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Solving Linear Programming Problems Using The Graphical Methods

A Research

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"You must not lose faith in humanity. Humanity is an ocean, if a few drops of the ocean are dirty, the ocean does not become dirty"

Gandhi

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Dedication

To all I love and respect,

Father, Mother, Wife, Brothers, and Sons

A special dedication to

Martyrs of the October Revolution

Acknowledgments

I would like to express my thanks to the Merciful God and then thanks my supervisor ***"Dr. Ahmed Sabah Al jilawi"*** for standing with me and supporting me to bring this work into big success. and thanks to all those who contributed to this work.

Abstract

This study presents "Solving Linear Programming Problem" by using one of the important methods of solution which is the graphical ones, then it shows different examples with mathematical ideal of everyday life.

The aim of this research is study of the linear programming problems which is part of the mathematical programming problems (linear and non-linear). By find the values of the variables that increase or decrease the objective function to reach the optimal solution, and the steps of the solution are shown in a graphical way. The different examples that are presented its constraints problem with two variables. Also, it presents how to determine the points and area of the optimal solution in which increases or decreases the objective function.

One of the practical applications to solve the problems of linear programming from daily life is studied, which is the problem of the diet to reach the best specific nutritional requirements at the lowest cost. A proposal for a new mathematical model is presented. It is one of the practical applications of daily life through the distribution of food aid and finding the lowest cost for distribution with an increase in profits per month while maintaining the value constants of the restrictions.

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List of Abbreviations

Abbreviation	The Full Form
LP	Linear Programming
LPP	Linear Programming Problems
WFP	World Food Program
NGO	Non-Governmental Organization

Chapter One

Linear Programming

1.1 Introduction

The goal of linear programming is to determine the values of decision variables that maximize or minimize a linear objective function, where the decision variables are subjects to linear constraints. A linear programming problem is a special case of a general constrained optimization problem. In the general setting, the goal is to find a point that minimizes the objective function and at the same time satisfies the constraints. We refer to any point that satisfies the constraints as a feasible point. In a linear programming problem, the objective function is linear, and the set of feasible points is determined by a set of linear equations or inequalities[9].

Thus, it is important to study methods for solving linear programming problems. Linear programming methods provide a way of choosing the best feasible point among the many possible feasible points. In general, the number of feasible points is infinitely large. However, as we shall see the solution to a linear programming problem can be found by searching through a particular finite number of feasible points, known as basic feasible solutions. Therefore, in principle, we can solve a linear programming problem simply by comparing the finite number of basic feasible solutions and finding one that minimizes or maximizes the objective function[3].

For most practical decision problems, even this finite number of basic feasible solutions is so large that the method of choosing the best solution by comparing them to each other is impractical. To get a feel for the amount of computation needed in a brute-force approach, the efficient methods for solving linear programming problems became available in the late 1930. In 1939 Kantorovich presented a number of solutions to some problems related to production and transportation planning. During World War II, Koopmans contributed significantly to the solution of transportation problems. Kantorovich and Koopmans were awarded a Nobel Prize in Economics in 1975 for their work on the theory of optimal allocation of resources[9].

Linear programming was developed during World War II, when a system with which to maximize the efficiency of resources was of utmost importance. New war related to the projects that demanded attention and spread resources thin. "Programming" was a military term that referred to activities such as planning schedules efficiently or deploying men optimally. George Dantzig, a member of the U.S. Air Force, developed the Simplex method of optimization in 1947 in order to provide an efficient algorithm for solving programming problems that had linear structures. Since then, experts from a variety of fields, especially mathematics and economics, have developed the theory behind "linear programming" and explored its applications[14].

The Graphical method of linear programming is an efficient and an elegant and it has been declared as one of the best algorithms with the greatest influence on the development and practice of science and engineering in the twentieth century. Graphical method of linear programming is used to solve the problems by finding the highest or lowest point of intersection between the objective function line and the feasible region on a graph.

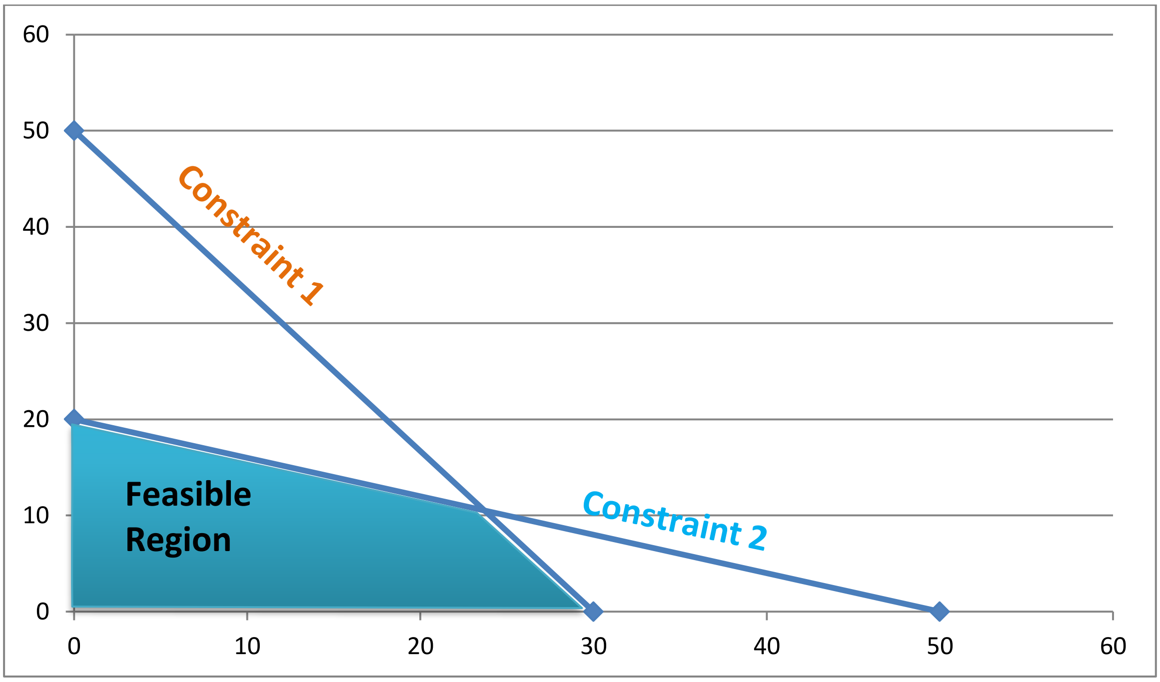

Figure 1.1: Feasible Region.

Although the noun is an optimization (finding the optimum), it does not always mean finding the optimal solution to a problem. This is often not possible and a heuristic algorithm should be used rather than optimization. The optimization is a general term in mathematics and computer science, used in several contexts including[15]:

1- In mathematics, the optimization is the branch concerned with finding the upper and lower limits of a mathematical function, sometimes with constraints. In other words, it means choosing the optimal element from a set of possible candidates for selection. The problem is forged on the basis of maximizing the objective function or minimizing the cost function.

2- In informatics, optimization is the process of improving system performance so that effective operating time is reduced.

3- In economics, it is the achievement of the best level of production that maximizes profit and the optimum volume of production, thus minimizing costs and maximizing production.

4- The topic of optimization is to find a maximum industrial production without exceeding the permissible limits of resources.

While, the optimization problem refers to maximizing or minimizing some function relative to some set, often representing a range of choices available in a certain situation. The function allows comparison of the different choices for determining which might be "best".

Common applications: minimal cost, maximal profit, minimal error, optimal design, optimal management, vibrational principles[19].

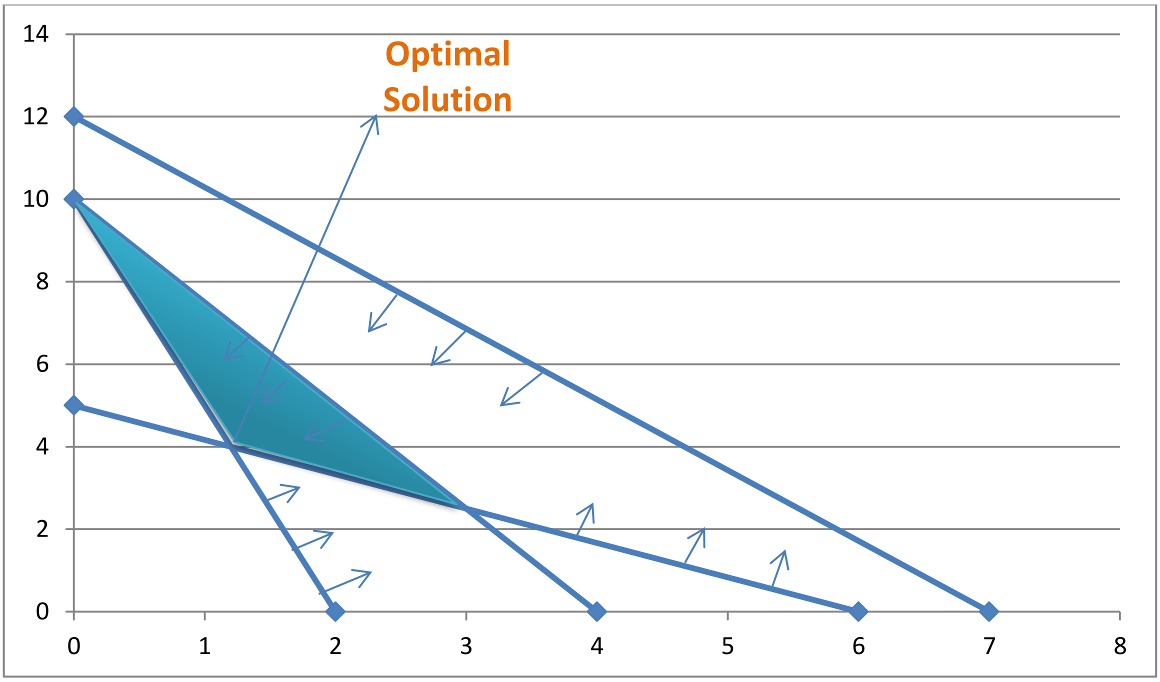

Figure 1.2: Optimal Solution.

Chapter Two

Definitions and Theorems

2.1 Linear Programming Problems

Linear programming is a basic and important method that helps decision makers to make correct and scientific decisions. Linear programming problems are part of the mathematical programming problems that include linear and nonlinear ones. Then, mathematical programming is in turn part of a more comprehensive subject, called operations research, which all relate to issues of organization and management, issues of transportation, agriculture, industry, and so on, and the goal of linear programming is to find the values of the variables that maximize or minimize the objective function[6].

Linear Programming Problems in math's is a system process of finding a maximum or minimum value of any variable in a function, it is also known by the name of optimization problem. (LPP) is helpful in developing and solving a decision making problem by mathematical techniques. The problem is generally given in a linear function which needs to be optimized subject to a set of different constraints. Major usage of LPP is in advising the management to make the most efficient and effective use of the scarce resources[7].

Linear programming allows researchers to find the best, most economical solution to a problem within all of its limitations, or constraints. Many fields use linear programming techniques to make their processes more efficient[6].

When you have a problem that involves a variety of resource constraints, linear programming can generate the best possible solution. Whether it's maximizing things like profit or space, or minimizing factors like cost and waste, using this tool is a quick and efficient way to structure the problem, and find a solution[6].

So, linear programming is a mathematical technique that determines the best way to use available resources. Managers use the process to help make decisions about the most efficient use of limited resources like money, time, materials, and machinery.

Advantages of Linear programming helps in attaining the optimum use of productive resources. It also indicates how a decision-maker can employ his productive factors effectively by selecting and distributing (allocating) these resources. Linear programming techniques improve the quality of decisions

2.2 Basic Facts Related to Linear Programming

Definition 2.2.1 Convex Set[1]

Let $S \subseteq R^n$. If the line segment between any two points in S lies in S,

i.e. $\lambda x + (1-\lambda)\, y \in S, \forall\, x, y \in S, \forall \lambda \in [0,1]$,

then S is said to be convex as shown in (Figure 2.1). It can be shown that a set $S \subseteq R^n$ is convex if and only if for any $x_1,\ldots\ldots, x_n \in S$, the convex combination is given by:

$$\sum_{i=1}^{n} \lambda_i x_i$$

when $\sum_{i=1}^{n} \lambda_i = 1$, $\lambda_i \geq 0$, $i = 1,\ldots\ldots,n$ belongs to S.

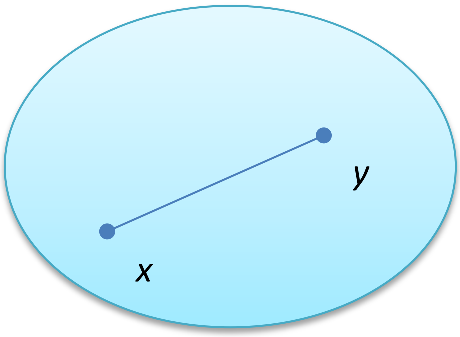

Figure 2.1: Convex Set.

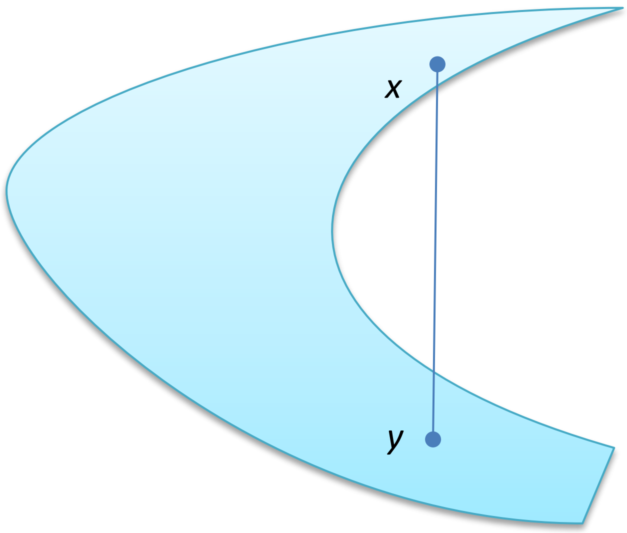

Figure 2.2**:** Non–Convex Set.

Example 2.2.1.1 Is the interval A= (1,3) is convex set?

Solution: $\forall\, x , y \in (1,3)$, $\lambda \in [0,1]$.

It needs to check that the: $\lambda x + (1-\lambda)\, y \in (1,3)$

$x \in (1,3) \rightarrow (1 < x < 3) \times \lambda \rightarrow \lambda < \lambda x < 3\lambda$ ……..(1)

and $y \in (1,3) \rightarrow (1 < y < 3) \times (1-\lambda) \rightarrow (1-\lambda) < (1-\lambda)y < 3(1-\lambda)$ ……..(2)

by sum (1) & (2), one can get

$\lambda + (1-\lambda) < \lambda x + (1-\lambda)\, y < 3\lambda + 3 - 3\lambda \rightarrow 1 < \lambda x + (1-\lambda)\, y < 3$

So, $\lambda x + (1-\lambda)\, y \in (1,3) = A$

Thus, (1,3) is a convex set.

Definition 2.2.2 Convex Function [1]

Let $S \subseteq R^n$ be a nonempty convex set. If $f : S \rightarrow R$ satisfies:

$$f(\lambda x_1 + (1-\lambda)\, x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2),\ \forall\, x_1, x_2 \in S,\ \forall \lambda \in [0,1],$$

then f is said to be a convex function on S. If the above inequality is true as a strict inequality for all $x_1 \neq x_2$ and for all $\lambda \in (0,1)$, then f is called a strictly convex function on S as shown in (Figure 2.1).

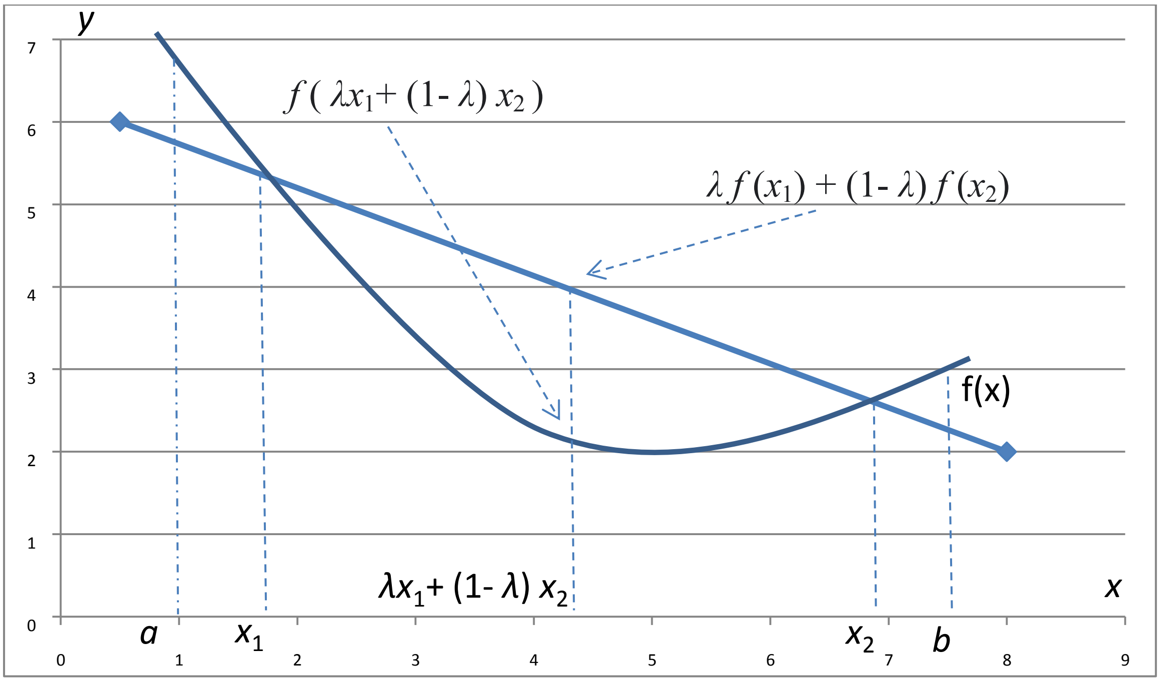

Figure 2.3: Convex Function.

Example 2.2.2.1 Show that the function $f(x) = x^2$ is convex function

Solution: Based on definition (2.2.2), one can obtain:

$f(\lambda x_1 + (1- \lambda) x_2) \leq \lambda f(x_1) + (1- \lambda) f(x_2)$

Since $f(x) = x^2$, So $f(\lambda x_1 + (1- \lambda) x_2)^2 \leq \lambda x_1^2 + (1- \lambda) x_2^2$

$\lambda x_1^2 + (1- \lambda) x_2^2 - \lambda^2 x_1^2 - (1- \lambda) x_2^2 - 2\lambda (1-\lambda) x_1 x_2$

$x(\lambda - \lambda^2) (x_1^2 + x_2^2 - 2x_1 x_2) = (\lambda - \lambda^2) (x_1 - x_2)^2$

$(x_1 - x_2)^2 \geq 0 \qquad 0 < \lambda < 1 \qquad \lambda^2 < \lambda \qquad \rightarrow 0 < \lambda - \lambda^2$

Definition 2.2.3 Vector Norm [1]

The function $\|.\| : R^n \rightarrow R$ is called a vector norm if it has the following properties:

1. $\|x\| \geq 0$ for any vector $x \in R^n$, and $\|x\| = 0$ if and only if $x = 0$
2. $\|ax\| = \|a\| \|x\|$ for any vector $x \in R^n$ and any scalar $\alpha \in R$;
3. $\|x + y\| \leq \|x\| + \|y\|$, for any vectors $x, y \in R^n$.

The last property is called the triangle inequality. It should be noted that when n = 1, the absolute value function is a vector norm.

Definition 2.2.4 Global Maxima and Minima[13]

If $f(a)$ is the largest value then it satisfies the inequality $f(x) \leq f(a)$, for all x in the domain of f. We call $f(a)$ the global or absolute maximum value of f and the point $(a, f(a))$ the global maximum point.

Similarly, if $f(a)$ is the smallest value of $f(x)$ then $f(a) \leq f(x)$, for all x in the domain of f. We call $f(a)$ the absolute or global minimum value of f and the point $(a, f(a))$ the global minimum.

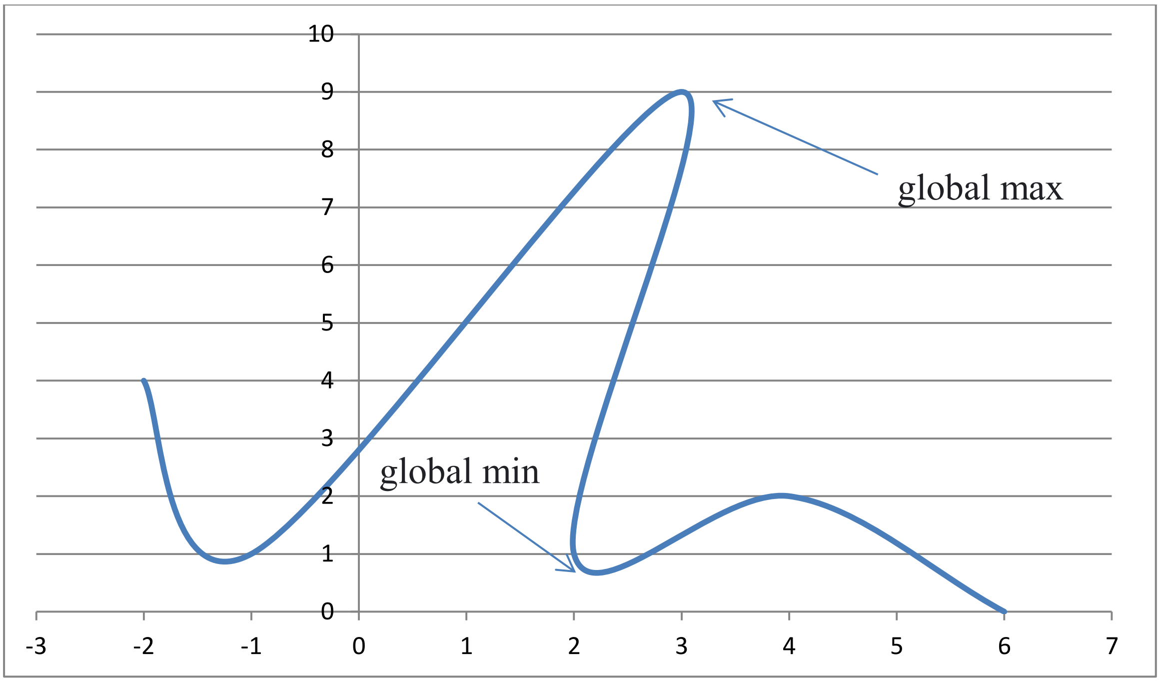

Figure 2.4: Global Maxima and Minima.

Definition 2.2.5 Local Maxima and Minima[13]

A local minimum/ maximum is a point in which the function reaches its lowest/ highest value in a certain region of the function. In formal words, this means that for every local minimum/ maximum x, there is an epsilon such that $f(x)$ is smaller/ greater than all values $f(y)$ for all y that have distance at most epsilon to x. That looks very complicated but it does mean as much as $f(x)$ is the smallest/ largest value for all points close to x. There might be values, however, that are smaller/larger than the local minimum/maximum.

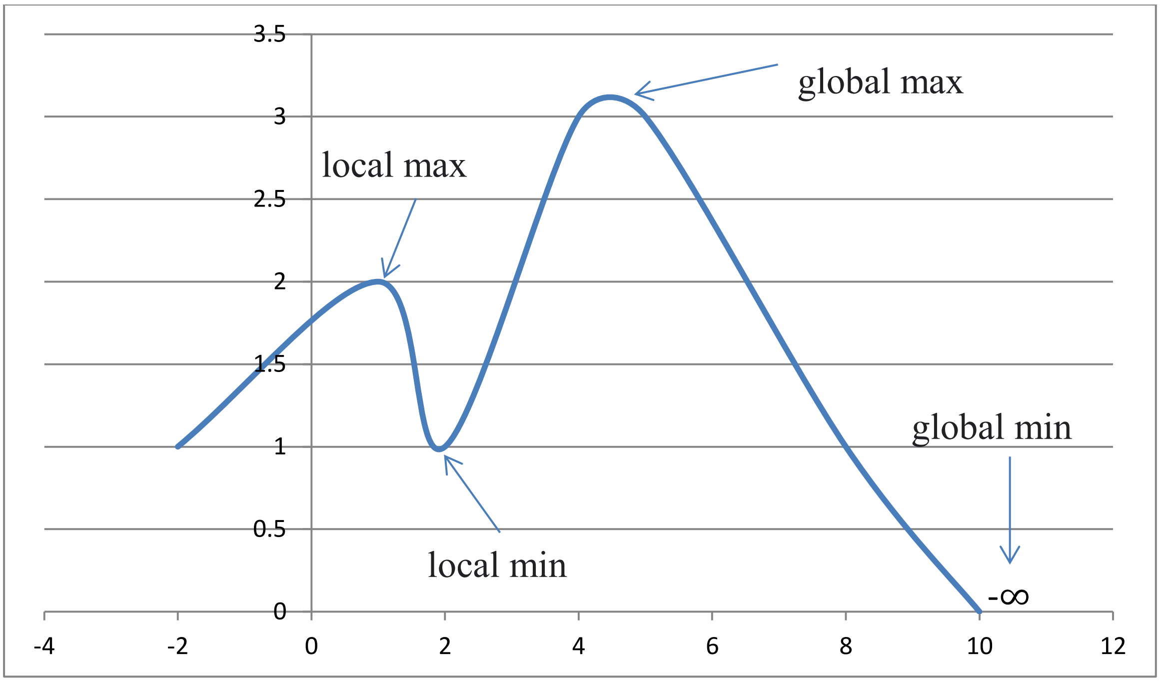

Figure 2.5: Local Maxima and Minima.

Definition 2.2.6 Continuous Function[11]

The function *f* is a continuous at a number *a* if

$$\lim_{x \to a} f(x) = f(a)$$

(i.e. we can make the value of $f(x)$ as close as we like to $f(a)$ by taking x sufficiently close to a).

A function $f : A \to R$ is continuous on a set $B \subseteq A$ if it is continuous at every point in B, and continuous if it is continuous at every point of its domain A.

Definition 2.2.7 Differentiable Function[11]

Suppose that $f : (a\ ,\ b) \to R$ and $a < c < b$. Then f is differentiable at c with derivative $f'(c)$

$$\lim_{h \to 0} \left[\frac{f(c+h) - f(c)}{h}\right] = f'(c)$$

The domain of f' is the set of points $c \in (a\ ,\ b)$ for which this limit exists. If the limit exists for every $c \in (a\ ,\ b)$ then we say that f is differentiable on $(a\ ,\ b)$.

2.3 The Optimization [15]

The optimization methods are used in many areas of study to find the solutions that maximize or minimize some study parameters, such as minimize costs in the production of a good or service, maximize profits, minimize raw material in the development of a good, or maximize production. Optimization is the process of making a trading system more effective by adjusting the variables used for technical analysis.

Optimization is everywhere from routine business transactions to important decisions of any sort, from engineering design to industrial manufacturing, and from choosing a career path to planning our holidays. In all these activities, there are always some things (objectives) that need to optimize and these objectives could be cost, profit, performance, quality, enjoyment, customer-rating and others. The formal approach to these optimization problems forms the major part of the mathematical optimization or mathematical programming. Optimization is everywhere, from business to engineering design, from planning your holiday to your daily routine. Business organizations have to maximize their profit and minimize the cost. Engineering design has to maximize the performance of the designed product while of course minimizing the cost at the same time. Even when we plan holidays, we want to maximize the enjoyment and minimize the cost. Therefore, the studies of optimization are of both scientific interest, practical implications and subsequently the methodology will have many applications [15].

A linear optimization problem is the task of minimizing a linear real-valued function of finitely many variables subject to linear constrains; in general there may be infinitely many constraints.

2.4 The Fundamental Theorem of Linear Programming

LP in standard form is given by:

$$\min c^T x \text{ subject to}$$

$$Ax = b \qquad x, b \geq 0$$

Theorem: Let A be an m x n matrix of rank m.

(i) If there is a feasible solution then there is a basic feasible solution.

(ii) If there is an optimal feasible solution then there is an optimal basic feasible solution [18].

2.5 The General Representation Theorem

One of the most important (and difficult) theorems in linear programming is the General Representation Theorem. This theorem not only provides a way to represent any point in a polyhedral set, but its proof also lays the groundwork for understanding the Simplex method, a basic tool for solving linear programs [8].

Theorem: Let X = $\{x : Ax \leq b; x \geq 0\}$ be a nonempty polyhedral set. Then the set of extreme points is not empty and is finite, say $\{x_1, x_2, \ldots, x_k\}$. Furthermore, the set of extreme directions is empty if and only if X is bounded. If X is not bounded, then the set of extreme directions is nonempty and is finite, say $\{d_1, d_2, \ldots, d_l\}$: Moreover, $\bar{x} \in$ X if and only if it can be represented as a convex combination of $x_1, x_2, \ldots, x_k$ plus a nonnegative linear combination of $d_1, d_2, \ldots, d_l$ that is:

$$\bar{x} = \sum_{j=1}^{k} \lambda_j x_j + \sum_{j=1}^{l} л_j d_j$$

$$\sum_{j=1}^{k} \lambda_j = 1$$

$$\lambda_j \geq 0 \qquad j = 1, 2, \ldots, k$$

$$л_j \geq 0 \qquad j = 1, 2, \ldots, l$$

Chapter Three

Solving Examples of Linear Programming Problems By Graphical Mothed

3.1. Introduction

Linear programming is the best optimization technique which gives the optimal solution for the given objective function with the system of linear constraints. The main goal of this technique is finding the variable values that maximize or minimize the given objective function. Here, the objective function defines the amount to be optimized, and the constraints define the range. The four main components of linear programming are:

1- Objective Function

2- Constraints

3- Data

4- Decision Variables

This format is sufficiently general to include all optimization problems (most of life's problems too for that matter). Since we are interested in mathematical methods for solving such problems, it is necessary that the statement be reduced to symbolic form. for example [1]:

$$\text{Maximize or minimize : } f(x_1, x_2)$$

$$\text{Subject to: } g(x_1, x_2) = 0$$

the above statement reads as follows: maximize some function f, of x_1, x_2 by setting $x_1$and x_2 subject to the requirement that another function g of x_1; x_2 takes on the value zero.

3.2 Methods to Solve Linear Programming Problems

We can solve linear programming problems using different methods, one of them is a graphical method, that is explained as follows:

3.2.1 Graphical Method to Solve the Linear Programming Problems

The graphical method for solving a linear programming problem can be used when there are decision variables [7].

A linear programming problem involves constraints that contain inequalities. An inequality is denoted with familiar symbols, $>$, $<$, $\geq$, and $\leq$. Due to difficulties with strict inequalities ($>$ and $<$), we will only focus on $\geq$ and $\leq$.

In order to have a linear programming problem, we must have:[2]

- Inequality constraints.
- An objective function, that is, a function whose value we either want to be as large as possible (want to maximize it) or as small as possible (want to minimize it)

There are some important definitions and concepts before moving on with the graphical method [7]:

1. Solution: A set of decision variables values which satisfy all the constraints of an LPP.
2. Feasible solution: Any solution which also satisfies the non-negativity limitations of the problem.
3. Optimal feasible solution: Any feasible solution which maximizes or minimizes the objective function.
4. Feasible Region: The common region determined by all the constraints and non-negativity limitations of an LPP.
5. Corner point: A point in the feasible region that is the intersection of two boundary lines.

The way for solving LPP through Graphical Method in detail is the following [7]:

1) Formulate the LPP problems and develop objective function along with all the constraints function.
2) Graph the feasible region and find the corner points. The coordinates of the corner points can be obtained by either inspection or by solving the two equations of the lines intersecting at that point.
3) Make a table listing the value of the objective function at each corner point.

4) Determine the optimal solution from the table in step 3. If the problem is of maximization or minimization type, the solution corresponding to the largest or smallest value of the objective function which is the optimal solution of the LPP.

The steps for solving uses a graphic method [2] are:

- Define the variables to be optimized.
- Write the objective function in words, then convert to mathematical equation.
- Write the constraints in words, then convert to mathematical inequalities.
- Graph the constraints as equations.

It can easily to note that [2]:

- If we have to find maximum output, we have to consider the innermost intersecting points of all equations.
- If we have to find minimum output, we consider the outermost intersecting points of all equations.
- If there is no point in common in the linear inequality, then there is no feasible solution.

Table 3.1 Step of Linear Programming Problems (LPP).

Linear programming problems provide the method of finding an optimized function along with the values which would optimize the required function.

Step	Formulation of LPP
1	Identify the number of decision variables which govern the behaviour of objective function
2	Identify the set of constraints on the decision variables and express them in the form of linear in equations or linear equations
3	Express the objective function in the form of a linear equation in the decision variable
4	Optimize the objective function either graphically or mathematically

Table 3.2 Steps for Graphical Method.

Step	Formulation of Graphical Method
1	Formulate the LPP
2	Construct a graph and plot the constraint lines
3	Determine the valid side of each constraint line
4	Identify the feasible solution region
5	Find the optimum points
6	Calculate the coordinates of optimum points
7	Evaluate the objective function at optimum points to get the required maximum/ minimum value of the objective function

3.3 Solved Example of Linear Programming Problems by Graphical with One Constraint

Example 3.3.1 Solve the following (LPP) by graphical method.

Maximize $Z = 5x + 3y$

Subject To: $4x + 5y \leq 40$

$$x \geq 0 \,,\, y \geq 0$$

Solution: $4x + 5y = 40$ (constraint)

When $x = 0$, $y = 8$ $\longrightarrow (0,8)$

When $y = 0$, $x = 10$ $\longrightarrow (10,0)$

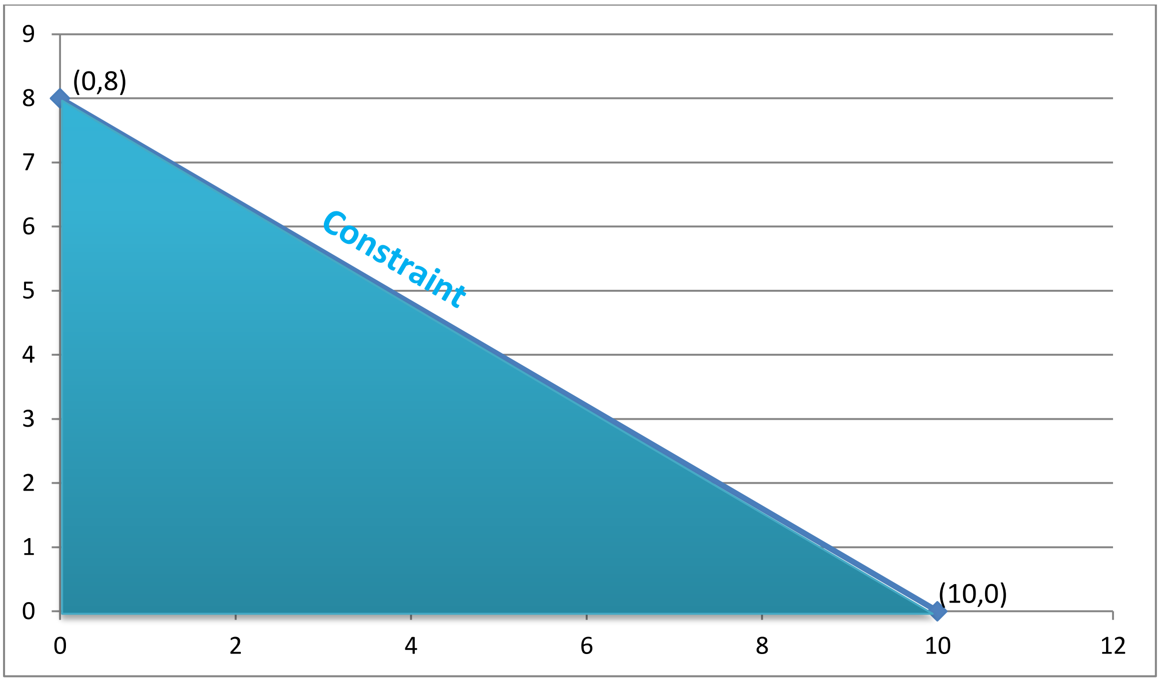

Figure 3.1: Feasible Region of Example (3.3.1).

Feasible Solution	Objective Function $Z = 5x + 3y$
(0,8)	$5(0) + 3(8) = 24$
(10,0)	$5(10) + 3(0) = 50$
(0,0)	$5(0) + 3(0) = 0$

The maximum value = 50

Maximum profit at point (10,0)

3.4 Solved Examples of Linear Programming Problems by Graphical with Two Constraints

Example 3.4.1 Solve the following (LPP) by graphical method.

Maximize $Z = 2x + 5y$

Subject To: $7x + 5y \leq 35$

$6x + 3y \leq 18$

$x \geq 0 , y \geq 0$

Solution: $7x + 5y = 35$ (constraint 1)

When $x = 0 , y = 7$ $\rightarrow (0,7)$

When $y = 0 , x = 5$ $\rightarrow (5,0)$

$6x + 3y = 18$ (constraint 2)

When $x = 0 , y = 6$ $\rightarrow (0,6)$

When $y = 0 , x = 2$ $\rightarrow (2,0)$

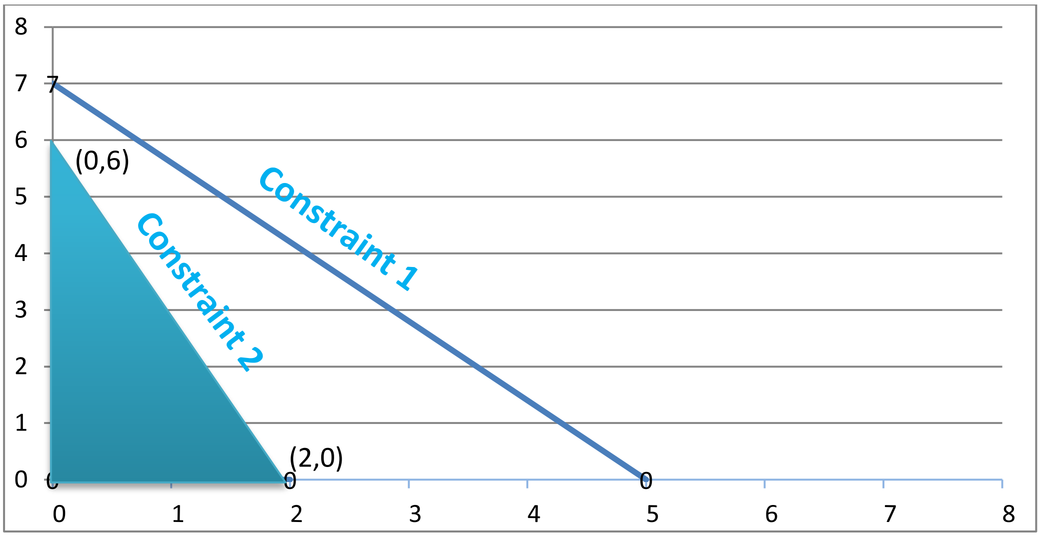

Figure 3.2: Feasible Region of Example (3.4.1).

Feasible Solution	Objective Function $Z = 2x + 5y$
(0,6)	$2(0) + 5(6) = 30$
(2,0)	$2(2) + 5(0) = 4$
(0,0)	$2(0) + 5(0) = 0$

The maximum value = 30

Maximum profit at point (0,6)

Example 3.4.2 Solve the following (LPP) by graphical method.

Minimize $Z = 60x + 20y$

Subject To: $2x + y \leq 80$

$6x + 3y \geq 120$

$x \geq 0 \text{ , } y \geq 0$

Solution: $2x + y = 80$ (constraint 1)

When $x = 0 \text{ , } y = 80$ $\longrightarrow (0,80)$

When $y = 0 \text{ , } x = 40$ $\longrightarrow (40,0)$

$6x + 3y = 120$ (constraint 2)

When $x = 0 \text{ , } y = 40$ $\longrightarrow (0,40)$

When $y = 0 \text{ , } x = 20$ $\longrightarrow (20,0)$

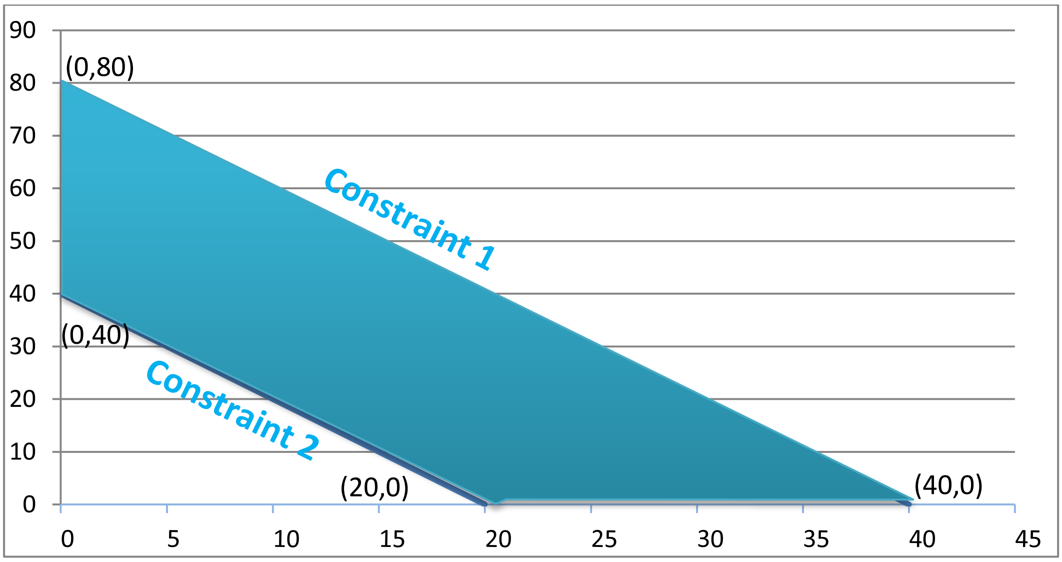

Figure 3.3: Feasible Region of Example (3.4.2).

Feasible Solution	Objective Function $Z = 60x + 20y$
(0,80)	$60(0) + 20(80) = 1600$
(40,0)	$60(40) + 20(0) = 2400$
(0,40)	$60(0) + 20(40) = 800$
(20,0)	$60(20) + 20(0) = 1200$

The minimize value = 800

Minimize profit at point (0,40)

3.5 Solved Examples of Linear Programming Problems by Graphical with Three Constraints

Example 3.5.1 Solve the following (LPP) by graphical method.

Maximum $Z = 20x + 15y$

Subject To: $x + 0.5y \leq 20$

$y = 50$

$2x + 3y \leq 60$

$x \geq 0 \, , y \geq 0$

Solution: $x + 0.5y = 20$ (constraint 1)

When $x = 0 \, , y = 40$ $\longrightarrow (0{,}40)$

When $y = 0 \, , x = 20$ $\longrightarrow (20{,}0)$

$y = 50$ (constraint 2)

$2x + 3y = 60$ (constraint 3)

When $x = 0 \, , y = 20$ $\longrightarrow (0{,}20)$

When $y = 0 \, , x = 30$ $\longrightarrow (30{,}0)$

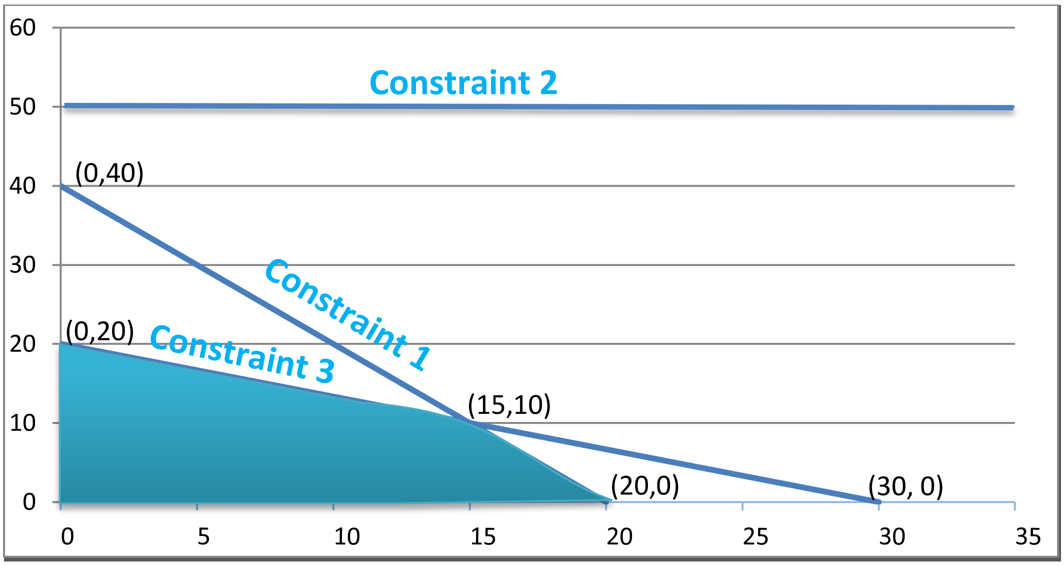

Figure 3.4: Feasible Region of Example (3.5.1).

The intersection between the line (1) and line (2) lends to:

$x + 0.5y = 20 \quad \times 2$
$2x + 3y = 60$

$-2y = -20 \rightarrow y = 10$
Substitute $y = 10$ in Equation (1) or (2) to find value of x
$x + 0.5(10) = 20 \rightarrow x = 15$
So, the point intersection is (15,10)

Feasible Solution	Objective Function $Z = 20x + 15y$
(20, 0)	20(20) + 15(0) = 400
(15,10)	20(15) + 15(10) = 450
(0,20)	20(0) + 15(20) = 300
(0,0)	20(0) + 15(0) = 0

The maximum value = 450

Maximum profit at point (15,10)

Example 3.5.2 Solve the following (LPP) by graphical method.

Minimum $Z = 5x + 10y$

Subject To: $x + y \leq 120$

$x + y \geq 60$

$x - 2y \geq 0$

$x \geq 0 , y \geq 0$

Solution: $x + y = 120$ (constraint 1)

When $x = 0 , y = 120 \rightarrow (0,120)$

When $y = 0 , x = 120 \rightarrow (120,0)$

$x + y = 60$ (constraint 2)

When $x = 0 , y = 60 \rightarrow (0,60)$

When $y = 0 , x = 60 \rightarrow (60,0)$

x - $2y = 0$ (constraint 3)

When $x = 0$, $y = 0$ $\longrightarrow$ (0,0)

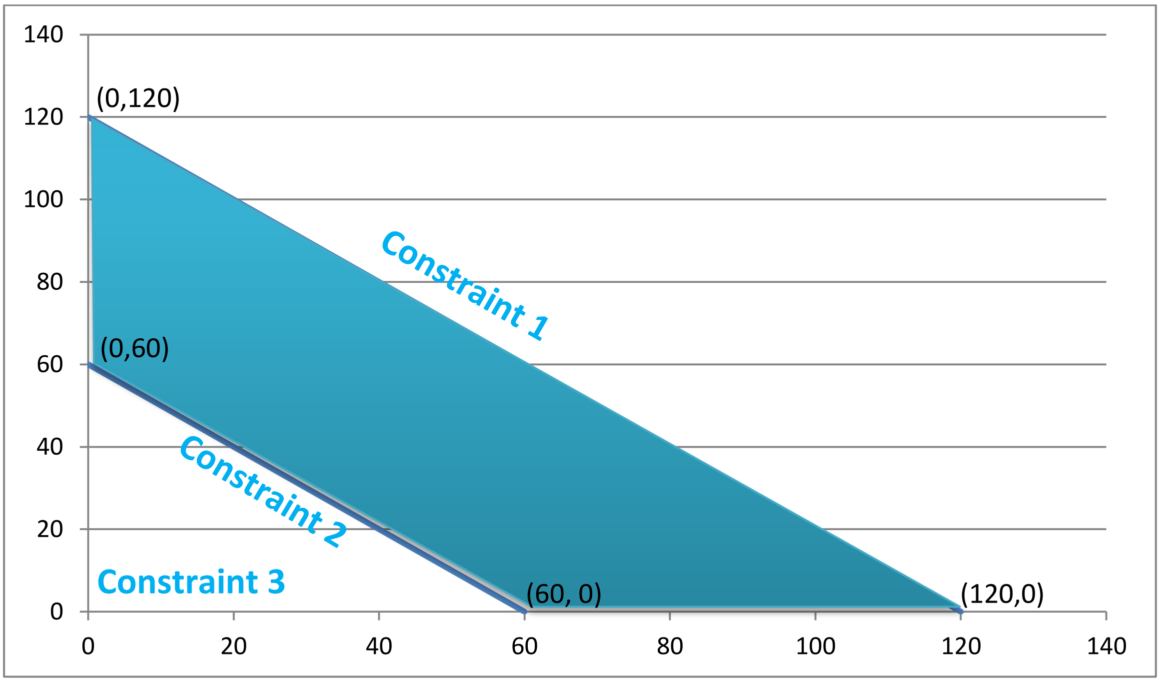

Figure 3.5: Feasible Region of Example (3.5.2).

Feasible Solution	Objective Function Z = $5x + 10y$
(0, 120)	5(0) + 10(120) = 1200
(0,60)	5(0) + 10(60) = 600
(60, 0)	5(60) + 10(0) = 300
(120,0)	5(120) + 10(0) = 600

The minimize value = 300

Minimize profit at point (60, 0)

Example 3.5.3 Solve the following (LPP) by graphical method.

Minimum $Z = 20x + 10y$

Subject To: $x + 2y \leq 40$

$3x + y \geq 30$

$4x + 3y \geq 60$

$x \geq 0 \, , y \geq 0$

Solution: $x + 2y = 40$ (constraint 1)

When $x = 0 \, , y = 20$ $\longrightarrow (0, 20)$

When $y = 0 \, , x = 40$ $\longrightarrow (40,0)$

$3x + y = 30$ (constraint 2)

When $x = 0 \, , y = 30$ $\longrightarrow (0,30)$

When $y = 0 \, , x = 10$ $\longrightarrow (10,0)$

$4x + 3y = 60$ (constraint 3)

When $x = 0 \, , y = 20$ $\longrightarrow (0,20)$

When $y = 0 \, , x = 15$ $\longrightarrow (15,0)$

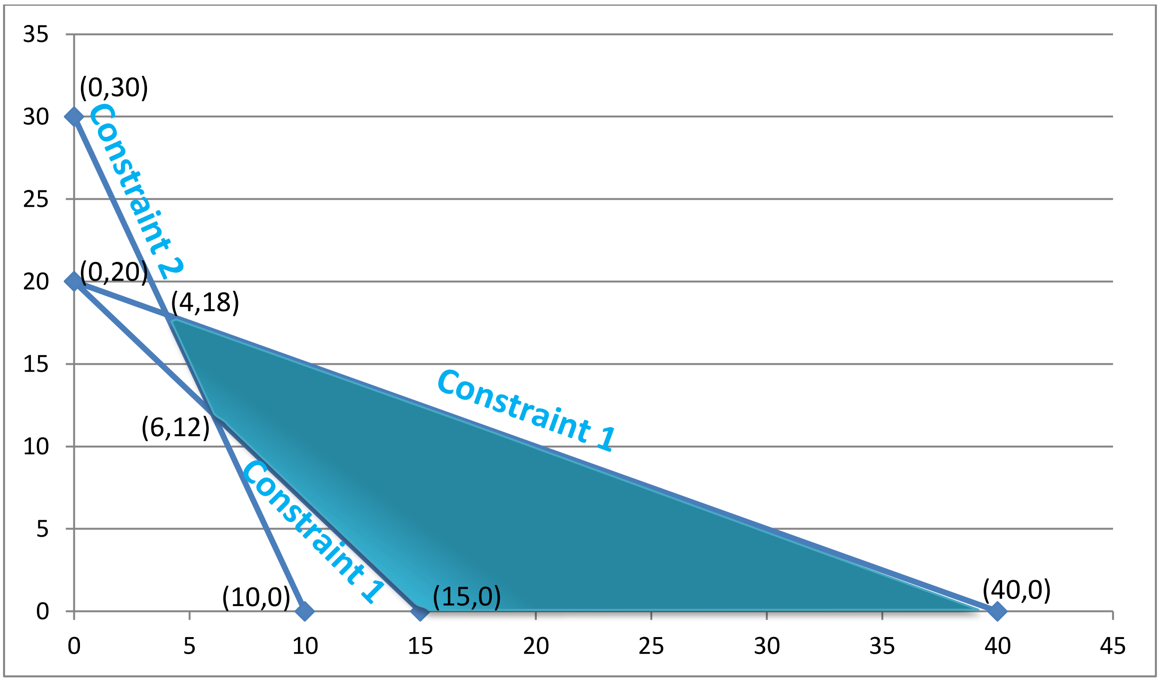

Figure 3.6: Feasible Region of Example (3.5.3).

The intersection of line (1) and line (2) leads to:

$x + 2y = 40 \quad \times -3$
$3x + y = 30$

$-5y = -90 \rightarrow y = 18$
Substitute $y = 18$ in Equation (1) or (2) to find value of x
$x + 2(18) = 40 \rightarrow x = 4$
So, the point intersection is (4,18).
The intersection of line (2) and line (3) leads to:

$3x + y = 30 \quad \times -3$
$4x + 3y = 60$

$-5x = -30 \rightarrow x = 6$
Substitute $x = 6$ in equation (2) or (3) to find value of y
$3(6) + y = 30 \rightarrow y = 12$
So, the point intersection is (6,12)

Feasible Solution	Objective Function $Z = 20x + 10y$
(4,18)	20(4) + 10(18) = 260
(6,12)	20(6) + 10(12) = 240
(15, 0)	20(15) + 10(0) = 300
(40,0)	20(40) + 10(0) = 800

The minimize value = 240.

Minimize profit at point (6,12).

Chapter Four
Linear Programming Problems and their Daily Life Applications

4.1 Applications of Linear Programming Problems

The linear programming problems are the ones which seek to optimize a quantity that is described linearly in terms of a few decision variables. Thus, the broad classification of the different types of linear programming problems can be encountered as (Diet Problem, Manufacturing Problems, Transportation Problems and Optimal Assignment Problems).

Table 4.1 Application of Linear Programming Problems.

Application of LPP	Constraints	Objective Function
Diet Problems	The specified nutritional requirement	The cost of food intake
Manufacturing Problems	Variables like work hour, material availability, production rate, etc.	The cost of production
Transportation Problems	The specific supply demand pattern	The cost of transportation
Optimal Assignment Problems	The number of employees, work hour of each employee, efficiency of employee etc.	The total assignment done

4.1.1 Diet Problem

As the name suggests in itself, such problems involve optimizing the intake of certain application of rich foods in certain nutrients that could help one follow a particular diet plan. More precisely, the goal of a diet problem is to select a set of foods that will satisfy a set of a daily nutritional requirement at a minimum cost. [4]

- Constraints – The specified nutritional requirements, that could be a specific calorie intake or the amount of sugar or cholesterol in the diet.
- Objective function – The cost of the food intake.

The "Diet Problem" (the search of a low - cost diet that would meet the nutritional needs of a US Army soldier) is characterized by a long history, whereas most solutions for comparable diet problems were developed in 2000 or later, during which computers with large calculation capacities became widely available and linear programming (LP) tools were developed. LP can be applied to a variety of diet problems, from food aid, national food programs, and dietary guidelines to individual issues. This review describes the developments in the search for constraints. After nutritional constraints, costs constraints, acceptability constraints and ecological constraints were introduced.

The studies that apply ecological constraints were analyzed and compared in detail. Most studies have used nutritional constraints and cost constraints in the analysis of dietary problems and solutions, Introducing acceptability constraints is recommended, but no study has provided the ultimate solution to calculating acceptability. Future possibilities lie in finding LP solutions for diets by combining nutritional, costs, ecological and acceptability constraints. LP is an important tool for environmental optimization and shows considerable potential as an instrument for finding solutions to a variety of very complex diet problems.

Example: A kitchen manager at Babylon Hospital has to decide the food mix for the patients. Dietary instructions are that each patient must get at least:

- One gram of protein
- One gram of fat
- Three grams. of carbohydrates

Additional guidelines mention that the carbohydrate content of any patient by any chance, shouldn’t exceed six grams. The availability of protein, fat, and carbohydrates in grams per kg of chicken, rice, and bread; along with the market costs of each of these food items is as given below:

Table 4.2 Dietary instructions.

	Protein	Fat	Carbohydrates	Price/kg (Rs.)
Chicken	10	2	1	30
Rice	2	1	15	5
Bread	2	0	10	4

Formulate a suitable diet mix by minimizing the cost, subject to the given constraints, assuming 100 patients on that day.

Linear Programming (LP) can be used to solve questions on matching diets to nutritional and other additional constraints with a minimum amount of changes. Linear programming is a mathematical technique that allows the generation of optimal solutions that satisfy several constraints at once [6].

The "Diet Problem" is a typical question of resource optimization or, in mathematical terms, of minimization of a linear function subject to multiple linear constraints, also called linear programming [4].

The goal of the diet problem is to select a set of foods that will satisfy a set of daily nutritional requirement at minimum cost. The problem is formulated as a linear program where the objective is to minimize cost and the constraints are to satisfy the specified nutritional requirements.

For the duration of World War II, the Air Force and other parts of the army were hiring mathematicians to solve the important diet problem and to plan affordable meals. Among the researchers involved in solving this problem was George Dantzig. He proposed a new algorithm he had developed. It took him until 1947, being the first to deliver the correct mathematical result [5].

Until now the approach has been used in many ways to design individual diets as well as population diets. The problem of the diet is interesting, because it is difficult to optimize the function of phenomenon like the diet, as it is composed of several variables: energy density, water content, macronutrients, micronutrients, bioactive substances, and contaminants [17].

Herforth et al. proposed a "simple framework based on three domains: nutritional quality, economic viability, and environmental sustainability". proposal by including the three domains in an integrated way . It is expected that LP makes it possible to model these domains across disciplines [12].

We reviews the application of linear programming to optimize diets with nutritional, economic, and environmental constraints. There are three main reasons for studying the application of LP to diets in greater depth:

- Linear programming is thought to be "the ideal tool to rigorously convert precise nutrient constraints into food combinations" [3].
- Maillot et al. stated that most food-based dietary guidelines assume that people eating according guidelines are receiving all recommended nutrients. However, in practice this is not always true. So, LP could be helpful to support development of dietary guidelines that fulfill all nutritional requirements.[17]
- Macdiarmid observed that healthy diets have not always lower environmental impacts. She assumed that LP is able to suggest diets and products with lower environmental impacts than the impacts of diets assessed through scenario type studies [16].

The goal of this review is to analyses if the application of LP since 2000 provided acceptable diet solutions in practice, especially when environmental constraints were introduced. That LP can be applied to a variety of diet problems: from food aid, national food programs, dietary guidelines, to individual solutions. In supporting dietary guidelines, LP has proven its value in many ways. Most studies have used nutritional constraints combined with cost constraints.

This review focus on optimization through the application of linear programming. The result of a LP problem shrinks to discover the optimum worth (maximum or minimum, liable to the problem) of the linear equation (named the "objective function"):

$$f = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$$

The function is conditional on different constraints, stated as inequalities "it is necessary to find the solution of the system of linear inequalities" (that is, the set of n-values of the variables x_i that simultaneously satisfies all the inequalities). The objective function is then evaluated by substituting the values of x_i in the equation that defines f.

4.1.1 The Aim of Studying Diet Problem

Nutrition is affected by numerous environmental and societal causes. Although the diet problems were already urgent during World War II, the challenge of feeding the world in a healthy and sustainable manner will only become more urgent [10].

4.2 Application Examples of Linear Programming Problems

Example 4.2.1 A company produces two models of calculators at two different places. In one day place A can produce 140 of models 1 and 35 models 2, while place B can produce 60 of models 1 and 90 models 2. If the company need to produce at least 420 models 1 and 315 models 2. Assume it costs 1200 per day to operate place A and 900 per day for place B. Consider x and y to be the number o operating days for place A and B respectively, Formulate a linear programming problem and solve it in order to minimize the total cost.

Solution: formulation of the problem

	Model 1	Model 2	Cost
A	140	35	1200
B	60	90	900
	≥ 420	≥ 315	

Minimize $C = 1200x + 900y$

Subject To: $140x + 60y \geq 420$

$$35x + 90y \geq 315$$

$$x \geq 0 \,,\, y \geq 0$$

$140x + 60\,y = 420$ (constraint 1)

When $x = 0$, $y = 7$ $\longrightarrow (0,7)$

When $y = 0$, $x = 3$ $\longrightarrow (3,0)$

$35x + 90y = 315$ (constraint 2)

When $x = 0$, $y = 3.5$ $\longrightarrow (0,3.5)$

When $y = 0$, $x = 9$ $\longrightarrow (9,0)$

The intersection of line (1) and line (2) leads to:

$140x + 60\,y = 420$ $\times\, 3$
$35x + 90y = 315$ $\times\, 2$

$350x = 630 \longrightarrow x = 1.8$

Substitute $x = 1.8$ in Equation (1) or (2) to find value of y
$35(1.8) + 90y = 315 \longrightarrow y = 2.8$
So, the point intersection is (1.8,2.8)

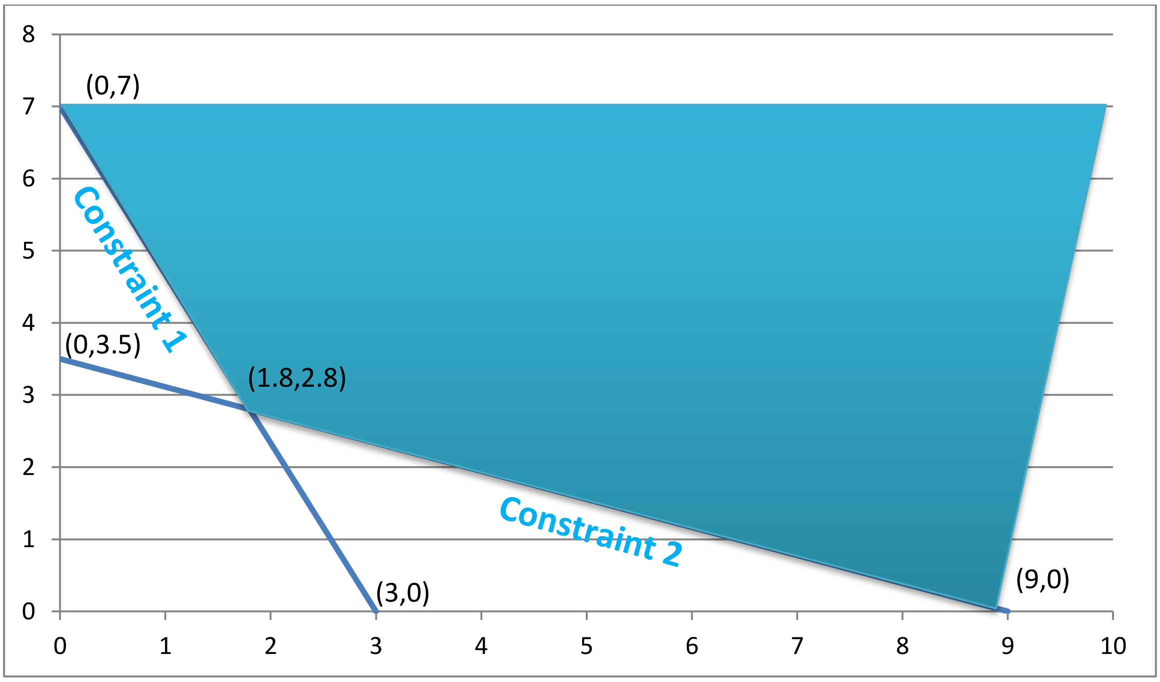

Figure 4.1: Feasible Region of Example (4.2.1).

Feasible Solution	Objective Function C = 1200x + 900y
(0,7)	1200(0) + 900(7) = 6300
(1.8,2.8)	1200(1.8) + 900(2.8) = 4680
(9, 0)	1200(9) + 900(0) = 10800

The minimize value = 4680.

Minimize profit at point (1.8,2.8)

Example 4.2.2 A plastic factory produces two types of plastic tools

1) The production of a unit of the first category requires 3 working hours and 4 kg of raw materials.

2) The production of the second category requires 5 working hours and 2 kg of raw materials.

The profits from these two categories are 10 and 8 dollars, respectively, for each production unit, and the factory's weekly capacity is 109 working hours and 80 kilograms of raw materials. Find the formulation of this problem in the form of linear programming to find the largest amount of profits.

.

Solution: formulation of the problem

	work hours	Produced material	Profits
A	3	4	10
B	5	2	8
	≤ 109	≤ 80	

Maximize $Z = 10x + 8y$

Subject To: $3x + 5y \leq 109$

$4x + 2y \leq 80$

$x \geq 0 , y \geq 0$

$3x + 5y = 109$ (constraint 1)

When $x = 0 , y = 21.8$ $\rightarrow (0,21.8)$

When $y = 0 , x = 36.3$ $\rightarrow (36.3,0)$

$4x + 2y = 80$ (constraint 2)

When $x = 0 , y = 40$ $\rightarrow (0,40)$

When $y = 0 , x = 20$ $\rightarrow (20,0)$

The intersection of line (1) and line (2) leads to:

$3x + 5y = 109$ $\times$ 2

$4x + 2y = 80$ $\times$ -5

$-14x = -182 \rightarrow x = 13$

Substitute $x = 13$ in Equation (1) or (2) to find value of y

$3(13) + 5y = 109 \rightarrow y = 14$

So, the point intersection is (13,14)

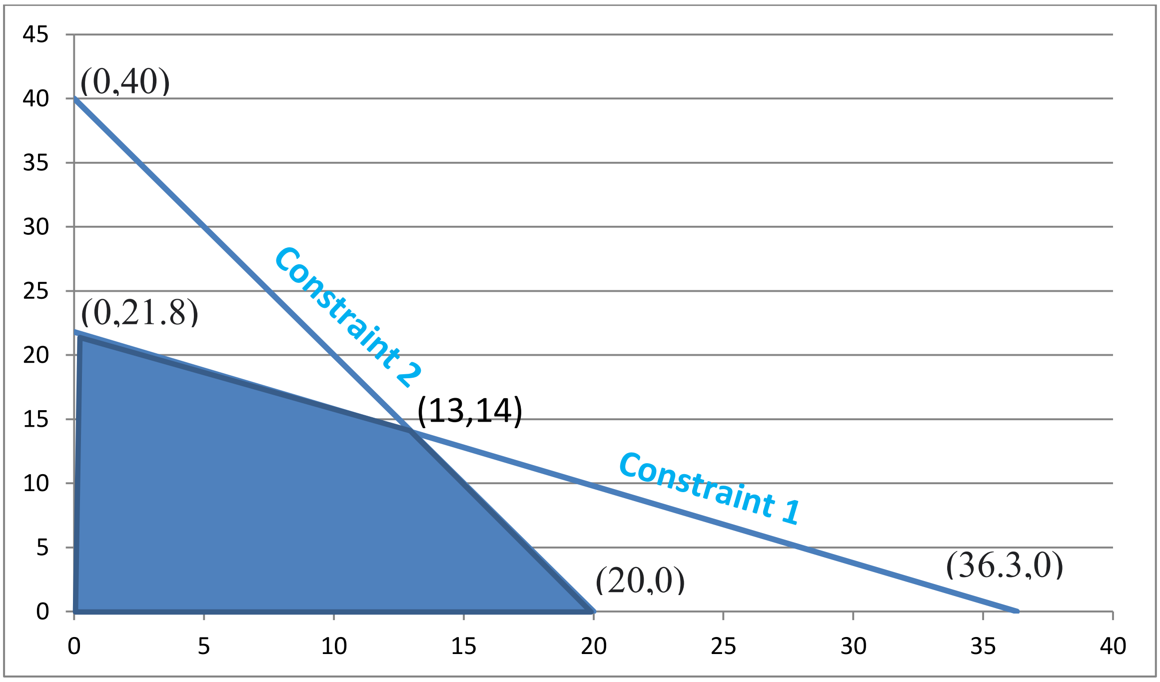

Figure 4.2: Feasible Region of Example (4.2.2).

Feasible Solution	Objective Function Z = 10x + 8y
(0,21.8)	10(0) + 8(21.8) = 174.4
(13,14)	10(13) + 8(14) = 242
(20,0)	10(20) + 8(0) = 200
(0,0)	10(0) + 8(0) = 0

The maximize value = 242

Minimize profit at point (13,14)

4.3 Suggested Solution by Linear Programming Problems

World Food Program (WFP) distributes meals for the forcibly displaced people from the stricken governorates to the relatively stable governorates, and among these governorates is Wasit governorate.

The distribution is carried out by an executive partner through the Iraqi Non-Governmental Organization (NGO), which represents the second party. While the first party is responsible for delivering foodstuffs to the stores of the second party.

The latter unloads the materials in the main warehouse and then delivers them to the secondary centers in the districts and townships, and then distributes them to the beneficiaries.

The first party calculates an amount (74 dollars) for each ton that the second party distributes monthly.

The second party is obligated to provide the following:

A- Main warehouse in the governorate center, and an administrative staff that includes store manager and ten loading and unloading workers.

B- Secondary warehouse in each district or township in which food is distributed and it includes a director of a distribution center, four distribution employees, four loading and unloading workers, and a driver with a car), noting that the number of distribution centers is nine.

C- Provided administrative cadres for the project includes project manager, project accountant, and a driver with a car.

Materials that are distributed for each individual:

-6 kilos of flour

- 2 kilos bean

So, the total will be 8 kilos for each individual.

The following table shows the number of officers, workers and their salary that NGO responsible for:

Table 4.3 Total Salaries.

No.	Job Title	Total Number	Salary $	Total Salary $
1	project manager	1	600	600
2	Project Accountant	1	400	400
3	Master Store Manager	1	400	400
4	Distribution center manager	9	350	3150
5	Distribution Officer	36	350	12600
6	Loading and unloading workers	46	300	13800
7	Driver with car	10	400	4000
Total				34950

The highest number of people to whom food is distributed is 64,000 people, with an average of 6 kg of flour and 2 kg of beans per person. The following table explains that:

Table 4.4 Final Cost.

No.	Material	Amount Kg	Number of Individual	Total Amount Kg	Total Amount ton	Cost per Ton $	Final Cost $
1	Flour	6	64000	384000	384	74	28.416
2	Bean	2	64000	128000	128	74	9.472
Total							37.888

The following table indicates the number of beneficiaries in each month who received their share of foodstuffs 6 kg of flour and 2 kg of beans, in addition to the amount of tons that were distributed minus the salaries of employees and workers for each person, the reported amount is 34950 dollars:

Table 4.5 Earnings in Dollars.

No.	The Month	The Number of Recipients	Total Amount of Distribution	Total Salaries in Dollars	Earnings in Dollars
1	January	63800	37.770	34950	2820
2	February	61025	36.127	34950	1177
3	March	61890	36.639	34950	1689
4	April	62134	36.783	34950	1833
5	May	63457	37.567	34950	2617
6	June	62763	37.156	34950	2206
7	July	63456	37.566	34950	2616
8	Father	59642	35.308	34950	358
9	September	63213	37.422	34950	2472
10	October	64100	37.947	34950	2997
11	November	62567	37.040	34950	2090
12	December	60743	35.960	34950	1010
Total					24477

Formulating the problem for the first and second months

	Flour	bean	Profits
January	382.8	127.6	2820
February	366.15	122.05	1769
	≤ 384	≤ 128	

Maximize $Z = 2820x + 1769y$

Subject To: $382.8x + 366.15y \leq 384$

$127.6x + 122.05y \leq 128$

$x \geq 0 , y \geq 0$

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الخلاصة

البحث يقدم دراسة حل مسائل البرمجة الخطية حيث استخدمت فيها احد اهم طرق الحل وهي الطريقة البيانية وعرضت امثلة متنوعة وتم طرح نموذج رياضي من الحياة اليومية.

الهدف من البحث هو عمل دراسة حول مسائل البرمجة الخطية التي هي جزء من مسائل البرمجة الرياضية (الخطية وغير الخطية) والهدف منها ايجاد قيم المتغيرات التي تزيد او تقلل من دالة الهدف للوصول الى الحل الامثل، وبينت خطوات الحل بالطريقة البيانية. تناولت الدراسة عدة امثلة لحل هذه المسائل ذات قيد واحد او اكثر مع متغيرين وكيفية تحديد نقاط ومنطقة الحل واختيار النقطة التي يكون الحل فيها هو الحل الامثل والتي تزيد او تقلل من دالة الهدف.

وتمت دراسة احد التطبيقات العملية لحل مسائل البرمجة الخطية من الحياة اليومية وهي النظام الغذائي للوصول الى افضل المتطلبات الغذائية المحددة وباقل كلفة. وتم عرض مقترح موديل رياضي جديد وهو احد التطبيقات العملية من الحياة اليومية من خلال توزيع المساعدات الغذائية وإيجاد اقل تكلفة للتوزيع مع زيادة بالأرباح شهريا مع الحفاظ على الثوابت القيمية للقيود.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

حل مسائل البرمجة الخطية باستخدام الطرق البيانية

بحث مقدم

الى مجلس كلية التربية للعلوم الصرفة/ جامعة بابل كجزء

من متطلبات نيل درجة الدبلوم العالي تربية/ رياضيات

مقدم من قبل الطالب

نوفل محسن هادي موسى

بإشراف الدكتور

احمد صباح احمد

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