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College of Education for Pure Sciences

ON a STRONG DIFFERENTIAL SUBORDINATION USING SOME OPERATORS

A project

Submitted to Mathematics Department, College of Education for Pure Science, University of Babylon as a Partial Fulfillment of the Requirment for the Degree of Higher Diploma Education/ Mathematics

By

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بسم الله الرحمن الرحيم

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صدق الله العلي العظيم

سورة طه الآية ١١٤

<u>Supervisor's Certification</u>

We certify that the preparation for this project entitled "**ON a STRONG DIFFERENTIAL SUBORDINATION USING SOME OPERATORS"** for the student **" Methaq Azeez Joudah Kadim "** was made under my supervision at University of Babylon, College of Education for Pure Sciences as a partial fulfillment for requirements of the Degree Higher Diploma Education/ Mathematics.

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الاهداء

الى امي وابي(رحمهما الله)

سبب وجودي في الحياة..............

الى اخي الشهيد احمد(رحمه الله).....

الى اخوتي واخواتي وعائلتي...........

الى كل من دعا لي بالتوفيق وشجعني ...

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We would appreciate hearing from anyone who stumbles upon this research and who has comments, corrections, and /or suggestions

*I wish to give my deepest gratitude to **my family**, especially to my parents, whose efforts I truly appreciate*

Thanks are also due to my friends for their encouragement and consideratio.

Abstract

functions and analytical functions in the complex plane that are examined in this project, a few interesting strong differential subordination findings arise. These strong differential subordinations are determined through a special case for extended multipliers.

Specifically, some of these strong subordination results discuss a number of interesting consequences. Relevant links to those in earlier works are also provided for some of the new results achieved in this research.

The squeeze (or sandwich) theorem is the most important theory of this project . It is typically used to confirm the limit of a function via comparison with two other functions whose limits are known or easily computed and we used strong Differential Subordination to show that by using kind of operators.

Mathematical Symbols

Symbol	***Name***
$\mathbb{C}$	The set of complex numbers.
U	the open unit disk of the complex plane $U:=\{z \in \mathbb{C}: \|z\| < 1\}$
$\bar{U}$	the close unit disk of the form $\bar{U}:=\{z \in \mathbb{C}: \|z\| \leq 1\}$
$\mathbb{D}$	the unit disk of the complex plane
$\mathbb{R}$	The set of real numbers.
$\mathbb{N}$	The set of natural numbers.
$\mathbb{N}_0$	The set of numbers $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
A_{ζ}	The class of functions of the form $$f(z,\zeta) = z + \sum_{k=2}^{\infty} a_k(\zeta) z^k, (z \in U, \zeta \in \bar{U})$$
Q_{ζ}	The set of functions that analytic and injective on $\bar{U}\times\bar{U}\setminus E(f,\zeta)$, where $E(f,\zeta) = \left\{r \in \partial U: \lim_{z\to r}(f,\zeta) = \infty\right\}$
Q_a	The sub class of Q_{ζ} with $f(0,\zeta)=a$
$\mathbb{A}(\mathrm{p})$	the class of functions of the form $$f(z) = z^{\mathrm{p}} + \sum_{k=1}^{\infty} a_{k+p} z^{k+\mathrm{p}}, z \in \mathbb{D}, \mathrm{p} \in \mathbb{N} := \{1,2,3,\dots\}$$
$a_k(\zeta)$	Holomorphic(analytic) functions in $\bar{U}$ for $k \geq 2$
$a_j(\zeta)$	Holomorphic (analytic) functions in $\bar{U}$ for $j \geq n$
w	analytic function in U with w(0)=0
$H(U \times \bar{U})$	The class of analytic functions in $(U \times \bar{U})$

$H[a, n, \zeta]$	The class of holomorphic functions with the form $\{f \in H(U \times \overline{U})$: $f(z, \zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots, z \in U, \zeta \in \overline{U}\}$
$f \prec\prec F$	f strongly subordinate to F
$\partial \mathrm{U}$	Boundary points
S, k, ω	parameters
$J^{s}_{\omega,k}$	The operator $J^{s}_{\omega,k}: \mathbb{A}_\zeta \to \mathbb{A}_\zeta, \mathrm{s} \in \mathbb{C}, \omega \in \mathbb{C}\backslash\mathbb{Z}0-, \mathrm{k} \in \mathbb{N}$
$Q(z, \zeta)$	analytic function in $U \times \overline{U}$ with $\psi(z, \zeta) = 1$
$S(\lambda, \alpha, \beta, m; \psi)$	The set of functions in the class satisfies the strong differential subordination for every $\zeta \in \overline{U}$ and $\lambda > 0, \alpha \epsilon R, \beta \geq 0, m \in N_0$.
$T(\lambda, \alpha, \beta, m; \psi)$	The set of functions in the class satisfies the strong differential superordination

Contents

Chapter 2 : Strong Subordination Results

وزارة التعليم العالي والبحث العلمي
جامعة القادسية كلية التربية

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إلى /الباحثة ميثاق عزيز جودة الطالبي المحترمة – أ.م.د. عقيل كتاب الخفاجي المحترم

م/ قبول نشر

تحية طيبة

يسرنا أن نعلمكم بقبول بحثكم الموسوم

(On a New Strong Differential Subordination and Supervision Defined by the Integral Operator)

للنشر في المجلة **AIP Conference Proceeding** ضمن فعاليات المؤتمر العلمي الدولي الثاني للعلوم الصرفة الذي تقيمه كلية التربية جامعة القادسية للمدة ٢٩-٣٠ حزيران ٢٠٢١.

شاكرين تعاونكم في إنجاح فعاليات المؤتمر

ا.م.د. خالد شياع خيرالله
رئيس محور الرياضيات
٢٠٢١/٦/٢٨

نسخة منه إلى /
لجنة الطباعة لاتخاذ ما يلزم

Introduction

In this project we study some geometrical properties of analytic univalent functions.

Also, we provide a comprehensive introduction to the differential subordination and how to be strong by using some operators.When we write this project we develop the project from some reserchs .

In[1] Wanas, Abbas Kareem; ALINA, Alb Lupas,. On a new strong differential subordinations and superordinations of analytic functions involving the generalized operator.
In[2] Wanas, Abbas Kareem. Two New Classes of Analytic Functions Defined by Strong Differential Subordinations and Superordinations.

. we introduce two new classes of analytic functions define by strong subordinations and superordinations involving the generalized operator.

For class

$\psi(\lambda, s, \zeta, \omega; Q), \lambda > 0, s \in \mathbb{C}, \omega \in \mathbb{C}\backslash\mathbb{Z}_0^- \zeta \in \overline{U} = \{z \in \mathbb{C}: |z| \leq 1\}$,

functions and analytical functions in $U \times \overline{U}$ that are examined

in this project, a few interesting strong differential subordination findings arise .

These strong differential. subordinations are determined through a special case for extended multipliers

$$J_{\omega,k}^{S} f(z,\zeta) = z + \sum_{k=2}^{\infty} \left(\frac{1+\omega}{k+\omega}\right)^{s} a_k\,(\zeta) z^k, \text{ where,} f \in \mathbb{A}_\zeta$$

$$\mathbb{A}_\zeta = \left\{ f \in H(U \times \overline{U}) : f(z,\zeta) = z + \sum_{k=2}^{\infty} a_k(\zeta) z^k;\ z \in U, \zeta \in \overline{U} \right\}.$$

Specifically, some of these strong subordination results discuss a number of interesting consequences. Relevant links to those in earlier works are also provided for some of the new results achieved in this project .

CHAPTER ONE

Fundamentals On Strong Differential Subordinations

1.1. Introduction

We include a very brief review of some of the basic facts from undergraduate complex analysis that we will need in this project see**[3]** .the methods of complex analysis are important in mathematics, science, and engineering.

Let x and y be real numbers and let i denote the imaginary unit having the property that $i^2 = -1$. A complex number is an expression of the form $x + yi$. Write $\mathbb{C} = \{z = x + iy: x \in \mathbb{R}, y \in \mathbb{R}\}$,

Where x the real part of z, denoted $\text{Re}\{z\}$, and y the imaginary part of z, denoted $\text{Im}\{z\}$.

1.2. Definition(simply connected domain) [3] let D be a domain $D \subset \mathbb{C}$ is an open and connected non-empty subset of the complex plane. The domain D is simply connected if both D and $\mathbb{C} \backslash D$ are connected.

1.3. Definition(complex differentiable) [3] A complex function $f: D \to \mathbb{C}$ defined for all $z \in D$ is said to be complex differentiable at $z_0 \in D$ if

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \text{ exists.}$$

<u>1.4.</u> <u>Definition (analytic function</u>) [3] The function $f: D \to \mathbb{C}$ is analytic at z_0(or holomorphic at z_0)if it is complex differentiable at every point in some neighborhood $\mathcal{N}(z_0;\epsilon)$of $z_0 \in D$.We say that f is analytic on D if f is analytic at z_0 for every $z_0 \in D$.

For example, let $f: D \to \mathbb{C}$, $f(z) = 1 + z$ is analytic function.

<u>1.5. Definition(univalent function)</u> [4] A function $f: D \to \mathbb{C}$ with property that $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in D$ with $z_1 \neq z_2$ is said to be one-to-one on D (or univalent,or injective). a function $f: D \to \mathbb{C}$ which both analytic on D and one-to-one on D is called conformal on D .

For example, a function $f: D \to \mathbb{C}$, $f(z) = z$ is univalent function.

<u>1.6.</u> <u>Definition (convex function</u>) [5] A set G in complex plane $\mathbb{C}$ is referred to as convex set when the line segment of any two interior points w_1 and w_2 of G lies in the

interior of G.,i.e.

$$(1-t)w_1 + tw_2 \in G \quad \text{for} \quad 0 \leq t \leq 1.$$

Let f ∈ S .then f convex function in U if and only if

$$\mathrm{Re}\left(1+\frac{z\,f''(z)}{f'(z)}\right)>0\,,(z\in U).$$

<u>1.7. Definition (Strongly Subordinate)</u> [**6**] Let $f(z,\zeta)$, $F(z,\zeta)$be analytic in $U\times\overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $F(z,\zeta)$ if there exists a function w analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$such that

$f(z,\zeta)=\ F(w(z),\zeta)$ for all $\zeta\in\overline{U}$.

In such a case we write

$$f(z,\zeta)\prec\prec F(z,\zeta),\ z\in U,\ \zeta\in\overline{U}.$$

<u>1.8. Definition(Hadamard product)</u> [5] the convolution or

(Hadamard product) of function f and g denoted by $f*g$ is define as following for the function in $\mathcal{A}(p)$ and $\mathcal{A}^{*}(p)$

respectively :

If $f(z)=z^{p}+\sum_{n=p+1}^{\infty}a_{n}\,z^{p}\,,g(z)=z^{p}+\sum_{n=p+1}^{\infty}b_{n}\,z^{p}$.

Then

$$(f*g)(z)=z^{p}+\sum_{n=p+1}^{\infty}a_{n}b_{n}\,z^{p}.$$

1.9. Definition(Beast dominant) [5] let $\psi: \mathbb{C}^3 \times U \times \overline{U} \rightarrow \mathbb{C}$, and let $h(z, \varepsilon)$ be univalent in U for all $\varepsilon \in \overline{U}$. If $p(z, \varepsilon)$ is analytic in $U \times \overline{U}$ and satisfies the (second –order) strong differential subordination:

$$\psi\big(p(z,\varepsilon) , z \acute{p}(z,\varepsilon) , z^2 \acute{\acute{p}}(z,\varepsilon); z , \varepsilon\big) \prec\prec h(z,\varepsilon) , z \in U, \varepsilon \in \overline{U}, \qquad (1)$$

Then $p(z, \varepsilon)$ is called a solution of the strong differential subordination. The univalent function $q(z, \varepsilon)$ is called a dominant of the solution of the strong differential subordination, or simply a dominant if

$$p(z,\varepsilon) \prec\prec q(z,\varepsilon) \quad , \text{for all } p(z,\varepsilon)$$

satisfies (1)

A dominant $\acute{q}(z, \varepsilon)$ that satisfies

$$\acute{q}(z,\varepsilon) \prec\prec q(z,\varepsilon)$$

For all dominant $q(z, \varepsilon)$ of (1) is said to be best dominant of (1). Note that the best dominant is unique up to a rotation of $U \times \overline{U}$.

1.10. Definition (The class A_ζ) [10]

Denote by $U = \{z \in \mathbb{C}: |z| < 1\}$ the open unit disk of the complex plane,

$\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ the close unit disk of the complex plane and

$H(U \times \overline{U})$ The class of analytic functions in $(U \times \overline{U})$.

For n a positive integer and $a \in \mathbb{C}$, let

$$H[a, n, \zeta] = \{f \in H(U \times \overline{U}) : f(z, \zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots, \quad z \in U, \zeta \in \overline{U}\} \qquad (2)$$

, where $a_j(\zeta)$ are holomorphic functions in $\overline{U}$ for $j \geq n$.

Let A_ζ the class of functions of the form:

$$f(z, \zeta) = z + \sum_{k=2}^{\infty} a_k(\zeta)z^k, (z \in U, \zeta \in \overline{U}) \qquad (3)$$

which are analytic in $U \times \overline{U}$ and $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$.

1.11. Definition(Subclass$Q_z(a)$) [5] We denote by Q_ζ the set of functions that are analytic and injective on $\overline{U} \times \overline{U} \setminus E(f, \zeta)$, where

$$E(f, \zeta) = \left\{r \in \partial U : \lim_{z \to r} (f, \zeta) = \infty\right\},$$

and

$$f_z'(r, \zeta \neq 0) \text{ for } r \in \partial U \times \overline{U} \setminus E(f, \zeta).$$

The subclass of Q_z with $f(0, \zeta) = a$ is denoted by $Q_z(a)$.

1.12. Remark [**5**]

(i) Since $f(z, \zeta)$ is analytic in $U \times \overline{U}$.,for all $\zeta \in \overline{U}$ and univalent in U ,

for all $\zeta \in \overline{U}$, Definition 1.11 is equivalent to

$f(o, \zeta) = F(0, \zeta)$ for all $\zeta \in \overline{U}$ and $f(U \times \overline{U}) \subset F(U \times \overline{U})$.

(ii) If $f(z, \zeta) = f(z)$ and $F(z, \zeta) = F(z)$, the strong subordination becomes the usual notion of subordination.

If $f(z, \zeta)$ is strongly subordinate to $F(z, \zeta)$, then $F(z, \zeta)$ is strongly superordinate to $f(z, \zeta)$.

As a dual notion of strong differential subordination, Oros [5] has introduced and developed the notion of strong differential superordinations.

1.13. Lemma [**7**]Let $h(z, \zeta)$ be an univalent function with

$h(0, \zeta) = a$ for every $\zeta \in \overline{U}$ and

$\mu\epsilon C\backslash\{0\}$ with $Re(\mu) \geq o$. If $p \in H[a,1,\zeta]$and

$$p(z,\zeta) + \frac{1}{\mu} zp'_z(z,\zeta) \prec\prec h(z,\zeta), (z \in U, \zeta \in \overline{U}) \qquad (4)$$

then

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta), (z \in U, \zeta \in \overline{U}),$$

where

$$q(z,\zeta) = \mu z^{-\mu} \int_0^z h(t,\zeta) t^{\mu-1}\, dt$$

is convex and it is the best dominant of (4).

1.14. Lemma [**5**] let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = a$ for every $\zeta \in \overline{U}$ and $\mu\epsilon C\backslash\{0\}$ with $Re(\mu) \geq o$. If

$$p \in H[a,1,\zeta]Q_\zeta, p(z,\zeta) + \frac{1}{\mu} zp'_z(z,\zeta)$$

is univalent in $U \times \overline{U}$

and

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\mu} zp'_z(z,\zeta), (z \in U, \zeta \in \overline{U}), \qquad (5)$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad (z \in U, \zeta \in \overline{U}),$$

where $q(z,\zeta) = \mu z^{-\mu} \int_0^z h(z,\zeta)\, t^{\mu-1}\, dt$ is convex and it is the best subordinant of (5)

The following operator was defined by Srivastava-Attiya [10] (see also [11], [12], [13]): For $f \in \mathbb{A}\zeta$, $s \in \mathbb{C}$,

$\omega \in \mathbb{C} \backslash \mathbb{Z}0 - k \in \mathbb{N} = \{2,3,\ldots\}$,

the operator $J^{S}_{\omega,k} : \mathbb{A}\zeta \to \mathbb{A}\zeta$ is defined by

$$J^{S}_{\omega,k}\, f(z,\zeta) = z + \sum_{k=2}^{\infty} \left(\frac{1+\omega}{k+\omega}\right)^{m} a_k(\zeta) z^k \qquad (6)$$

From Equation (6), we note that

$$z\left(J^{S}_{\omega,k}\, f(z,\zeta)\right)' = k(1+\omega) J^{S+1}_{\omega,k}\, f(z,\zeta) - \omega k J^{S}_{\omega,k}\, f(z,\zeta).$$

Very recently, many researchers [14–16] obtained the results for certain classes of analytic univalent (multivalent) functions, differential Subordination and strong differential Subordination.

1.15. Definition[15] Let $\mathcal{Q}(z,\zeta)$ be an analytic function in $U\times\overline{U}$ with $\psi(z,\zeta)=1$ for every $\zeta\in\overline{U}$ and $\lambda>0$, $\mathcal{S}\in\mathbb{C}$,$\omega\in\mathbb{C}\backslash\mathbb{Z}0-k\in\mathbb{N}=\{2,3,\ldots\}$, a function $f\in\mathbb{A}_{\zeta}$ is said to be in the class $\psi(\lambda,s,\zeta,\omega;\mathcal{Q})$ if it fulfills strong differential subordination.

$$\frac{1}{z}\left[\left(1-k(1+\omega)\right)\mathrm{J}_{\omega,k}^{S}\ f(z,\zeta)+k(1+\omega)\mathrm{J}_{\omega,k}^{S}\ f(z,\zeta)\right]\prec\prec\mathcal{Q}(z,\zeta).$$

A function $f\in\mathbb{A}_{\zeta}$ is said to be in the class $\varphi(\lambda,s,\zeta,\omega;\mathcal{Q})$ if it satisfies the strong differential superordination

$$\mathcal{Q}(z,\zeta)\prec\prec\frac{1}{z}\left[\left(1-k(1+\omega)\right)\mathrm{J}_{\omega,k}^{S}\ f(z,\zeta)+k(1+\omega)\mathrm{J}_{\omega,k}^{S+1}\ f(z,\zeta)\right].$$

CHAPTER TWO

strong subordination Results

2.1. Introduction After we define the operator we can get some theorem satisfies the strong differential subordination using some operator as follows.

2.2. Theorem Let $\mathcal{Q}(z,\zeta)$ be a function of convex in $U\times\bar{U}$ with $\mathcal{Q}(z,\zeta)=1$ for every $\zeta\in\bar{U}$and $\lambda>0$. If $f\in\psi(\lambda,s,\zeta,\omega;\mathcal{Q})$, There is then a convex function $q(z,\zeta)$:

$$\mathrm{q}(\mathrm{z},\zeta) \prec\prec \mathcal{Q}(\mathrm{z},\zeta)$$

$$and\ f \in \mathrm{S}(\lambda,\alpha,\beta,\mathrm{m};\mathrm{q}).$$

Proof.

$$\mathrm{p}(\mathrm{z},\zeta) = \frac{J_{\omega,k}^{S}\,\mathrm{f}(\mathrm{z},\zeta)}{z} = 1+\sum_{k=2}^{\infty}\left(\frac{1+\omega}{k+\omega}\right)^{s} a_k(\zeta)z^{k-1} \tag{7}$$

Then $p \in \mathrm{H}[1,1,\zeta]$.

Since $f \in \psi(\lambda,\mathrm{s},\zeta,\omega;\mathcal{Q})$, then we have

$$\frac{1}{z}\left[\left(1-k(1+\omega)\right)J_{\omega,k}^{S}\ f(\mathrm{z},\zeta)+(1+k(1+\omega)J_{\omega,k}^{S+1}\ f(\mathrm{z},\zeta)\right] = p(\mathrm{z},\zeta)+\mathrm{z}\,\lambda\grave{p}_z(\mathrm{z},\zeta) \prec\prec \mathcal{Q}(\mathrm{z},\zeta)\,.$$

An application of Lemma 1.13, with $\mu = \frac{1}{\lambda}$ yield

$$p(z,\zeta) \prec\prec q\,(z,\zeta) \prec\prec \mathcal{Q}(z,\zeta).$$

By using (6),we get on

$$\frac{J_{\omega,k}^{S}\,\mathrm{f}(z,\zeta)}{z} \prec\prec q\,(z,\zeta) \prec\prec \mathcal{Q}(z,\zeta).$$

where

$$q(z,\zeta) = \frac{1}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z \mathcal{Q}(\mathrm{t},\zeta)\mathrm{t}^{\frac{1}{\lambda}-1} dt.$$

is convex and it is the best dominant.

<u>2.3. Theorem</u> Let $\mathcal{Q}(z,\zeta)$ be a convex function in in $U \times U$ with $\mathcal{Q}(0,\zeta) = 1$ for every $\zeta \in \mathrm{U}$ and $\lambda > 0$. If

$$f \in \varphi(\lambda, \mathrm{s}, \zeta, \omega; \mathcal{Q}), \frac{J_{\omega,k}^{S} f(z,\zeta)}{z} \in \mathrm{H}[1,1,\zeta] \cap \mathrm{Q}\zeta$$

And

$$\frac{1}{z}\left[\left(1 - k(1+\omega)\right)J_{\omega,k}^{S}\ f(z,\zeta) + (1 + k(1+\omega)J_{\omega,k}^{S+1}\ f(z,\zeta)\right]$$

is univalent in $U \times U$, then there exists a convex function $q(z,\zeta)$ such that $f \in \varphi(0, \mathrm{s}, \zeta, \omega; \mathcal{Q})$.

Proof. Suppose that

$$p(z,\zeta)=\frac{J_{\omega,k}^{S}\,f(z,\zeta)}{z}=1+\sum_{k=2}^{\infty}\left(\frac{1+\omega}{k+\omega}\right)^{s}a_k(\zeta)z^{k-1} \qquad (8)$$

Then $p\in H[1,1,\zeta]\cap \mathcal{Q}_k$.

After a short calculation and considering $f\in\varphi(\lambda,s,\zeta,\omega;\mathcal{Q})$ we can conclude that

$$\mathcal{Q}(z,\zeta)\prec\prec p\,(z,\zeta)+\lambda\, z\, pz'(z,\zeta).$$

An application of Lemma 1.13, with $\mu=\frac{1}{\lambda}$ yields

$$q(z,\zeta)\prec\prec p\,(z,\zeta).$$

By using Equation(8), we obtain

$$q\,(z,\zeta)\prec\prec\frac{J_{\omega,k}^{S}\,f(z,\zeta)}{z},$$

is convex and it is the best subordinant.

Combining the results of Theorems 2.2, and 2.3, gives us the following strong differential.

2.4.Theorem Let $\mathcal{Q}_1(z,\zeta)$ and $\mathcal{Q}_2(z,\zeta)$ be convex functions in $U\times\bar{U}$ with

$$\mathcal{Q}_1(\mathrm{z},\zeta)=\mathcal{Q}_2(\mathrm{z},\zeta)=1 \text{ fore very } \zeta \in\bar{\mathrm{U}} \text{ and} \lambda>0.$$

If

$$f\in\psi(\lambda,\mathrm{s},\zeta,\omega;\mathcal{Q})\cap\varphi(\lambda,\mathrm{s},\zeta,\omega;\mathcal{Q}),$$

$$\frac{J_{\omega,k}^{S}\,f(\mathrm{z},\zeta)}{z}\in \mathrm{H}[1,1,\zeta]\cap \mathrm{Q}_{\zeta},$$

and

$$\frac{1}{z}\left[\left(1-k(1+\omega)\right)I_{\propto.\cdot\beta}^{m}\; f(\mathrm{z},\zeta)+(1+k(1+\omega)I_{\propto.\cdot\beta}^{m+1}\; f(\mathrm{z},\zeta)\right]$$

is univalent in U×$\bar{U}$, then

$$f\in\psi(0,\mathrm{s},\zeta,\omega;q_1)\cap\varphi(0,\mathrm{s},\zeta,\omega;q_2),$$

Where

$$q_1(\mathrm{z},\zeta)=)=\frac{1}{\lambda}z^{\frac{1}{\lambda}}\int_0^z \mathcal{Q}_1(\mathrm{t},\zeta)\mathrm{t}^{\frac{1}{\lambda}-1}dt$$

$$and\quad q_2(\mathrm{z},\zeta)=\frac{1}{\lambda}z^{\frac{1}{\lambda}}\int_0^z \mathcal{Q}_2(\mathrm{t},\zeta)\mathrm{t}^{\frac{1}{\lambda}-1}dt$$

The functions $q_1$and $q_2$are convex.

<u>2.5.Theorem</u>. Let $\mathcal{Q}(\mathrm{z},\zeta)$ be a convex function in $U\times\bar{U}$ with $\mathcal{Q}(0,\zeta)=1$ for every $\zeta\in\bar{\mathrm{U}}$

and

$$\mathrm{G(z,\zeta)} = \frac{\varepsilon + 2}{z^{\varepsilon+1}} \int_0^z t^z f(t,\zeta)\, \mathrm{dt}, (\mathrm{z \in U}, \zeta \in \overline{\mathrm{U}}, \mathrm{Re}(\varepsilon) > -2) \qquad (9)$$

If $f \in \psi(1, \mathrm{s}, \zeta, \omega; \mathcal{Q})$, a convex function is then available

$q(\mathrm{z},\zeta)$ such that

$$q(\mathrm{z},\zeta) \prec\prec \mathcal{Q}(\mathrm{z},\zeta)$$

and $\mathrm{G} \in \psi(1, \mathrm{s}, \zeta, \omega; \mathrm{q}).$

Proof. Suppose that

$$p(\mathrm{z},\zeta) = \left((J_{\omega,k}^{S}\ \mathrm{G(z,\zeta)} \right)_z' \ , (\mathrm{z \in U}\ , \zeta \in \overline{\mathrm{U}}) \qquad (10)$$

Then $p \in \mathrm{H}\,[1{,}1, \zeta].$

From (9) we have

$$z^{\varepsilon+1}\mathrm{G(z,\zeta)} = (\varepsilon + 2)\ \int_0^z t^{\varepsilon}\, \mathrm{f}\,(\mathrm{t},\zeta)\mathrm{dt} \qquad (11)$$

Differentiating both sides of(11) with respect to *z*, we get

$$(\varepsilon + 2\,) f\ (\mathrm{t},\zeta) = (\varepsilon + 1)\mathrm{G(z,\zeta)}\ +\ \mathrm{z}\, G_z'\, (\mathrm{z},\zeta)$$

and

$$(\varepsilon + 2\ )\, J_{\omega,k}^{S}\ (\mathrm{z},\zeta) = (\varepsilon + 1)\, J_{\omega,k}^{S}\ \mathrm{G}\,(\mathrm{t},\zeta) + \mathrm{z}(\, J_{\omega,k}^{S}\ \mathrm{G(z,\zeta)})_z'$$

The last relationship with *z* is differentiated, we get

$$(J_{\omega,k}^{S}\ f(\mathrm{z},\zeta))_{z}^{'} = (J_{\omega,k}^{S}\ \mathrm{G}(\mathrm{z},\zeta))_{z}^{'} + \frac{z}{\varepsilon+2} J_{\omega,k}^{S}\ \mathrm{G}(\mathrm{z},\zeta))_{z^{2}}^{''} \qquad (12)$$

Since $f \in \psi(1, \mathrm{s}, \zeta, \omega; \mathcal{Q})$, then we get

$$\frac{1}{z}\left[(\omega+1) J_{\omega,k}^{S}\, \mathrm{f}(\mathrm{z},\zeta) - \omega\, J_{\omega,k}^{S}\ \mathrm{f}(\mathrm{z},\zeta)\right] \prec\prec \mathcal{Q}(\mathrm{z},\zeta) \qquad (13)$$

Now, from (4),(13) is equivalent to

$$(J_{\omega,k}^{S}\ f(\mathrm{z},\zeta))_{z}^{'} \prec\prec \mathcal{Q}(\mathrm{z},\zeta) \qquad (14)$$

From(12) and (14), we get

$$(J_{\omega,k}^{S}\ \mathrm{G}(\mathrm{z},\zeta))_{z}^{'} + \frac{z}{\varepsilon+2} J_{\omega,k}^{S}\ \mathrm{G}(\mathrm{z},\zeta))_{z^{2}}^{''} \prec\prec \mathcal{Q}(\mathrm{z},\zeta) \qquad (15)$$

Replacing(10) in (15), we obtain

$$p(\mathrm{z},\zeta) + \frac{1}{\varepsilon+2}\, z p_{z}^{'} \prec\prec \mathcal{Q}(\mathrm{z},\zeta).$$

An application of Lemma 1.13, with $\mu = \varepsilon + 2$ yields

$$p(\mathrm{z},\zeta) \prec\prec q(\mathrm{z},\zeta)) \prec\prec \mathcal{Q}(z,\zeta).$$

By using(10), we obtain

$$(J_{\omega,k}^{S}\ \mathrm{G}(\mathrm{z},\zeta))_{z}^{'} \prec\prec \mathrm{q}(\mathrm{z},\zeta) \prec\prec \mathcal{Q}(\mathrm{z},\zeta).$$

is convex and it is the best dominant.

2.6. Theorem. Let $\mathcal{Q}(z,\zeta)$ be a convex function in $U\times\bar{U}$ with $\mathcal{Q}(z,\zeta) = 1$, $\forall\, \zeta \in \bar{U}$ and $G(z,\zeta)$ is given by (9).

If $f \in T(1,\alpha,\beta,m;\psi)$, $(I^{m}_{\alpha,\beta}\, G(z,\zeta))'_{z} \in H[1,1,\zeta] \cap Q_{\zeta}$

and

$$\frac{1}{z}\,[(\omega+1)\, J^{S}_{\omega,k}\, f(z,\zeta) - \omega J^{S}_{\omega,k}\,(z,\zeta)],$$

is univalent in $U\times\bar{U}$, a convex function $q(z,\zeta)$ is then available: $G\in\varphi(1,s,\zeta,\omega;\text{q})$

Proof. Assume it is

$$p(z,\zeta) = (J^{S}_{\omega,k}\ \text{G}(z,\zeta))'_{z}\ ,(z \in U, \zeta \in \bar{U}). \tag{16}$$

then

$$p \in H[1,1,\zeta] \cap Q_{\zeta}$$

After a short estimate and consideration $f \in \varphi(1,s,\zeta,\omega;\mathcal{Q})$, we can conclude that

$$\mathcal{Q}(z,\zeta)\ \prec\prec\ p(z,\zeta) + \frac{1}{\varepsilon+2}\, zp'_{z}(z,\zeta)$$

An application of Lemma 1.14 , with $\mu = \varepsilon + 2$ yields

$$q(z,\zeta) \prec\prec p(z,\zeta)$$

by using (16), we obtain

$$q(z,\zeta) \prec\prec (J_{\omega,k}^{S} G(z,\zeta))_z^{'}$$

where

$$q(z,\zeta) = (\varepsilon + 2) z^{(\varepsilon+2)} \int_0^z Q(z,\zeta) t^{\varepsilon+1} \, dt..$$

is convex and it is the best subordinant.

Conclusions

Combining the results of Theorems 2.4 and 2.5, gives us the following strong differential

Theorem Let $\mathcal{Q}_1(z,\zeta)$ and $\mathcal{Q}_2(z,\zeta)$ be convex functions in $U \times \overline{U}$ with $\psi_1(0,\zeta) = \psi_2(0,\zeta) = 1, \forall \zeta \in \overline{U}, G(z,\zeta)$

is given by (9). If

$$f \in \psi(1,s,\zeta,\omega;\mathcal{Q}_1) \cap \varphi(1,s,\zeta,\omega;\ \mathcal{Q}_2),$$

$$(J_{\omega,k}^{S}\ \mathrm{G(z,\zeta)})_z' \ \in H\ [1,1,\zeta] \cap Q_\zeta\ ,$$

and

$$\frac{1}{z}\ [(\omega+1)J_{\omega,k}^{S} f(z,\zeta) - \omega J_{\omega,k}^{S}\ (z,\zeta)],$$

is univalent in $U \times \overline{U}$. Then

$$f \in \psi(1,s,\zeta,\omega;q_1) \cap \varphi(1,s,\zeta,\omega;\ q_2),$$

Where

$$q_1(z,\zeta) = (\ \varepsilon+2)z^{-(\varepsilon+2)}\int_0^z \mathcal{Q}_1(\mathrm{z},\zeta)t^{\varepsilon+1}\,dt.$$

$$q_2(z,\zeta) = (\ \varepsilon+2)z^{-(\varepsilon+2)}\int_0^z \mathcal{Q}_2(\mathrm{z},\zeta)t^{\varepsilon+1}\,dt.$$

The functions q_1 and q_2 are convex.

Future Work

The Differential Subordination still one of the most effective approaches to solving external problems for analytic univalent functions. It is necessary to develop some of the theory of univalent functions .So ,It is possible to try Differential Subordination using another operators.

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ملخص.

الدوال و الدوال التحليلية في المستوى المعقد التي تم فحصها في هذا المشروع ، تظهر بعض نتائج التبعية التفاضلية القوية المثيرة للاهتمام. يتم تحديد هذه التبعيات التفاضلية القوية من خلال حال خاصة للمضاعفات الممتدة .و تناقش عددًا من النتائج المثيرة للاهتمام. كما يتم توفير روابط ذات صلة بتلك الموجودة في الأعمال السابقة لبعض النتائج الجديدة تحقيقها في هذا البحث.

تعتبر نظرية الضغط (أو الشطيرة) أهم نظرية في هذا البحث.

يتم استخدامها عادةً لتأكيد التبعيه عبر المقارنة مع دالتين أخريين تُعرف حدودهما أو تُحسب بسهولة ، وقد استخدمنا تبعية تفاضلية قوية لإظهار ذلك باستخدام بعض المؤثرات.

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قسم الرياضيات

حول التبعيه التفاضليه القويه باستخدام بعض المؤثرات

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من قبل

ميثاق عزيز جوده كاظم

بإشراف

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١٤٤٣هـ ٢٠٢١م