جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

المطويات المنمذجة على فضاءات المتجهات التبولوجية الزائفة

أطروحة

مقدمة الى مجلس كلية التربية للعلوم الصرفة

جامعة بابل كجزء من متطلبات نيل درجة الدكتوراه فلسفة في التربية - الرياضيات

من قبل

إنتصار حربي راضي فارس

بأشراف

أ.د. زاهر دبيس النافعي

١٤٤٣ هـ **2021م**

Republic of Iraq

Ministry of Higher Education and Scientific Research

University of Babylon

College of Education for Pure Sciences

Manifolds Modeled on Pseudo - Topological Vector spaces

A Dissertation

Submitted to the Council of the College of Education for Pure Sciences- University of Babylon in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Education/Mathematics

By

Intesar Harbi Radhi Faris

Supervised by

Prof. Zahir Dobeas Al-Nafie (Ph.D.)

2021 A.D. **1443 A.H.**

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Date: 19-9-2021

Name: Intesar Harbi Radhi Faris

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I certify that this dissertation “**Manifolds Modeled on Pseudo-Topological Vector Spaces**” by student **Intesar Harbi Radhi Faris** was prepared under my supervision at the University of Babylon, College of Education for Pure Sciences, in a partial fulfillment of requirements for the degree of Doctor of Philosophy in Education Mathematics.

Signature:

Name: **Dr. Zahir Dobeas Al-Nafie**

Title: Professor

Date:

In view of available recommendations, I forward this thesis for debate by the examining committee.

Signature:

Name: **Dr. Azal Jaafar Musa**

Title: Asst. Prof.

Head of Mathematics Department, Faculty of Education for Pure Sciences, University of Babylon.

Date:

Certification of Linguistic Expert

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Signature

Name: Dr. Wafa Mukhlis Faisal

Title: Assist. Prof.

Date: 5/9/2021

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We are the chairman and members of the examination committee; we certify that we have read this dissertation (**Manifolds Modeled on Pseudo-Topological Vector Spaces**) as Examining committee, examined the student (**Intesar Harbi Radhi Faris)** in its contents and it is qualified as dissertation for the degree of Doctor of Philosophy in Education/ Mathematics.

Signature:

Name: ***Dr. Ali Hussein Battor***

Title: Professor

Date: / 10 /202١

(Chairman)

Signature:

Name: ***Dr. Eman S. Bhaya***

Title: Professor

Date: / 10 /202١

(Member)

Signature:

Name: ***Dr. Auday Sabbri Abdul-Razzaq***

Title: Professor

Date: / 10 /202١

(Member)

Signature:

Name: ***Dr. Bushra Yousif Hussein***

Title: Professor

Date: / 10 /202١

(Member)

Signature:

Name: ***Dr.Reyadh Delphi Ali***

Title: Asst. Prof.

Date: / 10 /202١

(Member)

Signature:

Name: ***Dr. Zahir D. A. Al-Nafie***

Title: Professor

Date: / 10 /202١

(Supervisor)

Approved by the Faculty of the College of Education for Pure Sciences, University of Babylon

Signature:
Name: ***Bahaa Hussein Salih Rabee***
Title: Professor
Date: / /202١
Address: ***Dean of the College of Education for Pure Sciences***.

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I certify that I have read the scientific content of this dissertation "**Manifold Modeled on pseudo- topological vector space**" and I have approved this dissertation qualified for debate.

Signature

Name: **Dr. Ayed E. Hashoosh**

Title: Assist. Prof.

Date: 7/9/ 2021

مستخلص

تقدم هذه الاطروحة مفهوم المنطوي الزائف المنمذج على الفضاءات التبولوجية الزائفة ${T^{\#}}_{\tilde{p}VS}$.
النتائج التي توصلنا اليها هي:

قمنا ببناء مفهوم الفضاء التبولوجي الزائف الذي يمثل فضاء شبه متري، محدب محليا ومنفصل والتبولوجي المرتبط به متولد بواسطة عائلة معدودة منفصلة من شبه المعيارية الزائفة. كما ناقشنا عدة خصائص لهذا الفضاء مثل الانتظام، T_1 وشبه القيدية فضلا عن دراسة الاستمرارية للدوال بين هذه الفضاءات.

تعرف الدراسة قابلية الاشتقاق للدوال التي تربط بين اثنين من الفضاءات المعرفة. بعد إضافة خاصيتين الى فضاء $T_{\tilde{p}VS}$ تعرف الدراسة قابلية اشتقاق الدالة من الدرجة الثانية والدرجة ن من المرات الى ما لا نهاية له من المرات التي تسمى الدوال الملساء والتي تقودنا الى تعريف صف من الدوال $\tilde{C}^{\infty}(\Xi_1, \Xi_2)$ حيث تكون الدوال قابلة للاشتقاق على نحو لا منته. بعد ذلك تنتقل الدراسة الى بناء منطوي املس زائف معتمد على الفضاء ${T^{\#}}_{\tilde{p}VS}$ باعطاء بعض المفاهيم الاساسية والتعريفات مثل مفهوم جارت،التناسق الاملس، الاطلس الاملس، الدالة الملساء وبعض التعريفات الاخرى والنتائج المتعلقة بالموضوع.

بما ان فضاء المماسات وحقل المتجهات من بين المفاهيم الاساسية في المنطوي فقد تناولتهما الدراسة باعطاء بعض التعريفات والحصول على بعض النتائج.

Chapter One

Preliminaries

1.1. Introduction

This chapter consists of three sections the first section involves the basic information about the concept of a topological vector spaces. In section two we deals with the some types of topological vector spaces and relation between them. In section three we introduced the basic information about the pseudo topological vector space T_{PVS} Ξ that is separated and locally convex space.

1.2.Basic Definitions:

In this section, we used basic definitions of the topological vector spaces and some properties of these spaces.

Definition 1.2.1 [13]:

A vector space X over K is a set with two operations $+$ and $\bullet$ (addition and scalar multiplication) these operations are defined as follows.

A-The operation + must satisfy the following some conditions:

1- Closure: If $u, v \in X$, then $u + v \in X$.
2- Commutative law: For any vectors u and v in X, $u + v = v + u$.
3- Associative law: For any vectors u, v and w in X

$u + (v + w) = (v + u) + w.$

4- Additive identity: There exists an additive identity element in X, denoted by zero, such that for all vector v in X, $0 + v = v$, $v + 0 = v$.

5- Additive inverses: For any vectors u in X, the additive inverse of u is called a solution x of equations $u + x = 0$ *and* $x + u = 0$ and denoted by $-u$.

B- The operation (**.**) (Or called scalar multiplication) is defined between vectors, and satisfy the following conditions:

1- Closure: If $u \in X$ and, for any $a \in K$(the filed K is a field of real numbers or complex numbers), then $a.u \in X$.

2- A-Distributive law: For any $a \in K$ and all vectors u and v in X,

$a.(u + v) = a.u + a.v.$

B-Distributive law: For any $a, b \in K$ and the vector u in X,

$(a + b).u = a.u + b.u.$

3- Associative law: For any $a, b \in K$ and the vector u in X

$a.(b.u) = (a.b).u.$

4- Unitary law: For any vector u in X, $1.u = u.1 = u$.

Definition 1.2.2 [35]:

A topological vector space $(X$, T) (in short T.V.S.) is a vector space X over a field K(may be real numbers or complex numbers) and topology (T) such that: The two maps (addition and the scalar multiplication) are continuous.

That's mean:

1- For each a, b belong to X, and each open neighborhood W of $a + b$, there exist open neighborhood U of a and V of $b \in X$,such that
$U + V \subset W$.
2- For each λ belong to K, and $a \in X$ and each open neighborhood W in X containing $\lambda . a$ there exist open neighborhood U of λ in K and open neighborhood V of a in X such that $U . V \subset W$.

Examples 1.2.3 [35]:

1-Let X be any vector space over K, then it is with trivial topology (indiscrete topology) that form a topological vector spaces.
2-Let $X = \{0\}$ be a vector space over K, then it is with discrete topology that form a topological vector spaces.

Definition 1.2.4 [35]:

A subset A of a vector space Ξ is called convex set if contains line segment between any two points in the set A

(That is mean $\forall x_1, x_2 \in A, \exists \gamma \in [0, 1] \Rightarrow \gamma . x_1 + (1 - \gamma) . x_2 \in A$). Such that:

Line segment (between x_1 and x_2) $= \{x = \gamma . x_1 + (1 - \gamma) . x_2, \gamma \in [0,1]\}$.

Definition 1.2.5 [48]:

Let Ξ be a vector space over the field K. A subset A of a vector space Ξ is called balanced if $\gamma A \subseteq A$ for $|\gamma| \leq 1, \gamma \in K$.

Definition 1.2.6 [48]:

A subset A of a vector space Ξ is called absorbing set if for every $x \in \Xi$, there exists $\delta > 0$ such that $tx \in A$ for $\forall t, |t| < \delta$.

Definition 1.2.7 [25]:

A cover of a Ξ is a family of sets $\{u_\alpha\}_{\alpha\in A}$ such that: $\Xi \subseteq \bigcup_{\alpha\in A} u_\alpha$.

Definition 1.2.8 [25]:

Open cover $\{u_\alpha\}_{\alpha\in A}$ of Ξ is a cover of Ξ and all the sets u_α are open sets in Ξ.

Definition 1.2.9 [25]:

A refinement of a cover $\{u_\alpha\}_{\alpha\in A}$ of Ξ is another cover $\{v_i\}_{i\in I}$ such that every set from the second cover is contained in at least one set from the original cover (i.e. for all $i \in I$ there exist $\alpha(i) \in A$ such that

$v_i \subset u_{\alpha(i)}$).

Example 1.2.10:

Let, $\Xi = \{1,2,3\}$ and let T= {{1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, φ, Ξ} be topology. Take U= {{1, 2}, {1, 3}, {2, 3}} is a open cover of Ξ.

V= {{1}, {2}, {3}} is open refinement of a cover U of Ξ.

Definition 1.2.11 [25]:

A cover $\{v_i\}_{i\in I}$ of Ξ (not necessarily open sets) is locally finite cover if every point of Ξ has a neighborhood that intersects only a finite number of sets from the cover.

Definition 1.2.12 [25]:

A space Ξ is called paracompact if every open cover of Ξ has a refinement which is a locally finite open cover.

Definition 1.2.13 [42]:

If Ξ is a vector space, a metric on Ξ is a map $d\colon \Xi \times \Xi \to R^+ \cup \{0\}$, such that

$(i) d(x;\ y) = d(y;\ x) \forall x, y \in \Xi$.

$(ii) d(x;\ z) \leq d(x;\ y) + d(y;\ z) \forall x, y, z \in \Xi$.

$(iii) x \neq y \leftrightarrow d(x;\ y) \neq 0\ \forall x, y \in \Xi$.

A pair (Ξ, d) is called a metric space.If a metric space satisfies the condition:$d(x;\ y) = d(x + z;\ y + z)$,$\forall x, y, z \in \Xi$, it is called translation invariant metric.

Example 1.2.14 [42]:

Let Ξ be any vector space, a metric on Ξ is a map $d\colon \Xi \times \Xi \to \Xi$, such that

$d(x;\ y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$, then (Ξ, d) is translation metric space.

Definition 1.2.15 [5]:

Let Ξ a vector space over R. Then a map $\|.\|\colon \Xi \to R^+ \cup \{0\}, x \longmapsto \|x\|$ is said to be a norm if the following properties are satisfied:

1- For all $\alpha \epsilon R \Longrightarrow \|\alpha . x\| = |\alpha|\|x\|$for all $x \in \Xi$.
2- For all $x, y \in \Xi$, $\|x + y\| \leq \|x\| + \|y\|$.
3- $\|x\| \geq 0$ for all $x \in \Xi$.
4- $\|x\| = 0 \leftrightarrow x = 0$.

A pair $(\Xi, \|.\|)$ is called a normed space.

Definition 1.2.16 [40]:

Let Ξ be a topological space. Ξ is called first countable if for each point in Ξ there exists a countable neighborhood local base of its filter of neighborhoods.

Definition 1.2.17 [40]:

In any topological vector space Ξ the statements are hold:

1- Every set of one point is bounded.
2- Every neighborhood of 0 contains a neighborhood of 0 which is balanced.
3- For any neighborhood U of 0. Then there exist a sub neighborhood N of 0 such that, $N + N \subseteq U$.

1.3. Some Types of Topological Vector Spaces:

Let (Ξ, T) be a topological vector space, below are some types of topological vector space and some relations among them:

Definition 1.3.1[47]:

Let (Ξ, T) be a topological vector space, we say Ξ is locally convex if any local base that their components are convex.

Definition 1.3.2 [47]:

Let (Ξ, T) be a topological vector space. Any subset $B \subset \Xi$ is bounded if for each neighborhood U of 0 in Ξ there exists a scalar $s > 0$ such that $B \subset tU$ for all $t > s$.

Definition 1.3.3 [47]:

Let (Ξ, T) be a topological vector space. We say (Ξ, T) is locally bounded topological vector spaces if it admits a bounded neighborhood of 0.

Definition 1.3.4 [35]:

Any topological vector space which has a bounded neighborhood of zero is called locally bounded.

Definition 1.3.5 [35]:

Any topological vector space which has a neighborhood of zero which whose closure is compact, is called locally compact.

Definition 1.3.6 [35]:

Let (Ξ, T) be a topological vector space. We say (Ξ, T) is metrizable if it is generated by some metric d.

Definition 1.3.7[42]:

Let (Ξ, T) be a topological vector space, then (Ξ, T) is said to be F-space if T is generated by a complete invariant metric d.

Definition 1.3.8[42]:

Let (Ξ, T) be a topological vector space, we say it is Frechet space if Ξ is satisfied the two conditions locally convex space and F-space.

Definition 1.3.9[40]:

Let (Ξ, T) be a topological vector space, Ξ has Heine – Borel property if any bounded and closed $u \subseteq \Xi$ is compact set.

Definition 1.3.10[40]:

Let Ξ be a linear space. Then Ξ is said to be finite dimensional if there exists a finite subset $\{x_1, x_2, \dots, x_n\} \subseteq \Xi$ such that span $(x_1, x_2, \dots, x_n)$=Ξ, the span of $\{x_1, x_2, \dots, x_n\}$ is defined as:

span $(x_1, x_2, \dots, x_n)$= $\{ a_1 x_1 + a_2\, x_2 + \ \dots, +a_n x_n : a_1, a_2, \dots a_n \in K \}$.

Definition1.3.11[40]:

A topological vector space (Ξ, T) is called normable if there exists a norm $\|.\|$ on Ξ such that the canonical metric $(x, y) \longmapsto \|y - x\|$ induces the topology T on Ξ.

Theorem1.3.12[40]

The topological vector space (Ξ, T) is normable if and only if there is a bounded convex neighborhood of 0.

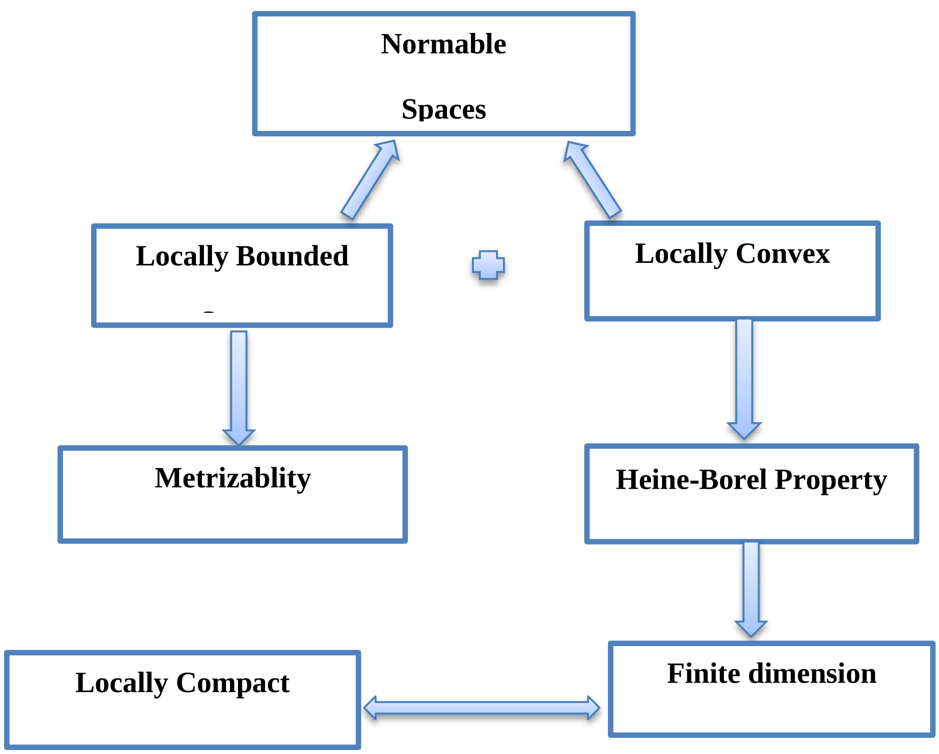

Fig.1.The Relation between some Properties of the Spaces.

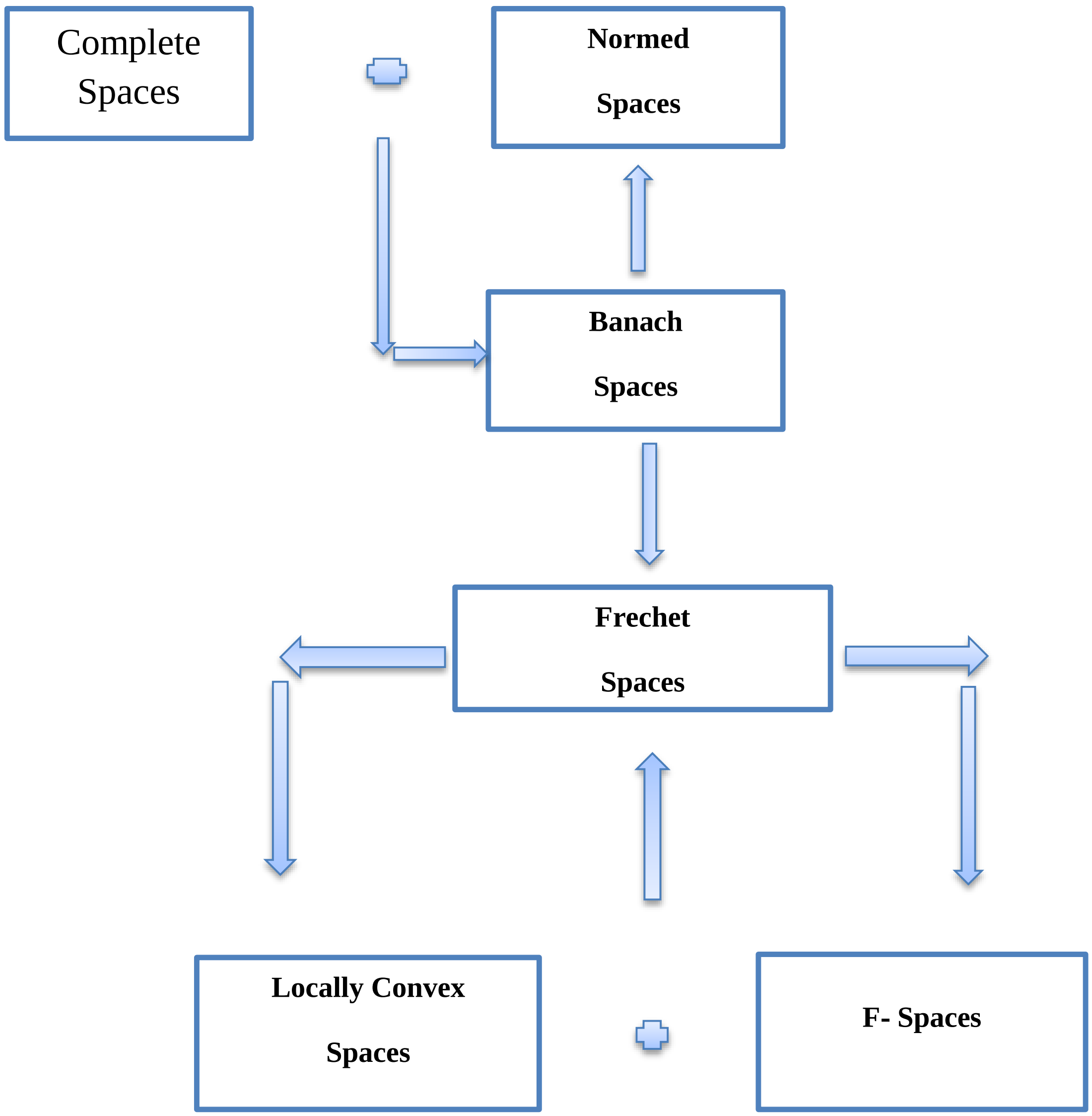

Fig.2.The Relation between some Spaces.

1.4. Pseudo Topological Vector Spaces

Basic information in this section will be introduced about the pseudo topological vector spaces and its built and its relation with topological vector spaces.

Definition 1.4.1[18]:

Let Ξ be a real vector space. A filter in Ξ is a non-empty set $\mathfrak{F}$ of subsets of Ξ, such that:

(1) The empty set is not belong to $\mathfrak{F}$.

(2) $X_1 \in \mathfrak{F}$ and $X_1 \subset X_2 \rightarrow X_2 \in \mathfrak{F}$.

(3) $X_1, X_2 \in \mathfrak{F} \rightarrow X_1 \cap X_2 \in \mathfrak{F}$.

Example 1.4.2:

Let $\Xi = \{1, 2, 3\}$ be vector space over R. $\mathfrak{F} = \{\{1\}, \{1, 2\}, \{1, 3\}, \Xi\}$ is filter on Ξ.

Definition 1.4.3 [18]:

A filter-basis in a space Ξ is a nonempty set β of subsets of Ξ that satisfies the conditions:

(1) $\emptyset \notin \beta$.

(2) For all $\mathcal{B}_1, \mathcal{B}_2 \in \beta$ there exists $\mathcal{B}_3 \in \beta$ such that $\mathcal{B}_1 \cap \mathcal{B}_2 \supset \mathcal{B}_3$.

Let $\digamma(\Xi)$ denotes the system of all filters on Ξ.

Definition 1.4.4 [18]:

The filter $\mathfrak{F}$ is said to be generated by a filter-basis β if it consists of all subsets of Ξ containing a set from β, and denoted as:

$\mathfrak{F} = [\beta] = \{A \subseteq \Xi : \beta \subseteq A\}$, and for any $a \in \Xi$, $[a]$ is a filter denoted by:
$[a] = \{A \subseteq \Xi : a \in A\}$.

Definition 1.4.5 [18]:

Let Ξ be a non-empty set. A pseudo topology (limit structure) τ on a space Ξ is a map: $\tau : \Xi \to 2^{\mathrm{F}(\Xi)}$, $m \mapsto \tau(m)$

If $\mathfrak{F} \in \tau(m)$, we say that $\mathfrak{F}$ converges to m (Every set $F \in \mathfrak{F}$ contain m) in τ and we write $\mathfrak{F} \downarrow_m$

(Or $\mathfrak{F} \in \tau(m)$)and the next conditions hold for any filters $\mathfrak{F}$ and p:

(1) $[m] \downarrow_m$, where $[m] = \{\mu \subset \Xi, m \in \mu\}$.
(2) $\mathfrak{F} \downarrow_m$, $\mathfrak{F} \subseteq \mathrm{p} \Rightarrow \mathrm{p} \downarrow_m$.
(3) $\mathfrak{F} \downarrow_m, \mathrm{p} \downarrow_m \Rightarrow \mathfrak{F} \cap \mathrm{p} \downarrow_m$.

Now let Ξ as above, we simply write $\mathfrak{F} \downarrow$ instead of $\mathfrak{F} \downarrow_0$(and also we can say $\mathfrak{F} \in \tau(0)$). In such a space, $\mathfrak{F} \downarrow_m$if and only if $\mathfrak{F} - m \downarrow$, where $\mathfrak{F} - m$ is the filter generated by the filter-basis

$\{A \subset \Xi, m \in \beta\}$, where β is the filter-basis which generates the filter $\mathfrak{F}$.

If Ξ is a topological space, we define:

$\mathfrak{F} \downarrow_m \Leftrightarrow \mathrm{N}_m(\Xi) \subseteq \mathfrak{F}$.

By $\mathfrak{r}$ we denote the filter of neighborhoods of 0 in the real line R with the natural topology. We shall use the notation $I_\epsilon \coloneqq \{r \in R: |r| \leq \epsilon\}$.

Definition 1.4.6 [18]:

Let $\mathfrak{F}_1, \mathfrak{F}_2$be two filters. Supermom of the two filters is defined as the filter consisting of all sets of the form $\cup_{i \in I}\ F_i$ where $F_i \in \mathfrak{F}_i$.

Definition 1.4.7 [12]:

Let Ξ be a space. Neighborhood filter of $m \in \Xi$ ($\mathrm{N}_m(\Xi)$) is defined as:
$\mathrm{N}_m(\Xi)\ = \cap \mathfrak{F}$ such that $\mathfrak{F} \downarrow_m$.We call a set $N \in \mathrm{N}_m$a neighborhood of m.

Proposition 1.4.8 [18]:

A pseudo topological vector space Ξ is topological vector space if and only if $(sup_{\mathfrak{F}\downarrow_0 \Xi}\mathfrak{F}) \downarrow_0 \Xi$, such that $(sup_{\mathfrak{F}\downarrow_0 \Xi}\mathfrak{F})$ is filter of neighborhood of 0.

Definition 1.4.9 [18]:

If $\mathfrak{F}_1, \mathfrak{F}_2$ are filters in a space , we say that $\mathfrak{F}_2$ is finer than $\mathfrak{F}_1$ if $\mathfrak{F}_2 \supseteq \mathfrak{F}_1$ and we can say also $\mathfrak{F}_1$ is coarser than $\mathfrak{F}_2$.

Definition 1.4.10 [18]:

The system of all filters $\mathrm{F}(\Xi)$ on a space Ξ is partially ordered by the included relation: $\mathfrak{F}_1 \leq \mathfrak{F}_2$ iff $\mathfrak{F}_1 \supseteq \mathfrak{F}_2$.

Definition 1.4.11 [18]:

Let $f: \Xi \longrightarrow \Xi_1$ be a mapping and $\mathfrak{F}$ a filter on a space Ξ. The set $\{ f(F): F \in \mathfrak{F}\}$ is a filter basis on a space Ξ_1, which generates a filter denoted by $f(\mathfrak{F})$ and called the image of $\mathfrak{F}$ under the mapping f.

1-If $f\colon \Xi \longrightarrow \Xi_1$, $g\colon \Xi_1 \longrightarrow P$ then $(g \circ f)(\mathfrak{F}) = g(f(\mathfrak{F}))$,

2-$f([m]) = [f(m)]$, for all $m \in \Xi$.

3-If $\mathfrak{F}_1 \leq \mathfrak{F}_2$ then, $f(\mathfrak{F}_1) \leq f(\mathfrak{F}_2)$.

4-Let $\mathfrak{F} \in \mathrm{F}(\Xi)$, then $\{f^{-1}(F)\colon F \in \mathfrak{F}\}$ is a filter base in Ξ if and only if $f^{-1}(F) \neq \emptyset$, $\forall F \in \mathfrak{F}$.

Definition 1.4.12 [18]:

Let $\mathfrak{F}$ be a filter on a space Ξ and $f\colon \Xi \to \Xi \times \Xi$ be the diagonal map

$f(x) = (x, x)$ for all $x \in \Xi$. Then, $f(\mathfrak{F}) \leq \mathfrak{F} \times \mathfrak{F}$.

Let $f_i\colon \Xi \to \Xi_i$, $i = 1, 2$ and $g\colon \Xi_1 \times \Xi_2 \to F$ be maps. If $K\colon \Xi \to F$ is the map denoted by $K(m) = g(f_1(m).f_2(m))$, $m \in \Xi$ then,

$K(\mathfrak{F}) \leq g(f_1(\mathfrak{F}), f_2(\mathfrak{F}))$.

Definition 1.4.13 [18]

Let $\mathfrak{F}_1, \mathfrak{F}_2$ be a two filters on a space Ξ_1, Ξ_2 respectively. Then $\mathfrak{F}_1 \times \mathfrak{F}_2$is a filter on $\Xi_1 \times \Xi_2$ generated by the filter basis $\{(x_1, x_2)\colon x_1 \in \Xi_1,\ x_2 \in \Xi_2\}$.

Lemma 1.4.14 [18]:

Let $\mathfrak{F}$ be a filter (on a space $\Xi_1 \times \Xi_2$). Then $\mathfrak{F} \downarrow_m \Xi_1 \times \Xi_2$ iff there exist $\mathfrak{F}_1, \mathfrak{F}_2$ with

1- $\mathfrak{F}_i \downarrow_{\prod_i(m)} \Xi_i, i = 1, 2$, and $\mathfrak{F} \leq \mathfrak{F} \times \mathfrak{F}$.

$\prod_\alpha$ is projection map defined on direct product $\times_{i\in I}\ \underline{\Xi_i}$.

2- $\mathfrak{F} \leq \prod_1 \mathfrak{F} \times \prod_2 \mathfrak{F}$.

Definition 1.4.15 [18]:

Let $\mathfrak{F}$ be a filter in $\varXi$. The filter $\mathfrak{F}$ is said to be equable if $\mathfrak{r}\mathfrak{F} = \mathfrak{F}$, such that $\mathfrak{r}$ is a filter neighborhood of a zero.

Remark 1.4.16 [18]:

Let $\mathfrak{F}_1, \mathfrak{F}_2$ be filters on $\varXi$, then $\mathfrak{r}.(\mathfrak{F}_1.\mathfrak{F}_2) \leq \mathfrak{r}.\mathfrak{F}_1 \times \mathfrak{r}.\mathfrak{F}_2$.

The pseudo topological vector space (briefly, T_{PVS}) can be defined in a natural way.

Definition 1.4.17 [18]:

A T_{PVS} is a vector space (the underlying space $\underline{\varXi}$) together with a compatible pseudo topology on it.

The pseudo topology is called compatible with the vector space structure if the two maps ($+: \varXi \times \varXi \to \varXi$, $\cdot: R \times \varXi \to \varXi$) are continuous.

Remark 1.4.18 [9]:

A topological space can be considered as special cases of pseudo topological space. Every topological space is pseudo topological space.

Conversely, if $\varXi$ is a pseudo space, then we can define a subset $\mathcal{U} \subset \varXi$ to be open if whenever $\mathfrak{F} \downarrow_m \varXi$ and $m \in \mathcal{U}$, we have $\mathcal{U} \in \mathfrak{F}$.

Definition 1.4.19 [5]:

Let $(\varXi, \tau)$ be a T_{PVS}. The interior $int(\mathcal{A})$ of a set $\mathcal{A} \subseteq \varXi$ can be defined by: $int(\mathcal{A}) = \{m \in \mathcal{A}: \mathfrak{F} \in \tau(m)$ implies $\mathcal{A} \in \mathfrak{F}\}$.

Definition 1.4.20 [5]:

Let (Ξ, τ) be a T_{PVS} and a set $\mathcal{A} \subseteq \Xi$ can be call τ-open set(for easily open set) if $\mathcal{A} = int(\mathcal{A})$.

The collection of this τ-open sets clearly defines a topology on Ξ such that every filter is converging to m in the original pseudo topology and in this topology.

Definition 1.4.21 [5]:

Let (Ξ , τ) be a T_{PVS} and $\mathcal{A} \subseteq \Xi$, then the closure of $\mathcal{A}$ is

$\bar{\mathcal{A}} = (\mathrm{CL}\, \mathcal{A}) = \{m \in \Xi : \exists\ \mathrm{p} \downarrow_m \Xi$ and $\mathcal{A} \in \mathrm{p}\}$.

Definition 1.4.22 [5]:

Let (Ξ, τ) be a T_{PVS} and $\mathcal{A} \subseteq \Xi$ can be called τ-closed set (for easily closed set) if $\mathcal{A} = \mathrm{CL}\, \mathcal{A}$.

Definition 1.4.23 [18]:

Let Ξ_1, Ξ_2 be two T_{PVS} with $\underline{\Xi_1} = \underline{\Xi_2}$. The structure of Ξ_1 is called finer than Ξ_2, and we write $\Xi_1 \leq \Xi_2$ if and only if $\mathfrak{F} \downarrow_m \Xi_1 \Longrightarrow \mathfrak{F} \downarrow_m \Xi_2$, we also say structure of Ξ_2 is coarser than Ξ_1.

Remark 1.4.24 [18]:

We can consider subspaces as a special case of structures induced by mappings: each subset A of a T_{PVS} Ξ has natural pseudo topology, namely the one induced by using the inclusion map $i: A \to \varXi$ together with this structure is called a subspace of $\varXi$.

[$\mathfrak{F} \downarrow_m \varXi \Leftrightarrow f_i(\mathfrak{F}) \downarrow_{f_i(m)} \varXi_i$ for all $i \in I$, $f_i: \varXi \to \varXi_i$]:

$\mathfrak{F} \downarrow_m \varXi_1 \Leftrightarrow i(\mathfrak{F}) \downarrow_m \varXi$.

Definition 1.4.25 [18]:

A filter $\mathfrak{F}$ in a T_{PVS} $\varXi$ is said to be quasi bounded if ꝛ $\mathfrak{F} \downarrow \varXi$.

Proposition 1.4.26 [18]:

A T_{PVS} $\varXi$ is called quasi bounded if and only if every filter in $\varXi$ is quasi bounded filter on T_{PVS} $\varXi$.

For each pseudo topology τ on a vector space $\varXi$, we define an associated pseudo topology $\tau^{\#}$. We say that a filter $\mathfrak{F}$ in $\varXi$ converges to zero in $\tau^{\#}$ if there exists an equable filter p $on\, \varXi$ such that p $\subset$ $\mathfrak{F}$ and p converges to zero in τ.

Remark 1.4.27 [18]:

If $\varXi$ is a topological vector space with topology τ, then $\tau = \tau^{\#}$.

In other words; a T_{PVS} $\varXi$ is called equable if and only if for each filter $\mathfrak{F}$ with $\mathfrak{F} \downarrow \varXi$, ($\exists$ an equable filter p) p $\geq \mathfrak{F}$ with p $\downarrow$ $\varXi$

(That's mean $\mathfrak{F} \downarrow \varXi \Longrightarrow \exists$p such that $\mathfrak{F} \leq$ p=ꝛp $\downarrow \varXi$) , and denoted by $\tau^{\#}$.

Remark 1.4.28 [18]:

$\tau^{\#} \leq \tau$ (Where $\tau^{\#}$ is equable pseudo topological vector space) and the equality satisfies when the T_{PVS} Ξ is equable.

Lemma 1.4.29 [18]

A- Let, , p be two an equable filters on T_{PVS} Ξ.

1- $\mathfrak{F} + \mathrm{p}$ is equable filter ;

2- Sup $(\mathfrak{F}, \mathrm{p})$ is equable filter.

B- Let $\mathfrak{F}$ respectively p be equable filters on T_{PVS} Ξ_1 respectively $T_{PVS}\Xi_2$, then $\mathfrak{F} \times \mathrm{p}$ is equable filter on $\Xi_1 \times \Xi_2$.

Definition 1.4.30 [18]:

A T_{PVS} (Ξ, τ) is called a separated, If $\mathfrak{F} \downarrow_m \Xi$ and $\mathfrak{F} \downarrow_n \Xi$, then $m = n$ [That's mean if $m \neq n$, then $\tau(m) \cap \tau(n) = \emptyset$] .

Definition 1.4.31 [44]:

A T_{PVS} (Ξ, τ) is said to be T_1: If $m \neq n$, then $[n] \notin \tau(m)$.

Definition 1.4.32 [18]:

Let Ξ be a T_{PVS} over R , and $\mathfrak{F}$ any filter on Ξ . We define a filter $\widehat{\mathfrak{F}}$ as a filter is generated by the convex sets of $\mathfrak{F}$.

Definition 1.4.33 [18]:

On a T_{PVS} Ξ define a convex filter $\mathfrak{F}^{\circ}$ as follows: $\mathfrak{F}^{\circ} = (\sup\{\mathfrak{F}, [0]\}\widehat{\ })$, $\mathfrak{F}^{\circ} \subseteq \widehat{\mathfrak{F}}$.

For each pseudo topology τ on a vector space Ξ, we define an associated pseudo topology τ°. We say that a filter $\mathfrak{F}$ in Ξ converges to zero in τ° if there exists convex filter $\mathfrak{F}^{\circ} \in \Xi$ such that $\mathfrak{F}^{\circ} \subseteq \mathfrak{F}$ and $\mathfrak{F}^{\circ}$ converges to zero in τ.

Remark 1.4.34 [18]:

Note that if Ξ is a topological vector space with topology τ, then $\tau = \tau^{\circ}$.

In other words; a T_{PVS} Ξ is called locally convex (denoted by Ξ°)if and only if for each filter $\mathfrak{F}$ with $\mathfrak{F} \downarrow \Xi$ there exists an convex filter $\mathfrak{F}^{\circ} \geq \mathfrak{F}$ with $\mathfrak{F}^{\circ} \downarrow \Xi$ (That's mean $\mathfrak{F} \downarrow \Xi^{\circ} \Longrightarrow \exists\ \mathfrak{F}^{\circ} \subseteq \mathfrak{F}, \ni \mathfrak{F}^{\circ} \downarrow \Xi$).

Remark 1.4.35 [18]:

Let Ξ be a T_{PVS}, then, $\Xi \leq \Xi^{\circ}$, and when the pseudo topological vector space is locally convex vector space the equality satisfies.

Definition 1.4.36 [18]:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two T_{PVS} and $f: \underline{\Xi_1} \rightarrow \underline{\Xi_2}$ a map, we say that a mapping f: $(\Xi_1, \tau_1)\rightarrow (\Xi_2, \tau_2)$ is called continuous at a point $m \in \Xi_1$ if $f(\mathfrak{F}) \in \tau_2\ (f(m))$for all filter $\mathfrak{F} \in\ \tau_1\ (m)$. The mapping is called continuous on Ξ_1 if it is continuous at each point of Ξ_1.

Definition 1.4.37 [18]:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two T_{PVS} and $f: \Xi_1 \rightarrow \Xi_2$ being any map, we denote that:$\Delta f: \Xi_1 \times \Xi_1 \rightarrow \Xi_2$ the map defined by

$\Delta f(a,b) = f(a+b) - f(a)\ \forall a,b \in \Xi_1$.

Definition 1.4.38 [18]:

The map $f: \Xi_1 \to \Xi_2$ is called equable continuous if and only if

$\mathfrak{r}\mathfrak{F} \downarrow \Xi_1$ and $\mathfrak{p} \downarrow \Xi_1 \Longrightarrow \Delta f(\mathfrak{F}, \mathfrak{p}) \downarrow \Xi_2$, for any filters $\mathfrak{F}, \mathfrak{p}$ on Ξ_1.

Proposition 1.4.39 [18]:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two T_{PVS}, if $f: (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$be an equable continuous, then it is continuous.

Definition 1.4.40 [18]:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two T_{PVS} and a map $f: (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ is called a quasi bounded map if and only if image of any quasi bounded filter on Ξ_1 is quasi bounded filter on Ξ_2

[that's mean $\mathfrak{r}\mathfrak{F} \downarrow \Xi_1 \Longrightarrow \mathfrak{r}f(\mathfrak{F}) \downarrow \Xi_2$].

Notation 1.4.41 [18]:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two T_{PVS}. A class $C_e^q(\Xi_1; \Xi_2)$ is class of all maps $f: \Xi_1 \to \Xi_2$ are equable continuous and quasi bounded maps.

Definition 1.4.42 [28]:

Let V, W be two vector spaces over the same field K. A map $T : V \to W$ is a linear map if the following two conditions are satisfied:

(i) $T(X + Y) = T(X) + T(Y)$ for any $X, Y \in V$,

$(ii)\ T(\lambda X) = \lambda T(X)$ for any $X \in V$ and $\lambda \in K$.

Remark 1.4.43 [18]:

For any f and $g \in C_e^q(\Xi_1;\Xi_2)$ and $\gamma \in k$ then $\gamma f + g \in C_e^q(\Xi_1;\Xi_2)$

, therefore: $C_e^q(\Xi_1;\Xi_2)$ is vector space.

Proposition 1.4.44 [9]:

If $\ell: \Xi_1 \to \Xi_2$ be a continuous at 0 and linear map, then ℓ is equable continuous and quasi bounded, (i.e. $\ell \in C_e^q(\Xi_1;\Xi_2)$).

Remark 1.4.45 [18]:

By above proposition $\underline{L}(\Xi_1;\Xi_2) \subset C_e^q(\Xi_1;\Xi_2)$ where $\underline{L}(\Xi_1;\Xi_2)$ is a vector space of all linear continuous maps.

Definition 1.4.46 [18]:

Let V, W, U be vector spaces over F. A function $\varphi : V \times W \to U$ is called bilinear if $\varphi(v + v_0, w) = \varphi(v,w) + \varphi(v_0, w)$ and $\varphi(c\,v, w) = c\,\varphi(v,w)$ $\varphi(v, w + w_0) = \varphi(v,w) + \varphi(v, w_0)$ and $\varphi(v, c\,w) = c\,\varphi(v,w)$.

Remark 1.4.47 [18]:

For bilinear (or multilinear) maps, continuity at 0 does not necessarily imply continuity at every point.

Proposition 1.4.48 [18]:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two T_{PVS}. If Ξ_1 is equable, then there are natural linear homeomorphisms as follows:

$L^{\#}{}_{m}(\Xi_1; L^{\#}{}_{q}(\Xi_1; \Xi_2)) \approx L^{\#}{}_{m+q}(\Xi_1; \Xi_2)$.

Chapter Two

Pseudo Seminorm with New Topological Structures

2.1. Introduction

Within this chapter, we will deal with two sections. The first section we built a space $T_{\tilde{p}VS}$, where we introduced a new definitions pseudo metric and pseudo seminorm (PSN), and from what we know, the pseudo topological vector space, and then we prove that space is pseudo metrizability. As for the second section will talk about some of the properties of this space, such as pseudo paracompactness and we gave some definitions by following the method of the researcher in the numbered source 18, as t he continuity and separated, etc., and we discussed some relationships and got some results about them.

2.2. Pseudo Metrizability:

In this section we will exhibit how to build the space $T_{\tilde{p}VS}$ depending on a space T_{PVS} (that is built on the definition of pseudo seminorm) and adding pseudo metrizability property and locally convexity to it.

Definition 2.2.1:

Let Ξ be a non- empty set. A pseudo metric space is a vector space Ξ together with a function

$\mathfrak{d}: \Xi \times \Xi \to R^{+}$ which is satisfies : $\forall\, x,\ y, z\ \in\ \Xi$

1. $\mathfrak{d}(x,\ y)\ >\ 0.$

2. $\mathfrak{d}(x,\ y) =\ \mathfrak{d}(y,\ x).$

3. $\mathfrak{d}(x, y) + \mathfrak{d}(y, z) \geq \mathfrak{d}(x, z)$.

Definition 2.2.2:

Let Ξ a vector space over R. The map $\|.\|_p: \Xi \to R^+, x \mapsto \|x\|_p$ is said to be a pseudo-seminorm (in brief, PSN) if the following properties are satisfied: For each $p \in \mathbb{N}$, and for all $x, y \in \Xi$,

1-For all $\alpha \in R - \{0\}, |\alpha| \leq 1 \Longrightarrow \|\alpha . x\|_p \leq \|x\|_p$.

2-$\|x + y\|_p \leq \|x\|_p + \|y\|_p$.

3-$\|x\|_p > 0$.

Any seminorm is PSN and if $\|x\|_p \geq 0$ for all $x \in \Xi$ then, $\|.\|_p$ will be called pseudo norm, which is of course is pseudo seminorm.

Definition 2.2.3:

A pseudo topological vector space (in brief, T_{PVS}) Ξ is said to be pseudo-metrizable if its topology induced by the pseudo metric $\mathfrak{d}: \Xi \times \Xi \to R^+$:

$\forall x, y \in \Xi, \mathfrak{d}(x, y) := \sum_{p=1}^{\infty} 2^{-p} \frac{\|x-y\|_p}{1+\|x-y\|_p}$, such that $\{\|x\|_p\}_{p\in\mathbb{N}}$ is a countable family of pseudo seminorms on Ξ.

It is clear that $\mathfrak{d}$ is a translation invariant pseudo metric.

The collection of all open balls:

$$B^p_{1/r}(x) = \left\{y \in \Xi : \|x - y\|_p < \frac{1}{r}\right\}, x \in \Xi, r \in \mathbb{N} - \{0\} \text{ and } p \in \mathbb{N},$$

is a base for the pseudo topology of Ξ.

Definition 2.2.4:

A T_{PVS} Ξ is said to be quasi-locally bounded if Ξ possess a base of quasi bounded filters of a neighborhood of 0, that is the base of Ξ consists filter $\mathfrak{F}$, which have the property $\mathfrak{r}\,\mathfrak{F} \downarrow \Xi$.

Proposition 2.2.5:

Let Ξ be a vector space over R. Then PSN on Ξ induces a topology that is compatible with vector space structure iff for all $x \in \Xi$ and $M > 0$ there exist $\alpha > 0$ such that $x \in \alpha. B_M(0)$.

Proof:

Suppose that a PSN on Ξ induced a compatible topology, then one can show that the stated property the PSN comes from (1) in Definition 1.2.15 (boundedness of points).

Conversely, let we have a PSN on Ξ, let N_0 be a filter of neighborhood of 0. Then it is clear that every $v \in N_0$ contains 0.

To show that N_0 is scale invariant.

We must show that for $M > 0$ and $\alpha \in R - \{0\}$, the scale ball $\alpha. B_M(0)$ is open.

Let αx be a point in $\alpha. B_M(0)$ let $N > 0$ such that, $\|x\| < M + N, N < M$.

Then, $B_N(0) + x \subseteq B_M(0)$.

If $|\alpha| \geq 1$ then $B_N(\alpha x) = B_N(0) + \alpha x \subseteq \alpha B_N(0) + \alpha x \subseteq \alpha B_M(0)$ and that mean $\alpha B_M(0)$ is open

Now put $|\alpha| \leq 1$ and take $m \in \mathbb{N}$ such that, $2^{-m} \leq |\alpha|$.

Then by the triangle inequality we get $B_{2^{-mN}}(0) \subseteq 2^{-m}B_N(0)$.

Hence, $B_{2^{-mN}}(\alpha x) = B_{2^{-mN}}(0) + \alpha x \subseteq \alpha B_N(0) + \alpha x \subseteq \alpha B_M(0)$.

This means that $\alpha B_M(0)$ is also open.

Now we have to show that, there exists a sub neighborhood v of 0 such that, $v + v \subseteq v_0$, let v_0 is defined as $v_0 \coloneqq B_M(0)$ that is an open ball with radius M and center zero, $M > 0$ then define $v \coloneqq B_{M/2}(0)$ then

$v + v \subseteq v_0$ and this follows from the triangle inequality.

Theorem 2.2.6:

A separated T_{PVS} Ξ is pseudo-metrizable iff it satisfies the first axiom of a countability. Moreover, if Ξ is pseudo-metrizable then it admits coinciding pseudo-semi norms.

Proof:

Clearly that, the pseudo metrizablity means the existence of a countable base of filter of neighborhood of 0.

Conversely, let $\{B_m\}_{m\in\mathbb{N}}$ be a filter base of the neighborhood of 0 such that B_m are balanced and $B_{m+1} + B_{m+1} \subseteq B_m$.

Now let M be finite subset of $\mathbb{N}$, then for each M define

$B_M \coloneqq \sum_{m\in M} B_m$ and $\alpha_M \coloneqq \sum_{m\in M} 2^{-m}$, observe that each B_M is balanced neighborhood of 0.

Now we define a function $\|.\|_p : \Xi \to R^+ : x \mapsto \|x\|_p$ by

$inf_M\{\alpha_M : x \in B_M\}$ if $x \in B_M$ for some M and $\|x\|_p = 1$ if not, then we show that $\|.\|_p$ defines a pseudo-semi norm generating the topology of Ξ.

Take $x \in \Xi$ and $\alpha \in R$ with$|\alpha| \leq 1$. Since B_M is balanced for each $M.\alpha x$ contained at least in the sets B_M as x. Since the definition of $\|.\|_p$ use infimum. $\|\alpha x\|_p \leq \|x\|_p$ and by the first property of the definition of pseudo-semi norm. To show the second property we first observe that for finite subset M.

$W$of $\mathbb{N}$ with property that $\alpha_M + \alpha_W < 1$ there is another finite subset L of $\mathbb{N}$ such that

$\alpha_L = \alpha_M + \alpha_W$ and $B_M + B_W \subseteq B_L$. Now put $x, y \in \Xi$,

if $\|x\|_p + \|y\|_p > 1$. The second property is trivial, choose $M > 0$ such that, $\|x\|_p + \|y\|_p + 2M < 1$.

Now take finite subsets M, W of $\mathbb{N}$ such that $x \in B_M$, $y \in B_W$

while $\alpha_M < \|x\|_p + M$ and $\alpha_W < \|y\|_p + M$. Let $L \subseteq \mathbb{N}$ such that

$\alpha_L = \alpha_M + \alpha_W$. Then $x + y \in B_L$

And therefore, $\|x + y\|_p \leq \alpha_L = \alpha_M + \alpha_W < \|x\|_p + \|y\|_p + 2M$.

And therefore $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ because, $M > 0$.

Now we have to prove that $\|x\|_p > 0$. For all $x \in \Xi$ by the separated property of Ξ, there exist some $m \in \mathbb{N}$ such that $B_m = B_{\{m\}}$ which is not containing x. Since $\alpha_M < \alpha_W$ we have, $B_M \subseteq B_W$.

Which means that $\|x\|_p \geq 2^{-m}$

That's mean $\|x\|_p > 0$.

It remains to show that the pseudo-semi norm $\|.\|_p$ generates the pseudo topology of Ξ and to make this we show that the open balls with centered by 0 form a base of the filter of neighborhood of 0. Choose $m \in \mathbb{N}$ and $M > 0$.

It is clear that, $B_{2^{-m}}(0) \subseteq B_m \subseteq B_{2^{-m}} + M$.

And since M is arbitrary we have, $B_m = B_{2^{-m}}(0)$. But

$\{B_m\}_{m\in\mathbb{N}}$ give a base of the filter of neighborhood of 0 such that the balls $\{B_{2^{-m}}(0)\}_{m\in\mathbb{N}}$ form a base of the pseudo topology of Ξ.

Example 2.2.7:

Let $(\Xi, \|.\|_p)$ be a seminormed space with a canonical metric given by

$\mathfrak{d}(x, y) = \|x - y\|_p$, then Ξ is pseudo-metrizable space.

Theorem 2.2.8:

Every separated pseudo-locally bounded T_{PVS} Ξ is pseudo-metrizable.

Proof:

Let $B = \{\mathfrak{F}_m : \mathfrak{r}\mathfrak{F}_m \downarrow \Xi\}_{m\in\mathbb{N}}$ be a base of the quasi bounded filter of neighborhood of 0 and let $\{\delta_n\}$ be a sequence of non-zero convergent elements to zero. If C is any circled of filter of neighborhood of 0, there exists $\delta \in \{\delta_n\}$ such that $B \subset \delta C$, since B is bounded; if $n \in N$ is such that

$|\delta_n \delta| \leq 1$, then, $\delta_n B \subset C$, since C is circled. It follows that $\{\delta_n B : n \in N\}$ is a 0-neighborhood filter base, which is mean Ξ is pseudo-metrizable.

Remark 2.2.9:

If we replace a pseudo space Ξ by Ξ° in the Definitions (2.2.1, 2.2.2, 2.2.3 & 2.2.4), we can find the following definitions:

Definition 2.2.10:

Let Ξ° a locally convex vector space over R.

Then a map $\|.\|_p : \Xi^\circ \to R^+, \ x \mapsto \|x\|_p$ is said to be a pseudo-seminorm

(in brief, PCN) if the following properties are satisfied:

For each $p \in \mathbb{N}$,

1- For all $\alpha \epsilon R - \{0\} \Longrightarrow \|\alpha . x\|_p \leq \|x\|_p$ for all $x \in \Xi^\circ$.

2- For all $x, y \in \Xi^\circ$, $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.

3- $\|x\|_p > 0$ for all $x \in \Xi^\circ$.

It is clear that any seminorm is pseudo seminorm for all $x \in \Xi^\circ$.

Definition 2.2.11:

A pseudo metric space is a set Ξ° together with a function

$d: \Xi^\circ \times \Xi^\circ \to R^+$ which assigns a real number $d(x, y)$ to satisfying the properties: $\forall\, x, y, z \in \Xi^\circ$

1. $d(x, y) > 0$.

2. $d(x, y) = d(y, x)$.

3. $d(x, y) + d(y, z) \geq d(x, z)$.

Definition 2.2.12:

A locally convex T_{PVS} Ξ° is said to be locally convex pseudo-metrizable if its topology induced by the following pseudo metric $d: \Xi^{\circ} \times \Xi^{\circ} \to R^{+}$:

$$d(x,y) \coloneqq \sum_{p=1}^{\infty} 2^{-p} \frac{\|x-y\|_p}{1+\|x-y\|_p}, x,y \in \Xi^{\circ}$$

,such that $\{\|x\|_p\}_{p\in\mathbb{N}}$, is a countable family of pseudo seminorms.

It is clear that a pseudo metric d defined above is a translation invariant.

The collection of all open balls:

$B^{p}_{1/r}(x) = \left\{ y \in \Xi^{\circ} : \|x-y\|_p < \frac{1}{r} \right\}, x \in \Xi^{\circ}, r \in \mathbb{N} - \{0\}$, $p \in \mathbb{N}$ is a base for the pseudo topology of Ξ° .

Definition 2.2.13:

A locally convex T_{PVS} Ξ° is said to be pseudo-locally bounded if Ξ° possess a base of pseudo bounded filters of a neighborhood of 0, that's mean the base of Ξ° consists a filters $\mathfrak{F}$, which have the property $\mathfrak{r}\, \mathfrak{F} \downarrow \Xi^{\circ}$.

And the results (2.2.5, 2.2.6, 2.2.7 and 2.2.8) will be true.

Notation 2.2.14:

For simplicity we will use the symbol ($T_{\tilde{p}VS}$) for a locally convex pseudo metrizable pseudo topological vector space.

Example 2.2.15 [41]:

Suppose $\mathbb{P} = \{\|.\|_i\}_{i\in I}$ is separating family of pseudo simenorms on a vector space Ξ. And suppose that, $\mathcal{v}_n(\|.\|_i) = \{m: \|m\|_i < \frac{1}{n}\}$, $n \in \mathbb{N} - \{0\}$ is a convex local base.

Let $\mathfrak{B}$ be the collection of all finite intersections of the set $\mathcal{v}_n(\|.\|_i)$. Then $\mathfrak{B}$ is a convex balanced local base for a pseudo topology τ on Ξ, which turns Ξ into a locally convex space such that every $\|.\|_i \in \mathbb{P}$ is continuous.

Example 2.2.16 [41]:

Let $\mathcal{O}$ be a non empty open set in some Euclidean space then $\mathcal{O}$ is the union of countably many compact sets $\mathcal{K}_n \neq \emptyset$ such that

$\mathcal{K}_n \subset \mathcal{K}_{n+1}$, $\forall\, n \in \mathbb{N} - \{0\}$, $\mathcal{C}(\mathcal{O})$ is the vector space of all complex valued continuous functions on $\mathcal{O}$, topologized by the separating family of PSN:

$\|f\|_n = \sup\{|f(x)|: x \in \mathcal{K}_n\}$, since $\mathbb{P} = \{\|.\|_i\}_{i\in I}$ is non-decreasing. The sets $\mathcal{v}_n(\|.\|_i) = \left\{f \in \mathcal{C}(\mathcal{O}): \|f\|_n < \frac{1}{n}\right\}, n \in \mathbb{N} - \{0\}$ form a convex local base for $\mathcal{C}(\mathcal{O})$, the pseudo topology of $\mathcal{C}(\mathcal{O})$ is compatible with the pseudo metric $d(f,g) = max_n \frac{2^{-n}\|f\|_n}{1+\|f\|_n}$, $\mathcal{C}(\mathcal{O})$ with compatible pseudo topology τ is a $T_{\tilde{p}VS}$.

Example 2.2.17[41]:

Let $\mathcal{O}$ be a non empty open subset of the complex plane, and let $\mathcal{H}(\mathcal{O})$ be a subspace of $\mathcal{C}(\mathcal{O})$ (Defined in the above example), that consists of the homomorphism functions in $\mathcal{O}$.

$\mathcal{H}(\mathcal{O})$ is also $T_{\tilde{p}VS}$.

2.3. Some New Properties of the Space $T_{\tilde{p}VS}$:

In this section, introduced some new properties of the space $T_{\tilde{p}VS}$. The first property is paracompactness; we showed that space $T_{\tilde{p}VS}$ is pseudo paracompact. The second property is separation property, where we have been given a definition of separated space and some results related to it. After that defined a T_1 −space and pseudo regularity spaces and we got some results about them.

Finally, we give a definition of the equable continuous maps and the relationship to continuous maps, the notion of the quasi bounded maps, the relationship to continuous maps and then we introduced the new class of maps that is equable continuous and quasi bounded $\tilde{C}_e^q(\Xi_1;\Xi_2)$ and then discuss some results about these notions.

Definition 2.3.1:

Let $(\varXi, \tau)$ be a $T_{\tilde{p}VS}$. The interior $[int(\mathcal{A})]$ of a set $\mathcal{A} \subseteq \varXi$ such that $\varXi$ is a $T_{\tilde{p}VS}$, can be denoted as:

$int(\mathcal{A}) = \{m \in \mathcal{A}:\ \mathfrak{F} \in \tau(m)$ implies $\mathcal{A} \in \mathfrak{F}\}$.

Definition 2.3.2:

Let $(\varXi, \tau)$ be a $T_{\tilde{p}VS}$ and a set $\mathcal{A} \subseteq \varXi$ can be call τ-open set (For easily open set) if $\mathcal{A} = int(\mathcal{A})$.

Definition 2.3.3:

Let $(\varXi, \tau)$ be a $T_{\tilde{p}VS}$. A cover of a $T_{\tilde{p}VS}$ $\varXi$ is a family of subsets of $\varXi$, $\{u_\alpha\}_{\alpha\in A}$ such that $\varXi = \bigcup_{\alpha\in A} u_\alpha$.

Definition 2.3.4:

Let $(\varXi, \tau)$ be a $T_{\tilde{p}VS}$. Open cover $\{u_\alpha\}_{\alpha\in A}$ of $\varXi$ is a cover of $\varXi$ and all the sets u_α are open sets in $\varXi$.

Definition 2.3.5:

A refinement of a cover $\{u_\alpha\}_{\alpha\in A}$ of a $T_{\tilde{p}VS}$ $\varXi$ is another cover $\{v_i\}_{i\in I}$ such that every set from the second cover is contained in at least one set from the original cover (i.e. for all $i \in I$ there exist $\alpha(i) \in A$ such that $v_i \subset u_{\alpha(i)}$).

Definition 2.3.6:

A cover $\{v_i\}_{i\in I}$ of $\varXi$ (not necessarily open sets) is locally finite cover if every point of $\varXi$ has a neighborhood that intersects only a finite number of sets from the cover.

Definition 2.3.7:

A $T_{\tilde{p}VS}$ $\varXi$ is called pseudo paracompact (for easily p -paracompact) if every open cover of $\varXi$ has a locally finite open refinement.

Definition (well – ordering principle) 2.3.8[23]:

Every non-empty set of non-negative integers has a smallest element.

Theorem 2.2.9:

Any $T_{\tilde{p}VS}$ $\varXi$ is p-paracompact.

Proof:

Let $(\varXi, d)$ be a pseudo metric space. Let $\mathcal{O} = \{O_\alpha\}_{\alpha\in\wedge}$ be an open cover of $\varXi$. By using the Well-Ordering Principle: there is a well-ordering $\prec$ on $\wedge$.

Recall that this means that every $\mathbb{S} \subseteq\wedge$ has a $\prec$-smallest element $min(\mathbb{S})$. In particular, for every $x \in \varXi$ there is a unique $\alpha \in\wedge$, such that $\alpha = \min(\{\beta \in\wedge : x \in O_\beta\})$, and $x \in O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta$. Take a filter $\mathfrak{F}$ as a filter of x such that the open balls with the centred via x form a base $\{B_\varepsilon(x)\}$ of the filter $\mathfrak{F}$.

For $\alpha \in\wedge$; $m \in \mathbb{N}$ define, inductively over m,

$w_{\alpha,m} = \bigcup_{x\in Z_{\alpha,m}} B_\varepsilon(x)$, take, $\varepsilon = 2^{-m}$,

Where, $Z_{\alpha,m} = \{x \in \varXi : B_{3.2^{-m}}(x) \subseteq O_\alpha$ (1),

$x \notin \bigcup_{\beta<\alpha} O_\beta \cup \bigcup_{\beta\in\wedge, n<m} w_{\beta,n}\}$ (2)

We need to prove $\mathcal{W} = \{w_{\alpha,m}\}_{\alpha\in\wedge}$, $m \in \mathbb{N}$, is a locally finite open refinement of $\mathcal{O}$ (Proving pseudo paracompactness).

Openness of the $w_{\alpha,m}$ is clear. From (1), $B_{2^{-m}}(x) \subseteq B_{3.2^{-m}}(x) \subseteq O_\alpha$ for all balls that constitute $w_{\alpha,m}$, therefore, $w_{\alpha,m} \subseteq O_\alpha$, and $\mathcal{W}$ is refinement to $\mathcal{O}$.

Now to prove $\mathcal{W}$ is covers Ξ:

Let, $x \in \Xi$, put $\alpha = \min(\{\beta \in \Lambda : x \in O_\beta\})$, and take $m \in \mathbb{N}$, such that (1) holds. Therefore by (2), either $x \in w_{\beta,n}$ for some $\beta \in \Lambda$ and $n < m$ or we have $x \in Z_{\alpha,m} \subseteq w_{\alpha,m}$. Thus $\mathcal{W}$ covers Ξ.

Now to prove $\mathcal{W}$ is locally finite:

Let $x \in \Xi$. Put $\alpha = \min(\{\beta \in \Lambda : x \in \bigcup_{m\in\mathbb{N}} w_{\beta,m}\})$.

Then we can take, $n \in \mathbb{N}$, such that $B_{2^{-n}}(x) \subseteq w_{\alpha,m}$ now we need to prove:

1- If $p \geq m+n$ then $B_{2^{-m-n}}(x)$ intersects no $w_{\beta,p}$;

2- If $p < m+n$ then $B_{2^{-m-n}}(x)$ intersects $w_{\beta,p}$ for at most one $\beta \in \Lambda$.

This implies that the open neighborhood ($B_{2^{-m-n}}(x)$) of x can meet at most $m+n-1$ elements of $\mathcal{W}$, then $\mathcal{W}$ is ; locally finite.

Proof (1): Let, $y \in Z_{\beta,p}$. In view of $p > m$, the (2) implies that $y \notin w_{\alpha,m}$ Together with $B_{2^{-n}}(x) \subseteq w_{\alpha,m}$ .This implies $d(x,y) \geq 2^{-n}$. We have $p \geq n+1$ (from assumption) and $m+n \geq n+1$ trivally.

This leads to $z \in B_{2^{-m-n}}(x) \cap B_{2^{-p}}(y)$ which imply:

$$d(x,y) \leq d(x,z) + d(z,y) < 2^{-m-n} + 2^{-p} \leq 2^{-n-1} + 2^{-n-1} = 2^{-n}$$

But which is contradiction, so

$B_{2^{-m-n}}(x) \cap B_{2^{-p}}(y) = \emptyset$.

(That's mean $B_{2^{-m-n}}(x)$ is disjoint from the all balls $B_{2^{-p}}(y)$, $y \in Z_{\beta,p}$, whose union is $w_{\beta,p}$) .

Proof (2): Let $x \in w_{\alpha,p}$, $y \in w_{\beta,p}$, where $\alpha < \beta$. From definition of w′s, there are z, u such that, $x \in B_{2^{-p}}(z) \subseteq w_{\alpha,p}$, $y \in B_{2^{-p}}(u) \subseteq w_{\beta,p}$, from(1), $B_{3.2^{-p}}(z) \subseteq O_\alpha$, but from (2) $u \notin O_\alpha$.

This implies, $d(z,u) \geq 3.2^{-p}$, and from the triangle inequality we have:

$3.2^{-p} \leq d(z,u) \leq d(z,x) + d(x,y) + d(y,u) < d(x,y) + 2^{-p} + 2^{-p}$,

Thus, $d(x,y) > 2^{-p}$.

Hence this implies: $w_{\alpha,p} \cap w_{\beta,p} = \emptyset$ whenever $\alpha \neq \beta$.

And also implies that every ball $B_{2^{-p-1}}$ intersects $w_{\alpha,p}$ for one α, and taking into account that $p < m+n$, that's mean $p+1 \leq m+n$, this conclusion a fortiori holds for each $B_{2^{-m-n}}(x)$.

Therefore Ξ is p-paracompact space.

Remark (Continuity on $T_{\tilde{p}VS}$)2.3.10:

We will introduce new class of continuity on $T_{\tilde{p}VS}$ spaces, by using the method of Frolicher and Walter [18].

Definition 2.3.11:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$ and f a map between (Ξ_1, τ_1) and

(Ξ_2, τ_2). The mapping f: (Ξ_1, τ_1)→ (Ξ_2, τ_2) is called continuous at a point $m \in \Xi_1$ if for all filter $\mathfrak{F} \in \tau_1(m)$ the filter $f(\mathfrak{F}) \in \tau_2(f(m))$.

The mapping f is called continuous on Ξ_1 if it is continuous at each point of Ξ_1.

Theorem 2.3.12:

Every topological structure T on Ξ yields a pseudo topology (limit structure) τ on Ξ, $\tau: \Xi \to 2^{\mathrm{F}(\Xi)}$.

Proof:

We define τ as: for all $m \in \Xi$, $\mathfrak{F} \in \tau(m) \leftrightarrow \mathfrak{F} \leq \mathrm{N}_m$ where N_m is the neighborhood filter of m in Ξ.

1- If $\mathfrak{F} \in \tau\ (m)$ and $\rho \leq \mathfrak{F}$ then, $\rho \leq \mathfrak{F} \leq \mathrm{N}_m$, therefore $\rho \in \tau\ (m)$.

2- If $\mathfrak{F}, \rho \in \tau\ (m)$ then $\rho, \mathfrak{F} \leq \mathrm{N}_m$, thus, $\rho \cap \mathfrak{F} \leq \mathrm{N}_m$, $\mathfrak{F} \cap \rho \in \tau\ (m)$.

3- $[m] \in \tau\ (m)$ as $[m] \leq \mathrm{N}_m \forall m \in \Xi$. Hence τ is limit structure.

The pseudo topological vector space τ is called the natural limit structure of the given topology T.

Remark 2.3.13:

For each pseudo topology (Ξ, τ) we can construct a topological space T_τ and is called the topology associated with the given limit structure τ.

Theorem 2.3.14:

Let (Ξ, τ) be a $T_{\tilde{p}VS}$ and let T_τ set of all open sets in Ξ, then T_τ is topology.

Proof:

1- $\emptyset \in T_\tau$ (Trivially).

2- $\Xi \in T_\tau$ since, $\Xi \in \mathfrak{F}, \forall\, \mathfrak{F} \in \tau\ (m)$, $m \in \Xi$.

3- Let $A, B \in T_\tau$, and $C = A \cap B \neq \emptyset$, let $m \in C$ then, $m \in A, B$ this implies $A, B \in \mathfrak{F}, \forall\, \mathfrak{F} \in \tau$, $C \in T_\tau$.

4- Let $\mathcal{A} \subseteq T_\tau$, and let $A_1 = \bigcup_{A \in \mathcal{A}} A$, let $m \in A_1$ then, there exist some $A \in \mathcal{A}$ such that $m \in A$, but $A \in \mathfrak{F}, \forall\, \mathfrak{F} \in \tau\ (m)$ and $A \subseteq A_1$. Hence, $A_1 \in \mathfrak{F}, \forall\, \mathfrak{F} \in \tau\ (m)$. Therefore, $A_1 \in T_\tau$.

Theorem 2.3.15:

Let Ξ be a set equipped with two pseudo topology τ_1 and τ_2. Then, $\tau_1 \geq \tau_2$ if and only if the identity mapping $i : (\Xi, \tau_1) \to (\Xi, \tau_2)$ is continuous.

Proof :

Suppose that, $\tau_1 \geq \tau_2$. Let $m \in \Xi$ and $\mathfrak{F} \in \tau_1\ (m)$.

Then $i(\mathfrak{F}) = \mathfrak{F} \in \tau_2\ (m)$, because, $\tau_1(m) \subseteq \tau_2(m), \forall m \in \Xi$.

Thus, i is continuous.

Conversely, if i is continuous then for all $\mathfrak{F} \in \tau_1(m)$ we get:

$i(\mathfrak{F}) = \mathfrak{F} \in \tau_2(m)$. Thus, $\tau_1(m) \subseteq \tau_2(m)$ for all $m \in \Xi$.

Hence, $\tau_1 \geq \tau_2$.

Corollary 2.3.16:

Let (Ξ,τ) be a $T_{\tilde{p}VS}$. Then the identity mapping i: $(\Xi,\tau) \to (\Xi,T_\tau)$ is continuous, and it is homeomorphism iff (Ξ, τ) is topological space.

Proof:

Since $T_\tau \leq \tau$ then the proof is satisfied by the Theorem (2.3.14).

Theorem 2.3.17:

Let (Ξ, τ_1) and (Ξ, τ_2) be two $T_{\tilde{p}VS}$, and let f: $(\Xi, \tau_1) \to (\Xi, \tau_2)$ be continuous and let $(\Xi, \tau_1^{\sim})$, $(\Xi, \tau_2^{\sim})$ be other $T_{\tilde{p}VS}$ such that $\tau_1^{\sim} \geq \tau_1$ and $\tau_2^{\sim} \leq \tau_2$. Then, f: $(\Xi, \tau_1^{\sim}) \to (\Xi, \tau_2^{\sim})$ is also continuous .

Proof :

Let $m \in \Xi$ and $\mathfrak{F} \in \tau_1^{\sim}(m)$.

Then $\mathfrak{F} \in \tau_1(m)$ and $f(\mathfrak{F}) \in \tau_2(f(m))$ as f is continuous from

(Ξ, τ_1) to (Ξ, τ_2).

Since, $\tau_2^{\sim} \leq \tau_2$, $f(\mathfrak{F}) \in \tau_2^{\sim}(f(m))$.

Hence, f: $(\Xi, \tau_1^{\sim}) \to (\Xi, \tau_2^{\sim})$ is continuous .

Theorem 2.3.18:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$ such that the mapping

f: $(\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ is continuous , then f^{-1}(B) is an open subset of Ξ_1 if B is an open subset of Ξ_2.

Proof :

Let B be an open subset of Ξ_2. If f^{-1}(B) =$\emptyset$, then, the proof is complete.

If f^{-1}(B) $\neq \emptyset$,let $m \in f^{-1}$(B), and let $\mathfrak{F} \in \tau_1(m)$.

Since f is continuous, then it is continuous at every point in $\Xi_1$as m,

$f(\mathfrak{F}) \in \tau_2(f(m))$.

But $f(m) \in$ B and B is open.

Therefore, B $\in f(\mathfrak{F})$(by definition of open set).

Thus, $\exists\, F \in \mathfrak{F}$ such that $f(F) \subseteq$ B . But, $F \subseteq f^{-1}(f(F)) \subseteq f^{-1}$(B).

Therefore, f^{-1}(B) $\in \mathfrak{F}$ (by definition of a filter).

Hence, f^{-1}(B) is a open subset of Ξ_1.

Remark 2.3.19:

The convers of theorem above 2.2.18 in general is not true.

Let (Ξ,τ) be any $T_{\tilde{p}VS}$ but not topological vector space and

i: $(\Xi, T_\tau)\rightarrow(\Xi,\tau)$ be the identity map. Then, i is not necessary continuous. Since, $T_\tau \leq \tau$. But i^{-1}(B) is open subset of Ξ in T_τ if B is open subset of Ξ in τ. But the converse theorem above is true iff f it is taken between two topological spaces.

Corollary 2.3.20:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$. Then if the mapping

f:$(\Xi_1,\tau_1)\rightarrow(\Xi_2,\tau_2)$ is continuous, then $f:(\Xi, T_{\tau_1})\rightarrow(\Xi, S_{\tau_2})$ is also continuous.

Proof:

Let, $\mathfrak{F} \in T_{\tau_1}$ at m, since T_{τ_1} and is the topology associated with limit structure τ_1 then $\mathfrak{F} \in \tau_1(m)$ and by continuity of f we have

$f(\mathfrak{F}) \in \tau_2(f(m))$. From Theorem 2.3.14, $f(\mathfrak{F}) \in S_{\tau_2}$.

Hence, $f: (\Xi, T_{\tau_1}) \rightarrow (\Xi, S_{\tau_2})$ is continuous map.

Theorem 2.3.21:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$ such that the mapping

$f: (\Xi_1, \tau_1) \rightarrow (\Xi_2, \tau_2)$ is continuous .

Then, $\mathrm{N}_{\tau_2}(f(m)) \subseteq f(\mathrm{N}_{\tau_1}(m))$ for all $m \in \Xi_1$, where $\mathrm{N}_{\tau_1}(m), \mathrm{N}_{\tau_2}(m)$ is filter neighborhood of m in τ_1 and τ_2 respectively.

Proof :

Let $u \in \mathrm{N}_{\tau_2}(f(m))$ and $\mathfrak{F} \in \tau_1(m)$. Since f is continuous, then $f(\mathfrak{F}) \in \tau_2(f(m))$. Thus, $\in f(\mathfrak{F})$. This implies that, $f^{-1}(u) \in \mathfrak{F}$, because $f(v) \subseteq u$ for some $v \in \mathfrak{F}$. Thus $f^{-1}(f(v)) \subseteq f^{-1}(u) \in \mathfrak{F}$ for all $\mathfrak{F} \in \tau_1(m)$.

Therefore, $f^{-1}(u) \in \mathrm{N}_{\tau_1}(m)$. Since $f(f^{-1}(u)) \subseteq u$ implies that $u \in f(\mathrm{N}_{\tau_1}(m))$. Hence, $\mathrm{N}_{\tau_2}(f(m)) \subseteq f(\mathrm{N}_{\tau_1}(m))$.

Theorem 2.3.22:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$ such that the mapping

$f: (\Xi_1, \tau_1) \rightarrow (\Xi_2, \tau_2)$ is continuous. Then, $f(cl(A) \subseteq cl(f(A))$ for all $A \subseteq \Xi_1$.

Proof :

Let $\in f(cl(A))$, there exists $m \in cl(A)$ such that $w = f(m)$.

Since $m \in cl(A)$, this implies there exists $\mathfrak{F} \in \tau_1(m)$ such that $A \in \mathfrak{F}$. So, $f(A) \in f(\mathfrak{F})$. Since f is continuous then $f(\mathfrak{F}) \in \tau_2(f(m))$.

Thus, $w = f(m) \in cl(f(A))$.

Theorem:2.3.23:

Let (Ξ_1, τ_1) , (Ξ_2, τ_2) and (Ξ_3 , τ_3) are $T_{\tilde{p}VS}$ spaces such that,

f: $(\Xi_1, \tau_1) \rightarrow (\Xi_2, \tau_2)$ is continuous at m and $g : (\Xi_2, \tau_2) \rightarrow (\Xi_3, \tau_3)$ is continuous at $f(m)$, then $g \circ f$: $(\Xi_1, \tau_1) \rightarrow (\Xi_3, \tau_3)$ is continuous at m.

Proof :

Let $m \in \Xi_1$ and $\mathfrak{F} \in \tau_1 (m)$ then, $f (\mathfrak{F}) \in \tau_2 (f(m))$,

and $g (f(\mathfrak{F})) \in \tau_3 (g(f(m))$, since f is continuous at m, and g is continuous at $f(m)$, so, $(g \circ f)(\mathfrak{F}) \in \tau_3 ((g \circ f)(m))$.

Hence, $g \circ f$ is continuous at m.

To construct product, and subspaces pseudo structures we will introduce the new concept of the initial pseudo structure (initial limit structure).

Definition 2.3.24:

Let Ξ be a set, and $\{(\Xi_i, \tau_i)\}_{i \in I}$ be a collection of $T_{\tilde{p}VS}$ and for each, $f_i: \Xi \rightarrow \Xi_i$ is a mapping. Define $\tau: \Xi \rightarrow 2^{\mathrm{F}(\Xi))}$by

$\mathfrak{F} \in \tau (m)$ if and only if $f_i(\mathfrak{F}) \in \tau_i(f_i(m))$ $\forall i \in I$.

We claim to: τ is a (limit structure) pseudo structure on Ξ .

1-Since $f_i([m]) = [f_i(m)] \in \tau_i(f_i(m))$ $\forall i \in I$, then, $[m] \in \tau(m)$.

2-If $\mathfrak{F} \in \tau (m)$ and $\mathfrak{F} \subseteq \rho$ then $f_i(\mathfrak{F}) \subseteq f_i(\rho)$ $\forall i \in I$

Since, $f_i(\mathfrak{F}) \in \tau_i(f_i(m))$, then $f_i(\rho) \in \tau_i(f_i(m))$ $\forall i \in I$.

Thus, $\rho \in \tau (m)$.

3-If $\mathfrak{F}, \rho \in \tau (m)$ then $f_i(\mathfrak{F}), f_i(\rho) \in \tau_i(f_i(m))$ $\forall i \in I$

Since, $f_i(\mathfrak{F}) \cap f_i(\rho) \subseteq f_i(\mathfrak{F} \cap \rho)$, then $f_i(\mathfrak{F} \cap \rho) \in \tau_i(f_i(m))$ $\forall i \in I$.

Thus, $\mathfrak{F} \cap \rho \in \tau (m)$.Therefore, τ is a pseudo structure on Ξ.

This pseudo structure is called the initial pseudo structure**.**

Definition 2.3.25:

Let (Ξ, τ) be a $T_{\tilde{p}VS}$ and $A \subseteq \Xi$. The subspace pseudo structure τ_A on A is the initial pseudo structure with respect to the inclusion mapping

$in: A \to \Xi$.

Definition 2.3.26:

Let $\mathfrak{F} \in$ Ƒ (A) and $m \in A$. We say that $\mathfrak{F} \in \tau_A (m)$ if $[\mathfrak{F}]_{\Xi} \in \tau\ (m)$.

Theorem 2.3.27:

Let (Ξ_1 , τ) and (Ξ_2 , τ^*) be two $T_{\tilde{p}VS}$ such that the mapping

$f: (\Xi_1 , \tau) \to (\Xi_2 , \tau^*)$ is continuous mapping.

Let, $A \subseteq \Xi_1$. Then the restriction map $f/_A: (A, \tau_A) \to (f(A) , {\tau^*}_{f(A)})$ is continuous.

Proof :

Let $m \in A$ and $\mathfrak{F} \in \tau_A (m)$, then $[\mathfrak{F}]_{\Xi_1} \in \tau\ (m)$,by Definition 2.3.26, and therefore,

$[f/_A(\mathfrak{F})\ ]_{\Xi_2} = f([\mathfrak{F}]_{\Xi_1}) \in \tau^*\ (f(m))$.

Hence, $f/_A(\mathfrak{F}) \in {\tau^*}_{f(A)}\ (f(m))$.

Therefore $f/_A$ is continuous map at m, and since m is arbitrary point in A then $f/_A$ is continuous map.

Theorem 2.3.28:

Let (Ξ, τ) be a $T_{\tilde{p}VS}$ and let $A \subseteq \Xi$, then, $\bar{B}_\tau^A = \overline{B_\tau} \cap A$ for each $B \subseteq A$. Where $\bar{B}_\tau^A$ is a closure of B in τ_A.

Proof:

Let $m \in \bar{B}_\tau^A$ then there exist a filter $\mathfrak{F} \in$ Ƒ(A) such that $B \in \mathfrak{F}$ and $\mathfrak{F} \in \tau_A(m)$. Clearly $[\mathfrak{F}]_{\Xi} \in \tau(m)$ and $B \in [\mathfrak{F}]_{\Xi}$.

Then, $m \in \overline{B_\tau}$. This implies that $m \in \overline{B_\tau} \cap A$.

Conversely let $m \in \overline{B_\tau} \cap A$ then $m \in A$ and there exist a filter $\mathfrak{F} \in \mathrm{F}(\Xi)$ such that, $B \in \mathfrak{F}$ and $\mathfrak{F} \in \tau(m)$. $A \in \mathfrak{F}$ so let $\mathfrak{F}_A = \{F \cap A : F \in \mathfrak{F}\}$.

$\mathfrak{F}_A \in \tau_A(m)$ by Definition 2.3.25 and $B \in \mathfrak{F}_A$. Thus, $m \in \bar{B}_\tau^A(B)$.

Therefore, $\bar{B}_\tau^A = \overline{B_\tau} \cap A$.

Theorem 2.3.29:

Let $(F, \mathcal{t})$ and (Ξ, τ) be two $T_{\tilde{p}VS}$ spaces and $\{(\Xi_i, \tau_i)\}_{i\in I}$ be a collection of $T_{\tilde{p}VS}$ spaces such that τ is the initial pseudo structure with respect to

continuous $f_i, \forall i \in I$ $(f_i : \Xi \to (\Xi_i, \tau_i))$ then, $f : (F, \mathcal{t}) \to (\Xi, \tau)$is continuous if and only if for each $i \in I, f_i \circ f : (F, \mathcal{t}) \to (\Xi_i, \tau_i)$ is continuous.

Proof:

If f is continuous, then $f_i \circ f$ is continuous for each $i \in I$, as f_i is continuous for each $i \in I$, and by the Theorem (2.3.24) the composition of two continuous maps is continuous.

Conversely, assume that $f_i \circ f$ is continuous for each $i \in I$.Let,

$\mathfrak{F} \in \mathcal{t}(m)$, $m \in F$. Since $(f_i \circ f)$ is continuous for each $i \in I$, we get $(f_i \circ f)(\mathfrak{F}) = (f_i(f(\mathfrak{F})) \in \tau_i(f_i(f(m)))$ for each $i \in I$.

Therefore, $f(\mathfrak{F}) \in \tau(f(m))$ by definition of initial pseudo space. Hence f is continuous at m, since m is arbitrary point in F , then f is continuous.

Definition 2.3.30:

A $T_{\tilde{p}VS}$ (Ξ, τ) is said to be separated if $\mathfrak{F} \downarrow_m \Xi$ and $\mathfrak{F} \downarrow_n \Xi$, then $m = n$ such that $m, n \in \Xi$.(That's mean if $m \neq n$, then $\tau(m) \cap \tau(n) = \emptyset$).

Definition 2.3.31:

A $T_{\tilde{p}VS}$ $(\varXi, \tau)$ is said to be T_1 if $m \neq n$, then $[n] \notin \tau(m)$, such that $m, n \in \varXi$.

Theorem 2.3.32:

Let $(\varXi_1, \tau_1)$ and $(\varXi_2, \tau_2)$ be two $T_{\tilde{p}VS}$ such that $f: (\varXi_1, \tau_1) \to (\varXi_2, \tau_2)$ is injective and continuous mapping. If $(\varXi_2, \tau_2)$ is separated then $(\varXi_1, \tau_1)$ is also separated space.

Proof :

Let $m, n \in \varXi_1$ such that $m \neq n$ and $\tau_1(m) \cap \tau_1(n) \neq \emptyset$ then there exist a filter $\mathfrak{F} \in \mathbb{F}(\varXi_1)$ such that $\mathfrak{F} \in \tau_1(m)$ and $\mathfrak{F} \in \tau_1(n)$.Since f is continuous , we get:

$f(\mathfrak{F}) \in \tau_2(f(m))$ and $f(\mathfrak{F}) \in \tau_2(f(n))$. Since f is injective we get

$f(m) \neq f(n)$. So $\tau_2(f(m)) \cap \tau_2(f(n)) \neq \emptyset$.

Contradiction, because the space $(\varXi_2, \tau_2)$ is separated.

Hence, $\tau_1(m) \cap \tau_1(n) = \emptyset$ and therefore, $(\varXi_1, \tau_1)$ is separated space.

Theorem 2.3.33:

Let $(\varXi_1 , \tau_1)$ and $(\varXi_2 , \tau_2)$ be two $T_{\tilde{p}VS}$ such that $f: (\varXi_1, \tau_1) \to (\varXi_2, \tau_2)$ is injective and continuous mapping. If $(\varXi_2, \tau_2)$ is T_1-space then $(\varXi_1, \tau_1)$ is also T_1-space.

Proof :

Let $m \in \varXi_1$ such that $[m] \in \tau_1(n), m \neq n$ then

$f([m]) = [f(m)] \in \tau_2(f(n))$ (Because f is continuous). Since f is injective, then $f(m) \neq f(n)$ as. Hence τ_2 is not T_1-space, but is a contradiction.

Hence, $(\varXi_1, \tau_1)$ is T_1-space.

Theorem: 2.3.34:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$. Let, $f, g: (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ be two continuous mappings from a pseudo space (Ξ_1, τ_1) into a separated pseudo space (Ξ_2, τ_2) .Then, the set $A = \{x \in \Xi_1: g(x) = f(x)\}$ is a closed subset of Ξ_1.

Proof:

A is closed subset in Ξ_1 if $A = CL(A)$, let $m \in A$, then $[m]$ is a filter and $A \in [m]$, therefore $m \in CL(A) \Longrightarrow A \subseteq CL(A)$, then remains to prove $CL(A) \subseteq A$. Let, $m \in CL(A)$, then there exist a filter $\mathfrak{F}_A \in Ḟ(A)$ such that,

$[\mathfrak{F}_A]_{\Xi_1} \in \tau_1(m)$.

$g([\mathfrak{F}_A]_{\Xi_1}) \in \tau_2(g(m))$ and $f([\mathfrak{F}_A]_{\Xi_1}) \in \tau_2(f(m))$ since f, g are continuous. Since $g([\mathfrak{F}_A]_{\Xi_1})$ and $f([\mathfrak{F}_A]_{\Xi_1})$ are generated by the filter bases $g(\mathfrak{F}_A)$ and $f(\mathfrak{F}_A)$ respectively and $g(n) = f(n)\ \forall\, n \in A$, we have

$$g([\mathfrak{F}_A]_{\Xi_1}) = f([\mathfrak{F}_A]_{\Xi_1}).$$

Since (Ξ_2, τ_2) is a separated space we get $g(m) = f(m)$.

Hence, $m \in A$, $CL(A) \subseteq A$, hence, $CL(A) = A$. Thus, A is a closed subset of Ξ_1.

Definition 2.3.35:

Let (Ξ,τ) be a $T_{\tilde{p}VS}$. The filter $CL_\tau(\mathfrak{F})$ is generated by the filter base $\{CL_\tau(F): F \in \mathfrak{F}\}$.

Definition 2.3.36:

A $T_{\tilde{p}VS}$ (Ξ,τ) is pseudo regular(p-regular) if $CL_\tau(\mathfrak{F}) \in \tau(m)$ whenever $\mathfrak{F} \in \tau(m)$.

Lemma 2.3.37:

Let (Ξ, τ) be a $T_{\tilde{p}VS}$, and $\mathfrak{F} \in \tau(m)$, then $m \in CL_\tau(F)$, for all $F \in \mathfrak{F}$.

Proof:

Since $\mathfrak{F} \in \tau(m)$ and $F \in \mathfrak{F}$ then, $m \in CL_\tau(F)$.

Theorem 2.3.38:

Let (Ξ, τ) be a $T_{\tilde{p}VS}$. If (Ξ, τ) is a p-regular T_1- pseudo space, then it is separated.

Proof :

Suppose that τ is not separated, then there exist a filter $\mathfrak{F} \in \mathrm{F}(\Xi)$ such that
$\mathfrak{F} \in \tau(m) \cap \tau(n)$ for some $m, n \in \Xi$ and $m \neq n$. .
$CL(\mathfrak{F}) \in \tau(m)$ and $\tau(n)$ by regularity of τ.
$m \in CL(F)$, for all $F \in \mathfrak{F}$ by Lemma 2.3.37.This means $CL(\mathfrak{F}) \subseteq [m]$. Hence, $[m] \in \tau(n)$, which is a contradiction as (Ξ, τ) is a T_1 – space.
Therefore, (Ξ, τ) is a separated space.

Theorem 2.3.39:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$, and f: $(\Xi_1, \tau_1) \rightarrow (\Xi_2, \tau_2)$ be continuous map , then:

1- If Ξ_2 is p-regular, then Ξ_1 is p-regular.
2- If Ξ_1 is p-regular and f is a surjection, then Ξ_2 is p-regular.

Proof :

1- Let $\mathfrak{F} \in \tau_1(m)$. $f(\mathfrak{F}) \in \tau_2(f(m))$ as f is continuous map. $CL(f(\mathfrak{F}))$ converges to $f(m)$ in Ξ_2.

Since $f(CL(F)) \subseteq CL(f(F))$ for all $F \in \mathfrak{F}$ by Theorem 2.3.22 we get
$CL(f(\mathfrak{F})) \subseteq f(CL(\mathfrak{F}))$ .Therefore, $f(CL(\mathfrak{F})) \in \tau_2(f(m))$.
Hence, $CL(\mathfrak{F})$ converges to m because f is continuous map.
Therefore, Ξ_1 is p-regular.

2- Let $\mathfrak{F} \in \tau_2(n)$ in Ξ_2 and since is surjective, then there exists

$m \in \Xi_1$ such that, $n = f(m)$ and $f\left(f^{-1}(\mathfrak{F})\right) = \mathfrak{F}$.

Thus $f^{-1}(\mathfrak{F})$ converges to m in Ξ_1 as f is continuous map.

Since Ξ_1 is p-regular, $CL(f^{-1}(\mathfrak{F})) \in \tau_1(m)$.

$f(CL(f^{-1}(\mathfrak{F})))$ converges to $f(m)$ in Ξ_2 as f is continuous.

We need to prove that $f(CL(f^{-1}(\mathfrak{F}))) \subseteq CL(\mathfrak{F})$

New we show that, $f^{-1}CL(B)) \subseteq CL(f^{-1}(B))$. Let, $b \in f^{-1}CL(B))$, then $f(b) \in CL(B)$, so $B \in \rho$ for some filter ρ converges to $f(b)$ in Ξ_2

$f\left(f^{-1}(\rho)\right) = \rho$ as f is surjective.

Since f is continuous map we have $f^{-1}(\rho)$ converges to b.

Since, $f^{-1}(B) \in f^{-1}(\rho)$, we get $b \in CL(f^{-1}(B))$,

$(f^{-1}(CL(B))) \subseteq CL(f^{-1}(B))$.

Note that, $(F) = f(f^{-1}(CL(F))) \subseteq f(CL(f^{-1}(F)))$.

Hence, $(CL(f^{-1}(\mathfrak{F}))) \subseteq CL(\mathfrak{F})$. Therefore, $CL(\mathfrak{F})$ converges to $f(m) = n$ in Ξ_2.

Hence Ξ_2 is p-regular.

Definition 2.3.40:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$, $f: \Xi_1 \to \Xi_2$ being any map , we denote by $\Delta f: \Xi_1 \times \Xi_1 \to \Xi_2$, the map defined by:

$$\forall x, y \in \Xi_1, \Delta f(x,y) = f(x+y) - f(x) .$$

The map $f: (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ is called equable continuous if and only if

$\mathfrak{F} \in \tau_1(0)$ and $\mathrm{p} \in \tau_1(0) \Longrightarrow \Delta f(\mathfrak{F}, \mathrm{p}) \in \tau_2(0)$, for any filters $\mathfrak{F}$, p on Ξ_1.

Proposition 2.3.41:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$. If $f: (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ is equable continuous, then it is continuous.

Proof:

Let $m \in \Xi_1$, and let $\mathfrak{F} \in \tau_1(m)$. Since $ɍ[m] \in \tau_1(0)$ and

$\mathfrak{F} - [m] \in \tau_1(0)$, then, $\Delta f\ ([m], \mathfrak{F} - [m]) \in \tau_2(0)$.

But, $\Delta f(x, y) = f(x + y) - f(x)$, then we get by Definition 1.4.12,

$\Delta f([m], \mathfrak{F} - [m]) \geq f(\mathfrak{F}) - f([m])$. Then, $f(\mathfrak{F}) - f([m]) \in \tau_2(0)$. This mean, $f(\mathfrak{F}) \in \tau_2(f(m))$, hence f is continuous map at m, therefore, f is continuous map.

Definition 2.3.42:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$. A map $f: (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ is called a quasi bounded map if and only if every quasi bounded filter $\mathfrak{F}$ in Ξ_1 then $f(\mathfrak{F})$ is quasi bounded filter in Ξ_2,

[i.e. $ɍ\mathfrak{F} \in \tau_1(0) \Longrightarrow ɍf(\mathfrak{F}) \in \tau_2(0)$].

Proposition 2.3.43:

Any linear continuous map $f: (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ between two $T_{\tilde{p}VS}$

(Ξ_1, τ_1) and (Ξ_2, τ_2) over a field R, is quasi bounded map.

Proof:

Let f be a continuous map, then for all convergent filter $\mathfrak{F}$ on Ξ_1 such that its convergent to a point x in Ξ_1, then $f(\mathfrak{F})$ is a convergent filter on Ξ_2 to the point $f(x)$.

Take $\mathcal{H}$ be a filter on $\Xi_1$such that $\mathcal{H}$ is quasi bounded filter, this means

$\mathcal{H} \in \tau_1(0)$, $\mathfrak{r}f(\mathcal{H}) = f(\mathfrak{r}\mathcal{H})$ (by linearity of f);

$f(\mathfrak{r}\mathcal{H}) \in \tau_2(0)$ (By continuity of f);

$\mathfrak{r}f(\mathcal{H}) \in \tau_2(0)$, therefore, f is quasi bounded map.

Remark 2.3.44:

The class of all equable continuous and quasi bounded maps $f: \Xi_1 \to \Xi_2$ such that Ξ_1 and Ξ_2 are $T_{\tilde{p}VS}$, is denoted by $\tilde{C}_e^q(\Xi_1; \Xi_2)$

Remark 2.3.45:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$. For any f and $g \in \tilde{C}_e^q(\Xi_1; \Xi_2)$ and $\gamma \in R$ then $\gamma f + g \in \tilde{C}_e^q(\Xi_1; \Xi_2)$. Therefore: $\tilde{C}_e^q(\Xi_1; \Xi_2)$ is vector space

Proposition 2.3.46:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$. For any two maps, $f \in \tilde{C}_e^q(\Xi_1; \Xi_2)$ and $g \in \tilde{C}_e^q(\Xi_2; \Xi_3)$. Then, the composite map $g \circ f \in \tilde{C}_e^q(\Xi_1; \Xi_3)$.

Proof:

Let $m, h \in \Xi_1$. Use the definition of Δ to get:

$\big(\Delta(g \circ f)\big)(m, h) = \Delta g(f(m), \Delta f(m, h))$. Then we get:

$(\Delta(g \circ f))(\mathfrak{f}, \mathfrak{p}) \leq \Delta g(f(\mathfrak{f}), \Delta f(\mathfrak{f}, \mathfrak{p}))$ $\forall \mathfrak{f}, \mathfrak{p}$ are filters on Ξ_1 (by Definition 1.4.12).

If we assume that $\mathfrak{r}\mathfrak{f} \in \tau_1(0)$ and $\mathfrak{p} \in \tau_1(0)$, it follows that the filter on right side of $((\Delta(g \circ f))(\mathfrak{f}, \mathfrak{p}) \leq \Delta g(f(\mathfrak{f}), \Delta f(\mathfrak{f}, \mathfrak{p}))$converges to 0 in Ξ_3, therefore, $(\Delta(g \circ f))(\mathfrak{f}, \mathfrak{p}) \in \tau_3(0)$.

Hence $g \circ f$ is equably continuous.

Now to prove it is quasi bounded:

We can see that it is following from the hypothesis that f and g are quasi bounded:

Let $\mathfrak{f}$ any filter on Ξ_1, since f is quasi bounded map then:

$\mathfrak{r}\mathfrak{f} \in \tau_1(0) \Longrightarrow \mathfrak{r} f(\mathfrak{f}) \in \tau_2(0)$.

Since $f(\mathfrak{f})$ is filter base generated other filter on Ξ_2, and since g is quasi bounded then $\mathfrak{r} f(\mathfrak{f}) \in \tau_1(0) \Longrightarrow \mathfrak{r}\, g(f(\mathfrak{f})) \in \tau_3(0)$

$\Longrightarrow$ $\mathfrak{r}\,(g \circ f)(\mathfrak{f}) \in \tau_3(0)$. Hence $g \circ f$ is quasi bounded map, therefore $g \circ f \in \tilde{C}_e^q(\Xi_1; \Xi_3)$.

Proposition 2.3.47:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$, and $l: (\Xi_1, \tau_1) \rightarrow (\Xi_2, \tau_2)$ is continuous at 0 and linear, then l is equable continuous and quasi bounded,

(That's mean $l \in \tilde{C}_e^q(\Xi_1; \Xi_2)$).

Proof:

From linearity of l , $\alpha. l(m) = l(\alpha. m)$ and

$\Delta\, l(a,b) = l(a+b) - l(a) = l(a) + l(b) - l(a) = l(b)$,

for all $a, b \in \Xi_1$.

To prove l equable continuous, let, $\mathfrak{r}.\mathfrak{p}, \mathfrak{q} \in \tau_1(0)$.

Since, $\Delta\, l(\mathfrak{p}, \mathfrak{q}) = l(\mathfrak{q}) \in \tau_2(0)$, therefore l is equable continuous map.

New, we prove l is quasi bounded:

Let, $\mathfrak{r}.\mathfrak{p} \in \tau_1(0)$, since $\mathfrak{r}.f(\mathfrak{p}) = f(\mathfrak{r}.\mathfrak{p}) \in \tau_2(0)$ by continuity of 0 we get for any filter $\mathfrak{p}$ on Ξ_1:

If $\mathfrak{p} \in \tau_1(0)$ then $f(\mathfrak{p}) \in \tau_2(0)$ so, $\mathfrak{r}.\mathfrak{p} \in \tau_1(0)$ then

$\mathfrak{r}.\mathrm{f}(\mathfrak{p}) = f(\mathfrak{r}.\mathfrak{p}) \in \tau_2(0)$. Hence, l is quasi bounded map.

Remark 2.3.48:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$, and let $\underline{L}(\Xi_1; \Xi_2)$ is a vector space of all linear and continuous maps between (Ξ_1, τ_1) and (Ξ_2, τ_2) as a $T_{\tilde{p}VS}$. Then $\underline{L}(\Xi_1; \Xi_2) \subseteq \tilde{C}_e^q(\Xi_1; \Xi_2)$.

Lemma 2.3.49:

Let (Ξ_1, τ_1), (Ξ_2, τ_2) and (Ξ_3, τ_3) be a $T_{\tilde{p}VS}$. A bilinear map

$\mathcal{L}$: $(\Xi_1, \tau_1) \times (\Xi_2, \tau_2) \to (\Xi_3, \tau_3)$ is equably continuous iff it satisfies the two conditions:

1- $\left.\begin{matrix} \mathfrak{p}_1 \in \tau_1(0) \\ \mathfrak{r}\mathfrak{q}_2 \in \tau_2(0) \end{matrix}\right\} \Longrightarrow \mathcal{L}(\mathfrak{p}_1, \mathfrak{q}_2) \in \tau_3(0)$;

2- $\left.\begin{matrix} \mathfrak{r}\mathfrak{q}_1 \in \tau_1(0) \\ \mathfrak{p}_2 \in \tau_2(0) \end{matrix}\right\} \Longrightarrow \mathcal{L}(\mathfrak{q}_1, \mathfrak{p}_2) \in \tau_3(0)$.

Proof:

By using the statement:

$\Delta\, \mathcal{L}\big((a_1, a_2), (b_1, b_2)\big) = \mathcal{L}\big((a_1 + b_1), (a_2 + b_2)\big) - \mathcal{L}(a_1, a_2)$

$= \mathcal{L}(a_1, a_2) + \mathcal{L}(a_1, b_2) + \mathcal{L}(b_1, a_2) + \mathcal{L}(b_1, b_2) - \mathcal{L}(a_1, a_2)$

$\Delta\, \mathcal{L}\big((a_1, a_2), (b_1, b_2)\big) = \mathcal{L}(a_1, b_2) + \mathcal{L}(b_1, a_2) + \mathcal{L}(b_1, b_2)$ …(1)

Suppose $\mathcal{L}$ is equably continuous and let $\mathfrak{p}_1 \in \tau_1(0)$ and ɍ$\mathfrak{q}_2 \in \tau_2(0)$.

Put $\mathfrak{q} = [0] \times \mathfrak{q}_2$ and $\mathfrak{p} = \mathfrak{p}_1 \times [0]$.

Then, ɍ $\mathfrak{q} \in (\tau_1 \times \tau_2)(0)$ and $\mathfrak{p} \in (\tau_1 \times \tau_2)(0)$ (Lemma 1.4.13 and Remark 1.4.14).

Hence from equable continuity of $\mathcal{L}$: $\Delta\, \mathcal{L}(\mathfrak{q}, \mathfrak{p}) \in \tau_3(0)$. This is a first condition.

Since, $\Delta\, \mathcal{L}(\mathfrak{q}, \mathfrak{p}) = \Delta\, \mathcal{L}(\mathfrak{p}_1, \mathfrak{q}_2)$.

Similarly: we can verify the second condition.

Conversely:

Let the conditions 1 and 2 are satisfied, and let

ɍ $\mathfrak{q} \in (\tau_1 \times \tau_2)(0)$ and $\mathfrak{p} \in (\tau_1 \times \tau_2)(0)$. Then the filters $\mathfrak{q}_i = \prod_i(\mathfrak{q})$, $\mathfrak{p}_i = \prod_i(\mathfrak{p})$ satisfy

ɍ $\mathfrak{q}_i \in \tau_i(0)$, $\mathfrak{p}_i \in \tau_i(0)$ such that $i = 1, 2$. Then by (Lemma1.3.13-1,and the equation 1) we get:

$\Delta\, \mathcal{L}(\mathfrak{q}, \mathfrak{p}) \leq \Delta\, \mathcal{L}(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{p}_1, \mathfrak{p}_2) \leq \mathcal{L}(\mathfrak{q}_1, \mathfrak{p}_2) + \mathcal{L}(\mathfrak{p}_1, \mathfrak{q}_2) + \mathcal{L}(\mathfrak{p}_1, \mathfrak{p}_2)$.

The value of right side is belonging to $\tau_3(0)$ (by assumption).

Therefore the left side is belonging to $\tau_3(0)$.

Hence the map $\mathcal{L}$ is equable continuous map.

Lemma 2.3.50:

Let $(\Xi_1\,,\tau_1)$, $(\Xi_2\,,\tau_2)$ and $(\Xi_3\,,\tau_3)$ are $T_{\tilde{p}VS}$. If the spaces Ξ_1 and Ξ_2 are equable spaces, then a bilinear map $\mathcal{L}$: $(\Xi_1\,,\tau_1) \times (\Xi_2\,,\tau_2) \to (\Xi_3\,,\tau_3)$

is equable continuous iff it is continuous at the $(0,0)$.

Proof:

The result directly follows from Proposition (2.3.42).

Conversely:

Let $\mathfrak{p}_1 \in \tau_1(0)$ and $\mathfrak{r}.\mathfrak{F} \in \tau_2(0)$. Since Ξ_1 is equable, then there exists $\mathfrak{q}_1$ with $\mathfrak{p}_1 \le \mathfrak{q}_1 = \mathfrak{r}.\mathfrak{q}_1 \in \tau_1(0)$. Then,

$$\mathcal{L}(\mathfrak{p}_1,\mathfrak{F}) \le \mathcal{L}\,(\mathfrak{r}.\mathfrak{q}_1,\mathfrak{F}) = \mathcal{L}(\mathfrak{q}_1,\ \mathfrak{r}.\mathfrak{F}) = \mathcal{L}(\mathfrak{q}_1 \times\ \mathfrak{r}.\mathfrak{F}).$$

But, $\mathfrak{q}_1 \times \mathfrak{r}.\mathfrak{F} \in (\tau_1 \times \tau_2)(0)$ (by Lemma 1.4.13), and by continuity of $\mathcal{L}$ at the zero we get:

$\mathcal{L}(\mathfrak{p}_1,\mathfrak{F}) \in \tau_3(0)$, therefore we obtain the first condition of above Lemma 2.3.49.

By the same way we can proof the other condition in this lemma, hence the map $\mathcal{L}$ is equable continuous.

Chapter Three

Smooth Maps of Class $\widetilde{C}^{\infty}$ on the Space $T_{\widetilde{p}VS}$

3.1. Introduction

This chapter consists of two sections, section one introduce the notion of remainder map such that we introduce the class of all remainder maps between $T_{\tilde{p}VS}$ spaces, and in the other section, we deal with the notion of first derivative, second derivative, higher derivative, and then we put the definition of $\tilde{C}^{\infty}$- derivatives of the maps on ${T^{\#}}_{\tilde{p}VS}$ spaces. Some definitions are constricted following A. Frölicher & B. Walter [18].

3.2. Class of Remainder Maps on $T_{\tilde{p}VS}$ Spaces:

Here, we introduced the notion of remainder map that is defined between two $T_{\tilde{p}VS}$ and has some properties. It is related with the other map θ_r in order to provide a definition of the first derivative. And there are some results about these notions.

Let $(\varXi_1,\tau_1)$ and $(\varXi_2,\tau_2)$ be two $T_{\tilde{p}VS}$. And r: $\varXi_1 \to \varXi_2$ be a mapping between two $T_{\tilde{p}VS}$ spaces. $\theta_r: R \times \varXi_1 \to \varXi_2$ defined by:

$\theta_r(\lambda, x) := \frac{r(\lambda x)}{\lambda}$ whenever $\lambda \neq 0$ and $\theta_r(\lambda, x) := 0$ whenever $\lambda = 0$.

<u>Definition 3.2.1:</u>

Let $(\varXi_1, \tau_1)$ and $(\varXi_2, \tau_2)$ be two $T_{\tilde{p}VS}$. $r: \varXi_1 \to \varXi_2$ is called a remainder iff $r(0) = 0$, $\mathfrak{r}.\mathfrak{F} \in \tau_1(0) \implies \theta_r(\mathfrak{r}, \mathfrak{F}) \in \tau_2(0)$ (θ_r is continuous at 0). And $\mathfrak{R}$ $(\varXi_1, \varXi_2)$ is called the class of all remainder maps from $(\varXi_1, \tau_1)$ to $(\varXi_2, \tau_2)$.

Proposition 3.2.2:

Let (Ξ_1 , τ_1) , (Ξ_2 , τ_2) and (Ξ_3 , τ_3) are $T_{\tilde{p}VS}$. If $r \in \mathfrak{N} (\Xi_2 ,\Xi_3)$ and

$\mathcal{L} \in \underline{L}(\Xi_1 ,\Xi_2)$ then: $r \circ \mathcal{L} \in \mathfrak{N} (\Xi_1 ,\Xi_3)$

Proof:

1- $(r \circ \mathcal{L})(0) = r(\mathcal{L}(0)) = r(0) = 0$.

2- Suppose that $\mathfrak{r}.\mathfrak{p} \in \tau_1(0)$ then, by assumption that, $\mathcal{L} \in \underline{L} (\Xi_1 ,\Xi_2)$ we get, $\theta_r(\mathfrak{r},\mathfrak{p}) \in \tau_2(0)$ and then, using the continuity at origin of $\mathcal{L}$, $\mathcal{L}(\theta_r(\mathfrak{r},\mathfrak{p})) \in \tau_3(0)$.

Hence, $\theta_{r\circ\mathcal{L}}(\mathfrak{r},\mathfrak{p}) \in \tau_3(0)$. Therefore, $r \circ \mathcal{L} \in \mathfrak{N} (\Xi_1, \Xi_3)$.

Lemma 3.2.3:

Let (Ξ_1, τ_1) and (Ξ_2 , τ_2) be two $T_{\tilde{p}VS}$.

1- For all $\alpha \in R$ and $r \in \mathfrak{N} (\Xi_1 ,\Xi_2)$, then $\alpha r \in \mathfrak{N} (\Xi_1 ,\Xi_2)$;

2- And for all $r , s \in \mathfrak{N} (\Xi_1 ,\Xi_2)$ then, $r + s \in \mathfrak{N} (\Xi_1 ,\Xi_2)$

, hence $\mathfrak{N} (\Xi_1 ,\Xi_2)$ is a vector space.

Proof:

1- It is a special case of Proposition 3.2.2.

2- From the definition of the operator θ_r it is obvious that

$$\theta_{r+s}(\alpha,x) = \frac{1}{\alpha} (r + s)(\alpha x) = \frac{1}{\alpha} r(\alpha x) + \frac{1}{\alpha} s(\alpha x)$$

$$= \theta_r(\alpha,x) + \theta_s(\alpha,x).$$

Theorem 3.2.4:

Let (Ξ_1, τ_1) , (Ξ_2, τ_2) and (Ξ_3, τ_3) are $T_{\tilde{p}VS}$, and let $r_1 \in \mathfrak{N} (\Xi_2, \Xi_3)$, if $r_2 \in \mathfrak{N}(\Xi_1, \Xi_2)$, then $r_1 \circ r_2 \in \mathfrak{N}(\Xi_1, \Xi_3)$.

Proof:

Since, $r_1 \in \mathfrak{N}(\Xi_2, \Xi_3)$ then there exist $\theta_{r_1}: R \times E_2 \to E_3$ such that

$\theta_{r_1}(\lambda, \mathfrak{F})=\begin{cases} \frac{r_1(\lambda\mathfrak{F})}{\lambda} & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases}$ is continuous at 0, and $r_1(0) = 0$, in the same a way since $r_2 \in \mathfrak{N}(\Xi_1, \Xi_2)$, then there exists a map

$\theta_{r_2}(\beta, y)=\begin{cases} \frac{r_2(\beta y)}{\beta} & \beta \neq 0 \\ 0 & \beta = 0 \end{cases}$ is continuous at 0, and $r_2(0) = 0$,

$(r_1 \circ r_2)(0) = r_1\big(r_2(0)\big) = r_1(0) = 0$, then $(r_1 \circ r_2)(0) = 0$

, take $\mathfrak{r}.\mathfrak{F} \in \tau_1(0)$:

By continuity of r_2 at 0 then, $\theta_{r_2}(\mathfrak{r}, \mathfrak{F}) \in \tau_2(0)$, $\mathfrak{r}.\theta_{r_2}(\mathfrak{r}, \mathfrak{F}) \in \tau_2(0)$.

By continuity of $r_1 \Rightarrow \theta_{r_1}(\mathfrak{r}, \theta_{r_2}(\mathfrak{r}, \mathfrak{F})) \in \tau_3(0)$

Therefore: $\theta_{r_1 \circ r_2}(\mathfrak{r}, \mathfrak{F}) \in \tau_3(0)$

$[\theta_{r_1}(\mathfrak{r}, \theta_{r_2}(\mathfrak{r}, \mathfrak{F})] = \frac{r_1(\mathfrak{r}.\theta_{r_2}(\mathfrak{r},\mathfrak{F}))}{\mathfrak{r}}$

$= \frac{r_1(\mathfrak{r}\frac{r_2(\mathfrak{r}.\mathfrak{F})}{\mathfrak{r}})}{\mathfrak{r}} = \frac{r_1(r_2(\mathfrak{r}.\mathfrak{F}))}{\mathfrak{r}} = \frac{(r_1 \circ r_2)(\mathfrak{r}.\mathfrak{F})}{\mathfrak{r}} = \theta_{r_1 \circ r_2}(\mathfrak{r}, \mathfrak{F})$.

Lemma 3.2.5:

Let (Ξ_1, τ_1), (Ξ_2, τ_2) and (Ξ_3, τ_3) are $T_{\tilde{p}VS}$. If $r_1 \in \mathfrak{N}(\Xi_1, \Xi_2)$ and

$\mathcal{L} \in \underline{L}(\Xi_1, \Xi_2)$ and $r_2 \in \mathfrak{N}(\Xi_2, \Xi_3)$ then,

$r = r_2 \circ (\mathcal{L} + r_1) \in \mathfrak{N}(\Xi_1, \Xi_3)$.

Proof:

From definition of θ_r, and the linearity of $\mathcal{L}$,

$\theta_r(\alpha, x) = \left(\theta_{(r_2\circ(\mathcal{L}+r_1))}\right)(\alpha, x) = \theta_{r_2}(\alpha, \mathcal{L}(x) + \theta_{r_1}(\alpha, x))$, therefore

$\theta_r(\mathfrak{r}, \mathfrak{B}) \leq \theta_{r_2}\left(\mathfrak{r}, \mathcal{L}(\mathfrak{B}) + \theta_{r_1}(\mathfrak{r}, \mathfrak{B})\right) = \theta_{r_2}(\mathfrak{r}, \mathbb{A})$

, such that $\mathbb{A} = \mathcal{L}(\mathfrak{B}) + \theta_{r_1}(\mathfrak{r}, \mathfrak{B})$, which satisfies:

$\mathfrak{r}.\mathbb{A} \leq \mathfrak{r}.\mathcal{L}(\mathfrak{B}) + \mathfrak{r}.\theta_{r_1}(\mathfrak{r}, \mathfrak{B})$.Assume that, $\mathfrak{r}.\mathfrak{B} \in \tau_1(0)$, we see that $\mathfrak{r}.\mathbb{A} \in \tau_2(0)$, therefore , $\theta_{r_1}(\mathfrak{r}, \mathbb{A}) \in \tau_3(0)$, so $\theta_r(\mathfrak{r}, \mathfrak{B}) \in \tau_3(0)$, hence

$r \in \mathfrak{N}(\Xi_1, \Xi_3)$.

Definition 3.2.6:

A $T_{\tilde{p}VS}$ (Ξ, τ) is called equable if and only if for each filter $\mathfrak{F}$ with

$\mathfrak{F} \in \tau(0)$,$\exists$ an equable filter p $\geq \mathfrak{F}$ with p $\in \tau(0)$

(That's mean $\mathfrak{F} \downarrow \Xi \Longrightarrow \mathfrak{F} \leq$ p=$\mathfrak{r}$p $\in \tau(0)$) , and denoted by $(\Xi^{\#}, \tau^{\#})$.

Proposition 3.2.7:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$. If $r: \Xi_1 \to \Xi_2$ is a remainder, then,

$r: \Xi_1^{\#} \to \Xi_2^{\#}$ is continuous at zero.

Proof:

Let, $\mathfrak{F} \in \tau_1^{\#}(0)$. Therefore, $\mathfrak{F} \leq \mathfrak{p} = \mathfrak{r}\mathfrak{p} \in \tau_1(0)$. From the definition of θ_r we have, 1-$r(0) = 0$;

2- $\mathrm{r}(\alpha.\mathrm{x}) = \alpha.\theta_{\mathrm{r}}(\alpha,\mathrm{x})$, and then by using (Definition1.4.11).

$r(\mathfrak{F}) \leq r(\mathfrak{p}) = r(\mathfrak{r}.\mathfrak{p}) = \mathfrak{r}.\theta_r(\mathfrak{r},\mathfrak{p})$. The filter $\xi = \mathfrak{r}.\theta_r(\mathfrak{r},\mathfrak{p})$ satisfies

$\xi = \mathfrak{r}.\xi \in \tau_2(0)$ and then, $r(\mathfrak{F}) \in \tau_2^{\#}(0)$, therefore

r: $(\Xi_1^{\#}, \tau_1^{\#}) \to (\Xi_2^{\#}, \tau_2^{\#})$ is continuous at the point zero.

Proposition 3.2.8:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) are $T_{\tilde{p}VS}$. If the linear map

$\mathcal{L}$: $(\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ is continuous, then is also the map

$\mathcal{L}$: $(\Xi_1^{\#}, \tau_1^{\#}) \to (\Xi_2^{\#}, \tau_2^{\#})$ is continuous.

Proof:

Let,$\mathfrak{q} \in \tau_1^{\#}(0)$.Then,$\mathfrak{q} \leq \mathfrak{p} = \mathfrak{r}\mathfrak{p} \in \tau_1(0)$,
And therefore, by (Definition1.4.11),
$\mathcal{L}(\mathfrak{q}) \leq \mathcal{L}(\mathfrak{p}) = \mathcal{L}(\mathfrak{r}\mathfrak{p}) = \mathfrak{r}\mathcal{L}(\mathfrak{p}) \in \tau_2(0)$.Hence, $\mathcal{L}(\mathfrak{q}) \in \tau_2^{\#}(0)$.
Therefore, $\mathcal{L}$: $(\Xi_1^{\#}, \tau_1^{\#}) \to (\Xi_2^{\#}, \tau_2^{\#})$ is continuous map.

Proposition 3.2.9:

Let (Ξ_1, τ_1) , (Ξ_2, τ_2) and (Ξ_3, τ_3) are $T_{\tilde{p}VS}$. If

B: $(\Xi_1, \tau_1) \times (\Xi_2, \tau_2) \to (\Xi_3, \tau_3)$ is bilinear and continuous at the $(0,0)$, then B: $(\Xi_1^{\#}, \tau_1^{\#}) \times (\Xi_2^{\#}, \tau_2^{\#}) \to (\Xi_3^{\#}, \tau_3^{\#})$ is continuous and so it is an equable continuous.

Proof:

It is sufficient to show that B: $(\Xi_1^{\#}, \tau_1^{\#}) \times (\Xi_2^{\#}, \tau_2^{\#}) \rightarrow (\Xi_3^{\#}, \tau_3^{\#})$ is continuous at the point (0,0). Let, $\mathfrak{F}_i \in \tau_i^{\#}(0), i = 1, 2$.

Then, $\mathfrak{F}_i \leq \mathfrak{p}_i = \mathfrak{r} . \mathfrak{p}_i \in \tau_i^{\#}(0)$,and

$B(\mathfrak{F}_1, \mathfrak{F}_2) \leq B(\mathfrak{p}_1, \mathfrak{p}_2) = B(\mathfrak{r} . \mathfrak{p}_1, \mathfrak{r} . \mathfrak{p}_2) = \mathfrak{r} . B(\mathfrak{p}_1, \mathfrak{p}_2) \in \tau_3(0)$.

Therefore, $B(\mathfrak{F}_1, \mathfrak{F}_2) \in \tau_3^{\#}(0)$.

Hence B is continuous at the origin (0,0).

Lemma 3.2.10:

Let (Ξ_1, τ_1) and (Ξ_2 , τ_2) be two $T_{\tilde{p}VS}$. The only remainder linear map

$r : \Xi_1 \rightarrow \Xi_2$, is the zero map.

Proof:

Let, $x \in \Xi_1$, $\mathfrak{r}[x] \in \tau_1(0)$, since r is remainder then there exist

$\theta_r(\mathfrak{r}, [x]) \in \tau_2(0)$ Whenever, $\mathfrak{r}[x] \in \tau_1(0)$

$$\theta_r(\lambda, [x])= \begin{cases} \frac{r(\lambda[x])}{\lambda} & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases}$$

$$= \begin{cases} \frac{\lambda r([x])}{\lambda} & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases} \text{ (by linearity of } r)$$

$= r([x]) = [r(x)]$.

$\theta_r(\lambda, [x]) \in \tau_2(0)$

Then, $[r(x)] \in \tau_2(0)$.

Hence, $r(x) = 0$ (Because Ξ_2 is separated).

Lemma 3.2.11:

If (Ξ, τ), (Ξ_1, τ_1), (Ξ_2, τ_2), …, (Ξ_n, τ_n) are $T_{\tilde{p}VS}$, then a map

$r: \Xi \to \prod_{k=1}^{n} \Xi_k$ is a remainder if and only if each of $r_k: \Xi \to \Xi_k$ are remainders, $1 \leq k \leq n$, where $r(x) = (r_1(x), \dots, r_n(x))$ for all $x \in \Xi$.

Proof:

Suppose there is some maps $\theta_r: R \times \Xi \to \prod_{k=1}^{n} \Xi_k$ such that

$\theta_r(\lambda, x) = \begin{cases} \frac{r(\lambda x)}{\lambda} & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases}$ is continuous at 0 and $r(0) = 0$,

$\theta_r(x) = (\theta_{r_1}(x), \dots, \theta_{r_n}(x))$,

$\theta_{r_i}(\lambda, x) = \begin{cases} \frac{r_i(\lambda x)}{\lambda} & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases}$ for all $x \in \Xi$, for all $\mathfrak{r}B \downarrow \Xi$.

By continuity of θ_r we have, $\theta_r(\mathfrak{r}, B) \downarrow \prod_{k=1}^{n} \Xi_k$.

Then, $\theta_{r_i}(\mathfrak{r}, B) \downarrow \Xi_i$ for all $i = 1,2,\dots,n$, this mean $\theta_{r_i}(\mathfrak{r}, B)$ is continuous at 0 for all $i = 1,2,\dots,n$, and finally $0 = r(0) = (r_1(0), \dots, r_n(0))$, then, $r_i(0) = 0$

, for all $i = 1,2,\dots,n$.

Therefore, $r_k: \Xi \to \Xi_k$ are remainders, for $1 \leq k \leq n$

Conversely:

Suppose $r_k \in \mathfrak{R}(\Xi_k, \Xi_k)$ for all $k = 1,2,\dots,n$, then there exists

$\theta_{r_k}(\lambda, x)=\begin{cases} \frac{r_k(\lambda x)}{\lambda} & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases}$ is continuous at 0

and $r_k(0) = 0$, for all $k = 1,2,\dots,n$, take,

$\theta_r(x) = (\theta_{r_1}(x), \dots., \theta_{r_n}(x))$,

For all $\text{ɍ}B \downarrow \Xi_i$, and by continuous of E_i for all $i = 1,2,\dots,n$.

$\theta_{r_i}(\text{ɍ}, B) \downarrow \Xi_i$ for all $i = 1,2,\dots,n$, hence:

$\theta_r(\text{ɍ}, B) \downarrow \prod_{k=1}^{n} \Xi_k$, and from $r_k(0) = 0$, for all $k = 1,2,\dots,n$,

Then, $r(0)= 0$. Therefore, $r: \Xi \to \prod_{k=1}^{n} \Xi_k$ is remainder.

$r \in \mathfrak{R}\,(\Xi, \prod_{k=1}^{n} \Xi_k)$.

Lemma 3.2.12:

Let (Ξ_1, τ_1) , (Ξ_2, τ_2) and (Ξ_3, τ_3) be a $T_{\tilde{p}VS}$. Let $\mathfrak{b}: \Xi_1 \times \Xi_2 \to \Xi_3$ be bilinear and continuous at the point zero. Then $\mathfrak{b}$ is quasi bounded.

Proof:

Let, $\text{ɍ}.\mathfrak{F} \in \tau_1(0) \times \tau_2(0)$. Since $\mathfrak{F}_i = \prod_i(\mathfrak{F})$ satisfied:

$\text{ɍ}.\mathfrak{F}_i \in \tau_i, i = 1, 2$ and $\mathfrak{F} \leq \mathfrak{F}_1 \times \mathfrak{F}_2$(From Definition1.3.11). By using $\text{ɍ} = \text{ɍ}.\text{ɍ}$ and by linearity of $(\mathfrak{b})$

$\alpha.\beta.\mathfrak{b}(m_1, m_2) = \mathfrak{b}(\alpha.m_1, \beta.m_2)$ then we get:

$\text{ɍ}.\mathfrak{b}(\mathfrak{F}) \leq \text{ɍ}.\mathfrak{b}(\mathfrak{F}_1, \mathfrak{F}_2) = \text{ɍ}.\text{ɍ}.\mathfrak{b}(\mathfrak{F}_1, \mathfrak{F}_2) = \mathfrak{b}(\text{ɍ}.\mathfrak{F}_1, \text{ɍ}.\mathfrak{F}_2)$then, from continuity of $\mathfrak{b}$ at the zero, the filter in the left side $\text{ɍ}.\mathfrak{b}(\mathfrak{F})$ belongs to $\tau_3(0)$ and also that the filter on the right side belongs to $\tau_3(0)$.

Definition 3.2.13:

1. The class $\tilde{C}^0(\Xi_1,\Xi_2)$ is a vector space $\tilde{C}_e^q(\Xi_1;\Xi_2)$ together with the pseudo topology induced by the inclusion, such that $\mathcal{X} \downarrow \tilde{C}^0(\Xi_1 ,\Xi_2)$ if and only if

($\forall\mathbb{B} \downarrow \Xi_1 \implies \mathcal{X}(\mathbb{B}) \downarrow \Xi_2$) where $\mathcal{X}$ is any filter on Ξ_1.

2. The class $L(\Xi_1 ,\Xi_2)$ is a vector space $\underline{L}(\Xi_1 ,\Xi_2)$ (linear and continuous maps) together with the pseudo topology induced by the inclusion, such that

$\mathcal{X} \downarrow L(\Xi_1;\Xi_2)$ if and only if ($\forall\mathcal{A} \downarrow E_1 \implies \mathcal{X}(\mathcal{A}) \downarrow E_2$).

3. Define the class $L_m(\Xi_1;\Xi)$ by the vector space $\underline{L}(\Xi_1,\Xi_2,\dots,\Xi_m;\Xi)$ consisting of the equable continuous multilinear maps from

$\Xi_1 \times \Xi_2 \times \dots \times \Xi_m$ into Ξ, together with the pseudo topology induced by the inclusion of $\underline{L}(\Xi_1,\Xi_2,\dots,\Xi_m;\Xi)$ such that $\Xi_1 = \Xi_2 = \dots = \Xi_m$, $m \in Z^+$, [$L_m(\Xi_1;\Xi) = L(\Xi_1,\Xi_2,\dots,\Xi_m;\Xi)$].We have,

$\mathcal{X} \downarrow L(\Xi_1,\Xi_2,\dots,\Xi_m;\Xi)$ if and only if

($\forall\mathcal{A}_i \downarrow E_i$ for $i = 1,2,\dots,m \implies \mathcal{X}(\mathcal{A}_1,\mathcal{A}_2,\dots,\mathcal{A}_m) \downarrow \Xi$).

4. $L^{\#}{}_m(\Xi_1;\Xi)$ is an equable a class $L_m(\Xi_1;\Xi)$.

3.3. The $\widetilde{C}^{\infty}$ - Differentiability and Derivatives

In this section we defined the first and second derivative and the derivative to m-time, and then defined the new class of maps that which contains all differential maps of the infinitive between the spaces $T^{\#}{}_{\tilde{p}VS}$ $\tilde{C}^{\infty}$-maps. Some definitions are constricted following A. Frölicher & B. Walter [18].

Definition 3.3.1:

Let (Ξ_1, τ_1) , (Ξ_2, τ_2) be a $T_{\tilde{p}VS}$. The map $f:(\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ is differentiable at the point a if there exist a map $\ell \in \underline{L}(\Xi_1; \Xi_2)$ such that the map r is defined by $f(a+h) = f(a) + \ell(h) + r(h)$ is remainder

The map $\ell \in \underline{L}(\Xi_1; \Xi_2)$ is uniquely determined, it is then called the derivative of f at the point a, and it is denoted by: $\ell = \acute{f}(a)$.

Proposition 3.3.2:

Let (Ξ_1,τ_1) , (Ξ_2, τ_2) be $T_{\tilde{p}VS}$. If f: $(\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ is a differentiable at the point a, then f: $(\Xi_1^{\#}, \tau_1^{\#}) \to (\Xi_2^{\#}, \tau_2^{\#})$ is continuous at the point a.

Proof:

From the above definition, $f\ (a+h) = f(a) + \ell(h) + r(h)$, where

$\ell \in \underline{L}(\Xi_1; \Xi_2)$ and $r \in \mathfrak{R}\ (\Xi_1, \Xi_2)$ then by the Proposition 3.2.8, we get,

f: $(\Xi_1^{\#}, \tau_1^{\#}) \to (\Xi_2^{\#}, \tau_2^{\#})$ is continuous at the point a.

Proposition 3.3.3:

If (Ξ_1, τ_1) and (Ξ_2, τ_2) are $T_{\tilde{p}VS}$ then the linear continuous map

$f: \Xi_1 \to \Xi_2$ is differentiable at each point in Ξ_1, and a remainder map is linear then, $\ell(h) = f(h)$.

Proof:

$f\,(a+h) = f(a) + \ell(h) + r(h)$

Since f is linear map then, $f\,(a+h) = f\,(a) + f(h), r(h) = 0$

$f\,(a) + f(h) = f(a) + \ell(h) + 0$

$f(h) = \ell(h)$.

Proposition 3.3.4:

If $\mathfrak{b}: \Xi_1 \times \Xi_2 \to \Xi_3$ is bilinear continuous, then $\mathfrak{b}$ is differentiable at each point $m = (m_1, m_2) \in \Xi_1 \times \Xi_2$, and $\mathfrak{b}'(m_1, m_2)(h_1, h_2)$= $\mathfrak{b}(h_1, m_2)$ + $\mathfrak{b}(m_1, h_2)$.

Proof:

Let $h = (h_1, h_2) \in \Xi_1 \times \Xi_2$, $\mathfrak{b}(m+h) = \mathfrak{b}(\,m_1 + h_1, m_2 + h_2)$

$= \mathfrak{b}(m_1, m_2) + \mathfrak{b}(h_1, m_2) + \mathfrak{b}(m_1, h_2) + \mathfrak{b}(h_1, h_2)$. Therefore we have, with $\ell(h) = \ell(h_1, h_2) = \mathfrak{b}(h_1, m_2) + \mathfrak{b}(m_1, h_2)$, and

$r(h) = r(h_1, h_2) = \mathfrak{b}(h_1, h_2)$.

$\mathfrak{b}(m+h) = \mathfrak{b}(m) + \ell(h) + r(h)$.

ℓ is linear, and since $\mathfrak{b}$ is bilinear, and also continuous, since $\mathfrak{b}$ is continuous. Therefore, $\ell \in \underline{L}(\Xi_1 \times \Xi_2;\ \Xi_3)$, and $r \in \mathfrak{R}\ (\Xi_1 \times \Xi_2;\ \Xi_3)$.

Proposition 3.3.5:

Let Ξ be a $T_{\tilde{p}VS}$ .If $f: R \to \Xi$ is differentiable at the point $a \in R$, then the following limit, which we denote by $f'(m)$, exists:

$f'(m) = \lim_{h \to 0} \frac{f(m+h)-f(m)}{h}$, and $f'(m)(h) = h. f'(m)$;

$f'(m) = f'(m)(1)$.

Proof:

We have, $f(m+h) = f(m) + \ell(h) + r(h)$, such that $\ell \in \underline{L}\ (R; \Xi)$ and

$r \in \mathfrak{R}\ (R; \Xi)$. Put $\ell(1) = f'(m). 1 = w$, and define the map $g: R \to \Xi$ as follows:

$$g(h) = \begin{cases} \frac{f(m+h)-f(m)}{h} & if\ h \neq 0 \\ w & if\ h = 0 \end{cases}, \quad \text{one has}$$

$\ell(h) = \ell(h. 1) = h. \ell(1) = h. w.$

$r(h) = f(m+h) - f(m) - h. w,$

$\theta_r(\beta, h) = \frac{f(m+\beta h)-f(m)}{\beta} - h. w,$

$\theta_r(\beta, 1) = g(\beta) - w.$

$ɍ. 1 \downarrow$ R we have $\theta_r(ɍ, 1) \downarrow \Xi$ (by definition of the θ_r , then,

$g(ɍ) - w \downarrow \Xi$ respectively $g(ɍ) \downarrow_w \Xi$, therefore $\lim_{h \to 0} g(h) = w$.

The proof is complete.

Remark 3.3.6:

If f is differentiable at a point m, then the uniqueness of the derivative implies the uniqueness of the remainder r.

Lemma 3.3.7:

Let (Ξ_1, τ_1), (Ξ_2, τ_2) are $T_{\tilde{p}VS}$, let $x_\circ \in \Xi_1$, if

$f, g: (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ are both differentiable at $x_\circ$ and if $a \in R$, then $af + g$ is differentiable at $x_\circ$ and $D(af + g)(x_\circ) = aDf(x_\circ) + Dg(x_\circ)$.

Proof:

Since f and g are differentiable at $x_\circ$ then there exists $r_1, r_2 \in \mathfrak{R}(\Xi_1, \Xi_2)$ such that, $f(x) = f(x_\circ) + l_1(x_\circ)h + r_1(h)$,

$g(x) = g(x_\circ) + l_1(x_\circ)h + r_2(h), , h \in \Xi_1$, and $l_1, l_2 \in L(\Xi_1; \Xi_2)$

$(af + g)(x) = af(x) + g(x)$

$= af(x_\circ) + al_1(x_\circ)h + ar_1(h) + g(x_\circ) + +l_2(x_\circ)h + r_2(h)$

$= (af(x_\circ) + g(x_\circ)) + (al_1(x_\circ)h + l_2(x_\circ)h) + (ar_1(h) + r_2(h))$

$= (af + g)(x_\circ) + (al_1 + l_2)(x_\circ)h + (ar_1 + r_2)(h)$

Since $al_1 + l_2 \in L(\Xi_1; \Xi_2)$ and $r_1 + r_2 \in \mathfrak{R}(\Xi_1, \Xi_2)$, then, $al_1 + l_2$ is derivative of $af + g$ at $x_\circ$.
Hence, $D(af + g)(x_\circ) = aDf(x_\circ) + Dg(x_\circ)$.

Theorem 3.3.8:

Let $(\Xi_1 \ , \ \tau_1)$, $(\Xi_2 \ , \ \tau_2)$ are $T_{\tilde{p}VS}$ and $f_1 , f_2 : (\Xi_1 , \tau_1) \rightarrow (\Xi_2 , \tau_2)$ are differentiable at $x_\circ \in \Xi_1$, then $f_1 \cdot f_2$ is differentiable at $x_\circ$, and its derivative at $x_\circ$ is

$$D(f_1 \cdot f_2)(x_\circ) = f_2(x_\circ)Df_1(x_\circ) + f_1(x_\circ)Df_2(x_\circ) .$$

Proof:

Since $f_1 , f_2$are differentiable at $x_\circ$, then there exists $r_1, r_2 \in \mathfrak{R} \ (\Xi_1 , \Xi_2)$

, such that, $f_1(x) = f_1(x_\circ) + l_1(x_\circ)h + r_1(h)$,

$f_2(x) = f_2(x_\circ) + l_2(x_\circ)h + r_2(h)$ for all $x, h \ \in \Xi_1$,

$(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) =$

$f_1(x_\circ) \cdot f_2(x_\circ)$+$f_1(x_\circ)l_2(x_\circ)h$+$f_1(x_\circ)r_2(h)$+$l_1(x_\circ)hf_2(x_\circ)$+ $l_1(x_\circ)hl_2(x_\circ)h$+ $l_1(x_\circ)hr_2(h)$+ $r_1(h)f_2(x_\circ)$+ $r_1(h)l_2(x_\circ)h$+ $r_1(h)r_2(h)$

Put: r^1= $f_1(x_\circ)r_2(h)$+ $l_1(x_\circ)hl_2(x_\circ)h$+ $l_1(x_\circ)hr_2(h)$+ $r_1(h)f_2(x_\circ)$+

$r_1(h)l_2(x_\circ)h$+ $r_1(h)r_2(h)$

To prove $r^1$is remainder map,

It's sufficient to prove $l_1(x_\circ)hl_2(x_\circ)h \in \mathfrak{R} \ (\Xi_1 , \Xi_2)$

Put, $r = l_1(x_\circ)hl_2(x_\circ)h$,

Define, $\theta_r(\lambda, x) = \begin{cases} \frac{r(\lambda x)}{\lambda} & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases}$

$$=\begin{cases}\frac{l_1(\lambda x)l_2(\lambda x)}{\lambda} & \lambda \neq 0 \\ 0 & \lambda = 0\end{cases}$$

$$=\begin{cases}\frac{\lambda^2 l_1(x)l_2(x)}{\lambda} & \lambda \neq 0 \\ 0 & \lambda = 0\end{cases}$$

$$= \lambda l_1(x)l_2(x)$$

If $ꝛ.B \in \tau_1(0)$, then $\theta_r(ꝛ, B) = ꝛ.l_1(B)l_2(B)$ for all $B \in \Xi_1$

$ꝛ.l_1(B)l_2(B) \in \tau_2(0)$.

Therefore $\theta_r(ꝛ, B) \in \tau_2(0)$, hence θ_r is continuous at 0, and so

$r \in \mathfrak{N}\ (\Xi_1, \Xi_2)$, then proof is completed.

Theorem(chain roul) 3.3.9:

Let (Ξ_1, τ_1) , (Ξ_2,τ_2) are $T_{\tilde{p}VS}$ and $f : (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ is differentiable at $x \in \Xi_1$, and $g : (\Xi_2, \tau_2) \to (\Xi_3, \tau_3)$ is differentiable at

$y = f(x) \in \Xi_2$, then, $g \circ f$ is differentiable at $x_\circ$, and its derivative at x is

$D(g \circ f)(x) = Dg(y).Df(x)$.

Proof:

By assumption one has:

$f(x+h) = f(x) + l_1(h) + r_1(h)$, $g(x+k) = g(y) + l_2(k) + r_2(k)$

Where:

$l_1 = Df(x) \in \underline{L}(\Xi_1; \Xi_2)$, $r_1 \in \mathfrak{N}\ (\Xi_1, \Xi_2)$

$l_2 = Dg(y) \in \underline{L}(\Xi_2; \Xi_3)$, $r_2 \in \mathfrak{R}\,(\Xi_2\,, \Xi_3)$.

Composing the mapping one gets, using the linearity of l_2:

$(g \circ f)(x+h) = (g \circ f)(x) + l(h) + r(h)$ where $l = l_2(\, l_1(h))$,

And $r(h) = l_2\big(\, r_1(h)\big) + r_2(l_1(h) + r_1(h))$.

Obviously, $l = l_2 \circ l_1 \in \underline{L}(\Xi_1; \Xi_3)$. By using Proposition 3.2.2, Lemma 3.2.5 and Lemma 3.1. 3, $r \in \mathfrak{R}\,(\Xi_1\,, \Xi_3)$, hence, the proof is complete.

Remark 3.3.10:

Let $\mathcal{S}: \Xi_1 \times \Xi_2 \to \Xi_3$ be a map of two variables, we can take a partial derivative with respect to just one or the other.

Fixed, $(x, y) \in \Xi_1 \times \Xi_2$, the partial mapping $a \mapsto \mathcal{S}(a, y)$ and $b \mapsto \mathcal{S}(x, b)$. If we get the differentiable of these maps at x, y respectively, $\mathcal{S}$ is called partially differentiable at $(x, y) \in \Xi_1 \times \Xi_2$, and the derivatives being are $D_x\mathcal{S}(x, y)$, $D_y\mathcal{S}(x, y)$ respectively.

Definition 3.3.11:

Let $\mathcal{S}: \Xi_1 \times \Xi_2 \to \Xi_3$ be a bilinear map, define the partially differentiable at $(x, y) \in \Xi_1 \times \Xi_2$ as follows:

$$\mathcal{S}(x+h, y) = \mathcal{S}(x, y) + D_x\mathcal{S}(x, y)h + r(h)$$

$$\mathcal{S}(x, y+k) = \mathcal{S}(x, y) + D_y\mathcal{S}(x, y)k + r(k).$$

Theorem 3.3.12:

Let $\mathcal{S}: \Xi_1 \times \Xi_2 \to \Xi_3$ be a map of two variables. Then $D\mathcal{S}(x,y)$ exists and is continuous iff $D_x\mathcal{S}(x,y)$ and $D_y\mathcal{S}(x,y)$exists and are continuous.

Proof:

Fixed, (h,k), $D\mathcal{S}(x,y)(h,k) = \mathcal{S}(x+h, y+k) - \mathcal{S}(x,y) - r(h,k)$,

Put $k = 0$ then, $D\mathcal{S}(x,y)h = \mathcal{S}(x+h,y) - \mathcal{S}(x,y) - r(h,) = D_x\mathcal{S}(x,y)$,

And in the same way if we put $h = 0$,

$D\mathcal{S}(x,y)k = \mathcal{S}(x, y+k) - \mathcal{S}(x,y) - r(\ k)$= $D_y\mathcal{S}(x,y)$.

Therefore, $D\mathcal{S}(x,y) = D_x\mathcal{S}(x,y)$ + $D_y\mathcal{S}(x,y)$. This expression achieves the wanted result.

Theorem 3.3.13:

If $\mathcal{S}(x)h$ is $\tilde{C}^1$ and linear in h then $D\mathcal{S}(x)\{h,k\}$ is bilinear, (That's mean linear separately in h and k).

Proof:

The total derivative is linear, then is the partial derivative with respect to x.

Thus, $D\mathcal{S}(x)\{h, k_1 + k_2\} =\ D\mathcal{S}(x)\{h,k_1\}$ + $D\mathcal{S}(x)\{h,k_2\}$.

To see the linearity in h, by the definition of the derivative:

$D\mathcal{S}(x)\{h_1 + h_2, k\} =\ \mathcal{S}(x + h_1 + h_2, y + k) - \mathcal{S}(x) - r(h_1 + h_2, k)$

$= \mathcal{S}(x + h_1, y + k) - \mathcal{S}(x) - r(h_1, k)$+$\mathcal{S}(x + h_2, y + k) - \mathcal{S}(x) - r(h_2, k)$

= $\mathcal{S}(x+k)(h_1)\ - \mathcal{S}(x)(h_1) - r(h_1,k) + \mathcal{S}(x+k)(h_2)\ - \mathcal{S}(x)(h_2)$

$-r(h_2,k)$

= $D\mathcal{S}(x)\{\ h_1, k\}$ + $D\mathcal{S}(x)\{\ h_2, k\}$.

Definition 3.3.14:

A $T_{\tilde{p}VS}\ \Xi$ is called admissible if and only if it satisfies the condition:

If $\mathfrak{F} \downarrow \Xi$ then, $CL(\mathfrak{F}) \downarrow \Xi$ (That is a $T_{\tilde{p}VS}\ \Xi$ is p-regular).

Notation 3.3.15:

Define the set of all equable and p-regular $T_{\tilde{p}VS}$ as ${T^{\#}}_{\tilde{p}VS}$, and from now on we suppose that $\Xi, \Xi_1, \Xi_2, \ldots, F, G, \ldots$ are always ${T^{\#}}_{\tilde{p}VS}$

Definition 3.3.16:

Let, (Ξ_1,τ_1) and (Ξ_2,τ_2) be a ${T^{\#}}_{\tilde{p}VS}$, a map $f: (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ is twice differentiable at a point $a \in \Xi_1$, then: $D^2 f(a)$ is derivative of the first derivative of f at the point a.

Definition 3.3.17:

Let $(\Xi_1,\ \tau_1)$ and $(\Xi_2,\ \tau_2)$ be a ${T^{\#}}_{\tilde{p}VS}$, a map $f: \Xi_1 \to \Xi_2$ is called twice differentiable at a point $a \in \Xi_1$, if and only if $Df: \Xi_1 \to L^{\#}(\Xi_1; \Xi_2)$ exists and is differentiable at a.

$D^2 f(a) \in L^{\#}(\Xi_1; L^{\#}(\Xi_1; \Xi_2)) = {L^{\#}}_2(\Xi_1; \Xi_2)$

We say f is $\tilde{C}^2$ if $D\mathcal{S}$ is $\tilde{C}^1$, which is true if and only if $D^2 f$ exists and it is continuous.

Definition 3.3.18:

Let $\mathcal{S}: \Xi_1 \times \Xi_2 \to \Xi_3$be a bilinear map on a ${T^{\#}}_{\tilde{p}VS}$ then,

$D\mathcal{S}(x)\{h, k\} = \mathcal{S}(x + k)h - \mathcal{S}(x) - r(k).$

Theorem 3.3.19:

If $\mathcal{S}: \Xi_1 \times \Xi_2 \to \Xi_3$, is $\tilde{C}^2$ then,

$D^2\mathcal{S}(x)\{h,k\} = \mathcal{S}(x+h+k) - \mathcal{S}(x+h) - \mathcal{S}(x+k) + \mathcal{S}(x) - r(k)$.

Proof:

Let, $x \in \Xi_1$,

$D^2\mathcal{S}(x)\{h,k\} = D[D\mathcal{S}(x)\{h,k\}]$

$= D[\mathcal{S}(x+k)h - \mathcal{S}(x) - r(k)]$

$= D\mathcal{S}(x+k)h - D\mathcal{S}(x) - Dr(k)$

$= \mathcal{S}(x+k+h) - \mathcal{S}(x+k) - r(h) - [\mathcal{S}(x+h) - \mathcal{S}(x) - r(h)] - r(k)$

$= \mathcal{S}(x+k+h) - \mathcal{S}(x+h) - \mathcal{S}(x+k) + \mathcal{S}(x) - r(k)$.

Corollary 3.3.20:

If $\mathcal{S}: \Xi_1 \times \Xi_2 \to \Xi_3$ is $\tilde{C}^2$ then the second derivative is symmetric,

$D^2\mathcal{S}(x)\{h,k\} = D^2\mathcal{S}(x)\{k,h\}$

Proof:

It is clear by the definition of derivative.

Corollary 3.3.21:

If $\mathcal{S}: \Xi_1 \times \Xi_2 \to \Xi_3$, is $\tilde{C}^2$, then $D^2\mathcal{S}(x)\{h,k\}$ is bilinear, (linear separately in h and k).

Proof:

We get the result immediately from Theorem 3.3.13 and Definition 3.3 .18.

Definition 3.3.22:

The derivative of the second derivative is a third derivative,

$D^2 f: \Xi_1 \to L^{\#}{}_2(\Xi_1; \Xi_2)$ exists and is differentiable at a.

$D^2 f(a) \in L^{\#}(\Xi_1, L^{\#}(\Xi_1; \Xi_2)) \cong L^{\#}{}_2(\Xi_1; \Xi_2)$.

$D(D^2 f(a)) = D^3 f(a) \in L^{\#}{}_3(\Xi_1; \Xi_2)$.

So we can generalize as follows:

A map $f\colon \Xi_1 \to \Xi_2$ of $\tilde{C}$ is called m-times differentiable at a, if and only if

$D^{(m-1)} f\colon \Xi_1 \to L^{\#}{}_{m-1}(\Xi_1; \Xi_2)$ exists and is differentiable at a.

$D^{(m-1)} f(a) \in L^{\#}(\Xi_1; L^{\#}{}_{m-1}(\Xi_1; \Xi_2)) = L^{\#}{}_m(\Xi_1; \Xi_2)$.

Theorem 3.3.23:

If $f, g\colon \Xi_1 \times \Xi_2 \to \Xi_3$ are $\tilde{C}^2$ then,

$D^2[g \circ f](m)\{h, k\} =$

$D^2 g\big(f\ (m)\big)\{\ D\ f(m)h, D\ f(m)k\} + D\ g\big(f\ (m)\big) D^2 f(m)\{h, k\}$.

Proof:

By applying the chain rule we get what is required.

$D[g \circ f](m)h = D\ g\big(f\ (m)\big) D\ f\ (m)h$ to itself.

Proposition3.3.24:

If $f\colon \Xi_1 \to \Xi_2$ is m-times differentiable, and $D^{(m)} f\colon \Xi_1 \to L^{\#}{}_m(\Xi_1; \Xi_2)$ is $n-times$ differentiable, then, f is $(m+n)-times$ differentiable.

Proof:

$D^{(m)} f\colon \Xi_1 \to L^{\#}{}_m(\Xi_1; \Xi_2)$ being $n-times$ differentiable,

$(D^{(m)})^{(n-1)} f\colon \Xi_1 \to L^{\#}{}_{n-1}(\Xi_1; L^{\#}{}_m(\Xi_1; \Xi_2))$ is differentiable. Hence,

$D^{(m+n-1)} f\colon \Xi_1 \to L^{\#}{}_{m+n-1}(\Xi_1; \Xi_2)$ is differentiable, so:

$D^{(m+n)} f \in L^{\#}{}_{m+n}(\Xi_1; \Xi_2)$. (That's mean f is $(m+n)-times$ differentiable).

Remark 3.3.25:

$\mathcal{F} \downarrow \tilde{C}^m(\Xi_1;\Xi_2)$ if and only if ($\mathcal{F}^{(n)} \downarrow \tilde{C}^0(\Xi_1; L^{\#}{}_m(\Xi_1;\Xi_2)$)

for $n = 0, 1, 2, \dots, m$.

Remark 3.3.26:

The class $\tilde{C}^m(\Xi_1,\Xi_2)$ is an admissible vector space, and

$\tilde{C}^{m+n}(\Xi_1,\Xi_2) \subset \tilde{C}^m(\Xi_1,\Xi_2)$.

Theorem 3.3.27:

If f , $g: \Xi_1 \to \Xi_2$ are $\tilde{C}^m(\Xi_1,\Xi_2)$, so $f \circ g$ is $\tilde{C}^m(\Xi_1,\Xi_2)$,

and $D^m(f \circ g) = (D^m f) \circ (D^m g)$.

Proof:

If $m = 1$, by chain rule the statement is true:

$D(f \circ g)(a) = (Df)(a) \circ (Dg)(a)$, for all $a \in \Xi_1$.

The higher cases follow from induction on m.

Definition 3.3.28:

The map f between Ξ_1 and Ξ_2 is $\tilde{C}^\infty(\Xi_1,\Xi_2)$ is called smooth map if it is $\tilde{C}^m(\Xi_1,\Xi_2)$ for all $m \in \mathbb{N}$.

Define $\tilde{C}^\infty(\Xi_1,\Xi_2)$ to be the projective limit of this system, $\tilde{C}^\infty(E_1,E_2)=\bigcap_{m=0}^{\infty} \tilde{C}^m(\Xi_1,\Xi_2)$,

$\mathcal{F} \downarrow \tilde{C}^\infty(\Xi_1,\Xi_2)$ iff $\mathcal{F} \downarrow \tilde{C}^m(\Xi_1,\Xi_2)$ for all $m \in \mathbb{N}$.

$\tilde{C}^\infty(\Xi_1,\Xi_2) \subset \tilde{C}^m(\Xi_1,\Xi_2) \subset \tilde{C}^0(\Xi_1,\Xi_2)$

Definition 3.3.29:

Let Ξ_1, Ξ_2 are $T^{\#}{}_{\tilde{p}VS}$, and let $\mathbb{f}: \Xi_1 \rightarrow \Xi_2$ be a bijection map, we say that $\mathbb{f}$ is a diffeomorphism if $\mathbb{f}$ and $\mathbb{f}^{-1}$ are differentiable of class $\tilde{C}^{\infty}(\Xi_1, \Xi_2)$.

Example 3.2.30:

If U, V are connected open subsets of R^n , a map $id : U \rightarrow V$ is identity map, id is diffeomorphism map.

Chapter Four

Pseudo Manifolds of Class $\widetilde{C}^{\infty}$

4.1. Introduction

The chapter includes three sections. In the first and second sections, we follow the traditional method of establishing the basic definitions of a manifold modeled on a model space ${T^{\#}}_{\tilde{p}VS}\ \Xi$, that is called pseudo manifold of class $\tilde{C}^{\infty}$ and it is the pair $(M, \mathcal{A})$, where M is pseudo topological manifold and $\mathcal{A}$ is a $\tilde{C}^{\infty}$-structure (maximal smooth structure).

In the third section, we deal with basic conceptions of differential geometry, such as the tangent space, vector field, and others.

4.2. Basic $\tilde{C}^{\infty}$-Structures

In this section we will define the basic structures of class- $\tilde{C}^{\infty}$ which define the $\tilde{C}^{\infty}$-manifolds like, $\tilde{C}^{\infty}$-chart, $\tilde{C}^{\infty}$-atlas and others. We will present these definitions in a similar way to the definitions present in [6, 11, 29, 31, 32].

Definition 4.2.1:

A ${T^{\#}}_{\tilde{p}VS}\ \Xi$ is connected if it is not the disjoint union of nonempty open subsets. A subset of a ${T^{\#}}_{\tilde{p}VS}\ \Xi$ is said to be connected if it is connected with the subspace ${T^{\#}}_{\tilde{p}VS}$.

Definition 4.2.2:

Let Ξ be a ${T^{\#}}_{\tilde{p}VS}$ and $m \in \Xi$. The component containing m, denoted $CO(m)$, is defined to be the union of all connected subsets of Ξ that contain m.
A subset of a ${T^{\#}}_{\tilde{p}VS}\ \Xi$ is a (connected) component if it is $CO(m)$ for some

$m \in \varXi$.

Definition 4.2.3:

A $T^{\#}{}_{\tilde{p}VS}\ \varXi$ is called locally connected if the pseudo topology has a basis consisting of connected sets. If a $T^{\#}{}_{\tilde{p}VS}\ \varXi$ is locally connected, then the connected components of each open set (considered with its subspace $T^{\#}{}_{\tilde{p}VS}$) are all open. In particular, each connected component of a locally connected space is open (and also closed).

Definition 4.2.4:

Let M be a pseudo topological space. A chart (or coordinate chart) in M is a pair (u, α) where u is an open subset of M and α is a homeomorphism map of u onto $\alpha(u)$ open subset of some $T^{\#}{}_{\tilde{p}VS}\ \varXi$.

For any $p \in M$, a chart (u, α), is a chart (coordinate chart) at p if and only if $p \in u$, (if $\alpha(p) = 0$, the chart is centered at p).

Definition 4.2.5:

A pseudo manifold M is a pseudo topological space modeled on some $T^{\#}{}_{\tilde{p}VS}\ \varXi$. (That's mean A pseudo topological space M is a pseudo manifold if it a demits an open covering $\{u_\alpha\}_{\alpha\in A}$ where each set u_α is homeomorphic to an open subset of a module space $\varXi$).

Definition 4.2.6:

Let (u, α) and (v, β) are two charts on M such that $u \cap v \neq \emptyset$ then the composite map $\beta \circ \alpha^{-1}: \alpha(u \cap v) \to \beta(u \cap v)$ (called transition map from α to β) is a composition of homeomorphisms, and is itself a homeomorphism.

Definition 4.2.7:

Two charts (u,φ), (v,θ) of a pseudo manifold are compatible if the two maps: $\theta \circ \varphi^{-1}$: $\varphi(u \cap v) \to \theta(u \cap v)$, $\varphi \circ \theta^{-1}$: $\theta(u \cap v) \to \varphi(u \cap v)$ are homeomorphism.

Definition 4.2.8:

Let $\mathcal{A} = \{(u_i, \alpha_i)\} i \in I$ be a collection of charts on M. We call $\mathcal{A}$ an atlas if the following conditions are satisfied:

1- $\bigcup_{i\in I} u_i = M$.

2- The sets of the form $\alpha_i(u_i \cap u_j)$ for $i, j \in I$ are all open in Ξ.

3- Whenever $u_i \cap u_j$ is not empty, the map $\alpha_j \circ {\alpha_i}^{-1}$: $\alpha_i(u_i \cap u_j) \to \alpha_j(u_i \cap u_j)$ is homeomorphism.

Definition 4.2.9:

Let M be a pseudo topological space. A chart (or coordinate chart) (u,α) in M is called $\tilde{C}^{\infty}$ $-$chart a if α is a diffeomorphism map of $u \subseteq M$ onto $\alpha(u)$ open subset of some ${T^{\#}}_{\tilde{p}VS}\ \Xi$.

Example 4.2.10:

Let M be a pseudo topological space, and $A \subseteq M$ open set then, (A, Id) is coordinate chart in M such that Id is identity map.

Definition 4.2.11:

Let (u,α) and (v,β) are two $\tilde{C}^{\infty}$ $-$charts on M such that $u \cap v \neq \emptyset$ then the composite map $\beta \circ \alpha^{-1}: \alpha(v \cap v) \to \beta(u \cap v)$ (called transition map from α to β) is a composition of diffeomorphism, and is itself a diffeomorphism.

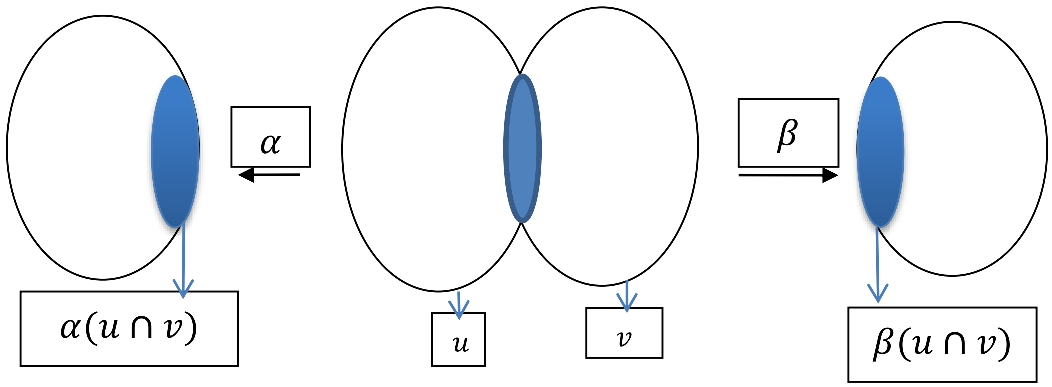

Fig 3. The transition map $\beta \circ \alpha^{-1}$ of class $\tilde{C}^{\infty}$ is defined on $\alpha(u\cap v)$.

Definition 4.2.12:

Two $\tilde{C}^{\infty}$ −charts (u,φ), (v,θ) of a pseudo manifold are smooth compatible if the two maps:

$\theta\circ\varphi^{-1}\colon \varphi(u\cap v)\to\theta(u\cap v)$, $\varphi\circ\theta^{-1}\colon \theta(u\cap v)\to\varphi(u\cap v)$ are smooth maps of class $\tilde{C}^{\infty}$.

Definition 4.2.13:

Let $\mathcal{A} = \{(u_i,\ \alpha_i)\} i\in I$ be a collection of $\tilde{C}^{\infty}$ − charts on M. We call $\mathcal{A}$ an atlas of class $\tilde{C}^{\infty}$(or $\tilde{C}^{\infty}$ −atlas) if the following conditions are satisfied:

1-$\bigcup_{i\in I} u_i = M$.

2- The sets of the form $\alpha_i(u_i \cap u_j)$ for $i,j\in I$ are all open in Ξ.

3- Whenever $u_i \cap u_j$ is not empty, the map

$\alpha_j\circ{\alpha_i}^{-1}\colon \alpha_i(u_i \cap u_j)\to\alpha_j(u_i \cap u_j)$ is a $\tilde{C}^{\infty}$ diffeomorphism.

Definition 4.2.14:

Two $\tilde{C}^{\infty}$ − atlases A and A_1 on M are compatible iff every $\tilde{C}^{\infty}$ −chart of one is compatible with the other $\tilde{C}^{\infty}$ −atlas.

This is equivalent to saying that the union $A \cup A_1$is still a smooth atlas.

And a $\tilde{C}^{\infty}$ $-$chart (u, α) is compatible with smooth atlas $A = \{(u_a, \alpha_a)_{a \in I}\}$ if and only if $A \cup \{(u, \alpha)\}$ is a $\tilde{C}^{\infty}$ $-$atlas.

Definition 4.2.15:

A smooth atlas $\mathcal{A}$ on M is maximal if every $\tilde{C}^{\infty}$ $-$chart that is smoothly compatible with every $\tilde{C}^{\infty}$ $-$chart in $\mathcal{A}$ is already in $\mathcal{A}$.

Definition 4.2.16:

A $\tilde{C}^{\infty}$ $-$ structure on M is a maximal $\tilde{C}^{\infty}$ $-$ atlas.

Remark 4.2.17:

Two smooth structures $\tilde{A}_1$ and $\tilde{A}_2$ on the same pseudo topological manifold M are said to be different if the map (Identity map)

id :$(M; \tilde{A}_1) \rightarrow (M; \tilde{A}_2)$ is not a diffeomorphism.

4.3. Pseudo Manifold of Class $\widetilde{C}^{\infty}$

In this section we will define the smooth pseudo manifold modeled on ${T^{\#}}_{\tilde{p}VS}$ spaces using the $\tilde{C}^{\infty}$–structures.

Definition 4.3.1:

A pseudo manifold of class $\tilde{C}^{\infty}$ is a pair $(M,\mathcal{A})$ where M is a pseudo topological manifold and $\mathcal{A}$ is a smooth structure ($\tilde{C}^{\infty}$–structure) on M.

Example 4.3.2:

Let $M = \mathcal{C}(\mathcal{O})$ be a pseudo topological vector space defined in Example 2.1.16. We defined $\tilde{C}^{\infty}$-chart (A,ϑ) on M as follows:

$A = M$ and ϑ is identity map, $\vartheta(m) = m\ , \forall\ m \in M$.

Then, (A,ϑ) satisfies the conditions of $\tilde{C}^{\infty}$-chart.

Let $\mathcal{A} = \{(A_i,\vartheta_i\}_{i\in I}$be atlas on pseudo topological vector space M, $\mathcal{A}$ is maximal $\tilde{C}^{\infty}$-atlas on M.

Therefore, $(M,\mathcal{A})$ is pseudo manifold of class $\tilde{C}^{\infty}$.

Remark 4.3.3:

If M is a pseudo manifold of class $\tilde{C}^{\infty}$, any chart contained in the given maximal $\tilde{C}^{\infty}$ −atlas will be called a $\tilde{C}^{\infty}$ −chart and the corresponding coordinate map will be called a $\tilde{C}^{\infty}$ − coordinate map.

Remark 4.3.4:

If $\mathcal{A}$ is some $\tilde{C}^{\infty}$ −atlas for pseudo manifold of class $\tilde{C}^{\infty}$ M, and (U,ϕ) is a smooth chart in $\mathcal{A}$, for any , nonempty open subset, $V \subseteq U$, we get a chart,

$(V, \phi /_V)$, and it is clear that this $\tilde{C}^\infty$ −chart is compatible with $\mathcal{A}$. Thus, $(V, \phi /_V)$ is also a chart for pseudo manifold of class $\tilde{C}^\infty M$.

This observation clarifies that if U is any open subset of a smooth M, then U is also a smooth whose $\tilde{C}^\infty$ −charts are the restrictions of $\tilde{C}^\infty$ −charts on M to U.

Proposition 4.3.5:

Let M be a pseudo manifold of class $\tilde{C}^\infty$, and let (u_1, α_1) be a smooth compatible with (u_2, α_2), and also (u_2, α_2) is smooth compatible with (u_3, α_3) in a $\tilde{C}^\infty -$ atlas $\mathcal{A}$, then

(u_1, α_1) be a smooth compatible with (u_3, α_3) on $\alpha_1(u_1 \cap u_2 \cap u_3)$.

Proof:

Since (u_1, α_1) be a smooth compatible with (u_2, α_2) then the map

$\alpha_2 \circ {\alpha_1}^{-1}: \alpha_1(u_1 \cap u_2) \to \alpha_2(u_1 \cap u_2)$ is smooth and also , since (u_2, α_2) be a smooth compatible with (u_3, α_3) then the map

$\alpha_3 \circ {\alpha_2}^{-1}: \alpha_2(u_2 \cap u_3) \to \alpha_3(u_2 \cap u_3)$ is smooth, and the open set

$u_1 \cap u_2 \cap u_3$ is subset of $u_1 \cap u_2$ and subset of $u_2 \cap u_3$, and the map

$\alpha_3 \circ {\alpha_1}^{-1} = (\alpha_3 \circ {\alpha_2}^{-1}) \circ (\alpha_2 \circ {\alpha_1}^{-1})$ is composite smooth maps then its smooth map on $\alpha_1(u_1 \cap u_2 \cap u_3)$, hence (u_1, α_1) be a smooth compatible with (u_3, α_3) on $\alpha_1(u_1 \cap u_2 \cap u_3)$.

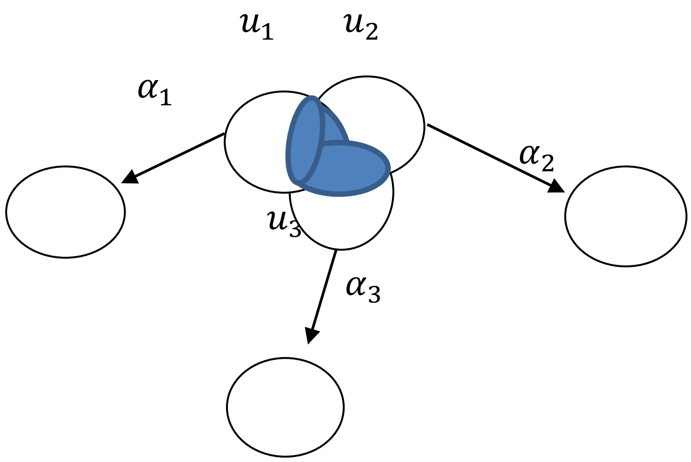

Fig.4: A Smooth Compatible between the $\tilde{C}^{\infty}$-Charts is a Transitive Relationship.

Lemma 4.3.6:

Let $\mathcal{A} = \{(u_\delta, \xi_\delta)\}$ be an $\tilde{C}^{\infty}$ −atlas on a pseudo manifold of class $\tilde{C}^{\infty}M$. If two $\tilde{C}^{\infty}$ −charts (υ, ψ) and (ω, φ) are both smooth compatible with the $\tilde{C}^{\infty}$ −atlas, $\mathcal{A} = \{(u_\delta, \xi_\delta)\}$, then they are smooth compatible with each other.

Proof:

Let $p \in \upsilon \cap \omega$. We need to show that $\varphi \circ \psi^{-1}$ is smooth at $\psi(p)$. Since $\mathcal{A} = \{(u_\delta, \xi_\delta)\}$ is an $\tilde{C}^{\infty}$ −atlas for M, $p \in u_\delta$ for some δ.

Then, $p \in \upsilon \cap \omega \cap u_\delta$. By the proposition above,

$\varphi \circ \psi^{-1} = (\varphi \circ \xi_\delta{}^{-1}) \circ (\xi_\delta \circ \psi^{-1})$ is smooth on $\psi(\upsilon \cap \omega \cap u_\delta)$, hence at $\psi(p)$. Since p is an arbitrary point of $\upsilon \cap \omega$, this proves that

$\varphi \circ \psi^{-1}$is smooth on $\psi(\upsilon \cap \omega)$. Similarly, $\psi \circ \varphi^{-1}$is smooth on $\varphi(\upsilon \cap \omega)$.

Therefore, (υ, ψ) and (ω, φ) are both smooth compatible.

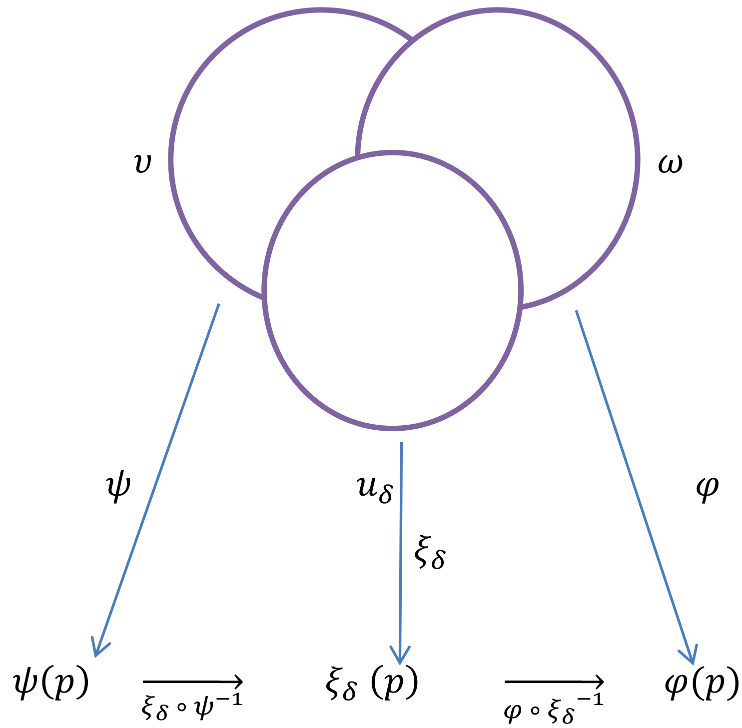

Fig.5: Two $\tilde{C}^\infty$ −Charts $(\upsilon\,,\psi)$ and (ω,φ) Compatible with the $\tilde{C}^\infty$ −Atlas $\mathcal{A}$.

Proposition 4.3.7:

Let $u \subset M$ be open set. If $\alpha: u \to \alpha(u) \subset \Xi$ is a diffeomorphism onto its image. Then (u,α) is a smooth chart in the smooth atlas of pseudo manifold of class $\tilde{C}^\infty$ M.

Proof:

Let (u_a,α_a) be a smooth chart in smooth atlas of M, then $\alpha \circ \alpha_a{}^{-1}$ and $\alpha_a \circ \alpha^{-1}$ are smooth maps. Then, (u,α) is smooth compatible with the smooth atlas. And from maximality of the smooth atlas of M , then the smooth chart (u,α) is in the smooth atlas of pseudo manifold of class $\tilde{C}^\infty M$.

Definition 4.3.8:

Let $f: M \to R$ is smooth function (or $\tilde{C}^{\infty}$ differentiable) on M if and only if the composite function $f \circ \alpha^{-1}: v \to R$ is smooth such that

$\alpha: u \to v \subset \Xi$, Ξ is $T^{\#}{}_{\tilde{p}VS}$.

Definition 4.3.9:

A map $F : U \to V$, $U \subset \Xi_1$ and $V \subset \Xi_2$ is said to be smooth if each of the component map of F has continuous partial derivatives of all orders.

Definition 4.3.10:

Let M and N be pseudo manifold of class $\tilde{C}^{\infty}$on a $T^{\#}{}_{\tilde{p}VS}$ Ξ_1 and Ξ_2 respectively, and let $F : M \to N$ be a map. If every point $p \in M$ has a neighborhood u such that the restriction $F/_{u}$ is smooth, then F is smooth. Conversely, if F is smooth, then its restriction to any open subset of M is smooth.

And we can say F is smooth if for every point $p \in M$ there exists a $\tilde{C}^{\infty}$-chart (u, θ) at p and a $\tilde{C}^{\infty}$-chart (v, μ) at $F(p)$ such that $F(u) \subset v$ and

$\mu \circ F \circ \theta^{-1}$ is smooth map from M into N.

Example 4.3.11:

Any open subset V of a pseudo manifold of class $\tilde{C}^{\infty}$ M is also a pseudo manifold. If $\{(u_\delta, \theta_\delta)\}$ is an smooth atlas for M, then $\{(u_\delta \cap V, \theta_\delta|_{u_\delta \cap V})\}_{\delta \in I}$ is an smooth atlas for V , where

$\theta_\delta|_{u_\delta \cap V} : u_\delta \cap V \to \Xi$ denotes the restriction of θ_δ to the subset $u_\delta \cap V$.

Lemma 4.3.12:

Let M be a pseudo topological manifold of class $\tilde{C}^\infty$.

1- Every $\tilde{C}^\infty$ −atlas for M is contained in a unique maximal $\tilde{C}^\infty$ −atlas.
2- Two $\tilde{C}^\infty$ −atlases for a pseudo manifold of class $\tilde{C}^\infty$ M determine the same maximal $\tilde{C}^\infty$ −atlas iff their union is a $\tilde{C}^\infty$ −atlas.

Proof:

1- Let $\mathcal{A}$ be a $\tilde{C}^\infty$ −atlas for M, and let A denote the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. To show that A is a smooth atlas, we need to show that any two charts of A are compatible with each other, which is to say that for any $(u,\theta),(v,\varphi) \in A$,
$\varphi \circ \theta^{-1}: \theta(u \cap v) \to \varphi(u \cap v)$ is smooth.
Let $x = \theta(p) \in \theta(u \cap v)$ be arbitrary.
Because the domains of the charts in $\mathcal{A}$ cover M, there is some chart $(w,\pi) \in \mathcal{A}$ such that $p \in w$. Since every chart in A is smoothly compatible with (w,π), both the maps $\pi \circ \theta^{-1}$ and $\varphi \circ \pi^{-1}$ are smooth where they are defined.
Since, $p \in u \cap v \cap w$, it follows that $\varphi \circ \theta^{-1} = (\varphi \circ \pi^{-1}) \circ (\pi \circ \theta^{-1})$ is smooth on a neighborhood of x.
Thus $\varphi \circ \theta^{-1}$ is smooth in a neighborhood of each point in $(u \cap v)$.
Therefore A is a smooth atlas.
Now to check that it is maximal, just note that any chart that is smoothly compatible with every chart in A must in particular be smoothly compatible with every chart in $\mathcal{A}$, so it is already in A.
This proves the existence of a maximal smooth atlas containing $\mathcal{A}$.

If B is any other maximal smooth atlas containing A, each of its charts is smoothly compatible with each chart in $\mathcal{A}$, so $B \subset A$.

By maximality of B, $B = A$.

2-

$\Rightarrow$ Let $\mathcal{A}_1, \mathcal{A}_2$ two $\tilde{C}^\infty -$atlases for a pseudo manifold M,

and let $\mathcal{A}_1 \cup \mathcal{A}_2$ be $\tilde{C}^\infty -$atlas, since $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}_1 \cup \mathcal{A}_2$ then

$\mathcal{A}_1 \cup \mathcal{A}_2$ is maximal of $\mathcal{A}_1$ and $\mathcal{A}_2$.

$\Leftarrow$ Suppose $\mathcal{A}_1 \cup \mathcal{A}_2$ is $\tilde{C}^\infty -$ maximal atlas then every atlas on a pseudo manifold M is contain in $\mathcal{A}_1 \cup \mathcal{A}_2$, from 1 the maximal

$\tilde{C}^\infty -$ atlas is unique then $\mathcal{A}_1$, $\mathcal{A}_2$ are specify the same maximal $\tilde{C}^\infty -$atlas.

Lemma 4.3.13:

Let, $\{(u_i, \theta_i)\}_{i\in I}$ is a $\tilde{C}^\infty -$atlas for M. If $f: M \to \Xi$ is a map such that

$f \circ {\theta_i}^{-1}$is smooth for each i, then f is also smooth.

Proof:

We need to prove that $f \circ {\theta_i}^{-1}$ is smooth for any $\tilde{C}^\infty -$chart (u, θ) on M. It suffices to show it is smooth in a neighborhood of each

point $x = \theta(x_0) \in \theta(u)$. For any $x_0 \in u$, there is a chart (u_i, θ_i) in the $\tilde{C}^\infty$-atlas whose domain contains x_0. Since $(u, \theta)$is smoothly compatible with (u_i, θ_i), the transition map $\theta_i \circ \theta^{-1}$ is smooth on its domain of definition, which includes x. Thus $f \circ \theta^{-1} = \left(f \circ {\theta_i}^{-1}\right) \circ (\theta_i \circ \theta^{-1})$

is smooth in a neighborhood of x.

Lemma 4.3.14:

Let $M_1, M_2$and M_3 are any three pseudo manifold of class $\tilde{C}^\infty$. And if the two maps $\mathbb{F}: M_1 \to M_2$ and $\mathbb{G}: M_2 \to M_3$ are smooth maps then, $\mathbb{G} \circ \mathbb{F}$ is also smooth map.

Proof:

Let (u,θ) and $(v,\ \vartheta)$ be any $\tilde{C}^{\infty}$ −charts for M_1 and M_3 respectively. We need to show that $\mathbb{G}\circ\mathbb{F}$ is smooth map (i.e. $\vartheta\circ(\mathbb{G}\circ\mathbb{F})\circ\theta^{-1}$ is smooth on its domain of definition, namely on $\theta(u\cap(\mathbb{G}\circ\mathbb{F})^{-1}(v))$.

For any point $x\in u\cap(\mathbb{G}\circ\mathbb{F})^{-1}(v)$, there is a chart (z,μ) for M_2 such that $\mathbb{F}(x)\in z$. By smoothness of $\mathbb{F}$ and $\mathbb{G}$ we get $\mu\circ\mathbb{F}\circ\theta^{-1}$ and $\vartheta\circ\mathbb{G}\circ\mu^{-1}$ are smooth on its domain of definition,

and therefore

$\vartheta\circ(\mathbb{G}\circ F)\circ\theta^{-1}=(\vartheta\circ\mathbb{G}\circ\mu^{-1})\circ(\mu\circ\mathbb{F}\circ\theta^{-1})$ is smooth.

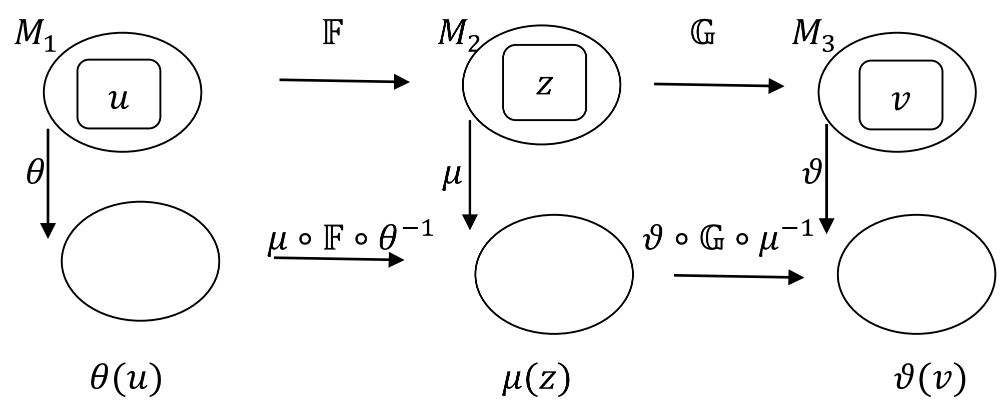

Fig.6: A Composition of Smooth maps is Smooth Map.

Lemma 4.3.15:

Let M_1, M_2 be any pseudo manifold of class $\tilde{C}^\infty$, then the product space $M_1 \times M_2$ has smooth structure such that the projection maps

$\pi_1: M_1 \times M_2 \to M_1$, $\pi_2: M_1 \times M_2 \to M_2$ are smooth. $M_1 \times M_2$ is called the product pseudo manifold of class $\tilde{C}^\infty$.

Proof:

Let, $\{(u_i, \theta_i)_{i \in I}\}$, $\{(v_j, \vartheta_j)_{j \in J}\}$be smooth atlases for a two pseudo manifolds M_1, M_2 respectively . Then $\{(u_i \times v_j)_{(i,j) \in I \times J}\}$ is an open cover of $M_1 \times M_2$ and the coordinate smooth map

$\theta_i \times \vartheta_j: u_i \times v_j \to \theta_i(u_i) \times \vartheta_j(v_j)$ define an smooth atlas. The compatibility and smoothness of structure are verified, hence $M_1 \times M_2$ is pseudo manifold of class $\tilde{C}^\infty$.

Definition4.3.16:

$\mathbb{F}: M_1 \to M_2$ is called a local diffeomorphism if every point $x \in M_1$ has a neighborhood u such that $\mathbb{F}(u)$ is open in M_2 and $\mathbb{F}: u \to \mathbb{F}(u)$ is a diffeomorphism.

Definition 4.3.17:

Let M_1 and M_2 are connected pseudo manifold of class $\tilde{C}^\infty$, a smooth covering map of class $\tilde{C}^\infty$, $\rho: M_1 \to M_2$ is a smooth bijective map with the property that every $p \in M_2$ has a neighborhood u such that each component of $\rho^{-1}(u)$ is mapped diffeomorphically on to u by ρ, we will also say that u is evenly covered.

The pseudo manifold M_2 is called the base of the covering, and M_1 is called a covering space of M_2.

Definition 4.3.18:

Let $\rho: M_1 \to M_2$ is any continuous map, a section of ρ is a continuous map $\sigma: M_2 \to M_1$ such that, $\rho \circ \sigma = id_{M_2}$, such that id_{M_2} is identity map on M_2.

Definition 4.3.19:

Let $\rho: M_1 \to M_2$ is any continuous map, a local section is a continuous map $\sigma: u \to M_1$ defined on some open set $u \subset M_2$ and satisfying the relation

$\rho \circ \sigma = Id_u$.

Lemma 4.3.20:

Let $\rho: M_1 \to M_2$ is a smooth covering map. Every point of $M_1$is in the image of a smooth local section of ρ.For any $y \in M_1$, there is a neighborhood u of $x = \rho(y)$ and a smooth local section $\sigma: u \to M_1$ such that $\sigma(x) = y$.

Proof:

Let $u \subset M_2$ be an evenly covered neighborhood of x. If v is the component of $\rho^{-1}(u)$ containing y, then $\rho/_v: v \to u$ is a diffeomorphism (by hypothesis). It follows that $\sigma = (\rho/_v)^{-1}: u \to v$ is a smooth local section of ρ and such that, $\sigma(x) = y$.

Proposition 4.3.21:

Suppose $\rho: M_1 \to M_2$ is a covering map of class $\tilde{C}^\infty$ and M_3 is any pseudo manifold of class $\tilde{C}^\infty$. A map $F: M_2 \to M_3$ is smooth if and only if $F \circ \rho: M_1 \to M_3$ is smooth:

$$\begin{tikzcd} M_1 \arrow[r, "\rho"] \arrow[dr, "F \circ \rho"'] & M_2 \arrow[d, "F"] \\ & M_3 \end{tikzcd}$$

Fig.7: Composition of Smooth Covering Maps.

Proof:

The first direction is clear. Suppose conversely that $F \circ \rho$ is smooth, and let $x \in M_2$ be arbitrary. By the preceding lemma, there is a neighborhood u of x and a smooth local section $\sigma: u \to M_1$, so that $\rho \circ \sigma = Id_u$.Then the restriction of F to u satisfies:

$F/_u = F \circ Id_u = F \circ (\rho \circ \sigma) = (F \circ \rho) \circ \sigma$; which is a composition of smooth maps. Thus F is smooth on u. Since F is smooth in a neighborhood of each point, it is smooth.

Lemma 4.3.22:

Let M and N be a pseudo manifold of class $\tilde{C}^\infty$, and let $F: M \to N$ be a any map. If $\{(u_i, \theta_i)_{i\in I}\}$, $\{(v_j, \vartheta_j)_{j\in J}\}$ are $\tilde{C}^\infty$ −atlases for a two pseudo manifolds M, N respectively, and if for each i, j , $\vartheta_j \circ F \circ \theta_i{}^{-1}$ is smooth on its domain of definition, then F is smooth.

Proof:

Let $p \in M$, and let a two $\tilde{C}^{\infty}$ −charts (u_i, θ_i) , $(v_j, \vartheta_j)$from the given atlases, such that $p \in u_i$ and $F(p) \in v_j$. From smoothness of $\vartheta_j \circ F \circ \theta_i{}^{-1}$, the set $u = F^{-1}(v_j) \cap u_i$ is open set in M, and $F(u) \subset v_j$. Hence the $\tilde{C}^{\infty}$ −charts $(u_i, \theta_i|_u)$, (v_j, ϑ_j) satisfy the conditions required in the definition of smoothness.

Proposition 4.3.23:

If M_2 is a pseudo manifold of class $\tilde{C}^{\infty}$and $\rho: M_1 \to M_2$ is any pseudo topological covering map, then M_1 has a smooth pseudo manifold structure such that ρ is a smooth covering map.

Proof:

Let $p,\ q \in M_1$` such that, $p \neq\ q$. Suppose $\tau\,(m)\ \cap\ \tau(\ n)\ \neq \emptyset$ then there exists a filter $\mathfrak{F}\ \in \tau\,(m)\ \cap\ \tau(\ n)$ , hence $\mathfrak{F} \downarrow_m$ and $\mathfrak{F} \downarrow_n$ on $M_1$`.

Therefore, $\rho(\mathfrak{F}) \downarrow_{\rho(m)}$ and $\rho\,(\mathfrak{F}) \downarrow_{\rho(n)}$, and since M_2 is separated then, $\rho(m) = \rho(n)$. But ρ is injective then it is contradiction.

Hence, $\tau\,(m) \cap\ \tau(\ n) = \emptyset$. Thus M_1` is separated.

Any point $p\ \in M_2$ has an evenly covered neighborhood u. Shrinking u if necessary, we may assume also that it is the domain of a coordinate map $\theta: u \to \Xi$. Letting v be a component of $\rho^{-1}(u)$ and

$\tilde{\theta} = \theta \circ \rho: \tilde{u} \to \Xi$, it is clear that $(\tilde{u}, \tilde{\theta})$ is a chart on M_2. If two such charts $(\tilde{u}\ , \tilde{\theta})$ and $(\tilde{v}, \tilde{\varphi})$ overlap, the transition map can be written

$\tilde{\varphi} \circ \tilde{\theta}^{-1} = (\ \varphi \circ \rho/_{\tilde{u} \cap \tilde{v}}) \circ (\ \theta \circ \rho\ /_{\tilde{u} \cap \tilde{v}})^{-1}$

$= \varphi \circ \rho\ /_{\tilde{u} \cap \tilde{v}} \circ (\ \rho\ /_{\tilde{u} \cap \tilde{v}})^{-1} \circ \theta^{-1}$

$= \varphi \circ \theta^{-1}$

Which is smooth. Thus the collection of all such charts defines a smooth structure on M_1.

4.4. Some New $\widetilde{C}^{\infty}$-Structures on Maps of Class $\widetilde{C}^{\infty}$

In this section we will introduce some new $\tilde{C}^{\infty}$-structures as tangent spaces, tangent bundles, vector spaces, vector bundles, cotangent spaces, convector spaces and others.

Definition 4.4.1:

Let M be a pseudo manifold of class $\tilde{C}^{\infty}$modeled on a pseudo space Ξ and a point $p \in M$, $\delta: I \to M$ (I is closed interval contain o) is smooth curve such that $\delta(0) = p$ then the

$\dot{\delta}(0) = \delta(t) - \delta(0) - r(t)$ is called tangent to the curve δ at p , $t \in I$, the tangent space $T_p M$ can be thought of as the set of all tangents to curves through the point $p \in M$.

Conversely: For all $v \in T_p M$ we can find a curve $\delta: I \to M$ such that

$\delta(0) = p$ and $\dot{\delta}(0) = v$.

Definition 4.4.2:

Let M be a pseudo manifold of class $\tilde{C}^{\infty}$and $p \in M$, and $\mathfrak{C}(p)$be the set of all smooth functions on an open neighborhood around p,

(i.e. $\mathfrak{C}(p) = \{f: U_f \to R: U_f \subset M, U_f$ open neighborhood containing $p\}$.

Now: we define an equivalence relation on $\mathfrak{C}(p)$by $f \sim_p g$ if and only if there exists an open neighborhood $V \subset U_f \cap U_g$ such that $f = g$ for all point

$p \in V$, denoted the set of all equivalence classes of elements in $\mathfrak{C}(p)$ by $\mathcal{E}(p)$.

Notation 4.4.3:

From now on we will denote the elements of $\mathcal{E}(p)$ by f instead of $[f]$.

Definition 4.4.4:

Let M be a pseudo manifold of class $\tilde{C}^{\infty}$. A tangent vector $\mathcal{V}_p$ at $p \in M$ is a map $\mathcal{V}_p : \mathcal{E}(p) \to R$ such that:

1- $\mathcal{V}_p\,(af + bg) = a\,\mathcal{V}_p\,(f) + b\,\mathcal{V}_p\,(g)$;

2- $\mathcal{V}_p\,(a) = 0$, a is constant.

3- $\mathcal{V}_p\,(f.g) = g(p)\mathcal{V}_p\,(f) + f(p)\mathcal{V}_p\,(g)$, for all $a, b \in R$ and $f, g \in \mathcal{E}(p)$.

Definition 4.4.5:

The set of all tangent vectors $\mathcal{V}_p$ at $p \in M$ (M is a pseudo manifold of class $\tilde{C}^{\infty}$) is denoted by T_pM and is called the tangent space of M at p.

The tangent space T_pM is turned into a real vector space by defining the operations (+) and (.) by:

1- $(\mathcal{V}_p + \mathcal{W}_p)(f) = \mathcal{V}_p\,(f) + \mathcal{W}_p\,(f)$;

2- $(a.\mathcal{V}_p\,)(f) = a.\mathcal{V}_p\,(f)$ for all $a \in R$ and $\mathcal{V}_p\,, \mathcal{W}_p \in T_pM$.

Definition 4.4.6:

For a point p in pseudo manifold of class $\tilde{C}^{\infty}$ M, if $v \in M$ and $f \in \mathcal{E}(p)$ then the derivative $\mathbb{v}_v f$ of f is differential operators at a point p (i.e. the set of mapping $\mathbb{v}_v : \mathcal{E}(p) \to R$ such that satisfies:

$\mathbb{v}_v\ f(p) = f(p + tv) - f(p) - r(t)v$

Remark 4.4.7:

The operator $\mathbb{v}$ has the properties:

1- $\mathbb{v}_v\,(a.f + b.g) = a.\mathbb{v}_v f + b.\mathbb{v}_v\ g$;

2- $\mathbb{v}_v\ (f.g) = \mathbb{v}_v\, f.g(p) + f(p).\mathbb{v}_v\ g$;

3- $\mathbb{v}_{(a.v+b.\mathcal{W})}\, f = a.\mathbb{v}_v f + b.\mathbb{v}_{\mathcal{W}} f$, for all $a, b\ \in R;\ v, \mathcal{W} \in M$

and $f, g\ \in\ \mathcal{E}(p)$.

Hence $\mathbb{v}_v$ is an element of T_pM

This is simply given by the addition (+) and the multiplication (·), by real numbers satisfying:

1- $(\mathbb{v}_v\ + \mathbb{W}_v)(f) = (\mathbb{v}_v\,)(f) + (\mathbb{W}_v)(f)$;

2- $(a.\mathbb{v}_v\,)(f) = a.\,(\mathbb{v}_v\,)(f)$

For all $\mathbb{v}_v\,, \mathbb{W}_v \in T_pM$, and $v \in M$

Theorem 4.4.8:

For a point p in a pseudo manifold of class $\tilde{C}^\infty\ M$, the map $\varphi: M \to T_pM$ defined by $v \longmapsto \mathbb{v}_v$ is linear vector space isomorphism.

Proof:

The linearity of the map φ follows directly from the fact

$\mathbb{v}_{(a.v+b.\mathcal{W})}\, f = a.\mathbb{v}_v f + b.\mathbb{v}_{\mathcal{W}} f$, for all $a, b\ \in R;\ v, \mathcal{W} \in M$

and $f, g\ \in\ \mathcal{E}(p)$.

Let v, w belong to M, and $v \neq\ w$ take the element $y \in M$ such that $\mathbb{v}_v\, y \neq \mathbb{v}_w\, y$, define the function $f: M \to R$ by $f(x) = \mathbb{v}_{\,x} y$.

Then:

$\mathbb{v}_v\ f(p) = f(p + tv) - f(p) - r(t)v \neq$

$f(p + tw) - f(p) - r(t)w = \mathbb{v}_w\ f(p)$

Therefore: $\mathbb{v}_v\ f(p) \neq \mathbb{v}_w\ f(p)$. Hence the map φ is injective.

Now to prove $\varphi: M\ \to T_pM$ is surjective.

Put, $v =\ \mathbb{v}(z), z \in M$. For the constant function $1: z \to 1$ we have

$\mathbb{v}(1) = \mathbb{v}(1.1) = 1.\mathbb{v}(1) + \mathbb{v}(1).1 = 2\mathbb{v}(1)$

So, $\mathbb{v}(1) = 0$

Then, $\mathbb{v}(k) = k.\mathbb{v}(1) = 0$ for all $k \in R$.

Since, $f(x) = f(p) + g(x)(x-p) + r(x-p)$ such that

$g(p) = \mathbb{v}_p\,(x)$

Apply the differential operator $\mathbb{v} \in T_pM$

$\mathbb{v}\big(f(x)\big) = \mathbb{v}[\, f(p) + g(p)(x-p) + r(x-p)\,]$

$= \mathbb{v}\big(\, f(p)\big) + \mathbb{v}\big(\, g(p)(x-p)\big) + \mathbb{v}(\, r(x-p))$

$= \mathbb{v}\big(\, f(p)\big) + \mathbb{v}\big(g(p)\big).(x-p) + g(p).\mathbb{v}(x-p) + \mathbb{v}(\, r(x-p))$

$= g(p).\mathbb{v}(x-p)$

Where, $v \in M$. This mean that the map $\varphi\,(v) = \mathbb{v}_v$, so the linear map $\varphi\colon M \to T_pM$ is surjective, hence, the linear map $\varphi\colon M \to T_pM$ is bijective.

Therefore φ is isomorphism.

Definition 4.4.9:

Let $F\colon M \to N$ be smooth map between two pseudo manifold of class $\tilde{C}^\infty$ M and N, $p \in M$. Then the differential of F at p is the mapping

$dF_p\colon T_pM \;\to\; T_{f(p)}N$

$dF_p([\delta]) \coloneqq [F \circ \delta] \coloneqq \mathcal{V}_p\,(f \circ F)$ for any $\mathcal{V}_p \in T_pM$, $f \in \varepsilon(p)$,

$\delta\colon I \to M,\, \delta(0) = p,$

$\left(dF_p\big(\mathcal{V}_p\big)\right)(f) = \mathcal{V}_p\,(f \circ F)$

$= \frac{d}{dt}(f \circ \delta(t))/_{t=0}$

Remark 4.4.10:

Let $M$and N be pseudo manifolds of class $\tilde{C}^{\infty}$, $p \in M$ and $F: M \to N$ be smooth map. Let $\delta: I \to M$ be a smooth curve in M such that $\delta(0) = p$ and $\dot{\delta}(0) = v_p$. And let $\gamma: I \to N$ be a smooth curve such that the smooth curve $\gamma = F \circ \delta$ in N with $\gamma(0) = F(p)$ and put $w_{F(p)} = \dot{\gamma}(0)$.

Then for each function $f \in \varepsilon(F(p))$ defined locally around $F(p)$, we have:

$$\left(dF_p\left(v_p\right)\right)(f) = v_p(f \circ F)$$

$$= \frac{d}{dt}(f \circ F \circ \delta(t))/_{t=0}$$

$$= \frac{d}{dt}(f \circ \gamma(t))/_{t=0}$$

$$= w_{F(p)}(f)$$

Therefore $\left(dF_p\left(v_p\right)\right)= w_{F(p)}$ or equivalently$\left(dF_p\left(\dot{\delta}(0)\right)\right) = \dot{\gamma}(0)$.

For all $f \in \varepsilon(F(p))$.

Remark 4.4.11:

The differential of the map (identity map) $id_M : M \to M$ at any point $p \in M$ is the identity map:

$id_{TpM} : TpM \to TpM$, because

$(d(id_M)\, v_p)(f) = v_p(f \circ id_M) = v_p f$, for any $v_p \in TpM$ and $f \in \varepsilon(M)$.

$$id_{TpM}(v_p))(f) = v_p(f \circ id_M) = v_p f.$$

Proposition 4.4.12:

Let $F: M_1 \to M_2$ be smooth map between two pseudo manifolds of class $\tilde{C}^\infty$. Then for each point $p \in M_1$ we get:

1- The map $dF_p: T_pM_1 \to T_{F(p)}M_2$ is linear;

2- If $id_{M_1}: M_1 \to M_1$ is the identity map, then, $d(id_{M_1})_p = id_{T_pM_1}$.

Proof:

1- Let $\alpha, \beta \in R$ and $v_p, w_p \in T_pM_1$ we have

$$dF_p\big(\alpha v_p + \beta w_p\big)(f) = \big(\alpha . v_p + \beta . w_p\big)(f \circ F)$$

$$= \alpha . v_p(f \circ F) + \beta . w_p(f \circ F)$$

$$= \alpha . dF_p\big(v_p\big)(f) + \beta . dF_p\big(w_p\big)(f)$$

Therefore the map dF_p is linear.

2- $id_{M_1}: M_1 \to M_1$

$$d(id_{M_1}): T_pM_1 \to T_pM_1$$

Then, $d(id_{M_1})_p = id_{T_pM_1}$.

Corollary 4.4.13:

Let $F: M \to N$ be a diffeomorphism between two pseudo manifolds of class $\tilde{C}^\infty$ M and N, with inverse $L = F^{-1}: N \to M$. If p is a point in M then the differential $dF_p: T_pM \to T_{F(p)}N$ of F at p is satisfies

$(dF_p)^{-1} = dL_{F(p)}$.

Proof:

The prove is direct result of the relations that follow

$dL_{F(p)} \circ dF_p = d(L \circ F)_p = d(id_M)_p = id_{T_pM}$

$dF_p \circ dL_{F(p)} = d(F \circ L)_{F(p)} = d(id_N)_{F(p)} = id_{T_{F(p)}N}$.

Therefore:

dF_p is inverse of $dL_{F(p)}$, this mean $(dF_p)^{-1} = dL_{F(p)}$.

Proposition 4.4.14:

If $F_1 \in \tilde{C}^\infty(M_1, N_1)$ and $F_2 \in \tilde{C}^\infty(M_2, N_2)$ then

$F_1 \times F_2 \in \tilde{C}^\infty(M_1 \times M_2, N_1 \times N_2)$.

Proof:

Let $(x_1, x_2) \in M_1 \times M_2$ and let (u_i, θ_i) is $\tilde{C}^\infty$-chart at x_i, and (v_i, μ_i) is $\tilde{C}^\infty$-chart at $F_i(x_i)$ such that $F_i(u_i) \subset v_i$ and

$\mu \circ F \circ \theta^{-1} \in \tilde{C}^\infty(\theta_i(u), \mu_i(v)), i = 1, 2.$

Then:

$(\mu_1 \times \mu_2) \circ (F_1 \times F_2) \circ (\theta_1 \times \theta_2)^{-1} =$

$[(\mu_1 \circ F_1) \times (\mu_2 \circ F_2)] \circ ({\theta_1}^{-1} \times {\theta_2}^{-1}) =$

$\left(\mu_1 \circ F_1 \circ {\theta_1}^{-1}\right) \times (\mu_2 \circ F_2 \circ {\theta_2}^{-1}) \in \tilde{C}^\infty(\theta_1(u_1) \times \theta_2(u_2), \mu_1(v_1) \times \mu_2(v_2)),$

$\tilde{C}^\infty(\theta_1(u_1) \times \theta_2(u_2), \mu_1(v_1) \times \mu_2(v_2)) =$

$\tilde{C}^\infty((\theta_1 \times \theta_2)(u_1 \times u_2), (\mu_1 \times \mu_2)(v_1 \times v_2)),$

Hence: $F_1 \times F_2 \in \tilde{C}^\infty(M_1 \times M_2, N_1 \times N_2)$.

Remark 4.4.15:

We can extension the above proposition to:

$F_1 \times F_2 \times \ldots \times F_n \in \tilde{C}^\infty(M_1 \times M_2 \times \ldots \times M_n, N_1 \times N_2 \times \ldots \times N_n),$

for all $F_i \in \tilde{C}^{\infty}(M_i, N_i), i = 1,2, \dots, n$. And proof it by induction on n.

Proposition 4.4.16:

Let M be a pseudo manifold of class $\tilde{C}^{\infty}$. Let $H: M \to M^n, (n \in Z^+)$ be the diagonal map defined by $H(x) = (x_1, \dots, x_n)$. Then, $H \in \tilde{C}^{\infty}(M, M^n)$

Proof:

Let $x \in M$ and (u, θ) be a $\tilde{C}^{\infty}$-chart at x, put $v = u \times \dots \times u(n - time)$

and $\varphi = \theta \times \dots \times \theta \; (n - time)$. Then (v, φ) is a $\tilde{C}^{\infty}$-chart at $H(x)$ and

$H(u) \subset v$. Let $D: E \to E^n$ be defined

$D(x) = (x, \dots, x)(n - time)$.

Clearly:

$\varphi \circ H \circ \theta^{-1} = D$ (Restricted to $\theta(u)$).

But, $D \in \tilde{C}^{\infty}(\theta(u), \varphi(v))$.

Therefore, $H \in \tilde{C}^{\infty}(M, M^n)$.

Proposition 4.4.17:

Let M be a pseudo manifold of class $\tilde{C}^{\infty}$, and $F_i \in \tilde{C}^{\infty}(M, \Xi)$,

$i = 1, 2, \dots, n \in Z^+$,

Where Ξ is pseudo space. Define $F: M \to \Xi$ by $F(x) = \sum_{i=1}^{n} F_i(x)$, for each $x \in M$. Then, $F \in \tilde{C}^{\infty}(M, \Xi)$.

Proof:

$F = H \circ (\sum_{i=1}^{n} F_i(x)) \circ G$, where $G: M \to M^n$ is the diagonal map

and $H: \Xi^n \to \Xi$ is defined by $H(y_1, \dots, y_n) = \sum_{i=1}^{n} y_i$,

Now $G \in \tilde{C}^{\infty}(M, M^n)$, and so the result follows.

Theorem 4.4.18:

If $F: M \to N$, $G: N \to O$ are smooth maps between a pseudo manifolds of class $\tilde{C}^{\infty}$ M, N and O, then so $d(G \circ F)_p = dG_{F(p)} \circ dF_p$.

Proof:

By previous definition, take the curve δ such that $\delta: I \to M$, and $\delta(0) = p$.

$$d(G \circ F)_p([\delta]) = [G \circ F \circ \delta] = dG_{F(p)}([F \circ \delta])$$

$$= dG_{F(p)} \circ dF_p([\delta]).$$

Definition 4.4.19:

Let M be a pseudo manifold of class $\tilde{C}^{\infty}$. The tangent bundle TM is the set:

$TM = \coprod_{p \in M} T_pM$

Where $\coprod$ stands for disjoint union.

The projection π extracts the root of a vector, that is, $\pi(v_p) = p$ if and only if $v_p \in T_pM$.

Definition 4.4.20:

Let Ξ be a vector space, a pseudo inner product $\langle .,. \rangle: \Xi \times \Xi \to R^+$ is a map satisfies the properties: For any $a, b, c \in \Xi, \beta \in R^+$,

1- $\langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$;

2- $\langle \beta . a, c \rangle = \beta . \langle a, c \rangle$;

3- $\langle a, b \rangle = \langle b, a \rangle$;

4- $\langle a, b \rangle > 0$.

Definition 4.4.21:

The cotangent space at a point $p \in M$ is the dual vector space to the tangent space T_pM and denoted by $(T_pM)^\wedge$ or $T_p{}^\wedge M$ ($\mathcal{C}_p: T_pM \to R$ is linear function such that $\mathcal{C}_p(v_p) = \langle \mathcal{C}_p, v_p \rangle$ for $v_p \in T_pM$) such that an element of the cotangent space is called a covector or cotangent vector.

This linear function has some properties:

1- $\langle \mathcal{C}_p, v_p + \alpha w_p \rangle = \langle \mathcal{C}_p, v_p \rangle + \alpha \langle \mathcal{C}_p, w_p \rangle$;

2- $\langle \mathcal{C}_p + \alpha U_p, v_p \rangle = \langle \mathcal{C}_p, v_p \rangle + \alpha \langle U_p, v_p \rangle$, for all $\alpha \in R$, and $\mathcal{C}_p, U_p$ are cotangent spaces, and $v_p, w_p \in T_pM$.

Definition 4.4.22:

A cotangent bundle is the set of all covectors at all points in M:

$T^\wedge M = \bigcup_{p \in M} T_p{}^\wedge M$.

Definition 4.4.23:

The natural projection is the map $\pi: TM \to M$ that takes a tangent vector v to the point $p \in M$ $\left(i.e.\ \pi(T_pM) = p\right)$ at which the vector v is attached.

The inverse image $(\pi^{-1}(p))$ of $p \in M$ is the tangent space T_pM the fiber of TM over the point $p \in M$.

Definition 4.4.24:

A vector field $\mathcal{X}$ on M is a smooth map $\mathcal{X}: M \to TM$ that assigns a vector $\mathcal{X}(p) = \mathcal{X}_p$ at the point $p \in M$; that is a section of the tangent boundle $\pi, \pi \circ \mathcal{X} = id$.

And also we can define a vector field $\mathcal{X}$ as a collection of tangent vectors X_p , $p \in M$ if and only if for all $f \in \varepsilon(M)$ the function $p \mapsto X_p(f)$ is a smooth then we get a linear map $\mathcal{X}: \varepsilon(M) \to \varepsilon(M)$ such that

$\mathcal{X}(f)(p) = \mathcal{X}(f)/_p = X_p(f)$.

Remark 4.4.25:

Since a tangent vector X_p satisfy product rule at a point p then $\mathcal{X}$ itself satisfies a product rule $[\mathcal{X}(f.g) = \mathcal{X}(f).g + f.\mathcal{X}(g)]$,

for all $f, g \in \varepsilon(M)$.

Definition 4.4.26:

A vector space of vector field on M is denoted by $\mathbb{X}(M)$.

Definition 4.4.27:

Define a projection map $\hat{\pi}$ for the cotangent bundle, $\hat{\pi}: T^{\wedge}M \to M$ such that if α is a covector at a point in $T^{\wedge}M$ then $\hat{\pi}(\alpha)$ is the point at which α is attached (i.e. $\hat{\pi}\left(T_p{}^{\wedge}M \right) = p$).

Definition 4.4.28:

A covector field is a smooth map $\eta: M \to T^{\wedge}M$ such that:

1- $\eta(p) \in T_p{}^{\wedge}M$;

2- $\eta(\mathcal{X}): M \to R$ is smooth function for all smooth vector field $\mathcal{X}$.

Remark 4.4.29:

If we take, $\mathcal{X}, \mathcal{Y} \in \mathbb{X}(M)$, $f, g \in \varepsilon(p)$ such that

$\mathcal{X}.\mathcal{Y}(f) = \mathcal{X}(\mathcal{Y}(f))$ then:

1- $\mathcal{X}.\mathcal{Y}(f+g) = \mathcal{X}(\mathcal{Y}(f+g))$

$= \mathcal{X}(\mathcal{Y}(f) + \mathcal{Y}(g))$

$= \mathcal{X}(\mathcal{Y}(f)) + \mathcal{X}(\mathcal{Y}(g))$

$= \mathcal{X}.\mathcal{Y}(f) + \mathcal{X}.\mathcal{Y}(g)$;

2- $\mathcal{X}.\mathcal{Y}(f.g) = \mathcal{X}(\mathcal{Y}(f.g))$

$= \mathcal{X}(\mathcal{Y}(f)g + f\mathcal{Y}(g))$

$= \mathcal{X}\big(\mathcal{Y}(f)\big)g + \mathcal{Y}(f)\mathcal{X}(g) + \mathcal{X}(f)\mathcal{Y}(g) + f.\mathcal{X}\big(\mathcal{Y}(g)\big).$

So this mean the product of two vectors fields, is not vector filed

We can structure vector field between two vector fields $\mathcal{X}, \mathcal{Y} \in \mathbb{X}(M)$ by commutetor $[\mathcal{X}, \mathcal{Y}](f) = \mathcal{X}\big(\mathcal{Y}(f)\big) - \mathcal{Y}(\mathcal{X}(f))$, it is called Lie bracket.

To prove $[.,.]$ is vector field:

For all $\mathcal{X}, \mathcal{Y} \in \mathbb{X}(M)$ and $f, g \in \varepsilon(p)$,

1- $[\mathcal{X}, \mathcal{Y}](f+g) = \mathcal{X}\big(\mathcal{Y}(f+g)\big) - \mathcal{Y}(\mathcal{X}(f+g))$

$= \mathcal{X}(\mathcal{Y}(f) + \mathcal{Y}(g)) - \mathcal{Y}(\mathcal{X}(f) + \mathcal{X}(g))$

$= \mathcal{X}(\mathcal{Y}(f)) + \mathcal{X}(\mathcal{Y}(g)) - \mathcal{Y}(\mathcal{X}(f)) - \mathcal{Y}(\mathcal{X}(g))$

$= [\mathcal{X}, \mathcal{Y}](f) + [\mathcal{X}, \mathcal{Y}](g).$

2- $[\mathcal{X}, \mathcal{Y}](f.g) = \mathcal{X}\big(\mathcal{Y}(f.g)\big) - \mathcal{Y}(\mathcal{X}(f.g))$

$= \mathcal{X}(\mathcal{Y}(f)g + f\mathcal{Y}(g)) - [\mathcal{X}(\mathcal{Y}(f)g + f\mathcal{Y}(g))]$

$= \mathcal{X}\big(\mathcal{Y}(f)\big)g + \mathcal{Y}(f)\mathcal{X}(g) + \mathcal{X}(f)\mathcal{Y}(g) + f.\mathcal{X}\big(\mathcal{Y}(g)\big) -$

$\mathcal{Y}\big(\mathcal{X}(f)\big)g - \mathcal{X}(f)\mathcal{Y}(g) - \mathcal{Y}(f)\mathcal{X}(g) - f.\mathcal{Y}\big(\mathcal{X}(g)\big)$

$= \big[\mathcal{X}\big(\mathcal{Y}(f)\big).g - \mathcal{Y}\big(\mathcal{X}(f)\big).g\big] + [f.\mathcal{X}\big(\mathcal{Y}(g)\big) - f.\mathcal{Y}\big(\mathcal{X}(g)\big)]$

$= \big[\mathcal{X}\big(\mathcal{Y}(f)\big) - \mathcal{Y}\big(\mathcal{X}(f)\big)\big].g + f.[\mathcal{X}\big(\mathcal{Y}(g)\big) - \mathcal{Y}\big(\mathcal{X}(g)\big)]$

$= [\mathcal{X}, \mathcal{Y}](f).g + f.[\mathcal{X}, \mathcal{Y}](g).$

Therefore [.,.] is vector field.

Properties of [.,.] 4.4.30:

For all $\mathcal{X},\mathcal{Y},\mathcal{Z} \in \mathbb{X}(M)$, $f,g,w \in \varepsilon(p)$ then:

1- $[\mathcal{X},\mathcal{X}] = 0$

2- $[\mathcal{X},\mathcal{Y}] = -[\mathcal{Y},\mathcal{X}]$;

3- $[\mathcal{X},a\mathcal{Y}+b\mathcal{Z}] = a[\mathcal{X},\mathcal{Y}] + b[\mathcal{X},\mathcal{Z}]$;

4- $[a\mathcal{X} + b\mathcal{Y},\mathcal{Z}] = a[\mathcal{X},\mathcal{Z}] + b[\mathcal{Y},\mathcal{Z}]$

5- $\big[\mathcal{X},[\mathcal{Y},\mathcal{Z}]\big] + [\mathcal{Y},[\mathcal{Z},\mathcal{X}]] + [\mathcal{Z},[\mathcal{X},\mathcal{Y}]] = 0$ (Jacobi identity).

6- $[f\mathcal{X},g\mathcal{Y}] = fg[\mathcal{X},\mathcal{Y}] + (f\mathcal{X}g)\mathcal{Y} - (g\mathcal{Y}f)\mathcal{X}$.

Proof:

1- $[\mathcal{X},\mathcal{X}](f) = \mathcal{X}\big(\mathcal{X}(f)\big) - \mathcal{X}\big(\mathcal{X}(f)\big) = 0$

2- $[\mathcal{X},\mathcal{Y}](f) = \mathcal{X}\big(\mathcal{Y}(f)\big) - \mathcal{Y}(\mathcal{X}(f))$

$= -[\mathcal{Y}\big(\mathcal{X}(f)\big) - \mathcal{X}\big(\mathcal{Y}(f)\big)]$

$= -[\mathcal{Y},\mathcal{X}](f)$.

3- $[\mathcal{X},a\mathcal{Y}+b\mathcal{Z}](f) = \mathcal{X}\big((a\mathcal{Y}+b\mathcal{Z})(f)\big) - (a\mathcal{Y}+b\mathcal{Z})(\mathcal{X}(f))$

$= \mathcal{X}\big((a\mathcal{Y}(f)+b\mathcal{Z}(f)\big) - \{a\mathcal{Y}(\mathcal{X}(f)+b\mathcal{Z}\big(\mathcal{X}(f)\big)\}$

$= \mathcal{X}\big(a\mathcal{Y}(f)) + \mathcal{X}(b\mathcal{Z}(f)\big) - a\mathcal{Y}(\mathcal{X}(f) - b\mathcal{Z}\big(\mathcal{X}(f)\big)$

$= a\{\mathcal{X}\mathcal{Y}(f) - \mathcal{Y}\mathcal{X}(f)\} + b\{\mathcal{X}\mathcal{Z}(f) - \mathcal{Z}\mathcal{X}(f)\}$

$= a[\mathcal{X},\mathcal{Y}] + b[\mathcal{X},\mathcal{Z}]$.

3- $a[\mathcal{X},\mathcal{Z}](f) + b[\mathcal{Y},\mathcal{Z}](f) =$

$$a\{\mathcal{X}(\mathcal{Z}(f)) - \mathcal{Z}(\mathcal{X}(f))\} + b\{\mathcal{Y}(\mathcal{Z}(f)) - \mathcal{Z}(\mathcal{Y}(f))\}$$

$$= a\mathcal{X}(\mathcal{Z}(f)) - a\mathcal{Z}(\mathcal{X}(f)) + b\mathcal{Y}(\mathcal{Z}(f)) - b\mathcal{Z}(\mathcal{Y}(f))$$

$$= a\mathcal{X}(\mathcal{Z}(f)) + b\mathcal{Y}(\mathcal{Z}(f)) - \{a\mathcal{Z}(\mathcal{X}(f)) + b\mathcal{Z}(\mathcal{Y}(f))\}$$

$$= (a\mathcal{X} + b\mathcal{Y})(\mathcal{Z}(f)) - \mathcal{Z}((a\mathcal{X} + b\mathcal{Y})(f))$$

$$= [a\mathcal{X} + b\mathcal{Y},\mathcal{Z}](f).$$

4- $[\mathcal{X},[\mathcal{Y},\mathcal{Z}](f)](f) + [\mathcal{Y},[\mathcal{Z},\mathcal{X}](f)](f) + [\mathcal{Z},[\mathcal{X},\mathcal{Y}](f)](f) =$

$$\mathcal{X}(([\mathcal{Y},\mathcal{Z}](f))(f)) - [\mathcal{Y},\mathcal{Z}](f)(\mathcal{X}(f)) + \mathcal{Y}(([\mathcal{Z},\mathcal{X}](f))(f)) -$$

$$[\mathcal{Z},\mathcal{X}](f)(\mathcal{Y}(f)) + \mathcal{Z}(([\mathcal{X},\mathcal{Y}](f))(f)) - [\mathcal{X},\mathcal{Y}](f)(\mathcal{Z}(f))$$

$$= \mathcal{X}((\mathcal{Y}(\mathcal{Z}(f)) - \mathcal{Z}(\mathcal{Y}(f)))(f)) - (\mathcal{Y}(\mathcal{Z}(f)) - \mathcal{Z}(\mathcal{Y}(f))(\mathcal{X}(f))$$

$$+\mathcal{Y}((\mathcal{Z}(\mathcal{X}(f)) - \mathcal{X}(\mathcal{Z}(f)))(f)) - [\mathcal{Z}(\mathcal{X}(f)) - \mathcal{X}(\mathcal{Z}(f)](\mathcal{Y}(f))$$

$$+\mathcal{Z}((\mathcal{X}(\mathcal{Y}(f)) - \mathcal{Y}(\mathcal{X}(f)))(f)) - [\mathcal{X}(\mathcal{Y}(f)) - \mathcal{Y}(\mathcal{X}(f))](\mathcal{Z}(f))$$

$$= \mathcal{X}\left(\mathcal{Y}(\mathcal{Z}(f)) - \mathcal{X}(\mathcal{Z}(\mathcal{Y}(f)))(f)\right) \quad -\mathcal{Y}(\mathcal{Z}(f))(\mathcal{X}(f) + \mathcal{Z}(\mathcal{Y}(f))(\mathcal{X}(f))$$

$$+\mathcal{Y}\left(\mathcal{Z}(\mathcal{X}(f)) - \mathcal{Y}(\mathcal{X}(\mathcal{Z}(f)))(f)\right) - \mathcal{Z}(\mathcal{X}(f))(\mathcal{Y}(f) + \mathcal{X}(\mathcal{Z}(f)(\mathcal{Y}(f)) +$$

$$\mathcal{Z}\left(\mathcal{X}(\mathcal{Y}(f)) - \mathcal{Z}(\mathcal{Y}(\mathcal{X}(f)))(f)\right)$$

$$-\mathcal{X}(\mathcal{Y}(f))(\mathcal{Z}(f) + \mathcal{Y}(\mathcal{X}(f))(\mathcal{Z}(f) = 0.$$

5- From definition of [.,.]:

$$[f\mathcal{X}, g\mathcal{Y}](w) = f\mathcal{X}(\ g\mathcal{Y})(w) - g\mathcal{Y}(f\mathcal{X})(w)$$

$$= f\{\mathcal{X}g\mathcal{Y}(w) + g\mathcal{X}(\ \mathcal{Y})(w)\} - g\{\mathcal{Y}f\mathcal{X}(w) + f\mathcal{Y}\mathcal{X}(w)\}$$

$$= f\mathcal{X}g\mathcal{Y}(w) + fg\mathcal{X}(\ \mathcal{Y})(w) - g\mathcal{Y}f\mathcal{X}(w) - gf\mathcal{Y}\mathcal{X}(w)$$

$$= fg\{\mathcal{X}\mathcal{Y}(w) - \mathcal{Y}\mathcal{X}(w)\} + f\mathcal{X}g\mathcal{Y}(w) - gf\mathcal{Y}\mathcal{X}(w)$$

$$= fg[\mathcal{X}, \mathcal{Y}](w) + \ (f\mathcal{X}g)\mathcal{Y}\ (w) - \ (g\mathcal{Y}f)\mathcal{X}(w).$$

Definition 4.4.31:

Let $F \in \ \tilde{C}^{\infty}(M, N)$ be a smooth map, vectors field $\mathcal{X} \in \ \mathbb{X}(M)$ and $\mathcal{Y} \in \mathbb{X}(N)$ are called related, written $\mathcal{X} \backsim_F \mathcal{Y}$ if $T_pF(\mathcal{X}_p) = \mathcal{Y}_{F(p)}$ for all $p \in M$.

Proposition 4.4.32:

Let $\mathcal{X} \in \ \mathbb{X}(M)$ and $\mathcal{Y} \in \mathbb{X}(N)$ be two smooth vector spaces, and let $F \in \ \varepsilon(M, N)$ be a smooth map, one has $\mathcal{X} \backsim_F \mathcal{Y}$ if and only if for all $g \in \varepsilon(N)$,

$\mathcal{X}(g \circ F) = \mathcal{Y}(g) \circ F,$

From diagram, with $\tilde{F} \circ g \ = \ g \ \circ F$ for $g \in \varepsilon(N)$, this means

$\mathcal{X} \circ \tilde{F} = \tilde{F} \circ \ \mathcal{Y}$:

$$\begin{tikzcd}
\varepsilon(M) \arrow[r, "\mathcal{X}"] & \varepsilon(M) \\
\varepsilon(N) \arrow[u, "\tilde{F}"] \arrow[r, "\mathcal{Y}"] & \varepsilon(N) \arrow[u, "\tilde{F}"']
\end{tikzcd}$$

Proof:

Let $\mathcal{X} \in \mathbb{X}(M)$ and $\mathcal{Y} \in \mathbb{X}(N)$ are F −related, $g \in \varepsilon(N)$

$(T_pF(\mathcal{X}_p))(g) = \mathcal{Y}_{F(p)}(g)$ for all $p \in M$, Definition 4.4.32.

$\Leftrightarrow \mathcal{X}_p(g \circ F) = \mathcal{Y}(g) \circ F(p)$, for all $p \in M$

$\Leftrightarrow \mathcal{X}(g \circ F) = \mathcal{Y}(g) \circ F$.

Theorem 4.4.34:

Let $F: M \to N$ be a smooth map between two pseudo manifolds of class $\tilde{C}^\infty$. Let $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{X}(M)$ and $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathbb{X}(N)$, if $\mathcal{X}_1 \backsim_F \mathcal{Y}_1$ and $\mathcal{X}_2 \backsim_F \mathcal{Y}_2$ Then, $[\mathcal{X}_1, \mathcal{X}_2] \backsim_F [\mathcal{Y}_1, \mathcal{Y}_2]$.

Proof:

From above proposition, since $\mathcal{X}_1 \backsim_F \mathcal{Y}_1$ then $\mathcal{X}_1(g \circ F) = \mathcal{Y}_1(g) \circ F$

and

Since $\mathcal{X}_2 \backsim_F \mathcal{Y}_2$ then $\mathcal{X}_2(g \circ F) = \mathcal{Y}_2(g) \circ F$ for all $g \in \varepsilon(N)$.

Then:

$$[\mathcal{X}_1, \mathcal{X}_2](g \circ F) = \mathcal{X}_1\big(\mathcal{X}_2(g \circ F)\big) - \mathcal{X}_2(\mathcal{X}_1(g \circ F))$$

$$= \mathcal{X}_1(\mathcal{Y}_2(g) \circ F) - \mathcal{X}_2(\mathcal{Y}_1(g) \circ F)$$

$$= \mathcal{Y}_1(\mathcal{Y}_2(g) \circ F - \mathcal{Y}_2(\mathcal{Y}_1(g) \circ F$$

$$= (\mathcal{Y}_1(\mathcal{Y}_2(g) - \mathcal{Y}_2(\mathcal{Y}_1(g)) \circ F$$

$$= [\mathcal{Y}_1, \mathcal{Y}_2](g) \circ F$$

Therefore: $[\mathcal{X}_1, \mathcal{X}_2] \backsim_F [\mathcal{Y}_1, \mathcal{Y}_2]$.

Corollary 4.4.35:

Let $F: M \to N$ be a diffeomorphism, and $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{X}(M)$,

Then, $\tilde{F}[\mathcal{X}_1, \mathcal{X}_2] = [\tilde{F}\mathcal{X}_1, \tilde{F}\mathcal{X}_2]$.

Proof:

This case is special case of the above theorem in which F is a diffeomorphism and $\mathcal{Y}_i = \tilde{F}\mathcal{X}_i\,, i = 1, 2$.

Definition 4.4.36:

Let $F: M \to N$ be a smooth map between two pseudo manifolds of class $\tilde{C}^\infty$. If $v_p \in T_pM$ then the function $\hat{F}v: \varepsilon(f(p)) \to R$ defined by

$\hat{F}\left(v_{f(p)}\right)(g) = v_p(g \circ F)$ for all $g \in \varepsilon(F(p))$ is called push forward

Theorem 4.4.37:

Let $F: M \to N$ be a smooth map between two pseudo manifolds of class $\tilde{C}^\infty$. If $v_p \in T_pM$ the push forward $\hat{F}v: \varepsilon(f(p)) \to R$ is tangent vector to N at $F(p)$.

Proof:

Let, $g_1, g_2 \in \varepsilon\big(F(p)\big)$, then:

1- $\hat{F}\left(v_{f(p)}\right)(g_1 + g_2) = v_p((g_1 + g_2) \circ F)$

$$= v_p(g_1 \circ F + g_2 \circ F)$$

$$= v_p(g_1 \circ F) + v_p(g_2 \circ F)$$

$$= \hat{F}\big(v_{f(p)}\big)(g_1) + \hat{F}\big(v_{f(p)}\big)(g_2).$$

2- $\hat{F}\big(v_{f(p)}\big)(g_1 . g_2) = v_p((g_1 . g_2) \circ F)$

$$= v_p((g_1 \circ F).(g_2 \circ F))$$

$$= v_p\big((g_1 \circ F)\big).(g_2 \circ F) + (g_1 \circ F).v_p((g_2 \circ F)$$

Therefore the vector $\hat{F}v$ is tangent vector to N at $F(p)$.

Future works:

- After we have established that space is a pseudo paracompact, the researcher can discuss the existence of partition of unity, as it has wide uses. It can be considered as a technical device for extending some local constructions to global constructions.

- In chapter 4 section 3 we explained some important concepts in smooth pseudo manifold like vector filed. Where, by its significance, the researcher can give a definition of the continuously differentiable curve $\gamma: [a, b] \to M$ within the period $[a, b]$ by using the integration for $\acute{\gamma}(t)$ in space $T_{\gamma(t)}M$, $t \in (a, b)$. Where it depends on the definition of the length of the curve to provide the definition of d_g (the metric function defined between two manifold points), to then define a concept Riemannian metric $(M,\ d_g)$, which often to solve problems of differential topology.

Conclusions

After giving the definition of the PNS we concluded that a pseudo metric space can be defined by countable family of pseudo simenorm to get a pseudo topological space as one of the convergence spaces and is considered an expansion of the topological space. Each separated and pseudo locally bounded space$T_{\tilde{p}VS}$ is a pseudo metrizable and the separated space is pseudo metrizable if and only if it realizes the first axiom of countable. Our space is p-paracompact. The first derivative of map between $T_{\tilde{p}VS}$ spaces and after adding the equable and admissible properties to the space, enables us to calculate the derivative to m-times, $m \in Z^{+}$. A new manifold based on ${T^{\#}}_{\tilde{p}VS}\ \Xi$ is wider than a manifold based on a topological space. The map between the pseudo manifold M and the tangent space T_pM is a linear map. The derivative of the map defined between two pseudo manifolds is a linear map. We conclude that Lie bracket is an example of a vector field. The push forward of the map $F: M \to N$ is an example of tangent vector to N at $F(p)$.

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Introduction

Our work starts with a review of the basic concepts of the theory of filters. Sequence is enough to describe the topology and its properties in metric spaces. More generally, the filter to be used to describe the pseudo topology. However, filters are needed in more abstract spaces. We started our work with the filter concept, which was first proposed in 1937 by the researcher Cartan [24] and an excellent treatment of the subject can be found in Bourbaki (1940)[8].

After ten years from Cartan of introducing the filters, Choquet used them to develop the theory of convergence spaces.

In 1948 Choquet [9] presented his theory of pseudo topological structures and pre-topological structures in which the concept of convergence of a filter is axiomatized. The basic convergence theory was developed by Fisher [26]. (Zurich) in 1959 introduced a convergence concept for filters. Some researchers have provided definitions of the convergence spaces based on the concept of filter, which are called the pseudo space as [4, 5, 12, 14, 20, 43].

Topological spaces considered as an extension of metric spaces [27], but a topological space is not necessary metrizable. Some mathematicians introduced some structures that are weaker than metric like Menger who provided the notion semi-metric space [38]. In the last few years, the study of non-symmetric topology has received a new derive as a consequence of its applications to the study of several problems in theoretical computer science and applied physics. An also Wilson [50] introduced a quasi-metric space. The other generalization of metric spaces is called pseudo-metric space that we provided by the definition of pseudo seminorm [19].

A pseudo topological space is a generalization of a topological space based on the concept of the filter and convergence of filters as fundamental.

Many topological concepts were easily generated into pseudo topological spaces.

The study of derivatives has received great interest from some researchers because of its importance as an introduction to the study of manifolds [1, 3, 13, 30, 46]

In the nineteenth century Carl Friedrich Gauss was a cartographer of many terms in modern differential geometry (chart, atlas, map, coordinate system, geodesic, etc.), it has also been studied by others such as [6, 11, 10, 15, 16, 28, 32].

In 1851 [41] Riemann introduced the notation of the manifold in mathematics as a Riemann surfaces, and then some researchers study the manifold physically and medically.

A manifold is a generalization of surfaces and curves to arbitrary dimension, while there are some various types of manifolds (topological manifolds), analytic manifolds, C^m- manifold and complex manifolds [22, 34, 36, 38, 51].

In this work we investigate information about pseudo space concepts such as interior of sets, open sets, closed set, first separation axiom, continuity, homeomorphism, paracompactness, regularity, T_{1-}spaces, and derivative and obtain some results about these concepts, and we benefited from what came in the sources of mathematical analysis of the space proof that has the property of pseudo paracompact.

A differentiable manifold model on a pseudo topological vector space is called pseudo manifold.

Outline of the Dissertation:

Under the concept of developing manifold spaces and opening wider horizons to study this important topic, after expanding the model space, from a topological vector space to a pseudo topological vector space, we review the broad lines that were addressed in this dissertation and bear the title" Manifold Modeled on Pseudo-Topological vector space" as follows:

This dissertation contains four chapters.

Chapter one: Includes three sections that deal with some of the basic information of the topology and pseudo topological vector space.

In section one we give a meaning of the topological vector space and we covered the basic concepts related to general topology as a basis, ball, open and closed set, cover, paracompact etc.

In the second section, we referred to some types of topological vector space, some properties of topological vector space and some relationships between them.

In the last section, we mentioned a basic information about filter, filter-basis, neighborhood filter, and pseudo topology, and about some properties as p- metrizability, p-paracompactness, continuity, equable continuous, quasi bounded map, T_1, p-regularity, separation and class of all equable continuous and quasi bounded maps.

Chapter Two: we went to build the model space on the next way, in section one, we provided a definition of pseudo seminorm and obtained a pseudo topology generated by countable family of pseudo seminorms, and we call it $T_{PVS}\ \Xi$, and we proved itis pseudo metrizable, and then saturated space with the locally convexity characteristic to call it next $T_{\tilde{p}VS}$.

In section two we heighted some characteristics of space $T_{\tilde{p}VS}$ such as, pseudo paracompactness, continuity, Separated, T_1 spaces, p-regularity and Equable continuous.

Chapter Three: This chapter consists of two sections, in section one we introduced the notion of class of remainder maps between $T_{\tilde{p}VS}$ $\varXi_1$ and $\varXi_2$ spaces, and in section two we put the notion of derivative (first, second, n-time and infinity) derivative of map between $T_{\tilde{p}VS}$ $\varXi_1$ and $\varXi_2$ spaces.

Chapter Four: This chapter has three sections. In the first section, we give some basic definitions of smooth structures like $\tilde{C}^{\infty}$ −chart, $\tilde{C}^{\infty}$ −atlas, transition maps, compatible, maximal smooth atlas. In section two, we defined the notion of pseudo manifold of class $\tilde{C}^{\infty}$ that is a pair of pseudo manifold and maximal smooth atlas, and discuss some of the results related to it. In section three, we discuss some new $\tilde{C}^{\infty}$-structures on maps of $\tilde{C}^{\infty}$-class, such that we deal with curve and tangent vector and how to define tangent space in terms of curve, and also define the differential operators. And then we move on to the definition of the differential map

F ($F: M \to N$) and the class $\tilde{C}^{\infty}(M, N)$ which consists of all smooth mapping from smooth pseudo manifold M into smooth pseudo manifold N such that for each $x \in M$, there exists a chart (u, θ) at x and a chart (v, μ) at $F(x)$such that $F(u) \subset v$ and $\mu \circ F \circ \theta^{-1} \in \tilde{C}^{\infty}(\theta(u), \mu(v))$. This section deals with other concepts as tangent bundleTM, cotangent space $(T_p M)^{\wedge}$ or $T_p{}^{\wedge}M$, the cotangent bundle $T^{\wedge}M$, vector field, vector space, convector filed, the Lie bracket as a vector filed and we also touched upon the concept of a relationship between two vector fields (related) and the function push forward as a tangent vector.

Dedication

To the memory of my dear father and to my dear mother. If everybody had parents like them, the world would be a much better place.

Acknowledgements

Thanks are to Allah Almighty for every blessing. I would like to express my gratitude and appreciation for my supervisor Professor Dr. Zahir Dobeas Al-Nafie whose guidance, support and encouragement have been invaluable throughout this study. I thank all my teachers who taught by them. I would also like to extend heartfelt thanks to my family and everyone who supported me psychologically to complete my academic journey.

Table of Contents

Symbols

The Symbol	The Meaning
Ξ	Any nonempty set
T	topological vector space
T_{PVS}	pseudo topological vector space
$\underline{\Xi}$	The underling space
K	Set of real numbers or set of complex numbers
$\mathbb{N}$	Natural numbers
R	Real numbers
R^+	Positive real numbers
$P(\Xi)$	Power set (collection of all subsets of Ξ)
$\mathfrak{F}$, p, $\mathcal{K}$	Any filters
$\mathfrak{F} \downarrow_m$	The filter ? converge to m
$T_{\tilde{p}VS}$	locally convex pseudo metrizable pseudo topological vector space
$T^{\#}{}_{\tilde{p}VS}$	Equable and admissible$T_{\tilde{p}VS}$
$d(.,.)$	Metric map
$\mathfrak{d}(x,\ y)$	Pseudo metric map
$\mathfrak{F}(\Xi)$	Set of all filters on Ξ
τ	pseudo topology
$\mathfrak{N}_m(\Xi)$	Neighborhood filter of $m \in \Xi$
$\leq$	Included relation
ꞙ	Filter neighborhood of a zero
$\tau^{\#}$	Equable pseudo topological vector space
$\widehat{\mathfrak{F}}$	Filter generated by the convex sets of $\mathfrak{F}$
$\mathfrak{F}^{\circ}$	A filter generated by the convex sets of supermom $\{\mathfrak{F}, [0]\}$
$C_e^q(\Xi_1, \Xi_2)$	Class of all equable continuous and quasi bounded maps from Ξ_1 to Ξ_2
$L^{\#}(\Xi_1, \Xi_2)$	Space of equable linear continuous map
Ξ°	A locally convex pseudo topological vector space
$\mathfrak{N}(\Xi_1, \Xi_2)$	A class of remainder maps from Ξ_1 to Ξ_2
$L(\Xi_1, \Xi_2)$	A class of linear and continuous maps from Ξ_1 to Ξ_2
$T^{\#}{}_{\tilde{p}VS}$	Equable and admissible $T_{\tilde{p}VS}$

$\tilde{C}^{\infty}(\Xi_1, \Xi_2)$	A class of smooth maps from Ξ_1 to Ξ_2
M, N, P	Any pseudo manifolds
T_pM	Tangent to curves through the point $p \in M$ (tangent space)
$\mathfrak{C}(p)$	The set of all smooth functions on an open neighborhood around p
$\mathcal{E}(p)$	The set of all equivalence classes of elements in $\mathfrak{C}(p)$
TM	The tangent bundle
$T_p{}^{\wedge}M$	The cotangent space at a point $p \in M$
$T^{\wedge}M$	The cotangent bundle
$\mathcal{X}$	A vector field on M
$\mathbb{X}(M)$	A vector space of vector field on M
π	The natural projection map
$\hat{\pi}$	Projection map for the cotangent bundle
η	A covector field
$[.,.]$	Lie bracket (vector field)

Table of Figures and Diagrams

Abstract

This dissertation introduces the notation of pseudo manifold which we model on pseudo topological vector space $T^{\#}{}_{\tilde{p}VS}$. The results we get are as follows: We construct the concept of pseudo topological vector space which represents a pseudo metrizable, locally convex and separated space, whose topology is induced by a countable separating family of pseudo seminorms. We also discuss several properties of this space such as the regularity, T_1, and quasi boundedness as well as the study of the continuity of maps between these spaces.

The study defines the differentiability of maps that connects two of the defined spaces. After adding two properties to the space $T_{\tilde{p}VS}$ we get space $T^{\#}{}_{\tilde{p}VS}$ and we introduce the notion of the differentiability of the map for twice and for n- times up to infinity which are called smooth maps leading to define a class of maps $\tilde{C}^{\infty}(\Xi_1, \Xi_2)$ where the maps are differentiable non-finitely. We use the pseudo topological vector space $T^{\#}{}_{\tilde{p}VS}$ as a model space to construct the concept of smooth pseudo manifold and introduce the definitions of smooth map and smooth covering map.

Since tangent space and vector filed are among the basic concepts in the manifold, the study deals with them and gives some definitions and gets some results.

Certification of Scientific Expert

I certify that I have read the scientific content of this dissertation "**Manifold Modeled on pseudo- topological vector space**" and I have approved this dissertation qualified for debate.

Signature

Name: **Dr. Ihsan Jabbar Kadhim**

Title: Assist. Prof.

Date: 23/9/ 2021