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Variational Problems in Fluid Flow and Heat Transfer

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By

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وزارة التعليم العالي والبحث العلمي
جامعة بابل
كلية التربية للعلوم الصرفة
قسم الرياضيات

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بحث مقدم الى
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من قبل الطالب

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بسم الله الرحمن الرحيم

﴿ اقْرَأْ بِاسْمِ رَبِّكَ الَّذِي خَلَقَ ﴿١﴾ خَلَقَ الْإِنسَانَ مِنْ عَلَقٍ ﴿٢﴾ اقْرَأْ وَرَبُّكَ الْأَكْرَمُ ﴿٣﴾
الَّذِي عَلَّمَ بِالْقَلَمِ ﴿٤﴾ عَلَّمَ الْإِنسَانَ مَا لَمْ يَعْلَمْ ﴿٥﴾ ﴾

صدق الله العظيم

سورة العلق

Dedication

To….

My father.. …

My Mother ……

My Dearest and Beloved Family:

Asid , Ossama , Mohammed, Zena

And My Friends.

With Respect and Love .

Acknowledgment

" In the Name of Allah , The Gracious , The Merciful "

Praise is to Allah , for providing me the willingness and strength to accomplish this work .

I would like to express my sincere gratitude and appreciation to my supervisor , Sahar Muhsen Jabbar , Department of Mathematics , College of Education for Pure Sciences, University of Babylon for her invaluable guidance and help throughout the preparation of this work . My deepest thanks are due to Dr. Azal Jaafar Musaa , Head of Mathematics Department . Finally , I would like to express my gratitude to all other individuals who contributed in a way or another to the fulfillment of this work.

Supervisor's Certification

We certify that the preparation for this project entitled " **Variational Problems in Fluid Flow and Heat Transfer"** for the student **" Ali Ahmed Kareem "** was made under my supervision at University of Babylon, College of Education for Pure Sciences as a partial fulfillment for requirements of the Degree Higher Diploma Education/ Mathematics.

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In view of available recommendation, I forward this project for debate by the examining committee.

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<u>Examining Committee Certification</u>

We certify that we have read this project entitled**" Variational Problems in Fluid Flow and Heat Transfer"** as examining committee examined the student **" Ali Ahmed Kareem"** in its contents and that in our opinion it is adequate for the partial fulfillment of requirement for the Degree Higher Diploma Education/ Mathematics.

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Abstract

The main objective of this research is studying the variational formulation of initial – boundary value problems .

This variational formula is an approximated technique with wide application throughout applied mathematics . So , some practical examples such as simple rectangular dam problem and heat transfer problem are represented with variational formulation.

الملخص

الهدف الرئيسي من هذا البحث هو دراسة الصياغة التغايرية لمسائل القيم الحدودية و الابتدائية .

هذه الصياغة التغايرية هي تقنية تقريبية مع تطبيق واسع في مجال الرياضيات التطبيقية لذلك تم تمثيل بعض الامثلة العملية مثل مسألة السد المستطيل البسيطة و مسألة انتقال الحرارة بالصيغة التغايرية .

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Introduction

The name calculus of variations comes from procedures of Lagrange involving an operator δ called a variation, but this restricted meaning has long been outgrown . The calculus of variations broadly interpreted includes all theory and practice concerning the existence and charac-terization of minima , maxima , and other critical values of a real-valued functional [16].

The calculus of variations may be said to begin with Newton's minimal resistance problem in 1687, followed by the brachistochrone curve problem raised by Johann Bernoulli (1696) [20].

It immediately occupied the attention of Jakob Bernoulli and the Marquis de l'Hôpital, but Leonhard Euler first elaborated the subject, beginning in 1733. Lagrange was influenced by Euler's work to contribute significantly to the theory. After Euler saw the 1755 work of the 19-year-old Lagrange, Euler dropped his own partly geometric approach in favor of Lagrange's purely analytic approach and renamed the subject the calculus of variations in his 1756 lecture Elementa Calculi Variationum [21].

Legendre (1786) laid down a method, not entirely satisfactory, for the discrimination of maxima and minima. Isaac Newton and Gottfried Leibniz also gave some early attention to the subject [22].

To this discrimination Vincenzo Brunacci (1810), Carl Friedrich Gauss (1829), Siméon Poisson (1831), Mikhail Ostrogradsky (1834), and Carl Jacobi (1837) have been among the contributors. An important general work is that of Sarrus (1842) which was condensed and improved by Cauchy

(1844). Other valuable treatises and memoirs have been written by Strauch (1849), Jellett (1850), Otto Hesse (1857), Alfred Clebsch (1858), and Carll (1885), but perhaps the most important work of the century is that of Weierstrass. His celebrated course on the theory is epoch-making, and it may be asserted that he was the first to place it on a firm and unquestionable foundation. The 20th and the 23rd Hilbert problem published in 1900 encouraged further development [22].

In the 20th century David Hilbert, Emmy Noether, Leonida Tonelli, Henri Lebesgue and Jacques Hadamard among others made significant contributions [22].

Marston Morse applied calculus of variations in what is now called Morse theory [23].

Lev Pontryagin, Ralph Rockafellar and F. H. Clarke developed new mathematical tools for the calculus of variations in optimal control theory[23]. The dynamic programming of Richard Bellman is an alternative to the calculus of variations .

Variational theory has connections with such fields as mathematical physics,differential geometry, mathematical statistics, conflict analysis, and the whole area of optimal design and performance of dynamical systems[16].

The phrase "variational formulation" has been used recently in connection with generalized formulations of boundary - or initial - value problems.

However, in the classical sense of the phrase, it also has to do with the minimization of a quadratic functional, which includes all of the intrinsic features of the problem, such as the governing equations, boundary and / or initial conditions , constraint conditions , and even jump conditions. Variational formulations , in either sense of the phrase, suggest new theories , provide a means for studying mathematical properties of solutions , and most importantly, provide natural means of approximation Variational formulations can be useful in three related topics. First, many problems of mechanics are posed in terms of finding the extremum , (i.e , minima or maxima) and thus by their nature can be formulated in terms of variational statements. Second , there are problems that can be formulated by her means , such as vector mechanics (e.g. Newton's laws), but these can also be formulated by means of variational principles (see[11]) Third , variational principles form a powerful basis for obtaining approximate solutions to practical problems, many of which are intractable otherwise .

This research consists of three chapters :

Chapter One gives some preliminary definitions and concepts about the calculus of variation . Chapter Two , in this chapter we discuss the variational technique of simple Rectangular dam problem .Chapter Three , in this chapter we discuss the variational technique for some heat transfer problems with example .

Chapter One

Preliminaries

1.1 Introduction :

A variational formulation is also called Magri's approach . It will be applied to the both types of integral equations Fredholm and Volterra integral equations . The intent of this chapter , is to give an explaination and representation of the variational formulation for solving integral equations . For more accuracy , integral equations is called the direct problem [9].

We should mention that this variational formulation also can be used for problem called the inverse problem which related to the integral equations , In another words , we have two problems , direct and inverse problem , the direct one is a problem consist of the determination of the mapping $D: S \to T$. from the set of all possible values into set of all possible data , whereas the inverse problem is the determination of the values S from the measured data T [9].

Symbolically , the direct problem is a mapping D:S→T while an inverse problem is a mapping $D^{-1}: T \to S$ [9] . The subject of inverse problem is discipline that provides mathematical tool for the efficient use of data in the estimation of parameters appearing in mathematical models , which may be in the form of algebraic , differential , or integral equations , and their associated initial and boundary conditions , so for aiding in modeling of phenomena .

Several mathematical methods have been developed to solve inverse problem , usually , the method utilized for solving inverse problem depends on the nature of direct problem such as the Bellman – Adomian method , the Epsilon method , and the pulse – Spectrum –

Technique (PST) method , that relative to partial differential equation for more details , see [1 , 4] .

In our work we just focus in the direct problem and use variational technique to solve these problems .

In this chapter , applications of Mgri's approach are discussed such as the vibrating spring equation and gas diffuses phenomenon .

1.2 Basic Definitions and Theoretical concepts:

Following are some of important definitions and basic notations that are related to our work in this area and they are given for completeness purpose .

1.2.1 Definition (Linear space) [10]:

A linear space over a field F is a collection of elements X with two defined operations . The first is an addition of elements in X and the second is a multiplication of elements in X by scalars in F . we stipulate the following conditions:

1. Additive operation for all $m, n \text{ and } w \in X$:

(a) The additive operation is closed , so that $(m + n)$ belongs to X .

(b) The operation is associative : $(m + n) + w = m + (n + w)$.

(c) There exists an identity 0 for which $m + 0 = m \text{ for all } m \in X$.

(d) For every m , there exists an inverse element denoted $(-m)$ such that , $(-m) + m = 0$.

(e) The additive operation is commutative : $m + n = n + m$

$for\ all\ m, n\ in\ X$.

2. Multiplication by scalars is closed that is ,

(a) $1.m = m \; for \; all \; m \; in \; X$.

(b) $\alpha.m \; is \; in \; X \; for \; all \; m \; in \; X \; and \; \alpha \; in \; F$.

(c) For all $\alpha, \beta \; in \; F \; and \; m \; in \; X : \alpha(\beta m) = (\alpha\beta)m$.

3. The following distributive laws hold .

$(a) \quad \alpha\,(m+n) = \alpha\, m + \alpha\, n \; for \; all \; \alpha \; in \; F \; and \; m, n \; in \; X$.

$(b) \quad (\alpha+\beta)\, m = \alpha\, m + \beta m \; for \; all \; \alpha, \beta \; in \; F \; and \; m \; in \; X$.

1.2.2 Definition (Normed Linear space) [13] :

A linear space M is said to be a **normed linear space** , if it is endowed with a non – negative number (real valued function) $\|m\|$, called a norm of m , such that :

(1) $\|m\| = 0 \; if \; and \; only \; if \; m = 0$.

(2) $\|\alpha m\| = |\alpha|\|m\| , \alpha \; is \; scalar \; (real/complex)$.

(3)$\|m_1 + m_2\| \leq \|m_1\| + \|m_2\| \; for \; all \; m_1, m_2 \in M$

$(tringle \; inequality)$.

1.2.3 Definition (Linear operator) [1]:

For any two normed linear space M and N , let a function $L : M \rightarrow N$, then L is called a linear operator if the domain D(L) is sup space of M and if the following conditions are satisfied :

(1) $L(m_1 + m_2) = L(m_1) + L(m_2)$

(2) $L(\alpha m) = \alpha L(m)$,

Where α is scalar and m_1 , m_2 are vectors in D(L) .

Note that : if D(L) = M , then L is a linear operator from M onto N .

1.2.4 Definition (Functional) [7] :

A functional is an operator $F : M \rightarrow N$ where its range lies in the real line R or in the complex C . A functional F on a linear space M be linear if :

(1) $F(m + n) = F(m) + F(n)$

(2) $F(\alpha\, m) = \alpha\, F(n). For\ any\ m, n \in M\ and\ any\ scalar\ \alpha$.

1.2.5 Definition (Bilinear Form) [6]

Let M and N be two normed linear spaces , a bilinear form defined on M and N is a functional F: M × N → R , which is linear in both M and N , where m and n are elements of M and N respectively , and the following properties are fulfilled :

(1) $F(m + wn) = F(m, n) + F(w, n)$.

(2) $F(\alpha m, n) = \alpha F(m, n)$.

(3) $F(m, n + w) = F(m, n) + F(m, w)$.

(4) $F(m, \alpha n) = \alpha F(m,n)$.

For any $\alpha \in R\ and\ m, n, w\ belonging\ to$ normed linear space M .

This Functional is usually denoted by symbol $\langle m, n \rangle$.

1.2.6 Definition (symmetric Bilinear Form) [6] :

Let M and N be two normed linear space and $\langle m, n \rangle$ be a bilinear form , then $F(m,n) = \langle m, n \rangle$ is said to be symmetric if :

$\langle m, n \rangle = \langle n, m \rangle, For\ all\ \langle m, n \rangle \in M \times N$.

It is clear that a linear operator L is said to be **symmetric linear operator** with respect to chosen bilinear form ⟨m,n⟩ , if :

$$\langle L\, m, n \rangle = \langle L\, n, m \rangle, \forall\, m \in M, n \in N$$

1.2.7 Definition (Non- Degenerate Bilinear form) [6] :

The bilinear form $\langle m, n \rangle$ is said to be non-degenerate on M and N if :

(1) $F(m, \bar{n}) = \langle m, \bar{n} \rangle$ =0 then $\bar{n} = 0$, $For\ every\ m \in M$.

(2) $F(\bar{m}, n) = \langle \bar{m}, n \rangle = 0\ then\ \bar{m} = 0, For\ every\ n \in N$.

The following are some examples of non-degenerates bilinear forms :

$$\langle m, n \rangle = \int_0^T m(t)n(t)dt\ For\ m, n : \mathrm{C}[0,T] \to R, T > 0 .$$

$$\langle m, n \rangle = \int_0^T \sum_n m(t)_n n(t)_n dt\ For\ m, n : \mathrm{C}[0,T] \to R^n, T > 0 .$$

$$\langle m, n \rangle = \int_0^T m(t)n(T-t)dt\ \ For\ m, n : C[0,T] \to R, T > 0 .$$

$\langle m,n\rangle = \int_0^T m(t)\left[\int_{s=0}^{s=T} k(s,t)n(s)ds\right]dt \; For \; m,n : C[0,T] \rightarrow R \; , T > 0 \; .$

1.3 calculus of variation [3, 5, 17] :

The subject of calculus of variation is an importance branch of applied mathematics of calculation (evaluation) the minimum or maximum value of functional .

In another words , the solution of any problem (ordinary differential equations , partial differential equations , integral equation , ect) will be equivalent to the solution of minimizing a functional F which corresponding to the linear equation in operator form $Lm = F$.

Basically , the idea of calculus of variation is the derivation of an equation is called the Euler-Lagrange equation that represents the linear problem that have to be solved in operator form ,

The derivation of this equation depends on the so called the first variational of functional as it is explained in the following definition .

1.3.1 Definition :[8]

A variational of a functional $F[m]$ which is defined on some normed linear space can be represented by :

$$\delta F[m] \equiv F[m+\delta m] - F[m]$$

Such that δ is the customary symbol of variational function .

1.3.2 Theorem [6]:

There is a variational problem which is given by :

$$F[m] = \frac{1}{2}\,(Lm, Lm) - (f, Lm)$$

$corresponding\ to\ linear\ equation\ \ Lm = f$, if and only if the operator L is symmetric relative to non-degenerate bilinear .

Proof : ⇐)

Since $\boldsymbol{F}[\boldsymbol{m}] = \frac{1}{2}\,(Lm, Lm) - (f, Lm), \quad then$

$$\delta F[m] = \frac{1}{2}\langle L(m+\delta m), (m+\delta m)\rangle - \langle f, m+\delta m\rangle - \frac{1}{2}\langle Lm, m\rangle + \langle f, m\rangle$$

$= \frac{1}{2}\langle Lm + L\delta m, m+\delta m\rangle - \langle f, m\rangle - \langle f, \delta m\rangle - \frac{1}{2}\langle Lm, m\rangle + \langle f, m\rangle$

$= \frac{1}{2}\langle Lm, m\rangle + \frac{1}{2}\langle Lm, \delta m\rangle - \frac{1}{2}\langle L\delta m, m\rangle + \frac{1}{2}\langle L\delta m, \delta m\rangle - \langle f, \delta m\rangle - \frac{1}{2}\langle Lm, m\rangle$

$= \frac{1}{2}\langle Lm, \delta m\rangle + \frac{1}{2}\langle L\delta m, m\rangle + \frac{1}{2}\langle L\delta m, \delta m\rangle - \langle f, \delta m\rangle$

Hence $\langle .\,, .\rangle$ is symmetric , then $\langle L\delta m, m\rangle = \langle Lm, \delta m\rangle$

So :

$\delta F[m] = \langle Lm, \delta m\rangle - \langle m, \delta m\rangle = \langle Lm - m, \delta m\rangle$ ….(1.1)

Now , if m^ is a solution (**critical point**) of (3.1) , leads to that

$\langle Lm\hat{} - f, \delta m\hat{}\rangle$=0 for an arbitrary δ , m^ since ⟨.,.⟩ is non-degenerate

bilinear from on $D(L)\ and\ R(L)$, therefore : $Lm\hat{}\ - f\ = 0$. And hence :

$Lm\hat{}\ = f$

⇒) suppose that m is a solution of $Lm = f$ which implies that

$Lm - f = 0$,

and hence :

$(Lm - f)^2 = 0$ …….(1.2)

$Integrating\ both\ sides\ of$ (1.2):

$$\int_0^T (Lm - f)^2 ds = 0 \Rightarrow \int_0^T ((Lm)^2 - 2fLm)^2 ds + \int_0^T f^2 ds = 0\ ,$$

$suppose$

$\int_0^T f^2 ds = k, since\ \langle m, n\rangle = \int_0^T m.n\ ds$

So :

$\langle Lm, Lm\rangle = \int_0^T Lm.Lm\ ds = \int_0^T (Lm)^2 ds$

And

$\langle f, Lm\rangle = \int_0^T f.Lm\ ds$.

Thus we have :

$\langle Lm, Lm\rangle - 2\langle f, Lm\rangle + k = 0$

$which\ means\ that$:

$$\frac{1}{2}\langle Lm, Lm\rangle - \langle f, Lm\rangle + c = 0\ , c = \frac{k}{2}.$$

i.e $\therefore F[m] + c = 0$,

this means that m is a critical point to $F[m]$.

It is clear that , for a symmetric linear operator L related to non-degenerate from $\langle m, n\rangle, there$ is a variational formulation for a given linear operator $Lm = f$.

However , if a given linear operator L is not symmetric with respect to the chosen bilinear from $\langle m,n\rangle$, then to find the variational formulation , we have two procedures .

The **first one** is to choose bilinear from $\langle m,n\rangle$ seeking for ways to modify the given equation to obtain a new equation that will be symmetric with respect to the chosen bilinear form .

The **second procedure** is to change the bilinear form so that the given linear operator will be symmetric with respect to the new bilinear form $[\mathbf{11}]$.

To illustrate the above discussion , let us consider an arbitrary symmetric bilinear form and a linear operator L as follows :

$$\langle m,n\rangle = \langle m,Ln\rangle \qquad \ldots\ldots(1.3)$$

$defined\ For\ n \in N\ and\ m \in D(L).$

$It\ is\ simpl$ to prove that the bilinear form (1.3) makes the given linear operator symmetric such as follows :

$$\langle Lm_1,Lm_2\rangle = \langle Lm_1,Lm_2\rangle = \langle Lm_2,Lm_1\rangle = \langle Lm_2,m_1\rangle \quad \ldots\ldots(1.4)$$

$So\ ,we\ can\ use\ the\ bilinear\ form$ (1.1) to get a variational formulation to the linear equation $Lm = f$. Using (Theorem 1.1) and because of the symmetry of L , the solution of the linear equation $Lm = f$ is the critical points of the functional :

$$\mathrm{F}[m] = \frac{1}{2}\langle Lm,m\rangle - \langle f,m\rangle$$

$$F[m] = \frac{1}{2}\,\langle Lm, m\rangle - \langle f, Lm\rangle \qquad \text{.........(1.5)}$$

In general , the bilinear form (1.3) is degenerated on $D(L)$ and $R(L)$ which leads to have more critical points than the solutions for a given linear operator $Lm = f$.

Therefore , all of these means that there are other requirements for the bilinear form in order to have all critical points of $F[m]$ be solutions for $Lm = f$, it is a necessary and sufficient condition for chosen bilinear form in which the bilinear form must be non-degenerated on the range $R(L)$ for the given linear operator L .

Hence , the functional (1.5) associated with non-degenerate and symmetric bilinear form $\langle m_1, m_2\rangle$ solves the problem of evaluating a variational formulation for a given linear equation $Lm = f$, **[6]**

For more details , see **[1, 2]** .

Chapter Two
Variational Technique of Simple Rectangular Dam Problem

2.1 Introduction

This chapter deals with variational formulation of the simple rectangular dam problem by using classical variational formulation . In this problem the governing PDE is the Laplace equation .

The objective of this chapter is to define the initial boundary value problems as well as the physical and mathematical interpretation and some properties of the problem .

2.2 Variational Formulation of Initial-Boundary Value Problems (I.B.V.P's.)

Consider the linear equation :

$Lu = f$ ……………………………….(2.1)

Where u denotes scalar vector valued function and L denotes a linear operator with domain D(L) in a linear space U and range R(L) in a second linear space V .

The aim here is to search for a functional F$[u]$ defined on the domain of the linear operator L , whose critical points are the solutions of the given equation (2.1) .

This problem may be called the " inverse problem " of calculus of variations , while the usual problem of finding the critical points of a pre-assigned functional may be called the " direct problem " .

2.3 symmetry Laplace's operator [4]:

Because of the importance of the operator used in the derivation of the variational formulation . We represented Laplace's operator and explained the symmetry Laplace's operator ,

$$\mathcal{L} = \frac{\partial^2}{\partial_{s^2}} + \frac{\partial^2}{\partial_{t^2}} ,$$

relative to the bilinear form :

$\langle \mathrm{m}, \mathrm{n} \rangle = \int_{\mathbf{0}}^{\boldsymbol{\tau}} m\, n\, ds$ **.** consider :

$\langle \mathcal{L}m, n \rangle = \iint_D (m_{ss} + n_{tt}) n\, ds\, dt \, \langle \mathcal{L}m, n \rangle = \iint_D (m_{ss}.n + m_{tt}.n) ds\, dt$

$\langle \mathcal{L}m, n \rangle = \iint_D \frac{\partial}{\partial s}((m_s.n) - m_s n_s + \frac{\partial}{\partial t}(m_t.n) - m_t.n_t) ds\, dt$

$\langle \mathcal{L}m, n \rangle = \iint_D \left[\frac{\partial}{\partial s}(m_s.n) + \frac{\partial}{\partial t}(m_t.n)\right] ds\, dt - \iint_D [(m_s.n_s) + (m_t.n_t)]\, ds\, dt$

And using **Green's Theorem* :**

$\langle \mathcal{L}m, n \rangle = \iint_D \frac{\partial p}{\partial s} + \frac{\partial q}{\partial t} = \int_C p\, ds + q\, dt \,, \ \ with\, p = m_s n\ and\ q = -m_t n$

We have :

$\int_C p\, ds + q\, dt = 0$

(since m and n equals to zero on the boundary c of the region D) .

Green's Theorem* : Let p(s,t) , q(s,t) be two real-valued function and $\frac{\partial p}{\partial s}, \frac{\partial q}{\partial t}$ are continuous everywhere on a closed area D which consists points with in and on the simple closed contour C , then :$\int_C (p(s,t)ds + q(s,t)dt\,) = \int\int_D \left(\frac{\partial p}{\partial s} + \frac{\partial q}{\partial t}\right) ds\, dt$.

Therefore ,

$\langle \mathcal{L}m, n \rangle = -\iint_D [(m_s . n_s) + (m_t . n_t)]\, ds\, dt$

From the above result , one can easily obtains that the linear operator L is symmetric relative them to the non- degenerate bilinear from chosen .

2.4 Variational Formulation Of Laplace Equations [1, 4] :

Theorem (1.1) could be applied to the example of evaluating the functional corresponding to the equation

$m_{ss} + m_{tt} = 0$.

Because of its importance . Consider the last PDE with the operator L given by :

$\mathcal{L} = \frac{\partial^2}{\partial_{s^2}} + \frac{\partial^2}{\partial_{t^2}}$ …. (1.6)

and the corresponding functional is given by :

$J(m) = \frac{1}{2}(Lm, m) - (f, m)$

$Similary :\ \langle \mathcal{L}n, m \rangle = -\iint_D [m_s n_s + m_t n_t] ds\, dt$

i.e : $\langle \mathcal{L}m, n \rangle = \langle \mathcal{L}n, m \rangle$.

And since $f = 0$ hence upon expanding the inner product , then the functional $J(m)$, take the form :

$J(m) = \frac{1}{2}\iint_D (m_{ss} + m_{tt}) m ds\, dt$

$J(m) = \frac{1}{2}\iint_D (m_{ss} . m + m_{tt} . m) m ds\, dt$

And recalling the following :

$$\frac{\delta}{\delta s}(m.m_s) = m.m_{ss} + m^2{}_s$$

And

$$\frac{\delta}{\delta s}(m.m_t) = m.m_{tt} + m^2{}_t$$

Therefore :

$$J(m) = \frac{1}{2}\iint_D \frac{\partial}{\partial s}((-m^2{}_s - m^2{}_t) + \frac{\partial}{\partial s}(m.m_s) - \frac{\delta}{\delta t}(m.m_t))ds\, dt$$

$$J(m) = \frac{1}{2}\iint_D \frac{\partial}{\partial s}(m^2{}_s + m^2{}_t)dsdt + \frac{1}{2}\iint_D \frac{\partial}{\partial s}(m.m_s) - \frac{\delta}{\delta t}(m.m_t))\, ds\, dt$$

By **Green's theorem :**

$$J(m) = \iint_D \frac{\partial p}{\partial s} + \frac{\partial q}{\partial t} = \int_C p\, ds + q\, dt$$

$with\ p = m_s n\ and\ q = -m_t n\ ,\ we\ have : \int_C p\, ds + q\, dt = 0$.

The final from of J , will be :

$$J(m) = \iint_D \frac{\partial}{\partial s}(m^2{}_s + m^2{}_t)dsdt\ .$$

2.5 Simple Rectangular Dam :

The simplest case of seepage through a dam separating two reservoirs at different levels can be regarded as the stationary saturated flow of an incompressible fluid through a homogeneous isotropic medium . The simple rectangular dam which has parallel vertical walls and an impervious horizontal base has been much studied and adopted by many authors as a model problem for assessing new methods of solving free-boundary problem .

2.5.1 Classical Formulation (Physical Derivation) [18]:

The physical derivation of the mathematical equations will first be given with reference to fig . (2.1) The dam is regarded as being long enough for the flow to be considered as two-dimensional in the xy – plane .

The flow is taken to be laminar (the layers of fluid slide smoothly , one upon another) and to be governed by Darcy's law which is expressed in the form :

$$q_1 = \frac{-k}{pg}\ grad \quad h = \frac{-k}{\mu}\ grad\ (p + pgy) \ \dots\dots\dots\dots\dots \quad (2.2)$$

Where q_1 is the velocity vector , p is the fluid pressure in the fluid , h is the hydraulic head , p is density , μ is the viscosity of the fluid , the scalar constant K is called the hydraulic conductivity , and k = (μK)/(pg) is the permeability of the medium .

The vertical coordinate y is positive up – words .

For the assumptions already made about the fluid k is constant and so the function

$$\emptyset_1 = \left(\frac{k}{\mu}\right)(p + pgy) \dots\dots\dots\dots\dots\dots\dots (2.3)$$

is a velocity potential and $q_1 = -grad\ \emptyset_1$

Assuming that the fluid is incompressible (the fluid has constant density ,

p = constant) , and that the medium is of uniform porosity and the flow is uniform (the cross section , shape and area , through which the flow occurs remains constant) .

The equation of continuity satisfied by the velocity field is

$div\, q_1 = 0$ and the velocity potential therefore satisfies Laplace's equation:

$div\,(grad\, \emptyset_1\,) = \nabla^2 \emptyset_1 = 0$ ………………………. (2.4)

i.e , $\emptyset_1$ satisfies Laplace's equation in the seepage region Ω in fig (2.1) .

It is convenient to introduce a modified velocity potential :

$\emptyset(x, y) = \frac{\mu \emptyset_1(x,y)}{Kpg} = \frac{p}{pg} + y$……………………….(2.5)

So that

$q = -grad\, \emptyset \;\; and \;\; \nabla^2 \emptyset = 0$ ……………..(2.6)

Where $q = \frac{\mu q_1}{Kpg}$ is a modified flow rate .

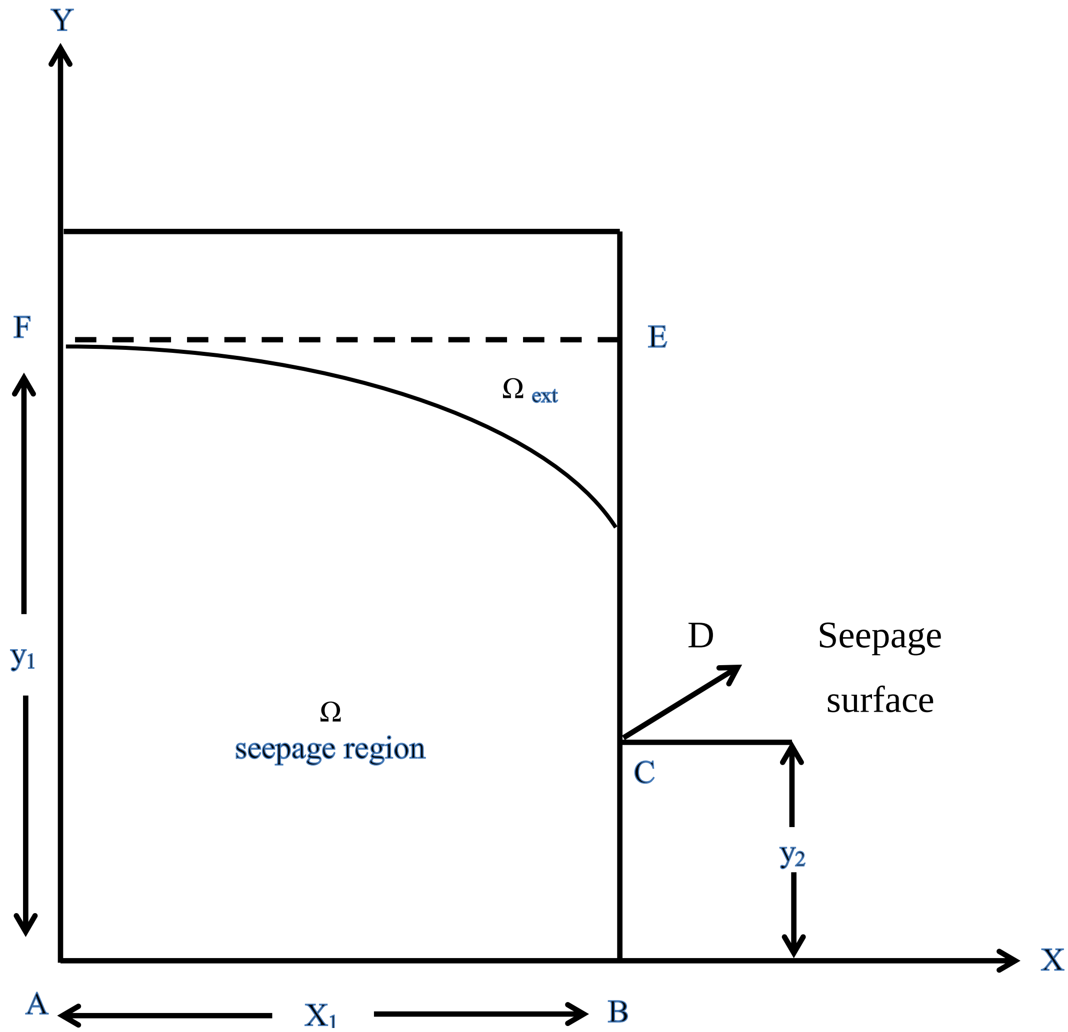

Fig . (2.1) Simple rectangular dam

2.5.2 Mathematical Formulation of the problem :

The seepage region is bounded by part of the walls of the dam Af and BD and its base AB but also by the free surface FD whose shape and position are to be determined , including the location of the " point of detachment " D on the wall BE . The part of the boundary CD , known as the " seepage surface" , must exist for physical reasons .

The conditions to be satisfied by $\emptyset(x, y)$ on the different parts of the boundary of the region Ω are derived as follows :

Since there can be no flow across an impervious surface the normal derivative $\frac{\partial \emptyset}{\partial n}$ must be zero on any such surface ,e.g . on the base AB , $\frac{\partial \emptyset}{\partial y} = 0$. Since the free surface FD is the interface between the eater in flow region Ω and the air above , into which no water penetrates , the condition $\frac{\partial \emptyset}{\partial n}$ holds on FD also . The second condition on the free boundary is that pressure must be continuous across it . But outside the flow region and above the two reservoirs the pressure is constant and may be taken to be zero . Putting p = 0 in (2.5) gives $\emptyset =$ y on FD .

Similarly on the seepage surface CD $\emptyset =$ y , since p = 0 , because again the water there is in contact with air . The condition along a porous boundary in contact with a reservoir is negligible compared to that in the porous medium . A zero velocity means the velocity potential $\emptyset =$ y+p/(pg) is constant and since p = 0 at a reservoir surface we have $\emptyset = y_1$ or $\emptyset = y_2$ along AF or BC respectively (Fig . (2.1)) .

As a summary , the mathematical formulation of the simple rectangular dam is therefore defined with reference to Fig . (2.1) by the following equation and conditions :

$\nabla^2\emptyset = 0 \;\; in\,\Omega\,, 0 < x < x_1 \;\;, 0 < y < f(x)$ ………….(2.7)

$\emptyset = y_1$ on AF , $\emptyset = y_2$ on BC …………………………(2.8)

$\emptyset = y \;, \frac{\partial\emptyset}{\partial n} = 0$ on FD ………………………………..(2.9)

$\emptyset = y$ on CD ……………………………………..(2.10)

$\frac{\partial\emptyset}{\partial y} = 0$ on AB ….. ………………………………(2.11)

$\frac{\partial\emptyset}{\partial x} \leq 0$ on CD ………………………………………..(2.12)

The condition (2.12) expresses the fact that some fluid leaves the porous medium on the seepage face CD , but none can enter the medium there .

The complete solution includes the determination of the free boundary

$y = f(x)$ such that :

$$\left.\begin{array}{l} f(0) = y_1 \\ f(x_1) > y_2 \\ \frac{d}{dx}f(0) = 0 \\ \frac{d}{dx}f(x_1) = \infty \end{array}\right\} \quad \text{…………………………} \; (2.13)$$

In this problem , orthogonal to the equipotential curves $\emptyset$ = constant there will be streamlines along which $\partial\emptyset/\partial n$ is zero . It follows that the impervious base and the free surface are both streamlines . Fluid particles move along a streamline always in the same direction from the upper to the lower reservoir which is the direction of x increasing in Fig .(2.1) .

Thus the fluid velocity is positive in the flow region and $\emptyset$ must decrease along any streamline . In particular , along the streamline free surface $\emptyset$ is a decreasing function of x , and since on FD $\emptyset = y = f(x)$, it follows that f(x) is monotone decreasing .

Mathematical properties of the free boundary $y = f(x)$ have been examined for the simple dam and other problem by several authors [**19**].

Finally , because the base and the free surface are streamlines they define a stream tube through any section of which the fluid flow is the same . In particular through any vertical section of the dam we have a constant flow rate given by

$$-\int_0^{f(x)} \frac{\partial}{\partial x}\emptyset(x,y)dy = \; constant \; = q \quad \ldots\ldots\ldots\ldots.. (2.14)$$

And sometimes referred to as the “ discharge”.

Variational Technique of the Simple Rectangular Dam :

As is known , variational method are one of the most important tools that can be used to solve many complicated problems of mathematical physics in general and free boundary value problem in particular . To solve the problem under consideration through variational approach , this problem must be formulated as a variational problem . To make such a formulation , first let $\langle u, v\rangle$ be the symmetric , non – degenerate ,
bilinear form defned by :

$$\langle u, v\rangle = \iint_{\Omega} u\, v\, r\, d\theta \quad \ldots \ldots \ldots \ldots \ldots \ldots (2.17)$$

Where

$u: \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$

Second , the linear operator ∇^2 must be symmetric with respect to the non-degenerate , bilinear from (2.17) .

Now , constact the functional

$$\mathrm{F}[u] = \frac{1}{2}\langle Lu, u\rangle - \langle f, u\rangle \quad \ldots \ldots \ldots \ldots \ldots (2.18)$$

The functional (2.18) may be simplified and written in the from

$F[p] = \frac{1}{2}\iint_{\Omega} \left\{\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}\right\} p\ r\ dr\ d\theta$ ……………(2.19)

and (2.19) can be reduced to a more familiar from as follows :

$$F[p] = \frac{1}{2}\left\{\frac{\partial^2 p}{\partial r^2}\,p\,r + \frac{\partial p}{\partial r}\ p + \frac{1}{r}\frac{\partial^2 p}{\partial \theta^2}\ p\right\} dr\, d\theta$$

$$= \frac{1}{2}\iint_{\Omega}\left\{\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}p\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial p}{\partial \theta}p\right)\right\} dr\, d\theta - \frac{1}{2}\iint_{\Omega}\left\{r\left(\frac{\partial p}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial p}{\partial \theta}\right)^2\right\} dr\, d\theta$$

Using Green’s theorem , we have :

$$F[p] = \frac{1}{2}\int_{\partial\Omega}\left\{r\frac{\partial p}{\partial r}p d\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}p\ dr\right\} - \frac{1}{2}\iint_{\Omega}\left\{r\left(\frac{\partial p}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial p}{\partial \theta}\right)^2\right\} dr\ d\theta$$

Since p is constant on the boundary $\partial\Omega$, then the above line integral is equal to zero , and the final version of the functional F(p) is given by :

$F[p] = -\frac{1}{2}\iint_{\Omega} \left\{r P_r^2 + \frac{1}{r}\ P_\theta^2\right\} dr\ d\theta$ …………………………..(2.20)

As a notation , it is important to notice that the critical points of the functional (2.20) are the solution of the simple rectangular dam problem (2.7) – (2.12) .

Chapter Three
Variational Technique for some Heat Transfer problems

3.1 Introduction [14]

Variational formulations corresponding to boundry value problems for ordinary and elliptic partial differential equations have been of common use in continuum mechanics and other branches of physics .

For non-elliptic eqations (diffusion and wave equations) , the Dirichlet problem is improperly posed and different conditions on parts of the boundary only have to be assigned .

3.2 The Divergence Theorem

Let $f \equiv f\,(x_1, x_2, \dots\dots, x_n) \in R^n$

be a vector function , and if R is a simple region in R^n

Then

$$\int_R div\, f\, dR = \int_B f\,.n\, dB$$

Where B is a surface on R , and $\vec{n}$ is a unit vector on B .

Where

$$div\, f = \sum_{k=1}^{n} \frac{\partial f_k}{\partial x_k}$$

Such that :

$$\int_R \sum_{k=1}^{n} \frac{\partial f_k}{\partial x_k} dR = \int_B \sum_{k=1}^{n} f_k n_k dB$$

3.3 Variational Principles for some Heat Transfer Problems[15]

It is useful to recall some well-lcnown results of the entropy analysis of thermo-fluid-dynamic processes , e.g., heat (or mass)transfer in fluid flows .

The two principles which are of great interest in this context are the principle of least dissipation of energy and the principle of minimum rate of production of entropy .

Recently , Sciubba showed that if one assumes that the unsteady flow of aviscous fluid assumes at any instant and at every point of therelevant domain a flow configuration such that the exergy destruction rate is minimum , then the solution of this variational problem gives a solution of Navier-Stokes equations compatible with suitable boundary and initial condition .

There is another approach to the above flow and heat transfer problem advanced by Biot using the Lagrangian thermodynamics .This approach is not strictly variational and may be regarded as some sort of a quasi-variational principle .

Consider the energy equation (diffusion equation) . which have to find the solution U for the following diffusion problem in $[0, \mathrm{T}]$

$$-\nabla^2 U + k\frac{\partial U}{\partial t} = \sigma \quad \ldots\ldots\ldots\ldots.\ldots\ldots (3.1)$$

Subject to the initial condition

$$U(x,0) = 0 \qquad , \quad x \in R \quad \ldots\ldots\ldots\ldots (3.2)$$

And the boundary condition

$$U(3,t) = 0 \qquad \ldots\ldots\ldots\ldots (3.3)$$

Where

$\sigma(x,t)$ is defined on V , and K is a positive real constant .

We choose Integral bilinear form

$$(v_1, v_2) = \int_0^T v_1(t)\overline{v_2}(t)dt \ldots\ldots\ldots\ldots\ldots (3.4)$$

Such that

$$\overline{v_2}(t) = \int_0^T k(t,s)v_2(s)ds \ldots\ldots\ldots\ldots\ldots\ldots\ldots. (3.5)$$

Where $k(t,s)$, (kernel function) is symmetric , smooth and non – degenerate function .

So let $k(t,s) \equiv h(t,s) = h(s,t)$ ……………………… (3.6)

Where $0 \leq s \leq T$ and $0 \leq t \leq T$

Because of the non-degeneracy of (3.4) , then the integral transform (3.5) is invertible

So ,

$$\int_0^T h(t,s)v_2(s)ds = 0 \quad \text{……………… (3.7)}$$

has the solution

$$v_2(t) \equiv 0$$

We choose symmetric kernel function from Voltera type , we defined as follow :

$$h(t,s) = \begin{cases} q(t,s) & 0 \leq s \leq T-t \\ 0 & T-t < s \leq T \end{cases}$$

Because of the following Volterra equation

$$\int_{0}^{T-t} q(t,s)v_2\,ds = 0 \quad \dots\dots\dots\dots\dots \quad (3.9)$$

Is kernel – free for any function $q(t,s)$ it is mean , the eq . (3.9) has the solution

$$v_2(t) \equiv 0$$

Therefore , we choose

$$(v_1, v_2) = \int_{R} \int_{0}^{T} v_1(x,t)\overline{v_2}(x,t)dt\,dR$$

Such that

$$\overline{v_2}(x,t) = \int_{0}^{T-t} q(t,s)v_2(x,s)ds$$

Where $q(t,s)$ is symmetric function on

$$0 \leq s \ \leq T - t$$

Therefore , we choose the bilinear from as follows :

$$\langle v,u\rangle = \int_{R} \int_{0}^{T} v(x,t) \left[\int_{0}^{T-t} q(t,s) \frac{\partial u(x,s)}{\partial s} \right] dt \; dR \; \ldots \ldots . (3.10)$$

So , the diffusion operator will be symmetric with respect to all bilinear forms which defined in (3.10) and for all arbitrary symmetric function $q(t,s)$.

So , by using (3.10) , we find the corresponding functional to diffusion equation (3.1) .

$$\therefore \quad F[u] = \frac{1}{2} \langle Lu,u\rangle - \langle f,u\rangle$$

And $\quad f \equiv \sigma \quad , \quad Lu \equiv -\nabla^2 u + k \frac{\partial u}{\partial t}$

Then ,

$$F[u] = \frac{1}{2}\int_R \int_0^T \left[-\nabla^2 u(x,t) + k\frac{\partial u(x,t)}{\partial t}\right]\left[\int_0^{T-t} q(t,s)\frac{\partial u(x,s)}{\partial s}\, ds\right] dt\, dR$$

$$-\int_R \int_0^T \sigma(x,t)\left[\int_0^{T-t} q(t,s)\frac{\partial u(x,s)}{\partial s} ds\right] dt\, dR$$

$$= \frac{1}{2}\int_R \int_0^T \int_0^{T-t} q(t,s)\left[-\nabla^2 u(x,t)\frac{\partial u(x,t)}{\partial s} + k\frac{\partial u(x,t)}{\partial t}\frac{\partial u(x,s)}{\partial s}\right.$$

$$\left. - 2\sigma(x,t)\frac{\partial u(x,t)}{\partial s}\right] ds\, dt\, dR$$

By using (the divergence theorem) and the boundary condition (3.3)

We get

$$\int_R -\nabla^2 u(x,t)\frac{\partial u(x,t)}{\partial s}dR = \int_R \nabla u(x,t).\nabla\frac{\partial u(x,s)}{\partial s}dR$$

$$F[u] = \frac{1}{2}\int_R \int_0^T \int_0^{T-t} q(t,s)\left[\nabla u(x,t).\nabla\frac{\partial u(x,t)}{\partial s} + k\frac{\partial u(x,t)}{\partial t}\frac{\partial u(x,s)}{\partial s} - 2\sigma(x,t)\frac{\partial u(x,t)}{\partial s}\right]ds\,dt\,dR \dots\dots\dots..(3.11)$$

Thus , the critical points of the functional (3.11) are solutions of the diffusion equation which satisfied the conditions (3.2) and (3.3) .

For example **[15]**

If $q(t,s) = 1$, then the bilinear from (3.10) is

$$\langle v,u\rangle = \int_R \int_0^T v(x,t)\left[\int_0^{T-t} \frac{\partial u(x,s)}{\partial s} ds\right] dtdR$$

$$= \int_R \int_0^T v(x,t)[u(x,T-t) - u(x,0)]dtdR$$

By using the condition (3.2) , we get

$$\langle v,u\rangle = \int_R \int_0^T v(x,t)u(x,T-t)dt\, dR \dots\dots\dots\dots (3.12)$$

Which is the convolution bilinear form from (3.11) , we get the corresponding functional .

$$F[u] = \frac{1}{2}\int_R \int_0^T \int_0^{T-t} \left[\nabla u(x,t).\nabla \frac{\partial u(x,s)}{\partial s} + k\frac{\partial u(x,t)}{\partial t}\frac{\partial u(x,s)}{\partial s} - 2\sigma(x,t)\frac{\partial u(x,s)}{\partial s}\right] ds\, dt\, dR$$

$$= \frac{1}{2}\int_R \int_0^T \left[\nabla u(x,t).\nabla u(x,s) + k\frac{\partial u(x,t)}{\partial t} u(x,s) - 2\sigma(x,t)u(x,s) \right]_{S=0}^{S=T-t} dt\, dR$$

$$\therefore \; u(x,0) = 0$$

$$\therefore \; \nabla u(x,0) = 0$$

Therefore ,

$$F[u] = \frac{1}{2}\int_R \int_0^T \left[\nabla u(x,t).\nabla u(x,T-t) + k\frac{\partial u(x,t)}{\partial t}u(x,T-t) - 2\sigma(x,t)u(x,T-t)\right] dt\, dR \;\ldots\ldots\ldots (3.13)$$

We see that the functional has the derivative with respect to time .

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٢-المؤهلات الأكاديمية والعلمية

بكالوريوس رياضيات –كليه التربيه للعلوم الصرفه –جامعه البصره

سنه التخرج   ٢٠٠٦ - ٢٠٠٧

٣- الخبرات الوظيفية

-تعينت مدرس في وزاره التربيه تربيه ميسان سنه   ٢٠٠٨

- مدير في ثانويه علي الغربي للبنين   حاليا سنوات الخدمه : ١٣سنه

-الدورات التدريبيه : التحقت بالعديد من الدورات تطويريه لمدرسي الرياضيات في تربيه ميسان

حاصل على العديد من كتب الشكر من تربيه ميسان