \to \infty} f_X (x) = \lim\limits_{x \to \infty} \left(\lambda^2 e^{\lambda (b-a)x - \lambda b} e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)x - \lambda b} -a \right]} \right)$

$= \lambda^2 e^{\lambda (b-a)(\infty)-\lambda b} e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)(\infty)-\lambda b} -a \right]}$

$= \lambda^2 e^{-\infty} e^{-\infty} = 0$

From Figure 11assarsion this fact.
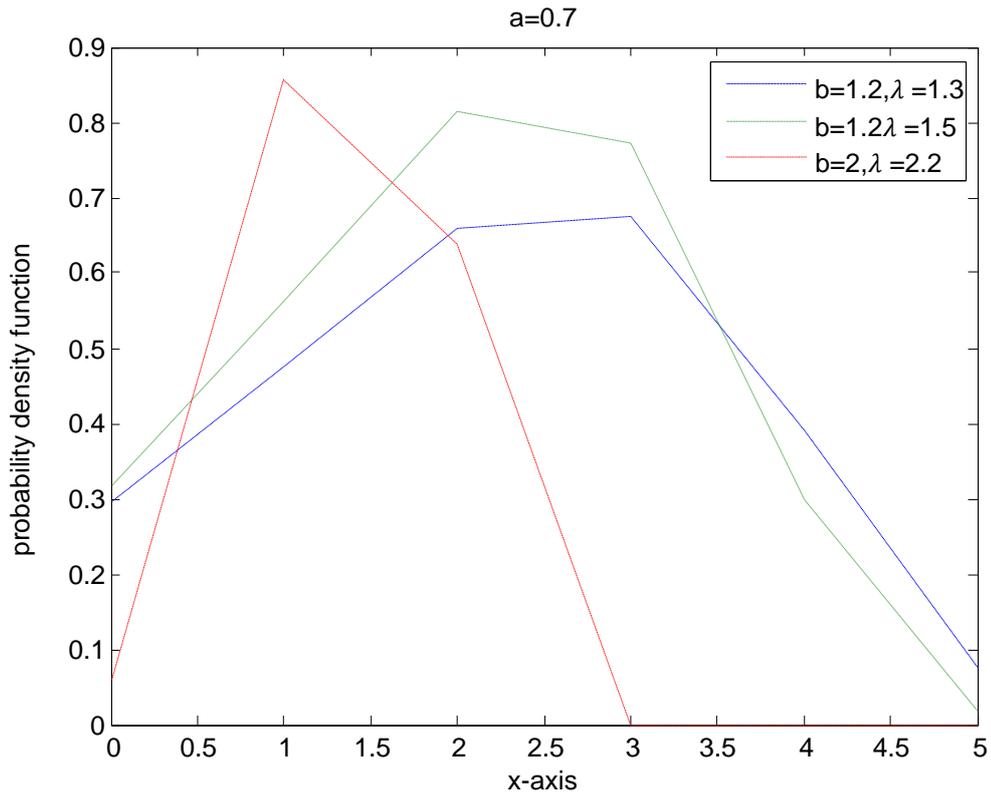
Figure.9:This graph showed pdf of E-GURQD with the parameters

$a = 0.7$; $b = (1.2, 1.2, 2),$ $\lambda = (1.3, 1.5, 2.2)$.

Figures (9) indicate that the E-GURQ distribution generates different shapes such as symmetric, right-skewed.

3.2.3. Some Reliability function

The **reliability function** is given by the following function

$R_X(x) = 1 - F_X(x)$

$= 1 - \left[1 - e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)x - \lambda b} - a \right)} \right]$

$R_X(x) = e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)x - \lambda b} - a \right)}$ \hfill (3.35)

Taking the limit to $R_X(x)$, that is

$$\lim_{x \to \left(\frac{\ln a + \lambda b}{\lambda(b-a)}\right)} R_X(x) = \lim_{x \to \left(\frac{\ln a + \lambda b}{\lambda(b-a)}\right)} \left\{ e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)x - \lambda b} - a\right)} \right\}$$

$$= e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)\left[\frac{\ln a + \lambda b}{\lambda(b-a)}\right] - \lambda b} - a\right)}$$

$$= e^{\frac{-\lambda}{(b-a)}\left(e^{\ln a + \lambda b - \lambda b} - a\right)} = 1$$

$$\lim_{x \to \infty} R_X(x) = \lim_{x \to \infty} \left\{ e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)x - \lambda b} - a\right)} \right\}$$

$$= e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)\infty - \lambda b} - a\right)}$$

$$= e^{-\infty} = 0$$

The **hazard function** of the random variable X can be written as

$$h_X(x) = \frac{f_X(x)}{1 - F_X(x)} = \frac{\lambda^2 e^{\lambda(b-a)x - \lambda b} \, e^{-\lambda\left[\frac{e^{\lambda(b-a)x - \lambda b} - a}{b-a}\right]}}{1 - \left[1 - e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)x - \lambda b} - a\right)}\right]}$$

$$h_X(x) = \lambda^2 e^{\lambda(b-a)x - \lambda b} \tag{3.36}$$

The **reverse hazard** it is function of a random variable X for the E-GURQD is provided by

$$h_r(x) = \frac{f_X(x)}{F_X(x)}$$

$$h_r(x) = \frac{\lambda^2 e^{\lambda(b-a)x - \lambda b} e^{-\lambda\left[\frac{e^{\lambda(b-a)x - \lambda b} - a}{b-a}\right]}}{1 - e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)x - \lambda b} - a\right)}}$$

(3.37)

The **cumulative hazard function** for the E- GURQD is provided by

$$\bar{H}_U(x) = -\ln\left(R_X(x)\right)$$

$$= -\ln\left[e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)x - \lambda b} - a\right)}\right]$$

$$\bar{H}_U(x) = \frac{\lambda}{(b-a)}\left(e^{\lambda(b-a)x - \lambda b} - a\right)$$

(3.38)

3.2.4. Mode:

To find the mode of the E- GURQD, it must be

$$\frac{d \ln f_X(x)}{dx} = 0$$

$$\frac{d}{dx}\left(\ln f_X(x)\right) = 0$$

$$\frac{d}{dx}\left(\ln\left\{\lambda^2 e^{\lambda(b-a)x - \lambda b} e^{-\lambda\left[\frac{e^{\lambda(b-a)x - \lambda b} - a}{b-a}\right]}\right\}\right) = 0$$

$$\frac{d}{dx}\left(\ln\lambda^2 + \ln e^{\lambda(b-a)x - \lambda b} + \ln e^{-\lambda\left[\frac{e^{\lambda(b-a)x - \lambda b} - a}{b-a}\right]}\right) = 0$$

$$\frac{d}{dx}\left(\ln\lambda^2 + \lambda(b-a)x - \lambda b - \frac{\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]\right) = 0$$

$$\lambda(b-a)-\lambda^2 e^{\lambda(b-a)x_{\text{mode}}-\lambda b}=0$$

$$e^{\lambda(b-a)x_{\text{mode}}-\lambda b}=\frac{(b-a)}{\lambda}$$

$$\lambda(b-a)x_{\text{mode}}-\lambda b=\ln\frac{(b-a)}{\lambda}$$

$$x_{\text{mode}}=\frac{\ln\left[\frac{(b-a)}{\lambda}\right]+\lambda b}{\lambda(b-a)} \tag{3.39}$$

3.2.5. Moments:

The moments of the E- GURQD It is given by the following theorem.

Theorem 3.2.5.6.: The rth order moment $E\left(X^r\right)$ about the origin of E- GURQD is given by:

$$\mu'_r(x)=\frac{1}{(\lambda(b-a))^r}\sum_{k=0}^{r}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{c}\frac{(-1)^j\left[\lambda b\right]^k a^{i-j}(\ln a)^{c-m}}{i\,!\,g^{i+1}\,d^{m-1}}\binom{r}{k}\binom{i}{j}$$
$$\binom{c}{m}e^{-d\,\ln a}\,\Gamma(m+1) \tag{3.40}$$

Proof. The rth moment of X is obtained by:

$$\mu'_r(x)=E\left(X^r\right)=\int_{\left(\frac{\ln a+\lambda b}{\lambda(b-a)}\right)}^{\infty}x^r f_X(x)\,dx$$

$$=\int_{\left(\frac{\ln a+\lambda b}{\lambda(b-a)}\right)}^{\infty}x^r\left\{\lambda^2 e^{\lambda(b-a)x-\lambda b}\,e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x-\lambda b}-a\right]}\right\}dx$$

Let $y=\frac{\lambda}{(b-a)}\left(e^{\lambda(b-a)x-\lambda b}-a\right)$ then $x=\dfrac{\ln\left(\frac{(b-a)}{\lambda}y+a\right)+\lambda b}{\lambda(b-a)}$

If $x = \left| \dfrac{\ln a + \lambda b}{\lambda(b-a)} \right|$ then $y = 0$, if $x = \infty$ then $y = \infty$

$$dy = \lambda^2 e^{\lambda(b-a)x - \lambda b} \, dx$$

$$\mu_r'(x) = \int_0^\infty \left\{ \frac{\ln\left(\frac{(b-a)}{\lambda} y + a\right) + \lambda b}{\lambda(b-a)} \right\}^r e^{-y} \, dy$$

$$= \frac{1}{(\lambda(b-a))^r} \int_0^\infty \left\{ \ln\left(\frac{(b-a)}{\lambda} y + a\right) + \lambda b \right\}^r e^{-y} \, dy$$

Using the binomial series of $\left\{ \ln\left(\frac{(b-a)}{\lambda} y + a\right) + \lambda b \right\}^r$ yields:

$$\left\{ \ln\left(\frac{(b-a)}{\lambda} y + a\right) + \lambda b \right\}^r = \sum_{k=0}^r \binom{r}{k} \left[\ln\left(\frac{(b-a)}{\lambda} y + a\right) \right]^{r-k} [\lambda b]^k$$

$$\mu_r'(x) = \frac{1}{(\lambda(b-a))^r} \sum_{k=0}^r \binom{r}{k} [\lambda b]^k \int_0^\infty [\ln(\vartheta y + a)]^{r-k} e^{-y} \, dy \, , \vartheta = \frac{(b-a)}{\lambda}$$

From Taylor series of e^y get on $e^{-y} = \sum_{i=0}^\infty \frac{(-y)^i}{i!}$ then

$$\mu_r'(x) = \frac{1}{(\lambda(b-a))^r} \sum_{k=0}^r \sum_{i=0}^\infty \frac{(-1)^i [\lambda b]^k}{i!} \binom{r}{k} \int_0^\infty [\ln(\vartheta y + a)]^{r-k} y^i \, dy$$

Let $e^x = a + \vartheta y$ and $y = \dfrac{a - e^x}{\vartheta}$ then $\dfrac{e^x}{\vartheta} dx = dy$ and if $y = 0$, $x = \ln a$

and $y = \infty$, $x = \infty$

$$\mu_r'(x) = \frac{1}{(\lambda(b-a))^r} \sum_{k=0}^r \sum_{i=0}^\infty \frac{(-1)^i [\lambda b]^k}{i!} \binom{r}{k} \int_{\ln a}^\infty [x]^{r-k} \left(\frac{a - e^x}{\vartheta}\right)^i \frac{e^x}{\vartheta} dx$$

$$\mu_r'(x) = \frac{1}{(\lambda(b-a))^r} \sum_{k=0}^r \sum_{i=0}^\infty \frac{(-1)^i [\lambda b]^k}{i! \, \vartheta^{i+1}} \binom{r}{k} \int_{\ln a}^\infty x^{r-k} e^x \left(a - e^x\right)^i dx$$

Using the binomial series of $\left(a - e^x\right)^i$ yields:

$$\left(a-e^x\right)^i =(-1)^i \left(e^x -a\right)^i =(-1)^i \sum_{j=0}^{i}(-1)^j \binom{i}{j}a^j \left(e^x\right)^{i-j}$$

From substituting from Eq.

$$\mu_r'(x)=\frac{1}{(\lambda(b-a))^r}\sum_{k=0}^{r}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^j \left[\lambda b\right]^k a^j}{i\,!\,9^{i+1}}\binom{r}{k}\binom{i}{j}\int_{\ln a}^{\infty} x^{r-k}\left[e^x\right]^{i-j+1}dx$$

Let $x =t +\ln a$ and $0\leq t <\infty$, $dx =dt$

$$\mu_r'(x)=\frac{1}{(\lambda(b-a))^r}\sum_{k=0}^{r}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^j \left[\lambda b\right]^k a^{i-j}}{i\,!\,9^{i+1}}\binom{r}{k}\binom{i}{j}$$

$$\int_{0}^{\infty}(t +\ln a)^{r-k}\left[e^{t+\ln a}\right]^{i-j+1}dt\,,c =r -k\,,d =-\left[(j -1)-i\right]$$

$$\mu_r'(x)=\frac{1}{(\lambda(b-a))^r}\sum_{k=0}^{r}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^j \left[\lambda b\right]^k a^{i-j}}{i\,!\,9^{i+1}}\binom{r}{k}\binom{i}{j}$$

$$\int_{0}^{\infty}(t +\ln a)^{c}\left[e^{t+\ln a}\right]^{-d}dt\,,c =r -k\,,-d =\left[(j -1)-i\right]$$

$$\mu_r'(x)=\frac{1}{(\lambda(b-a))^r}\sum_{k=0}^{r}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^j \left[\lambda b\right]^k a^{i-j}}{i\,!\,9^{i+1}}\binom{r}{k}\binom{i}{j}$$

$$\int_{0}^{\infty}(t +\ln a)^{c}\left[e^{t+\ln a}\right]^{-d}dt\,,c =r -k\,,-d =\left[(j -1)-i\right]$$

Using the binomial series of $(t +\ln a)^c$ yields:

$$(t +\ln a)^c = \sum_{m=0}^{c}\binom{c}{m}(\ln a)^{c-m}t^m$$

$$\mu_r'(x)=\frac{1}{(\lambda(b-a))^r}\sum_{k=0}^{r}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{c}\frac{(-1)^j \left[\lambda b\right]^k a^{i-j}(\ln a)^{c-m}}{i\,!\,9^{i+1}}\binom{r}{k}\binom{i}{j}$$

$$\binom{c}{m}e^{-d\ln a}\int_{0}^{\infty}t^m e^{-t\,d}dt$$

Let $-z =-d\,t\,,0\leq z <\infty$ and $\dfrac{dz}{d}=dt$

$$\mu_r'(x)=\frac{\left[\lambda b\right]^k}{(\lambda(b-a))^r}\sum_{k=0}^{r}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{c}\frac{(-1)^j a^{i-j}(\ln a)^{c-m}}{i\,!\,9^{i+1}\,d^{m-1}}\binom{r}{k}\binom{i}{j}\binom{c}{m}$$

$$e^{-d\ln a}\int_{0}^{\infty}z^m e^{-z}dz$$

Then the moment

$$\mu_r'(x) = \frac{1}{(\lambda(b-a))^r} \sum_{k=0}^{r} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{m=0}^{c} \frac{(-1)^j [\lambda b]^k a^{i-j} (\ln a)^{c-m}}{i! \, 9^{i+1} \, d^{m-1}} \binom{r}{k}\binom{i}{j}$$
$$\binom{c}{m} e^{-d \ln a} \Gamma(m+1)$$

Proposition 3.2.5.7.: The mean (μ) for a random variable X~ E-GURQD is given by:

$$\mu(x) = \frac{1}{(\lambda(b-a))} \sum_{k=0}^{1} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{m=0}^{c} \frac{(-1)^j [\lambda b]^k a^{i-j} (\ln a)^{c-m}}{i! \, 9^{i+1} \, d^{m-1}} \binom{1}{k}\binom{i}{j}$$
$$\binom{c}{m} e^{-d \ln a} \Gamma(m+1) \tag{3.41}$$

Proof. The mean ($\boldsymbol{\mu}$) of X~ E-GURQD is obtained by putting r=1 in Eq. (3.40).

Remark: r =2,3,4 we will obtain the (second , third and fourth) moments

$$\mu_2(x) = \frac{1}{(\lambda(b-a))} \sum_{k=0}^{2} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{m=0}^{c} \frac{(-1)^j [\lambda b]^k a^{i-j} (\ln a)^{c-m}}{i! \, 9^{i+1} \, d^{m-1}} \binom{2}{k}\binom{i}{j}$$
$$\binom{c}{m} e^{-d \ln a} \Gamma(m+1)$$

$$\mu_3(x) = \frac{1}{(\lambda(b-a))} \sum_{k=0}^{3} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{m=0}^{c} \frac{(-1)^j [\lambda b]^k a^{i-j} (\ln a)^{c-m}}{i! \, 9^{i+1} \, d^{m-1}} \binom{3}{k}\binom{i}{j}$$
$$\binom{c}{m} e^{-d \ln a} \Gamma(m+1)$$

$$\mu_4(x) = \frac{1}{(\lambda(b-a))} \sum_{k=0}^{4} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{m=0}^{c} \frac{(-1)^j [\lambda b]^k a^{i-j} (\ln a)^{c-m}}{i! \, 9^{i+1} \, d^{m-1}} \binom{4}{k}\binom{i}{j}$$
$$\binom{c}{m} e^{-d \ln a} \Gamma(m+1)$$

Theorem 3.2.5.8. : The rth moment about the mean for a random variable X~ E-GURQD is given by:

$$\mu_r(x) = \frac{1}{(\lambda(b-a))^r} \sum_{k=0}^{r} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{m=0}^{c} \sum_{g=0}^{r} \frac{(-1)^{j+r+g} [\lambda b]^k a^{i-j} (\ln a)^{c-m} \delta^{r-g}}{i! \, g^{i+1} \, d^{m-1}}$$

$$\binom{r}{k}\binom{i}{j}\binom{c}{m}\binom{r}{g} e^{-d \ln a} \Gamma(m+1) \tag{3.42}$$

Proof: The rth moment about the mean μ_r of a random variable X with pdf $f(x)$ is called Central moments ,stated by

$$\mu_r = E\left([X - \mu_x]^r\right)$$

The $(x - \mu_x)^r$ is equal to using the binomial series

$$(x - \mu_x)^r = \sum_{g=0}^{r} (-1)^{r+g} \binom{r}{g} (\mu_x)^{r-g} x^g$$

Then

$$\mu_r = E\left(\sum_{g=0}^{r} (-1)^{r+g} \binom{r}{g} (\mu_x)^{r-g} x^g\right)$$

$$= \left(\sum_{g=0}^{r} (-1)^{r+g} \binom{r}{g} (\mu_x)^{r-g}\right) E\left(x^g\right)$$

$$\mu_r = \int_{\frac{\ln a + \lambda b}{\lambda(b-a)}}^{\infty} \left(\sum_{g=0}^{r} (-1)^{r+g} \binom{r}{g} (\mu_x)^{r-g}\right) x^g f(x)dx \quad \text{for r = 0, 1, 2, . .}$$

.Where $\mu_x = \mu = \delta$ is the mean of E-GURQD and $f(x)$ is the pdf .

$$\mu_r = \left(\sum_{g=0}^{r} (-1)^{r+g} \binom{r}{g} \delta^{r-g}\right) \int_{\frac{\ln a + \lambda b}{\lambda(b-a)}}^{\infty} x^g f(x)dx$$

$$\mu_r = \left(\sum_{g=0}^{r} (-1)^{r+g} \binom{r}{g} \delta^{r-g}\right) \mu_g'$$

Here, μ'_g denotes the gth moment of the E-GURQD distribution, (by replacing r by g) into equation (3.40), we obtain the rth moment around the mean of the **E-GURQD** distribution as follows:

$$\mu_r(x) = \frac{1}{(\lambda(b-a))^r} \sum_{k=0}^{r} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{m=0}^{c} \sum_{g=0}^{r} \frac{(-1)^{j+r+g}[\lambda b]^k a^{i-j}(\ln a)^{c-m} \delta^{r-g}}{i!\, g^{i+1}\, d^{m-1}}$$
$$\binom{r}{k}\binom{i}{j}\binom{c}{m}\binom{r}{g} e^{-d\ln a}\Gamma(m+1)$$

Proposition. 3.2.5.9. : The variance for a random variable X~ E-GURQD is given by:

$$var(x) = \frac{1}{(\lambda(b-a))^r} \sum_{k=0}^{2} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{m=0}^{c} \sum_{g=0}^{2} \frac{(-1)^{j+r+g}[\lambda b]^k a^{i-j}(\ln a)^{c-m} \delta^{2-g}}{i!\, g^{i+1}\, d^{m-1}}$$
$$\binom{2}{k}\binom{i}{j}\binom{c}{m}\binom{2}{g} e^{-d\ln a}\Gamma(m+1) \qquad\qquad ...(3.43)$$

Proof: We know the variance of a random variable X is defined by defined by

$$var(X) = E\left[(X-\mu)^2\right]$$

As a result, we can calculate the variance of a random variable $X \sim GURQD$ by plugging r=2 into equation (3.42). This is the end of the proof.

Remark: Standard Deviation $(\sigma) = \sqrt{var(X)}$ \qquad\qquad (3.44)

3.2.6. Coefficient of Skewness , Kurtosis and Variation

In this subsection, we use the moment about the mean to drive the skewness, kurtosis, and variation of the **E-GURQ** distribution, as shown in the following:

Coefficient of Variation of the **E-GURQD** is given by:

$$(C.V.) = \frac{\sigma}{\mu} \qquad (3.45)$$

By substituting Equation No. (3.41) and Equation No. (3.44) into Equation No. (3.45) we can get (CV.)

The coefficient of skewness K_a of E- GURQD is given by

$$K_a = \frac{E\left[(X-\mu)^3\right]}{E\left[(X-\mu)^2\right]^{\frac{3}{2}}} \qquad (3.46)$$

The K_a can be found by substituting r=3 into equation (3.42), and substituting in the numerator, as well as r=2, and substituting in the denominator.

The coefficient of kurtosis L of E- GURQD is given by

$$L = \frac{E\left[(X-\mu)^4\right]}{E\left((X-\mu)^2\right)^2} \qquad (3.47)$$

The L can be found by substituting r=4 into equation (3.42), and substituting in the numerator, as well as r=2, and substituting in the denominator.

3.2.7. Moment Generating Function:

The MGF (moment generating function) for the **E- GURQD** has been discussed.

Theorem 3.2.7.10. : The moment generating function of **E-GURQD** distribution is given by the following:

$$M(t) = e^{\left(\frac{t\lambda b}{\lambda(b-a)}\right)} \sum_{k=0}^{\varphi} \binom{\varphi}{k} a^{\varphi-k} \left(\frac{(b-a)}{\lambda}\right)^k \Gamma(k+1) \qquad (3.48)$$

Proof. The moment generating function $M_X(t)$ of the random variable X is given by:

$$M(t) = E\left(e^{tx}\right) = \int_{\left(\frac{\ln a + \lambda b}{\lambda(b-a)}\right)}^{\infty} e^{tx} f_X(x) dx$$

$$= \int_{\frac{\ln a + \lambda b}{\lambda(b-a)}}^{\infty} \left(e^{tx} \lambda^2 e^{\lambda(b-a)x - \lambda b} e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]} \right) dx$$

$$M(t) = \int_{\frac{\ln a + \lambda b}{\lambda(b-a)}}^{\infty} \left(e^{tx} \lambda^2 e^{\lambda(b-a)x - \lambda b} e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]} \right) dx$$

Let $y = e^{\lambda(b-a)x - \lambda b}$ than $x = \frac{\ln y + \lambda b}{\lambda(b-a)}$

If $x = \left|\frac{\ln a + \lambda b}{\lambda(b-a)}\right|$ then $y = a$, if $x = \infty$ then $y = \infty$

$$dy = \lambda(b-a)e^{\lambda(b-a)x - \lambda b} dx$$

$$dx = \frac{dy}{\lambda(b-a)e^{\lambda(b-a)x - \lambda b}}$$

$$M(t) = \frac{\lambda}{(b-a)} \int_a^\infty \left[e^{t\left(\frac{\ln y + \lambda b}{\lambda(b-a)}\right)} e^{\frac{-\lambda}{(b-a)}[y-a]} \right] dy$$

$$M(t) = \frac{\lambda}{(b-a)} e^{\left(\frac{tb}{(b-a)}\right)} \int_a^\infty \left[e^{\frac{t\ln y}{\lambda(b-a)}} e^{\frac{-\lambda}{(b-a)}[y-a]} \right] dy$$

$$M(t) = \frac{\lambda}{(b-a)} e^{\left(\frac{tb}{(b-a)}\right)} \int_a^\infty \left[e^{\ln y^{\frac{t}{\lambda(b-a)}}} e^{\frac{-\lambda}{(b-a)}[y-a]} \right] dy$$

$$M(t) = \frac{\lambda}{(b-a)} e^{\left(\frac{tb}{(b-a)}\right)} \int_a^\infty \left[y^{\frac{t}{\lambda(b-a)}} e^{\frac{-\lambda}{(b-a)}[y-a]} \right] dy$$

Let $I = \int_a^\infty \left[y^{\frac{t}{\lambda(b-a)}} e^{\frac{-\lambda}{(b-a)}[y-a]} \right] dy$, $\varphi = \frac{t}{\lambda(b-a)}$

And $z = \frac{\lambda}{(b-a)}[y-a]$, $y = \frac{(b-a)}{\lambda} z + a$ then $dy = \frac{(b-a)}{\lambda} dz$

If y=a than z=0, $y = \infty$ then $z = \infty$

$$I = \frac{(b-a)}{\lambda} \int_0^\infty \left[\left(\frac{(b-a)}{\lambda} z + a\right)^\varphi e^{-z} \right] dz$$

$$I = \frac{(b-a)}{\lambda} \int_0^\infty \left[\sum_{k=0}^\varphi \binom{\varphi}{k} a^{\varphi-k} \left(\frac{(b-a)}{\lambda} z\right)^k e^{-z} \right] dz$$

$$I = \frac{(b-a)}{\lambda} \sum_{k=0}^\varphi \binom{\varphi}{k} a^{\varphi-k} \left(\frac{(b-a)}{\lambda}\right)^k \int_0^\infty z^k e^{-z} \, dz$$

$$I = \frac{(b-a)}{\lambda} \sum_{k=0}^\varphi \binom{\varphi}{k} a^{\varphi-k} \left(\frac{(b-a)}{\lambda}\right)^k \int_0^\infty z^{(k+1)-1} e^{-z} \, dz$$

$$I = \frac{(b-a)}{\lambda} \sum_{k=0}^{\varphi} \binom{\varphi}{k} a^{\varphi-k} \left(\frac{(b-a)}{\lambda} \right)^k \Gamma(k+1)$$

Than the moment generating function

$$M(t) = e^{\left(\frac{t\lambda b}{\lambda(b-a)} \right)} \sum_{k=0}^{\varphi} \binom{\varphi}{k} a^{\varphi-k} \left(\frac{(b-a)}{\lambda} \right)^k \Gamma(k+1)$$

3.2.8. Quantile:

Theorem 3.2.8.11. : The quantile x_p of the E- GURQD where ϴ= (a ;b ;λ) random variable X is given by

$$x_p = \frac{\ln\left[\ln(1-p)^{\frac{-(b-a)}{\lambda}} + a \right] + \lambda b}{\lambda(b-a)} \tag{3.49}$$

Proof:

Let $p = F_X(x)$ then $p = 1 - e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)x_p - \lambda b} - a \right)}$

$$e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)x_p - \lambda b} - a \right)} = 1 - p$$

$$e^{\lambda(b-a)x_p - \lambda b} - a = \ln(1-p)^{\frac{-(b-a)}{\lambda}}$$

$$e^{\lambda(b-a)x_p - \lambda b} = \ln(1-p)^{\frac{-(b-a)}{\lambda}} + a$$

$$\lambda(b-a)x_p - \lambda b = \ln\left[\ln(1-p)^{\frac{-(b-a)}{\lambda}} + a \right]$$

$$x_p = \frac{\ln\left[\ln(1-p)^{\frac{-(b-a)}{\lambda}} + a \right] + \lambda b}{\lambda(b-a)}$$

The median of this distribution is obtained when p=0.5

$$Median = x_{0.5} = \frac{\ln\left[\ln(0.5)^{\frac{-(b-a)}{\lambda}} + a\right] + \lambda b}{\lambda(b-a)} \qquad (3.50)$$

3.2.9. Order Statistics :

Let $X_1, X_2, ..., X_n$ is a simple random sample with distribution function F(x) and density function f(x) respectively .

Let $a < Y_1 \leq Y_2 \leq .... \leq Y_n < b$, $-\infty < a < b < \infty$ indicate the order statistics obtained from this sample. The density function of $X_{(i:n)}$, $1 \leq i \leq n$ is known by:

$$f_{i:n}(y_i) = \frac{n!}{(i-1)!(n-i)!}\left[1 - e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x-\lambda b} - a\right]}\right]^{i-1}$$

$$\left[e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x-\lambda b} - a\right]}\right]^{n-i}\left[\lambda^2 e^{\lambda(b-a)x-\lambda b} e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x-\lambda b} - a\right]}\right]$$

$, e^{-\lambda b} < y_i < e^{-\lambda a}$

The maximum order statistics X_n of pdf is given by

$$g(y_n) = n\left[F_X(y_n)\right]^{n-1} f_X(y_n)$$

$$g\left(y_n\right)=n\left[1-e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)x-\lambda b}-a\right)}\right]^{n-1}$$

$$\left[\lambda^2 e^{\lambda(b-a)x-\lambda b}\; e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x-\lambda b}-a\right]}\right],e^{-\lambda b}<y_i<e^{-\lambda a}\quad(3.51)$$

And the pdf of the smallest X_n order statistics is:

$$g\left(y_1\right)=n\left[e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)x-\lambda b}-a\right)}\right]^{n-1}$$

$$\left[\lambda^2 e^{\lambda(b-a)x-\lambda b}\; e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x-\lambda b}-a\right]}\right]\qquad(3.52)$$

3.2.10. Entropy:

Theorem 3.2.10.12.: If X a random variable has a E-GURQ distribution ,then the entropy of X is given by:

$$Entropy = \lambda b +-\log\lambda^2 -\lambda(b-a)\mu \qquad(3.53)$$

Proof.

Entropy$=E\left(-\log f_X\left(x\right)\right)$

$$=\int_{\left(\frac{\ln a+\lambda b}{\lambda(b-a)}\right)}^{\infty} f_X\left(x\right)\left[-\log f_X\left(x\right)\right]dx$$

$$= \int_{\left(\frac{\ln a + \lambda b}{\lambda (b-a)}\right)}^{\infty} \left\{ \left[\lambda^2 e^{\lambda (b-a)x - \lambda b} \; e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)x - \lambda b}-a\right]} \right] \right.$$

$$\left. \left[-\log\left(\lambda^2 e^{\lambda (b-a)x - \lambda b} \; e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)x - \lambda b}-a\right]} \right) \right] \right\} dx$$

$$= -\int_{\left(\frac{\ln a + \lambda b}{\lambda (b-a)}\right)}^{\infty} \left\{ \left[\lambda^2 e^{\lambda (b-a)x - \lambda b} \; e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)x - \lambda b}-a\right]} \right] \right.$$

$$\left. \left[\left(\log \lambda^2 + \log e^{\lambda (b-a)x - \lambda b} + \log e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)x - \lambda b}-a\right]} \right) \right] \right\} dx$$

$$= -\int_{\left(\frac{\ln a + \lambda b}{\lambda (b-a)}\right)}^{\infty} \left\{ \left[\lambda^2 e^{\lambda (b-a)x - \lambda b} \; e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)x - \lambda b}-a\right]} \right] \right.$$

$$\left. \left[\left(\log \lambda^2 + \left[\lambda (b-a)x - \lambda b\right]\log e + \log e^{\left(\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)x - \lambda b}-a\right]\right)} \right) \right] \right\} dx$$

$$Ent = -\int_{\left(\frac{\ln a + \lambda b}{\lambda (b-a)}\right)}^{\infty} \left\{ \left(\log \lambda^2\right) \lambda^2 e^{\lambda (b-a)x - \lambda b} \; e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)x - \lambda b}-a\right]} + \right.$$

$$\left[\left(\lambda (b-a)x - \lambda b\right)\log e\right] \lambda^2 e^{\lambda (b-a)x - \lambda b} \; e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)x - \lambda b}-a\right]} + \lambda^2 e^{\lambda (b-a)x - \lambda b}$$

$$e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)x - \lambda b}-a\right]} \left(\log e^{\left(\frac{-\lambda}{(b-a)}\left[e^{\lambda (b-a)x - \lambda b}-a\right]\right)} \right) \right\} dx$$

Let

$$Ent = -\left(I_1 + I_2 + I_3\right)$$

Such that

$$I_1 = \int_{\left(\frac{\ln a + \lambda b}{\lambda(b-a)}\right)}^{\infty} \left(\log \lambda^2\right) \lambda^2 e^{\lambda(b-a)x - \lambda b} \, e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]} \, dx$$

$$= -\left(\log \lambda^2\right) \left(e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]} \Big|_{\left(\frac{\ln a + \lambda b}{\lambda(b-a)}\right)}^{\infty} \right)$$

$$I_1 = \log \lambda^2$$

$$I_2 = \int_{\left(\frac{\ln a + \lambda b}{\lambda(b-a)}\right)}^{\infty} \left[(\lambda(b-a)x - \lambda b)\log e\right] \lambda^2 e^{\lambda(b-a)x - \lambda b} e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]} \, dx$$

$$= \lambda(\log e)(b-a)\int_{\left(\frac{\ln a + \lambda b}{\lambda(b-a)}\right)}^{\infty} \left\{ x\lambda^2 e^{\lambda(b-a)x - \lambda b} e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]} \right\} dx -$$

$$\lambda(\log e)b \int_{\left(\frac{\ln a + \lambda b}{\lambda(b-a)}\right)}^{\infty} \left\{ \lambda^2 e^{\lambda(b-a)x - \lambda b} e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]} \right\} dx$$

$$= \lambda(\log e)(b-a)\mu + \lambda(\log e)b\left(e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]} \Big|_{\left(\frac{\ln a + \lambda b}{\lambda(b-a)}\right)}^{\infty} \right)$$

$$I_2 = \lambda(\log e)(b-a)\mu - \lambda(\log e)b$$

$$I_3 = \int_{\left(\frac{\ln a + \lambda b}{\lambda(b-a)}\right)}^{\infty} \lambda^2 e^{\lambda(b-a)x - \lambda b} e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]} \left(\log e^{\left(\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]\right)} \right) dx$$

Let $e^{-y} = e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]}$

$$\frac{\lambda b + \ln\left(a + \frac{(b-a)}{\lambda} y\right)}{\lambda(b-a)} = x$$

$$-e^{-y} \, dy = -\lambda^2 e^{\lambda(b-a)x - \lambda b} e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x - \lambda b} - a\right]} dx$$

If $x = \left(\frac{\ln a + \lambda b}{\lambda(b-a)}\right)$ then $y = 0$ *and* $x = \infty$ then $y = \infty$

$$I_3 = (\log e) \int_0^\infty -y \, e^{-y} \, dy$$

$$= -(\log e) \int_0^\infty y \, e^{-y} \, dy$$

$$= -(\log e) \Gamma(2) \, , \Gamma(2) = 1! = 1$$

$$I_3 = -(\log e)$$

Then

$$Ent = -\left(\log \lambda^2 + \lambda(\log e)(b-a)\mu - \lambda(\log e)b - (\log e) \right)$$

$$Entropy = \lambda b + -\log \lambda^2 - \lambda(b-a)\mu$$

3.2.11. Estimation Methods

We will address in this section several methods for estimating unknown parameters Generalizing Uniform Distribution Using the Quantile Function

3.2.11. 1. Maximum Likelihood Method

We will use maximum likelihood method for estimating the parameters of Exponential- Generalizing Uniform Distribution Let $X_1, X_2, ..., X_n$ indicate the sample size n at random from the Exponential- Generalizing Uniform Distribution. Then the likelihood function is given by

$$L\left(x_i; \lambda, b, a\right) = \prod_{i=1}^{n} \lambda^2 e^{\lambda(b-a)x - \lambda b} \, e^{\frac{-\lambda}{(b-a)}\left[e^{\lambda(b-a)x-\lambda b} - a\right]}$$

$$= \sum_{i=1}^{n}\left(\ln \lambda^2 + \lambda(b-a)x_i - \lambda b - \frac{\lambda}{(b-a)}\left[e^{\lambda(b-a)x_i - \lambda b} - a\right]\right)$$

$$\ln L\left(x_i; \lambda, b, a\right) = n \ln \lambda^2 + \lambda(b-a)\sum_{i=1}^{n} x_i - n\lambda b - \frac{\lambda}{(b-a)}\sum_{i=1}^{n} e^{\lambda(b-a)x_i - \lambda b} +$$

$$\frac{\lambda n a}{(b-a)} \qquad\qquad (3.54)$$

By differentiating equation (3.54) with respect to λ, a, b, and equality to zero.

$$\frac{\partial}{\partial \lambda} \ln L\left(x_i; \lambda, b, a\right) = 2n\frac{1}{\hat{\lambda}} + (b-a)\sum_{i=1}^{n} x_i - nb - \frac{1}{(b-a)}\sum_{i=1}^{n} e^{\hat{\lambda}(b-a)x_i - \hat{\lambda}b} - \frac{\hat{\lambda}}{(b-a)}$$

$$\sum_{i=1}^{n}\left[(b-a)x_i - b\right]e^{\hat{\lambda}(b-a)x_i - \hat{\lambda}b} + \frac{na}{(b-a)} = 0$$

$$2n\frac{1}{\hat{\lambda}} + (\hat{b}-\hat{a})\sum_{i=1}^{n} x_i + \frac{n\hat{a}}{(\hat{b}-\hat{a})} = n\hat{b} + \frac{1}{(\hat{b}-\hat{a})}\sum_{i=1}^{n} e^{\hat{\lambda}(\hat{b}-\hat{a})x_i - \hat{\lambda}\hat{b}} + \frac{\hat{\lambda}}{(\hat{b}-\hat{a})}$$

$$\sum_{i=1}^{n}\left[(\hat{b}-\hat{a})x_i - \hat{b}\right]e^{\hat{\lambda}(\hat{b}-\hat{a})x_i - \hat{\lambda}\hat{b}} \qquad\qquad (3.55)$$

$$\frac{\partial}{\partial a}\ln L\left(x_i;\lambda,b,a\right)=-\hat{\lambda}\sum_{i=1}^{n}x_i-\frac{\hat{\lambda}}{\left(\hat{b}-\hat{a}\right)^2}\sum_{i=1}^{n}e^{\hat{\lambda}(\hat{b}-\hat{a})x_i-\hat{\lambda}\hat{b}}+\frac{\hat{\lambda}}{\left(\hat{b}-\hat{a}\right)^2}$$

$$\sum_{i=1}^{n}\hat{\lambda}x_i\,e^{\hat{\lambda}(\hat{b}-\hat{a})x_i-\hat{\lambda}\hat{b}}+\frac{\hat{\lambda}n\hat{b}}{\left(\hat{b}-\hat{a}\right)^2}=0$$

$$\hat{\lambda}\sum_{i=1}^{n}x_i+\frac{\hat{\lambda}}{\left(\hat{b}-\hat{a}\right)^2}\sum_{i=1}^{n}e^{\hat{\lambda}(\hat{b}-\hat{a})x_i-\hat{\lambda}\hat{b}}=\frac{\hat{\lambda}}{\left(\hat{b}-\hat{a}\right)^2}\sum_{i=1}^{n}\hat{\lambda}x_i\,e^{\hat{\lambda}(\hat{b}-\hat{a})x_i-\hat{\lambda}\hat{b}}$$

$$+\frac{\hat{\lambda}n\hat{b}}{\left(\hat{b}-\hat{a}\right)^2}\tag{3.56}$$

$$\frac{\partial}{\partial b}\ln L\left(x_i;\lambda,b,a\right)=\hat{\lambda}\sum_{i=1}^{n}x_i-n\hat{\lambda}-\frac{\hat{\lambda}}{\left(\hat{b}-\hat{a}\right)^2}\sum_{i=1}^{n}e^{\hat{\lambda}(\hat{b}-\hat{a})x_i-\hat{\lambda}\hat{b}}-$$

$$\frac{\hat{\lambda}}{\left(\hat{b}-\hat{a}\right)}\sum_{i=1}^{n}\left[\hat{\lambda}x_i-\hat{\lambda}\right]e^{\hat{\lambda}(\hat{b}-\hat{a})x_i-\hat{\lambda}\hat{b}}-\frac{\hat{\lambda}n\hat{a}}{\left(\hat{b}-\hat{a}\right)^2}=0$$

$$\hat{\lambda}\sum_{i=1}^{n}x_i=n\hat{\lambda}+\frac{\hat{\lambda}}{\left(\hat{b}-\hat{a}\right)^2}\sum_{i=1}^{n}e^{\hat{\lambda}(\hat{b}-\hat{a})x_i-\hat{\lambda}\hat{b}}+\frac{\hat{\lambda}}{\left(\hat{b}-\hat{a}\right)}$$

$$\sum_{i=1}^{n}\left[\hat{\lambda}x_i-\hat{\lambda}\right]e^{\hat{\lambda}(\hat{b}-\hat{a})x_i-\hat{\lambda}\hat{b}}+\frac{\hat{\lambda}n\hat{a}}{\left(\hat{b}-\hat{a}\right)^2}\tag{3.57}$$

3.2.11. 2. Method of Moments

The E-GURQ distribution's moment estimate technique is defined as such that $E\left(X^r\right)$ the rth moment about origin.

$$E\left(X^r\right)=\frac{1}{n}\sum_{k=1}^{n}X_k^r$$

When $E\left(X^r\right)=\frac{1}{n}\sum_{k=1}^{n}X_k^r$

$$\frac{1}{(\lambda(b-a))^r}\sum_{k=0}^{r}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{c}\frac{(-1)^j[\lambda b]^k a^{i-j}(\ln a)^{c-m}}{i!\,\vartheta^{i+1}\,d^{m-1}}\binom{r}{k}\binom{i}{j}$$

$$\binom{c}{m}e^{-d\,\ln a}\,\Gamma(m+1)=\frac{1}{n}\sum_{k=1}^{n}X_k^r \qquad (3.58)$$

Considering three cases from (3.57):

$$\frac{1}{(\lambda(b-a))}\sum_{k=0}^{1}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{c}\frac{(-1)^j[\lambda b]^k a^{i-j}(\ln a)^{c-m}}{i!\,\vartheta^{i+1}\,d^{m-1}}\binom{1}{k}\binom{i}{j}$$

$$\binom{c}{m}e^{-d\,\ln a}\,\Gamma(m+1)=\frac{1}{n}\sum_{k=1}^{n}X_k \qquad (3.59)$$

$$\frac{1}{(\lambda(b-a))^2}\sum_{k=0}^{2}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{c}\frac{(-1)^j[\lambda b]^k a^{i-j}(\ln a)^{c-m}}{i!\,\vartheta^{i+1}\,d^{m-1}}\binom{2}{k}\binom{i}{j}$$

$$\binom{c}{m}e^{-d\,\ln a}\,\Gamma(m+1)=\frac{1}{n}\sum_{k=1}^{n}X_k^2 \qquad (3.60)$$

$$\frac{1}{(\lambda(b-a))^3}\sum_{k=0}^{3}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{c}\frac{(-1)^j[\lambda b]^k a^{i-j}(\ln a)^{c-m}}{i!\,\vartheta^{i+1}\,d^{m-1}}\binom{3}{k}\binom{i}{j}$$

$$\binom{c}{m}e^{-d\,\ln a}\,\Gamma(m+1)=\frac{1}{n}\sum_{k=1}^{n}X_k^3 \qquad (3.61)$$

We may get estimates for the parameters a, b and λ by solving the equations (3.59),(3.60), (3.61) for a, b and λ using numerical methods .

3.2.11. 3. Least Squares of Method

This method is based on minimizing the total squares of errors and can be formulated as follows :

$$Ls=\sum_{\tau=1}^{m}\left[F(x)-\left(\frac{\tau}{m+1}\right)\right]^2 \qquad (3.62)$$

$$Ls=\sum_{\tau=1}^{m}\left[1-e^{\frac{-\lambda}{(b-a)}\left(e^{\lambda(b-a)x-\lambda b}-a\right)}-\left(\frac{\tau}{m+1}\right)\right]^2 \qquad (3.63)$$

By differentiating equation (3.63) with respect to λ, a, b, and equality to zero.

$$\sum_{\tau=1}^{m}\left[\left[\left(\frac{1}{(\hat{b}-\hat{a})}\left(e^{\hat{\lambda}(\hat{b}-\hat{a})x-\hat{\lambda}\hat{b}}-\hat{a}\right)+\frac{\hat{\lambda}\left((\hat{b}-\hat{a})x-\hat{b}\right)}{(\hat{b}-\hat{a})}e^{\hat{\lambda}(\hat{b}-\hat{a})x-\hat{\lambda}\hat{b}}\right)\right.\right.$$
$$\left.e^{\frac{-\hat{\lambda}}{(\hat{b}-\hat{a})}\left[e^{\hat{\lambda}(\hat{b}-\hat{a})x-\hat{\lambda}\hat{b}}-\hat{a}\right]}\right]\left[1-e^{\frac{-\hat{\lambda}}{(\hat{b}-\hat{a})}\left[e^{\hat{\lambda}(\hat{b}-\hat{a})x-\hat{\lambda}\hat{b}}-\hat{a}\right]}\right]-\left(\frac{\tau}{m+1}\right)\right]=0 \quad (3.64)$$

$$\sum_{\tau=1}^{m}\left[\left[\left(\frac{-\hat{\lambda}}{(\hat{b}-\hat{a})^2}\left(e^{\hat{\lambda}(\hat{b}-\hat{a})x-\hat{\lambda}\hat{b}}-\hat{a}\right)-\frac{\hat{\lambda}\left((\hat{\lambda}x+1)\right)}{(\hat{b}-\hat{a})}e^{\hat{\lambda}(\hat{b}-\hat{a})x-\hat{\lambda}\hat{b}}\right)\right.\right.$$
$$\left.e^{\frac{-\hat{\lambda}}{(\hat{b}-\hat{a})}\left[e^{\hat{\lambda}(\hat{b}-\hat{a})x-\hat{\lambda}\hat{b}}-\hat{a}\right]}\right]\left[1-e^{\frac{-\hat{\lambda}}{(\hat{b}-\hat{a})}\left[e^{\hat{\lambda}(\hat{b}-\hat{a})x-\hat{\lambda}\hat{b}}-\hat{a}\right]}\right]-\left(\frac{\tau}{m+1}\right)\right]=0 \quad (3.65)$$

$$\sum_{\tau=1}^{m}\left[\left[\left(\frac{\hat{\lambda}}{(\hat{b}-\hat{a})^2}\left(e^{\hat{\lambda}(\hat{b}-\hat{a})x-\hat{\lambda}\hat{b}}-\hat{a}\right)-\frac{\hat{\lambda}\left((\hat{\lambda}x-\hat{\lambda})\right)}{(\hat{b}-\hat{a})}e^{\hat{\lambda}(\hat{b}-\hat{a})x-\hat{\lambda}\hat{b}}\right)\right.\right.$$
$$\left.e^{\frac{-\hat{\lambda}}{(\hat{b}-\hat{a})}\left[e^{\hat{\lambda}(\hat{b}-\hat{a})x-\hat{\lambda}\hat{b}}-\hat{a}\right]}\right]\left[1-e^{\frac{-\hat{\lambda}}{(\hat{b}-\hat{a})}\left[e^{\hat{\lambda}(\hat{b}-\hat{a})x-\hat{\lambda}\hat{b}}-\hat{a}\right]}\right]-\left(\frac{\tau}{m+1}\right)\right]=0 \quad (3.66)$$

it is clear that the equations (3.64), (3.65), (3.66) are non-linear related, but it can be numerically solved.

Application

We presented an application on real data by finding the estimation of the parameters for the given data, as well as comparing between these distributions, namely GUEQD, GURQD and E-GURQD using the measures, the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Hannan-Quinn Information Criterion (HQIC), Kolmogorov-Smirnov test (KS) .

$$AIC = -2\hat{\ell} + 2q,$$
$$BIC = -2\hat{\ell} + 2q\log(n),$$
$$HQIC = -2\hat{\ell} + 2q\log\big(\log(n)\big),$$

Whereas $\left(\hat{\ell}\right)$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, q the number of parameters, and n the sample size.

Data set: These are dataset corresponding to remission times (in months) of a random sample of (128) for sick bladder cancer [6, 31].
0.08,2.09,3.48,4.87,6.94,8.66,13.11,23.63,0.20,2.23,3.52,4.98,6.97,9.02,13.29,0.40,2.26,3.57,5.06,7.09,9.22,13.80,25.74,0.50,2.46,3.64,5.09,7.26,9.47,14.24,25.82,0.51,2.54,3.70,5.17,7.28,9.74,14.76,26.31,0.81,2.62,3.82,5.32,7.32,10.06,14.77,32.15,2.64,3.88,5.32,7.39,10.34,14.83,34.26,0.90,2.69,4.18,5.34,7.59,10.66,15.96,36.66,1.05,2.2.69,4.23,5.41,7.62,10.75,16.62,43.01,1.19,2.75,4.26,5.41,7.63,17.12,46.12,1.26,2.83,4.33,5.49,7.66,11.25,17.14,79.05,1.35,2.87,5.62,7.87,11.64,17.36,1.40,3.02,4.34,5.71,7.93,11.79,18.10,1.46,4.40,5.85,8.26,11.98,19.13,1.76,3.25,4.50,6.25,8.37,12.02,2.02,3.31,4.51,6.54,8.53,12.03,20.28,2.02,3.36,6.76,12.07,21.73,2.07,3.36,6.93,8.65,12.63,22.69 .

In Table (3.1) the MLEs of the model parameters for the data and the numeric values of the model selection statistics AIC, HQIC and BIC are given.

Table (3.1) shows that the E- GURQD model has the lowest values for the criteria AIC,HQIC, and BIC, indicating that it is better than the other to model this data set.

Table 3.1. Parameters Estimates for the Data and The Values of Statistics

AIC,**HQIC and BIC** for the first data set where $\omega = (\alpha, \beta, \lambda)$.

Model	parameters	AIC	BIC	HQIC	KS
GUEQD(ω)	â = 0.001 b̂ =1 λ̂ = 79.0471	**-42.7926**	**-34.2365**	**-39.3162**	**0.9719**
GURQD(ω)	â = 0.001 b̂ =1 λ̂ = 79.0471	**-41.6989**	**-33.1428**	**-38.2225**	**0.969**
E- GURQD(ω)	â = 0.0499 b̂ =0.1928 λ̂ = 1.16	**-6.4221e+06**	**-6.4221e+06**	**-6.4221e+06**	**0.9979**

Conclusion
and
Future Works

Conclusion:

Conclusion:

In this dissertation

1- Three continuous distributions each of them with three parameters can be constructed called generalized uniform distribution using the exponential quantile function (GUEQD), generalization of the uniform distribution using the quantile function (GURQD) and exponential- generalizing uniform distribution (E- GURQD. Where the cdfs of the four distributions was found based on two families, T-X{Y} family and T-normal families of distributions. Then we found the pdf files corresponding to these distributions by deriving the cdfs for these distributions.

2- We have derived explicit expressions for quantile and median, moments, mean, variance, skew modulus, kurtosis and variance, moment generation functions, order statistics, and Shannon entropy.

3- The three parameters of the three proposed models were estimated using estimation methods(Maximum Likelihood Method, Method of moments ,Least Squares of Method).

4- Their usefulness have demonstrated by mean of application to real data set and compared our proposed distributions among themselves using criteria such as AIC, HQIC and BIC. The E- GURQ model is found better than the GUEQ and GURQ models.

Future works:

1. Using other estimation methods to estimate the three parameters of GUEQ and GURQ and then compare between these methods to indicate which method is the most efficient methods.

2. Finding new distribution by T-normal families of distributions using the quantity function for the distribution E-GURQ.

3. Estimate of the reliability function for distributions GUEQ and GURQ.

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─────────────────────**Appendix**─────────────────────

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المستخلص:

لقد قدمنا ثلاثة توزيعات جديدة من ثلاثة معلمات وهي التوزيع الموحد المعمم باستخدام الدالة الكمية الأسية ، وتعميم التوزيع المنتظم باستخدام الدالة الكمية والتوزيع المنتظم الأسي. لذلك ناقش بعض خصائصها الرياضية والإحصائية كدالة كثافة الاحتمال (pdf) ، دالة التوزيع التراكمي (cdf) ، بعض دوال الموثوقية ، المنوال ، العزوم ، التباين ، معامل (الالتواء ، الانحراف ، التفرطح) ، دالة توليد العزوم ، إحصائيات الطلب ، الكمي ، الانتروبيا. تم تقدير المعلمات باستخدام طرق التقدير (طريقة الامكان الاعظم ، طريقة العزوم ، المربعات الصغرى للطريقة).

يتم شرح فائدة هذه التوزيعات من خلال تطبيق بيانات حقيقية ، ثم مقارنة التوزيعات المقترحة (GUEQ و GURQ و E-GURQ) فيما بينها باستخدام مقاييس مثل **AIC** و BIC و HQIC.

تم استخدام برنامج Matlab (2013b) لمقارنة وتقدير المعلمات الواردة في الجدول بالإضافة إلى رسم الدوال الخاصة لكل توزيع.

بناء نماذج التوزيع الاحصائي مع التطبيق

أطروحة مقدمة الى مجلس كلية التربية للعلوم الصرفة في جامعة بابل

كجزء من متطلبات نيل درجة الدكتوراه فلسفة في التربية / الرياضيات

من قبل

سرمد رحمان حسين عبد الله

بأشراف

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