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Department of Materials Engineering

Micromechanics Analysis of Unidirectional Fiber–Reinforced Materials

A Thesis
Submitted to the College of Engineering of the University of Babylon in Partial Fulfillment of the Requirements for the Degree of Master of Science in Materials Engineering

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NOMENCLATURE

The following symbols are used generally throughout the text other are defined as when used .

Symbol	Description	Unit
A	Cross section area of element	$m^{٢}$
b	Length of the model	m
$[B]^{(e)}$	Element strain matrix	-
C	Contiguity factor	-
C_{ij}	Elasticity matrix in material co-ordinate	$N/m^{٢}$
D_f	The flexural stiffness	N.m
[D]	Elasticity matrix in global co-ordinates	$N/m^{٢}$
dA	Increment area unit	$m^{٢}$
dv	Increment volume unit	$m^{٣}$
dW	Increment work done per unite volume	$J/m^{٣}$
dx	Increment distance in x-direction	m
dy	Increment distance in y-direction	m
dz	Increment distance in z-direction	m
$E_{١}$	Modulus of elasticity in longitudinal direction	$N/m^{٢}$
$E_{٢}$	Modulus of elasticity in transverse direction	$N/m^{٢}$
$E_{٣}$	Modulus of elasticity in normal transverse direction	$N/m^{٢}$
E_f	Modulus of elasticity of fiber	$N/m^{٢}$
E_m	Modulus of elasticity of matrix	$N/m^{٢}$
{F}	Global force vector	N
$\{f^{(e)}\}$	Element force vector	N
Fx	Body forces in x-direction	$N/m^{٣}$
Fy	Body forces in y-direction	$N/m^{٣}$
Fz	Body forces in z-direction	$N/m^{٣}$
$G_{٢١}$,$G_{٢٣}$,$G_{٣١}$	Shear modulus in ٢-٣,٣-١,and ١-٢ planes respectively	$N/m^{٢}$
G_f	Shear modulus of fiber	$N/m^{٢}$
G_m	Shear modulus of matrix	$N/m^{٢}$
H	Fiber spacing	m
h_f	Width of fiber	m
h_m	Width of matrix	m
i,j	Tensor subscripts	-
[K]	Global stiffness matrix	N.m
$[K^{(e)}]$	Element stiffness matrix	N.m
k	Fiber misalignment factor	-
K_f	Bulk modulus of fiber	$N/m^{٢}$

K_m	Bulk modulus of matrix	$N/m^{٢}$
M	Composite modulus $E_{٢}$,$G_{١٢}$,or $\upsilon_{٢٣}$	
M_f	Corresponding fiber modulus E_f,G_f,and υ_f	
M_m	Corresponding matrix modulus E_m,G_m,and υ_m	
$N_{١}$,$N_{٢}$,$N_{٣}$	Shape functions for nodes ١,٢ and ٣ respectively	-
Pi	Line load	N
Q_{ij}	Reduced stiffness	$N/m^{٢}$
S_{ij}	Compliance matrix in material co-ordinate	$m^{٢}/N$
t	Surface couple per unit length	N/m
U	Strain energy density for isotropic body	$N/m^{٢}$
$U^{(e)}$	Element strain energy	J
$U_f^{\ k}$	Strain energy density per unit length of the kth fiber	$N/m^{٣}$
U_m	Strain energy density for matrix	$N/m^{٢}$
U_f	Strain energy density for fiber	$N/m^{٢}$
Uc	Strain energy density of composite	$N/m^{٢}$
u_i	Overall displacement or displacement at center line	m
$\bar{u}_i$	Displacement within the fibers	m
V_f	Fiber volume fraction	-
V_m	Matrix volume fraction	-
Vmin	Minimum fiber volume fraction	-
Vcrit	Critical fiber volume fraction	-
$W_f^{(e)}$	Element work done by body force	J
$W_p^{(e)}$	Element work done by distributed loads	J
x,y,z or $x_{١}$,$x_{٢}$,$x_{٣}$	Cartesian co-ordinates	
١ , ٢ , ٣	Material principle co-ordinates	-

Greek Symbols

Symbol	Description	Unit
σ	Normal stress	N/m^2
ε	Normal strain	-
τ	Shear stress	N/m^2
γ	Shear strain	-
υ_{ij}	Poisson's ratio for transverse strain in j-direction on stressed in the i-direction	-
υ_f	Poisson's ratio of fiber	-
υ_m	Poisson's ratio of matrix	-
σ_c	Overall stress in the composite	N/m^2
σ_f	Tensile stress of fiber	N/m^2
σ_m	Tensile stress of matrix	N/m^2
σ_{cu}	Ultimate stress of the composite	N/m^2
σ_{fu}	Ultimate stress of the fiber	N/m^2
σ_{mu}	Ultimate stress of the matrix	N/m^2
ρ_c	Density of a composite	Kg/m^3
ρ_f	Density of fiber	Kg/m^3
ρ_m	Density of matrix	Kg/m^3
λ	Lame's material constant	N/m^2
μ	Lame's material constant or shear modulus	N/m^2
$\boldsymbol{\mu_f}, \lambda_f$	Lame's material constant for fibers	N/m^2
$\boldsymbol{\mu_m}, \lambda_m$	Lame's material constant for matrix	N/m^2
$\boldsymbol{\mu_{121}}$	Couple stress	N/m^2
ζ	Reinforcement geometry factor	-
η	Reduced factor	-
γ_{ij}	The component of the relative strains	-
ε_{ij}	The component of the overall symmetric strain	-
$\{\varepsilon\}^{(e)}$	Element strain vector	-
ε_T	Thermal strain	-
$\{\sigma\}^{(e)}$	Element stress vector	N/m^2
Ψ_{21}	Measure of the shear deformation in the fiber direction	
Ψ_{22}	Measure of the shear deformation in the normal direction of the fiber	-
$\boldsymbol{\chi_{121}}$	The derivative of Ψ_{21} with respect to x_1	-
Φ_i, Φ_j, Φ_k	Element nodal displacement	m
$\Phi^{(e)}$	Displacement with in element	m
$\{\Phi\}$	Global displacement	m
π	Potential energy	J
$\pi^{(e)}$	Element potential energy	J

Superscript

Symbol	**Description**
(e)	Element
T	Transpose
-T	Inverse Transpose

Subscript

Symbol	**Description**
c	Composite
f	fiber
m	matrix
٣i-٢,٣j-٢,٣k-٢	Nodal in axial direction
٣i-١,٣j-١,٣k-١	Nodal in vertical direction
٣i,٣j,٣k	Nodal in rotation direction
١,٢,٣	Longitudinal, transverse and out of plane direction respectively

Abbreviations

Abbr.	**Description**
B.C.	Boundary Conditions
CS	Couple Stress
Eqn.	Equation
Eqns.	Equations
Fig.	Figure
Figs.	Figures
FMSS	Fiber-Matrix Shear Stress
HOM	Homogeneous Orthotropic Materials
HTE	Halpin-Tsai Equations
max.	Maximum
min.	Minimum
No.	Number
RVE	Representative Volume Element
UDFRM	Unidirectional Fiber- Reinforced Material
UDL	Unidirectional laminate
Tab.	Table

Abstract

The micromechanical approaches playing a major role in studying the behaviour of composite materials specially unidirectional fiber-reinforced materials (UDFRM) by studying the behaviour of their constituents (fiber-matrix and sometimes interface). Therefore, this work has adopted microstructure approach which has considered one of their approaches. But this approach didn't have a good chance in the study among them and the reason was ascribed to find number of parameters and variables in its equations. In spite of this reason the microstrcture approach shows improved its capability to calculate the shear stress between fiber and matrix in an excellent way.

In this work, UDFRM in x_1-direction. Then a representative volume element (RVE) which includes two fibers embedded in a matrix in (x_1- x_2) plane was also taken. And thus it was assumed that:

- The fiber has a rectangular cross-section with thickness equal to matrix thickness.
- There was perfect bond between fiber and matrix.

By employing strain energy method in plane elasticity the equations for this material were derived which represent the relationship between stress-and strain. Finite element method has been used taking a triangular element with ٣-degrees of freedom per node based on displacement formulation. Using quick basic language the computer program (SAOUDL) which comprises all the above dependent manner was designed.

To verify the validity, the designed computer program (SAOUDL) was applied on four cases, so as to study the following parameters:

١- The type of load and supports, to study their effect on all vertical displacement (ϕ_2), local shear deformation (ϕ_3), couple stress ($\mu_{١٢١}$) and finally the fiber matrix shear stress ($\tau_{12} + \sigma_{21}$).

٢- Fiber size which is studied as a ratio between fiber spacing and the length of studying model (H/b).

٣- The ratio between elastic modulus of fibers and matrix (E_f/E_m).

٤- Poisson's ratio for fiber (ν_f) and matrix (ν_m).

From cases one and two all the program results were compared with the exact solution results for the same problem under study. Likewise in case three the comparison was applied with the studying case from another author on designed computer program and the comparison of the results of this program has a good agreement. This work has arrived to many conclusions such as:

- The increment in fiber size cause increment in reduction of ϕ_2 and ϕ_3.
- Whenever the fiber size increased the reduction in $\mu_{١٢١}$ and ($\tau_{12} + \sigma_{21}$) are incremented also.
- Whenever (E_f/E_m) increased the ϕ_2 and ϕ_3 reduced and this reduction increased when fiber size are high and likewise the increment in E_f/E_m decreased all $\mu_{١٢١}$ and ($\tau_{12} + \sigma_{21}$).
- ν_m has a major effect on ϕ_2 and ϕ_3 and a low effect on $\mu_{١٢١}$ and ($\tau_{12} + \sigma_{21}$) and the effect of ν_m are higher than the effect of ν_f.

From all these conclusions, it is shown that the previous parameters have the same effect but this state change with the changing of the type of the loading and supports.

الخلاصة

لعبت طرق الآلية الدقيقة دورا كبيرا في دراسة سلوك المواد المركبة و خاصة المواد المقواة بألياف موحدة الاتجاه (Unidirectional fiber-reinforced materials) بواسطة دراسة سلوك مكوناتها (الليف (Fiber)، المادة الأساس (Matrix) و الحد البيني (interface)). لذلك فلقد تبنى هذا العمل طريقة البنية الدقيقة (Microstructure approach) والذي يعتبر أحد طرق هذه الآلية لكن هذه الطريقة لم يكن لها الحظ الكبير في الدراسة من بين طرقها والسبب يعزى إلى وجود العديد من العناصر والمتغيرات التي تتضمنها معادلاتها ورغم ذلك فأن هذا الطريقة قد اثبتت وبكل جداره قدرتها على حساب إجهاد القص المتولد بين الليف والمادة الأساس وبشكل ممتاز.

في هذا العمل تم اخذ مادة مقواة بألياف موحدة الاتجاه (باتجاه محور x_1) . ثم اخذ منها عنصر ممثل الحجم (Representative Volume Element) المتكون من ليفين مزروعين داخل المادة الأساس في مستوي (x_1, x_2). كذلك تم فرض:-

- الليف ذو مقطع مستطيل مع سمك مساوي إلى سمك المادة الأساس.
- ترابط مثالي بين الليف والمادة الأساس.

وبواسطة استخدام نظرية طاقة الانفعال في المرونة المستوية (Strain energy method in plane elasticity) تم اشتقاق المعادلات الخاصة بهذه المادة والتي تمثل العلاقة بين الإجهاد والانفعال. استخدمت طريقة العناصر المحددة (Finite Element Method) كطريقة عددية وتم اعتماد العنصر المثلث (Triangular element) كأساس للتحليل بثلاث درجات للحرية لكل عقدة لحساب الأزاحه واستخدمت لغة بيسك السريعة (Quick Basic Language) لأعداد البرنامج (SAOUDL) الذي يشمل كل الأساليب المعتمدة أعلاه .

ومن اجل التأكد من دقة البرنامج المعد مسبقا تم تطبيق أربعة حالات عليه ودراسة العوامل التاليه:-

١- نوع التحميل والمساند المفروضه وتم دراسة تاثيرهم على كلا من المسافة العمودية (Φ_2) ، تشوه القص الموضعي (Φ_3) ، الإجهاد المزدوج (μ_{121}) و إجهاد القص بين الليف والمادة الأساس ($\tau_{12}+\sigma_{21}$) .

٢- حجم الليف والذي درس كنسبة بين المسافة بين الألياف وطول الليف أو النموذج المدروس . (H/b)

٣- النسبة بين معامل المرونة للّيف والمادة الأساس (E_f/E_m).

٤- نسبة بواسون لكل من الليف (υ_f) والمادة الأساس (υ_m) .

وفي الحالتين الأولى والثانية تم مقارنة نتائج البرنامج مع نتائج الحل المضبوط (Exact solution results) لنفس المسألة المدروسة . وكذلك في الحالة الثالثة تم تطبيق أحد الحالات المدروسة من قبل باحث أخر على البرنامج المعد ومقارنة نتائج البرنامج معه. وقد اتضح بان نتائج البرنامج جيده بشكل كبير . خلال هذا العمل تم الوصول إلى استنتاجات عديدة و منها :

- الزيادة بحجم الليف تسبب نقصان في كلا من Φ٢ و Φ٣ .
- كلما ازداد حجم الليف ازداد الاختزال في كلا من $\mu_{١٢١}$ و ($\tau_{١٢}+\sigma_{٢١}$) .
- كلما ازدادت (E_f/E_m) كلما قل Φ٢ وΦ٣ وهذا النقصان يزداد عندما يكون حجم الليف كبير وكذلك بزيادة (E_f/E_m) يقل كلا من $\mu_{١٢١}$ و ($\tau_{١٢}+\sigma_{٢١}$).
- υ_m لها تأثير كبير على Φ٢ و Φ٣ وقليل على $\mu_{١٢١}$ و ($\tau_{١٢}+\sigma_{٢١}$). و التأثير υ_m هذا يكون اكبر من تأثير υ_f.

من خلال هذه الاستنتاجات يتضح بان جميع العوامل السابقة لها نفس التأثير ولكن هذه الحالة تتغير بتغير نوع الحمل والمسند .

الإهداء

إلى... بضعة الرسول المصطفى (ص)

فاطمة الزهراء (ع)

تقديرا و إعتزازا .

إلى...روح والدي الطاهرة احتراما لذكراه

إلى...ينبوع الحنان الدافئ...والدتي

إلى...رفاق دربي...اخوتي و أخواتي .

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٢٠٠٢

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حنين

زهير

٢٠٠٢

Chapter One

Introduction

١-١ General

A composite material can be defined as a macroscopic combination of two or more distinct materials to form a new material system. Composite materials unlike metal are man-made, and therefore, the constituents of composite materials can be selected or combined so as to produce a useful material that has desired properties. **David R. ٢٠٠٠**.

Thus by forming a composite material many properties can be improved. These properties include:-

١- Strength.
٢- Stiffness.
٣- Corrosion Resistance.
٤- Wear Resistance.
٥- Attractiveness.
٦- Fatigue Life.
٧- Weight.
٨- Thermal Insulation.
٩- Thermal Conductivity.
١٠- Acoustical Insulation.
١١- Temperature-Dependent Behaviour.

Naturally, not all of the above properties are improved at the same time nor is there usually any requirement to do so. **Jones, ١٩٧٥**.

The constituents of the composite material are the matrix and the structural constituents:-

- The matrix is the part that gives the composites its bulk form and encloses the other parts. Therefore, the function of the matrix is to support and protect the fibers and to transfer stress when the fiber is broken. Typically, the matrix has relatively lower stiffness, strength, and density than the structural constituents. However the combination of fibers and a matrix can have very high strength and high stiffness with low density. The material matrices are polymers, metals and their alloys, intermetallic, glasses, glass-ceramics, and crystalline ceramics. In fact, most of composite matrices are polymers. One classification that can be made is between thermoplastic materials and thermosetting materials. Thermoplastics can be softened repeatedly by heating with no change in properties or chemical composition. Conversely, after initial curing of thermosetting resins, they can not be resoftened. A chemical change occurs during curing with heat and pressure.

  Some examples of thermoplastics include nylon, polyethylene, polystyrene and polyvinylchloride (PVC). Thermosetting resins include polyesters, alkyds, epoxies resins, and polyamides resins **Robert, L., ١٩٩٦**.

- The structural constituents (or the reinforcement materials) determine the internal structure of the composite. They may be formed as continuous fibers, i.e. in lengths running the full length of the composite, or discontinuous,(i.e. in short lengths). They may be aligned so that they are all lying in the same direction or randomly orientated (Fig.(١-١)). Aligning them all in the same direction gives a directionality to the properties of the composite. However, the fibers have high tensile strength, high elastic modulus, low weight and much stiff and hard than matrix. The effect of the fibers is to increase the tensile strength and tensile modulus, the amount of change depending on both the form the fibers take and the amount. **Bolton, ١٩٩٨** and **Krishan, ١٩٩٩**. Some of the important ones are listed in Table (١-١), along with a summary of their salient characteristics. Reinforcements include organic fibers such as polyethylene and aramid, metallic fibers, and ceramic fibers and particles.

Table (١-١): Properties of some important reinforcement **Bolton, ١٩٩٨**

Fiber	**Density** ρ **(Mg/m^٣)**	**Tensile modulus E (GPa)**	**Tensile strength** σ_t **(MPa)**
Alumina	٣.٢	١٧٠	٢١٠٠
Silicon Carbide (Nicalon)	٢.٦	٢٥٠	٢٢٠٠
Boron	٢.٦٥	٤٢٠	٣٥٠٠
Carbon	١.٨	٢٥٠	٢٧٠٠
E-glass	٢.٥	٧٠	٢٢٠٠ *
Polyethylene (Spectra ١٠٠٠)	٠.٩٧	١٧٢	٢٩٦٤
Polyamide (Aramid) (Kevlar ٤٩)	١.٤٥	١٢٥	٣٠٠٠

Note:- * ٣٥٠٠ MPa freshly drawn.

(a) Continuous, aligned **(b)** Discontinuous, aligned **(c)** Discontinuous, random

Fig. (١-١): Form of fiber reinforcements Bolton, ١٩٩٨.

In constituent materials of composite there is also an important role played by the interface or interphase, the first being a common boundary between two distinct materials, the latter is a distinct added phase, due to chemical reaction of the matrix with the structural constituents. **De Wilde, ١٩٨٨**.

١-٢ Classification of Composite Materials

Composite materials can be classified into four categories according to the structural constituents:-

a) Fiber reinforced composite (fibrous composite) which consists of fibers in a matrix.

b) Laminated composite, which consist of layers of various materials.

c) Particulate composite, which are composed of particles in a matrix.

d) Dispersion strengthened composite which consists of small particles dispersed throughout the matrix.

There is another type of composite materials named as laminated fiber reinforced composite which is composed of laminated and fibrous composites **Jones, ١٩٧٥**.

Most common engineering materials are homogeneous which have the same properties at every points (position), and isotropic where the properties are the same in every direction at a point in the body. In contrast, composite materials are often heterogeneous and anisotropic. **Leknitskii, ١٩٨١**.

When a component is manufactured from a reinforced plastic, the forming process does more than simply shape the component; it also positions the reinforcing particles, and fixed their orientation.

The methods McCrum **et. al., ١٩٩٧** commonly used for the manufacture of fiber reinforced polymer matrix composites are: -

١- Pultrusion .
٢- Filament winding.
٣- Hand lay-up.
٤- Hand spray-up.
٥- Compression moulding.
٦- Reinforced reaction injection moulding (RRIM).
٧- Reinforced thermoplastics.

In this work the pultrusion products are considered. In this process continuous fiber rovings are hauled through a bath of resin (polymer precursor) and then pulled through a heated die to give the required shape product as shown in Fig.(١-٢). The resin then cross-links and hardens, producing along prismatic component with fibers aligned uniaxially parallel to its long axis. This method is used for the production of long lengths of uniform cross-section rods, tubes or I-sections.

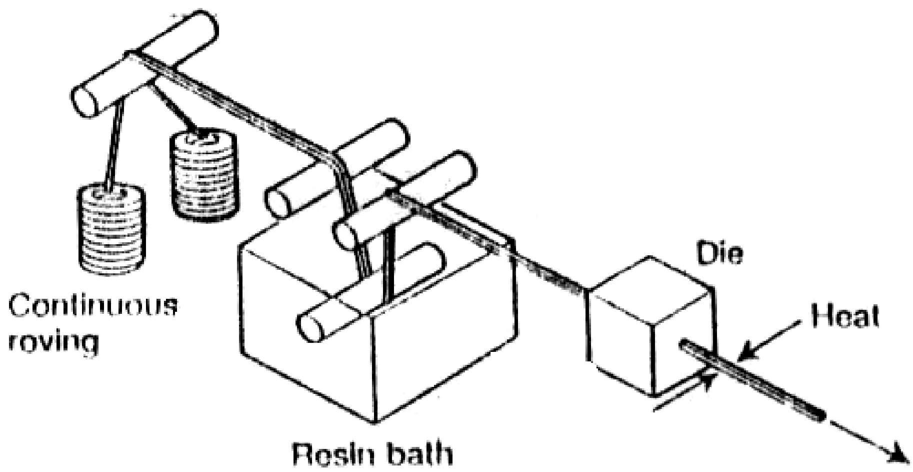


Fig.(١-٢): Pultrusion process. McCrum **et. al., ١٩٩٧**

١-٣ Application of composite materials

The development of composite materials has been a subject of intensive interest for at least ١٥ years, but the concept of a composite solid has been employed over since materials were first used.

More recently, fiber reinforced resin composites that have high strength to weight or high stiffness to weight ratios have become important in weight sensitive applications such as aircraft, space vehicles, sports, boats and medical applications. Such composite materials technology began to emerge about (١٩٦٠) with the advent of modern fiber composites consisting of very stiff and strong aligned fiber (glass, carbon, boron, and graphite) in polymeric matrix and later also in a light weight metal matrix. **Hashin-Survey, ١٩٨٣**.

It is convenient to divide the applications of all composites into aerospace and non-aerospace categories. In the category of aerospace applications such as aircraft, helicopters, missiles, reentry, and other aerospace vehicles low density coupled with other desirable features, such as a tailored thermal expansion and conductivity, and high stiffness and strength, are the main drivers. Performance, rather than cost, is an important item as well. The main advantages for using composite in non aerospace applications like sporting goods industry are safety, less weight, and higher strength than conventional materials. Examples of non aerospace applications are automobile industry, rocket engines, and civil construction. **Krishan, ١٩٩٩**.

١-٤ Mechanics of composite materials

In the composite materials fields there are two approaches represented the behaviour of composite materials, macromechanics and micromechanics.

Macromechanics is the study of composite material behaviour wherein the material is presumed homogeneous and the effects of the constitutive materials are detected only as averaged apparent macroscopic properties of the composite material. **David,** ٢٠٠٠ and **Jones,** ١٩٧٥.

In macromechanics study, strength, stiffness and other properties are based on fiber orientation, number of lamina (layer), thickness of each layer and all other structural design parameters.

Micromechanics is the study of composite material wherein the interaction of the constituent materials is examined in detail, as apart of definition of the behaviour of the heterogeneous composite materials. This means that micromechanics is the study of mechanical properties of unidirectional composite in terms of those of constituent materials. **Stephen,** ١٩٧٤.

Fig.(١-٣) has been constructed based on the following observations. First, fiber is linear elastic up to fracture. Second, matrix is linear initially; however, it behaves nonlinearly as strain increases. The strain at which non-linearity starts to appear is greater than the fracture strain of fiber.

Commonly, the important parameters in the micromechanics study of the composite materials are filament (fiber)-matrix array, void inclusion and fiber volume fraction.

In this work, analysis of unidirectional fiber-reinforced materials (unidirectional laminates) based on micromechanics approach is presented taking into consideration the parameters: fiber size, elastic modulus ratio between fiber and matrix (E_f / E_m) and Poisson's ratio of matrix and fiber. Literature survey indicates that a substantial amount of work has been done on micromechanics approach of the composite materials. Some of works studied the mechanical behaviour and fracture of discontinuous fibers and the other studied the mechanical behaviour of continuous fibers. Few amounts of works have been done on a circular hole in plate composed from unidirectional fiber-reinforced materials by macromechanics or micro mechanics approach.

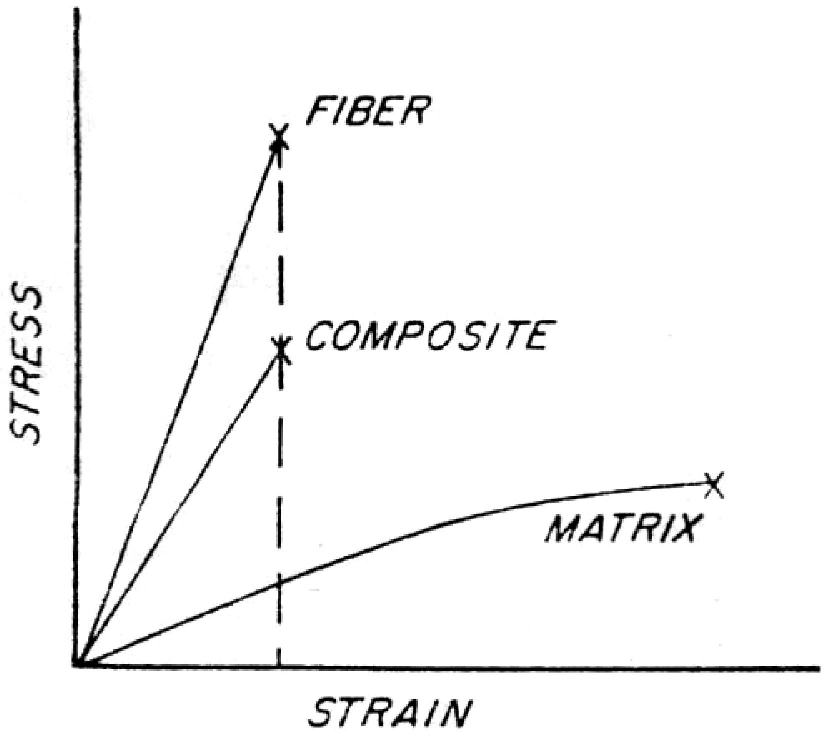


Fig. (١-٣): Typical stress- strain relations of fiber, matrix and composite.

Stephen, ١٩٧٤.

١-٥ The objective of the present work

The objective of the present work is to study static behaviour of unidirectional fiber-reinforced composite when subjected to mechanical symmetric loading by using finite-element method. In this investigation the following effective parameters are studied in detail:-

١- Fiber size (with constant fiber volume fraction).

٢- Elastic modulus ratio between fiber and matrix.

٣- Poisson’s ratio of matrix and fiber.

٤- Boundary condition of the model under consideration such as the type of loading and the types of supports.

In this study, finite element computer program (SAOUDL) is designed to achieve the previous objectives. The results are comparing first with an exact solution results of a case of pure shear in which the ends of the fibers are rotationally restrained. And second the results are comparing well with other work for the case of a circular hole in a uniform (plane stress) tension field.

١-٦ Layout of the thesis

This thesis consists of six chapters. After this introductory, chapter two is concerned with a brief of literature review and back-groundwork on the micromechanics approaches. Chapter three is devoted to the theoretical analysis applied to unidirectional fiber- reinforced materials in plane stress. Chapter four is concerned with finite element and computer program, which are constructed to perform all calculations, based on theoretical analysis. The results of this work are presented and discussed in chapter five. Chapter six presents the conclusions to be drawn from this work in addition to many further recommendations to develop this work.

Chapter Two

Literature Review

٢-١ General

Modern technology has found extensive use for unidirectionally fiber-reinforced composite materials. To make effective use of these materials, knowledge of their properties and performances when subjected to loads is essential. Many aspects of their behaviour are directly associated with the microscopic structure of these materials. The desire to understand these materials drives the research in this field into the micromechanics of this type of materials **Hashin,** ١٩٨٣ and **Aboudi,** ١٩٨٩. However, when the micromechanical behaviour of such a composite material is of interest, for example, the initiation and development of a damage process well before a macroscopic failure, one has to take account of the heterogeneity between fibers and matrix.

All the above features necessitate the need for a more realistic prediction of the structural behaviour of unidirectional composite materials, one form of these studies the current study of unidirectional fiber reinforced materials subjected to mechanical symmetric loading. Which has a very important interest in the micromechanics fields.

٢-٢ Review of micromechanical approach of composite materials

With rapidly growing computational modeling capability, the micromechanical analysis of fiber-reinforced composite materials has been become an important means of understanding the behaviour of these materials. Thus the works that have been done on the micromechanical study of the composite structures can be divided into four groups according to the length of the fiber and bond between fiber and matrix that in the first three groups. The last group involved the works that have been done on a circular hole in plate composed from unidirectional fiber-reinforced materials by macromechanical or micromechanical approach.

The first group represented the works that have been done on composite materials with continuous fiber and by assuming a perfect bond between fiber and matrix. This group contains the following works:

Barry, P. W. (١٩٧٨), presented a model which used to predict the range of possible composite strengths. The model considers both static and dynamic stress concentration effects on intact fibers which result from a fiber failure. The model results are used to predict the range of strength for composite materials prepared from three types of carbon fiber and these are compared with experimental results.

Malcolm, D. J. (١٩٧٨), completed the analogy between the linear elastic behaviour of a unidirectional composite and an isotropic material with an oriented microstructure. The physical interpretation of the additional stresses and the constitutive constants presented in the theory of micro-elasticity is given and the shear stress between matrix and fiber is presented in terms of these stresses. A constant strain finite element is formulated and the stiffness matrix is presented in full. Finite element results are presented for the case of a circular hole in a uniform (plane stress) tension field when load is applied normal to the fibers where the maximum tensile stress is shown to decrease as the fiber size is increased.

Hashin, Z. (١٩٧٩), derived an expressions and bounds for the five effective elastic moduli, thermal expansion coefficient and conductivities of unidirectional fiber composites. Consisting of transversely isotropic phases (i.e. fibers and matrix). The expressions have been obtained on the basis of analogies between isotropic and transversely isotropic elasticity equations. Application results for determination of the five elastic moduli of graphite fiber were discussed. Thus, the results were of an importance for carbon and graphite fiber composites since such fibers are highly anisotropic.

Zhang, W.C. and **Evans, K. E. (١٩٨٨)**, presented a numerical method to predict the mechanical properties of composite materials with anisotropic constituents and used this method to predict the properties of fiber- reinforced composite. The fiber- reinforced composite was treated as an anisotropic but homogeneous continuum and the elastic constants were determined by using an energy equivalence method. Finite element method was used to calculate the strain energies (or complementary energies) of the components (i.e. fiber and matrix). Comparison was made with previous techniques to determine the longitudinal, transverse moduli and Poisson’s ratios for isotropic and transversely isotropic fiber in isotropic matrices.

Sideridis, E. (١٩٨٨), described a model to find the approximate equations for determining the in- plane shear modulus of a unidirectional fiber-reinforced composite from the constituent material properties. Classical elasticity theory has been applied to the simplified model of a composite unit cell in which the concept of an interphase between fiber and matrix has been taken into account. Thus the model considers that the composite materials consist of three phases, that is the fiber, the matrix and the interphase. Thermal analysis was used for the determination of the thickness and volume fraction of the interphase. The model had introduced to be an improvement for the shear modulus.

Chamis, C. C. (١٩٨٩), presented composite mechanics disciplines and described them at their various levels of sophistication and attendant scales of applications correlation with experimental data was used as the prime discriminator between alternative methods and level of sophistication. The discussion was developed by using selected, but typical, examples of each composite mechanics discipline identifying the degree of success, with respect to correlation with experimental data, and problems remaining.

Shuguang Li (١٩٩٩), presented the unit cell for micromechanical analysis of unidirectionally fiber-reinforced composites. A systematic consideration has been made of the symmetries presented in the idealized fiber-matrix systems. Appropriate boundary conditions of the unit cell have been derived from these symmetry considerations for micromechanical analysis. The loads on the unit cells and the responses of it in terms of macroscopic stress or strains have been addressed in such a way that the effective properties of the material can be obtained from micromechanical analysis of the unit cell in a standard manner.

Shuguang Li (٢٠٠٠), employed two typical idealized packing systems for unidirectionally fiber-reinforced composites, square and hexagonal ones. Only the translational symmetry transformation has been used. The unit cells so derived are capable of accommodating fibers of irregular cross-section and imperfections symmetrically distributed around fibers. Such as micro cracks and local debonding in the system, provided the regularity of the packing and imperfections, is present. All the unit cell subjected to arbitrary combinations of macroscopic stresses or strains. The unit cells boundary conditions have been derived from appropriate considerations of the conditions of symmetry transformations.

The expressions of the effective material properties of the composite represented by the unit cell are then obtained in terms of the loads applied to those extra degrees of freedom and the nodal displacements at those extra degrees of freedom which are available from the output of an appropriate analysis of such a unit cell. The results of this work, validating the unit cells established, draw interesting comparisons between the two unit cells representing the square and hexagonal packing systems.

Shuguang Li and **Zhenmin Zou (٢٠٠٠)**, demonstrated the use of the unit cells in finite element analyses of unidirectionally fiber-reinforced composites. Both square and hexagonal fiber- matrix systems have been included. The appropriate boundary conditions for each unit cell have been provided under all the possible loads corresponding to uniaxial macroscopic longitudinal and transverse tension/compression and shear stress states. The results obtained from the unit cells have been discussed in such a manner which can provide a systematic series of simple but necessary benchmark cases for correct use of such unit cells in finite element analysis of unidirectionally fiber- reinforced composites.

The second group included the works that have been done on the short fibers by micromechanical analysis and the effect of broken fibers on the strength of unidirectional composite materials. Thus, this group contains the following works:-

Law, n. and **Mclaughlin, R. (١٩٧٨)**, gave an application of the self-consistent method (S.C.M.) to the problem of determining overall moduli for short fiber-reinforced composites, assuming that the fibers can be considered to be spheroids. For fully-aligned fibers, the numerical results are presented in graphical form and show the dependence of the compliances on aspects ratio and volume fraction. By making use of some ideas on how to handle the misalignment of fibers the S.C.M results are shown to compare favorably with experiment.

Schultrich, B. et al (١٩٧٨), attempt to calculate the σ-(ε)curve of short fiber composites by considering regular arrays of plates in a ductile matrix-several quantities of interest, such as variation along the fiber, Young’s modulus, and yield stress, are calculated as functions of the parameters and structure of the composite. Among the latter, the overlap of the fibers may affect the properties strongly. The change of composite behaviour from mainly elastic to yield may occur in several ways depending on the parameters.

Akbarzadeh, A. (١٩٧٨), studied the effect of broken fibers on the strength of unidirectional composite materials. The breaking of a fiber has a negligible effect on the axial strength but that the void caused at the breaking point has a considerable effect on the transverse strength of the composite body. The stress intensity in the vicinity of the broken fiber has been compared with a similar case without a broken fiber and it has been shown that breaking of fibers can substantially increase the stress intensity. Curves for predicting the maximum stresses in the vicinity of the broken fibers are presented for unidirectional composite materials loaded transversely.

Göran Tolf (١٩٨٣), performed a theoretical investigation of the stress field in a short-fiber composite. The concept of a typical region is introduced and the boundary conditions for such region are derived by using these boundary conditions, the stresses are calculated and knowing this stress-field, macroscopic properties are calculated. Conclusions are drawn about the mechanical behaviour of the composite, like critical fiber length and fracture toughness.

Lauke, B. et. al. (١٩٨٥), presented a theoretical model for calculating the work of fracture in such composites of short, subcritical fibers in a ductile matrix with relatively weak interface. Starting from a micromechanical analysis of the debonding and sliding length, the fracture energies are calculated in general terms. Depending on the relative contributions to the total energy which itself depended on loading rate, the composite fracture energy varies with volume fraction in a qualitatively different manner.

The third group involved the works that have been taken considerably the interface between fiber and matrix and by a different way like the following works:-

Agarwal, B. D. and **Bansal, R. K. (١٩٧٩)**, performed a study by single fiber model and by using an axisymmetric finite element analysis which has been carried out to study the effect of interfacial conditions on the properties of discontinuous fiber composites. This study has been possible to take into account the interaction between fibers by appropriately selecting the boundary conditions. The influence of interfacial conditions on load transfer length, critical attraction length, elastic modulus of the composite and composite strength were established by presenting results.

Laws, V. (١٩٨٢), extended Lawrence's theory to calculate the load/displacement curve during pullout, the crack spacing and strength of an aligned short fiber composite. The effect of the bonds, interfacial and frictional on fiber pullout, crack spacing and strength is outlined, and the calculation of the strength of the bonds is discussed.

Shirazi-Adl, A. (١٩٨٨), using a penalty function. Proposed a displacement-based modified potential energy, which enforced the continuity of stresses at a two-material interface. The finite element formulation has been developed and applied for the stress analysis of a number of structures made of highly dissimilar materials. His results were compared with those obtained from the conventional finite element analysis and the exact solution. On the contrary, the usual finite element formulations have been resulted in significantly discontinuous stresses at the two-material interface.

Lepetitcorps, Y. et. al. (١٩٨٩), studied the mechanical adhesion between filaments (B, Sic) and titanium matrices. Because a single fiber composite was chosen for this purpose, the critical length measurement and the shear strength were calculated. Using a statistical analysis, the study indicated the role played by the surface treatments of the fibers on the reinforcement/matrix adhesion. The conclusions obtained on model materials are in agreement with the result obtained on real composites.

Finally, **the forth group** included the works that have been studied the stress concentration factor and stresses due to a circular hole in uniform plate composed from unidirectional fiber-reinforced composite. This group involved the following works:

Shastry, B. P. and **Rao, G.V. (١٩٧٧)**, studied the effect of fiber orientation on stress concentration in a finite width composite laminate using finite element techniques. The stress concentration factor was found to be maximum when the fiber directions are parallel to the loading and minimum when they make an angle of ٤٥^{o}. In all cases the maximum stress concentration occurs on the hole boundary at the minimum cross-section.

Paul, T.K. and **Rao, K.M. (١٩٨٩)**, evaluated stresses and stress concentration factor due to a circular hole in fiber-reinforced composite laminates subjected to transverse loads and presented the variation of stress concentration factor with plate thickness, hole size and nature of load distribution.

٢-٣ Summary:

The behaviour of composites depends on variety parameters such as the properties of the components, bonding between the components, alignment of fibers and so on. Thus, the micromechanical finite element analysis, which has been employed increasingly for fiber-reinforced composites in the past decade, has become an important means of understanding this behaviour of composites. The theoretical and experimental methods usually applied in micromechanical analysis in order to tackle the problems that caused by the above parameters. The theoretical method can be divided into two complementary families:

First the numerical method which provides more or less exact solutions of the stress fields for special sets of parameters. Thus, there are numerous detailed results for elastic or elastic-plastic fibrous composites. The information that one can get from them is limited, of course due to special geometry, etc.

Second the analytical approach which makes use of comparatively crude models which enable the problem to be treated in a more general manner and to attain finally some mathematical expression which contains all the parameters of interest **Schultrich, B. (١٩٧٨)**. From the literature review, it is clear that a very few works have been done on the mechanical behaviour of unidirectional fiber-reinforced materials by experimental micromechanical approach due to many factors which influence the situation: (١) fiber/fiber and fiber/matrix interaction is not usually taken into account, (٢) perfect alignment of the fibers is difficult to achieve during specimen preparation, (٣) processing variables such as incomplete bonding, fiber defects and void entrapment, which tend to reduce the effective module, are often neglected, (٤) furthermore the experimental method was expensive and needs high instrument, time and cost. **Sideridies, E. (١٩٨٨)**.

However, a substantial amount of work has been done on the mechanical behaviour of fiber- reinforced composites by micromechanical finite element analysis, based on plane elasticity theory in order to predict the properties of constituent materials, the stresses and strains on them and failure that occur in unidirectional fiber-reinforced materials.

In other way, no works have been done on the fiber-reinforced composite subjected to mechanical and thermomechanical loading based upon three dimensional elasticity theory and axisymmetric finite element method by using micromechanics analysis due to the difficulty and complexity of such work. So the present work studies the behaviour of unidirectional fiber-reinforced materials subjected to the mechanical loading based upon two dimensional plane elasticity theory and two dimensional finite element method using displacement formulation, also a computer program is designed to achieve our requirements.

Chapter Three

Theoretical Analysis of Unidirectional Fiber-Reinforced Materials (UDFRM)

٣-١ Introduction:-

Unidirectional lamina is the basic form of continuous fiber composites, orientated in the same direction. The stiffness and strength in the fiber direction are typically much greater than in the transverse directions. **Bolton,** ١٩٩٨ and **McCrum et. al.,** ١٩٩٧. Thus, the knowledge of the mechanical behaviour of lamina constituents, the fiber and the matrix, is essential to study of the unidirectional fiber-reinforced composite structures.

Predicting accurately the mechanical behaviour of composite material is not an easy task. Differences between the properties of reinforcement and the matrix cause complex distribution of stress and strain at the microscopic level, when load is applied. Reasonably accurate predictions can be made for unidirectional fiber composite, by employing simplified assumptions about the stress and strain distributions.

In this work, analysis of unidirectional fiber-reinforced materials have been done by assuming that longitudinal stress in the fiber varies linearly across its width whereas the transverse stress is uniform across the fiber.

In this chapter, stress equations of equilibrium in Cartesian co-ordinate are described, the stress-strain relations for anisotropic and orthotropic materials are summarized. The stress-strain relation for orthotropic materials in plane stress, the micromechanics approaches of composite materials, and a microstructure approach (which is considered in this work) for unidirectional laminate are derived and finally the general view about this work is limited in the last of this chapter.

٣-٢ Stress Equations of Equilibrium in Cartesian Co-ordinate :-

The element shown in Fig.(٣-١) displays only the stress components in the x-direction together with the body-force stress components. **Hearn,** ١٩٧٧.

Thus, for equilibrium of forces in the x-direction:

$$\left[\sigma_{xx}+\frac{\partial}{\partial x}\sigma_{xx}\,dx-\sigma_{xx}\right]dy.dz+\left[\tau_{xy}+\frac{\partial}{\partial y}\tau_{xy}\,dy-\tau_{xy}\right]dx.dz+$$
$$\left[\tau_{xz}+\frac{\partial}{\partial z}\tau_{xz}\,dz-\tau_{xz}\right]dx.dy+F_x\,dx.dy.dz=0$$

dividing through by dx. dy. dz and simplifying:-

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+F_x=0 \qquad \ldots\ldots (٣\text{-}١)$$

Similarly, for equilibrium in the y and z directions respectively :

$$\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+F_y=0 \qquad \ldots\ldots(٣\text{-}٢)$$

$$\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+F_z=0 \qquad \ldots\ldots(٣\text{-}٣)$$

The equilibrium of forces does not represent a complete check on the equilibrium of the system. This can only be achieved by an additional consideration of the moments of the forces which must also be in balance. The element shown in Fig. (٣-٢), shows only the stresses which produce moments about the y-axis. For equilibrium of moment about the y-axis,

$$\left[\tau_{xz} + \frac{\partial}{\partial z}(\tau_{xz})\frac{dz}{2}\right] dx.dy\ \frac{dz}{2} + \left[\tau_{xz} - \frac{\partial}{\partial z}(\tau_{xz})\frac{dz}{2}\right] dx.dy.\ \frac{dz}{2} -$$

$$\left[\tau_{zx} + \frac{\partial}{\partial x}(\tau_{zx})\frac{dx}{2}\right] dy.dz\ \frac{dx}{2} - \left[\tau_{zx} - \frac{\partial}{\partial x}(\tau_{zx})\frac{dx}{2}\right] dy.dz\ \frac{dx}{2} = 0$$

dividing through (dx dy dz) and simplifying, this reduces to:

$$\tau_{xz} = \tau_{zx} \qquad \text{...... (٣-٤)}$$

similarly the equilibrium of moments about the x-axis and z-axis gives that:-

$$\left.\begin{aligned} \tau_{zy} &= \tau_{yz} \\ \tau_{xy} &= \tau_{yx} \end{aligned}\right\} \qquad \text{......(٣-٥)}$$

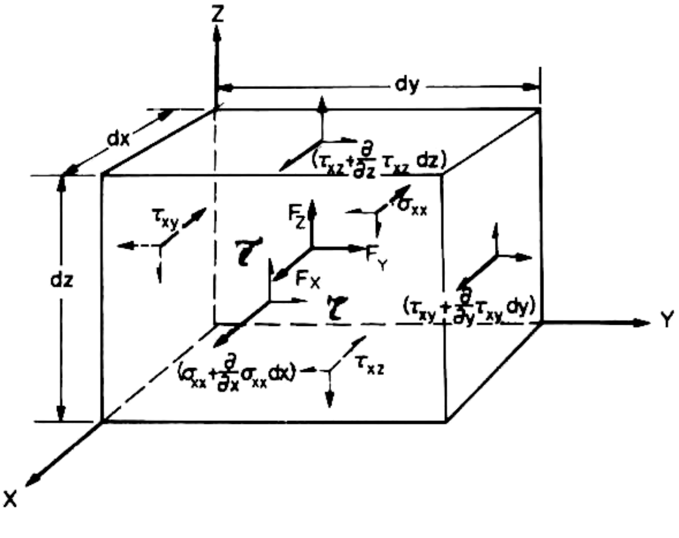


Fig (٣-١): Small element showing body force stresses and other stresses in the x-direction only.

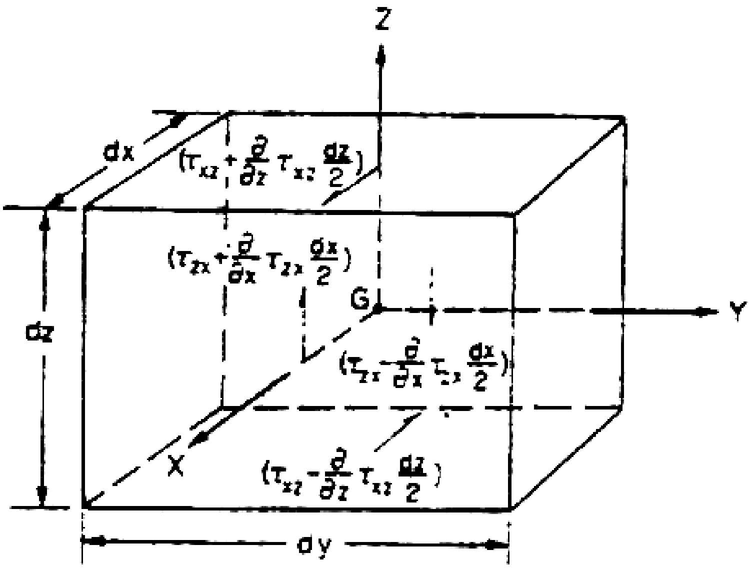


Fig. (٣-٢): Element showing only stresses which contribute to a moment about y-axis.

٣-٣ Stress-Strain Relation For Anisotropic Materials :

In the generalized case Hooke's law may be expressed as:-

$$\varepsilon_{ij} = S_{ij}\,\sigma_j \qquad \text{......(٣-٦)}$$

and $\sigma_{ij} = C_{ij}\,\varepsilon_j$ i, j = ١, ..., ٦ ......(٣-٧)

where Sij is the compliance tensor and Cij is the stiffness matrix (often called just the elastic constant). **Dieter,** ١٩٨٨.

The Eqns.(٣-٦) and (٣-٧) show that the components of Cij can be determined by the matrix inversion of Sij. At this stage Cij or Sij have ٣٦ independent constants, but further reduction in the number of independent constants is possible when the strain energy is considered. Elastic materials for which an elastic potential or strain energy density function exists have incremental work per unit volume of. **Jones,** ١٩٧٥

$$dW = \sigma_i\, d\varepsilon_i \qquad \text{...(٣-٨)}$$

when the stresses σ_i act through strains $d\varepsilon_i$. However, because of the stress-strain relations, Eqn.(٣-٧), the incremental work becomes :-

$$dW = C_{ij}\,\varepsilon_j\, d\varepsilon_i \qquad \text{...(٣-٩)}$$

The work done per unit volume is: -

$$W=\frac{1}{2}C_{ij}\ \varepsilon_i\ \varepsilon_j \qquad ......(٣-١٠)$$

However, Hooke's Law, Eqn.(٣-٧), can be derived from Eqn.(٣-٩):

$$\frac{\partial W}{\partial \varepsilon_i}=C_{ij}\ \varepsilon_j \qquad ......(٣-١١)$$

Where upon derivative

$$\frac{\partial^2 W}{\partial \varepsilon_i \partial \varepsilon_j}=C_{ij} \qquad ......(٣-١٢)$$

similarly , $\frac{\partial^2 W}{\partial \varepsilon_j \partial \varepsilon_i}=C_{ji}$ ......(٣-١٣)

But the order of differential of W is immaterial, so

$$C_{ij}=C_{ji} \qquad ......(٣-١٤)$$

In general, $C_{ij}=C_{ji}$ and $S_{ij}=S_{ji}$. Therefore, for the general anisotropic linear elastic solid there are twenty-one independent elastic constants. From the previous equation, stress-strain relations in the material coordinates for anisotropic materials are:-

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} . \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \qquad ...(٣-١٥)$$

Eqns.(٣-١٥) show that anisotropic materials have no planes of symmetry for the material properties. An alternative name for such anisotropic material is a triclinic material. **Dieter,** ١٩٨٨

If there is one plane of material property symmetry, the independent elastic constants reduced to thirteen and such material which contains this No. of independent constant is termed monoclinic.

An important class of engineering materials are orthotropic materials. An orthotropic material has three mutually perpendicular planes of the material property symmetry. Thus, the stress-strain relations in co-ordinates aligned with principal material directions which are parallel to the intersections of three orthogonal planes of material symmetry, are: -

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} . \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \qquad ....(٣-١٦)$$

Such material is called orthotropic material. There is no interaction between normal stresses $\sigma_1, \sigma_2, \sigma_3$ and shearing strains $\gamma_{23}, \gamma_{31}, \gamma_{12}$, such as occurs in anisotropic materials. There are only nine independent constants in the elasticity matrix. **Jones,** ١٩٧٥.

If at every point of a material there is one plane in which the mechanical properties are equal in all directions, then the material is termed transversely isotropic. For (١-٢) plane of isotropy, the stress-strain relations then have only five independent constants.

An isotropic material has only two independent constants in the stiffness matrix due to the infinite number of planes of material property symmetry: **Timoshenko & Goodier,** ١٩٧٠.

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{31}\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{(C_{11}-C_{12})}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{(C_{11}-C_{12})}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{(C_{11}-C_{12})}{2}\end{bmatrix}.\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{Bmatrix}$$

.... (٣-١٧)

The forward items show that the number of independent constants decreases with increasing planes of material property symmetry. , **Dieter,** ١٩٨٨.

The elasticity matrix Cij for an orthotropic material in terms of engineering constants is obtained as follows: **Barker et. al.,** ١٩٧٢.

$$\left.\begin{aligned}
C_{11} &= \frac{(1-\nu_{23}\,\nu_{32})*E_{11}}{F}\\
C_{12} &= \frac{(\nu_{12}+\nu_{13}\,\nu_{32})*E_{22}}{F}\\
C_{13} &= \frac{(\nu_{13}+\nu_{12}\,\nu_{23})*E_{33}}{F}\\
C_{22} &= \frac{(1-\nu_{13}\,\nu_{31})*E_{22}}{F}\\
C_{33} &= \frac{(1-\nu_{12}\,\nu_{21})*E_{33}}{F}\\
C_{23} &= \frac{(\nu_{23}+\nu_{21}\,\nu_{13})*E_{33}}{F}\\
C_{٤٤} &= G_{٢٣}\,,\ C_{٥٥} = G_{١٣}\,,\ C_{٦٦} = G_{١٢}
\end{aligned}\right\}$$

....(٣-١٨)

$$F=1-\nu_{12}\,\nu_{21}-\nu_{13}\,\nu_{31}-\nu_{32}\,\nu_{23}-\nu_{12}\,\nu_{23}\,\nu_{31}-\nu_{21}\,\nu_{13}\,\nu_{32}$$

٣-٤ Stress-Strain Relations For Plane Stress in an Orthotropic Material :

Referring to Eqns.(٣-٦) and (٣-١٦) the three dimensional strain-stress relation for orthotropic material can be written as:-

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} . \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} \quad \ldots(٣\text{-}١٩)$$

For a lamina in the ١-٢ plane as shown in Fig.(٣-٣), a plane stress state is defined by setting. , **Jones,** ١٩٧٥.

$$\sigma_3 = 0 \quad , \quad \tau_{23} = 0 \quad , \quad \tau_{13} = 0 \quad \ldots(٣\text{-}٢٠)$$

For orthotropic materials, such a procedure results in implied of

$$\begin{gathered} \varepsilon_3 = S_{13}\,\sigma_1 + S_{23}\,\sigma_2 \\ \gamma_{23} = 0 \quad , \quad \gamma_{31} = 0 \end{gathered} \quad \ldots(٣\text{-}٢١)$$

Moreover, the strain-stress relations in Eqns.(٣-١٩) reduce to:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \quad \ldots(٣\text{-}٢٢)$$

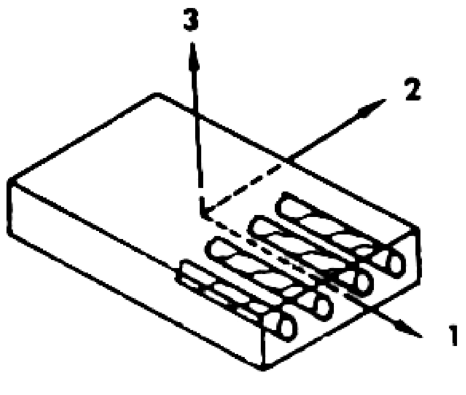


Fig. (٣-٣): Unidirectionally reinforced lamina.

Supplemented by Eqn. (٣-٢١) where:

$$S_{11} = \frac{1}{E_1} \quad , \quad S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2}$$
$$S_{22} = \frac{1}{E_2} \quad , \quad S_{66} = \frac{1}{G_{12}} \qquad \text{....(٣-٢٣)}$$

Note that in order to determine ε_3 in Eqn. (٣-٢١), ν_{13} and ν_{23} must be known in addition to those engineering constants shown in Eqn.(٣-٢٣). That is ν_{13} and ν_{23} arise from S١٣ and S٢٣ in Eqn.(٣-٢١).

The strain-stress relation in Eqn. (٣-٢٢) can be inverted to obtain the stress-strain relations:-

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \qquad \text{...(٣-٢٤)}$$

Where the Q_{ij} , the so-called reduced stiffnesses, are

$$\left.\begin{aligned} Q_{11} &= \frac{S_{22}}{S_{11} S_{22} - S_{12}^2} \\ Q_{12} &= \frac{-S_{12}}{S_{11} S_{22} - S_{12}^2} \\ Q_{22} &= \frac{S_{11}}{S_{11} S_{22} - S_{12}^2} \\ Q_{66} &= \frac{1}{S_{66}} \end{aligned}\right\} \qquad \text{...(٣-٢٥)}$$

in terms of the engineering constants:

$$\left.\begin{aligned} Q_{11} &= \frac{E_1}{1-\nu_{12}\,\nu_{21}} \\ Q_{12} &= \frac{\nu_{12}\,E_2}{1-\nu_{12}\,\nu_{21}} = \frac{\nu_{21}\,E_2}{1-\nu_{12}\,\nu_{21}} \\ Q_{22} &= \frac{E_2}{1-\nu_{12}\,\nu_{21}} \\ Q_{66} &= G_{12} \end{aligned}\right\} \qquad \ldots(٣\text{-}٢٦)$$

Note that there are four independent material properties, E١, E٢, ν_{12}, and G١٢ in Eqns.(٣-٢٢) and (٣-٢٤) when Eqns.(٣-٢٤) and (٣-٢٦) are considered in addition to the reciprocal relation. **Dieter,** ١٩٨٨.

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \qquad \ldots(٣\text{-}٢٧)$$

<u>٣-٥ Micromechanics Approaches of Composite Materials:</u>

As cited in the first chapter that the micromechanics is the study of properties of a monolayer in a composite material in the microscopic or constituent level in order to determine the principal strength, elastic and thermal properties of the monolayer. **Grinius, et. al.,** ١٩٧٥.

However, the micromechanical analysis has inherent limitations. For example, a perfect bond between fiber and matrix is a usual analysis restriction that might well not be satisfied by some composites. An imperfect bond would presumably yield a material with properties degraded from those of the micromechanics analysis.

Basic to the discussion of micromechanics is the representative volume element, which can be defined as the smallest region to show the microscopic structural details, yet large enough to represent the overall behaviour of composite. A simple representative volume element (RVE) can consist of a fiber embedded in a matrix block. Once a representative volume element is chosen, proper boundary conditions are prescribed, the prescribed boundary conditions must be the same as those if (RVE) actually in the composite. **Stephen W. Tsai** and **Thomas,** ١٩٨٠.

The micromechanics of composite materials can be divided into two basic approaches.

١- Mechanics of materials approach.

٢- Elasticity approach.

٣-٥-١ Mechanics of Materials Approaches:

This approach embodies the usual concept of vastly simplifying assumptions regarding the hypothesized behaviour of the mechanical system. The mechanics of material approach can be divided into:-

A) Mechanics of Material Approach to Stiffness :

This approach used to predict the elastic moduli and the strength of unidirectional composite in terms of those of constituent materials. Also this approach was done by simplifying assumptions for the mechanical behaviour of composite materials. For example, the most prominent assumption is that the strains in the fiber direction of a unidirectional fibrous composite are the same in the fiber as in the matrix as shown in Table (٣-١), case of $E_{١}$.

B) Mechanics of Materials Approach to Strength :

For a composite containing continuous fiber and that is unidirectionally aligned and loaded in the fiber direction, the overall stress in the composite follows a similar relation in Table (٣-١). That is called rule of mixture.

$$\sigma_c = \sigma_f V_f + \sigma_m V_m \qquad \text{......(٣-٢٨)}$$

Figure (٣-٤), illustrates this relationship on a stress-strain diagram.

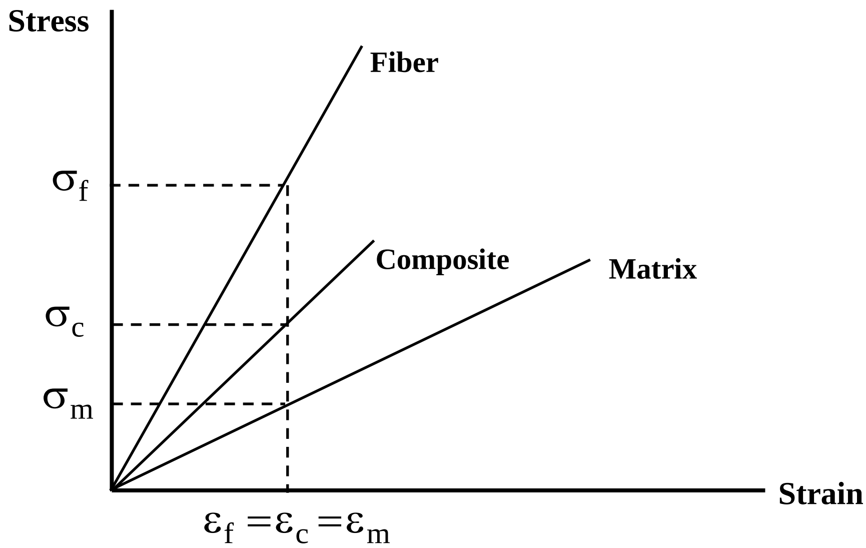


Fig. (٣-٤): Relationship among stresses and strains for a composite and its fiber and matrix materials **Robert, ١٩٩٦** and **Jones, ١٩٧٥**.

The rule of mixture in Eqn.(٣-٢٨) ignores any negative deviations due to any misalignment of the fibers or due to the formation of a reaction product between fiber and matrix. Also, it is assumed that the components do not interact during straining and that these properties in the composite state are the same as those in the isolated state. Then, for a series of composites with different fiber volume fractions, σ_c would be linearly dependent on V_f. Since, **Krishan, ١٩٩٩**.

$$V_f + V_m = ١ \qquad \text{...(٣-٢٩)}$$

Eqn. (٣-٢٨) can be written as:

$$\sigma_c = \sigma_f V_f + \sigma_m (1 - V_f) \qquad ...(٣-٣٠)$$

Certain restrictions on V_f can be put in order to have real reinforcement. For this a composite must have a certain in minimum fiber (continuous) volume fraction, V_{min}. Assuming that the fiber are identical and uniform (that is, all of them have the same ultimate tensile strength), the ultimate strength of the composite will be attained ideally, at a strain equals to the strain corresponding to the ultimate stress of the fiber. Then,

$$\sigma_{cu} = \sigma_{fu} V_f + \sigma'_{mu} (1 - V_f) \qquad V_f \geq V_{min} \qquad ...(٣-٣١)$$

where

σ_{fu} = ultimate (maximum) fiber tensile stress

σ'_{mu} = matrix stress at the strain corresponding to the fiber's ultimate tensile stress.

At low volume fractions the fibers may not be enough to control the matrix elongation. Thus, the fibers would be subjected to high strains with only small loads and would fracture. If all fiber break at the same strain, then the composite will fracture unless the matrix can take the entire load imposed on the composite, that is.

$$\sigma_{cu} < \sigma_{mu} V_m \qquad ...(٣-٣٢)$$

where

σ_{mu} = ultimate tensile stress of the matrix.

Finally, the entire composite fails after fracture of the fibers if :

$$\sigma_{cu} = \sigma_{fu} V_f + \sigma'_{mu} (1 - V_f) \geq \sigma_{mu} (1 - V_f) \qquad ....(٣-٣٣)$$

The equality in this expressions serves to define, V_{min}, that must be surpassed in order to have real reinforcement, in that case,

$$V_{min} = \frac{\sigma_{mu} - \sigma'_{mu}}{\sigma_{fu} + \sigma_{mu} - \sigma'_{mu}} \qquad ...(٣-٣٤)$$

The value of V_{min} increases with decreasing fiber strength or increasing matrix strength.

If fiber reinforcement is to effect a greater strength than can be obtained by the matrix alone, then

$$\sigma_{cu} > \sigma_{mu} \qquad \ldots(٣\text{-}٣٥)$$

Thus, Vcrit, can be given by the equation:

$$\sigma_{cu} = \sigma_{fu} V_f + \sigma'_{mu}(1 - V_f) > \sigma_{mu}$$

in this case,

$$V_{crit} = \frac{\sigma_{mu} - \sigma'_{mu}}{\sigma_{fu} - \sigma'_{mu}} \qquad \ldots.(٣\text{-}٣٦)$$

Vcrit increases with increasing degree of matrix work-hardening $(\sigma_{mu} - \sigma'_m)$. Figure (٣-٥) shows graphically the determination of V_{min} and V_{crit}. Its clear that V_{crit} will always be greater than V_{min}.

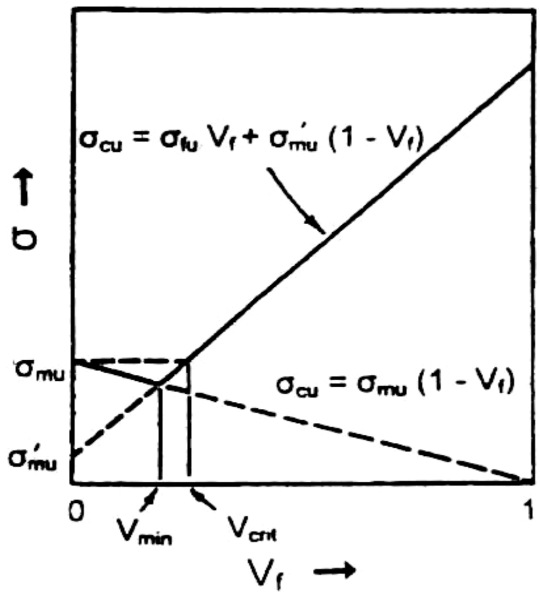


Fig. (٣-٥): Determination of Vmin and Vcrit **Krisham,** ١٩٩٩ and **Jones,** ١٩٧٥.

٣-٥-٢ Elasticity Approach :

The objective of all of the micromechanics approaches (mechanics of materials approach and elasticity approach) is to determine the elastic moduli or stiffness or compliances of composite materials in terms of the elastic moduli of the constituent materials. An additional objective of the micromechanics approaches to composite materials is to determine the strength of the composite materials in terms of the strength of the constituent materials.

The important approach is the elasticity approach that can be classified as: -

- Bounding techniques of elasticity (the variational calculus method).
- Exact solutions.
- Approximate solutions, including:
  A- Elasticity solutions with contiguity.
  B- The Halpin-Tsai Equations.
  C- Microstructure theory.

The first method focuses on the upper and lower limits of the properties of the composite and does not predict those properties directly. Only when the upper and the lower bounds coincide a particular property is determined. Frequently, the upper and lower bounds are well separated. **Krishan, ١٩٩٩**.

The exact solution method is appropriate to indicate the types of solutions that are available and to compare them with the mechanics of material results.

Because of the resulting complexity of the problem, many advanced analytical techniques and numerical solution procedures are necessary to obtain a good solution. However, the assumptions made in such analyses regarding the interaction between the fiber and the matrix are not entirely realistic. An interesting approach to more realistic fiber-matrix interaction is the contiguity approach, which will be examined in the next section. Also the Halpin-Tsai equations are widely used which will be also discussed later. Both the contiguity approach and Halpin-Tsai equations are used in this work in order to examine the accuracy of the finite element computer program for Unidirectional Fiber-Reinforced Materials (UDFRM). Finally, the microstructure approach which was adopted in this work was derived in details at the last of this chapter.

٣-٥-٢-١ Approximate Solutions:

As cited in the forward sections the most interesting and important approaches that are widely used in elasticity approaches of unidirectional composite materials are:-

A) Elasticity Solutions with Contiguity :-

From an analytical point of view, a linear combination of

١) a solution in which all fibers are isolated from one another, and

٢) a solution in which all fibers contact each other provides the correct modulus. If C denotes degree of contiguity, then C = ٠ corresponds to no contiguity (isolated fibers) and C = ١ corresponds to perfect contiguity (all fibers in contact).

Naturally, with high volume fractions of fibers **C** would be expected to approach C = ١. **Jones,** ١٩٧٥.

For the elasticity approach in which the contiguity is considered, Tsai **Tsai S. W.,** ١٩٦٤ obtained for the modulus transverse to the fibers.

$$E_2 = 2\left[1-\nu_f+\left(\nu_f-\nu_m\right)V_m\right]\left[\begin{array}{l}\left(1-C\right)\dfrac{K_f\,(2K_m+G_m)-G_m\left(K_f-K_m\right)V_m}{(2K_m+G_m)+2\left(K_f-K_m\right)V_m}\\ +C\dfrac{K_f\,(2K_m+G_f)+G_f\left(K_m-K_f\right)V_m}{(2K_m+G_f)-2\left(K_m-K_f\right)V_m}\end{array}\right]$$

...(٣-٣٧)

Tsai also obtains

$$\nu_{12} = \left(1-C\right)\frac{K_f\,\nu_f\,(2K_m+G_m)V_f+K_m\nu_m\left(2K_f+G_m\right)V_m}{K_f\,(2K_m+G_m)-G_m\left(K_f-K_m\right)V_m} + C\frac{K_m\,\nu_m\,(2K_f+G_f)V_m+K_f\nu_f\left(2K_m+G_f\right)V_f}{K_f\,(2K_m+G_m)+G_f\left(K_m-K_f\right)V_m}$$

....(٣-٣٨)

$$G_{12} = \left(1-C\right)G_m\,\frac{2G_f-\left(G_f-G_m\right)V_m}{2G_m+\left(G_f-G_m\right)V_m}+C\,G_f\,\frac{\left(G_f+G_m\right)-\left(G_f-G_m\right)}{\left(G_f+G_m\right)+\left(G_f-G_m\right)}$$

(٣-٣٩)

where

$$\left.\begin{array}{l} K_f = \dfrac{E_f}{2\left(1-\nu_f\right)} \\ K_m = \dfrac{E_m}{2\left(1-\nu_m\right)} \\ G_f = \dfrac{E_f}{2\left(1+\nu_f\right)} \\ G_m = \dfrac{E_m}{2\left(1+\nu_m\right)} \end{array}\right\}$$

...(٣-٤٠)

For the modulus in the directions of the fibers, Tsai **Tsai,** ١٩٦٤ modified the rule of mixture to account for imperfections in fiber alignment:

$$E_1 = k\left(V_f\,E_f+V_mE_m\right)$$

...(٣-٤١)

Where **k** is the fiber misalignment factor ordinarily varies from ٠.٩ to ١. This factor is an experimentally determined constant and is highly dependent on the manufacturing process.

B) **The Halpin-Tsai Equations:-**

Halpin and Tsai developed an interpolation procedure that is an approximate representation of more complicated micromechanics results. The beauty of the procedure is twofold. First, it is simple so it can readily be used in the design process. Second, it enables the generalization of usually limited, although more exact, micromechanics results. , **Halpin, J. C.** and **S. W. Tsai,** ١٩٦٩.

Halpin and Tsai developed the following approximate form:

$$E_1 \approx E_f V_f + E_m V_m$$ ...(٣-٤٢)

$$\nu_{12} = \nu_f V_f + \nu_m V_m$$ ...(٣-٤٣)

and

$$\frac{M}{M_m} = \frac{1+\xi\eta V_f}{1-\eta V_f}$$ ...(٣-٤٤)

where

$$\eta = \frac{(M_f / M_m) - 1}{(M_f / M_m) + \xi}$$ ...(٣-٤٥)

in which

M = composite modulus E٢. G١٢, or ν_{23}

M_f = corresponding fiber modulus, E_f, G_f, or ν_f

M_m = corresponding matrix modulus E_m, G_m, or ν_m

and ζ is a measure of fiber reinforcement of the composite which depends on the fiber geometry, packing geometry, and loading conditions. The value of ξ can be obtained from Eqns.(٣-٤٤) and (٣-٤٥).

The term η is called reduced factor and its values $\leq$ ١. Moreover, it is apparent from Eqn. (٣-٤٥) that η is affected by the constituent material properties as well as by the reinforcement geometry factor ξ.

C) Microstructure Approach :

Properties of UDFRM are dependent on various characteristics of the fibers, the matrix and the fiber/matrix interface. In general, the behaviour of the fiber and matrix are studied well by micromechanics approaches which have a good significance in this field. But from micromechanics approach the microstructure approach is known in considerably less detail. Thus this has led to a substantial volume of research over the past few years, which has been aimed to study the behaviour of the composite materials by microstructure theory. So the present work studies the behaviour of (UDFRM) by microstructure approach as shown in the next section.

٣-٥-٢-٢ A Microstructure Approach for Unidirectional Fiber-Reinforced Materials (UDFRM):

In this approach the UDFRM can be examined microscopically by considering a unidirectional composite with reinforcing layers (fibers) in ($x_١$-direction) as shown in Fig.(٣-٦). The representative volume element of this composite is chosen to be two fibers embedded in a matrix plate. The fibers are assumed homogeneous and have a rectangular cross section with the same thickness as the matrix plate. Therefore, the width ratio h_f / ($h_f + h_m$) is chosen to be the same as the fiber volume fraction of the composite itself.

Fig. (٣-٧) shows (RVE) for (UDFRM) with fiber spacing (H) that is equal to the sum of the width of fiber and matrix. The $\bar{u}_i$ represent displacements within the fibers and these displacements are independent of displacement, u_i , which define the deformation of a classical continuum. Both displacement are related to Cartesian co-ordinates $\mathbf{x_i}$.

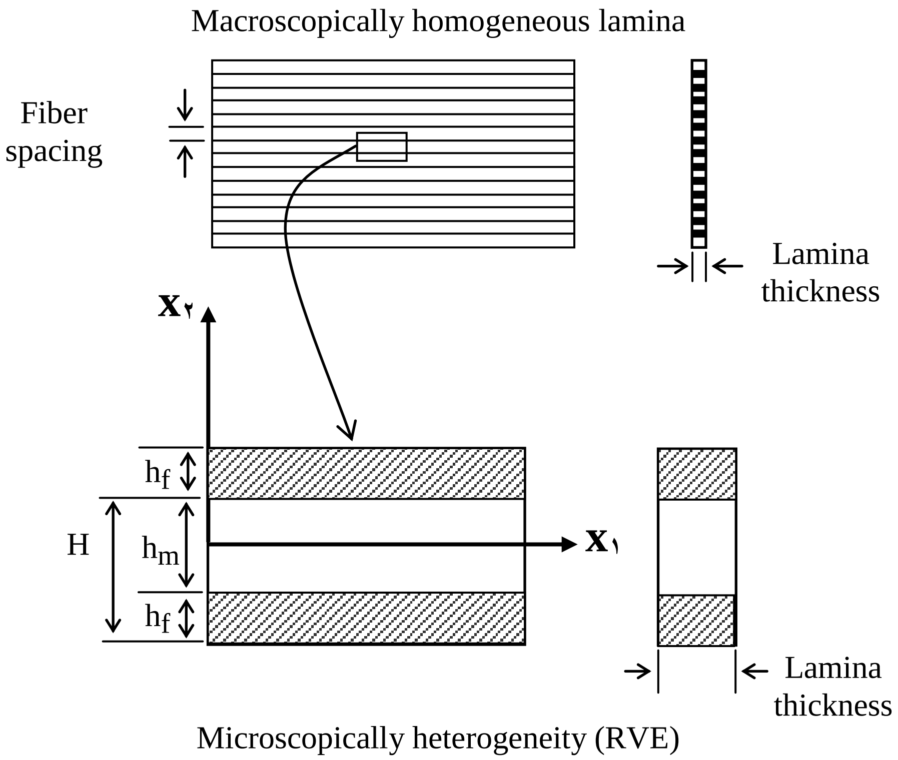


Fig. (٣-٦): Representative volume element (RVE) A perfect bond is assumed between fiber and matrix.

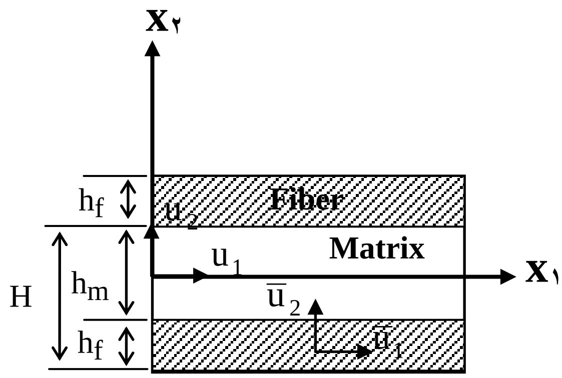


Fig. (٣-٧) Arrangement of fiber and matrix

The strain energy **Herrman, ١٩٧٦** and **Malcolm, ١٩٧٨** of **n** discrete reinforcing layers (fiber) can be supposed by a weighted integral over the entire region. Therefore :-

$$\sum_{k=1}^{n} U_f^k \approx \frac{V_f}{h_f} \int U_f \, dx_2 \qquad (٣\text{-}٤٦)$$

where $V_f = h_f / H$ indicates the fiber volume fraction or density of the fibers and U_f^k is the strain energy density per unit length of the k_{th} fiber.

For an isotropic elastic solid in plane strain the strain energy density. "U" may be written as: **Dieter ,** ١٩٨٨ and **Herrman,** ١٩٧٦

$$U = (\lambda / 2)(\varepsilon_{11} + \varepsilon_{22})^2 + \mu\left(\varepsilon_{11}^2 + 2\varepsilon_{12}^2 + \varepsilon_{22}^2\right) \qquad (٣\text{-}٤٧)$$

where: λ and μ are the Lame's material constant which are equal to :-

$$\mu = \text{shear modulus} = G = \frac{E}{2(1+\nu)} \; ; \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad (٣\text{-}٤٨)$$

Consider, at first, alternating plane of parallel sheets of two homogeneous, isotropic materials Fig. (٣-٨).

The lame's elastic constants, and the width of the fiber layers and the matrix layers are denoted by λ_f, μ_f , h_f and λ_m, μ_m, h_m respectively.

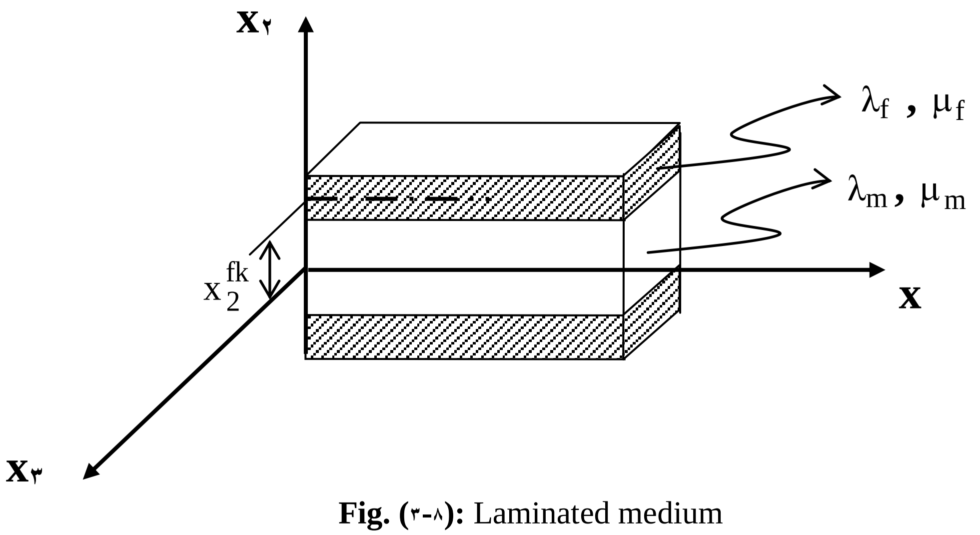


Fig. (٣-٨): Laminated medium

where the k-th reinforcing sheet, whose midplane position is defined by x_2^{fk}. The displacements in plane strain may be expressed in the form:-

$$\left.\begin{aligned} u_1 &= \bar{u}_1 + \bar{x}_2^{fk}\,\psi_{21} \\ u_2 &= \bar{u}_2 \end{aligned}\right\} \qquad \ldots(٣-٤٩)$$

where ψ_{21} is local shear deformation.

and $\bar{x}_2^{fk}$ is the coordinate in a local coordinate system.

Expression for the strain energy density may be obtained by employing the displacement distribution in Eqn.(٣-٤٩) together with energy expressions in Eqn.(٣-٤٧) and integrating through the width $\mathbf{h_f}$, and by assuming that longitudinal stress in the fiber varies linearly across its width and that transverse normal stress is uniform across the fiber.

When :-

The component of overall strain tensor

$$\varepsilon_{ji} = \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \ldots(٣-٥٠)$$

and the component of relative strains

$$\gamma_{ij} = \frac{\partial u_j}{\partial x_i} - \psi_{ij} \qquad \ldots(٣-٥١)$$

then $\varepsilon_{11} = \dfrac{\partial u_1}{\partial x_1} = \dfrac{\partial \bar{u}_1}{\partial x_1} + \dfrac{h_f}{2}.\dfrac{\partial \psi_{21}}{\partial x_1}$

The term $\dfrac{h_f}{2}.\dfrac{\partial \psi_{21}}{\partial x_2}$ is very infinitesimal, so equal to zero

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \ldots(٣-٥٢)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = \frac{\partial \bar{u}_2}{\partial x_2} = \psi_{22} \qquad \ldots(٣-٥٣)$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial \bar{u}_2}{\partial x_1} + \frac{\partial \bar{u}_1}{\partial x_2} + \frac{h_f}{2} \cdot \frac{\partial \psi_{21}}{\partial x_2}\right)$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \psi_{21}\right) \qquad \ldots(٣-٥٤)$$

Where $$\psi_{21} = \frac{\partial \bar{u}_1}{\partial x_2}, \ \psi_{22} = \frac{\partial \bar{u}_2}{\partial x_2} \qquad \ldots.(٣-٥٥)$$

By substituting Eqn.(٣-٥٢) to (٣-٥٤) in Eqn.(٣-٤٧) and integrating it to dx٢ the strain energy density of the fiber may be written as :-

$$U_f^k = \frac{1}{2} D_f \left(\frac{\partial \psi 21}{\partial x_1}\right)^2 + \frac{1}{2}\mu_f \, h_f \left(\psi_{21} + \frac{\partial u_2}{\partial x_1}\right)^2 + \frac{1}{2}\lambda_f h_f \left(\frac{\partial u_1}{\partial x_1} + \psi_{22}\right)^2$$
$$+ \mu_f \, h_f \left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \psi_{22}^2\right] \qquad \ldots.(٣-٥٦)$$

where, Lame's materials constant for fiber λ_f, μ_f are equal to

$$\lambda_f = \frac{\nu_f E_f}{(1+\nu_f)(1-2\nu_f)}, \quad \mu_f = \frac{E_f}{2(1+\nu_f)}$$

and the flexural stiffness, $D_f = \mu_f \, h_f^3 / 6(1-\nu_f)$

$$\chi_{121} = \frac{\partial \psi_{21}}{\partial x_1} \qquad \ldots.(٣-٥٧)$$

Within the matrix the normal strains in the x١ directions are assumed identical to those in the fiber; the transverse normal and shear strains within the matrix, however, are not dependent solely on the overall displacement field u_i , but are also influenced by the strains within, and the thickness of the fiber the strain energy per unit volume of the matrix may be shown to be :-

$$U_m = \frac{1}{2}\lambda_m\left[\varepsilon_{11} + \left(\varepsilon_{22} + \frac{V_f}{1-V_f}\gamma_{22}\right)\right]^2 +$$

$$\mu_m\left[\varepsilon_{11}^2 + \left(\varepsilon_{22} + \frac{V_f}{1-V_f}\gamma_{22}\right)^2 + 2\left(\varepsilon_{12} + \frac{1}{2}\frac{V_f}{1-V_f}\gamma_{21}\right)^2\right]$$

....(٣-٥٨)

Where, λ_m and μ_m are Lame's material constants for the matrix. Thus the total strain energy per unit volume of the composite may be written as :-

$$U_c = \frac{V_f}{h_f}U_f + (1-V_f)U_m \qquad \text{....(٣-٥٩)}$$

which, using Eqns. (٣-٥٦), (٣-٥٨) and (٣-٥١) can be expressed as :-

$$U_c = \varepsilon_{11}^2\left(\frac{1}{2}\lambda + \mu\right) + \varepsilon_{22}^2\left(\frac{1}{2}\lambda + \mu\right) + \varepsilon_{11}\varepsilon_{22}\lambda + \varepsilon_{11}\gamma_{22}V_f(\lambda_m - \lambda_f) +$$
$$\varepsilon_{22}\gamma_{22}V_f(\lambda_m + 2\mu_m - \lambda_f - 2\mu_f) +$$
$$\gamma_{22}^2 V_f\left[\frac{1}{2}\lambda_f + \mu_f + \frac{V_f}{1-V_f}\left(\frac{1}{2}\lambda_m + \mu_m\right)\right] + 2\varepsilon_{12}^2\mu + 2\varepsilon_{12}\gamma_{21}V_f(\mu_m - \mu_f)$$
$$+\frac{1}{2}\gamma_{21}^2 V_f\left(\mu_f + \frac{V_f}{1-V_f}\mu_m\right) + \frac{V_f D_f}{2h_f}\chi_{121}^2$$

....(٣-٦٠)

where

$$\left.\begin{aligned} \lambda &= V_f\lambda_f + (1-V_f)\lambda_m \\ \mu &= V_f\mu_f + (1-V_f)\mu_m \\ \chi_{121} &= \psi_{21,1} = \partial\psi_{21}/\partial x_1 \end{aligned}\right\} \qquad \text{....(٣-٦١)}$$

This strain energy may be compared with the strain energy of an isotropic material with microstructure to result in the following relations:-

$$\tau_{11} = \frac{\partial U_c}{\partial \varepsilon_{11}} = (\lambda + 2\mu)\varepsilon_{11} + \lambda\varepsilon_{22} + V_f\left(\lambda_m - \lambda_f\right)\gamma_{22}$$

$$\tau_{22} = \frac{\partial U_c}{\partial \varepsilon_{22}} = \lambda\varepsilon_{11} + (\lambda + 2\mu)\varepsilon_{22} + V_f\left(\lambda_m + 2\mu_m - \lambda_f - 2\mu_f\right)\gamma_{22}$$

$$\tau_{12} = \tau_{21} = \frac{\partial U_c}{\partial \varepsilon_{12}} = 2\mu\ \varepsilon_{12} + V_f\left(\mu_m - \mu_f\right)\gamma_{21}$$

$$\sigma_{22} = \frac{\partial U_c}{\partial \gamma_{22}} = V_f\left(\lambda_m - \lambda_f\right)\varepsilon_{11} + V_f\left[\lambda_f + 2\mu_f + \frac{V_f}{1 - V_f}\left(\lambda_m + 2\mu_m\right)\right]\gamma_{22}$$
$$+ V_f\left[\lambda_m + 2\mu_m - \lambda_f - 2\mu_f\right]$$

$$\sigma_{21} = \frac{\partial U_c}{\partial \gamma_{21}} = V_f\left(\mu_f + \frac{V_f}{1 - V_f}\mu_m\right)\gamma_{21} + 2V_f\left(\mu_m - \mu_f\right)\varepsilon_{12}$$

$$\mu_{121} = \frac{\partial U_c}{\partial x_{121}} = \frac{V_f D_f}{h_f}\chi_{121}$$

Eqns.(٣-٦٢)

where μ_{121} is the couple stress.

The above equations are valid in a state of plane strain where the overall strain ε_{33} is zero. Assuming that the stress τ_{33} has a form similar to the value of τ_{11} in Eqns.(٣-٦٢) of plane stress may be derived as:-

$$\begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \\ \sigma_{21} \\ \sigma_{22} \\ \mu_{121} \end{Bmatrix} = \begin{bmatrix} d_1 & d_2 & 0 & 0 & d_8 & 0 \\ d_2 & d_1 & 0 & 0 & d_9 & 0 \\ 0 & 0 & d_3 & d_5 & 0 & 0 \\ 0 & 0 & d_5 & d_4 & 0 & 0 \\ d_8 & d_9 & 0 & 0 & d_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_7 \end{bmatrix} . \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ \gamma_{21} \\ \gamma_{22} \\ \chi_{121} \end{Bmatrix}$$

…(٣-٦٣)

where:-

$$d_1 = 4\mu(\mu + \lambda)/(\lambda + 2\mu)$$
$$d_2 = 2\mu\ \lambda/(\lambda + 2\mu)$$
$$d_3 = \mu$$
$$d_4 = V_f \left(\mu_f + \frac{V_f}{1 - V_f} \mu_m \right)$$
$$d_5 = V_f \left(\mu_m - \mu_f\right)$$
$$d_6 = V_f \left[\lambda_f + 2\mu_f + \frac{V_f}{1 - V_f}(\lambda_m + 2\mu_m) - V_f \left(\lambda_m - \lambda_f\right)^2 / (\lambda + 2\mu) \right]$$
$$d_7 = V_f\ \mu_f\ h_f^2 / 6(1 - \nu_f)$$
$$d_8 = V_f^2\ \mu\left(\lambda_m - \lambda_f\right)/(\lambda + 2\mu)$$
$$d_9 = d_8 + 2V_f\left(\mu_m - \mu_f\right)$$

…(٣-٦٤)

From the strain energy expression contained in Eqns.(٣-٦٠) and (٣-٦٢) equations of equilibrium of the following form are obtained by standard variational techniques. **Appendix A**.

$$\tau_{11,1} + \left(\tau_{12} + \sigma_{21}\right)_{,2} = 0 \qquad \text{…(٣-٦٥ a)}$$

$$\tau_{22,2} + \tau_{12,1} = 0 \qquad \text{...(٣-٦٥ b)}$$

$$\mu_{121,1} + \sigma_{21} = 0 \qquad \text{…(٣-٦٥ c)}$$

٣-٦ General View about Present Work:

The last two chapters mentioned that fibers could be continuous or discontinuous. Generally the highest strength and stiffness of composite materials are obtained when fibers are continuous. Also when the fibers are very long the effect of their ends can be ignored whereas for the short fibers (discontinuous fibers) this effect can not be neglected because they are weak points in the composite sites of high stress concentration in the matrix. It must be assumed that negligible stress gets transferred to the fibers across their end faces. Stress builds up along each fiber from zero at its ends to a maximum at the center. **McCrum, ١٩٩٧**.

This work used equation (٣-٦٣) in a plane stress which is obtained from using microstructure approach, to design the computer program (SAOUDL) by using finite element method which will be discussed in the next chapter in details. Due to the complexity of these Eqns. as they contain large number of parameters and symbols few works were done on microstructure approach of UDFRM. Most works were done by micromechanics approaches to calculate the composite properties from its constituents (fiber and matrix), compared with experimental results. But the works that base on microstructure approach are difficult to compare with other done in the same region. The reasons were ascribed to the difficulties that encountered during experimental, which made this method very expensive and complex. So in this work the comparison will be done with exact solution results for the same example used to valid the degree of accuracy for this equations.

Chapter Four

Finite Element Method and Computer Program

٤-١ General:

The finite element method is a widely accepted numerical procedure for solving the differential equations of engineering and physics and is the computational basis of many computer-aided design systems [**Larry, ١٩٨٤**]. The finite element method is rapidly becoming a necessity in those curricula which solve problems in the general areas of structural analysis, continuum mechanics, heat transfer, seepage, magnetic flux, and other problems.

In this chapter, the finite element for unidirectional fiber-reinforced materials is described. The stiffness matrix formulation for (UDFRM) is derived. The main program is summarized and finally, the descriptions of subroutines are focused.

٤-٢ The Finite Element For Unidirectional Fiber-Reinforced Materials (UDFRM):-

In this work, UDFRM has contributed with the elements system of equations to obtain the element stiffness matrix and element force vector from equilibrium stress state of UDFRM under symmetric boundary condition given in Eqns.(٣-٦٥a) to (٣-٦٥c) which had been written in the previous chapter basing on minimization potential energy principle.

The element that has been depended on the present work is a triangular element which has straight sides and three nodes, one at each corner as shown in Fig.(٤-١), like wise, with three degree of freedom per node. Thus, the nodal displacement values for this element are denoted as ϕ_i, ϕ_j, and ϕ_k for nodes ١, ٢ and ٣ respectively.

Also the shape functions which corresponding to nodes ١, ٢ and ٣ are N_i , N_j and N_k or $N_١$, $N_٢$ and $N_٣$ respectively. So the displacement within an element can be:

$$\phi^{(e)} = N_i\phi_i + N_j\phi_j + N_k\phi_k \qquad (٤\text{-}١)$$

where:

$$N_i = \frac{1}{2A}(a_i + b_i x + c_i y) \qquad (٤\text{-}٢)$$

$$N_j = \frac{1}{2A}(a_j + b_j x + c_j y) \qquad (٤\text{-}٣)$$

$$N_k = \frac{1}{2A}(a_k + b_k x + c_k y) \qquad (٤\text{-}٤)$$

and

$$\left.\begin{array}{l} a_i = x_j y_k - x_k y_j \;,\; b_i = y_j - y_k \quad \text{and} \quad c_i = x_k - x_i \\ a_j = x_k y_i - x_i y_k \;,\; b_j = y_k - y_i \quad \text{and} \quad c_j = x_i - x_k \\ a_k = x_i y_j - x_j y_i \;,\; b_k = y_i - y_j \quad \text{and} \quad c_k = x_j - x_i \end{array}\right\} \qquad (٤\text{-}٥)$$

$$2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} \qquad (٤\text{-}٦)$$

Note that $N_i + N_j + N_k =$ ١ at any point.

Also "e" in Eqn.(٤-١) refers to element and both ϕ^e, ϕ_i, ϕ_j, and ϕ_k and N_i , N_j, and N_k are function of (x, y) only. Therefore from Fig.(٤-٢) Eqn. (٤-١) can be written as:

$$\phi^{e}(x,y)=\begin{bmatrix}\phi_1(x,y)\\ \phi_2(x,y)\\ \phi_3(x,y)\end{bmatrix}=\begin{bmatrix}N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0\\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0\\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3\end{bmatrix}\begin{bmatrix}\phi_{3i-2}\\ \phi_{3i-1}\\ \phi_{3i}\\ \phi_{3j-2}\\ \phi_{3j-1}\\ \phi_{3j}\\ \phi_{3k-2}\\ \phi_{3k-1}\\ \phi_{3k}\end{bmatrix}$$

....(٤-٧)

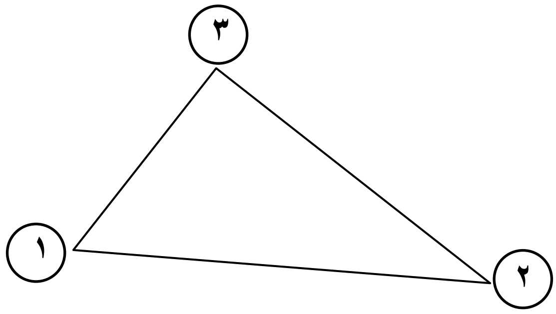


Fig.(٤-١): Linear triangular element.

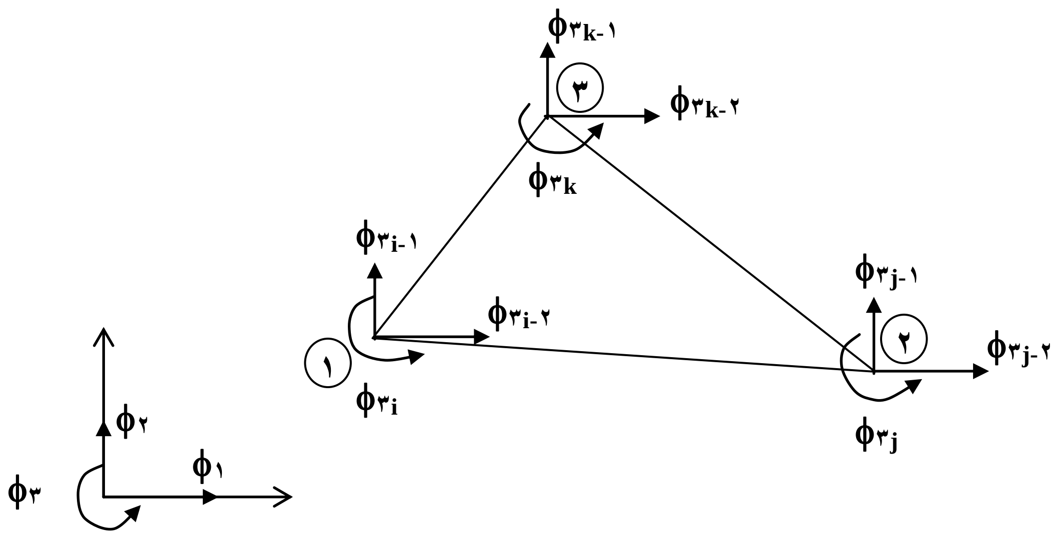


Fig.(٤-٢): Numbering of finite element node and degree of freedom.

The potential energy π on the descritized region (element) is defined as :

$$\pi = \sum_{e=1}^{n} U^{(e)} - \left(W_f^{(e)} + W_p^{(e)}\right) - \sum \{\phi_i\}^T . \{P_i\} \qquad (٤\text{-}٨)$$

where $U^{(e)}$ is the strain energy in the element. $W_f^{(e)}$ is the work done by body forces and $W_p^{(e)}$ is the work done by distributed surface loads that resulting from the stress components acting on the outside surface, where the product $\{\phi_i\}^T.\{P_i\}$ refer to the work done by distributed load. When the subscripts **i** indicates the line application of a line load Pi.

The strain energy $U^{(e)}$ can be written as:

$$U^{(e)} = \frac{1}{2}\int_v \{\sigma\}^T.\{\varepsilon\}.dv$$
$$= \frac{1}{2}\int_v \{\varepsilon\}^T.[D].\{\varepsilon\}.dv \qquad (٤\text{-}٩)$$

as [D] is symmetric, $[D]^T = [D]$ in the above equation.

But the strain energy $U^{(e)}$ must be written in terms of the displacement. The nodal displacements, however, are related to the total strain components not the elastic strain components. [**Logan, ١٩٩٢**].

So the total strain: $\{e\} = \{\varepsilon\} + \{\varepsilon_T\}$

$$\{\varepsilon\} = \{e\} - \{\varepsilon_T\} \qquad (٤\text{-}١٠)$$

$$\{e\} = [B].\{\phi^{(e)}\} \qquad (٤\text{-}١١)$$

where [B] is the strain matrix and the derivative of the shape functions [N١, N٢ , N٣] for the element. The vector $\{\sigma\}, \{\varepsilon\}, \{\varepsilon_T\}$ and $\{e\}$ means

$$\left.\begin{aligned}\{\sigma\}^T &= (\sigma_x \;\; \sigma_y \;\; \tau_{xy})\\ \{\varepsilon\}^T &= (\varepsilon_x \;\; \varepsilon_y \;\; \gamma_{xy})\\ \{\sigma\}^T &= (\alpha\,\delta T \;\; \alpha\,\delta T \;\; 0)\\ \{e\}^T &= (e_x \;\; e_y \;\; 0)\end{aligned}\right\} \qquad (٤\text{-}١٢)$$

From Eqns. (٤-١٠), (٤-١١), and (٤-١٢) the strain energy in Eqn. (٤-٩) becomes:-

$$U^{(e)} = \frac{1}{2}\int_V \{\phi^{(e)}\}^T [B]^T [D][B]\{\phi^{(e)}\}dv - \int_V \{\phi^{(e)}\}^T [B]^T [D]\{\varepsilon_T\}dv \qquad (٤\text{-}١٣)$$

The displacement equations are left in the general form as:

$$\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{Bmatrix} = [N]\{\phi^{(e)}\} \qquad (٤\text{-}١٤)$$

Thus the work done by the body forces and by distributed surface loads are given as follows:-

$$W_f^{(e)} = \int_v \{\phi^{(e)}\}^T [N]^T \begin{Bmatrix} x\\ y\end{Bmatrix} dv \qquad (٤\text{-}١٥)$$

$$W_p^{(e)} = \int_A \{\phi^{(e)}\}^T [N]^T \begin{Bmatrix} P_x\\ P_y\end{Bmatrix} dA \qquad (٤\text{-}١٦)$$

Taking into consideration that the product $[N].\{\phi^{(e)}\}$ in Eqn.(٤-١٤) is the column vector. So :-

$$[N]\{\phi^{(e)}\} = \{\phi^{(e)}\}^T [N]^T \qquad (٤\text{-}١٧)$$

In this work, the effect of body forces and distributed surface loads are neglected due to their small effects compared with external mechanical loads. This work has been studied the linear behaviour of UDFRM only because of large difficulties would appear if the thermal effect is to be taken into account.

Then the Eqn.(٤-٨) can be written as follows:

$$\pi^{(e)}=\frac{1}{2}\{\phi^{(e)}\}^{T}\left(\int_{V}[B]^{T}[D][B].dv\right)\{\phi^{(e)}\}-\sum_{i}\{\phi_i\}^{T}\{P_i\} \qquad (٤-١٨)$$

The principle of minimum potential energy which is taken into consideration is stated as follows:- (for conservative systems of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable). [**Chandrupalta & Belegundu, ١٩٩٧**]. So,

$$\frac{\partial \pi}{\partial\{\phi\}}=0 \qquad (٤-١٩)$$

The final system of equations in the element can be evaluated from:

$$\frac{\partial \pi^{(e)}}{\partial\{\phi^{(e)}\}}=0 \qquad (٤-٢٠)$$

But,

$$\pi^{(e)}=U^{(e)}-W_P^{(e)} \qquad (٤-٢١)$$

Then $$\frac{\partial \pi^{(e)}}{\partial\{\phi^{(e)}\}}=[K^{(e)}]\{\phi^{(e)}\}-\{f^{(e)}\}=0 \qquad (٤-٢٢)$$

From the above equations the element stiffness matrix $[K^{(e)}]$ and element force vector $\{f^{(e)}\}$ can be written as:

$$[K^{(e)}]=\int_{V}[B]^{T}[D][B].dv \qquad (٤-٢٣)$$

and this equations leads to

$$[K^{(e)}]=\int_{A}[B]^{T}[D][B]t.dA=[B]^{T}[D][B]t.A \qquad (٤-٢٤)$$

and $\{f^{(e)}\} = \sum P_i$ (٤-٢٥)

In finite element formulation the general form of Eqn.(٤-٢٢) is:

$$\lfloor k^{(e)} \rfloor \{\phi^{(e)}\} = \{f^{(e)}\} \quad (٤-٢٦)$$

so the global stiffness matrix K, and the global force vector {F} can be given for the overall structure as:-

$$[K]\{\phi\} = \{F\} \quad (٤-٢٧)$$

In Eqn.(٤-٢٥), $\{P_i\}$ is the concentrated force that was observed to be used with the structural applications, and it is represented as an applied load on the system (unidirectional fiber reinforced materials). In this work the concentrated load has to be applied at a point on the cross-sectional area or on the surface area of the element and the nodes are located and add load components.

٤-٣ The Stiffness Matrix Formulation For Unidirectional Fiber-Reinforced Materials :

The previous section has cited that the thermal strain was neglected therefore the total strain is equal to elastic strain and Eqn.(٤-١٠) becomes:

$$\{\varepsilon\} = \{e\} = [B].\{\phi^{(e)}\} \quad (٤-٢٨)$$

In the total strains {e} the strain components and the displacement are related. These relationships are called the strain-displacement equations and can be derived in plane stress as:

$$e_{11} = \frac{\partial u_1}{\partial x_1} = \varepsilon_{11} \ , \quad e_{22} = \frac{\partial u_2}{\partial x_2} = \varepsilon_{22} \ , \quad e_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \quad (٤-٢٩)$$

In the present work it has been considered that the material principal co-ordinates ١, ٢, and ٣ are corresponding to the Cartesian co-ordinates x١ , x٢ , and x٣ respectively. In chapter three the strain displacement equations for UDFRM are derived from Eqns.(٣-٥٠) and (٣-٥١). Therefore, the component of overall symmetric strain tensor and components of the relative strains can be written as:-

$$\varepsilon_{11}=\frac{\partial u_1}{\partial x_1} \quad , \quad \varepsilon_{22}=\frac{\partial u_2}{\partial x_2} \quad , \quad 2\varepsilon_{12}=\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}=2\varepsilon_{21} \qquad (٤\text{-}٣٠)$$

$$\gamma_{21}=\frac{\partial u_1}{\partial x_2}-\psi_{21} \quad ; \quad \gamma_{22}=\frac{\partial u_2}{\partial x_2}-\psi_{22} \qquad (٤\text{-}٣١)$$

and the couple strain $\chi_{121}=\frac{\partial \psi_{21}}{\partial x_1}$ (٤-٣٢)

Within the matrix the normal strains in the x١ directions are assumed identical to those in the fibers because of the perfect bond between fiber and matrix. The local shear deformation ϕ_3 that denoted by $(\phi_{3i}, \phi_{3j}, \text{and}\, \phi_{3k})$ at each node as shown in Fig.(٤-٢) is equal to ψ_{21}, so by using an expansion of the displacements $\phi_1, \phi_2, \text{and}\, \phi_3$ the strain matrix [B] can be derived as:-

$$\left.\begin{aligned}\phi_1 &= c_1 + c_2 x_1 + c_3 x_2 \\ \phi_2 &= c_4 + c_5 x_1 + c_6 x_2 \\ \phi_3 &= c_7 + c_8 x_1 + c_9 x_2 = \psi_{21}\end{aligned}\right\} \qquad (٤\text{-}٣٣)$$

where c١ to c٩ are constants. So the element strain-displacement relations are given from Eqn.(٤-٢٨) as:-

$$\{\varepsilon^{(e)}\}=\left[B^{(e)}\right]\{\phi^{(e)}\} \qquad (٤\text{-}٣٤)$$

and the matrix [B] can be obtained by differentiating the displacement equations for ϕ with respect to x١. From Eqn.(٤-٧) $\frac{\partial \phi_1}{\partial x_1}$ is the derivative of shape functions (N١ , N٢ , and N٣) with respect to x١ in the first row and in the second row $\frac{\partial \phi_2}{\partial x_2}$ is the derivative of shape functions (N١ , N٢ , and N٣) with respect to x٢ and so on. So the Eqn.(٤-٣٤) in UDFRM becomes:-

$$\left\{\varepsilon^{(e)}\right\}=\begin{bmatrix} \frac{\partial N_1}{\partial x_1} & 0 & 0 & \frac{\partial N_2}{\partial x_1} & 0 & 0 & \frac{\partial N_3}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial N_1}{\partial x_2} & 0 & 0 & \frac{\partial N_2}{\partial x_2} & 0 & 0 & \frac{\partial N_3}{\partial x_2} & 0 \\ \frac{\partial N_1}{\partial x_2} & \frac{\partial N_1}{\partial x_1} & 0 & \frac{\partial N_2}{\partial x_2} & \frac{\partial N_2}{\partial x_1} & 0 & \frac{\partial N_3}{\partial x_2} & \frac{\partial N_3}{\partial x_1} & 0 \\ \frac{\partial N_1}{\partial x_2} & 0 & N_1 & \frac{\partial N_2}{\partial x_2} & 0 & N_2 & \frac{\partial N_3}{\partial x_2} & 0 & N_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial N_1}{\partial x_1} & 0 & 0 & \frac{\partial N_2}{\partial x_1} & 0 & 0 & \frac{\partial N_3}{\partial x_1} \end{bmatrix} \begin{bmatrix} \phi_{3i-2} \\ \phi_{3i-1} \\ \phi_{3i} \\ \phi_{3j-2} \\ \phi_{3j-1} \\ \phi_{3j} \\ \phi_{3k-2} \\ \phi_{3k-1} \\ \phi_{3k} \end{bmatrix}$$

(٣-٣٥)

In the above equation the relative strains (γ_{22}) are equal to zero and this can be interpreted by the Eqn.(٣-٤٩) in previous chapter which equal to:-

$$U_2 = \overline{U}_2$$

Thus from Eqn.(٤-٣١) :

$$\gamma_{22} = \frac{\partial u_2}{\partial x_2} - \psi_{22} = \frac{\partial u_2}{\partial x_2} - \frac{\partial \overline{u}_2}{\partial x_2} = \frac{\partial u_2}{\partial x_2} - \frac{\partial u_2}{\partial x_2} = 0$$

The Eqn.(٤-٣٥) may be used to evaluate the stiffness matrix from Eqn.(٤-٢٤) as follows:

$$[K] = [B]^T [D] [B]\ t\ .\ A \qquad (٤\text{-}٣٦)$$

where:

$$[B]=\begin{bmatrix} \frac{\partial N_1}{\partial x_1} & 0 & 0 & \frac{\partial N_2}{\partial x_1} & 0 & 0 & \frac{\partial N_3}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial N_1}{\partial x_2} & 0 & 0 & \frac{\partial N_2}{\partial x_2} & 0 & 0 & \frac{\partial N_3}{\partial x_2} & 0 \\ \frac{\partial N_1}{\partial x_2} & \frac{\partial N_1}{\partial x_1} & 0 & \frac{\partial N_2}{\partial x_2} & \frac{\partial N_2}{\partial x_1} & 0 & \frac{\partial N_3}{\partial x_2} & \frac{\partial N_3}{\partial x_1} & 0 \\ \frac{\partial N_1}{\partial x_2} & 0 & N_1 & \frac{\partial N_2}{\partial x_2} & 0 & N_2 & \frac{\partial N_3}{\partial x_2} & 0 & N_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial N_1}{\partial x_1} & 0 & 0 & \frac{\partial N_2}{\partial x_1} & 0 & 0 & \frac{\partial N_3}{\partial x_1} \end{bmatrix}$$

(٤-٣٧)

[D] is the elasticity matrix for UDFRM given in Eqn.(٣-٦٣) in previous chapter. The term **t** and **A** in Eqn.(٤-٣٦) are the thickness of composite and area of the element respectively.

٤-٤ Introduction to The Main Computer Program:-

The computer program has been developed to study the behaviour of UDFRM based on the effects of fiber size, elastic modulus ratio between fiber and matrix, and the Poisson's ratio of matrix and fiber. These studies are based on the theoretical analysis presented in chapter three.

This computer program, which is called (SAOUDL) (Stress Analysis of Unidirectional Laminate) consists of a main program and four subroutines written in Quick Basic language.

It stores the coefficients of $\{\phi\}$, $\{f\}$ and [K] in single vector $\{A\}$ denoted by A( ) in the program. When using vector storage the nodal values $\{\phi\}$ are located at the top followed by $\{f\}$ and then columns of [K].

This program has many facilities which can be written as:

- It is used to predict fiber-matrix shear stress at a high and low fiber size, and also to predict the local shear deformation, (ϕ_3).
- It can be used to predict the behaviour of composite comprising by reinforcing layers with the same materials (homogeneous orthotropic materials) or different materials (hybrid materials).
- It can be used for UDFRM under different loads such as tension, pure shear, etc…).
- This program can be used to determine the total stress that act on the composite with good acceptable accuracy.

The main program steps are outlined in the flowchart shown in Fig.(٤-٣), and its subroutines are described in the following section.

٤-٥ Description of Subroutines:-

The computer program (SAOUDL) has four subroutines which are described as follows:-

٤-٥-١ Subroutine EISTMX

This subroutine evaluates the element stiffness matrix $[K^{(e)}]$ from Eqn.(٤-٢٤). In this equation, the strain matrix [B] is evaluated by using Eqn.(٤-٣٧) in the loop that calculates the stress components in each element. Also this subroutine evaluates the element force vector from Eqn.(٤-٢٦). The element matrices can be printed to allow the users to check by a hand calculation.

٤-٥-٢ Subroutine MODIFY

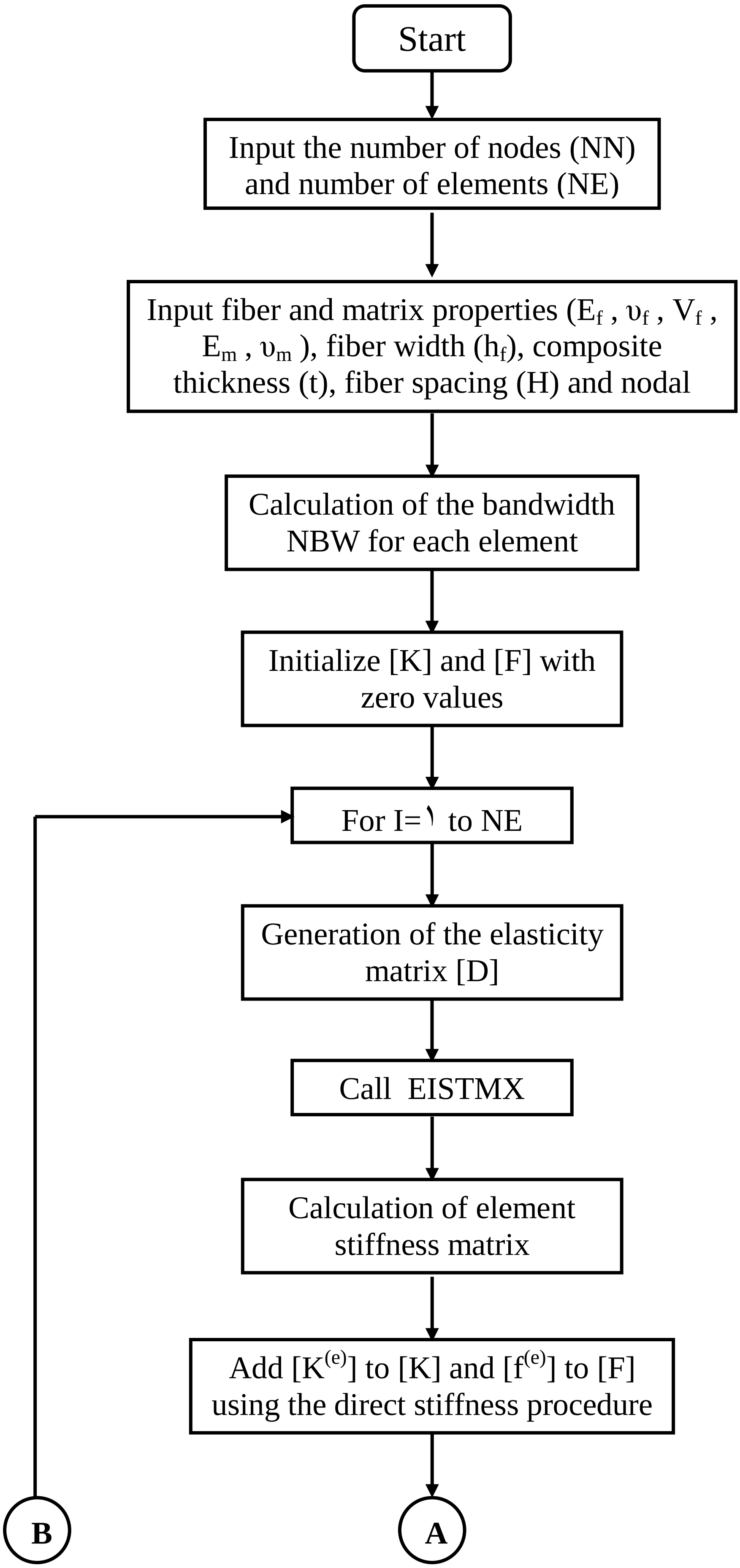
It incorporates the specified nodal values into the system of equations using the method of deletion of rows and columns (known values of displacement $\{\phi\}$).

٤-٥-٣ Subroutines DCMPBD:

It decomposes the global stiffness matrix [K] into an upper triangular form using the method of Gaussian elimination. This subroutine assumes that [K] is symmetric and only those elements within the band-width and on or above the main diagonal are stored. The programming logic is not easy to follow because the coefficients of [K] are stored in a vector rather than in a two-dimensional array.

٤-٥-٤ Subroutines SLVBD

It is a companion program to DCMPBD , which decomposes the global force vector, {f}, and solves the system of equations by using back substitution. The solution of the system of equations is separated into two subroutines so that they can be used to solve time-dependent problem.


Start
Input the number of nodes (NN) and number of elements (NE)
Input fiber and matrix properties (Ef , υf , Vf , Em , υm), fiber width (hf), composite thickness (t), fiber spacing (H) and nodal
Calculation of the bandwidth NBW for each element
Initialize [K] and [F] with zero values
For I=١ to NE
Generation of the elasticity matrix [D]
Call EISTMX
Calculation of element stiffness matrix
Add [K(e)] to [K] and [f(e)] to [F] using the direct stiffness procedure
B
A

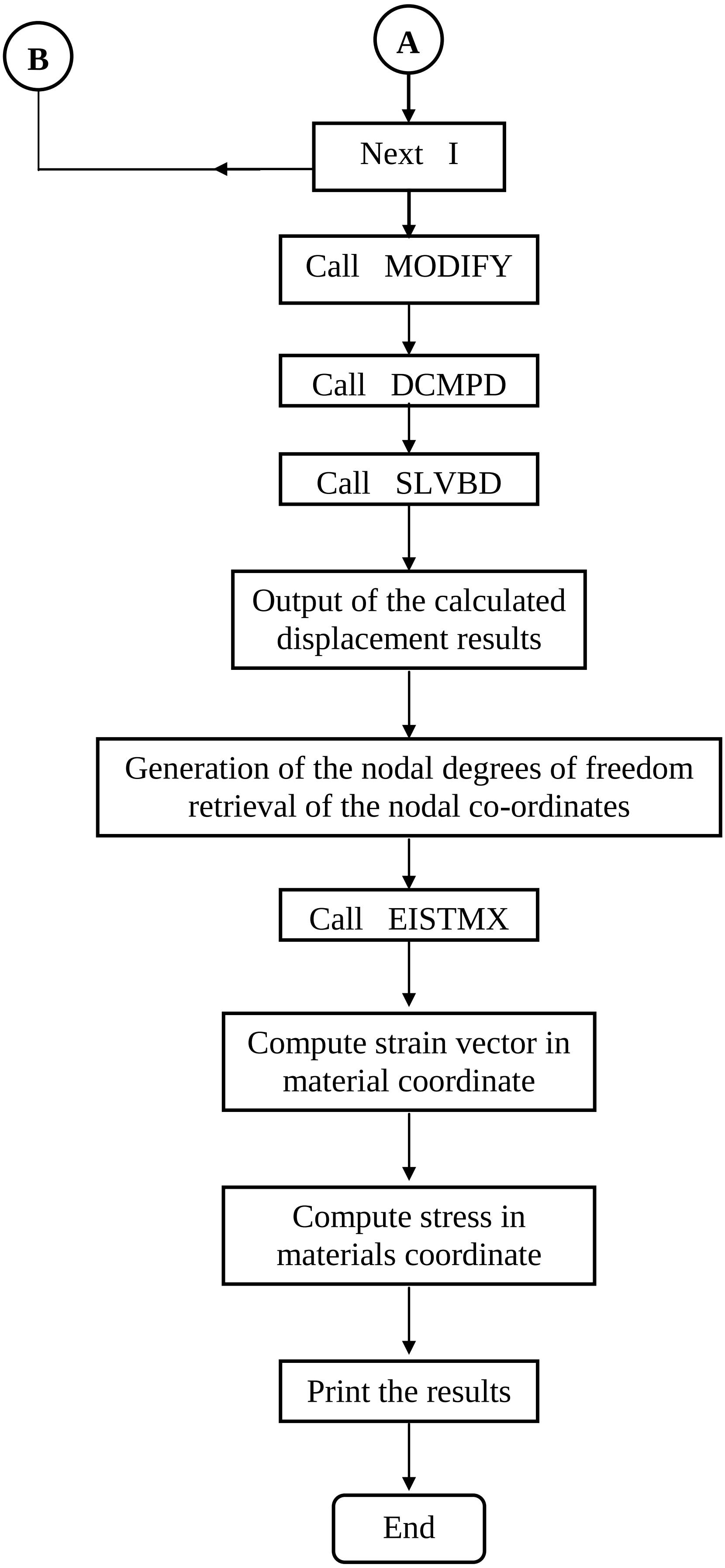


Fig.(٤-٥): Flow chart of the program SAOUDL

Chapter Five

Results and Discussion

٥-١ General: -

This chapter discussed the results obtained by using finite element computer program [SAOUDL] (Stress Analysis of Unidirectional Laminates), and the stress-strain relation for unidirectional fiber-reinforced materials, which was derived in chapter three. Due to the lack of researches that studied the micromechanical behaviour of UDFRM especially by microstructure theory, the present work has been confirmed its results with exact solutions results.

The numerical computation of the present work can be divided into four sections according to the type of loading and supports. The first two sections have the same loading and supports but consider different composite that is homogeneous orthotropic materials as in the first and UDFRM in the second. The last two sections have different loading and supports but the same materials as in section ٥-٣ (that is UDFRM). The following parameters fiber size, elastic modulus ratio between fiber and matrix (E_f / E_m) and the fiber and matrix Poisson's ratio will be discussed in the last three sections when the fiber volume fraction is constant (V_f = ٠.٢٥).

٥-٢ Pure shear and homogeneous orthotropic materials (HOM) :

This section can be considered as a general case to be applied on the finite element computer program (SAOUDL) by the model shown in [Fig.(٥-١-c)] with ٢١ nodes and ٢٤ elements. Thus by assuming HOM instead of UDFRM, the plate in pure shear under plane stress can be used in this section and the next section. An exact solution to this problem may be obtained by the use of the two equilibrium equations [٣-٦٥ b and c] and the constitutive relations:

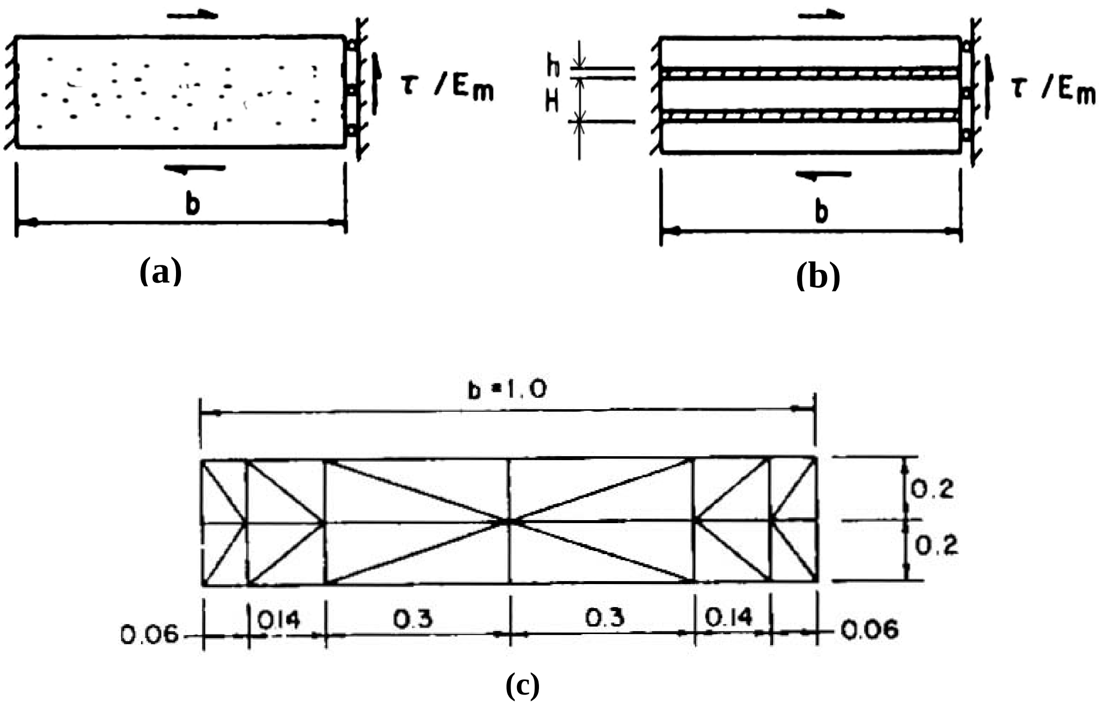


Fig.(٥-١): Plate in pure shear under plane stress, **(a)** Homogeneous orthotropic materials and, **(b)** Unidirectional fiber reinforced materials, **(c)** Arrangement of finite elements.

$$\tau_{12} = d_3\, 2\varepsilon_{12} + d_5 \gamma_{21} = \tau \qquad (٥\text{-}١)$$

$$\sigma_{21} = d_5\, 2\varepsilon_{12} + d_4 \gamma_{21} \qquad (٥\text{-}٢)$$

$$\mu_{121} = d_7\, \chi_{121} \qquad (٥\text{-}٣)$$

where the strain measures are defined by Eqns.(٣-٥٠) and (٣-٥١) in conjunction with the kinematic restraints:-

$\phi_1 = 0$ for all x١ and x٢ values (٥-٤)

$\phi_2 = 0$ at x١ = ٠ (٥-٥)

$\phi_3 = 0$ at x١ = ٠ (٥-٦)

$\phi_3 = 0$ at x١ = b (٥-٧)

$\phi_2 = \phi_2(x_1)$ (٥-٨)

Solutions for the displacements and stresses may be shown to have the following forms:-

$$\phi_2 = \left(x_1 - x_1 \frac{d_5 C}{k^2} + cd_5 \left(B\left(\cosh(k x_1) - 1\right) + \sinh(k x_1) / k^3\right)\right) \Big/ d_3 \qquad (٥\text{-}٩)$$

$$\phi_3 = c\left(B\sinh\,(k x_1) + \cosh(kx_1) - 1\right) / k^2 \qquad (٥\text{-}١٠)$$

$$\chi_{121} = d_7\; c\left(B\cosh(k x_1) + \sinh(kx_1)\right)/k \qquad (٥\text{-}١١)$$

$$\sigma_{21} = -d_7\, c\left(B\sinh(k x_1) + \cosh(kx_1)\right) \qquad (٥\text{-}١٢)$$

where

$$c = -d_5 / (d_3 * d_7) \qquad (٥\text{-}١٣)$$

$$k^2 = \left(d_4 - d_5^2 / d_3\right) / d_7 \qquad (٥\text{-}١٤)$$

The results which obtained from Eqns.(٥-٩) to (٥-١٢) would be compared with the results of finite element computer program (SAOUDL) for each value of x١ .

Fig.(٥-١-a) shows the plate composed from HOM, Fig.(٥-١-b) shows the plate composed from UDFRM. And (٥-١-c) shows the arrangement of finite element. The stress-strain relations of HOM are given by Eqns. (٣-٢٤) and (٣-٢٦) in chapter three. The engineering constants (E١, E٢, υ٢١, υ١٢, and G١٢) can be calculated by using first contiguity approach with contiguity factor C = ٠ and fiber misalignment factor k =١ (these had been shown in Eqns.(٣-٣٧) to (٣-٤١)), and second Halpin-Tsai equations with a measure of fiber reinforcement ζ =١ (These equations had been written as Eqns.(٣-٤٢) to (٣-٤٤)). Both cases are assumed to be ideal, this means when C = ٠ then all fibers are isolated but when C = ١ all fibers are contiguous. These equations and the model shown in Fig.(٥-١-c), are applied on the program (SAOUDL). Its results are compared well with the exact solution results for vertical displacement "ϕ_2" in Eqn.(٥-٩) of the same plate, but composed from UDFRM as shown in Fig.(٥-١-b). Table (٥-١) shows the exact solution results and program (SAOUDL) results of ϕ_2 for this case.

Table (٥-١): The exact solution results of ϕ_2 for plate consisted for UDFRM and the program results (SAOUDL) for plate consisted from HOM, by contiguity approach and Halpin-Tsai equations.

Values of x١	**Exact solution results from Eqn.(B-٩)**	**F.E. results by contiguity approach**	**F.E. results by Halpin-Tsai equation**

٠	٠	٠	٠
٠.٠٦	٠.٠٣١	٠.٠٨٧٣	٠.٠٨٢
٠.٢	٠.٣٨٥	٠.٣	٠.٢٩١
٠.٥	٠.٧٣٢	٠.٧٢٨	٠.٧٢٧
٠.٨	١.٢٥	١.٢٣	١.١٦٤
٠.٩٤	١.٤٢٧	١.٣٦٨	١.٤
١.٠	١.٤٥٣	١.٤٥٦	١.٤٥٥

From Table (٥-١) and Fig.(٥-٢) and (٥-٣) its clear that the vertical displacement ϕ_2 has a good agreement with the exact solution results. This can be considered as a strong confirmation to the computer program (SAOUDL). In Fig.(٥-٢) the contiguity factor C = ٠ as cited in the beginning of this chapter. Also Fig.(٥-٤) and (٥-٥) show the effect of "C" on the ϕ_2 at constant fiber volume fraction. From these figures it is clear that whenever the contiguity factor approaches the value of one the results of ϕ_2 would be far from the exact solution results. These results are very acceptable because in UDFRM the fibers are not contact each other but isolated by matrix.

٥-٣ Pure Shear and Unidirectional Fiber-Reinforced Materials:-

In this section the computer program (SAOUDL) is applied on the model shown in Fig.(٥-١-c) at section (٥-٢), using the stress-strain equations of UDFRM, i.e. Eqns.(٣-٦٣). The results obtained from the computer program (SAOUDL) are compared well with the exact solutions results which has been done on the same loading on plate (pure shear) at section (٥-٢) and according to the following affecting parameters.

٥-٣-١ Effect of Fiber Size:-

The effect of fiber size on the ϕ_2, local shear deformation ϕ_3, couple stress μ_{121}, and fiber-matrix shear stress $(\sigma_{21}+\tau_{12})$ can be studied by assuming a constant volume fraction ($h_f/H = V_f =$ ٠.٢٥) with a different values of fiber size (this can be done by taking two values of fiber size; low when H/b = ٠.٢ so $h_f =$ ٠.٠٥, and high when H/b = ١ so $h_f =$ ٠.٢٥). Also a constant elastic modulus ratio between fiber and matrix

($E_f/E_m =$ ١٠٠), and a constant Poisson's ratio of fiber and matrix $(\nu_f = \nu_m = 0.2)$ will be assumed.

Concisely, Fig.(٥-٦) and (٥-٧) show that the increment of fiber size (fiber width), causes a reduction in vertical displacement $(\phi_2)$and the local shear deformation ϕ_3.

Thus, when the fiber size is very small, the ϕ_2 approaches from HOM, briefly at low (H/b) ratios these deformations both approached from values for conventional homogeneous materials. Fig.(٥-٦) also shows that ϕ_2 has a maximum value at the end of plate and zero at the start of it whereas ϕ_3 has zero value at both ends and maximum value at the middle of plate, this can be explained to the boundary condition (B. C.) given at section (٥-٢). Fig.(٥-٨) shows the effect of fiber size on couple stress μ_{121}, obviously the increment of fiber size cause increment of couple stress μ_{121} this can be ascribed to the Eqns.(٣-٦٣) and specially to the factor d٧ in equation of couple stress.

The shear stress between fiber and matrix has been represented by the combination of shear stress τ_{12} and relative stress σ_{21}. Fig.(٥-٩) shows that the fiber-matrix shear stress (FMSS) tends to be a minimum. at the restrained ends and a maximum at the center of the plate. So the fiber size has an important effect in reducing this inter-component shear which is often a mode of weakness and potential failure at regions of stress concentration. Both results of vertical displacement ϕ_2, local shear deformation ϕ_3, couple stress, and FMSS, which are plotted in Figs.(٥-٦), (٥-٧), (٥-٨), and (٥-٩) respectively, are all show a good agreement with exact solution results, excepting the results of FMSS $(\tau_{12}+\sigma_{21})$ at low value of fiber size (H/b = ٠.٢). The reason for this discrepancy is found in the very high gradient of the relative stress, σ_{21}, in the region of the end restraint which the constant strain elements were unable to simulate. According to the equilibrium Eqn.(٣-٦٥-c) in chapter three, the relative stress is proportional to the second derivative of the local rotation, ϕ_3 , and a finite element arrangement approximation should, therefore, be able to model this curvature.

٥-٣-٢ Effect of Elastic Modulus Ratio (E_f / E_m):

The effect of elastic modulus ratio between fiber and matrix (E_f / E_m), can be studied in the two case of constant fiber size first high value and second low value. Three values of the elastic modulus ratio (E_f / E_m) were assumed (E_f / E_m = ١, ١٠, and ١٠٠). Again this effect can be done by assuming constant fiber volume fraction (V_f = ٠.٢٥), and constant fiber and matrix Poisson's ratio $(\nu_f = \nu_m = 0.2)$, Tab.(٥-٢) and (٥-٣) show these assumptions.

Table (٥-٢): Assumptions have been taken to study the effect of elastic modulus ratio at low value of fiber size (H/b = ٠.٢).

No. of case	E_f / E_m	V_f	V_m	ν_f	ν_m
١	١	٠.٢٥	٠.٧٥	٠.٢	٠.٢
٢	١٠	٠.٢٥	٠.٧٥	٠.٢	٠.٢
٣	١٠٠	٠.٢٥	٠.٧٥	٠.٢	٠.٢

Table (٥-٣): Assumptions have been taken to study the effect of elastic modulus ratio at high value of fiber size (H/b = ١.٠).

No. of case	E_f / E_m	V_f	V_m	ν_f	ν_m
١	١	٠.٢٥	٠.٧٥	٠.٢	٠.٢
٢	١٠	٠.٢٥	٠.٧٥	٠.٢	٠.٢
٣	١٠٠	٠.٢٥	٠.٧٥	٠.٢	٠.٢

Applying the assumptions in table (٥-٢) on the computer program (SAOUDL) the effect of E_f / E_m on the ϕ_2 and ϕ_3 are plotted in Fig. (٥-١٠) and (٥-١١) respectively. From these figures, its obvious that the computer program (SAOUDL) results for ϕ_2 and ϕ_3 have a good agreement with exact solution results. Also, whenever the elastic modulus ratio decreases, the ϕ_2 and ϕ_3 approach from HOM specially when E_f / E_m = ١ the ϕ_3 has zero values at every position of plate.

Briefly the results that are plotted in Fig.(٥-١٠), (٥-١١), (٥-١٤), and (٥-١٥) are calculated at constant low fiber size as shown in table (٥-٢) whereas the results in Fig.(٥-١٢), (٥-١٣), (٥-١٦), and (٥-١٧) are calculated at constant high fiber size as shown in Table (٥-٣). In Fig.(٥-١٠), (٥-١١), (٥-١٢) and (٥-١٣) the increment of E_f / E_m caused reduction in both ϕ_2 and ϕ_3 and this reduction is increased when the fiber size increased because when the elastic modulus of fiber are higher than that of matrix this will give a good properties of UDFRM given when fiber size is low.

Fig.(٥-١٤), (٥-١٦), and (٥-١٥), (٥-١٧) show the effect of E_f / E_m on couple stress μ_{121} and FMSS $(\sigma_{21} + \tau_{12})$respectively. From these figures, it is clear that the increment in E_f / E_m caused a little reduction in couple stress and FMSS when fiber size has a low value. But at a higher fiber size the increment in E_f / E_m caused increment in FMSS and decrement first and then decrement of couple stress. Also when E_f / E_m =١ the couple stress equals zero and FMSS equals one and this behaviour is corresponded to the behaviour of HOM.

All the results that obtained from computer program (SAOUDL) have a good agreement with exact solutions results for the effect of E_f / E_m on ϕ_2, ϕ_3, μ_{121}, and $(\sigma_{21} + \tau_{12})$, except the FMSS have some values different from the exact solution results and the reason is the same as that cited in the last section.

٥-٣-٣ Effect of Matrix and Fiber Poisson's Ratio:-

The effect of matrix Poisson's ratio (ν_m) or fiber Poisson's ratio (ν_f) on ϕ_2, ϕ_3, μ_{121}, and $(\sigma_{21}+\tau_{12})$, can be studied by changing (ν_m) or ν_f and fixed the other parameters (see tables (٥-٤) and (٥-٥)).

Table (٥-٤): Assumptions to study the effect of matrix Poisson's ratio (ν_m) on ϕ_2, ϕ_3, μ_{121}, and $(\sigma_{21}+\tau_{12})$ at low fiber size (H/b = ٠.٢).

No. of case	ν_m	ν_f	V_f	V_m	E_f	E_m
١	٠	٠.٢	٠.٢٥	٠.٧٥	١٠٠	١
٢	٠.٢	٠.٢	٠.٢٥	٠.٧٥	١٠٠	١
٣	٠.٣٥	٠.٢	٠.٢٥	٠.٧٥	١٠٠	١

Note:- To study the effect of fiber size and (ν_m) or (ν_f) on ϕ_2, ϕ_3, μ_{121}, and $(\sigma_{21}+\tau_{12})$, the elastic modulus ratio E_f/E_m has been taken equal to ١٠٠ because this value gives a good properties for UDFRM.

Table (٥-٤): Assumptions to study the effect of fiber Poisson's ratio (ν_f) on ϕ_2, ϕ_3, μ_{121}, and $(\sigma_{21}+\tau_{12})$ at low fiber size (H/b = ٠.٢).

No. of case	ν_f	ν_m	V_f	V_m	E_f	E_m
١	٠	٠.٢	٠.٢٥	٠.٧٥	١٠٠	١
٢	٠.٢	٠.٢	٠.٢٥	٠.٧٥	١٠٠	١
٣	٠.٣٥	٠.٢	٠.٢٥	٠.٧٥	١٠٠	١

By applying table (٥-٤) and (٥-٥) in computer program (SAOUDL), its results and exact solution results are plotted in Fig.(٥-

١٨), (٥-١٩), (٥-٢٠), (٥-٢١) and (٥-٢٢), (٥-٢٣), (٥-٢٤), (٥-٢٥) respectively.

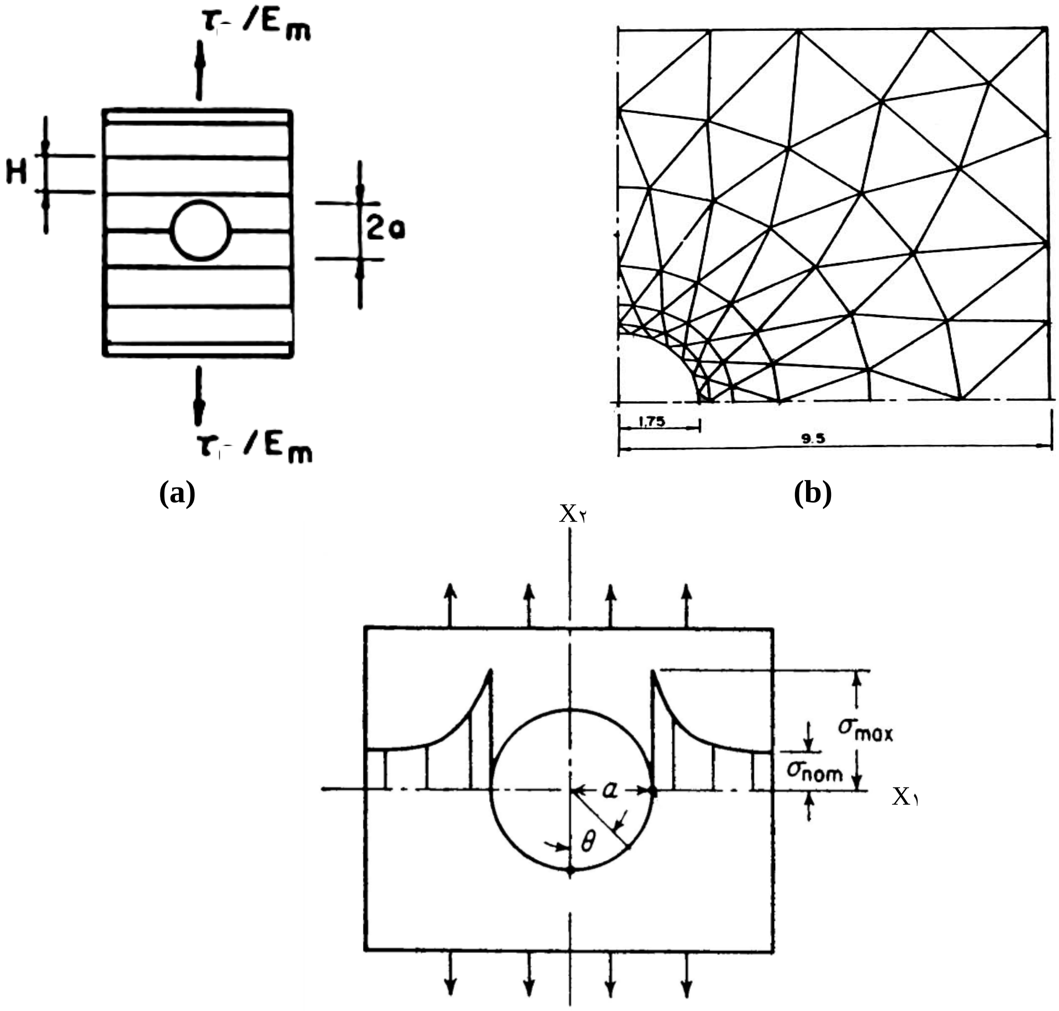
Figs.(٥-١٨) and (٥-١٩), show that the increment in ν_m will cause increment in ϕ_2 and ϕ_3, likewise this effect of ν_m on ϕ_2 and ϕ_3, is greater than the effect of ν_f on them as shown in Fig.(٥-٢٢) and (٥-٢٣).

While the increment in ν_m causes a very little sensitive increment in μ_{121} and FMSS as shown in Fig.(٥-٢٠) and (٥-٢١) respectively. Inspite of the small effect of ν_m on μ_{121} and FMSS this effect is very sensitive when compared with effect of ν_f on μ_{121} and FMSS which are shown in Fig.(٥-٢٤) and (٥-٢٥).

From all above figures the computer program (SAOUDL) results are compared well with exact solution results in case to study the effect of ν_m or ν_f on ϕ_2, ϕ_3, μ_{121}, and $(\sigma_{21} + \tau_{12})$.

٥-٤ Tensile Stress and Stress Distribution Around Circular Hole :

All the results obtained from computer program (SAOUDL) in the last two sections have a good agreement with exact solutions results. To more validation the numerical results in this section is the further examined accuracy of the (SAOUDL) that studies the stress distribution around a circular hole in the uniform tension field, under load normal to the fiber direction as shown in Fig.(٥-٢٦), Fig.(٥-٢٦-c) shows the stress distribution due to circular hole.



(c)

Fig.(٥-٢٦): (a) Circular hole in plate in uniform tension field

(b) Arrangement of finite elements. **(c)** Stress distribution around circular hole.

Stresses in the region of a circular hole are typical of the stress concentration that may occur in an otherwise uniform tension field and application of the microstructure theory of UDFRM behaviour was therefore made to this problem. The field equations for this boundary value problem have so far proved intractable and an exact solution is therefore not available. However, a finite element analysis of the problem was attempted by dividing a quarter of the field into the mesh arrangement as shown in Fig.(٥-٢٦-b) where the grid pattern was considered fine enough to reproduce all but the steepest stress and strain gradients.

The mesh in Fig.(٥-٢٦-b) is used in the computer program (SAOUDL) with the following properties [E_f/E_m =١٠٠, ν_m = ν_f =٠.٢] and constant fiber volume fraction V_f = ٠.٢٥. The results obtained by studying the effect of fiber size for two values high and low on the ϕ_3, μ_{121}, and shear stress (τ_{12}), and FMSS were compared with Malcolm [**Malcolm, D. J. ,** ١٩٧٨] results for the same case but different manner of computer program. The events from this comparison were found in Figs.(٥-٢٧), (٥-٢٨), (٥-٢٩) and (٥-٣٠), which have a good agreement with Malcolm results.

Figs.(٥-٢٧) and (٥-٢٨) show the variation of the local rotation ϕ_3 and the couple stress μ_{121}, around the edge of the hole for two different value of fiber size. The couple stress is greater for the larger fibers and peak at the center line where the gradient of ϕ_3 is also a max.. The average shear stress and the FMSS are presented in Figs.(٥-٢٩) and (٥-٣٠) respectively. This emphasises the very high gradient and curvature of this quantity near the center line of the hole which the finite element model can not aspire to simulate. Nevertheless, it is important to note that while an increase in fiber size may result in a greater max. average shear stress the FMSS in the same material can be very considerably reduced.

٥-٥ Distributed Load and Unidirectional Fiber-Reinforced Materials :

The last three sections show a good validity of the numerical computation that has been done during this work. To know the effect of types of loading and supports, the following case has been taken. Fig.(٥-٣١-a) shows this case which represents a plate has rigid supports at both ends so the local shear deformation ϕ_3 are zero in these ends. This plate composed from UDFRM and has properties resembling to the properties of the plate used in sections (٥-٣), these are (ν_m = ν_f = ٠.٢ , E_f/E_m=١٠٠) and with constant fiber volume fraction V_f = ٠.٢٥. Also it is subjected to distributed concentrated load. In the same manner which was used in section (٥-٣), the effect of fiber size, E_f/E_m , and ν_f or ν_m on ϕ_2, ϕ_3, μ_{121}, and $(\sigma_{21} + \tau_{12})$ must be studied also. Fig.(٥-٣١-b) shows the finite element arrangement, briefly the style of this case has the same style that was done in section (٥-٣) with the same assumption but different loading and supports.

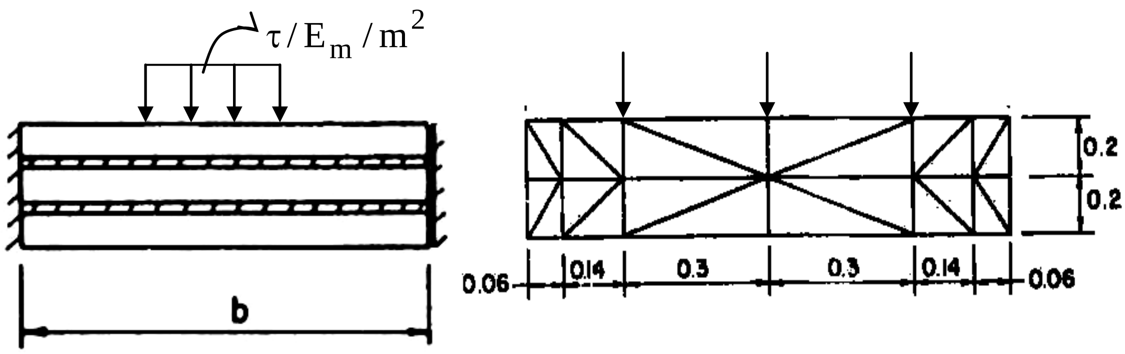


Fig.(٥-٣١): Plate under distributed load

(a) UDFRM **(b)** Finite element arrangement.

<u>٥-٥-١ Effect of Fiber Size :</u>

This effect studied by changing (H/b) and fixed the other parameters according to table (٥-٦).

Table (٥-٦): Assumptions for studying the effect of fiber size on ϕ_2, ϕ_3, μ_{121}, and $\left(\sigma_{21}+\tau_{12}\right)$.

No. of case	**H/b**	V_f	h_f	V_m	E_f	E_m	ν_f	ν_m
١	٠.٢	٠.٢٥	٠.٠٥	٠.٧٥	١٠٠	١	٠.٢	٠.٢
٢	١.٠	٠.٢٥	٠.٢٥	٠.٧٥	١٠٠	١	٠.٢	٠.٢

By applying table (٥-٦) and B.C. of this problem in computer program (SAOUDL) the results which obtained are plotted in figures. Fig.(٥-٣٢) shows that the vertical displacement ϕ_2 has a max. value in the middle of plate at the position when the max. load is effected, while local shear deformation ϕ_3 has a min. value at this position. The behaviour of ϕ_3 gives a wave profile shown in Fig.(٥-٣٣). Both deformations are reduced when fiber size increased, likewise Fig.(٥-٣٤) the effect of (H/b) on μ_{121} which has a max. value at each end and a min. at the position of the max. load is effected and this value of couple stress increased with the increment of (H/b). Also the effect of H/b on FMSS is plotted in Fig.(٥-٣٥) and this figure shows that FMSS has max. value at min value of applied load and fiber size.

The event from all these figures is the fiber size which has a large effect on ϕ_2, ϕ_3, μ_{121}, and $\left(\sigma_{21}+\tau_{12}\right)$ and this effect depended well on types of loading and supports which give B.C. of problem. This dependence can be explained according to Eqns. (٣-٦٣) in chapter three which is affected by the width of fiber.

٥-٥-٢ Effect of Elastic Modulus Ratio (E_f/E_m):

By using the assumptions, given in tables (٥-٢) and (٥-٣) and with the same manner that done in section (٥-٣), the (SAOUDL) computer program is used to study the effect of (E_f/E_m) on ϕ_2, ϕ_3, μ_{121}, and $(\sigma_{21}+\tau_{12})$. Figs.(٥-٣٦), (٥-٣٧), (٥-٣٨) and (٥-٣٩) show the effect of (E_f/E_m) at low fiber size , while Figs.(٥-٤٠), (٥-٤١), (٥-٤٢), and (٥-٤٣) show the effect at high fiber size.

Fig.(٥-٣٦) shows that the ϕ_2 increased with increasing the elastic modulus ratio (E_f/E_m) and this increment increased when fiber size has high value (H/b =١.٠) as shown in fig.(٥-٤٠). All the figures of ϕ_2 start from zero and decreased until it has a min. value at the position of max. load and then begins to increase until reaching zero value again at the end of plate because the rigid supports in both end given a zero value of ϕ_2 and ϕ_3.

From Fig.(٥-٣٧) and (٥-٤١) which show the effect of E_f/E_m on ϕ_3 its clear that ϕ_3 has a little change when E_f/E_m increases at low H/b (see Fig.(٥-٣٧)). While at high value of H/b, ϕ_3 has a sensitive change or sensitive reduction when E_f/E_m increases (see Fig.(٥-٤١)).

Thus, the types of loading and supports have a large effect on the behaviour of ϕ_3 because the changing on them causes changing in B.C. for the problem.

Fig.(٥-٣٨) and (٥-٤٢) show the effect of E_f/E_m on μ_{121} which has a large changing when E_f/E_m increases and also this changing increased at high fiber size as shown in Fig. (٥-٤٢).

Finally Fig.(٥-٣٩) and (٥-٤٣) show the effect of E_f/E_m on FMSS which is increased when E_f/E_m increased and has a max. value at the end of plate also this increment depended on fiber size thus it is higher at low fiber size than high fiber size.

٥-٣-٣ Effect of Poisson's Ratio of Fiber and Matrix :

Basing on the assumptions given in Tables (٥-٤) and (٥-٥) and with the same manner that done in Sec.(٥-٣), the effect of ν_m or ν_f on ϕ_2, ϕ_3, μ_{121}, and $(\sigma_{21}+\tau_{12})$ can be studied using the computer program (ASOUDL). So Figs. (٥-٤٤), (٥-٤٥), (٥-٤٦), and (٥-٤٧) show the effect of ν_m and Fig.(٥-٤٨), (٥-٤٩), (٥-٥٠), and (٥-٥١) show the effect of ν_f on ϕ_2, ϕ_3, μ_{121}, and $(\sigma_{21}+\tau_{12})$ respectively. These figures show that ν_m has a greater effect on ϕ_2, ϕ_3, μ_{121}, and $(\sigma_{21}+\tau_{12})$ than ν_f. So when ν_m increases the rate of decreasing of ϕ_2 increases also. Fig.(٥-٤٦) shows the effect of ν_m on μ_{121}. It can be noticed that increasing ν_m decreases the value of μ_{121}. While FMSS has a little decreasing with increasing ν_m as shown in Fig.(٥-٤٧).

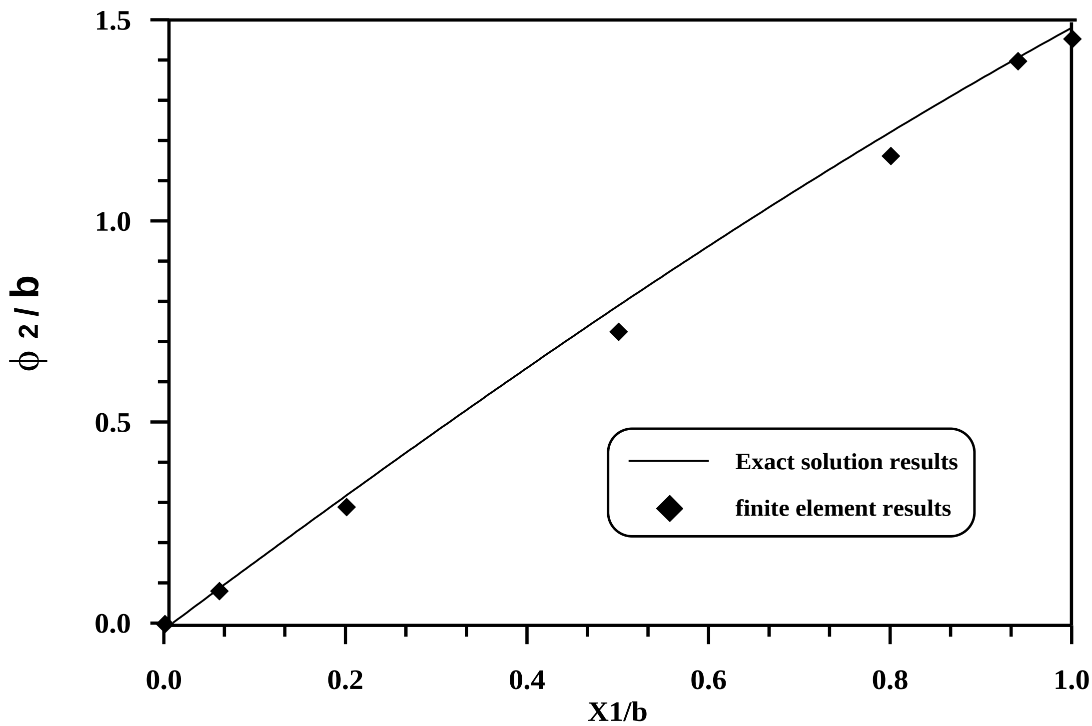


Fig.(5-2):Displacement for both finite element results of homogeneous orthotropic materiales by contiguity approach and exact solution.

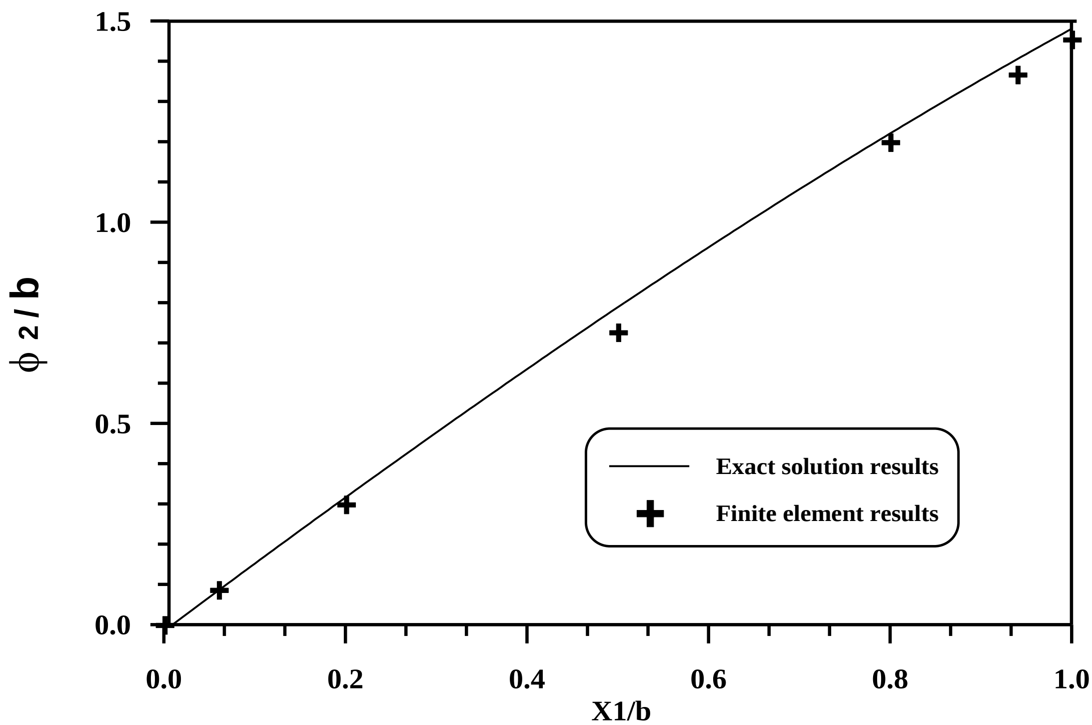


Fig.(5-3):Displacement for both finite element results in homogeneous orthotropic materia by HTS equation with (zeta=1) and exact solution.

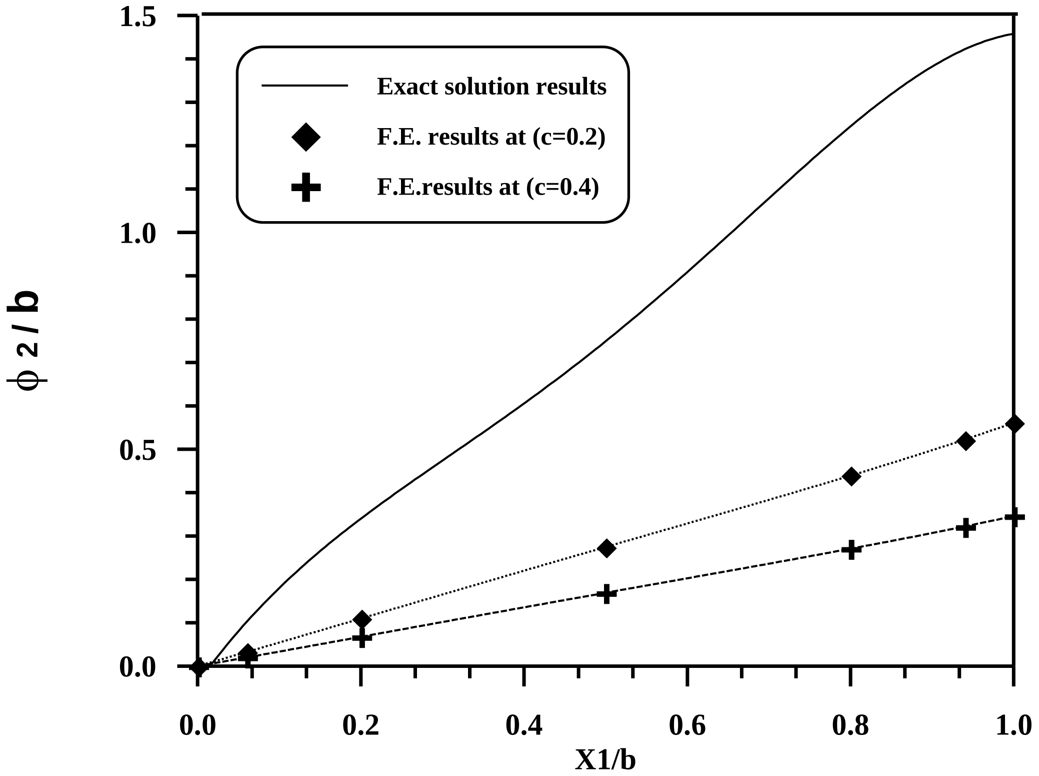


Fig.(5-4):The effect of contiguity factor on the displacement comparison with exact solution .

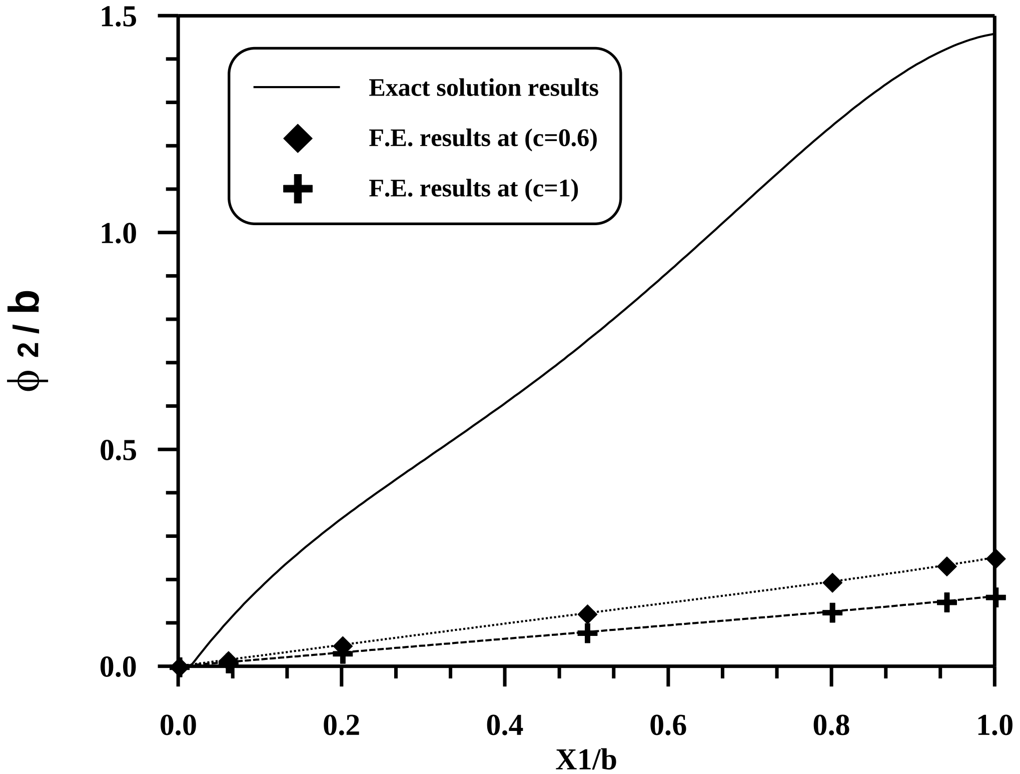


Fig.(5-5):The effect of contiguity facor on the displacement comparison with exact solution results.

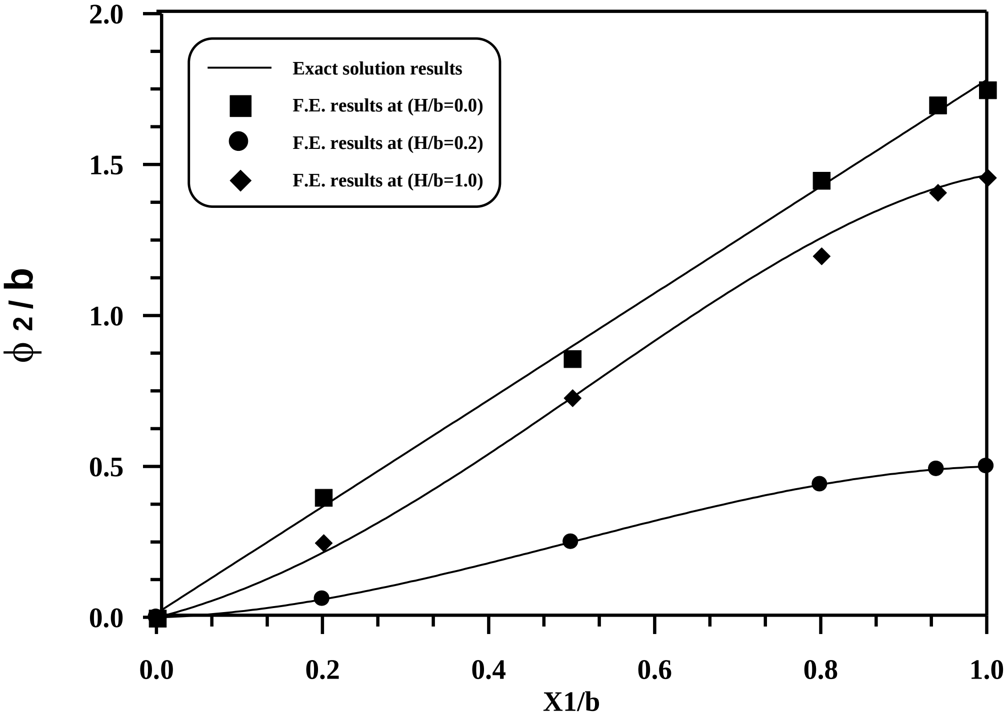


Fig.(5-6):Effect of fiber size on vertical displacement.

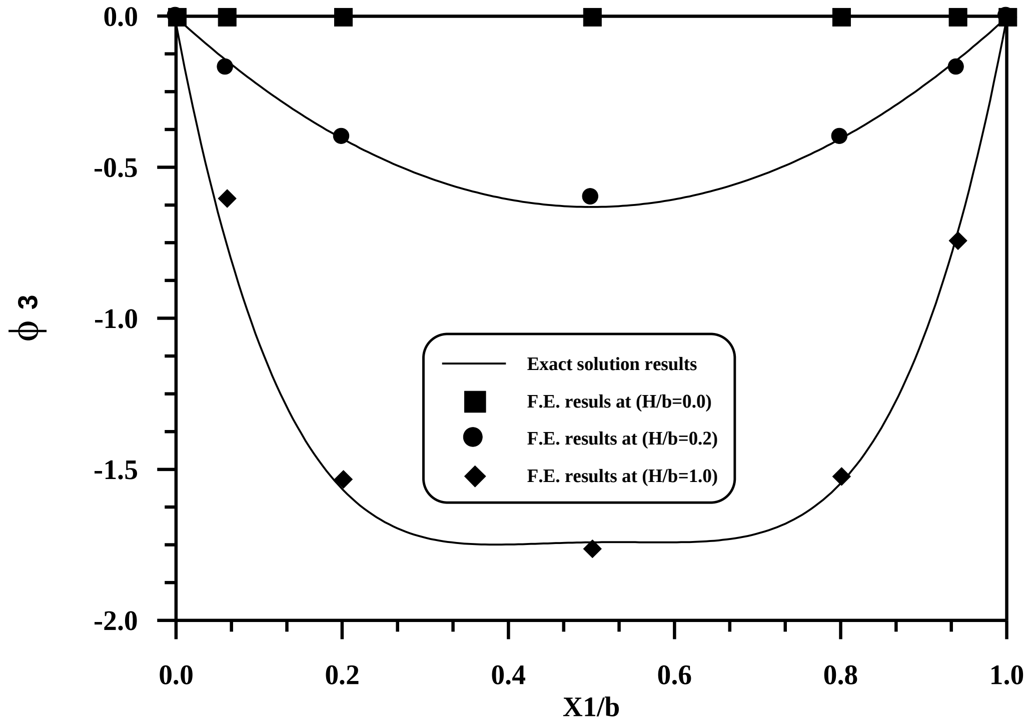


Fig.(5-7):Effect of fiber size on local shear deformation.

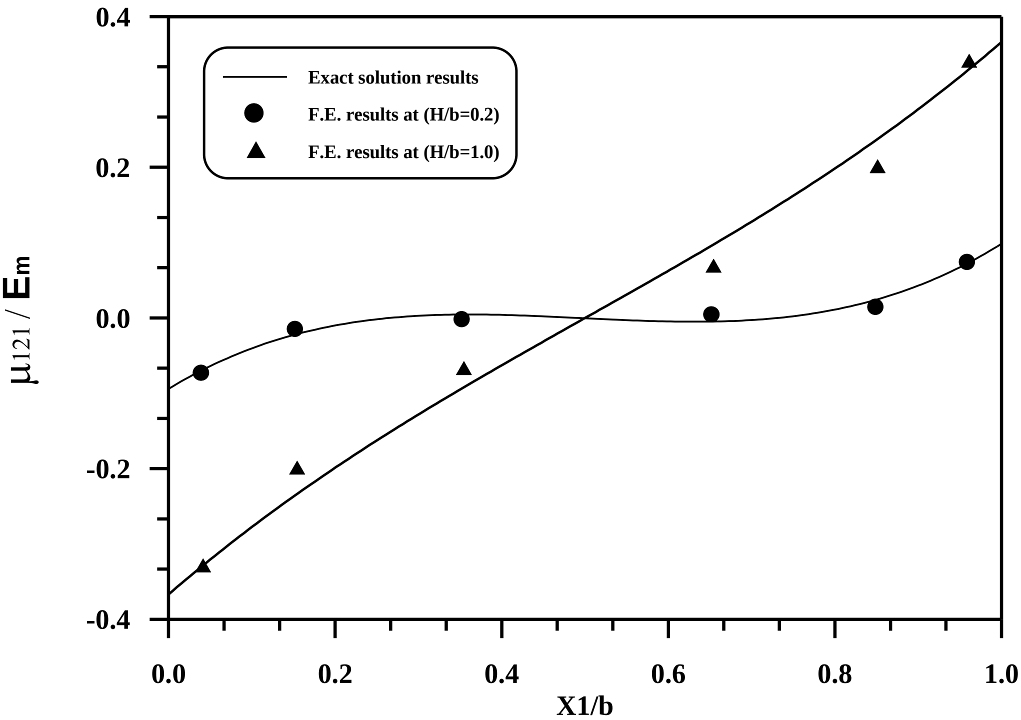


Fig.(5-8):Effect of fiber size on couple stress.

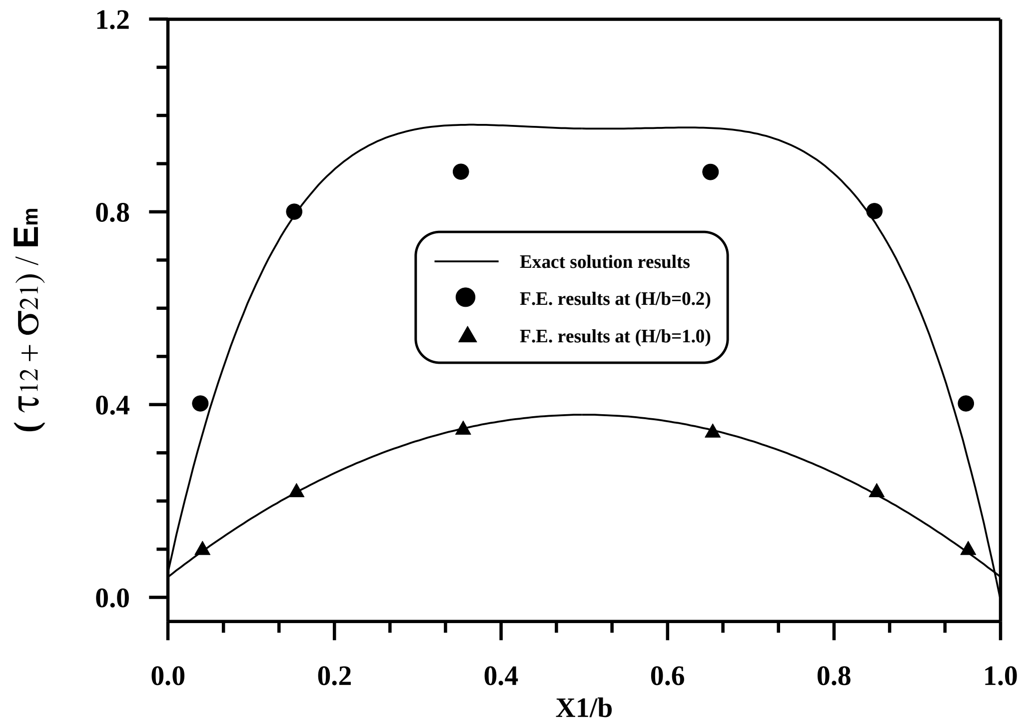


Fig.(5-9):Effect of fibre size on fibre-matrix shear stress

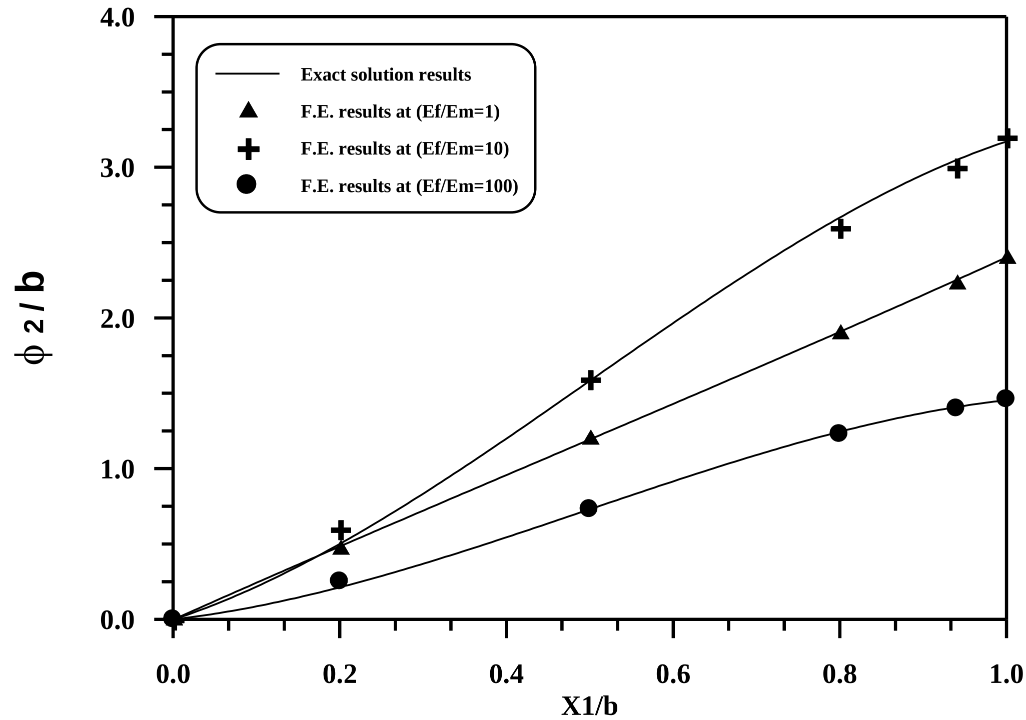


Fig.(5-10):Effect of elastic modulus ratio between fiber and matrix on vertical displacement.
at low fiber size (H/b=٠.٢) .

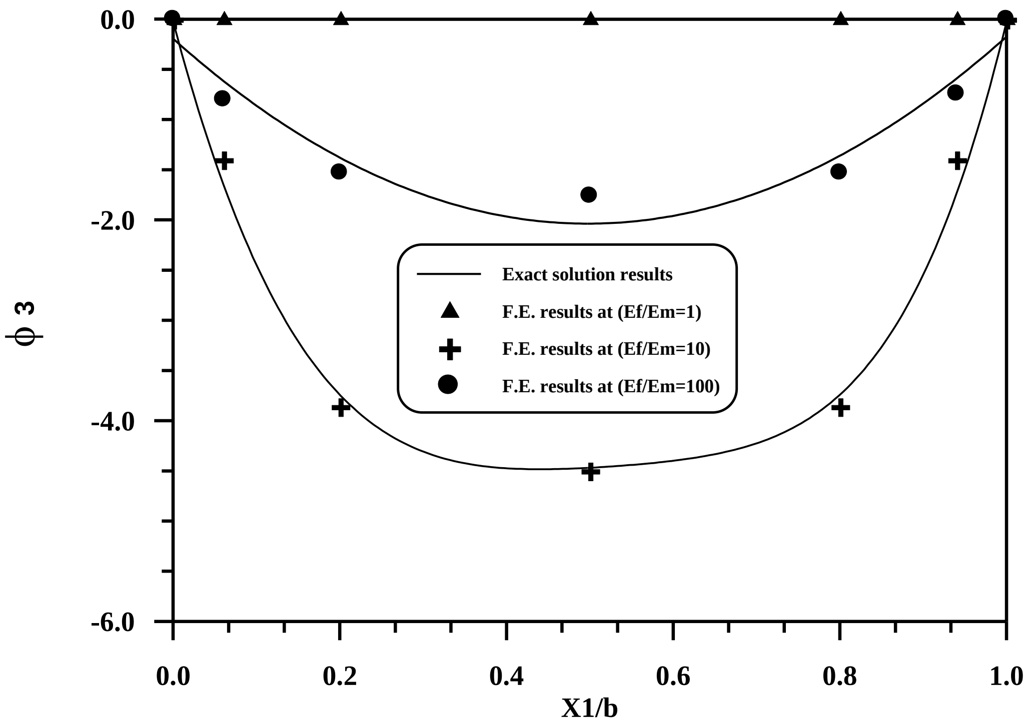


Fig.(5-11):Effect of elastic modulus ratio between fiber and matrix on local shear deformation .
at low fiber size (H/b=٠.٢)

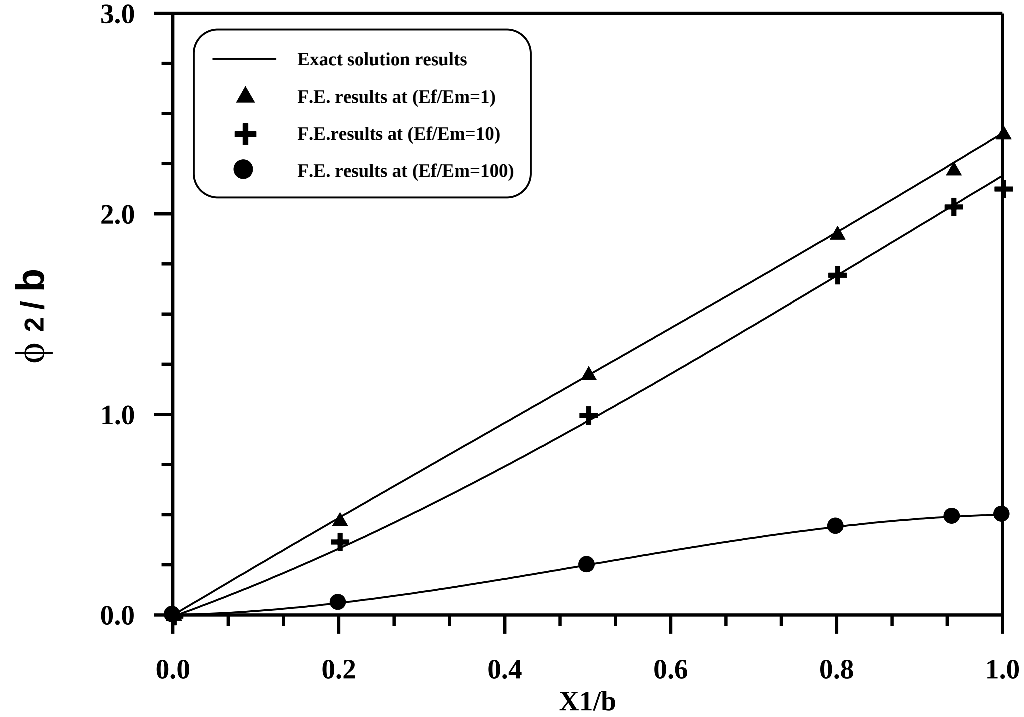


Fig.(5-12):Effect of elastic modulus ratio between fiber and matrix on vertical displacement . at high fiber size (H/b=٠.٢) .

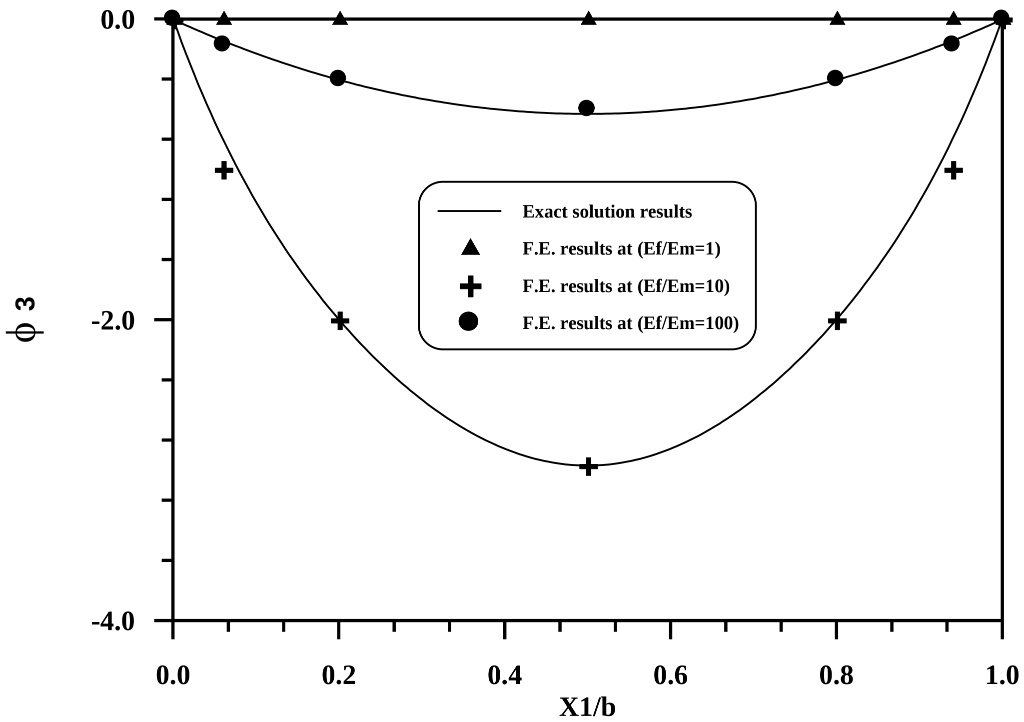


Fig.(5-13):Effect of elastic modulus ratio between fiber and matrix on local deformation at high fiber size (H/b=٠.٢) .

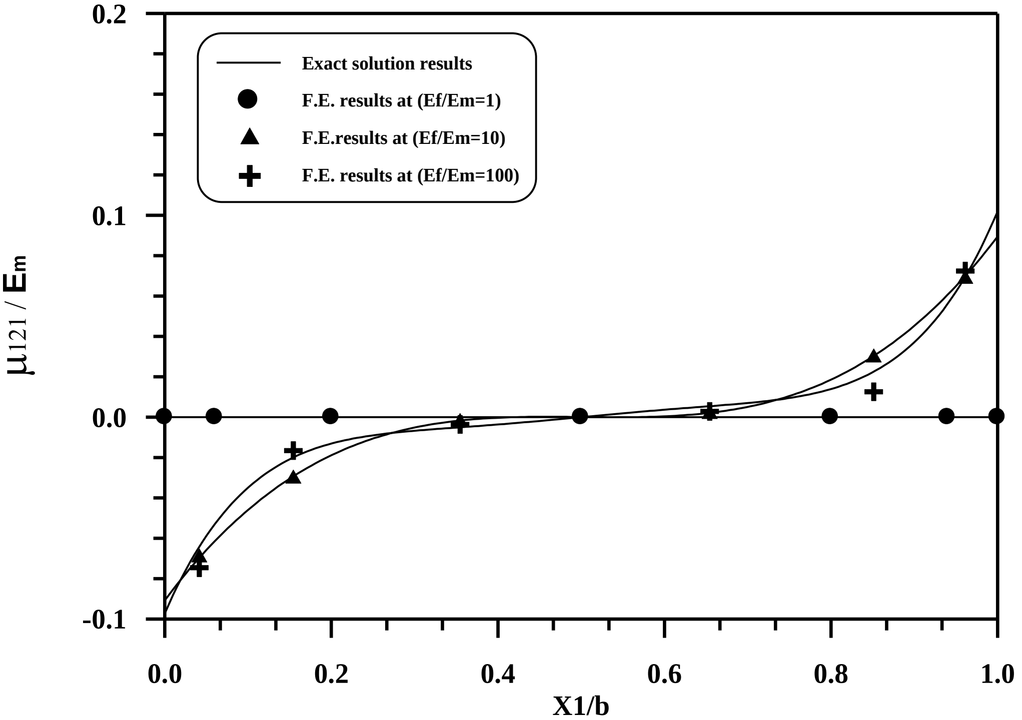


Fig.(5-14):Effect of elastic modulus ratio between fiber and matrix on couple stress at low fiber size (H/ b=0.2) .

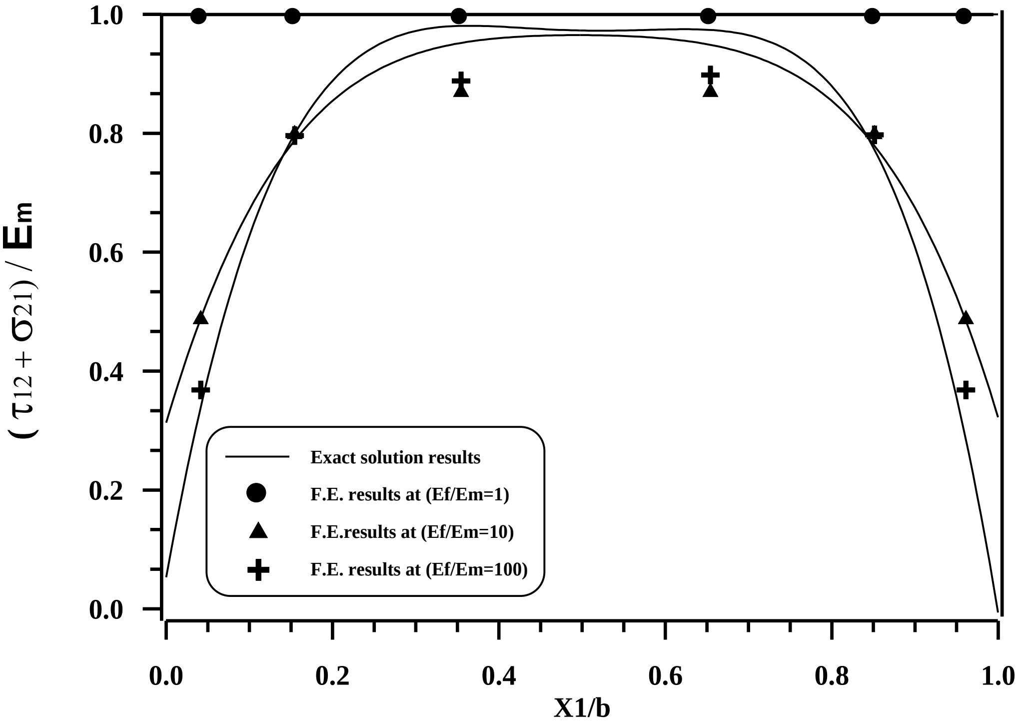


Fig.(5-15):Effect of elastic modulus ratio between fiber and matrix on fiber-matrix shear stress at low fiber size (H/ b=0.2) .

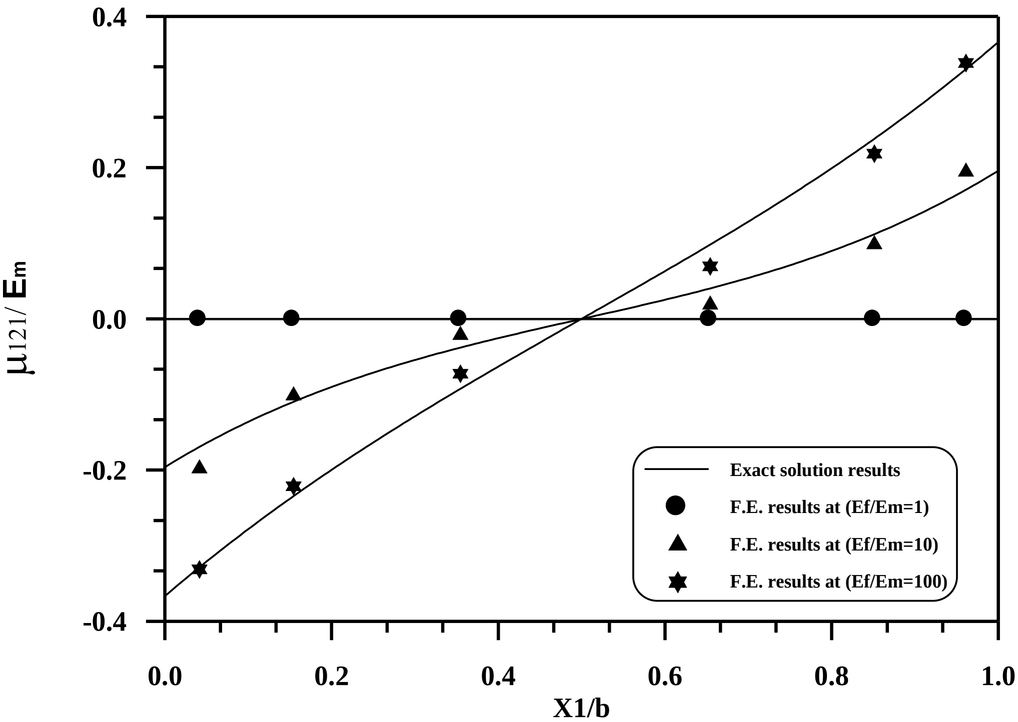


Fig.(5-16):Effect of elastic modulus ratio between fiber and matrix on couple stress at high fiber size (H/ b=1.0) .

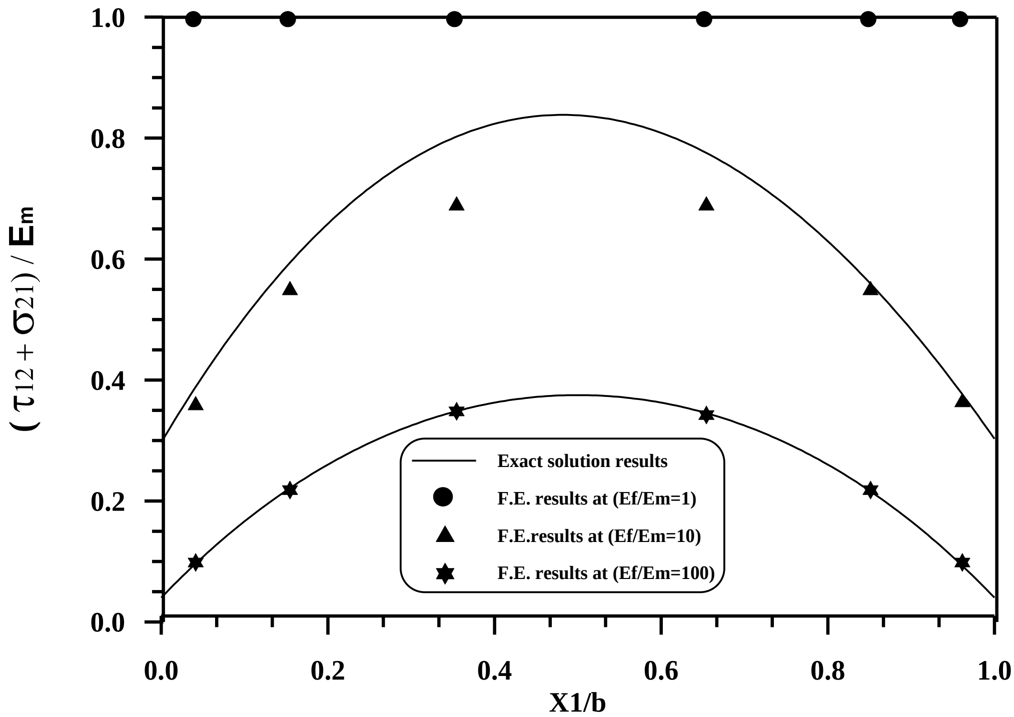


Fig.(5-17):Effect of elastic modulus ratio between fiber and matrix on fiber-matrix shear stress at high fiber size (H/ b=1.0) .

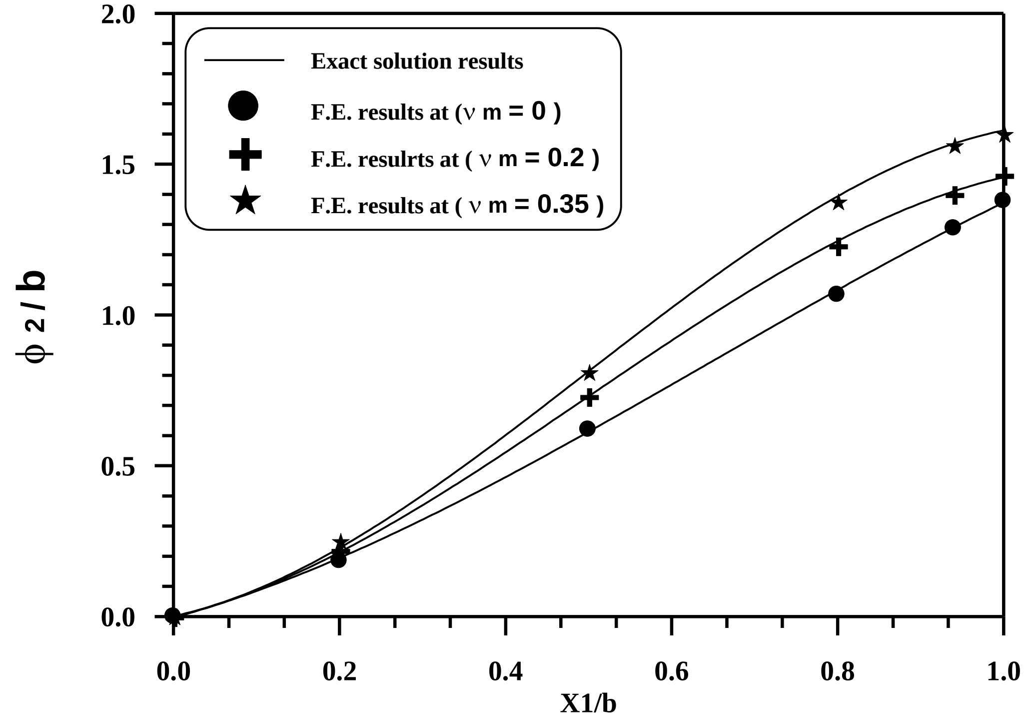


Fig.(5-18):Effect of matrix Poisson's ratio on vertical displacement

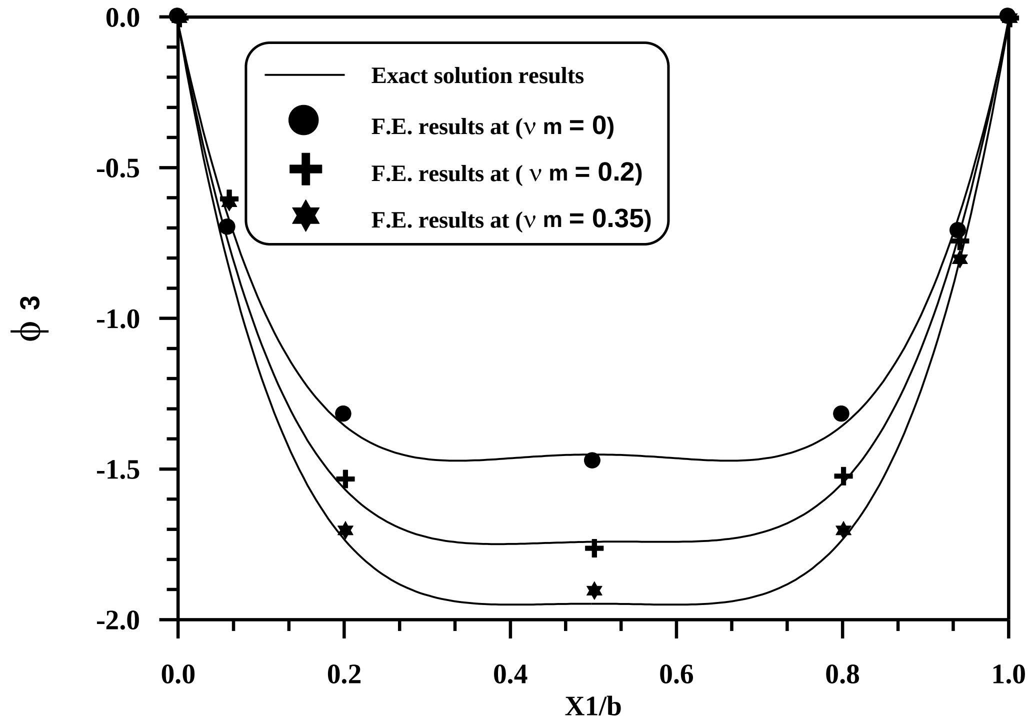


Fig.(5-19):Effect of matrix Poisson's ratio on local shear deformation.

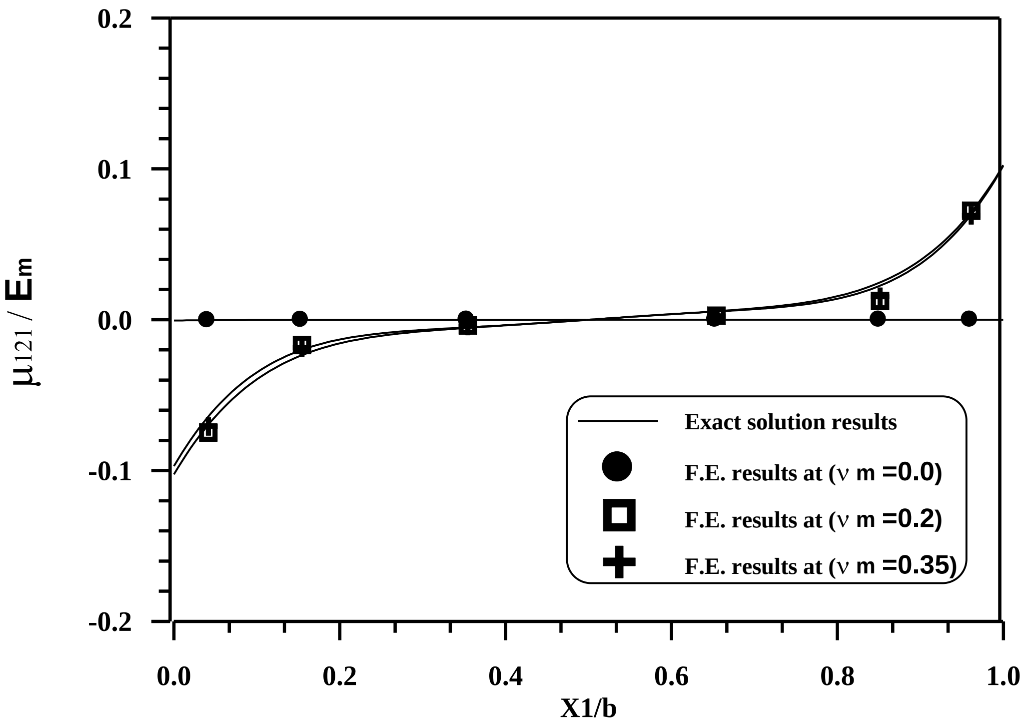


Fig.(5-20):Effect of matrix Poisson's ratio on couple stress.

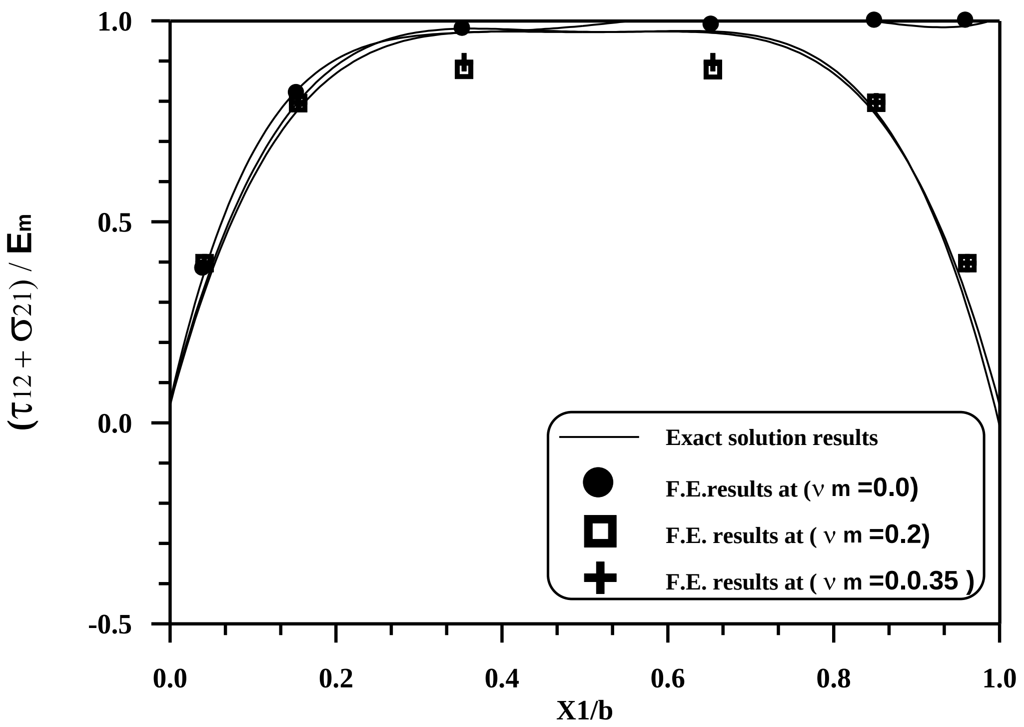
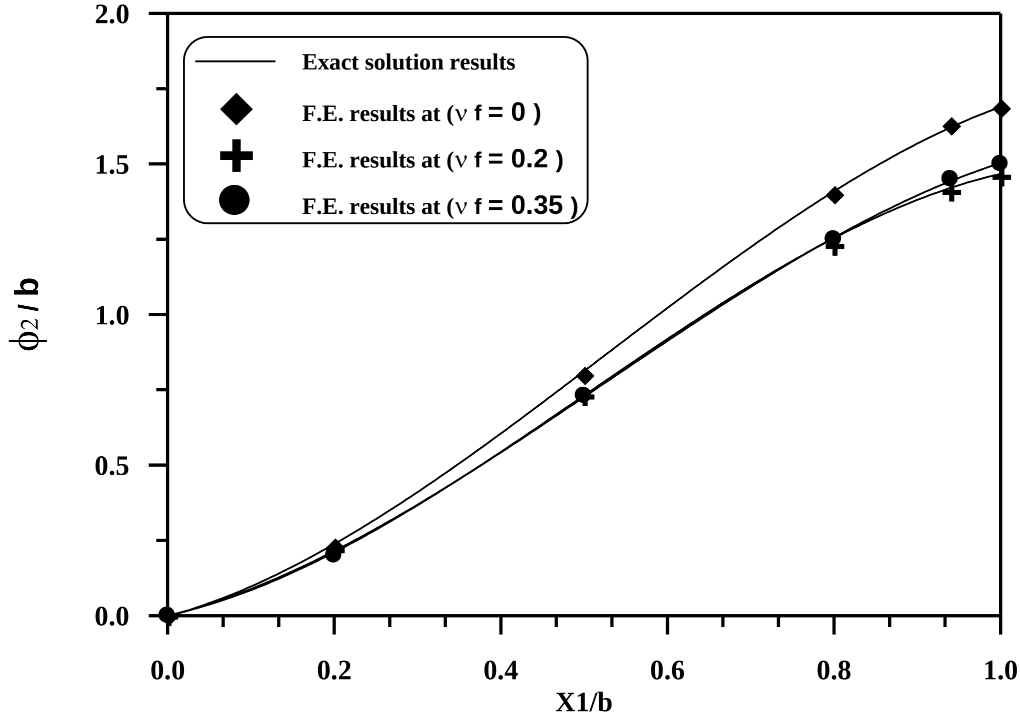


Fig.(5-21):Effect of matrix Poisson's ratio on fiber-matrix shear stress.



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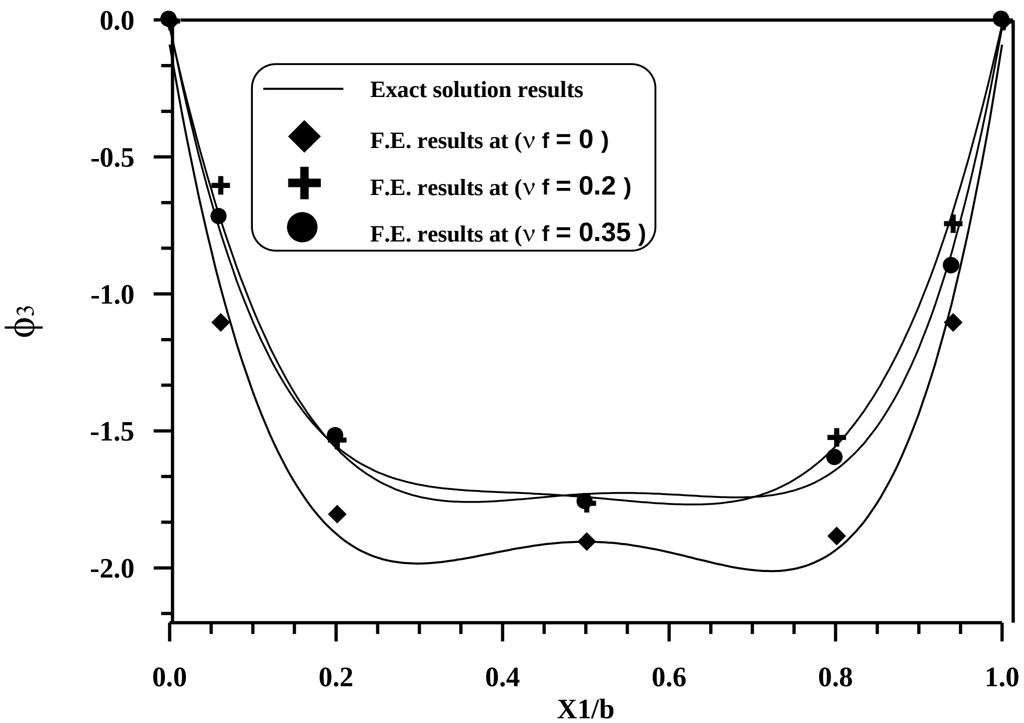

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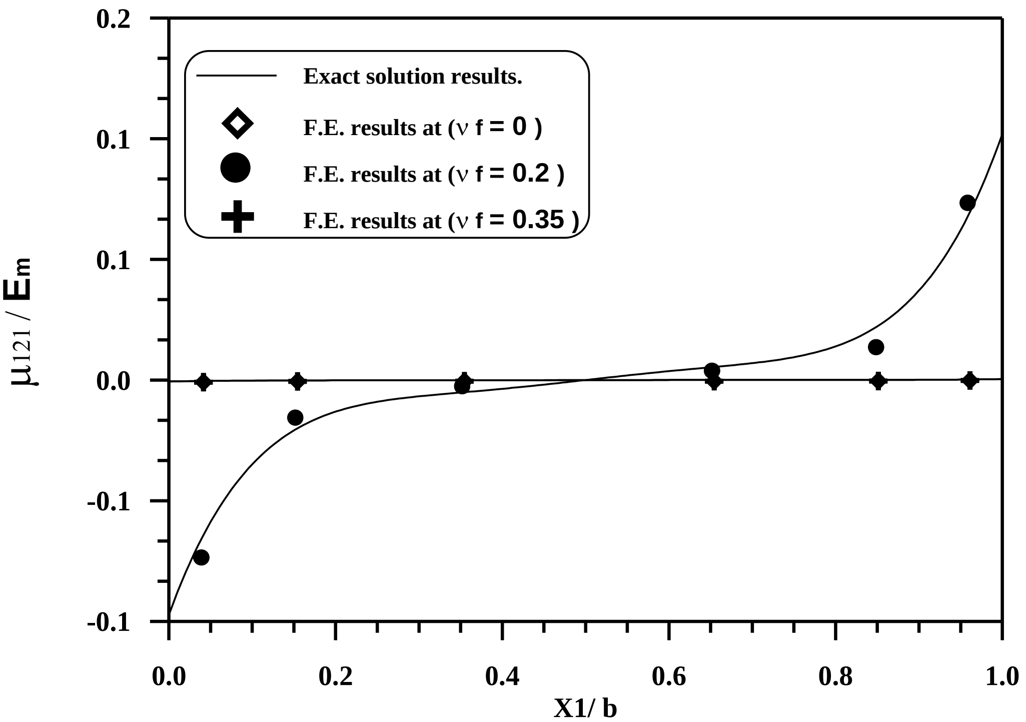

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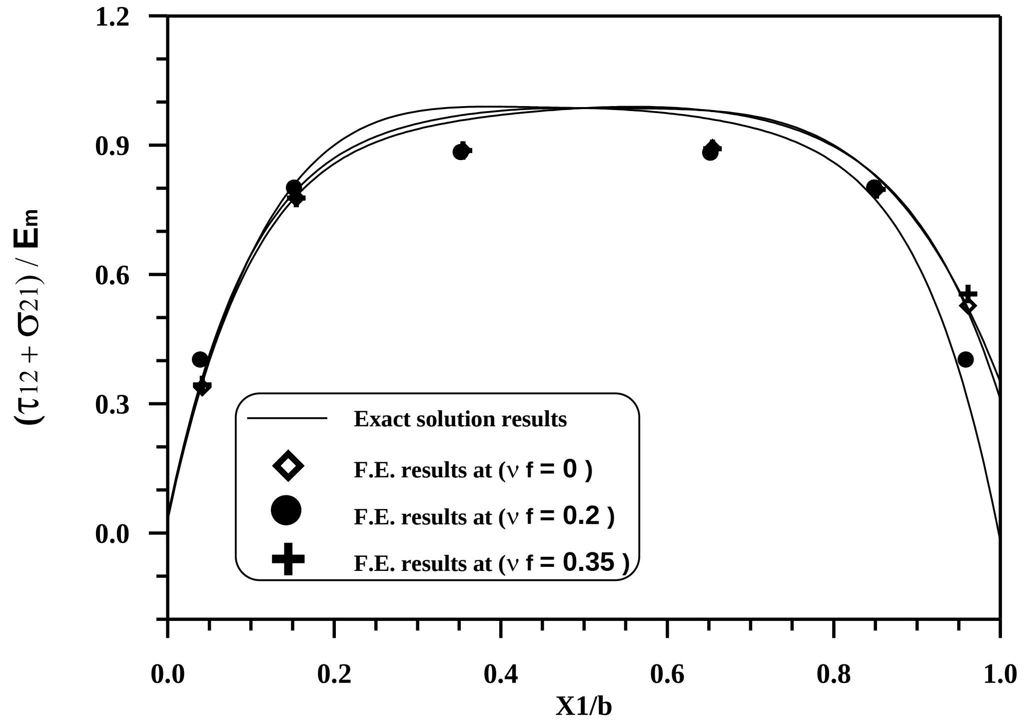
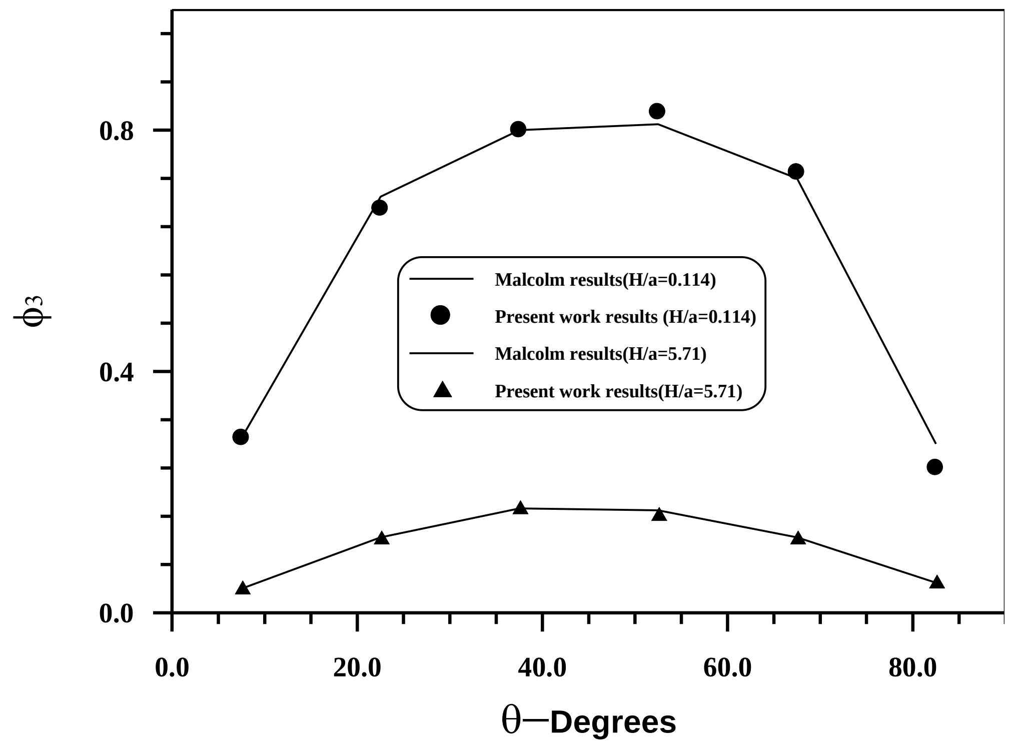


Fig.(5-25):Effect of fiber Poisson's ratio on fiber -matrix shear stress.



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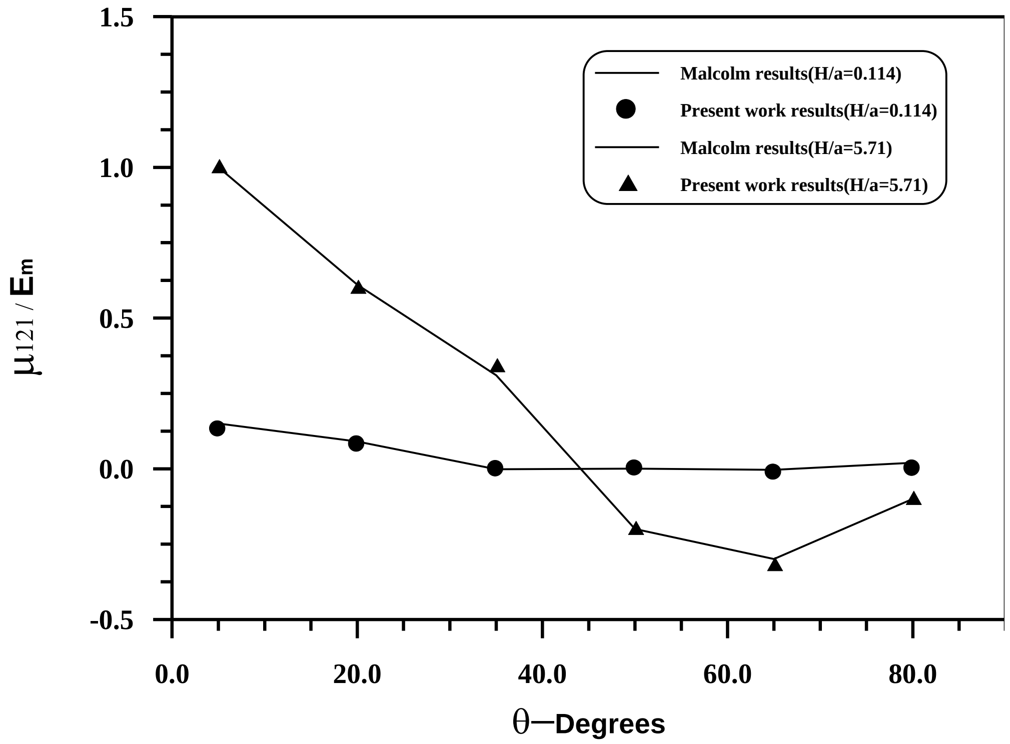

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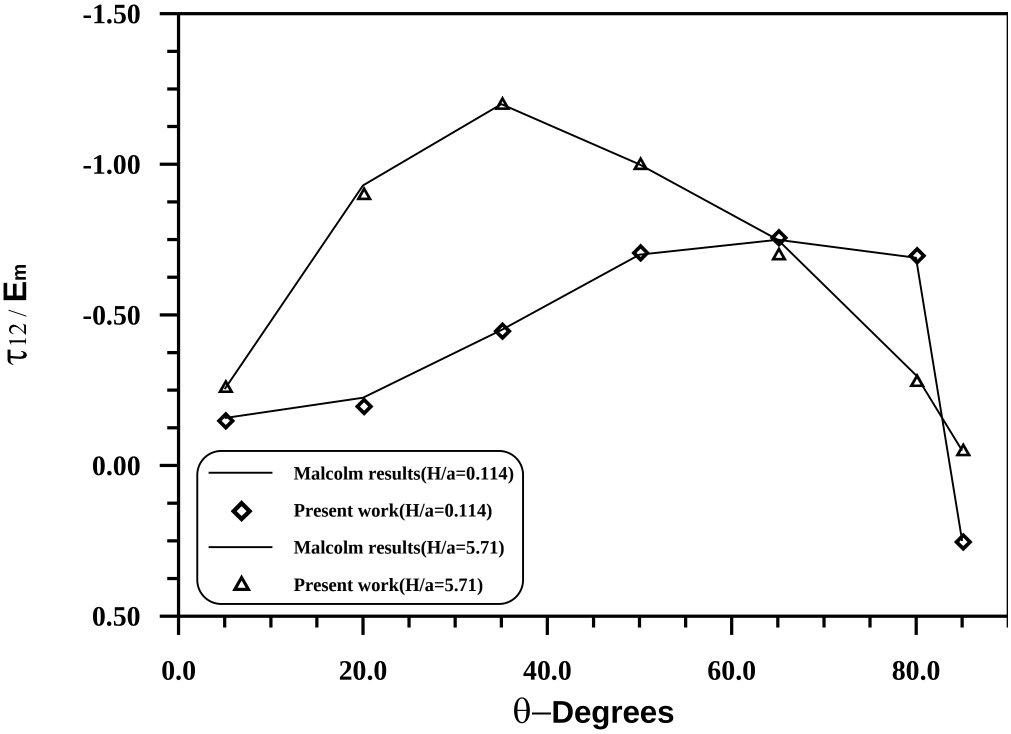

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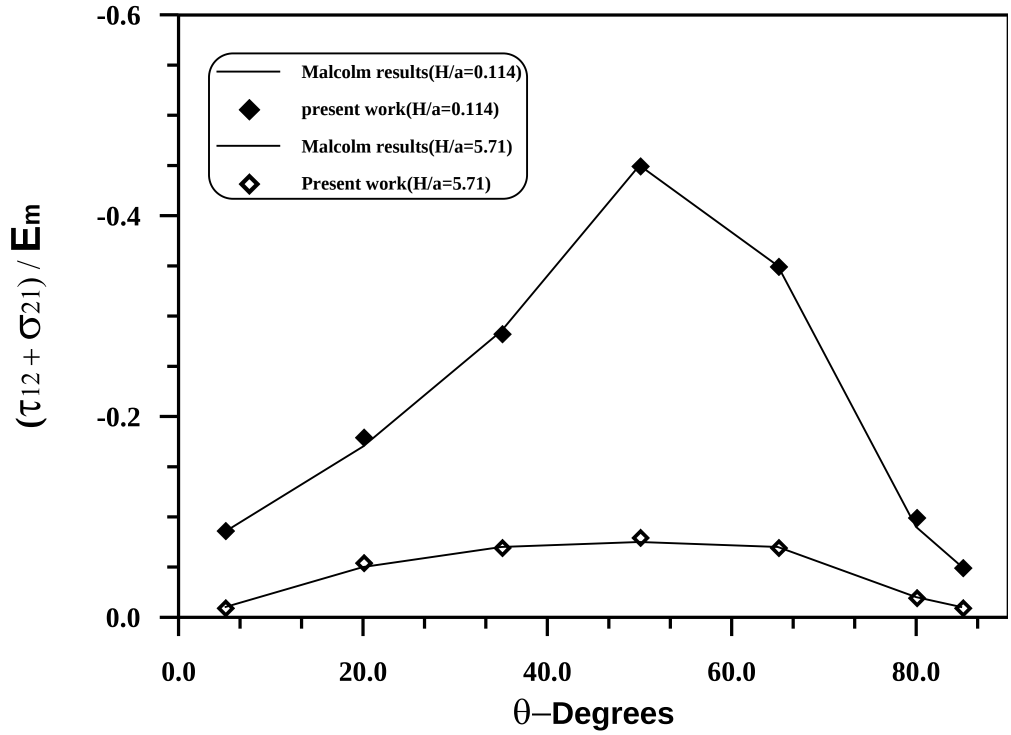
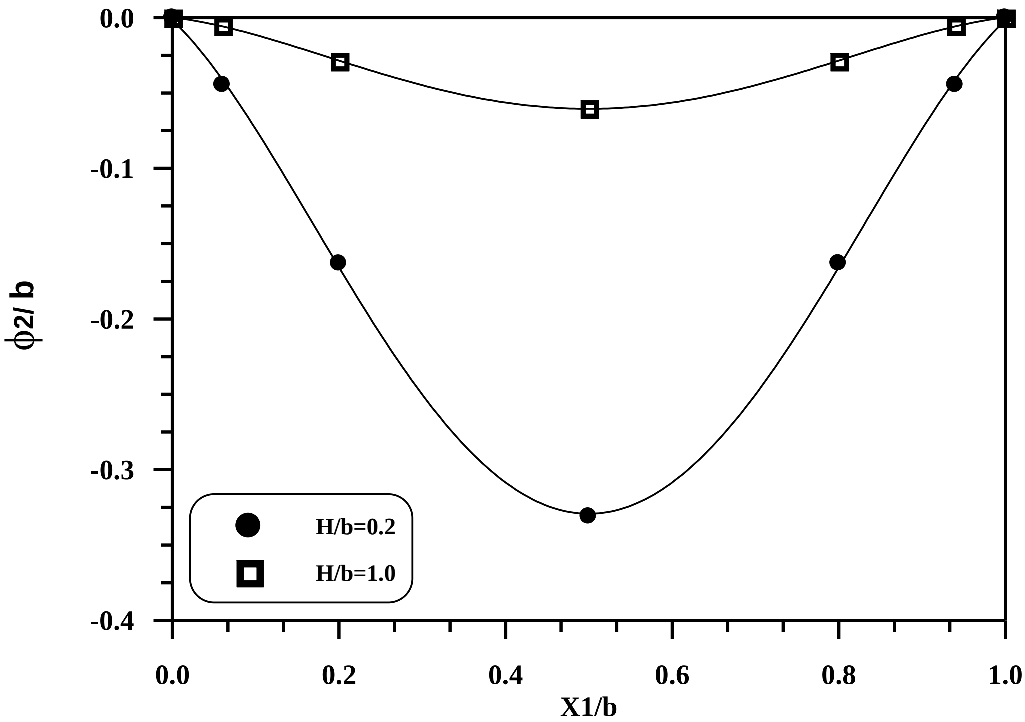


Fig.(5-30):Effect of fiber size on fiber-matrix shear stress.



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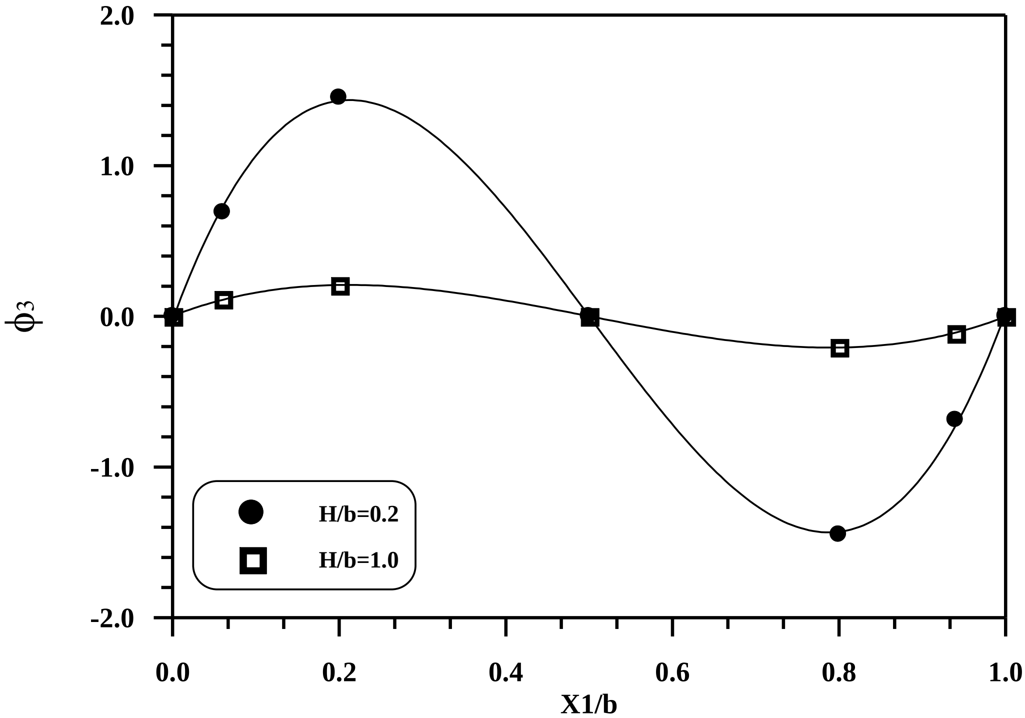

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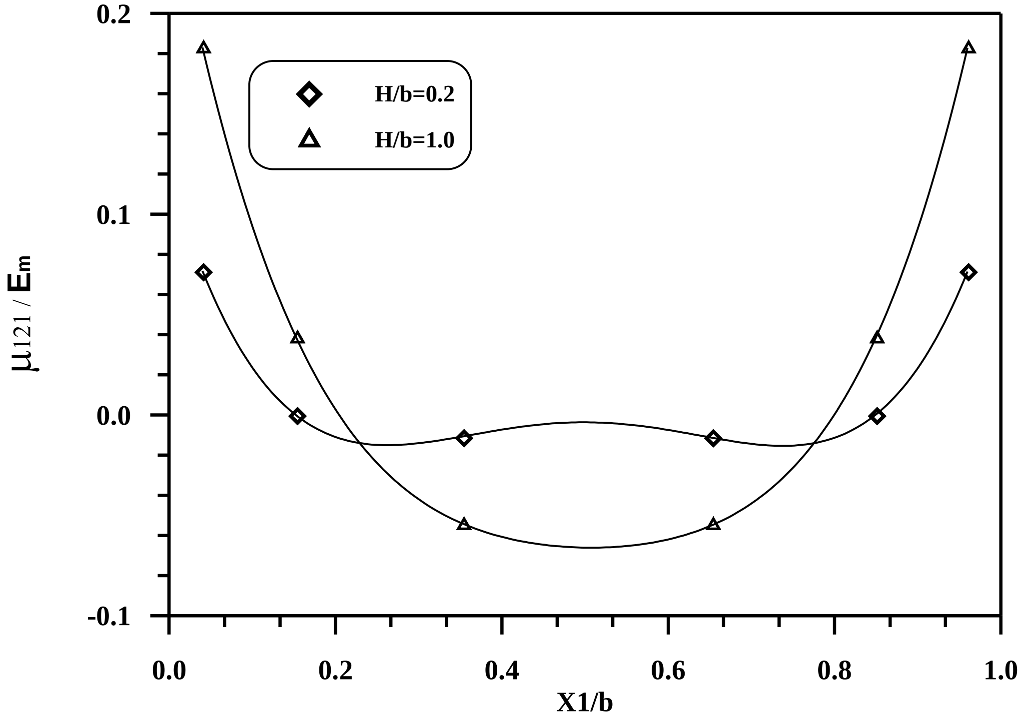


Fig.(5-34):Effect of fiber size on couple stress.

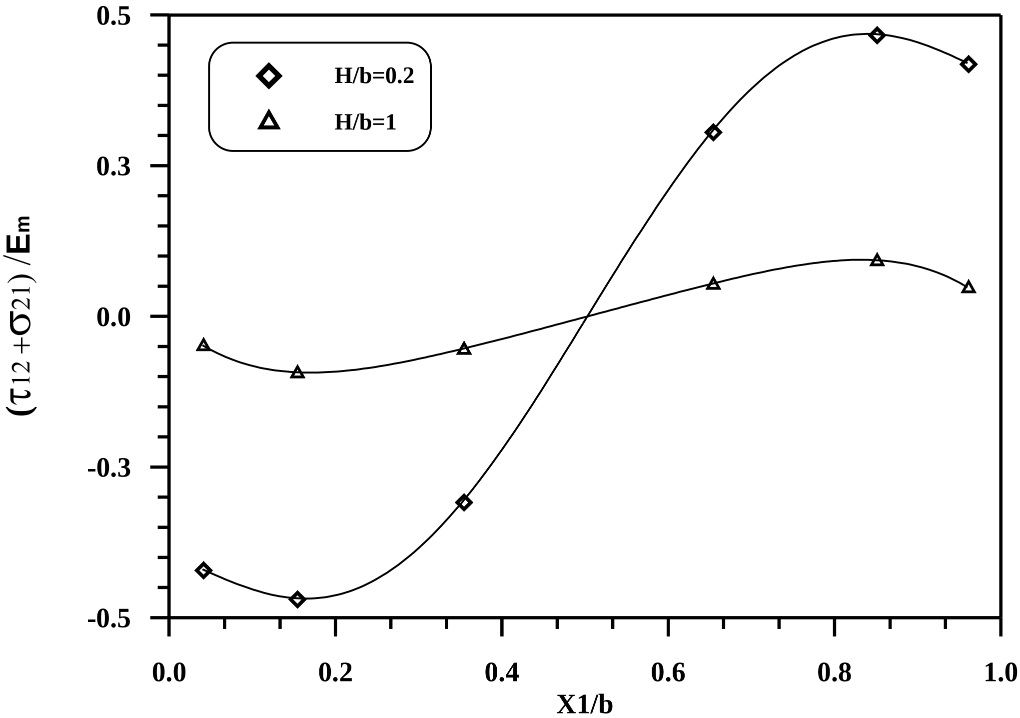


Fig.(5-35):Effect of fiber size on fiber-matrix shear stress.

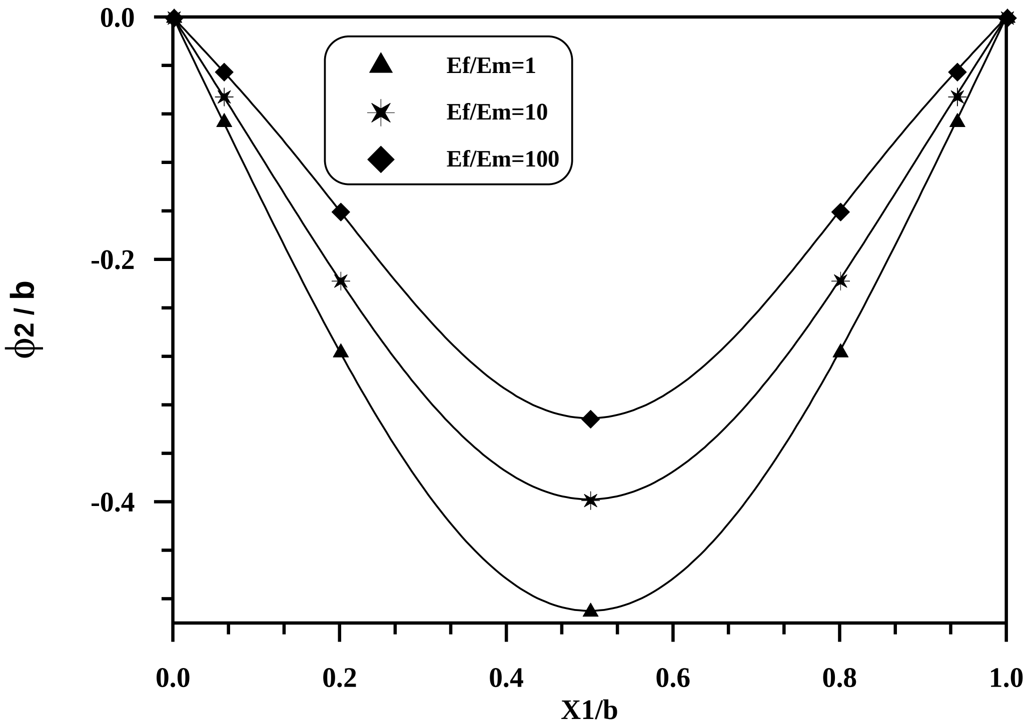
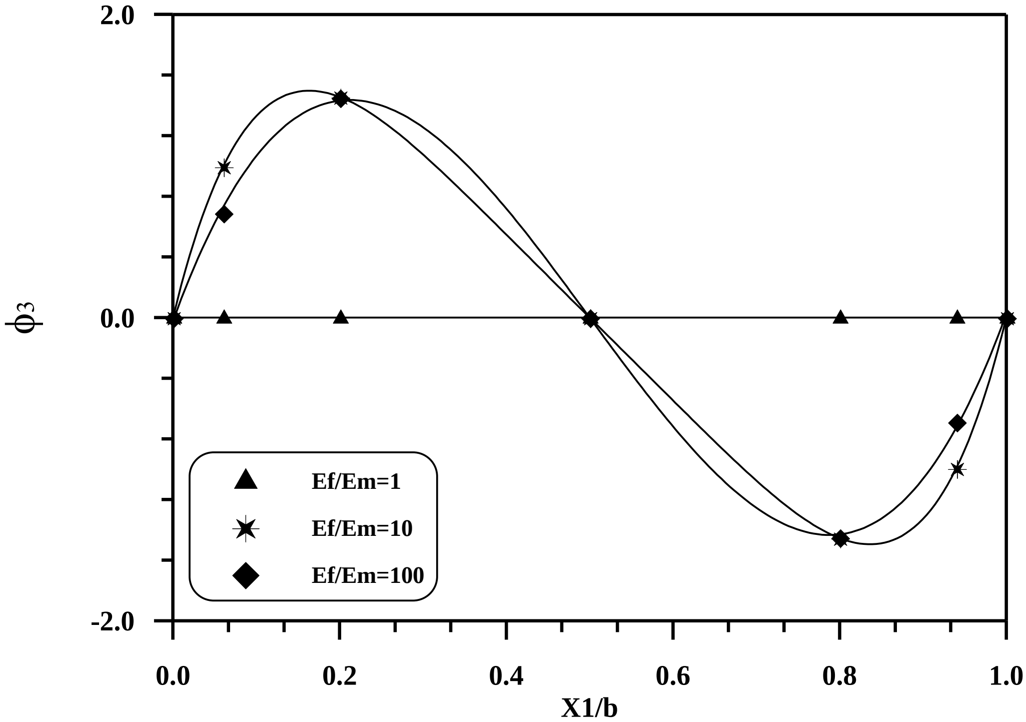


Fig.(5-36):Effect of elastic modulus ratio between fiber and matrix on vertical dislacement at low fiber size (H/ b=0.2) .



Fig(5-37)-Effect of elastic modulus ratio between fiber and matrix on local shear deformation at low fiber size (H/ b=0.2) .

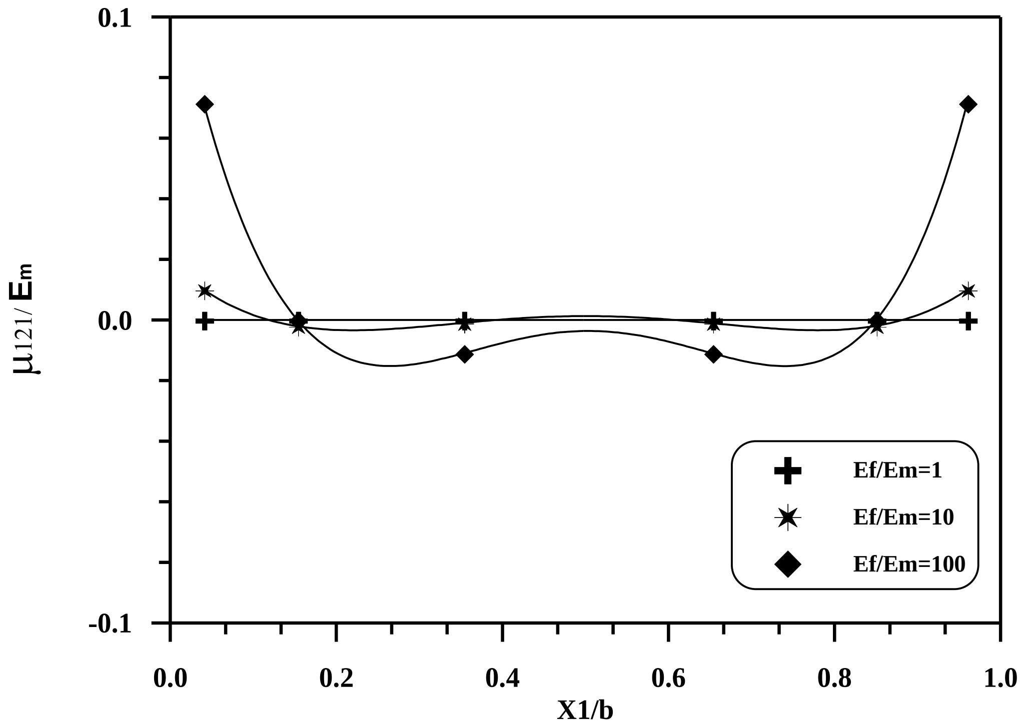
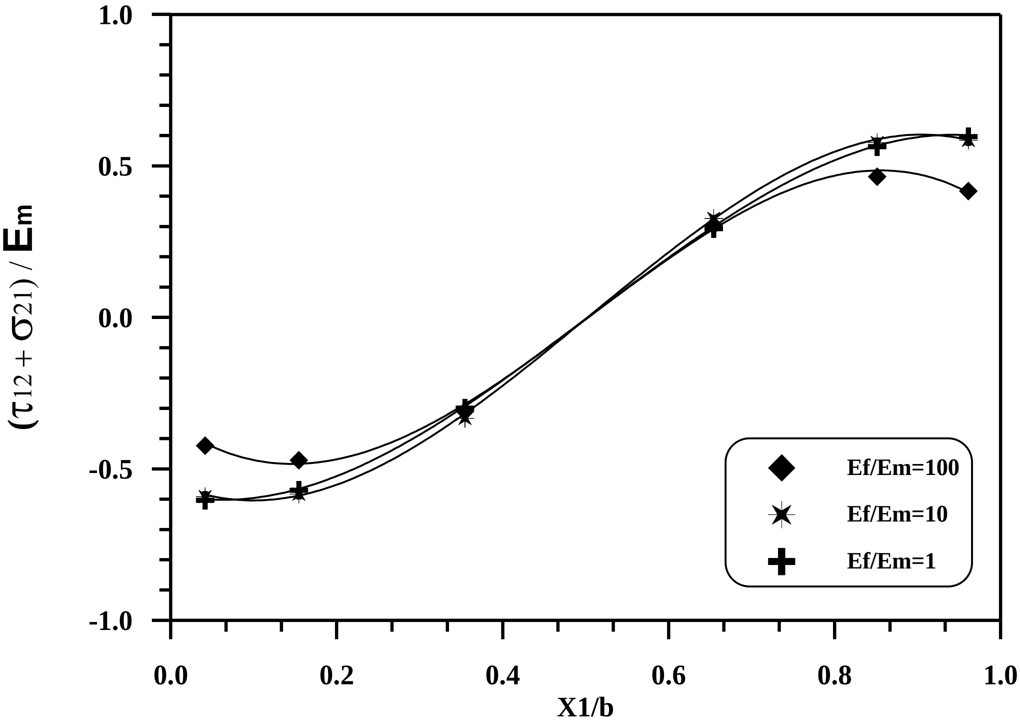


Fig.(5-38):Effect of elastic modulus ratio between fiber and matrix on couple stress at low fiber size (H/ b=0.2) .



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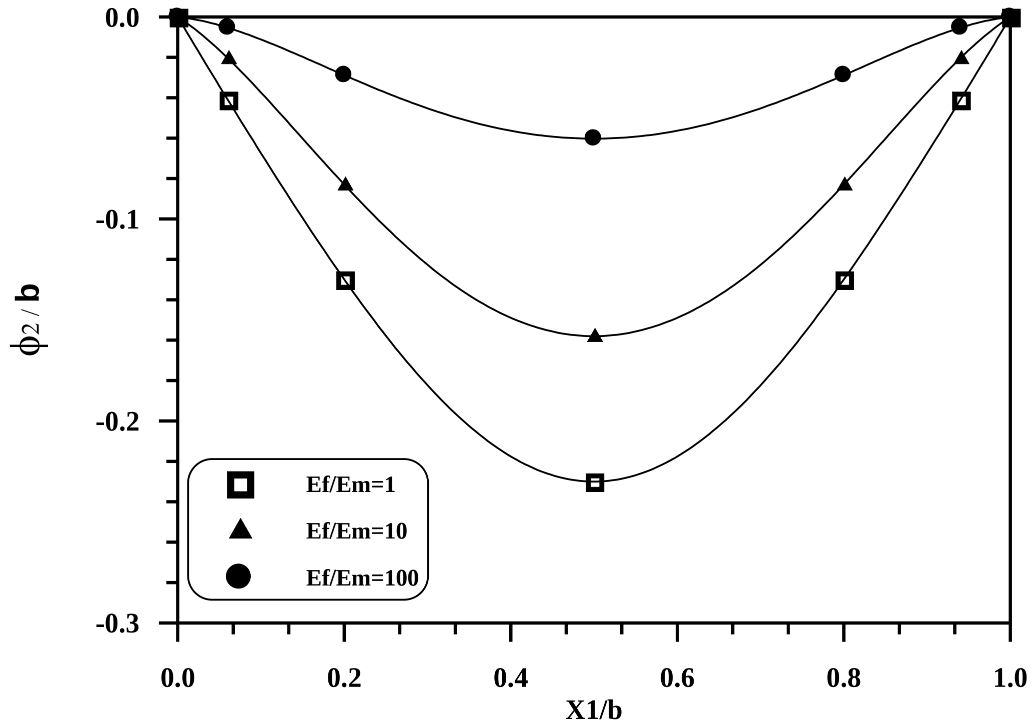

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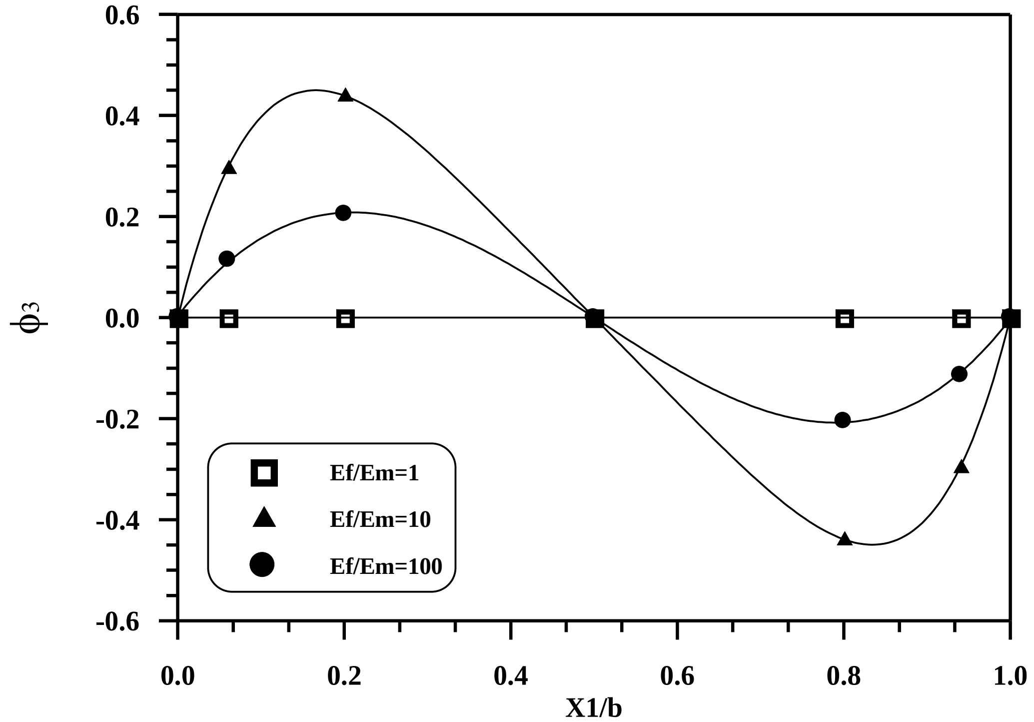

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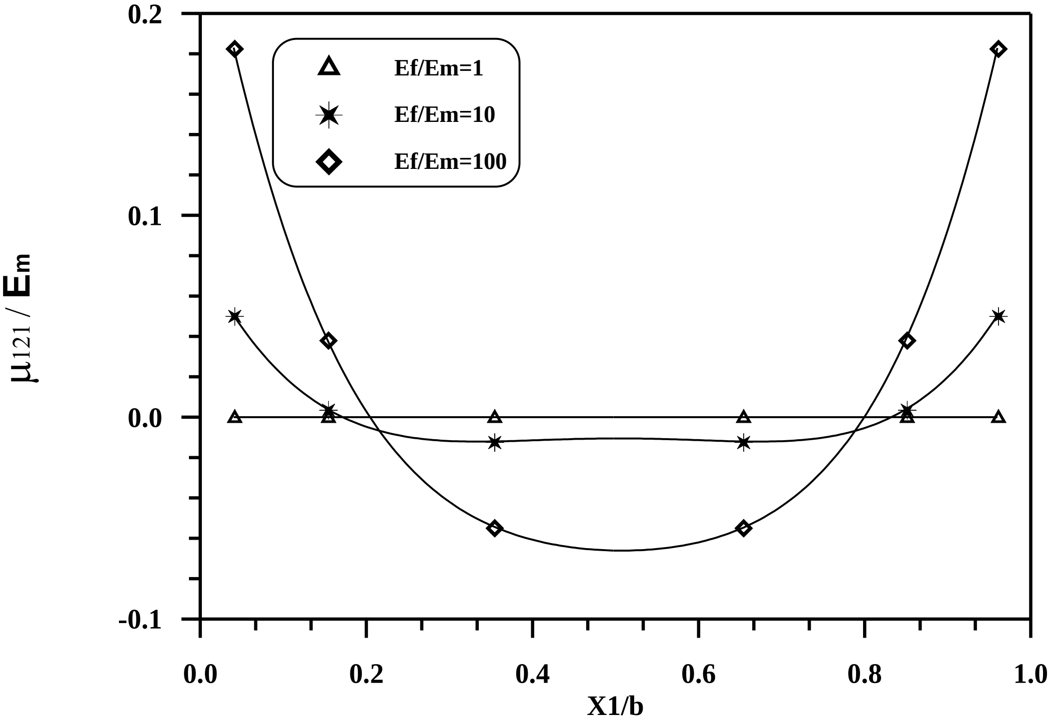

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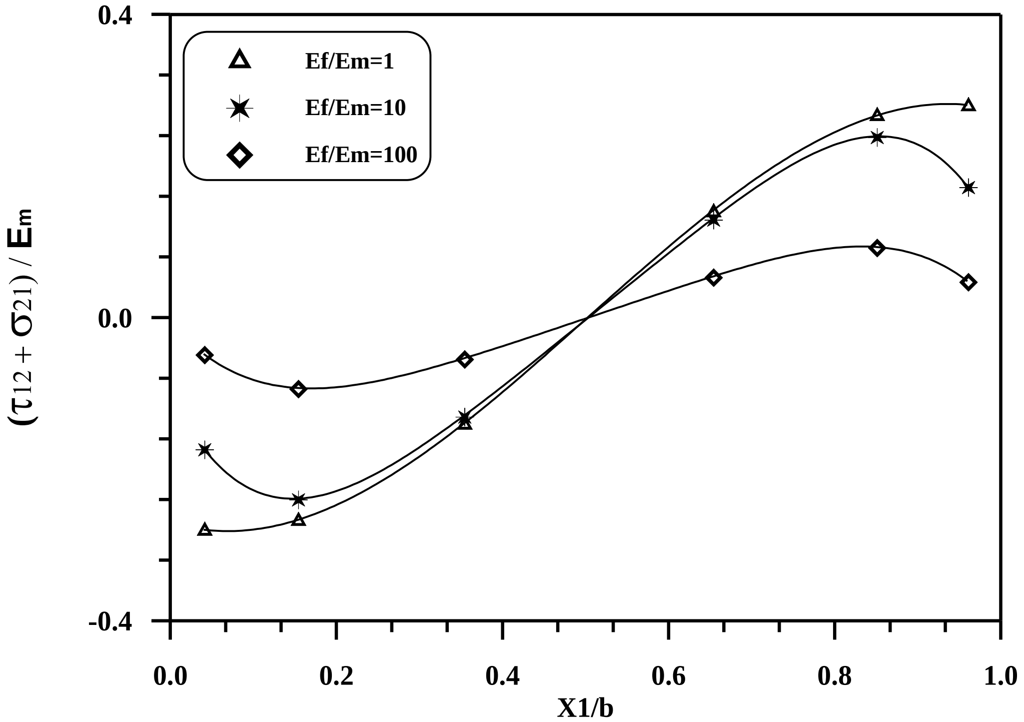

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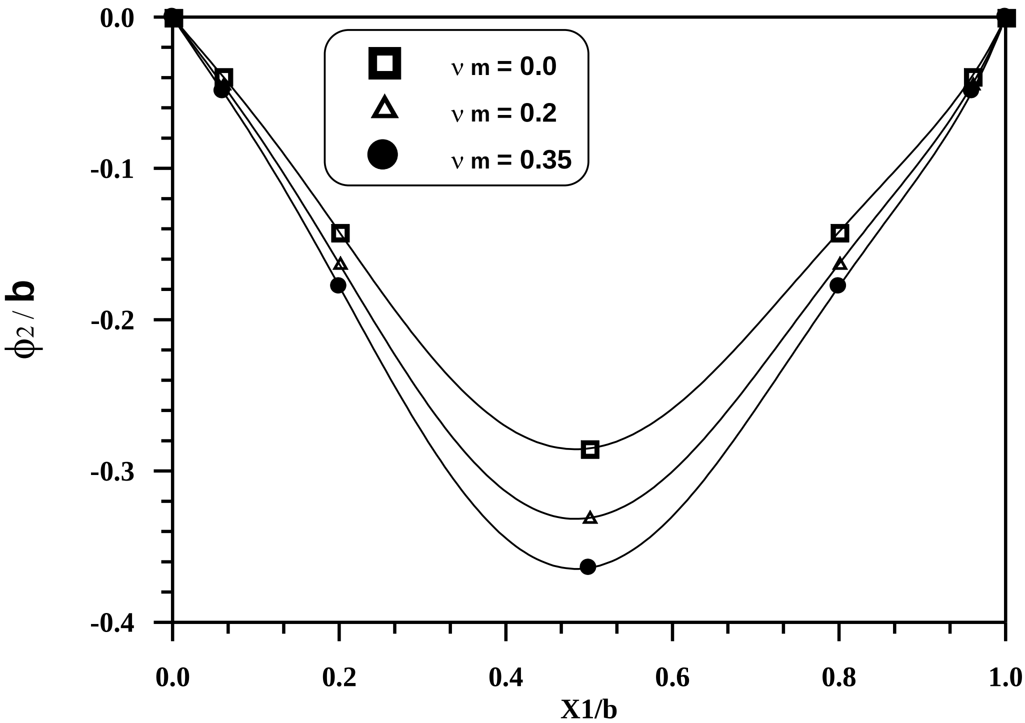
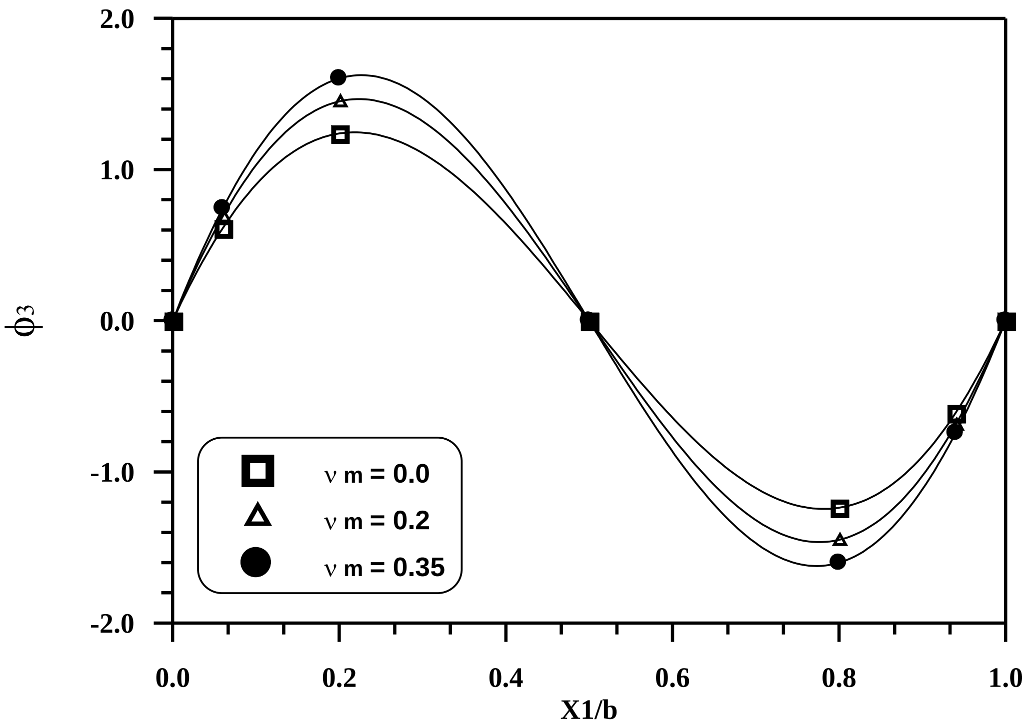


Fig.(5-44):Effect of matrix Poisson's ratio on vertical displacement.



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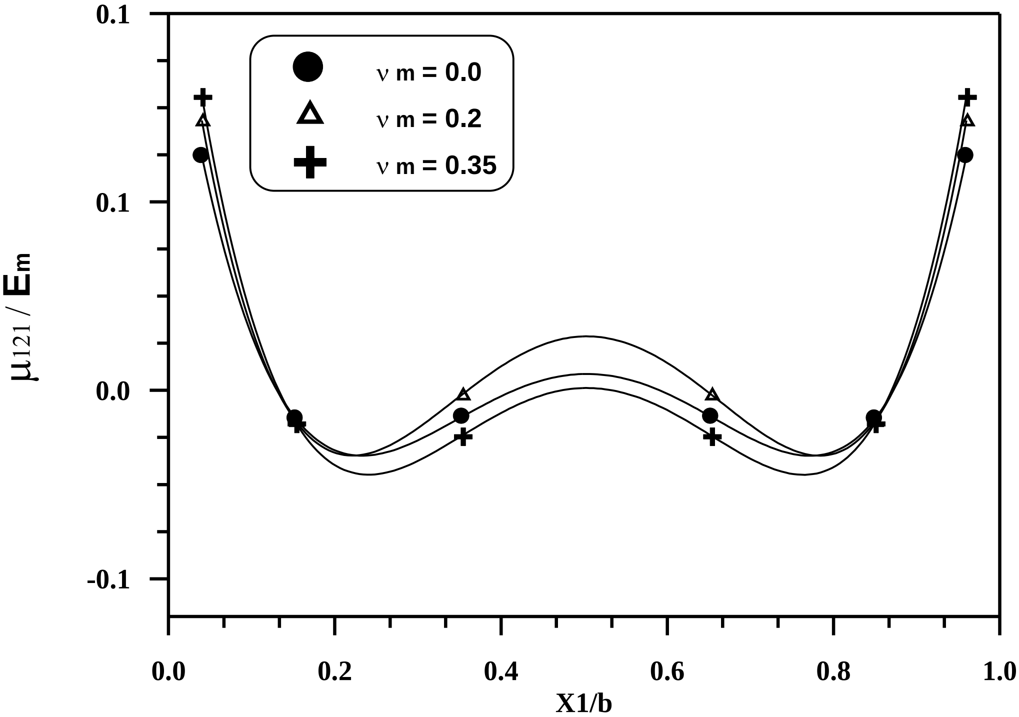

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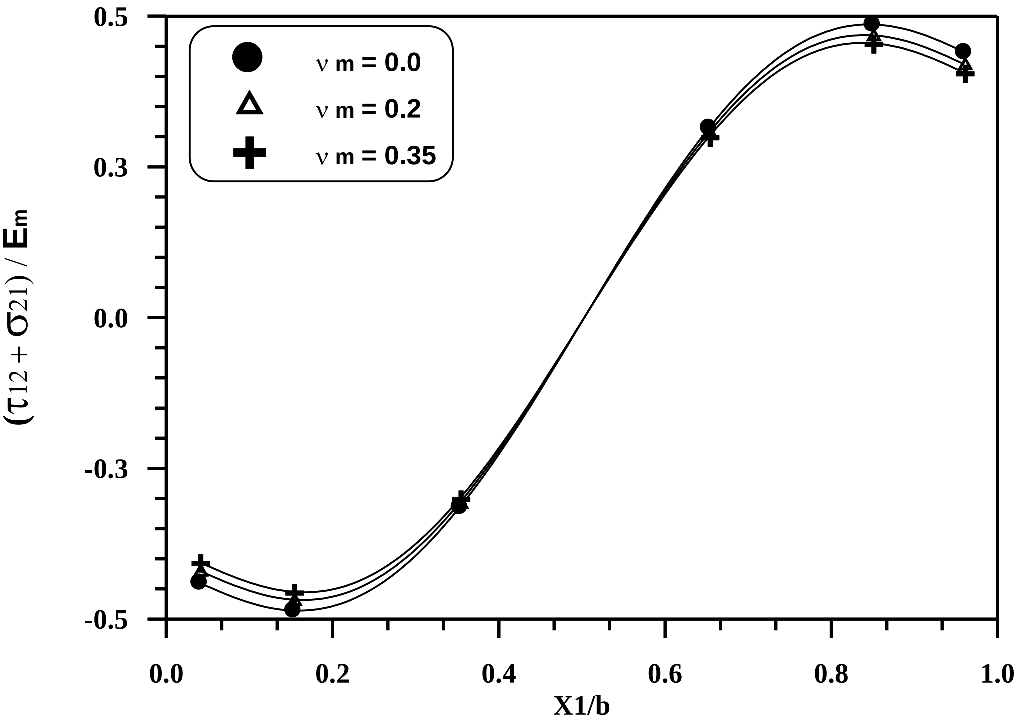
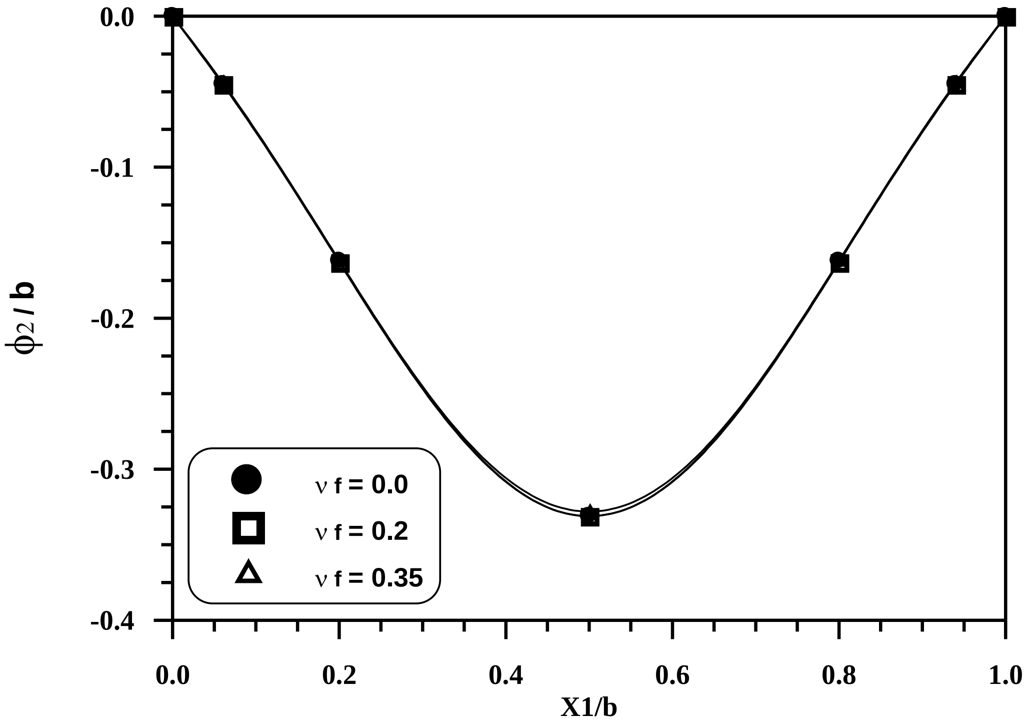


Fig.(5-47):Effect of matix Poisson's ratio on fiber-matrix shear stress



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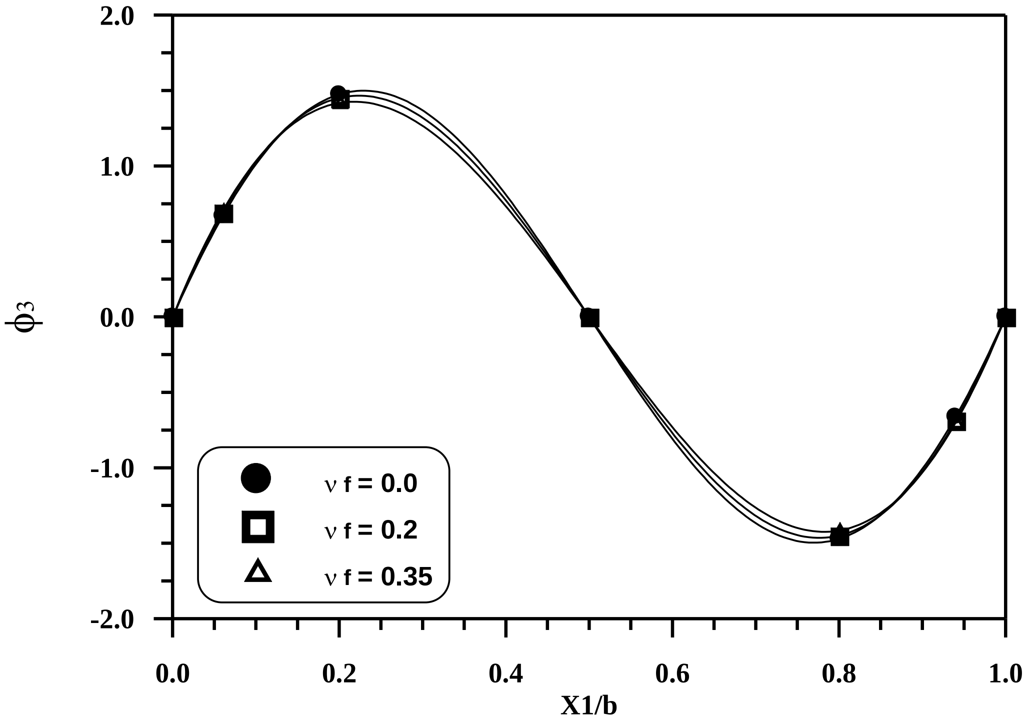

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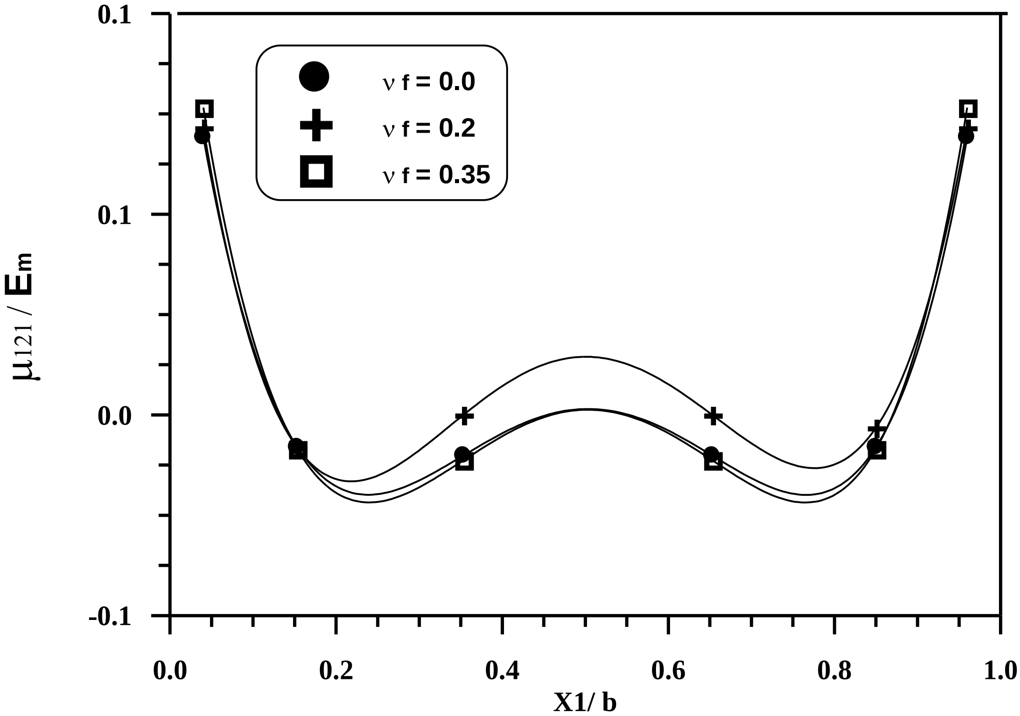

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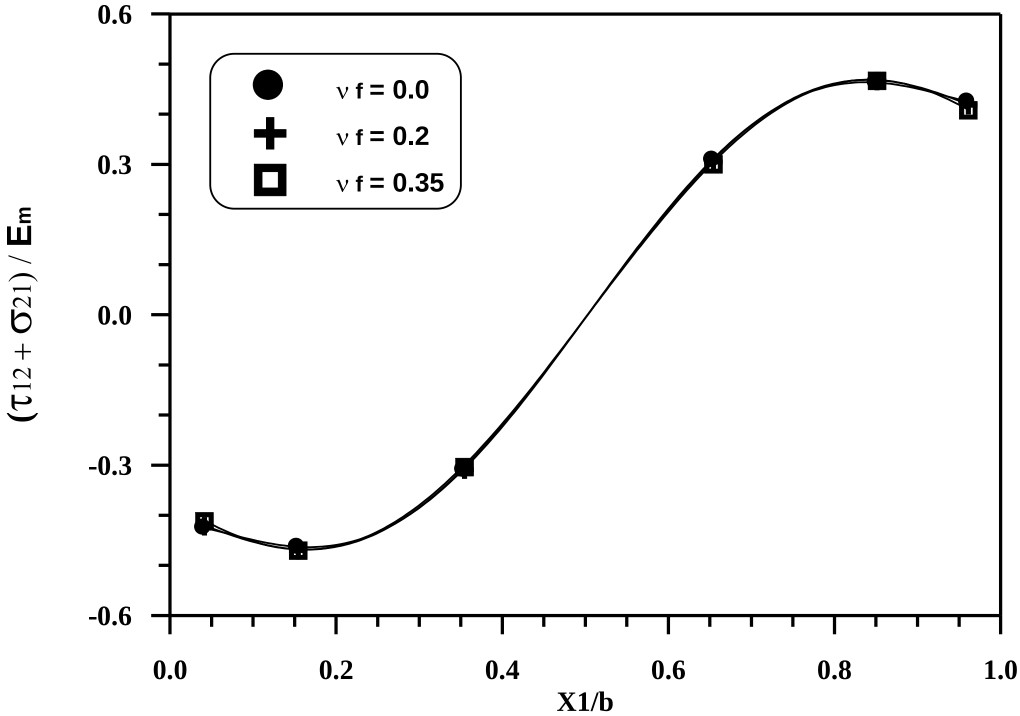

Fig.(5-51):Effect of fiber Poisson's ratio on fiber -matrix shear stress.

Chapter six

Conclusions and Recommendation

٦-١ Conclusions:-

On the basis of the results of the present investigation, the following conclusions can be drawn:-

١) The plate has rigid support in one side both the vertical displacement (ϕ_2) and the local shear deformation (ϕ_3) are equal zero at rigid supports. At plate which has rigid supports in both side the (ϕ_3) has the wave profile.

٢) The perfect bound between fiber and matrix which has been assumed in this work led to the same horizontal displacement (ϕ_1) in both fiber and matrix and also a zero value for the relative shear strain $\gamma_{٢٢}$.

٣) At the plate composed from UDFRM and restrained by rigid supports in one side and roller in other, and subjected to pure shear loading, all the ϕ_2 , ϕ_3 , μ_{121}, and ($\sigma_{٢١}$ + $\tau_{١٢}$) have a reduction with the increment of fiber size (H/b).

٤) In the same plate the increment in E_f/E_m causes reduction in ϕ_2 and ϕ_3 for all high or low constant values of H/b and also causes a little reduction in μ_{121} and ($\sigma_{٢١}$ + $\tau_{١٢}$) at constant low value of H/b this reduction increases at high constant value of H/b.

٥) The increment at both Poisson's ratio for matrix or fiber causes an increment in ϕ_2 and ϕ_3 and a little change in μ_{121} and fiber-matrix shear stress.

٦) In plate composed from UDFRM and contains circular hole under tension fields the couple stress around the edge of hole are greater for the larger fibers and peak at the center line where the gradient of ϕ_3 is also maximum.

٧) Increasing in fiber size for the same plate in No.٦ leads to that the average shear stress ($\tau_{١٢}$) around circular hole has a greater max. value whereas the ($\sigma_{٢١}$ + $\tau_{١٢}$) for the same material reductions of this increase.

٨) In the plate composed from UDFRM and has rigid supports in both ends and subjected to the distributed concentrated load. The vertical displacement (ϕ_2) has a max. values at the region of max. concentrated load, whereas the ϕ_3 has a minimum value at max. concentrated load.

٩) In the same plate in No.٨ the μ_{121} has a min. value at max. concentrated load and a minimum H/b.

١٠) The ($\sigma_{٢١}$ + $\tau_{١٢}$) has max. value at min. concentrated load and min. H/b.

١١) Also in this plate the ϕ_2 reduction with E_f/E_m increase whereas ϕ_3 has a little reduction due to this increment of Ef/Em.

١٢) When the Poisson's ratio of matrix or fiber increases, the ϕ_2 and ϕ_3 increase also but ($\sigma_{٢١}$ + $\tau_{١٢}$) has a little effect by this increment while couple stress increases due to this increment especially at the region under applied load.

١٣) All the ν_m has a large effect on ϕ_2, ϕ_3, μ_{121} and ($\sigma_{٢١}$ + $\tau_{١٢}$) than ν_f .

٦-٢ Recommendations for further work:

The following recommendation for furthre work:

١) Considering the thermal behaviour of UDFRM by using the microstructure theory too instead of microstructure theory at steady state which is using in the present work.

٢) Studying the microstructure of UDFRM in three-dimension.

٣) Assumed imperfect bound instead of perfect bound.

٤) Studying the micromechanical behaviour of UDFRM taking into consideration the inclusion between fiber and matrix.

٥) Studying the UDFRM behaviour by assuming change fiber volume fraction instead of assumed it constant as done in this work.

List of contents

CERTIFICATION

We certify that this thesis entitled ***"Micromechanics Analysis Of Unidirectional Fiber-Reinforced Materials"*** was prepared by ***"Haneen Zuhair Najee"*** under our supervision at Babylon University, College of Engineering in partial fulfillment of the requirements for the degree of Master of Science in Materials Engineering.

Signature:	**Signature:**
Name: **Asst. Prof. Dr.Tahseen A. AL-Hattab**	**Name**: **Asst. Prof. Dr. Haydar A. Hussain**
(Supervisor)	**(Supervisor)**
Date: / / ٢٠٠٢	**Date**: / / ٢٠٠٢

CERTIFICATION

We certify as an examining committee that we have read this thesis, entitled ***"Micromechanics Analysis Of Unidirectional Fiber-Reinforced Materials"***, and as examining the student ***"Haneen Zuhair Najee"*** in its content and what related to it, and found it meets the standard of a thesis for the degree of Master of Science in Materials Engineering.

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Date:

Approval of the Deanery of the College of Engineering
Dean of the College of Engineering

Signature
Prof. Dr. Abdul Amir S. Resen
Dean of the College of Engineering
University of Babylon

Date: / / ٢٠٠٢

Appendix A

Equations of Equilibrium of "UDFRM"

From the strain energy expression contained in Eqns.(٣-٥٩) and (٣-٦٢) in chapter three, the equations of equilibrium of UDFRM can be derived by using standard variational techniques based on the principle of virtual work.

Principle of virtual work means a body is in equilibrium if the internal work equals the external virtual work for every kinematically admissible displacement field (u_i, ε_{ij}). Since the body is in equilibrium then:

$$\sigma_{ij,j} + f_i = 0 \qquad \text{(A-١)}$$

where f_i are the body forces, the external virtual work (i.e., the work of the real loads M_i moving through the virtual displacements δ_{mi}) is denoted by δ_{WE} as shown in Fig.A١. the virtual work of the tractions T_i acting on the surface and the body forces f_i acting within the elastic body, given by (See **Davis & Selvadurai, ١٩٩٦**)

$$\delta_{WE} = \int_s T_i \left(\delta_{ui}\right) ds + \int_v f_i \left(\delta_{ui}\right) dv \qquad \text{(A-٢)}$$

with absence of body forces Eqn.(A-٢) can be written as:

$$\delta_{WE} = \int_s T_i \left(\delta_{ui}\right) ds \qquad \text{(A-٣)}$$

Consider the traction vector T_i at the location x_i and referred to a plane the unit out-ward normal to which has components n_i , the state of stress at the same location is specified by σ_{ij} . So

$$T_i = \sigma_{ij} n_j \qquad \text{(A-٤)}$$

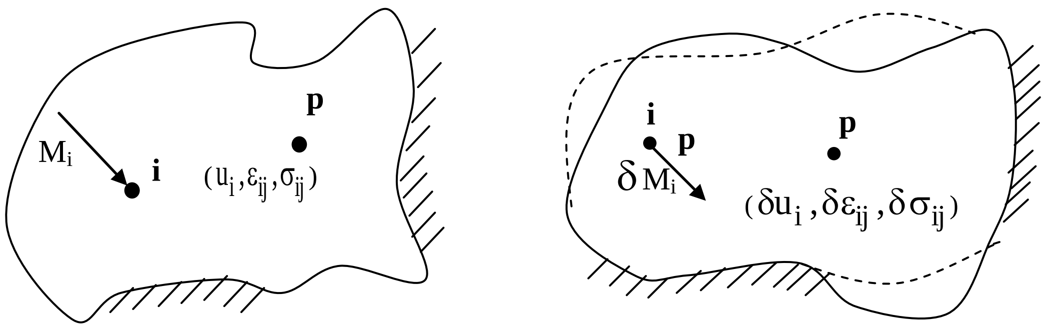

Fig. A١ : Loading of a continuum region by generalized forces M_i and by virtual displacements δ_{ui}.

To represent the surface reaction of RVE of UDFRM, the traction vectors on the plane surface through the two coordinates planes can be represented in terms of the stress components acting on these planes (Fig.A٢-a) the traction vector on the plane surface ABCD has components T١ and T٢ . In addition to these tractions vectors, it is assumed that the RVE region also subjected to couple traction (**t**) per unit length.

Consider the equilibrium of forces acting on the RVE. In the x١-direction this yields to:-

$$T_1(\Delta s\ \ )-\tau_{11}(\Delta s_1)-\tau_{12}(\Delta s_2)-\sigma_{21}(\Delta s_2)=0$$

where $\Delta s =$ is the area of plane ABCD

$$\Delta s_1 = n_1\,\Delta s \qquad ; \qquad \Delta s_2 = n_2\,\Delta s$$

where $n_1 = n_2 = \cos(\angle BOC)$

so

$$T_1 - \tau_{11}\,n_1 - (\tau_{12} + \sigma_{21})n_2 = 0$$

$$\therefore T_1 = \tau_{11}\,n_1 + (\tau_{12} + \sigma_{21})n_2 \qquad \text{(A-٥)}$$

by the same way have obtain:

$$\therefore T_2 = \tau_{22}\, n_2 + \tau_{12}\, n_1 \qquad \text{(A-٦)}$$

also, $t = \mu_{121}\; n_1$

where (**t**) is a surface couple per unit length. T١ and T٢ are surface tractions per unit length and n١ and n٢ are component of the unit outward vector normal to the surface.

Now by sub. Eqn.(A-٤) in (A-٣) and derivative with respect to x١ first and x٢ second and then summation the two equations as follows:

$$W_E = T_i(\varepsilon_i) \;\Rightarrow\; W_E = \tau_{11}\, n_1 + (\tau_{11} + \sigma_{21})n_2$$

$$\frac{\partial W_E}{\partial x_1} = \frac{\partial \tau_{11}}{\partial x_1} = 0$$

$$\frac{\partial W_E}{\partial x_2} = \frac{\partial(\tau_{12} + \sigma_{21})}{\partial x_2} = 0$$

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial(\tau_{12} + \sigma_{21})}{\partial x_1} = 0 \qquad \text{(A-٧)}$$

by the same manner:

$$\frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{12}}{\partial x_1} = 0 \qquad \text{(A-٨)}$$

and

$$\frac{\mu_{121}}{\partial x_1} + \sigma_{21} = 0 \qquad \text{(A-٩)}$$

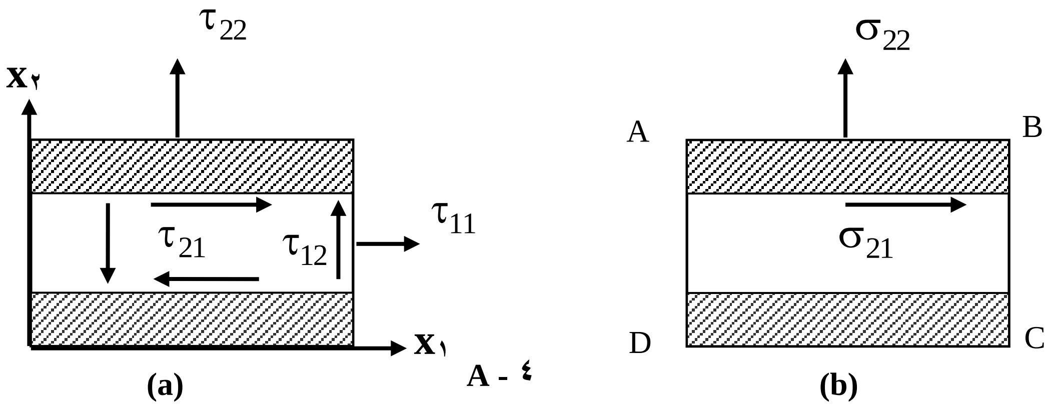

Fig.(A٢): Stress components on the faces of the RVE perpendicular to the co-ordinate direction.

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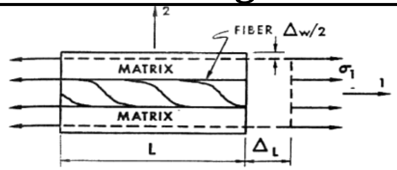
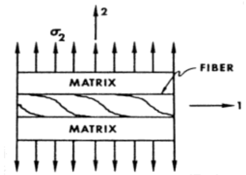
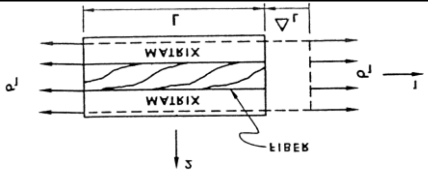
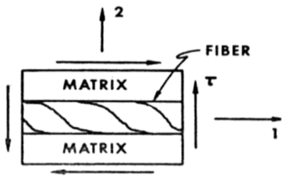
Table (٢-١): The properties and strength which can be calculated by the mechanics of material approach

The case	RVE with direction of loading	The results	References
E١		Rule of mixtures E١= $E_f V_f + E_m V_m$	Robert,١٩٩٦ and Krishan , ١٩٩٩
E٢		Apparent young's modulus in the transverse direction of fiber $\frac{E_2}{E_m} = \frac{1}{V_m + V_f \left(E_m / E_f\right)}$	Jones,١٩٧٥ and Krishan, ١٩٩٩
ν_{12}		Rule of mixtures $\nu_{12} = \nu_m V_m + \nu_f V_f$	Jones, ١٩٧٥ and Bolton, ١٩٩٧
G١٢		Apparent Shear modulus $\frac{G_{12}}{G_m} = \frac{1}{V_m + V_f \frac{G_m}{G_f}}$	Jones, ١٩٧٥
σ_c	As same in case E١	Rule of mixtures $\sigma_C = \sigma_f V_f + \sigma_m V_m$	Robert, ١٩٩٦
ρ_c		Rule of mixtures $\rho_c = \rho_f V_f + \rho_m V_m$	Robert, ١٩٩٦ and Bolton, ١٩٩٧