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Ministry of Higher Education
and Scientific Research
University of Babylon
College of Education of pure sciences

On subclasses of harmonic multivalent functions defined by a differential operator

A project
Submitted to Mathematics Department, College of Education of pure sciences
University of Babylon as a partial Fulfillment of the Requirements for the
Degree of Higher Diploma Education / Mathematics

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بسم الله الرحمن الرحيم

يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنْكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ وَاللَّهُ

بِمَا تَعْمَلُونَ خَبِيرٌ

صدق الله العلي العظيم

سورة المجادلة 11

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I certify that this research entitled " **On subclasses of harmonic multivalent functions defined by a differential operator** " for the student **Sarah Adil Hamzah**, was prepared under my supervision in University of Babylon, 66 College of Education for pure Science as a partial requirement for the Degree of Higher Diploma Education /Mathematics.

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We certify that we have read this research entitled "**On subclasses of harmonic multivalent functions defined by a differential operator**" as examining committee examined the student **Sarah Adil Hamzah** in its contents and that in our opinion it is adequate for the partial fulfillment of the requirement for the Degree of Higher Diploma Education/ Mathematics

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Dedication

I dedicate my work to

- ✶ My lovely father and mother
- ✶ My brothers
- ✶ My husband and my kids
- ✶ My friends

For they always being here for me through all of my ups and downs, without even a sigh.

Acknowledgement

All praise and glory to Almighty ALLAH for providing me with the health and strength to finish this work and do something that will benefit humanity.

My thanks and appreciations go to my supervisors **Dr. Aqeel Ketab Al-khafaji** for his guidance, patience, motivation, support, and advice during the research.

List of Contents

List of Symbols

No.	Symbol	Description
1	$\mathbb{C}$	Complex number
2	R	Real number
3	$\mathbb{N}$	Natural Number
٤	$f(z)$	Harmonic function
٥	h	Valued harmonic function
٦	γ, μ	Real harmonic function
٧	ψ_q^n	Operator of the harmonic function
٨	S_{Hp}	Set of allp valent harmonic function
9	U	Unit disk
10	$D_q\, f(z)$	q-derivative operator Dq of function
١1	$\mathcal{MH}_{\mu,p}^{n,m}(\gamma, \delta, \mu)$	Sub class of harmonic function
١2	$\mathcal{H}_{\mu,p}^{n,m}(\gamma, \delta, \mu)$	Class of harmonic function
١3	$clco\ \mathcal{MH}_{\mu,p}^{n,m}(\gamma, \delta, \mu)$	Extreme points convex hull of $\mathcal{MH}_{\mu,p}^{n,m}(\gamma, \delta, \mu)$
١4	$\Omega_{q,p}^{n,m}(k, \mu, \gamma, \delta)$	Operator of harmonic function

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تحية طيبة

يسرنا أن نعلمكم بقبول بحثكم الموسوم

(Certain Properties of a New Class of Harmonic P-Valent Functions Specified by a Differential Operator Based on q-Derivative Operator)

للنشر في المجلة **AIP Conference Proceeding** ضمن فعاليات المؤتمر العلمي الدولي الثاني للعلوم الصرفة الذي تقيمه كلية التربية جامعة القادسية للمدة ٢٩-٣٠ حزيران ٢٠٢١.

شاكرين تعاونكم في إنجاح فعاليات المؤتمر

أ.م.د. خالد شياع خيرالله
رئيس محور الرياضيات
٢٠٢١ /٦ / ٢٩

نسخة منه إلى /
لجنة الطباعة لاتخاذ ما يلزم

Abstract

In our work, we introduced and investigated some properties of subclass of multivalent harmonic functions involving the generalized derivative operator, such as extreme points and closure under an integral operator for this class are obtained.

Introduction

In this project we study some geometrical properties of analytic multivalent functions.

Also, we provide a comprehensive introduction to the subclasses of harmonic multivalent function by using some operators. When we write this project we develop the project from some researches.

In [1] Wanas, Abbas Kareem; AlINA, AIb lupas, on a new class of harmonic multivalent function.

In [2] Wanas, Abbas Kareem; Two New Classes of Analytic Function Defined by Strong Differential Subordinations and Superordinations.

We introduce new classes of analytic function defined by operator.

For class.

$$\psi(\lambda, \delta, M, P), \lambda > 0, \quad u = \{z \in C : |z| \leq 1\}$$

CHAPTER 1

Definitions and Examples

1.1 Introduction

This research contains two chapters, in the first chapter study the univalent and analytic functions, multivalent function (P- value), simply connected domain, complex differentiable and some simples.

Chapter two, we deal with an investigation in some the properties of a subclass of P-value function, which are defined by the generalized derivative operator to obtain some results such as extreme points to those subclasses. The choice of the open unit disk u={z∈C:|z|<1}.

Definition 1.2 [17] , suppose that u={z∈C:|z|<1} indicates for open unit disk at the complex plane ℂ.

A function f of complex variable is said to be analytic in a point $z_{\circ}$ if it is derivative is found not only in $z_{\circ}$ but also at all points z in some neighborhoods for z_0. Where be analytical in open unit disk U if is analytic in all points in U. It is an entire function if it is an analytic function in all points at complex plane ℂ.

Definition (simply connected domain) 1.3 [17] let D be a domain D ⊂ C is an open and connected non-empty subset of the complex plane. The domain D is simply connected if both D and C/D are connected.

Definition (complex differentiable) 1.4 [17] A complex function f:D→C defined for all z ∈ D is said to be complex differentiable at $z_\circ$ ∈ D if

$$\lim_{z\to z_\circ} \frac{f(z)-f(z_\circ)}{z-z_\circ} \text{ exists}$$

Definition (analytic function) 1.5 [17] The function f:D → C is analytic at $z_\circ$ (or holomorphic at $z_\circ$) if it is complex differentiable at every point in some neighborhood N($z_\circ$; ∈) of $z_\circ$ ∈ D. We say that f is analytic on D if f is analytic at $z_\circ$ for every $z_\circ$ ∈ D.

For example, let f: D→C, f (z) =1+ z is analytic function.

Definition (univalent function) 1.6 [7] A function f:D → C with property that $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in D$ with $z_1 \neq z_2$ is said to be one-to-one on D (or univalent, or injective) A function f: D → C which both analytic on D and one-to-one on D is called conformal on D.

For example, a function f:D → C, f(z) = z is univalent function.

Extreme Points for the Class $\widehat{\boldsymbol{SH}}^{\boldsymbol{n},\boldsymbol{1}}_{\boldsymbol{p},\lambda}(\alpha, \beta)$ 1.7

We recall that, a function $f \in \mathcal{F} \subset S_H^0$ is called an extreme point of $\mathcal{F}$ if, for $0 < \mu < 1, f = \mu f_1 + (1-\mu)f_2$ implies $f_1 = f_2 = f$ for all f_1 and f_2 in F. The intersection of closed convex subsets of S_H^0 which contain $\mathcal{F}$ is called the closed convex hull of $\mathcal{F}$. We denote by $cl\,\mathcal{F}$, the set of all closed convex hull of $\mathcal{F}$.

Example 1.8 [1]: If a function $\boldsymbol{f(z)} = \frac{\mathbf{i}}{\boldsymbol{z^2}} = \frac{\boldsymbol{2xy+i\,(x^2-y^2)}}{\boldsymbol{(x^2-y^2)^2}}$.

The two functions

$$u(x,y) = \frac{2xy}{(x^2+y^2)^2}\ , \qquad v(x,y) = \frac{(x^2-y^2)}{(x^2+y^2)^2}$$

Have continuous partial derivatives u_x, u_y, v_x, v_y are in all values of (*x*,*y*) ≠ (0,0). Cauchy-Riemann equations are satisfied because.

$$u_x = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}, \qquad u_y = \frac{2x^3 - 6xy^2}{(x^2+y^2)^3},$$

$$v_x = \frac{6xy^2 - 2x^3}{(x^2+y^2)^3}, \qquad v_y = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

Then

$$u_x = v_y = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}, \qquad u_y = -v_x \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$$

Hence *f* is analytic for all z ≠ 0.

Examples 1.9, function *f* (z) = z is a univalent in U, while *f* (z) = $\boldsymbol{z^2}$ is the not univalent function in U. Also

$$f\ (z) = z + \frac{z^{(2n+1)}}{(2n+1)}$$

is a univalent function in U of all positive integers *n*. We will refer to all function *f* as belonging to class *B* an analytic function in an open unit disk U and normalized by the conditions.

$$f(0) = 0\ ,\ f'(0) = 1$$

Example 1.10 [12]:consider a domain

$$D = \{z \in \mathbb{C} : 1 < |z| < 2,\ 0 < \arg z < \frac{3\pi}{2}\}$$

As well the function f: $D \rightarrow \mathbb{C}$ given by $f(z) = z^2$. It is clear that f is analytic on D and locally univalent at every point $z_\circ \in D$ since $f'(z_\circ) = 2z_\circ \neq 0$ for all $z_\circ \in D$.

However, f not univalent on D, since

$$f\left(\frac{3}{2\sqrt{2}} + \mathbf{i}\frac{3}{2\sqrt{2}}\right) = f\left(\frac{-3}{2\sqrt{2}} + \mathbf{i}\frac{3}{2\sqrt{2}}\right) = \frac{9}{4}\mathbf{i}.$$

CHAPTER 2

The Subclass of Harmonic Multivalent $\mathcal{H}_{\mu,p}^{n,m}$ generated by the differential Operator $G_{n,p,q}^{n,m}$

2.1 Introduction

The harmonic functions are well known to have numerous applications in seemingly different areas of medicine, engineering, electronics, physics, aerodynamics, operational research and others. In a simply connected domain D ⊆ ℂ, harmonic maping is univalent of valued harmonic functions $h = v + i\mu$, where v and μ are real harmonic functions in D. We can write $h = r + is$ in any connected domain where r and s are analytical in D. We call v the analytical part and μ the co-analytical part of h.

Ahuja and Jahangiri [11] defined class $S_{Hp}, (p, \in N = \{1,2,3, ...\})$, which consists of all p-valent harmonic functions, $v = r + is$, that orientation preserving in U (open unit disk u= {z ∈ ℂ : $|z| < 1$ }) where r and s are of the form

$$r(z) = z + \sum_{k=p+1}^{\infty} a_k z^k\,, \qquad s(z) = \sum_{k=p}^{\infty} b_k z^k, \qquad |b_p| < 1, p \in \mathbb{N} = \{1,2,3, ...\} \tag{2.1}$$

Note that when the co-analytical part s is zero, S_{Hp} reduces to class $\mathcal{M}_p$ of normalized multivalent analytical functions. As a result, function v, can be expressed for this class as

$$v(z) = z^p + \sum_{k=p+1}^{\infty} a_k\, z^k, \tag{2.2}$$

For $0 < q < 1$,in [5] Jackson defined the q-derivative operator $D_q$of a function f by

$$D_q\, f(z) = \begin{cases} \dfrac{f(z) - f(q_z)}{(1-\mathrm{p})z}, & if\ z \neq 0, \\ f\grave{(}0). & if\ z = 0, \end{cases} \tag{2.3}$$

From (2.3) it follows that if, $v \in \mathcal{M}_p$ has the form (2.2),then

$D_q v(z) = D_q\left(z^{\mathrm{p}} + \sum_{k=1}^{\infty} a_{k+\mathrm{p}}\ z^{k+\mathrm{p}}\right) = [\mathrm{p}]_q z^{\mathrm{p}-1} + \sum_{k=1}^{\infty}\ [k+\mathrm{p}]_q\ a_{k+\mathrm{p}}\ z^{k+\mathrm{p}-1}, z \in \mathbb{E},$

where $[k]_q := \frac{1-q^k}{1-q}$ and thus$\lim_{q\to 1}[k]_q = k.$

Using the above Jackson q-derivative we will define the operator $\partial_{\mathrm{p},q}^{n}: \mathcal{M}_p \to \mathcal{M}_p, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, by

$$\partial_{\mathrm{p},q}^{0} v(z) := v(z), \qquad \partial_{\mathrm{p},q}^{n} v(z) := z D_q\left(\ \partial_{\mathrm{p},q}^{n-1} v(z)\right), n \in \mathbb{N}.$$

Therefore, if $v \in \mathcal{M}_p$ has the form (2.2)it follows that

$$\partial_{\mathrm{p},q}^{n} v(z) = \left(v * \mathrm{G}_{\mathrm{p},q}^{n}\right)(z), \qquad z \in \mathbb{E} \quad , \mathrm{p} \in \mathbb{N}_0\, ,$$

where

$$G_{\mathrm{p},q}^{n}(z) := z^{\mathrm{p}} + \sum_{k=1}^{\infty}\left([k+\mathrm{p}]_q\right)^n z^{k+\mathrm{p}}, \quad z \in \mathbb{E} \quad , \mathrm{p} \in \mathbb{N}\, , n \in \mathbb{N}_0.$$

Moreover $\partial_{\mathrm{p},q}^{n} v(z) = z^{\mathrm{p}} + \sum_{k=1}^{\infty}\left([k+\mathrm{p}]_q\right)^n a_{k+\mathrm{p}}\ z^{k+\mathrm{p}}, \quad z \in \mathbb{E},$

$$\partial_{\mathrm{p},q}^{n} v(z) = z^{\mathrm{p}} + \sum_{k=1}^{\infty}\left([k+\mathrm{p}]_q\right)^n a_{k+\mathrm{p}}\ z^{k+\mathrm{p}}, \quad z \in \mathbb{E},$$

For,$\eta \geq 0$, with the aid of the operator $\partial_{\mathrm{p},q}^{n}$we will define the new q-differential *operator* $\mathfrak{S}_{\eta,\mathrm{p},q}^{n,m} : \mathcal{M}_p \to \mathcal{M}_p$ by

$$\mathfrak{S}_{\eta,\mathrm{p},q}^{n,0}\ v(z) := \ \partial_{\mathrm{p},q}^{n} v(z),$$

From the above definition it follows easily that if $f \in \mathcal{A}$ is of the form (1), then

$$\mathfrak{S}_{\eta,p,q}^{n,m}\ v(z) := \ z^p + \sum_{k=1}^{\infty}\left([k+p]_q\right)^n \ \left(\frac{p+\eta k}{p}\right)^m a_{k+p}\ z^{k+p}, \quad z \in \mathbb{E},\ \ m \in \mathbb{N}_0. \tag{2.4}$$

we modified the operator $\mathfrak{S}_{\eta,p,q}^{n,m}\, v(z)$ of harmonic multivalent function $v = r + \bar{s}$ as

$$\mathfrak{S}_{\eta,p,q}^{n,m}\, v(z) = \mathfrak{S}_{\eta,p,q}^{n,m}\, r(z) + \overline{\mathfrak{S}_{\eta,p,q}^{n,m}\, s(z)}, \tag{2.5}$$

where

$$\mathfrak{S}_{\eta,p,q}^{n,m}\, r(z) = z^p + \sum_{k=2}^{\infty} \left([k+p]_q\right)^n \left(\frac{p+\eta k}{p}\right)^m a_{k+p-1}\, z^{k+p-1}$$

and

$$\mathfrak{S}_{\eta,p,q}^{n,m}\, s(z) = \sum_{k=1}^{\infty} \left([k+p]_q\right)^n \left(\frac{p+\eta k}{p}\right)^m b_{k+p-1}\, z^{k+p-1}.$$

With Operator $\mathfrak{S}_{\eta,p,q}^{n,m}\, v(z)$ we present the class $\mathcal{H}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$, of multivalent harmonic functions as follows:

Notation 2.1

For $\mathbf{0 \le \boldsymbol{\gamma}, \boldsymbol{\mu} < 1}$, $\mathbf{0 \le \boldsymbol{\delta} < \frac{1}{2}}$, the function $\boldsymbol{v = r + \bar{s}}$ of the form (1.2) is in the class $\boldsymbol{\mathcal{H}_{\mu,p}^{n,m}(\gamma,\delta,\mu)}$ if satisfy the inequality

$$Re\left\{\frac{2\delta z^p + \mathfrak{S}_{\eta,p,q}^{n,m}\, v(z)}{\gamma \mathfrak{S}_{\eta,p,q}^{n,m+1}\, v(z) + (1-\gamma) z^p}\right\} > \mu\,, \tag{2.6}$$

where $n,k,p \in N$, $z \in U$. and $\mathfrak{F}_{\eta,p,q}^{n,m} h$ is defined by (1.5).

Let $\mathcal{MH}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$ describe the subclass of $\mathcal{H}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$ of harmonic functions $v = r + \bar{s}$ to make r and s of the shape

$$r(z) = z - \sum_{k=p+1}^{\infty} |a_k| z^k\,, \qquad s(z) = \sum_{k=p}^{\infty} |b_k| z^k, \qquad |b_p| < 1, p \in \mathbb{N} = \{1,2,3,\dots\}. \tag{2.7}$$

Many researchers have investigated subclasses of univalent and multivalent harmonic functions. [13، 8، 18، 10، 14، 15، 4، 19، 16، 6، 20، 2، 21، 22، 23، 24، 9، 3].

In this research, we study a class of harmonic multivalent functions defined by a differential operator based on q-derivative operator. Coefficient bounds, distortion bounds, extreme points, inclusion results and closure under an integral operator for functions in the class $\mathcal{MH}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$ are obtained.

Main Results 2.٢

We provide sufficient coefficient conditions for harmonic functions of class $\mathcal{H}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$ in our first theorem.

Theorem 2. 1 For $0 \le \gamma,,\ \ 0 \le \delta < \frac{1}{2}, \mu < \delta$. Let $v = r + is$ in S_{Hp}

be of the form Eq. (1.1). If

$$\sum_{k=2}^{\infty} \Omega_{q,p}^{n,m}(k,\mu,\gamma,\delta)\left|a_{k+p-1}\right| + \sum_{k=1}^{\infty} \mathcal{Q}_{q,p}^{n,m}(k,\mu,\gamma,\delta)\left|b_{k+p-1}\right| \le 1\,, \tag{2.8}$$

where,

$$\Omega_{q,p}^{n,m}(k,\mu,\gamma,\delta) = \frac{\psi_q^n(k,m)\left(1-\mu\gamma\left(\frac{p}{k+p-1}\right)\right)}{(1+\delta-\mu)}$$

$$\mathcal{Q}_{q,p}^{n,m}(k,\mu,\gamma,\delta) = \frac{\psi_q^n(k,m)\left(1-\mu\gamma\left(\frac{p}{k+p-1}\right)\right)}{(1+\delta-\mu)},$$

$$and\ \psi_q^n(k,m) = \left([k+p]_q\right)^n\ \left(\frac{p+\eta k}{p}\right)^m$$

$$Then\ f \in \mathcal{H}_{\mu,p}^{n,m}(\gamma,\delta,\mu).$$

Proof. To prove that $f \in \mathcal{H}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$ by the condition (2.6), we only need to show that if (1.8) holds, then

$$Re\left\{\frac{2\delta z^p + \mathfrak{S}_{\eta,p,q}^{n,m}\ v(z)}{\gamma\mathfrak{S}_{\eta,p,q}^{n,m+1}\ v(z) + (1-\gamma)z^p}\right\} = Re\frac{G(z)}{H(z)} \ge \mu\,,$$

where $z = re^{i\theta}$, $0 \le \theta \le 2\pi,\ 0 \le r < 1$

Note that

$$G(z) = 2\delta z^p + \mathfrak{S}_{\eta,p,q}^{n,m}\, v(z),$$

and

$$H(z) = \gamma \mathfrak{S}_{\eta,p,q}^{n,m+1}\, v(z) + (1-\gamma) z^p.$$

Using the fact that, $\mu \ge 0$. Then $\mathrm{Re}(w) > \alpha$ if and only if $|w-(1+\mu)| < |w+(1-\mu)|$, we have

$$|G(z) + (1-\mu)H(z)| - |G(z) - (1+\mu)H(z)| \ge 0\,. \qquad (2.9)$$

Substituting G(z) and H(z) in (1.9) we obtain

$$|G(z) + (1-\mu)H(z)| - |G(z) - (1+\mu)H(z)|$$

$$= \left|2\delta z^p + \mathfrak{S}_{\eta,p,q}^{n,m}\, v(z) + (1-\mu)\left[\gamma \mathfrak{S}_{\eta,p,q}^{n,m+1}\, v(z) + (1-\gamma)z^p\right]\right| - \left|2\delta z^p + \mathfrak{S}_{\eta,p,q}^{n,m}\, v(z) - (1+\mu)\left[\gamma \mathfrak{S}_{\eta,p,q}^{n,m+1}\, v(z) + (1-\gamma)z^p\right]\right|$$

$$= \left| z^p + \sum_{k=2}^{\infty} \psi_q^n(k,m) a_{k+p-1} z^{k+p-1} + \sum_{k=1}^{\infty} \psi_q^n(k,m)\, \overline{b_{k+p-1} z^{k+p-1}} + 2\delta z^p + (1-\mu)\left[(1-\gamma)z^p + \gamma z^p + \gamma \sum_{k=2}^{\infty} \psi_q^n(k,m+1) a_{k+p-1} z^{k+p-1} + \gamma \sum_{k=1}^{\infty} \psi_q^n(k,m+1)\, \overline{b_{k+p-1} z^{k+p-1}}\right]\right|$$

$$-\left|z^p+\sum_{k=2}^{\infty}\psi_q^n(k,m)a_{k+p-1}z^{k+p-1}+\sum_{k=1}^{\infty}\psi_q^n(k,m)\,\overline{b_{k+p-1}z^{k+p-1}}+\delta z^p\right.$$

$$-(1+\mu)\left[(1-\gamma)z^p+\gamma z^p\right.$$

$$+\gamma\sum_{k=2}^{\infty}\psi_q^n(k,m+1)a_{k+p-1}z^{k+p-1}$$

$$\left.\left.+\gamma\sum_{k=1}^{\infty}\psi_q^n(k,m+1)\,\overline{b_{k+p-1}z^{k+p-1}}\right]\right|$$

$$=\left|(2+2\delta-\mu)z^p\right.$$

$$+\sum_{k=2}^{\infty}\psi_q^n(k,m)[1+(1-\mu)\gamma\rho]a_{k+p-1}z^{k+p-1}$$

$$\left.-\sum_{k=1}^{\infty}\psi_q^n(k,m)[1-(1-\mu)\gamma\rho]\,\overline{b_{k+p-1}z^{k+p-1}}\right|$$

$$-\left|\mu z^p+\sum_{k=2}^{\infty}\psi_q^n(k,m)[1-(1+\mu)\gamma\rho]a_{k+p-1}z^{k+p-1}\right.$$

$$\left.-\sum_{k=1}^{\infty}\psi_q^n(k,m)[1+(1+\mu)\gamma\rho]\,\overline{b_{k+p-1}z^{k+p-1}}\right|$$

$$\geq(2+2\delta-\mu)|z|^p$$

$$-\sum_{k=2}^{\infty}\psi_q^n(k,m)[1+(1-\mu)\gamma\rho]\left|a_{k+p-1}\right||z|^{k+p-1}$$

$$-\sum_{k=1}^{\infty}\psi_q^n(k,m)[1-(1-\mu)\gamma\rho]\left|b_{k+p-1}\right||z|^{k+p-1}-\mu|z|^p$$

$$-\sum_{k=2}^{\infty}\psi_q^n(k,m)[1-(1+\mu)\gamma\rho]\left|a_{k+p-1}\right||z|^{k+p-1}$$

$$-\sum_{k=1}^{\infty}\psi_q^n(k,m)[1+(1+\mu)\gamma\rho]\left|b_{k+p-1}\right||z|^{k+p-1}$$

$$=2(1+\delta-\mu)|z|^p$$

$$-\sum_{k=2}^{\infty}\psi_q^n(k,m)[1+(1-\mu)\gamma\rho+1-(1+\mu)\gamma\rho]\left|a_{k+p-1}\right||z|^{k+p-1}$$

$$-\sum_{k=1}^{\infty}\psi_q^n(k,m)[1-(1-\mu)\gamma\rho+1+(1+\mu)\gamma\rho]\left|b_{k+p-1}\right||z|^{k+p-1}$$

$$=2(1+\delta-\mu)|z|^p$$

$$-\sum_{k=2}^{\infty}2\psi_q^n(k,m)[1-\mu\gamma\rho]\left|a_{k+p-1}\right||z|^{k+p-1}$$

$$-\sum_{k=1}^{\infty}2\psi_q^n(k,m)[1+\mu\gamma\rho]\left|b_{k+p-1}\right||z|^{k+p-1}$$

$$=2(1+\delta-\mu)|z|^p$$

$$\left[1-\left\{\sum_{k=2}^{\infty}\frac{\psi_q^n(k,m)[1-\mu\gamma\rho]}{(1+\delta-\mu)}\left|a_{k+p-1}\right||z|^{k+p-1}+\sum_{k=1}^{\infty}\frac{2\psi_q^n(k,m)[1+\mu\gamma\rho]}{(1+\delta-\mu)}\left|b_{k+p-1}\right||z|^{k+p-1}\right\}\right]$$

$$\left[1-\left\{\sum_{k=2}^{\infty}\frac{\psi_q^n(k,m)[1-\mu\gamma\rho]}{(1+\delta-\mu)}\left|a_{k+p-1}\right|+\sum_{k=1}^{\infty}\frac{2\psi_q^n(k,m)[1+\mu\gamma\rho]}{(1+\delta-\mu)}\left|b_{k+p-1}\right|\right\}\right].$$

Or equivalently to

$$|G(z)+(1-\mu)H(z)|-|G(z)-(1+\mu)H(z)|=2(1+\delta-\mu)$$

$$\left[1-\left\{\sum_{k=2}^{\infty}\frac{\psi_q^n(k,m)\left[1-\mu\gamma\left(\frac{p}{k+p-1}\right)\right]}{(1+\delta-\mu)}\left|a_{k+p-1}\right|\right.\right.$$
$$\left.\left.+\sum_{k=1}^{\infty}\frac{\psi_q^n(k,m)\left[1+\mu\gamma\left(\frac{p}{k+p-1}\right)\right]}{(1+\delta-\mu)}\left|b_{k+p-1}\right|\right\}\right]\geq 0. \quad ☐$$

The harmonic multivalent function

$$f(z)=z^p+\sum_{k=2}^{\infty}\frac{1}{\Omega_{\mathrm{q,p}}^{\mathrm{n,m}}(\mathrm{k},\mu,\gamma,\delta)}x_k z^{k+p-1}$$
$$+\sum_{k=1}^{\infty}\frac{1}{\mathcal{Q}_{\mathrm{q,p}}^{\mathrm{n,m}}(\mathrm{k},\mu,\gamma,\delta)}y_k z^{k+p-1}, \tag{2.10}$$

Where n∈ N and $\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k=1$, indicate that the bound of coefficient (1.8) is sharp. Since

$$\sum_{k=2}^{\infty}\Omega_{\mathrm{q,p}}^{\mathrm{n,m}}(\mathrm{k},\mu,\gamma,\delta)\left|a_{k+p-1}\right|+\sum_{k=1}^{\infty}\mathcal{Q}_{\mathrm{q,p}}^{\mathrm{n,m}}(\mathrm{k},\mu,\gamma,\delta)\left|b_{k+p-1}\right|$$
$$=\sum_{k=2}^{\infty}\Omega_{\mathrm{q,p}}^{\mathrm{n,m}}(\mathrm{k},\mu,\gamma,\delta)\frac{1}{\Omega_{\mathrm{q,p}}^{\mathrm{n,m}}(\mathrm{k},\mu,\gamma,\delta)}|x_k|$$
$$+\sum_{k=1}^{\infty}\mathcal{Q}_{\mathrm{q,p}}^{\mathrm{n,m}}(\mathrm{k},\mu,\gamma,\delta)\frac{1}{\mathcal{Q}_{\mathrm{q,p}}^{\mathrm{n,m}}(\mathrm{k},\mu,\gamma,\delta)}|y_k|$$
$$=\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1.$$

We now show that condition (2.8) is also required for the functions $v=r+\bar{s}$ where r and S are specified (2.7).

Theorem 2.2 Let $v=r+\bar{s}$, where r *and* s be of the form (2.7). Then $v\in\mathcal{MH}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$ if and only if satisfies the condition

$$\sum_{k=2}^{\infty} \Omega_{q,p}^{n,m}(k,\mu,\gamma,\delta)\left|a_{k+p-1}\right| + \sum_{k=1}^{\infty} \mathcal{Q}_{q,p}^{n,m}(k,\mu,\gamma,\delta)\left|b_{k+p-1}\right| \le 1\,, \qquad (2.11)$$

where,

$$\Omega_{q,p}^{n,m}(k,\mu,\gamma,\delta) = \frac{\psi_q^n(k,m)\left(1-\mu\gamma\left(\frac{p}{k+p-1}\right)\right)}{(1+\delta-\mu)}$$

$$\mathcal{Q}_{q,p}^{n,m}(k,\mu,\gamma,\delta) = \frac{\psi_q^n(k,m)\left(1-\mu\gamma\left(\frac{p}{k+p-1}\right)\right)}{(1+\delta-\mu)},$$

$b_{k+p-1} > a_{k+p-1}$, for *every* $k \ge 2$.

Proof. Because $\mathcal{MH}_{\mu,p}^{n,m}(\gamma,\delta,\mu) \subset \mathcal{H}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$=B, we just need to prove the theorem "only if. For functions $v = r + \bar{s}$ of the form (2.7), we notice that the condition

$$Re\left\{\frac{2\delta z^p + \mathfrak{S}_{\eta,p,q}^{n,m}\, v(z)}{\gamma \mathfrak{S}_{\eta,p,q}^{n,m+1}\, v(z) + (1-\gamma)z^p}\right\} > \mu\,,$$

is equivalent to

$$Re\left\{\frac{\begin{array}{c}(1+\delta-\mu)z^p\\ -\sum_{k=2}^{\infty}\psi_q^n(k,m)[1-\mu\gamma\rho]a_{k+p-1}z^{k+p-1} + \sum_{k=1}^{\infty}\psi_q^n(k,m)[1+\mu\gamma\rho]b_{k+p-1}\overline{z^{k+p-1}}\end{array}}{z^p - \gamma\sum_{k=2}^{\infty}\psi_q^n(k,m+1)a_{k+p-1}z^{k+p-1} + \gamma\sum_{k=1}^{\infty}\psi_q^n(k,m+1)b_{k+p-1}z^{k+p-1}}\right\}$$

$$\ge 0. \qquad (2.12)$$

Now, the last inequality (9) must hold for all values of z in U. Choosing the value of z on the positive real axis where $0 \le z = r < 1$, we have $b_{k+p-1} > a_{k+p-1}$, *for every* $k \ge 2$,

$$\frac{\left[\begin{array}{c}(1+\delta-\mu)\\ -\sum_{k=2}^{\infty}\psi_q^n(k,m)[1-\mu\gamma\rho]a_{k+p-1}r^{k+p-1} - \sum_{k=1}^{\infty}\psi_q^n(k,m)[1+\mu\gamma\rho]b_{k+p-1}r^{k+p-1}\end{array}\right]}{\left[1-\gamma\sum_{k=2}^{\infty}\psi_q^n(k,m+1)a_{k+p-1}r^{k+p-1} + \gamma\sum_{k=1}^{\infty}\psi_q^n(k,m+1)b_{k+p-1}r^{k+p-1}\right]}$$

$$\ge 0. \qquad (2.13)$$

We notice that the expression in (10) is negative for r sufficiently closed to 1 when the condition (8) does not hold. Hence there exist $z_0 = r_0$ in (0,1) for which the quotient in (1.14) is negative, therefore there is a contradicts the required condition for $v \in \mathcal{MH}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$. ▯

Extreme Points 2.٣

Here, we consider the extreme points of closed convex hull of $\mathcal{MH}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$, denoted by $clco\mathcal{MH}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$ for functions in $5(\gamma,\delta,\mu)$.

Theorem 2.٣

Let v given by (2.7). Then $v \in \mathcal{MH}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$ if and only if

$$v(z) = \sum_{k=1}^{\infty}\left[x_{k+p-1}h_{k+p-1}(z) + y_{k+p-1}g_{k+p-1}(z)\right],$$

$$h_p(z) = z^p, \qquad h_{k+p-1}(z) = z^p - \frac{1}{\Omega_{q,p}^{n,m}(k,\mu,\gamma,\delta)} z^{k+p-1}, \qquad k \geq 2,3,\ldots,$$

and

$$g_{k+p-1}(z) = z^p + (-1)^{n-1}\frac{1}{\mathcal{Q}_{q,p}^{n,m}(k,\mu,\gamma,\delta)} z^{k+p-1}, \qquad k \geq 2,3,\ldots$$

$$x_{k+p-1} \geq 0, \qquad y_{k+p-1} \geq 0,\ x_p = 1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}.$$

In particular, the extreme points of $\mathcal{MH}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$ are $\{h_{k+p-1}\}$ *and* $\{g_{k+p-1}\}$.

Proof. Suppose

$$v(z) = \sum_{k=1}^{\infty}\left[x_{k+p-1}h_{k+p-1}(z) + y_{k+p-1}g_{k+p-1}(z)\right]$$

$$= \sum_{k=1}^{\infty} (x_{k+p-1} + y_{k+p-1}) z^p - \sum_{k=2}^{\infty} \frac{1}{\Omega_{q,p}^{n,m}(k,\mu,\gamma,\delta)} x_{k+p-1} z^{k+p-1}$$
$$+ \sum_{k=1}^{\infty} \frac{1}{\mathcal{Q}_{q,p}^{n,m}(k,\mu,\gamma,\delta)} y_{k+p-1} z^{k+p-1}$$

$$= z^p - \sum_{k=2}^{\infty} \frac{1}{\Omega_{q,p}^{n,m}(k,\mu,\gamma,\delta)} x_{k+p-1} z^{k+p-1}$$
$$+ \sum_{k=1}^{\infty} \frac{1}{\mathcal{Q}_{q,p}^{n,m}(k,\mu,\gamma,\delta)} y_{k+p-1} z^{k+p-1}.$$

On the other hand we have

$$\sum_{k=2}^{\infty} \Omega_{q,p}^{n,m}(k,\mu,\gamma,\delta) |a_{k+p-1}| + \sum_{k=1}^{\infty} \mathcal{Q}_{q,p}^{n,m}(k,\mu,\gamma,\delta) |b_{k+p-1}|$$

$$= \sum_{k=2}^{\infty} \Omega_{q,p}^{n,m}(k,\mu,\gamma,\delta) \left(\frac{1}{\Omega_{q,p}^{n,m}(k,\mu,\gamma,\delta)} x_{k+p-1} \right)$$
$$+ \sum_{k=1}^{\infty} \mathcal{Q}_{q,p}^{n,m}(k,\mu,\gamma,\delta) \left(\frac{1}{\mathcal{Q}_{q,p}^{n,m}(k,\mu,\gamma,\delta)} y_{k+p-1} \right)$$

$$= \sum_{k=2}^{\infty} x_{k+p-1} + \sum_{k=1}^{\infty} y_{k+p-1} = 1 - x_p \leq 1 .$$

Therefore, $v \in clco\mathcal{MH}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$.

Conversely, if $v \in clco\mathcal{MH}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$. Assume , we have

$$x_p = 1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1} .$$

$$x_{k+p-1} = \Omega_{q,p}^{n,m}(k,\mu,\gamma,\delta) a_{k+p-1}, \; k = 2,3, \ldots$$

and

$$y_{k+p-1} = \mathcal{Q}_{q,p}^{n,m}(k,\mu,\gamma,\delta) b_{k+p-1}, \; k = 2,3, \ldots$$

Now, consider the following

$$v(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + \sum_{k=1}^{\infty} b_{k+p-1}\, , \overline{z^{k+p-1}}$$

$$= z^p - \sum_{k=2}^{\infty} \frac{1}{\Omega_{q,p}^{n,m}(k,\mu,\gamma,\delta)} x_{k+p-1} z^{k+p-1} + \sum_{k=1}^{\infty} \frac{1}{Q_{q,p}^{n,m}(k,\mu,\gamma,\delta)} y_{k+p-1} z^{k+p-1}$$

$$= z^p - \sum_{k=2}^{\infty} \left[z^p - h_{k+p-1}(z)\right] x_{k+p-1} + \sum_{k=1}^{\infty} \left[z^p - g_{k+p-1}(z)\right] y_{k+p-1}$$

$$= \left[1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}\right] z^p + \sum_{k=2}^{\infty} x_{k+p-1} h_{k+p-1}(z)$$

$$+ \sum_{k=1}^{\infty} x_{k+p-1} h_{k+p-1}(z)$$

$$= \sum_{k=2}^{\infty} x_{k+p-1} h_{k+p-1}(z) + \sum_{k=1}^{\infty} x_{k+p-1} h_{k+p-1}(z).$$

CHAPTER 3

Conclusions

The conclusions of this work are as follows:-

- Section (2.2) defined inequalities based on parameters v, e, a and b_1 and any changes in their values leading to large changes in the classes
- We can discuss it with a negative coefficient, the operator of analytical multivalent functions with an adverse coefficient.
- The class $\mathcal{H}_{\mu,p}^{n,m}(\gamma,\delta,\mu)$ of all multivalent harmonic functions dependent on the values of μ and b.

  Also, if p=1, then we get the class $\mathcal{H}_{m,1}(n,m,y)$ of all harmonic univalent function.

Future Works

The suggestions for future work as well as ideas on future work are as follows :

- Some multivalent function properties based on the generalized Mittag-Leffler functions.
- In the multivalent functions class define through Sigmoid Activation Function with Chebyshev polynomials.
- Certain properties of a subclass of meromorphic p-valent functions generalized derivative operator.
- An investigation of some properties of subclass of univalent functions defined by generalized integral operator.

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الملخص

في عملنا ، قدمنا بحثا و تحققنا عن خصائص من الفئة الفرعية متعددة التكافؤات التوافقية التي تتضمن الحصول على مؤثر تفاضلي نهايات النقاط والانغلاق تحت عامل مؤثر في ذلك الصنف.

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بحث مقدم الى
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من قبل
سارة عادل حمزة فرج

بأشراف
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