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ON THE SHAPE PRESERVING APPROXIMATION

A thesis submitted to the Department of Mathematics College of Education Babylon University as a partial fulfillment of requirements for the degree of the Master of Science in mathematics

By

Halgwrd Mohammed Darwesh

Supervised by

Dr. Eman Samir Bhaya

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التقريب الحافظ للشكل

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كجزء من متطلبات نيل درجة الماجستير في علوم الرياضيات

تقدم بها

هلكورد محمد درويش

بإشراف

د. إيمان سمير بهية

ذي العقدة 1946هـ كانون الأول 2005 م

كۆمارى عيَراق
وةزاروتى خويَندنى بالأ و تويَذينةوةى زانستى
زانكۆى بابـــــل
كــۆلـــــيَذى ثةروةردة
بةشــــــى ماتــماتيـــك

نزيك كردنةوةى شـــــيَوة ثــاريَــزران

نامةيةكى ثيَشكةشكراوة بة
كۆلـــيَذى ثةروةردة، بةشــــى ماتـــماتيــك، لة زانـــكۆى بابــــل
وةك بةشـــيَكى لة تةواوكةر بۆ بةدةستهيَنانى ثلةى ماستةر لة زانستى متماتيك دا.

لــــــةلايةن
هةلَطورد محمد دةرويَش

بة ســـةر ثةرشــــــــتى
د. إيمان سمير بهية

سةرماوةرزي 2705 كوردي ذي القعدة ى 1426 كۆضي كانوني يةكةمي 2005 زاينى

Abstract

Sometimes one may desire to approximate a function f defined on a finite interval (for example $[-1,1]$), subject to the conservation of so called shape properties (positivity, monotonicity and convexity).

Our first contribution is that we have approximated a function from a space $L_p[-1,1]$, $0 < p < \infty$, by a number of piecewise linear functions and we have obtained global estimate of each of them using the second order of Ditzian – Totik modulus of smoothness ω_2^φ. Furthermore, these piecewise linear functions preserves the positivity of the function.

We have also proved the rate of coconvex approximation in the $L_p[-1,1]$ - spaces with $0 < p < \infty$, in terms of smoothness ω_3^φ, where the constants involved depend on the location of the points of change of convexity. We have thus filled up a gap due to the uncertainty between previously known estimates involving ω_2^φ and the impossibility of having such estimates involving ω_4.

We have constructed a negative theorem that we could not have any Jackson-type estimates for some cases where the constants involved independent of the points of convexity change which emphasize the essentiality of this dependency in some cases. We also proved that in some cases the constants may be taken independent of the location of convexity change points. But mostly, we obtained such estimates for functions of that themselves are continuous piecewise polynomials on the Chebyshev partition, which form a single polynomial in a small neighborhood of each point of change of convexity. These estimates involved the k^{th} Ditzian – Totik modulus of smoothness ω_k^{φ} of the piecewise polynomials when they themselves are of order k .

المُستَخْلَص

نرغب أحياناً بتقريب دالة ما معرفة على الفترة I ، بحيث يكون ذلك التقريـب حافظـاً لبعض الخواص الشكلية لتلك الدالة. بعبارةٍ أخرى في بعض الأحيان نحتاج أن يكون تقريب دالـةٍ ما حافظاً لخواصها، كأن تكون موجبة، رتيبة أو محدبة في كل الفترة I أو في أجزاءٍ منها. هذا ما نسميه *بالتقريب الحافظ للشكل*.

لرسمِ صورةٍ متكاملةٍ في أذهانِنا عن نظريةِ التقريب. و من اجلِ أن نهيئَ بعـض المعلومـات الأساسيةِ التي نحتاجُها حتى يُصبحَ العملُ أكثرَ وضوحاً، استذكرنا بعضِ تطبيقـات تلـك النظريـة بالإضافةِ إلى بعض التعاربف و المعلومات الأساسية واهم النتائج المعروفة التي لهـا علاقـة بموضـوع البحث.

إن أولَ النتـائج الـتي حصـلنا عليهـا في هـذا العمـل هـي تقريـب الدالـة f في الفضـاء $L_p[-1,1]$ لكل $0 < p < \infty$ باستخدام بعض الدوال متقطعة الخطية الموجبة و حصلنا على نتيجة مباشرة لدرجة التقريب لتلك الدوال بدلالة مقياس النعومة ω_2^{φ} .

كـذلك قمنـا باسـتخراج رتبـة التقريـب الحـافظ لـتغير تحـدب الدالـة، للـدوال في فضـاء $L_p[-1,1]$ و $0 < p < \infty$ بدلالة المقياس ω_3^{φ} مضروباً بثابت يعتمد على عدد النقـاط الـتي تغير فيها تحدب الدالة ، لكل (n) أكبر أة تساوي ثابـت أخـر يعتمـد علـى موقـع النقـاط الـتي

تغيرفيهـا تحـدب الدالـة . و بهـذه نحـن غَلقنَـا هـذا الفـراغَ بـين التخمينـاتِ المعروفـةِ سـابقاً باستخدام ω_2^{φ} وإستحالةِ إمـتِلاك مثـل هـذهِ التخمينـاتِ باسـتخدام ω_4 . و للتأكيـد ضـرورية الإعتماد واحد من الثابتان على موقع النقاط التي تغير فبها تحدب الدلـة، برهننـا أنـه في بعـض الحالات لايمكننا الحصول على مبرهنةمباشرة لتلك الـدوال حتـى بدلالـة ω ، بشـرط ان هـذان الثابتان لايعتمدان على موقع النقاط التي تغير فيها تحدب الدالة. كمـا برهننـا في بعـض الحـالات أخرى انه يمكن اخذ الثابتان بحيث لايعتمدان على موقع النقاط التي تغير فبهـا تحـدب الدالـة. لكن غالباً حَصلنَا على مثل هذه التخميناتِ تلك الدوال التي أنفسـهم متعـددات الحـدود المسـتمرة قطعياً على تجزئة شيفيشيف بدلالة مقياس النعومة ω_k^{φ} .

بــوخـــــــــتـة

هەندآ كات دەويستريَت نەخشەيەك نزيك بكريَتەوە، كە ثيَناسە كراوە لەسەر ماوەيەكى كؤتادار (بؤ نمونە$[-1,1]$)، بە مەرجى ثاراستنى تايبەتمەنديەكانى شيَوەى نەخشەكە وەك(Positivity، Monotonicity وە Convexity).

يەكەم بەدەستهيَنانمان ئەوەية كە هەر نەخشەيەكى ناو بؤشاييەكانى - $L_p[-1,1]$ ($L_p[-1,1]-$Spaces) نزيك دەكەينەوە بە ذمارەيەك لە نەخشەى هيَلَى ثارضە ثارضە (Piecewise Linear Function)، وە هەر يەكەشيان دەخەملَيَنرآ بە ثيَوانەى لووسى ω_2^{φ}. لەوەش زياتر، هەريەكە لەو نەخشە هيَلَيية ثارضە ثارضانە ثاريَزطارى لە (Positivity) نەخشەكان دەكەن.

سەلماندومانە رِيَذەى نزيك كردنەوەى فرەرِادەدارى قوثاوى هاوبەش (Coconvex Polynomial Approximation) لە ناو بؤشاييەكانى- $L_p[-1,1]$ بؤ $0<p<\infty$، دەتوانريَت بثيَوريَت بە ω_3^{φ} لــــيَكدانى نەطؤرِيَك كە بەندە لەسەر ذمارەى ئەو خالآنەى كە طؤرِانى قوثاوى نەخشەكەى تيا رِوودەدات، بؤ هەر(n)يَك كە طەورەتر و يەكسانە لە نەطؤرِيَكى دى، كە بەندە لەسەر شويَنى ئەو خالآنەى كە طؤرِانى قوثاوى نەخشەكەى تيا رِوودەدات. بەمەش توانيمان ئەو كەلەبەرە دابخەين كە هەبوو لە نيَوان توانينى خەملآندنى نزيك كردنەوەى فرەرِادەدارى هاوبەش بە ω_2^{φ} وە مەحالأبوونى بوونى ئەو خەملآندنانە بە ω_4.

بؤ دلَنيابوون لەسەر ثيَويستى بەند بوونى يەكيَك لەو نەطؤرِانە لەسەر شويَنى ئەو خالآنەى كە طؤرِانى قوثاوى نەخشەكەى تيا رِووداوە، تيؤريَكى سلبيمان دروست كردووە و تيايدا ثيشــا نمان داوە كە هەرطيز ناطونجآ لە هەندآ حالَەتدا خەملآندنى جؤرى جاكسؤن بەدەست بهيَنريَت بؤ ئەم جؤرە لە نزيك كردنەوە، بە مەرجآ ئەو نەطؤرِانەى هەن بەند نەبن لەسەر شويَنى خالَەكانى طؤرِانى قوثاوى نەخشەكە، بەلَكو بەند بن لەسەر ذمارەيان. لە هەمان كاتدا سەلماندوومانە كە دەتوانرآلە

هةندآ حالَةتى ديدا، ئةو نةطؤرِانة وةربطيريَن بآئةوةى بةندبن لةسةر شويَنى خالَةكانى طؤرِانى قوثاوى نةخشةكة. بةلآم زؤرينةى ئةم جؤرة خةملآندنانةمان بةدةست هيَناوة بؤ ئةو نةخشانةى كة خؤيان فرةترِادةدارى بةردةوامى ثارضة ثارضة (Continuous Piecewise Polynomial)ن لةسةر بةش بةشى ضيبايشيظ(Chebyshev Partition)، وة لة تةنها فرةترِادةداريَك ثيَكهاتوون لة دراوسآيةكى بضووكى (Small Neighborhood)خالَةكانى طؤرِانى قوثاوى نةخشةكة، ئةو خةملآندنانةش خةملَيَنراون بة (k)ةم ثيَوانةى لووسى لة جؤرى ديتيزيان – توتيك ω_k^{φ}، بؤ ئةو فرةترِادةدارة ثارضانة كة خؤيان دةرةجةيان $(k-1)$ة.

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Halgwrd M. D.

2005

بِسْمِ اللهِ الرَّحمنِ الرَّحيمِ

مَا يَفتَحِ اللهُ لِلنَّاسِ مِنْ رَحْمَةٍ فَلا مُمْسِكَ لَهَا وَ مَا يُمْسِكْ فَلا مُرسِلَ لَهُ مِنْ بَعْدِهِ وَ هُوَ العَزِيزُ الْحَكِيمُ

صَدَقَ اللهُ العليُّ العظيم

فاطر آية ٢

I certify that this thesis was prepared under my supervision at the Babylon University, College of Education as a partial fulfillment of the requirements of degree Master of Science in Mathematics.

Signature:
Name: *Dr. Eman Samir Bhaya*
Scientific Grade: *Lecture*
Date: *11 / 12 / 2005*

In view of the available recommendation, I forward this thesis for debate by the examining committee.

Signature:
Name: *Dr.Iftichar M. Al Shara'a*
Chairwoman of the Departmental committee of Graduate Studies in Mathematics
Scientific Grade: *Lecture*
Date: *11 / 12 / 2005*

We certify that we have read this thesis as examining committee examined the student in its contents and that in our opinion it is adequate for the partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

Signature:
Name: *Dr. Ali Hussain Batoor*
Scientific Grade: Asist. Professor
Date: *28 / 1 / 2006*
Chairman

Signature:
Name: *Dr. Sahib K. Jassim*
Scientific Grade: Asist. Professor
Date: *28 / 1 / 2006*
Member

Signature:
Name: *Dr.Iftichar M. Al Shara'a*
Scientific Grade: *Lecture*
Date: *28 / 1 / 2006*
Member

Signature:
Name: *Dr. Eman Samir Bhaya*
Scientific Grade: *Lecture*
Date: *28 / 1 / 2006*
Supervisor

Approved by the University Committee on Graduate Studies.

Signature:
Name: Dr. Luay A.A. Al-Swidi
Dean of the College of Education
Scientific Grade: Asist. Professor
Date: *28 / 1 / 2006*

Chapter One

Introduction and Preliminaries

1.1 Approximations Theory and Overview

Our interest in approximation theory stems from its beauty, its utility, and its rich history. There are also many connections that can be drawn to questions in both classical and modern analysis.

The main problem of approximation consists in finding for a complicated f function from a large space X a close – by, a simple function ϕ from a small subset Ω of X. There are three elements here. The large space X is usually a normed space, such as C, L_p with $o < p < \infty$ or one of the other Banach spaces of functions. The distance from ϕ to f can be measured by $\|f - \phi\|$ in X. Finally, we have to define the special functions of Ω.

There are many possibilities, but the following three are basic in the theory. For functions on a compact interval $[a,b]$, one often chooses $\Omega := \Pi_n$ to be the space of all algebraic polynomials P_n of degree $\leq n$,

$$P_n(x) := \sum_{k=0}^{n} a_i x^i ,$$

where a_i's are constants independent of x.

For the $\mathcal{T}$ (which we will usually understand to be $\Re$ with the identifications of modulo 2π), natural is the class $\mathcal{T}_n$ of trigonometric polynomials of degree $\leq n$,

$$T_n(x) := \frac{a_0}{2} + \sum_{j=1}^{n} \left(a_j \cos jx + b_j \sin jx\right),$$

where a_j and b_j's are real constants independent of the variable x.

Finally, the third important class is the piecewise polynomials; especially splines:

Definition (A).[14] A function $S:[a,b] \to \Re$ is a ***piecewise polynomial*** of degree $\leq n$ (or equivalently of order $n+1$), if there exists a partition of the interval $[a,b]$, such that S is a polynomial of degree $\leq n$, on each interval of these partition, and the end points of these subintervals are called the breakpoints (or knots) of S. However, a piecewise polynomial s is said to be a ***spline*** of degree $\leq n$, if $s, s', s'', \ldots, s^{(n-1)}$ are all continuous on $[a,b]$, where $\Re$ is the set of all real number.

A complete theory of approximation was formed based on the result of S. N. Bernstein and D. Jackson that became the main part of the report presented by S. N. Bernstein at International Mathematics Congress in Cambridge [2]. Which are them results are so called Direct and Inverse theorems. The second is returned to the Ph.D dissertation of a great mathematician S. N. Bernstein [3], in which he proved (if we have a certain degree of approximation of a function, then the function possesses a certain smoothness). The first is returned to the American mathematician D. Jackson [30], in which he proved a series of theorems that are converse to Bernstein theorems.

There are many important applications of approximation theory, as it was begun itself by application, by the Great Russian scientist P. L. Chebyshev, and he said about the relationship between a mathematical theory and its applications illustrate quite clearly not just the source of creativity, but also scientific and philosophical positions. In this work, we give some of it's applications in mathematics, which are:

(i) ***Solving algebraic equations***. In the memoir [9], there are about ten theorems (6-10, 15-19) that are derived from the basic propositions on best approximation. These theorems state that, under certain conditions the polynomial of interest has at least one zero in some interval. The length of the interval depends on the one hand, on the value of the polynomial at the center of the interval, on the other hand, on specific assumption on the coefficients or on the zeros of the polynomial. For example theorem 10 states that: if the polynomial $P(x) = x^{2n+1} + \ldots + k$ does not contain any even power of x, then it has at least one zero in interval $|x| \le 2\left(\frac{|k|}{2}\right)^{\frac{1}{2n+1}}$. Let us also formulate theorem 9 which uses a different assumption: if a polynomial $P(x)$ of degree n with leading coefficient equal to one, and has only real roots, then for any t there is a real root in the interval $|x - t| \le 4\left(\frac{P(t)}{4}\right)^{\frac{1}{n}}$. Later in his 1872 memoir, using a result on monotone polynomials, Chebyshev narrowed the interval by replacing with $|x - t| \le 4\left(\frac{P(t)}{2(n-1)\pi}\right)^{\frac{1}{n}}$. (A further improvement of this result is due to A. A. Markove [52]).

(ii) ***Interpolation (remainder estimate)***. To minimize the error in the Lagrange interpolation formula, Chebyshev suggests taking the nodes of the interpolation (say in the interval $(-1,1)$) to be the zero of the polynomial $T_n(x) = \cos(n \arccos(x))$, since for a given function $f(x)$ the remainder has the form $R(x) = \frac{f^{(n+1)}(\lambda)}{(n+1)!} P_n(x)$, where $\lambda \in (-1,1)$, $P_n(x) = \prod_{k=1}^{n}(x - x_k)$ and x_k are the nodes. Therefore with $R(x) \le \max \left|\frac{f^{(n+1)}(\lambda)}{(n+1)!}\right| \max |P_n(x)|$, the choice $P_n(x) = T_n(x)$ is the most profitable. Here Chebyshev partly envisions the later result of Runge that say's that as $n \to \infty$ Chebyshev interpolation converges for any function that is regular in the basic interval, while this is not true for Newton interpolation with equally space nodes.

(iii) ***Finding a numerical approximation values to*** $b^{\frac{1}{m}}$ ***, where*** $b \in \Re$ ***and*** $m \in N$ ***(the set of all natural number)*** [69].

Consider the geometric series $\sum_{k=0}^{\infty} ar^k$. It is well known that this series diverges, if $|r| \ge 1$, and converges to the sum $\frac{a}{1-r}$, if $|r| < 1$.

Substituting $r = x$, and $a = 1$ in the geometric series makes it easy to think of the series

$$P(x)=\sum_{k=0}^{\infty}x^k=1+x+x^2+x^3+\cdots,$$

as a function of x, and it's converges only for $|x|<1$; moreover, it is well known that $P(x)=\frac{1}{1-x}$ for all $x\in(-1,1)$.

Therefore, the two functions $P(x)$ and $g(x)=\frac{1}{1-x}$ are closely related: they have different domain but where the series $P(x)$ converges, the two functions are equal. Let's recall what it means for an infinite series to converge to a sum. It means that the partial sums of the series converge to the sum. So the partial sum of the above series is $P_n(x)=1+x+\cdots+x^n$, and for every $x\in(-1,1)$, $\lim_{n\to\infty}P_n(x)=g(x)$.

Now, $P_0(x)=1$ is the value of $g(x)$ at $x=0$, $P_1(x)=1+x$ is the tangent line to $g(x)$ at $x=0$, (since $P_1(0)=1=g'(0)$). The tangent line $y=P_1(x)$ can be thought of as the best linear approximation to the function to the graph $y=g(x)$ at the point $x=0$. Then $P_2(x)$ can be thought of as best quadratic polynomial approximate $g(x)$ at $x=0$, (since $P_2''(0)=2=g''(0)$, $P_2'(0)=1=g'(0)$ and $P_2(0)=1=g(0)$). Similarly, $P_n(x)$ is the n^{th} degree polynomial approximate $g(x)$, (since $P_n^{(k)}(0)=k!=g^{(k)}(0)$, $k=1,2,\ldots,n$, and $P_n(0)=1=g(0)$), in this case we say P_n and g agree to order n. Notice that $P_n(x)$

and $g(x)$ very well close to $x=0$, but they do not at all for any value of x with $|x|\geq 1$. This makes sense because the partial sums $P_n(x)$ converge to $g(x)$ only for x with $|x|<1$. This suggests that for any function f (we assume we can differentiate f as many times, as we like), we can associate polynomials to a function at a given point $x=a$ which approximate the function well near $x=a$. In particular, the n^{th} degree polynomial will have the same value, and all derivatives up to the n^{th} one, equal to those of f at the point a.

In fact there is a formula for such polynomial, which we call the n^{th} degree Taylor polynomial of $f(x)$ at $x=a$:

$$\begin{aligned} P_n(x) &= f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k , \end{aligned} \qquad [33]$$

where by $f^{(0)}$ we mean f itself.

Note that Taylor polynomial of degree n of f at $x=0$ is the unique polynomial of degree $\leq n$ which is agree with f, of order n, at $x=a$, however, we known that if f is identically zero near a, then we can not hope for the Taylor polynomials to Converge to f at a point where f is non zero [33]. We have already the linear Taylor polynomial P_1 is the linear approximation to f at x=a. Also linear approximations are useful to approximate the value of functions, as in this example.

Example: By using the linear approximation, find the approximate value of $\sqrt[3]{9}$.

Choose $f(x)=\sqrt[3]{x}$ and $a=8$, since $\sqrt[3]{8}=2$ is a nice number and 9 is close to 8 , Also compute $P_1(x)=2+\frac{1}{12}(x-8)$, then $P_1(9)\approx 2.083$. Now since $\sqrt[3]{9}\sim 2.0800838$, then the error in this approximation is
$\left|\sqrt[3]{9}-P_1(9)\right|\sim 0.0032495$.

This is a pretty good approximation, but we can do better by taking higher-order Taylor polynomials, since if we use the quadratic Taylor polynomial, then we obtain the error is $\left|\sqrt[3]{9}-P_2(9)\right|\sim 0.00022272$. Hence if we use the Taylor polynomials of higher degree, then the error is quite small.

Example: Approximate $\sqrt{103}$ by using the Taylor polynomials of degree 2 and 3.

Choose $f(x)=\sqrt{x}$ and $a=100$, then compute
$P_2(x)=10+0.05(x-100)-0.000125\,(x-100)^2$, and $P_2(x)=10.148875$,

so the error is $\left|\sqrt{103}-P_2(x)\right| \sim 1.65651 * 10^{-5}$. It's clear that $P_3(x)=P_2(x)+6.25*10^{-5}(x-100)^3$. Hence $P_3(103)=10.148894875$ and the error is $\left|\sqrt{103}-P_3(x)\right| \sim 3.0991 * 10^{-7}$.

1.2 The Spaces L_p, $p<1$.

We are going to study the degree of constrained and unconstrained approximation of a function f in either the uniform norm or in the $L_p[-1,1]$, $0<p<\infty$ quasi-norm. The degree of approximation will be measured by the appropriate quasi-norm which we denote by $\|.\|_p := \|.\|_{L_p(I)}$, where $I:=[-1,1]$. Since we need the L_p quasi-norm on other intervals we will in all cases of an interval $J \neq I$, indicate that by writing $\|.\|_{L_p(J)}$. Also we refer to ($\|.\|_p$ with $p=\infty$), the uniform norm on I and we denote by $\|.\|$ and on the interval J by $\|.\|_J$. Furthermore, $\|.\|_p$ is a norm for $1 \leq p < \infty$. Characteristic for $L_p, p \geq 1$ are the inequalities of Holder and Minkowski.

The dual Space of $L_p, 1 \le p < \infty$, is the spaces L_q with the conjugate exponent q of p. Thus the spaces L_p with $1 \le p < \infty$ are reflexive.

The different structure of the spaces L_p with $0 < p < 1$ and the numerous questions by others lead us to understand the need for the following few facts about L_p for $p < 1$.

We consider the space $L_p(I)$, consisting of all measurable functions f on I, for which

$$\|f\|_p^p := \int_I |f(x)|^p \, dx < \infty .$$

Recall that $\|f\|_p \le 2^{\frac{1}{p}-1} \|f\|_1$, that is $L_1 \subset L_p$.

As we will see shortly, the L_p norm is not actually a norm for $p < 1$. Nevertheless, it is not hard to see that $L_p(I)$ is a complete metric space:

It is easy to prove the following theorem

Theorem B. $\|f+g\|_p \le \left(\|f\|_p^p + \|g\|_p^p\right)^{\frac{1}{p}} \le 2^{\frac{1}{p}} \left(\|f\|_p + \|g\|_p\right)$, for any $f, g \in L_p(I)$.

Thus $d(f,g) = \|f-g\|_p^p$ defines a translation invariant metric on L_p. It is a complete metric, because convergence (respectively Cauchy) in L_p implies convergence (respectively Cauchy) in measure since $\varepsilon^p \text{meas } \{t : |f(t)| > \varepsilon\} \le \int_{-1}^{1} |f(t)|^p \, dt$.

What this means to us is that [4]

(i) A linear map on L_p is continuous if and only if it is bounded (continuous at zero).

(ii) The open mapping and closed graph theorems still apply.

(iii) The Hahn Banach theorem may fail!

Indeed, as we will see shortly, L_p is not locally convex. In fact, it is impossible to define a norm on L_p which gives the same topology as the usual metric. There are several ways to see that $L_p(I)$ is not normable for $0 < p < 1$. Most useful from our point of view is

Theorem C. [10] L_p, $0 < p < 1$ has a trivial dual.

The Hahn Banach theorem allows a much fancier sounding version of this result:

Corollary D. [10] There are no non zero continuous linear map from L_p into any normed space.

In any event, it should now be clear that there can be no norm on L_p which gives the same topology as the usual metric and that the Hahn Banach theorem evidently fails in $L_p(I)$, for $0 < p < 1$. The fact that L_p has a trivial dual with the theorem from [46] states that [there exists a nonzero continuous linear function on a linear space X, if and only if there is at least one convex set that is not all of X.], imply another rather strange result that would be hard to believe otherwise.

Corollary E. [10] If U is any neighborhood of zero in $L_p(I)$, then

$$L_p(I) = \text{conv}\,(U)$$

In particular

$$L_p(I) = \text{conv}\,\{f : \|f\|_p^p < 1\},$$

where $\text{conv}\,(U)$ is a smallest convex neighborhood of zero contains U.

Now, we will settle the question of which $L_p, p < 1$ embed into L_q for $q \geq 1$. Or which subspaces of $L_p(I)$ on which all of the various $L_p(I)$ quasi-norms for $0 < p < q$ are equivalent. The key in this article is due to Kadec and Pelzynyski from [31]:

For $0 < \varepsilon < 1$ and $0 < p < \infty$, consider the following subset of $L_p(I)$

$$M(p,\varepsilon) := \{f \in L_p(I) : \text{meas } \{x : |f(x)| \geq \varepsilon \|f\|_p\} \geq \varepsilon\},$$

where by "meas" , we mean the measure of a set.

Notice that if $\varepsilon_1 < \varepsilon_2$, then $M(p,\varepsilon_2) \subset M(p,\varepsilon_1)$. Also $\bigcup_{\varepsilon>0} M(p,\varepsilon) = L_p(I)$, since for any nonzero $f \in L_p(I)$ we have meas $\{|f| \geq \varepsilon\} \to \{f \neq 0\}$ as $\varepsilon \to 0$. In fact, any finite subset of $L_p(I)$ is contained in an $M(p,\varepsilon)$ for same $\varepsilon > 0$. Finally note that meas $\{|f| \geq \|f\|_p\} \geq 1$ implies $|f| = \|f\|_p$ almost every where.

The following theorem puts this observation to good use

Theorem F. [8] For a subset S of $L_p(I)$, the following are equivalent

(i) $S \subset M(p,\varepsilon)$ for some $\varepsilon > 0$.

(ii) For each $0 < p < q$, there exists a constant $c(q) < \infty$ such that $\|f\|_q \leq \|f\|_p \leq c(q)\|f\|_q$ for all $f \in S$.

(iii) For some $0 < p < q$, there exists a constant $c(q) < \infty$ such that $\|f\|_q \leq \|f\|_p \leq c(q)\|f\|_q$, for all $f \in S$.

In our thesis, we will use the notation c and C to denote such absolute constants which are of no significance to us and may differ on different occurren-ces, even in the same line. In order to emphasize that c or C depends only in parameters $\nu_1, \nu_2, ..., \nu_k$, the notation $c(\nu_1, \nu_2, ..., \nu_k)$ (respectively $C(\nu_1, \nu_2, ..., \nu_k)$) is used, if one of those v_i is $= p$, we mean that constant depends on p, only in the case $p < 1$. Also we will have constants $c_1, c_2, ...$ when we have a reason to keep trace of them in the computations that we have to curry in the proofs.

1.3 Moduli of Smoothness.

Moduli of smoothness are measurements criteria used mostly in approximation theory, numerical and real analysis. Such measurement of a function done by differentiability is not reliable in many aspects in approximation theory. The moduli of smoothness can provide more subtle measurements. Whenever a function f is defined on a metric space X, its modulus can be defined too, but here, we are only concerned with $X = J := [a, b]$. We will use moduli of smoothness which are connected with difference of higher orders.

The r^{th} symmetric difference of f is given by:

$$\Delta_h^r(f,x,J):=\Delta_h^r(f,x):=\begin{cases}\sum_{i=0}^{r}(-1)^i\binom{r}{i}f\left(x+\frac{r}{2}h-ih\right) & \left(x\pm\frac{r}{2}h\right)\in J\\ 0 & \text{o.w}\end{cases}.$$

The forward and backward r^{th} differences of f are defined respectively by:

$$\vec{\Delta}_h^r(f,x,J):=\vec{\Delta}_h^r(f,x):=\begin{cases}\sum_{i=0}^{r}(-1)^i\binom{r}{i}f(x+(r-i)h) & x,(x+rh)\in J\\ 0 & \text{o.w}\end{cases},$$

and

$$\overleftarrow{\Delta}_h^r(f,x,J):=\overleftarrow{\Delta}_h^r(f,x):=\begin{cases}\sum_{i=0}^{r}(-1)^i\binom{r}{i}f(x-ih) & x,(x-rh)\in J\\ 0 & \text{o.w}\end{cases}.$$

Then the r^{th} ***usual (ordinary) modulus of smoothness*** defined by:

$$\omega_r(f,t,J)_p:=\sup_{0<h\le t}\left\|\Delta_h^r(f,.)\right\|_{L_p(J)},\ t\ge 0.$$

A new way of measuring smoothness was moduli of smoothness with weighted, a modification of the moduli of smoothness based on differences Δ_u^r in which the step is of the form $u:=\varphi(x)h$ and is therefore allowed to depend up on the position of x in the interval $J:=[a,b]$. The motivation for this lies in the properties of algebraic polynomial approximation. The requirements on the smoothness of f can be relaxed if x is close to a and b, without impairing the error of approximation. Several authors, among them Ivanov [28], have introduced moduli of this type, but the most useful proved to be the last version, which is introduced by Ditzian and Totik [19]. Thus the need for this new concept arises to solve some basic problems, such as characterizing the behavior of best polynomial approximation in $L_p(J)$.

The ***Ditzian – Totik modulus of smoothness*** which is defined for $f \in L_p(J)$ as follows:

$$\omega_r^{\varphi}(f,t,J)_p := \sup_{0<h\le t} \left\| \Delta_{h\varphi(.)}^r (f,.) \right\|_{L_p(J)}, \ t \ge 0 .$$

If the interval $I := [-1,1]$ is used in any of the above notations, it will be omitted for the sake of simplicity, for example

$$\omega_r(f,t)_p := \omega_r(f,t,I)_p \ \text{ and } \ \omega_r^{\varphi}(f,t)_p := \omega_r^{\varphi}(f,t,I)_p .$$

Also we will denote

$$\omega_r(f,t) := \omega_r(f,t,I)_\infty \ \text{ and } \ \omega_r^{\varphi}(f,t) := \omega_r^{\varphi}(f,t,I)_\infty .$$

In the application the φ usually used

$\varphi(x) := \sqrt{\left(1-x^2\right)}$ for the interval $[-1,1]$

$\varphi(x) := \sqrt{x(1-x)}$ and $\varphi(x) := \sqrt{x}\,(1-x)$ for $[0,1]$

$\varphi(x) := x$, $\varphi(x) := \sqrt{x}$ and $\varphi(x) := \sqrt{x(1+x)}$ for $(0,\infty)$.

For the forward and backward differences, $\overline{\Delta}_h^r$, the ordinary modulus of smoothness is defined by:

$$\overline{\omega}_r(f,t,J)_p := \sup_{0<h\le t} \left\| \overline{\Delta}_h^r (f,.) \right\|_{L_p(J)}, \ t \ge 0 .$$

We also recall the ***averaged modulus of smoothness*** which is defined for such functions f on the interval J as follows:

$$w_r(f,t,J)_p := \left(t^{-1} \int_0^t \int_J \left| \Delta_h^r(f,.) \right|^p \right)^{\frac{1}{p}} \quad , \quad t \ge 0 \ .$$

These moduli of smoothness are equivalent to the ordinary modulus of smoothness (see [14] and [19], that is

$$C_1(r,p)\,\overline{\omega}_r(f,t,J)_p \le \omega_r(f,t,J)_p \le C_2(r,p)\,\overline{\omega}_r(f,t,J)_p ,$$

and

$$c_1(r,p)\,w_r(f,t,J)_p \le \omega_r(f,t,J)_p \le c_2(r,p)\,w_r(f,t,J)_p ,$$

in this case we write $\overline{\omega}_r \sim \omega_r \sim w_r$.

We always have that $\omega_r^{\varphi}(f,t,J)_p \le \omega_r(f,t,J)_p \ , 0 < p \le \infty$. But the converse is not true in general. However in E. Bhaya [4], there has been proved that the moduli ω_r^{φ} and ω_r for a function f defined on $J := [a,b] \subseteq I$ are equivalent, if $|J| \approx \Delta_n(a)$, where $\Delta_n(a) := \frac{1}{n}\sqrt{(1-a^2)} + \frac{1}{n^2}$, namely

$$\omega_r(f,\Delta_n(a),J)_p \sim \omega_r^{\varphi}(f,n^{-1},J)_p .$$

Note that in [4] only the case in which $p \ge 1$ is considered, but the proof is the same for $0 < p < 1$. Furthermore, we borrow from [49] the notation of the length of the interval $J := [a,b] \subseteq I$, relative to its position in I. Namely,

$$/J/ := \frac{|J|}{\varphi((a+b)/2)},$$

where $|J| := b - a$ is the usual length of the interval J. Then we will prove the following lemma:

Lemma 1. For any, $f \in L_p(J)$, $0 < p < \infty$ and $J \subset I$, we have

$$\omega_r(f, |J|, J)_p \le \omega_r^{\varphi}(f, /J/)_p \;, \tag{2}$$

and for $t \ge 0$

$$\omega_r(f, t^2)_p \le C(r,p)\; \omega_r^{\varphi}(f,t)_p \,. \tag{3}$$

Proof:

We have $x \pm \frac{r}{2}h \in J$ if and only if $h \le \frac{|J|}{r}$, (4)

and we observe that, if $x \pm \frac{r}{2}h \in J$ and $x \pm \frac{r}{2}h\,\varphi(x) \in I$, we have $\left[x - \frac{r}{2}h, x + \frac{r}{2}h\right] \subseteq J$, then by using the following inequality from [49], $/J_1/ \le /J/$ for $J_1 \subseteq J$, we obtain

$$h \le \frac{/J/}{r}\varphi(x). \tag{5}$$

Hence, by using the above two inequalities (4), (5) and the property (b) of ω_r^{φ} (bellow), we obtain

$$\omega_r(f,|J|,J)_p \le \sup_{0<h<\frac{/J/}{r}\varphi(x)} \| \Delta_h^r(f,x) \|_{L_p(J)}$$

$$= \sup_{0<h<\frac{/J/}{r}} \| \Delta_{h\varphi(x)}^r(f,x) \|_{L_p(J)}$$

$$\le \sup_{0<h<\frac{/J/}{r}} \| \Delta_{h\varphi(x)}^r(f,x) \|_p$$

$$= \omega_r^{\varphi}\left(f,\frac{/J/}{r}\right)_p \le \omega_r^{\varphi}(f,/J/)_p .$$

Thus the proof of (2) is complete.

For prove (3), we recall the following inequality from[49]

$h \le \frac{2}{r}\varphi(x)$ for any h such that $x \pm \frac{r}{2}h\varphi(x) \in I$, then we get

$h^2 \le ch\,\varphi(x)$, where $c \ge 1$.

Put $\bar{h} : h^2$, then

$$\omega_r(f,t^2)_p = \sup_{0<\bar{h}<t^2} \| \Delta_{\bar{h}}^r(f,x) \|_p$$

$$\le \sup_{0<h<t} \| \Delta_{ch\,\varphi(x)}^r(f,x) \|_p$$

$$= \sup_{0<h<ct} \| \Delta_{h\varphi(x)}^r(f,x) \|_p = \omega_r^{\varphi}(f,ct)_p$$

$$\le C(r,p)\omega_r^{\varphi}(f,/J/)_p . \Diamond$$

Now let us make a comparison between the classical modulus of smoothness and its extension to ω_r^{φ} by the following properties from [19]:

The following most basic facts about $\omega_r(f,t,J)_p$, satisfied for the two moduli

(a) $\lim_{t\to 0} \omega_r(f,t,J)_p = 0$ for all $f \in L_p(J), 0 < p < \infty$.

(b) $\omega_r(f,t,J)_p$ is an a nondeceasing function of t.

(c) $\omega_r(f,\lambda t,J)_p \le C(p)\ \lambda^r \omega_r(f,t,J)_p$ for $\lambda \ge 1$.

Another basic properties of the classical modulus of smoothness is

(d) $\omega_{r+1}(f,t,J)_p \le 2\,\omega_r(f,t,J)_p$,

which is generalized using the following inequality in [19]

($\bar{d}$) $\omega^{\varphi}_{r+1}(f,t,J)_p \le c(p)\ \omega^{\varphi}_r(f,t,J)_p$.

Another inequality about the classical modulus of smoothness is

(e) $\omega_{r+k}(f,t,J)_p \le c\, t^k\ \omega_r(f^{(k)},t,J)_p$, $p \ge 1$,

which is valid if $f^{(k)} \in L_p(J)$ and $f^{(k-1)}$ is absolutely continuous in every closed subinterval of J .

The inequality (e) is not true for $0 < p < 1$. But the inequality [19]

($\bar{e}$) $\omega^{\varphi}_{r+k}(f,t,J)_p \le c(p)\ t^k \omega^{\varphi}_r\left(f^{(k)},t,J\right)_p$,

is true if $f \in W^k_p(J)$ and $p > 0$, where $W^k_p(J)$ is the sobolove space and its consists of all functions in $L_p(J)$ such that $f^{(k-1)}$ are absolutely continuous and $f^{(k)} \in L_p(J)$. From the above discussion, we convinced that in order to estimate the rate of approximation f in $L_p(J)$, the measurement of smoothness ω_r^{φ} is the appropriate tool in the case $0<p<1$. Furthermore, it is well known that, from (e) if $f \in W_p^k(I)$, with $1 \le p < \infty$, then $\omega_m(f,t)_p \le c\, t\, \omega_{m-1}(f',t)_p \le \cdots$, but its inverse with a constant independent of f is not true in general. Hu [21], proved that for splines S with any knot, $\omega_m(S,t)_p \sim t\,\omega_{m-1}(S',t)_p \sim t^2\omega_m(S'',t)_p \sim t^k\omega_{m-k}(S^{(k)},t)_p$, where $k \le m$ and in general, by $\omega_0(f^{(k)},t)_p$, we mean $\left\| f^{(k)} \right\|_p$.

1.4 Shape Preserving Approximation

The connection between structural properties of a function and its degree of approximation is a significant problem that approximation theory considers. It aims at relating the smoothness of a function to the rate of decrease of the degree of approximation to zero. In this thesis, we examine these questions for algebraic polynomials as well as a particular case of piecewise polynomials approximation, that is, approximation by piecewise linear functions. Positive approximation by piecewise linear functions is also considered.

The study will be restricted to only one aspect of the problem which is the derivation of estimates for the degree of approximation in terms of smoothness. These are the direct estimates of approximation which are called Jackson – type estimates or Jackson's theorems, accredited to D. Jackson who reached such results for Trigonometric approximation [30].

The conservation of certain geometric properties of the data by the designed mathematical might be the main point of view in many applications. These properties include positivity, monotonicity, convexity and in general, k-convexity. This is the Topic that so called ***shape preserving approximation or constrained approximation*** is concerned with. Whenever constraints emerge, the situation becomes more difficult to get direct estimates, but the researches concentrating on such point have been widely acquired attention in recent times for the theory of non-constrained problems is still useful.

We intend to refer to those modifications which are essential in making a break-through approach. We do this for coconvex approximation by algebraic polyno-mials. The main objective of this thesis is to provide the answer of the question that whether the constraint cost anything or can we achieve the same degree of approximation as in the non-constrained case?

First: In Positive (or monotone) approximation, we have given a function f which is positive (nondecreasing) in I . We wish to approximate f by an algebraic polynomials, splines or piecewise polynomials which is also positive (nondecreasing) in I . We denote the class of all positive functions in I , by Δ^0 and by Δ^1 the class of all nondecreasing function in I , and for $f \in L_p(I) \cap \Delta^0$ with $0 < p \leq \infty$, we have the following notations:

$$E_n^{(0)}(f)_p := \inf_{P_n \in \Pi_n \cap \Delta^0} \|f - P_n\|_p ,$$

the ***degree of positive polynomial approximation***, and

$$E_n(f)_p := \inf_{P_n \in \Pi_n} \|f - P_n\|_p ,$$

the ***degree of unconstrained polynomial approximation***.

To be more specific, we discuss some of those results concluded in this topic as well as our results.

If $p = \infty$ (i.e., in the uniform normed space), the uniform estimates for positive approximation are not of much significance, since the rate of positive polynomial approximation has the same degree as that of nonconstrained polynomial approximation, that is

$$E_n(f) \leq E_n^{(0)}(f) \leq 2E_n(f).$$

At the same time, if $0 < p < \infty$, the situation would be completely different. It was shown by Hu, Kopotun and Yu [22] (see also[23], [29], [41] and [65]) that for any $f \in L_p(I) \cap \Delta^0$ with $o < p < \infty$

$$E_n^{(0)}(f)_p \leq C(p)\omega^{\varphi}\left(f, n^{-1}\right)_p \quad , \tag{6}$$

on the other hand, for every $n \in \mathbb{N}$, $0 < p < \infty$ and $A > 0$, there exists a function $f \in L_p(I) \cap \Delta^0$ such that

$$E_n^{(0)}(f)_p > A\,\omega_2^{\varphi}(f,1)_p \, .$$

That is, ω^{φ} is best possible rate, because it can not be replaced by $\omega_2^{\varphi}(f,1)_p$. Also, in [22] it has been asserted and proved that the same thing is sitill true for positive spline approximation with the rate drops to ω . However, these results had been obtained previously, but for splines with equally spaced knots, but Hu, Kopotun and Yu [22] proved if $T_n := \{z_0, z_1, \ldots, z_n \mid -1 := z_0 < z_1 < \ldots < z_n := 1\}$ with $n \geq 1$ be a given knot sequence, not necessary that the distance between them to be equal, that

Theorem G. For $f \in L_p(I) \cap \Delta^0$, with $0 < p < \infty$, there is a spline $S \in C^{r-2}(I) \cap \Delta^0$ of order $r \geq 2$ with Knot T_n such that

$$\| f - S \|_p \leq C(\rho, p)\omega\left(f, n^{-1}\right)_p ,$$

where $\rho := \max\left\{\frac{z_{i+2}-z_{i+1}}{z_{i+1}-z_i} \mid i=0,1,\dots,n-1\right\}$ is the scale of the knot T_n.

Also they proved that ω can not be replaced by $\omega_2(f,1)_p$, for any $0<p<\infty$, even if splines of any order on any given (fixed) knot sequence are used and no continuity is desired.

In Chapter 2, we improve Hu, Kopotun and Yu's [22] result, we mean, we try to obtain an estimate in which ω_2^{φ} is involved by using a piecewise linear approximation. The first piecewise linear function, we have used is $S_n f$ which is introduced by DeVore and Yu [15] on a specific partition of interval I, and the existence of this partitioning is guaranteed by them, using a constructive proof. The proof establishes a pointwise estimate for monotone polynomials approximation. This idea of DeVore and Yu was successfully applied by Leviatan [47] to give a global estimate on monotone approximation. Both proofs are based on a two – stages approximation, in the first stage, a function $f \in L_\infty(I)$ is approximated by a piecewise linear function $S_n f$ which interpolates f in the extreme points of the sub intervals of the partition of the interval I. While the Second stage is approximate this piecewise linear function $S_n f$ by an appropriate algebraic polyno-mial. However we applied the same idea to prove one of our results on coconvex polynomial approximation in Chapter 3. While in the second stage we use the same piecewise linear function in the first stage to define an appropriate algebraic polynomial. Because of this reason, we interest in “approximation by a piecewise linear functions”. In Chapter 2, we approximate any function $f \in L_p(I)$, (not necessary to be positive on I) by a piecewise linear functions and we obtain estimates in term of the second Ditzian – Totik modulus of smooth-ness, and at first we prove

Theorem 7. For $n \geq 2$ and $f \in L_p(I)$ with $0 < p < \infty$ we have

$$\| f - S_n f \|_p \leq C(p)\, \omega_2^{\varphi}\left(f, \frac{1}{n} \right)_p \quad .$$

The estimate emphasizes that, if no continuity desired we can obtain estimates for positive approximation by splines of order 2, which involved the second Ditzian – Totik modulus of smoothness, hence involved the second order ordinary modulus of smooth-ness. Then we use another two piecewise linear functions $U_n f$ and $V_n f$ which introduced by Zoltan [67], and we obtain the same estimate for $U_n f$ as we have for $S_n f$ in Theorem 7. Despite the different between $U_n f$ and $S_n f$ in the definition, they also differ in the aspect that $U_n f$ interpolates f only at the end points of the whole interval I. The following theorem stated that the estimates for both $U_n f$ and $S_n f$ are the same.

Theorem 8. For $f \in L_p(I), 0 < p \leq \infty$ and $n \geq 2$, then

$$\| f - U_n f \|_p \leq C(p)\, \omega_2^{\varphi}\left(f, \frac{1}{n}\right)_p \ .$$

Thereafter, we approximate those functions $f \in L_p(I)$ with $1 \leq p < \infty$ by another type of piecewise linear functions which is $V_n f$ that interpolates f at each of those points which $S_n f$ interpolates f at them, and the difference between them besides that in their definitions, is $S_n f$ will continuous in case f is continuous, but this is not necessarily true for $V_n f$. Then we prove

Theorem 9. Let $f \in L_p(I)$ with $1 \leq p < \infty$ and $n > r \geq 2$, then

$$\| f - V_n f \|_p \leq C(r)\left(\omega_r^{\varphi}\left(f, n^{-1}\right)_p + \frac{1}{n} \int_0^{n^{-1}} \frac{\omega_r^{\varphi}(f, t)_p}{t^2}\, dt + E_0(f)_p \right) .$$

As an immediate consequence of the above theorems we have the following corollary.

Corollary 10. For any $f \in L_p(I)$ with $1 \le p < \infty$ and $n \ge 2$, we have

$$\| f - V_n f \|_p \le C \left(\frac{1}{n} \int_0^{n^{-1}} \frac{\omega_2^{\varphi}(f,t)_p}{t^2} dt + \| f \|_p \right) .$$

Second: In coconvex polynomial approximation, we are given a function f changes its convexity finitely many times in the interval I. We are interested in estimating the degree of approximation of f by polynomials which are coconvex with it, namely, polynomials that change their convexity exactly at the points where f does. Question of this nature first appeared in the work of D. J. Newman and et al (see [54], [55] and [56]). They dealt not with a function f changes convexity, but rater with f changes monotonicity finitely many times in I and they were able to obtain weaker Jackson type estimates on the degree of approximation of that function by polynomials which were truly comonotone with it as well as some proper Jackson estimates when the polynomials were comonotone with f except near the points where a change of monotonicity of f occurred. Later Newman [53], and also Ilive [27], obtained the proper Jackson estimate involving the modulus of continuity of f, ω, for the approximation by polynomials that were truly comonotone with f. Then Beatson and Leviatan [1] obtained the desired estimates under the assumption that f possesses a continuous derivative in I. Afterwards, in general this topic was developed by a number of good and convincing results were accomplished (for more detail see references). Regarding "Coconvex Polynomial approximation" several results have been obtained in the uniform space, but in $L_p(I)$ with $0 < p < \infty$, it seems that nothing like have been achieved.

To be specific, Let $s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and let $Y_s = \{y_i\}_{i=1}^s$ be the set of points such that $-1 := y_0 < y_1 < y_2 < \ldots < y_s < y_{s+1} := 1$, where $s = 0$, $Y_0 := \phi$. For Y_s we set

$$\pi(x) := \pi(x, Y_s) := \prod_{i=1}^{s} (x - y_i) \text{ and } \delta(x) := \operatorname{sgn}(\pi(x)), \tag{11}$$

where the empty product is equal to 1.

Let $\Delta^2(Y_s)$, be the set of all functions f that change convexity at the points $y_i \in Y_s$, and are convex near 1. In particular, if $s = 0$, then f is convex on I, and write $f \in \Delta^2$, that is (the ***divided differences*** $[x_0, x_1, x_2; f]$ are nonnegative for all choices of three distinct points x_0, x_1 and x_2), where the divided difference of a function f at the points $x_0, x_1, \ldots, x_n$ are defined by(see [4])

$$[x_0, x_1, \ldots, x_n; f] := \sum_{j=0}^{n} \frac{f(x_i)}{\prod_{\substack{i=0 \\ i \neq j}}^{n} (x_j - x_i)}.$$

Moreover, if f is twice differentiable function on I (i.e., $f \in C^2(I)$), then

$$f \in \Delta^2(Y_s), \text{ if and only if } f''(x)\pi(x) \ge 0, \quad \forall x \in I,$$

$$(\text{or equivalently, if and only if } f''(x)\delta(x) \ge 0, \ \forall x \in I).$$

In Chapter 3, the approximation will be carried out by polynomials $P_n \in \Pi_n$. Now, for $f \in L_p(I) \cap \Delta^2(Y_s)$, we denote by

$$E_n^{(2)}(f, Y_s)_p := \inf_{P_n \in \Pi_n \cap \Delta^2(Y_s)} \| f - P_n \|_p,$$

the ***degree of Coconvex polynomial approximation***. If $s = 0$ ($Y_0 := \phi$), then we write $E_n^{(2)}(f)_p := \inf_{P_n \in \Pi_n \cap \Delta^2} \| f - P_n \|_p$, which usually referred to us the ***degree of convex polynomial approximation***.

Interest in the subject began in 1980's with the work on convex polynomial approximation by Shvedove [58] who proved that for a given convex function $f \in L_p(I)$, $0 < p \le \infty$ and $n \ge 2$, there exists a convex polynomial P_n of degree not exceeding n, such that

$$\| f - P_n \|_p \le C(p)\omega_2\left(f, \frac{1}{n}\right)_p, \tag{12}$$

where $C(p)$ independent of both n and f. At the same time, he went on to prove that ω_2 in (12) can not be replaced by ω_4, while keeping the constant independent
of n and f.

In resent years (12) has been improved in a sequence of researches by DeVore, Leviatan and Yu, (for example, see [12], [48] and [64]) who were able to replace ω_2 by ω_2^{φ} , for $n \geq 1$.

In 1994 Hu, Leviatan and Yu [26] have shown that ω_2^{φ} can be replaced by ω_3 in the uniform space.

In the same year, Kopotun [34] had proved that ω_3 can be replaced by ω_3^{φ} .

For the quasi-norm spaces L_p with $0 < p < \infty$, the problem remained unsolved till DeVore, Hu and Yu [11], in 1996, were proved that ω_2 in (12) can be replaced by ω_3^{φ} with the constant $C(p)$ remaining independent of f and n , then closing the gap which was left by Shvedov [58].

But for Coconvex polynomial approximation, what has been known yet restricted to the uniform norm space, besides some few results which are known for quasi-norm spaces L_p ,($0 < p < \infty$) .

First of all in 1981, Beatson and Leviatan gave a remark in [1] it might be possible to obtain a Jackson – type estimate of coconvex approximation of a function with only one regular convexity – turning point, and Yu [64] obtained a Jackson – type estimate of coconvex approximation of functions with one regular convexity – turning point also quoted her result of functions $f \in C^k(I)$ and $k \geq 3$ (the space of all function such that $f^{(k-1)}$ are absolutely continuous in I and $f^{(k)} \in C(I)$), with some extra conditions on convexity turning points.

In 1993, Wu and Zhou [63] and Zhou [66], they proved that for $0 < p \le \infty$, it is impossible to get a Jackson – type estimate of coconvex approximation involving $\omega_4(f,1)_p$ with constants independent of n and f.

Afterwards, in 1995 Kopotun [37] obtained the following result for twice differentiable functions.

Theorem H. For a function $f \in C^2(I) \cap \Delta^2(Y_s)$ with $1 \le s < \infty$, there is a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$ such that

$$\| f - P_n \| \le C(s) \frac{1}{n^2} \omega^{\varphi}\left(f'', \frac{1}{n} \right),$$

$$\| f' - P_n' \| \le C(s) \frac{1}{n} \omega^{\varphi}\left(f'', \frac{1}{n} \right),$$

and

$$\| f'' - P_n'' \| \le C(s) \omega^{\varphi}\left(f'', \frac{1}{n} \right),$$

for all $n \ge N := N(Y_s)$ is a constant depending on the location of points of Y_s in I, and $C(s)$ is a constant depend only on s - the number of convexity change.

In 2003 E. Bhaya [4] improved Kopotun's result for functions $f \in L_p^1(I) := \{ f ; f, f' \in L_p(I) \}$ with $1 \le p < \infty$ (the space of all of those function which are 1-fold integrals in $L_p(I)$ with $1 \le p < \infty$). Namely, she proved the following theorem.

Theorem I. For a function $f \in L_p^1(I) \cap \Delta^2(Y_s)$ with $1 \le p < \infty$, there is a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$ such that

$$\| f - P_n \|_p \le C(s)\omega_2^{\varphi}\left(f', \frac{1}{n}\right)_p ,$$

and

$$\| f' - P_n' \|_p \le C(s)\omega_2^{\varphi}\left(f', \frac{1}{n}\right)_p ,$$

for all $n \ge N := N(Y_s)$.

Also, in 1999 Kopotun, Leviatan and Shevchuk [43] improved Koputon's result Theorem H, for the uniform norm space, but not simultaneously. Namely, they proved

Theorem J. If $f \in C(I) \cap \Delta^2(Y_s)$ with $1 \le s < \infty$, then there is a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$ such that

$$\| f - P_n \| \le C(s, p)\omega_3^{\varphi}\left(f, \frac{1}{n}\right) \tag{13}$$

for all $n \ge N := N(Y_s)$.

Thereafter, several other results have been achieved for coconvex polynomial approximation throughout a number of researches by Leviatan and Shevchuk [50], [51] and by Kopotun, Leviatan and Shevchuk [44] [45].

Our first achievement in this area is to emphasize that the estimate (13) is exact in the quasi-norm spaces L_p with $0 < p < \infty$. Namely, we prove

Theorem 14. Let $f \in L_p(I)$ with $0 < p < \infty$ have s changes of convexity at $Y_s := \{y_i\}_{i=1}^s$, and denote $d(Y_s) := \min\{1 + y_1, y_2 - y_1, \ldots, y_s - y_{s-1}, 1 - y_s\}$. Then there exists a constant $A(s)$ such that for $n > N := N(Y_s) := \frac{A(s)}{d(Y_s)}$, there is a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$, such that

$$\|f - P_n\|_p \le C(s,p)\,\omega_3^{\varphi}\left(f, \frac{1}{n}\right)_p, \tag{15}$$

hence

$$\|f - P_n\|_p \le C(s,p)\,\omega_3\left(f, \frac{1}{n}\right)_p. \tag{16}$$

The estimate (15) is best possible in that one can not replace $\omega_3^{\varphi}(f, 1/n)_p$ by any higher modulus of smoothness, even not with the larger ordinary modulus of smoothness. This is due to the work of Shvedov [58] in case $s=0$, and to Wu and Zhou [63], Zhou [66] in case $s>0$, (as we mentioned above).

As an immediate consequence of Theorem (16), we have the following corollary

Corollary 17. Let $f \in W_p^r(I)$ with $r=1,2,3$ have s changes of convexity at $Y_s := \{y_i\}_{i=1}^s$, and denote $d(Y_s) := \min\{1+y_1, y_2-y_1, \ldots, y_s - y_{s-1}, 1-y_s\}$. Then there exists a constant $A(s)$ such that for $n > N := N(Y_s) := \frac{A(s)}{d(Y_s)}$, there is a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$, such that

$$\| f - P_n \|_p \le C(s,p) n^{-r} \| f^{(r)} \|_p .$$

This corollary is obtained directly from (15). However, we can not obtain it from (16), in the case $p<1$, since the inequality $\omega_{r+k}(f,t)_p \le c(p) t^k \omega_r(f^{(k)},t)_p$ is not satisfied in that case. Moreover, there is no non-zero continuous linear functional, estimate

$$E_n(f)_p \le C(r,p) n^{-1} \omega_r(f', 1/n)_p ,$$

is not true in general for any $r \in \mathbb{N}_0$. In [16] Ditzian proved the rate of approximations of simultaneous approxim-ations of a function and its derivatives is very bad (see Kopotun [36] also). This pathological nature of the spaces L_p, $p < 1$ might become the reason of lag of the achievements in these spaces. Furthermore, with this bad properties, it is well known that for unconstrained approximation the usual Jackson – Type estimates, involve the first derivative is not longer valid if $p < 1$ (see Kopotun [39], for example). However, it does not guarantee that the same is true in constrained case, since the functions satisfy some shape preserving constraint from a proper subset of W_p^k . In fact, it was shown in [39] that for convex function one can get estimates which are not true in the general (Unconstrained) case.

Now, to emphasize that the dependence of constant N in Theorem (14) is essential we construct a negative theorem, which we prove for $s \geq 4$ and $1 \leq p < \infty$, we can not have any Jackson – type estimates, even involving modulus of continuity ω. Namely, we prove

Theorem 18. For each $n \geq 3$, $A > 0$, $s \geq 4$ and any $1 \leq p < \infty$, there exist a points $-1 < y_1 < y_2 < \ldots < y_s < 1$ and a function $f \in L_p(I) \cap \Delta^2(Y_s)$ Such that

$$\left\| f - P_n \right\|_p > A\,\omega\left(f, \frac{1}{n} \right)_p ,$$

for any polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$.

At the same time, we will prove that, if $f \in L_p(I)$, $0 < p < \infty$, which has only one inflection point it would be possible for the constants to be independent of the location of inflection point in the interval I, which is stated in the following theorem

Theorem 19. Let $f \in L_p(I) \cap \Delta^2(\{y_1\})$ and $0 < p < \infty$, that is f has only one inflection point in I . Then

$$E_n^{(2)}\left(f, \{y_1\} \right)_p \leq C(p)\,\omega_2^{\varphi}\left(f, \frac{1}{n} \right)_p , \ \forall n \geq 1 .$$

In the following theorem we show that, for $0 \leq s < \infty$ we can take constants N and C to be independent of the location of convexity change points, if f itself continuous piecewise polynomials of any order, on the Chebyshev partition.

Theorem 20. For every $k,n \in \mathbb{N}$, $s \in \mathbb{N}_0$ and $0 < p < \infty$, if S is a continuous piecewise polynomial , of order k ,on the Chebyshev partition of the interval I , belonging to $\Delta^2(Y_s)$, which form a single polynomial near the points of convexity change, then there is a polynomial $P_n \in \Delta^2(Y_s)$, of degree not exceeding $c(s)n$. Such that

$$\| S - P_n \|_p \le C(s,p)\omega_2\left(S, \frac{1}{n} \right)_p ,$$

and

$$\| S - P_n \|_p \le C(k,s,p)\omega_2^{\varphi}\left(S, \frac{1}{n} \right)_p .$$

We will strength our result in the above theorem in the following two theorems,

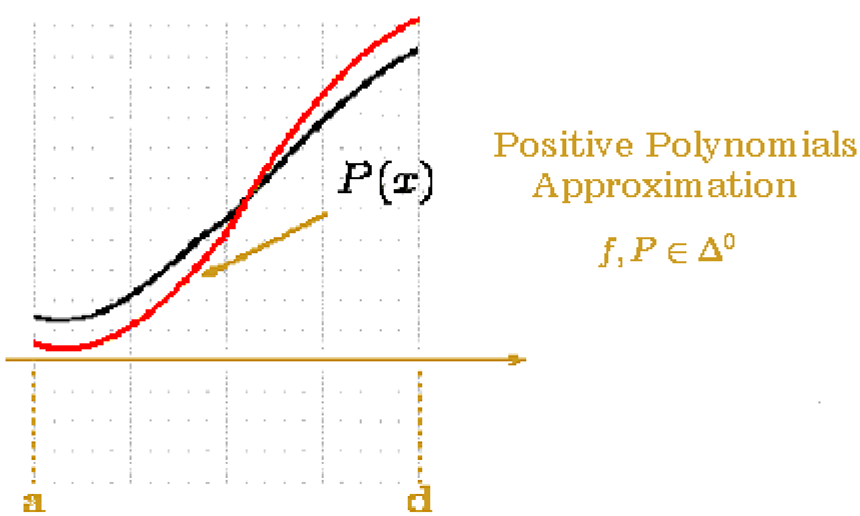
Theorem 21. For every $s \in \mathbb{N}_0$, $0 < p < \infty$ and $k,n \in \mathbb{N}$ there are constants $C(k,s,p)$ and $c(s,p)$, such that if S is a continuous piecewise polynomial, of order k , on the Chebyshev partition of the interval I , which has continuous first derivative in I and belong to $\Delta^2(Y_s)$, such that S form a single polynomial near convexity change points, then there is a polynomial $P_n \in \Delta^2(Y_s)$ of degree not exceeding $c(s,p)n$, satisfies

$$\| S - P_n \|_p \le C(k,s,p)\omega_k^{\varphi}\left(S, \frac{1}{n} \right)_p .$$

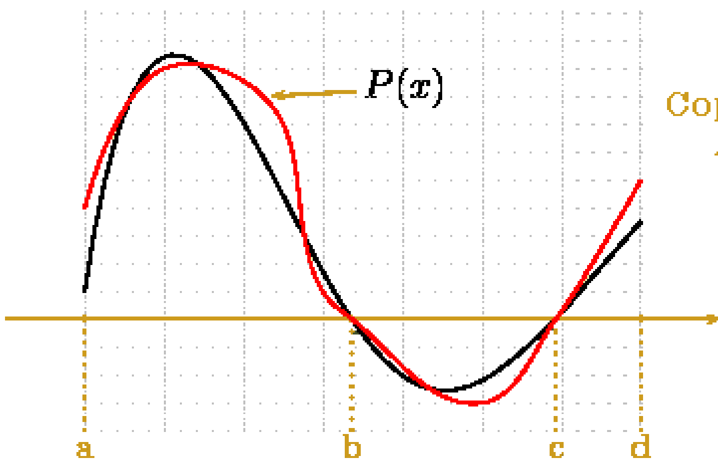
Theorem 22. For every $s \in \mathbb{N}_0$, $0 < p < \infty$ and $k, n \in \mathbb{N}$ there are constants $C(k,s,p)$ and $c(s,p)$, such that if S is a continuous piecewise polynomial, of order k , on the Chebyshev partition of the interval I , belonging to $\Delta^2(Y_s)$ and S form a single polynomial near y_i's, then there is a polynomial $P_n \in \Delta^2(Y_s)$ of degree not exceeding $c(s,p)n$, satisfies

$$\| S - P_n \|_p \le C(k,s,p)\,\omega_k^{\varphi}\left(S, \frac{1}{n}\right)_p .$$

1.5 Geometric means of Shape Preserving Approximation



$$E_n^{(0)}(f)_p := \inf\{\|f - P\|_p \mid P(x) > 0\}$$

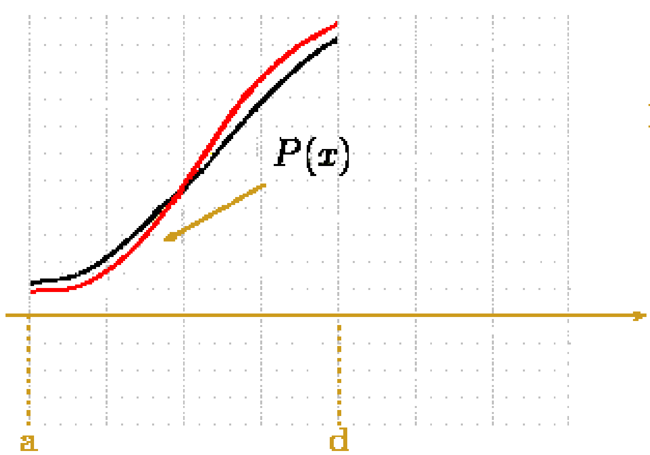


Copositive Polynomials Approximation

$Y_s = \{b, c\}, s = 2$

$f, P \in \Delta^0(Y_s)$

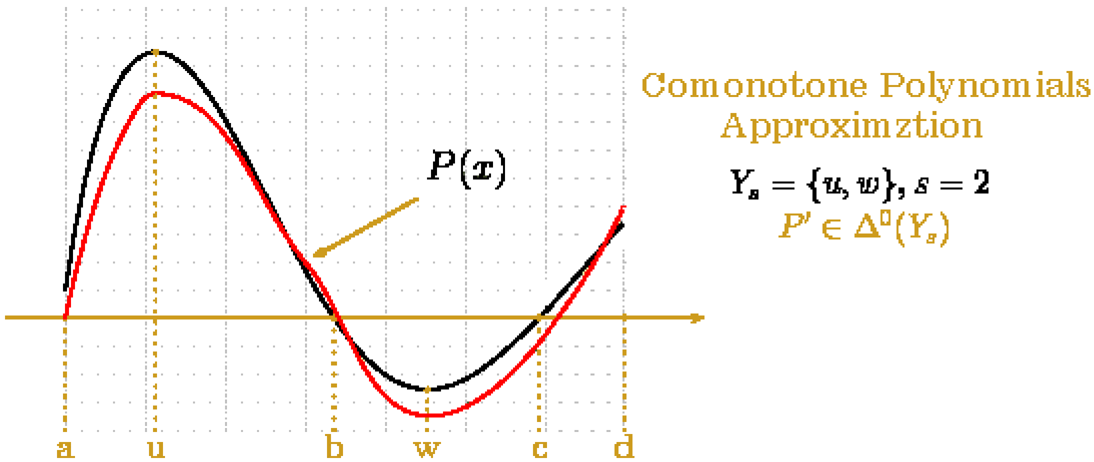
$$E_n^{(0)}(f, Y_s)_p := \inf\{\|f - P\|_p \mid f(x) \cdot P(x) > 0\}$$



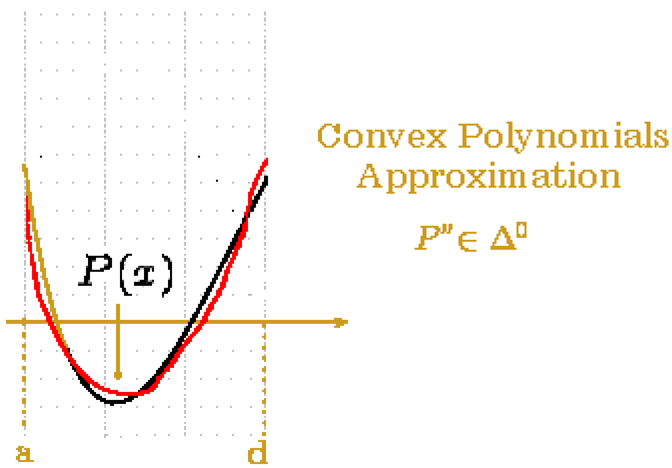
Monotone Polynomials Approximation

$P' \in \Delta^0$

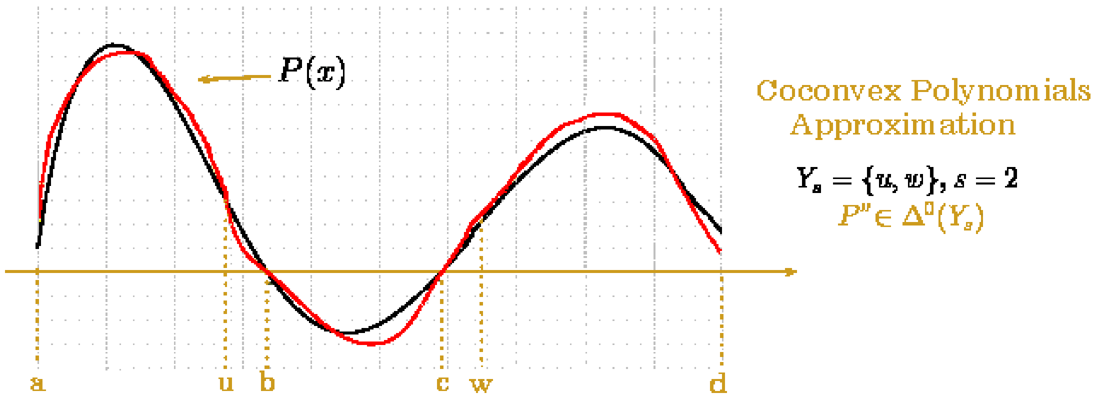
$$E_n^{(1)}(f)_p := \inf\{\|f - P\|_p \mid P'(x) \geq 0\}$$



$$E_n^{(1)}(f, Y_s)_p := \inf\{\|f - P\|_p \mid P'(x) \cdot f'(x) \geq 0\}$$



$$E_n^{(2)}(f)_p := \inf\{\|f - P\|_p \mid P''(x) \geq 0\}$$



$$E_n^{(2)}(f, Y_s)_p := \inf \{ \|f - P\|_p \mid P''(x) \cdot f'(x) \geq 0 \}$$

Note: if we take a look at the above pictures, we can notice that the error in the positive (or Copositive) polynomial approximation is greater than monotone (or Commonotone) polynomial approximation, rspectivly, as well as this greater convex (or Coconvex) polynomial approximation respectively .

Chapter Three

Coconvex Polynomial Approximation

After introduction and preliminaries, we are now in the heart of the matter. Let $f \in L_p(I)$ change its convexity finitely many times in the interval I , say s times at Y_s , We estimate the degree of approximation of f by polynomials which change convexity exactly at the points where f does. We discuss some Jackson type estimates where the constants involved depending on the location of the points of change of convexity. Also we show that in some cases the constants may be taken independent of the points of change of convexity, but that in some other cases this dependence is essential, but mostly we obtain such estimates for functions f that themselves are continuous piecewise polynomials on the Chebyshev partition, which form a single polynomial in a small neighborhood of each point of change of convexity.

3.1 Introduction and Main Results

Throughout this chapter we use the following notations, given $n \in \mathbb{N}$, we set $x_{-1} = 1$, $x_n = -1$ and

$x_j := x_{j,n} := \cos\left(\frac{j\pi}{n}\right)$, the Chebyshev partition of interval I,

we denote $I_j := I_{j,n} := [x_j, x_{j-1}]$, $h_j := |I_j| := x_{j-1} - x_j$,

and $\psi_j := \psi_{j,n} := \dfrac{h_j}{|x - x_j| + h_j}$, $j = 0, 1, \ldots, n$.

Let $\sum_{k,n}$ be denote the collection of all piecewise polynomials of degree $k-1$, on the Chebyshev partition and let $\sum_{k,n}^1 \subseteq \sum_{k,n}$, be the subset of all continuously differentiable piecewise polynomials on the Chebyshev partition. That is, if $S \in \sum_{k,n}$, then

$$S \mid I_j := P_j \ , \ j = 1, 2, \ldots, n \quad \text{where} \quad P_j \in \Pi_{k-1},$$

$$P_j(x_j) = P_{j+1}(x_j) \ , \ j = 1, 2, \ldots, n-1,$$

and if $S \in \sum_{k,n}^1$, then in addition,

$$P_j'(x_j) = P_{j+1}'(x_j) \ , \ j = 1, 2, \ldots, n-1.$$

Given $0 \le s < \infty$, let

$$O_i := O_{i,n}(Y_s) := (x_{j+1}, x_{j-2}) \ , \text{ if } \ y_i \in [x_j, x_{j-1}) \ ,$$

and

$$O := O(n, Y_s) := \bigcup_{i=1}^{s} O_i \quad , O(n, \phi) := \phi .$$

We write

$$j \in H := H(n, Y_s) \text{ , if } I_j \cap O = \phi .$$

Finally, Let $\sum_{k,n}(Y_s) \subseteq \sum_{k,n}$ and $\sum_{k,n}^{1}(Y_s) \subseteq \sum_{k,n}^{1}$ be denote the subsets of such piecewise polynomials for which

$$P_j \equiv P_{j+1} \text{ , whenever both } j, j+1 \notin H .$$

Our prime interest in this chapter is to generalize and emphasize that the estimate (13) is exact in the L_p - spaces with $0 < p < \infty$, and then we closed the gap due to the uncertainty between the previously known estimates involved ω_2^{φ} and the impossibility of having such estimates involving ω_4 . For this we prove the following theorem:

Theorem 3.1.1.

Let $f \in L_p(I)$ with $0 < p < \infty$ have s changes of convexity at $Y_s := \{y_i\}_{i=1}^{s}$, and denote $d(Y_s) := \min\{1 + y_1, y_2 - y_1, \ldots, y_s - y_{s-1}, 1 - y_s\}$. Then there exists a constant $A(s)$ such that for $n > N := N(Y_s) := \frac{A(s)}{d(Y_s)}$, there is a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$, such that

$$\| f - P_n \|_p \leq C(s,p)\,\omega_3^{\varphi}\left(f, \frac{1}{n}\right)_p , \qquad (3.1.2)$$

hence

$$\| f - P_n \|_p \leq C(s,p)\,\omega_3\left(f, \frac{1}{n}\right)_p . \qquad (3.1.3)$$

Note that, the estimate (3.1.2) is best possible, in the sense that one can not replace $\omega_3^{\varphi}\left(f, \frac{1}{n}\right)_p$, by any higher moduli of smoothness, this is due to the work Shvedov [58] in case f is convex on I, and to Wu and Zhou [63], Zhou [66] in case f has convexity change point.

Now, as immediate consequences of the above theorem we have the following corollary.

Corollary 3.1.4.

Let $f \in W_p^r(I)$ with $r = 1,2,3$ with $0 < p < \infty$ have s changes of convexity at $Y_s := \{y_i\}_{i=1}^s$, and denote $d(Y_s) := \min\{1 + y_1, y_2 - y_1, \ldots, y_s - y_{s-1}, 1 - y_s\}$. Then there exists a constant $A(s)$ such that for $n > N := N(Y_s) := \frac{A(s)}{d(Y_s)}$, there is a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$, such that

$$\| f - P_n \|_p \leq C(s,p)\, n^{-r} \| f^{(r)} \|_p .$$

The proof of Corollary 3.1.4, is follows directly from (3.1.2) and the property ($\bar{e}$) of Ditzian – Totik modulus of smoothness.◊

One may hope that to obtain a similar estimate such as (3.1.2) or (3.1.3) with constants independent to the location of convexity change points of Y_s with the same or lower moduli of smoothness. In Theorem 3.1.6, we will show that in some cases it is not possible to obtain a direct estimate which involves at least moduli of continuity with constants independent of Y_s. In other words we can not have any Jackson – type estimates when we replace $N(Y_s)$ by $N(s)$ with $s \geq 4$. Namely, we prove that, for any $A > 0$, $n \geq 3$ and $1 \leq p < \infty$, a function $f \in C(I)$ which has at least 4 – convexity change points exists, such that if $P_n \in \Pi_n$ is convex with f , then

$$\left\| f - P_n \right\|_p > A\,\omega\left(f, \frac{1}{n} \right)_p . \tag{3.1.5}$$

Or, we state it, in the following theorem:

Theorem 3.1.6.

For each $n \geq 3$, $A > 0$, $s \geq 4$ and any $1 \leq p < \infty$, there exists a set $Y_s := \{y_i\}_{i=1}^{s}$ and a function $f \in C(I) \cap \Delta^2(Y_s)$, such that for any polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$, satisfies

$$E_n^{(2)}(f, Y_s)_p > A\,\omega\left(f, \frac{1}{n}\right)_p .$$

While, the same inequality also holds in comonotone polynomial approximation with change the function to a function which has at least one monotonicty change point. In other word for each $n \geq 4$, $A > 0$, $s \geq 1$ and any $1 \leq p < \infty$, there exists a set $Y_s := \{y_i\}_{i=1}^{s}$ and a function $f \in C(I) \cap \Delta^2(Y_s)$, such that for any polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$, satisfies(3.1.5). To prove it, we can use the same example in the proof of Theorem 2.1.6 of [4], with the same technique of our proof of Theorem 3.1.6.

On the other hand, we will prove that, if $s = 1$ we can obtain the estimate similar to (3.1.3) for any $0 < p < \infty$ with constants independent of the location of convexity change point and the ordinary modulus of smoothness of second order. Namely, we prove

Theorem 3.1.7.

Let $f \in L_p(I) \cap \Delta^2(\{y_1\})$ and $0 < p < \infty$, that is f has only one inflection point in I . Then

$$E_n^{(2)}(f, \{y_1\})_p \leq C(p)\,\omega_2\left(f, \frac{1}{n}\right)_p , \ \forall n \geq 1 .$$

However, we will show that, we can have the above estimate with constants $N=1$ and C depend only on $s \geq 0$ (the number of convexity change points) and p, as $p \to 0$, if we replace f by a piecewise polynomial $S \in \sum_{k,n}(Y_s)$. Furthermore, we can replace ω_2 by ω_2^{φ}, if we take the above constant C to be depend on $k \geq 1$ (the order of the piecewise polynomial S). This is stated in the following theorem.

Theorem 3.1.8.

For every $k,n \in \mathbb{N}$, $s \in \mathbb{N}_0$ and $0 < p < \infty$, if $S \in \sum_{k,n}(Y_s) \cap \Delta^2(Y_s)$ then there is a polynomial $P_n \in \Delta^2(Y_s)$, of degree not exceeding $c(s)n$. Such that

$$\|S - P_n\|_p \leq C(s,p)\omega_2\left(S, \frac{1}{n}\right)_p, \tag{3.1.9}$$

and

$$\|S - P_n\|_p \leq C(k,s,p)\omega_2^{\varphi}\left(S, \frac{1}{n}\right)_p \tag{3.1.10}$$

It is quite natural to ask whether one can strength (3.1.10) in the sense of being able to replace ω_2^{φ} by Ditzian – Totik moduli of smoothness of higher order. This indeed turns out to be possible for ω_k^φ as we will show in Theorem 3.1.11. However this theorem is one of our main positive results

Theorem 3.1.11.

For every $s \in \mathbb{N}_0$, $0 < p < \infty$ and $k, n \in \mathbb{N}$ there are constants $C(k,s,p)$ and $c(s,p)$, such that if $S \in \sum_{k,n}(Y_s) \cap \Delta^2(Y_s)$, then there is a polynomial $P_n \in \Delta^2(Y_s)$ of degree not exceeding $c(s,p)n$, satisfies

$$\|S - P_n\|_p \le C(k,s,p)\,\omega_k^\varphi\left(S, \frac{1}{n}\right)_p .$$

A particular case of Theorem 3.1.11, is shown in Theorem 3.1.8, and by virtue of Lemma 3.2.34 below, to conclude the proof of Theorem 3.1.11, it suffices to prove

Theorem 3.1.12.

For every $s \in \mathbb{N}_0$, $0 < p < \infty$ and $k, n \in \mathbb{N}$ there are constants $C(k,s,p)$ and $c(s,p)$, such that if $S \in \sum_{k,n}^1(Y_s) \cap \Delta^2(Y_s)$, then there is a polynomial $P_n \in \Delta^2(Y_s)$ of degree not exceeding $c(s,p)n$, satisfies

$$\|S - P_n\|_p \le C(k,s,p)\,\omega_k^\varphi\left(S, \frac{1}{n}\right)_p .$$

Theorem 3.1.12, is trivial for the cases $k = 1,2$, since $\sum_{1,n}^{1} \subseteq \sum_{2,n}^{1} \subseteq \Pi_1$, thus it remains only to show it for $k \geq 3$.

3.2 Auxiliary results and lemmas

Firstly we begin with the following notations, for a given s and $n \in \mathbb{N}$, let

$$\Delta_n(x) := \frac{1}{n}\sqrt{\left(1 - x^2\right)} + \frac{1}{n^2} := \frac{1}{n}\varphi(x) + \frac{1}{n^2} \ ,$$

and

$$\Pi(x) := \prod_{i=1}^{s} \frac{|x - y_i|}{|x - y_i| + \Delta_n(x)} \ . \tag{3.2.1}$$

The following lemma contains simple but important inequalities which are used in almost all our proofs, (see [13], [22], [47], [49], [51] and [64])

Lemma 3.2.2.

The following inequalities hold for all $x, y \in I$ and $j = 1,2,\ldots,n$:

$$h_{j\pm1} < 3h_j , \tag{3.2.3}$$

$$h_j \leq \Delta_n(x) \leq 5h_j , \quad \text{for } x \in I_j , \tag{3.2.4}$$

$$\Delta_n(y)^2 \leq 4\Delta_n(x)\left(|x - y| + \Delta_n(x)\right) , \tag{3.2.5}$$

$$\frac{1}{2}\left(|x-y|+\Delta_n(x)\right)\le |x-y|+\Delta_n(y)\le 2\left(|x-y|+\Delta_n(x)\right). \qquad (3.2.6)$$

In particular,

$$\frac{1}{10}\left(\left|x-x_j\right|+h_j\right)<\left|x-x_j\right|+\Delta_n(x)<2\left(\left|x-x_j\right|+h_j\right). \qquad (3.2.7)$$

If $0\le j\le i<J\le n$, then

$$\frac{1}{2}(J-j)\le \frac{x_j-x_J}{x_i-x_{i+1}}\le (J-j)^2. \qquad (3.2.8)$$

Furthermore, if either $J\le 3j$ or $n-j\le 3(n-J)$, then

$$\frac{1}{2}(J-j)\le \frac{x_j-x_J}{x_i-x_{i+1}}\le 2(J-j). \qquad (3.2.9)$$

If $j\in H$, then

$$3\left|x_j-y_i\right|\ge h_j. \qquad (3.2.10)$$

Now, for $n\ge 10$, we have

$$C_1\varphi(x)n^{-1}\le x_{j-4}-x_{j+3}\le C_2\varphi(x)n^{-1}, \quad x\in[x_{j+3},x_{j-4}] \qquad (3.2.11)$$
$$j=5,6,\ldots,n-4,$$

$$\begin{aligned} 1+x &\le C_2\varphi(x)n^{-1}, \quad -1\le x\le x_{n-7}\\ 1-x &\le C_2\varphi(x)n^{-1}, \quad x_7\le x\le 1, \end{aligned} \qquad (3.2.12)$$

$$\begin{aligned} C_1\varphi(x)n^{-1} &\le x_{n-1}+1, \quad -1\le x\le x_{n-7}\\ C_1\varphi(x)n^{-1} &\le 1-x_1, \quad x_7\le x\le 1, \end{aligned} \qquad (3.2.13)$$

$$\int_{-1}^{1}\psi_j^{\alpha}(x)\,dx\le ch_j, \quad \alpha\ge 2, \qquad (3.2.14)$$

and

$$\sum_{j=1}^{n}\psi_j^{\alpha}(x)\le c, \ \alpha\ge 2. \qquad (3.2.15)$$

To prove our results we need the following lemmas.

Lemma 3.2.16. [50]

Let P_k be a polynomial of degree $\le k$ and $a < b$.

If Max $\{x \in [a,b]; P_k''(x) \le 0\} < \frac{b-a}{16k^3}$, then for every $\chi_0 \in [a,b]$

$$[a, \chi_0, b, P_k] \ge 0 .$$

Lemma 3.2.17. [50]

Let $b := 6(s+1)$, then for each $j \in H$, there exist polynomials τ_j and $\bar{\tau}_j$ of degree $\le c(s)n$, satisfying

$$\begin{aligned} &\tau_j''(x)\pi(x)\pi(x_j) \ge 0, \quad x \in I \\ &\bar{\tau}_j''(x)\pi(x)\pi(x_j) \le 0, \quad x \in I \setminus I_j \end{aligned} \tag{3.2.18}$$

$$\left|\bar{\tau}_j''(x)\right| \le c(s) h_j^{-1} \psi_j^{2b}(x) \frac{|\pi(x)|}{|\pi(x_j)|} \le c(s) \left|\bar{\tau}_j''(x)\right|, \quad x \in I \tag{3.2.19}$$

and

$$\begin{aligned} &\left|(x - x_j)_+ - \tau_j(x)\right| \le c(s) h_j \psi_j^{2(b-1)-s}(x) \\ &\left|(x - x_j)_+ - \bar{\tau}_j(x)\right| \le c(s) h_j \psi_j^{2(b-1)-s}(x), \quad x \in I \end{aligned} \tag{3.2.20}$$

Lemma 3.2.21. [26]

For $f \in L_p(I)$, $0 < p < \infty$ and $r \in N$ we have

$$\omega_r^{\varphi}(f, t)_p \leq c(r, p) \| f \|_p .$$

Also, we need the following lemma from [11], [51] and [36]. However there is a special case of this lemma in [22].

Lemma 3.2.22.

For a function $f \in L_p(I)$ with $0 < p < \infty$ and $k \in N$, the following inequality holds

$$\left(\sum_{j=1}^{n} \omega_r (f, h_j, \Im_j)_p^p \right)^{\frac{1}{p}} \leq C(B_0, r, p)\, \omega_r^{\varphi}(f, n^{-1})_p \quad ,$$

where, for every j, $I_j \subseteq \Im_j$ is such that $|\Im_j| \leq B_0 |I_j|$.

Proof:

$$\sum_{j=1}^{n}\omega_r(f,h_j,\Im_j)_p^p \le \sum_{j=1}^{n} C(r,p) W_r(f,|\Im_j|,\Im_j)_p^p$$

$$= C(r,p)^p \sum_{j=1}^{n} |\Im_j|^{-1} \int_0^{|\Im_j|}\int_{\Im_j} |\Delta_h^r(f,x,\Im_j)|^p \, dxdh$$

$$\le C(r,p)^p \sum_{j=1}^{n} \int_{\Im_j}\int_0^{\frac{|\Im_j|}{\varphi(x)}} \frac{\varphi(x)}{|\Im_j|} \left|\Delta_{h\varphi(x)}^r(f,x,\Im_j)\right|^p dxdh$$

$$\le C(r,p)^p \sum_{j=1}^{n} Cn \int_0^{Cn^{-1}}\int_{\Im_j} \left|\Delta_{h\varphi(x)}^r(f,x,\Im_j)\right|^p dxdh$$

$$\le C(r,p)^p Cn \int_0^{Cn^{-1}}\int_{-1}^{1} \left|\Delta_{h\varphi(x)}^r(f,x,)\right|^p dxdh$$

$$\le C(B_0,k,p)^p \, \omega_r^{\varphi}(f,n^{-1})_p^p ,$$

where we changed the order of integration and made a simple change of variables to arrive the second inequality, and to obtain the third inequality we used the following technique

Since $|\Im_j| \le B_0 h_j$, hence $\frac{1}{h_j} \le \frac{B_0}{|\Im_j|} \le \frac{B_0}{h_j}$, then for each $j = 5,6,\ldots,n-4$, from (3.2.11) we have

$$\frac{|\Im_j|}{\varphi(x)} \le \frac{B_0 h_j}{\varphi(x)} \le \frac{C_2 B_0}{n} \text{ and } \frac{\varphi(x)}{|\Im_j|} \le \frac{\varphi(x)}{h_j} \le C_1^{-1} n ,$$

but for $j = 1,2,3,4$, we have $\left|\Delta_{h\varphi(x)}^r(f,x,\Im_j)\right| = 0$, if $x + \frac{r}{2} h\varphi(x) > 1$, that is, if $h > 2(1-x)\frac{\varphi(x)}{r}$ and this mean that the inner integration in the second inequality is taken over $0 < h \le 2\frac{(1-x)\varphi(x)}{r} \le 2C_2 n^{-1}$ by (3.2.12), and by (3.2.13), we have $\frac{\varphi(x)}{|\Im_j|} \le \frac{\varphi(x)}{h_j} \sim \frac{\varphi(x)}{h_1} \le C_1^{-1} n$. Thus the third inequality is also holds for $j = 1, 2, 3, 4$, and for $j = n-5, n-6, \ldots, n$, we can use the same technique to obtain the third inequality. ◊

The following theorem about the property of convex function is important to our work.

Theorem 3.2.23. [57]

Suppose that $f : I \to \Re$ is convex function, then f satisfies Lipshchitz condition on any closed subinterval $[a, b]$ of I^{o} (interior of I), f is absolutely continuous on $[a, b]$ and in particular it's continuous on I^{o}, f has left and right nondecreasing derivatives, $f'_{-}(x)$ and $f'_{+}(x)$ on I. Furthermore, the set E where f' fail to exist is countable, and f' is continuous on $I \setminus E$.

The following theorem is well known, and is now called the Whitney (or Whitney type) theorem. It was proved by Burkill [7] ($k = 2, p = \infty$), Whitney [61] and [62] ($p = \infty$), Brudnyi [6] ($1 \le p \le \infty$), and Storozhenko [60] ($0 < p < 1$). It has application in many areas, and has been generalized to various classes of functions and other approximating spaces, but here we write only the "classical version" of it, which we needed to prove of our results.

Theorem 3.2.24. [40]

Let $f \in L_p[a,b]$, $0 < p < \infty$. Then there exists $q_{k-1} \in \Pi_{k-1}$, such that

$$\|f - q_{k-1}\|_{L_p[a,b]} \le C(k,p)\omega_k(f, b-a, [a,b])_p .$$

Lemma 3.2.25. [42]

Let J_1 and J_2 be subintervals such that $J_1 \subset J_2$. If $q_k \in \Pi_k$, then for $0 < p \le \infty$

$$\|q_k\|_{L_p(J_2)} \le c(k,p)\left(\frac{|J_2|}{|J_1|}\right)^{k+\frac{1}{p}} \|q_k\|_{L_p(J_1)} .$$

Let $I_{i,j} := I_{i,j,n}$ denote the smallest closed interval containing both I_i and I_j, and $h_{j,j} := |I_{i,j}|$.

For $S \in \sum_{k,n}$, set

$$a_{i,j}(S)_p := \|P_i - P_j\|_{L_p(I_i)}\left(\frac{h_j}{h_{i,j}}\right)^k ,$$

where P_i denote the polynomial defined by $P_i := P_i \mid I_i := S \mid I_i$.

We are going to call an interval A a proper interval, if its end points belong to the Chebyshev partition, that is, are among the x_j's.

For any proper interval A, let

$$a_k(S,A)_p := \max \left(a_{i,j}(S)_p\right),$$

where the maximum is taken over all i and j, such that $I_i \subseteq A$ and $I_j \subseteq A$. Finally, write

$$a_k(S)_p := a_k(S,I)_p .$$

Observe that, for each i and j, if ν between i and j, then we have the following inequality from [49],

$$\frac{h_{i,\nu}}{h_\nu} \le \frac{h_{i,j}}{h_j}, \tag{3.2.26}$$

and for the Chebyshev partition of the interval I, also from [49], we have for each $j = 1,2,\ldots,n$

$$\frac{2}{n} \le /I_j/ = \frac{|I_j|}{\varphi((x_j + x_{j+1})/2)} \le \frac{\pi}{n} . \tag{3.2.27}$$

Hence, we can obtain the following lemmas, which we needed to prove of our theorems.

Lemma 3.2.28.

For any $0 < p \le \infty$ and $S \in \sum_{k,n}$, we have

$$a_k(S)_p \le C(k,p)\omega_k^\varphi\left(S,\frac{1}{n}\right)_p \le C(k,p)a_k(S)_p . \tag{3.2.29}$$

Proof:

Let $S \in \sum_{k,n}$, to show the left hand side of (3.2.29). It's enough when we show for each i , j

$$\left\| P_i - P_j \right\|_{L_p(I_i)} \le C(k,p)\left(\frac{h_{i,j}}{h_j}\right)^k \omega_k^{\varphi}\left(S,\frac{1}{n}\right)_p . \tag{3.2.30}$$

To this end, let us first assume that $j = i \pm 1$, and then by Whitney's theorem 3.2.24, there exists a polynomial $q_{k-1} \in \Pi_{k-1}$, such that

$$\left\| S - q_{k-1} \right\|_{L_p(I_{i,j})} \le C(k,p)\omega_k\left(S, h_{i,j}, I_{i,j}\right)_p , \tag{3.2.31}$$

then

$$\begin{aligned}\left\| P_i - q_{k-1} \right\|_{L_p(I_i)} = \left\| S - q_{k-1} \right\|_{L_p(I_i)} &\le \left\| S - q_{k-1} \right\|_{L_p(I_{i,j})} \\ &\le C(k,p)\omega_k\left(S, h_{i,j}, I_{i,j}\right)_p .\end{aligned}$$

Now, we observe that $P_j - q_{k-1}$ is a polynomial of degree $\le k-1$, then by using (3.2.3), (3.2.31) and Lemma 3.2.25, we obtain

$$\begin{aligned}\left\| P_j - q_{k-1} \right\|_{L_p(I_i)} &\le \left\| P_j - q_{k-1} \right\|_{L_p(I_{i,j})} \\ &\le C(k,p)\left\| P_j - q_{k-1} \right\|_{L_p(I_j)} \le C(k,p)\left\| S - q_{k-1} \right\|_{L_p(I_{i,j})} \\ &\le C(k,p)\omega_k\left(S, h_{i,j}, I_{i,j}\right)_p .\end{aligned}$$

Thus

$$\left\| P_i - P_j \right\|_{L_p(I_i)} \le C(k,p)\omega_k\left(S, h_{i,j}, I_{i,j}\right)_p . \tag{3.2.32}$$

Hence, by using (2) and 3.2.27, we obtain

$$\left\| P_i - P_j \right\|_{L_p(I_i)} \le C(k,p)\omega_k^{\varphi}\left(S, n^{-1}\right)_p \ . \qquad (3.2.33)$$

This implies (3.2.30), in the case $j = i \pm 1$, otherwise assume that $i < j$, then for each ν, such that $i < \nu < j$, it follows from Lemma 3.2.25, the inequalities (3.2.26) and (3.2.33) that

$$\begin{aligned}
\left\| P_\nu - P_{\nu\mp1} \right\|_{L_p(I_i)} &\le \left\| P_\nu - P_{\nu\mp1} \right\|_{L_p(I_{i,\nu})} \\
&\le C(k,p)\left(\frac{h_{i,\nu}}{h_\nu}\right)^{k-1+\frac{1}{p}} \left\| P_\nu - P_{\nu\mp1} \right\|_{L_p(I_\nu)} \\
&\le C(k,p)\left(\frac{h_{i,j}}{h_j}\right)^{k-1+\frac{1}{p}} \left\| P_\nu - P_{\nu\mp1} \right\|_{L_p(I_\nu)} \\
&\le C(k,p)\left(\frac{h_{i,j}}{h_j}\right)^{k-1} \omega_k^{\varphi}\left(S, \frac{1}{n}\right)_p .
\end{aligned}$$

Therefore

$$\begin{aligned}
\left\| P_j - P_i \right\|_{L_p(I_i)} &\le C(p)\sum_{l=1}^{j-i} \left\| P_{j+1-l} - P_{j-l} \right\|_{L_p(I_i)} \\
&\le C(k,p)\sum_{l=1}^{j-i}\left(\frac{h_{i,j}}{h_i}\right)^{k-1} \omega_k^{\varphi}\left(S, \frac{1}{n}\right)_p \\
&\le C(k,p)\left| j-i \right|\left(\frac{h_{i,j}}{h_i}\right)^{k} \omega_k^{\varphi}\left(S, \frac{1}{n}\right)_p .
\end{aligned}$$

Thus (3.2.30) is proved.

We turn to prove the right hand estimate of (3.2.29) we take x and $h < \frac{1}{n}$ satisfying $x \pm \frac{k}{2} h\varphi(x) \in I$, and for each

$i=0,1,\ldots,k$ we let v_i be such that $x+\left(i-\frac{k}{2}\right)h\,\varphi(x)\in I_{v_i}$. Then by using the fact that for each $i=1,2,\ldots,k$, $h_{v_i,v_0}\le C(k)h_{v_0}$ which follows from (3.2.27), we obtain

$$\left\|\Delta^k_{h\varphi(x)}(S,x)\right\|_p\le\left\|\Delta^k_{h\varphi(x)}(S-P_{v_0},x)\right\|_p$$

$$=\left\|\sum_{i=0}^{k}(-1)^i\binom{k}{i}\left(S-P_{v_0}\right)_{\left(x+\left(i-\frac{k}{2}\right)h\varphi(x)\right)}\right\|_p$$

$$=\left\|\sum_{i=0}^{k}(-1)^i\binom{k}{i}\left(P_{v_i}-P_{v_0}\right)_{\left(x+\left(i-\frac{k}{2}\right)h\varphi(x)\right)}\right\|_p.$$

Now, since by our chosen to x and h, the integration is taken over all of those x's, such that $x+\left(i-\frac{k}{2}\right)h\,\varphi(x)\in I_{v_i}$,

then

$$\left\|\Delta^k_{h\varphi(x)}(S,x)\right\|_p\le C(k,p)\sum_{i=0}^{k}\max_{i=0,1,\ldots,k}\left\|P_{v_i}-P_{v_0}\right\|_{L_p\left(I_{v_i}\right)}$$

$$\le C(k,p)\max_{i=0,1,\ldots,k}\left(\frac{h_{v_0}}{h_{v_i,v_0}}\right)^k\left\|P_{v_i}-P_{v_0}\right\|_{L_p\left(I_{v_i}\right)}$$

$$\le C(k,p)a_k(S)_p.\ \Diamond$$

Note: the above lemma was proved by Leviatan and Shevchuk [49] in the case $p=\infty$. However all Lemmas in the rest of this section they have been proved in [50] for such case.

Lemma 3.2.34.

Let $k \ge 3$, then for each $S \in \sum_{k,n}(Y_s) \cap \Delta^2(Y_s)$, there is $\tilde{S} \in \sum_{k,n}^1 (Y_s) \cap \Delta^2(Y_s)$ such that

$$\left\| S - \tilde{S} \right\|_p \le C(k,p)\omega_k^{\varphi}\left(S, n^{-1}\right)_p . \tag{3.2.35}$$

In particular,

$$\omega_k^{\varphi}\left(\tilde{S}, n^{-1}\right)_p \le C(k,p)\omega_k^{\varphi}\left(S, n^{-1}\right)_p . \tag{3.2.36}$$

Proof:

Leviatan and Shevchuk [50] had constructed a piecewise polynomial $\tilde{S} \in \sum_{k,n}^1 (Y_s) \cap \Delta^2(Y_s)$ with pieces defined as

$$\left(\tilde{S} \mid I_j\right)(x) := P_j(x) + \alpha_j(x) + \beta_j(x) + \theta(x),$$

where θ is a piecewise constant function with jumps at most the $2s$ points x_j's near the y_i exactly, the jumps at these x_j's are

$$\theta\left(x_{j_+}\right) - \theta\left(x_{j_-}\right) := \begin{cases} \frac{1}{2}\left[P_j'\left(x_j\right) - P_{j+1}'\left(x_j\right)\right]\left(x_j - x_{j+1}\right), & \text{if } j \notin H,\ (j+1) \in H \\ \frac{1}{2}\left[P_j'\left(x_j\right) - P_{j+1}'\left(x_j\right)\right]\left(x_j - x_{j-1}\right), & \text{if } j \in H,\ (j+1) \notin H, \end{cases}$$

and $\alpha_j(x)$, $\beta_j(x)$ are defined as follows

For each $1 < j \le n$,

$$\alpha_j(x) := \begin{cases} \frac{1}{2}\frac{x_{j-1}-x_{j-2}}{x_{j-1}-x_j}\frac{P'_{j-1}(x_{j-1})-P'_j(x_{j-1})}{x_j-x_{j-2}}(x-x_j)^2, & \text{if } j,(j-1)\in H \\ \frac{1}{2}\frac{P'_{j-1}(x_{j-1})-P'_j(x_{j-1})}{x_{j-1}-x_j}(x-x_j)^2, & \text{if } j\in H,(j-1)\notin H \\ 0 & \text{if } j\notin H, \end{cases}$$

and $\alpha_1(x) := 0$.

Also, for each $1 \le j < n$,

$$\beta_j(x) := \begin{cases} \frac{1}{2}\frac{x_j-x_{j+1}}{x_{j-1}-x_j}\frac{P'_j(x_j)-P'_{j+1}(x_j)}{x_{j+1}-x_{j-1}}(x-x_{j-1})^2, & \text{if } j,(j-1)\in H \\ \frac{1}{2}\frac{P'_j(x_j)-P'_{j+1}(x_j)}{x_{j-1}-x_j}(x-x_{j-1})^2, & \text{if } j\in H,(j-1)\notin H \\ 0 & \text{if } j\notin H, \end{cases}$$

and $\beta_n(x) := 0$, and the above $\tilde{S}$ is the required function.

Then from the above definitions of α_j, β_j and θ , we have

$$\|\alpha_j + \beta_j + \theta\|_{L_p(I_j)} \le C(p)\|P_j - P_{j\pm1}\|_{L_p(I_j)},$$

so by Markov's type inequality (2.2.7) and using (3.2.32), we obtain

$$\begin{aligned}(x_{j-1}-x_j)\|P'_j(x_j)-P'_{j+1}(x_j)\|_{L_p(I_j)} &\le C(k,p)\|P_j - P_{j+1}\|_{L_p(I_j)} \\ &\le C(k,p)\omega_k(S, h_{j,j+1}, I_{j,j+1})_p,\end{aligned} \tag{3.2.37}$$

then by virtue of Lemma3.2.22 and the inequality (3.2.37), we obtain

$$\|S - \tilde{S}\|_p^p = \sum_{j=1}^{n}\|P_j - (P_j + \alpha_j + \beta_j + \theta)\|_{L_p(I_j)}^p$$

$$= \sum_{j=1}^{n} \left\| \alpha_j + \beta_j + \theta \right\|_{L_p(I_j)}^p$$
$$\le C(k,p)^p \sum_{j=1}^{n} \omega_k \left(S, h_{j,j+1}, I_{j,j+1}\right)_p^p$$
$$\le C(k,p)^p \, \omega_k^{\varphi} \left(S, h_j, I_{j,j+1}\right)_p^p$$
$$\le C(k,p)^p \, \omega_k^{\varphi} \left(S, n^{-1}\right)_p^p \ . \Diamond$$

If $S \in \sum_{k,n}^{1}$, S' is absolutely continuous in I, then we have the following inequality from [50]

$$\left| \Delta_{h\varphi(x)}^2 (S,x) \right| \le 4 \left\| \left(\Delta_n(x)\right)^2 S''(x) \right\|, \text{ where } h < \frac{1}{n}.$$

Then by using (2.2.4) and the fact that S is a piecewise polynomial of degree $\le k-1$ on the Chebyshev partition, we obtain

$$\left\| \Delta_{h\varphi(x)}^2 (S,x) \right\|_p \le C(k,p) \left\| \left(\Delta_n(x)\right)^2 S''(x) \right\|_p, \text{ where } h < \frac{1}{n}.$$

Recalling the definition of the second order Ditzian – Totik modulus of smoothness, we can get

$$\omega_2^{\varphi}\left(S, n^{-1}\right)_p \le C(k,p) \left\| \left(\Delta_n(x)\right)^2 S''(x) \right\|_p \qquad (3.2.38)$$

So, we obtain the following lemmas.

Lemma 3.2.39.

If $S \in \sum_{k,n}^{1}$, $n \in N$, $k \ge 2$ then

$$a_k(S)_p \le C(k,p) \left\| \left(\Delta_n(x)\right)^2 S''\left(x^2\right) \right\|_p .$$

The proof of the lemma is follows, by using the left hand side inequality (3.2.29), the inequality (3.2.38) and the fact that $\omega_k^{\varphi}(f,t)_p \le c(k,p)\omega_2^{\varphi}(f,t)_p$ which follows from property $(\overline{d})$. ◊

Lemma 3.2.40.

For $k \ge 3$, $0 < p < \infty$ and $S \in \sum_{k,n}^{1}$ is such that

$$a_k(S)_p \le 1. \tag{3.2.41}$$

If an interval $I_{\mu,\upsilon}$ contains at least $(2k-5)$ intervals I_i, and points $\hat{x}_i \in I_i^o$ (interior of I_i) such that

$$(\Delta_n(\hat{x}_i))^2 \,|\, S''(\hat{x}_i)| \le 1, \tag{3.2.42}$$

then for every $0 \le j \le n$, we have

$$\left\| (\Delta_n(x))^2 S''(x) \right\|_{L_p(I_j)} \le C(k,p)\left((j-\mu)^{4k} + (j-\upsilon)^{4k}\right). \tag{3.2.43}$$

Proof:

Fix j and $x \in I_j^o$. It follows by definitions $a_{i,j}(S)_p$ and (3.2.41). that

$$a_{i,j}(S)_p \le a_k(S)_p \le 1,$$

then

$$\left\| P_i - P_j \right\|_{L_p(I_j)} \le \left(\frac{h_{i,j}}{h_j}\right)^k, \tag{3.2.44}$$

and since $P_i - P_j$ is a polynomial of degree $\le k-1$, so from (2.2.4) and the Markov's type inequality (2.2.7), we have

$$\begin{aligned} \|P_i - P_j\|_{I_i} &\le C(k,p)\, h_i^{-\frac{1}{p}} \|P_i - P_j\|_{L_p(I_i)}, \\ \|P_i'' - P_j''\|_{I_i} &\le C(k,p)\, h_i^{-2-\frac{1}{p}} \|P_i - P_j\|_{L_p(I_i)}, \end{aligned} \tag{3.2.45}$$

in turn (3.2.42) implies that,

$$(\Delta_n(\hat{x}_i))^2 |P_i''(\hat{x}_i)| \le 1 .$$

Hence

$$h_i^{1/p} (\Delta_n(\hat{x}_i))^2 |P_i''(\hat{x}_i)| \le 1,$$

then by using (3.2.4), we conclude that

$$|P_i''(\hat{x}_i)| \le h_i^{-2-\frac{1}{p}} .$$

Also, by using the above inequalities and (3.2.8), we can get

$$\begin{aligned} |P_j''(\hat{x}_i)| &\le |P_j''(\hat{x}_i) - P_i''(\hat{x}_i)| + |P_i''(\hat{x}_i)| \\ &\le \|P_j'' - P_i''\|_{I_i} + |P_i''(\hat{x}_i)| \\ &\le C(k,p)\, h_i^{-2-\frac{1}{p}} \left(\frac{h_{i,j}}{h_j}\right)^k + h_i^{-2-\frac{1}{p}} \\ &\le C(k,p)\, h_i^{-2-\frac{1}{p}} \left(\frac{h_{i,j}}{h_j}\right)^k \\ &\le C(k,p)\, h_j^{-2-\frac{1}{p}} (|j-i|+1)^{2k} \end{aligned} \tag{3.2.46}$$

By supposition, there are $k-2$ points $\hat{x}_{i_\ell} \in I_{\mu,\upsilon}$, $(\ell = 1,2,\ldots,k-2)$, each two being separated by an interval $I_i \subseteq I_{\mu,\upsilon}$. Recalling that $x \in I_j$, so by virtue of (3.2.3) and (3.2.8), we have for each $1 \le \ell \ne m \le k-2$,

$$\frac{\left|x - \widehat{x}_{i_\ell}\right|}{\left|\widehat{x}_{i_m} - \widehat{x}_{i_\ell}\right|} \le c\,\frac{\left|x_j - \widehat{x}_{i_\ell}\right|}{\left|\widehat{x}_{i_m} - \widehat{x}_{i_\ell}\right|} \le c(k)\frac{h_{j,i_\ell}}{h_{i_\ell}} \tag{3.2.47}$$
$$\le c(k)\left(\left|j - i_\ell\right| + 1\right)^2 \le c(k)\left(\left|j-\mu\right| + \left|j - \upsilon\right|\right)^2.$$

Now, by virtue of the representation, from [50], we have

$$P_j''(x) \equiv \sum_{m=1}^{k-2} P_j''\left(\widehat{x}_{i_1}\right) \prod_{\substack{\ell=1\\ \ell\ne m}}^{k-2} \frac{\left|x - \widehat{x}_{i_\ell}\right|}{\left|\widehat{x}_{i_m} - \widehat{x}_{i_\ell}\right|},$$

We obtain from (3.2.4), (3.2.46) and (3.2.47),

$$\left\| \left(\Delta_n(x)\right)^2 S''(x)\right\|_{L_p(I_j)} = \left\| \left(\Delta_n(x)\right)^2 P_j''(x)\right\|_{L_p(I_j)}$$
$$\le \left\| 25\, h_j^2 \sum_{m=1}^{k-2} P_j''\left(\widehat{x}_{i_1}\right) \prod_{\substack{\ell=1\\ \ell\ne m}}^{k-2} \frac{\left|x - \widehat{x}_{i_\ell}\right|}{\left|\widehat{x}_{i_m} - \widehat{x}_{i_\ell}\right|}\right\|_{L_p(I_j)}$$

$$\le C(k,p)\left\| 25\, h_j^2 \sum_{m=1}^{k-2} h_i^{-2-\frac{1}{p}} \left(\left|j - i_{i_m}\right| + 1\right)^{2k} \prod_{\substack{\ell=1\\ \ell\ne m}}^{k-2} \left(\left|j - i_{i_\ell}\right| + 1\right)^2 \right\|_{L_p(I_j)}$$

$$\le \left\| 25\, h_j^2 \sum_{m=1}^{k-2} C(k,p) h_j^{-2-\frac{1}{p}} \left(\left| j - i_{i_m} \right| + 1 \right)^{2k} \prod_{\substack{\ell=1 \\ \ell \ne m}}^{k-2} \left(\left| j - i_{i_\ell} \right| + 1 \right)^2 \right\|_{L_p(I_j)}$$

$$\le C(k,p) h_j^{-\frac{1}{p}} \left\| \sum_{m=1}^{k-2} \left(| j-\mu | + | j-\upsilon | \right)^{2k} \prod_{\substack{\ell=1 \\ \ell \ne m}}^{k-2} \left(| j-\mu | + | j-\upsilon | \right)^2 \right\|_{L_p(I_j)}$$

$$\le C(k,p) h_j^{-\frac{1}{p}} \left\| \frac{(k-2(k-3))}{2} \left(| j-\mu | + | j-\upsilon | \right)^{2(2k-3)} \right\|_{L_p(I_j)}$$

$$\le C(k,p) h_j^{-\frac{1}{p}} h_j^{\frac{1}{p}} \left(| j-\mu | + | j-\upsilon | \right)^{2(2k-3)}$$

$$\le C(k,p) \left((j-\mu)^{4k} + (j-\upsilon)^{4k} \right). \Diamond$$

Also, we need the following Lemma.

Lemma 3.2.48. [50]

Let E be an interval which is the union of $m \ge 12s$ of the intervals I_j, let a set $J \subseteq E$ be the union of $1 \le \mu \le \frac{m}{4}$ of these intervals. Then there exists a polynomial $Q_n(x) = Q_n(x, E, J)$ of degree $\le c(s)n$, satisfying

$$Q_n''(x)\delta(x) \ge c_1 \frac{1}{\mu} \left(\frac{\Delta_n(x)}{\max \{\Delta_n(x), \operatorname{dist}(x, E)\}} \right)^{25(s+1)} \frac{\Pi(x)}{\Delta_n(x)}, \qquad x \in J \cup (I \setminus E), \tag{3.2.49}$$

(we may take $c_1 \le 1$)

$$Q_n''(x)\delta(x) \ge -\frac{\Pi(x)}{\Delta_n(x)}, \quad x \in E \setminus J, \tag{3.2.50}$$

and

$$\left|Q_n(x)\right| \le c(s)m^6\Delta_n(x)\sum_{I_j\subseteq E}\frac{h_j}{\left(\left|x-x_j\right|+\Delta_n(x)\right)}, \quad x\in I. \qquad (3.2.51)$$

For the rest of this section we assume that $s>0$, because otherwise many of statements are vacuous and there is nothing to prove. For $j\notin H$, let j^* be denote the closest element to it from H (if there are two such elements, then we take the bigger one), and denote by I_j^* the connected component of $\overline{O}$ (the closure of O), that contains x_j. Since the interval I_j^* contains at most $3s$ intervals I_υ, then we have from (3.2.8) that

$$h_j \le \left|I_j^*\right| \le (3s)^2 h_j. \qquad (3.2.52)$$

In order to unified notation we denote if $j\in H$, $j^*=j$ and $I_j^*=I_j$.

Now, the following polynomials $\tilde{T}_{j,n_1}(x,b,Y_s)$, $(j=1,,2\ldots,n)$ and D_{n_1} of degrees $\le C(s)n_1$, with n_1 a natural number which is divisible by n, and b constant depend on s, were created by Leviatan and Shevchuk [50], will be useful, and the polynomial $\tilde{T}_{j,n_1}$ satisfy the following lemma, for $b\ge 6(3s+1)$ and $b_1=2b-3s-1$.

Lemma .3.2.53. [50]

The following relations hold.

$$\sum_{j=1}^{n} \tilde{T}_{j,n_1}(x) \equiv 1, \qquad (3.2.54)$$

$$\tilde{T}'_{j,n_1}(y_i) = \tilde{T}''_{j,n_1}(y_i) = 0,\ 1 \le i \le s \text{ and } 1 \le j \le n$$
$$\tilde{T}_{j,n_1}(y_i) = 0,\ 1 \le i \le s,\ 1 \le j \le n \text{ and } y_i \in I_j^*, \qquad (3.2.55)$$

and

$$\left|\tilde{T}^{(q)}_{j,n_1}(x)\right| \le \frac{C(s)}{(\Delta_{n_1}(x))^q}\left(\frac{\Delta_{n_1}(x)}{\Delta_{n_1}(x) + \operatorname{dist}(x, I_j)}\right)^{b_2}, \qquad (3.2.56)$$
$$1 \le j \le n,\ 0 \le q \le s+2, \text{ and } x \in I,$$

where $b_2 = \frac{1}{2}(b_1 - 1)$.

Now, for $S \in \sum_{k,n}$, and n_1 divisible by n , the polynomial D_{n_1} is defined as following

$$D_{n_1}(x) := D_{n_1}(x, S) := \sum_{j=1}^{n} P_j(x) \tilde{T}_{j,n_1}(x, b, Y_s), \qquad (3.2.57)$$

and let us we denote by

$$O_e := \{u \in \overline{O} : [u - (1/2)\Delta_n(u), u + (1/2)\Delta_n(u)] \subseteq \overline{O}\} \cup (\overline{O} \cap (I_1 \cup I_n)).$$

Then, we can obtain the following lemma.

Lemma 3.2.58.

Let $b_3 = b_2 - s - 2k - 6 > 0$, and let A be a proper interval, for $S \in \sum_{k,n}^{1}(Y_s)$, then for $x \neq x_j$, $0 \le j \le n$, and $0 < p \le \infty$, we have

$$h_{\upsilon}^{\frac{1}{p}} \left| S''(x) - D''_{n_1}(x) \right| \le \frac{C(k,s,p)}{(\Delta_n(x))^2} \Bigg(a_k(S,A)_p + \\ + a_k(S)_p \left(\frac{\Delta_n(x)}{\text{dist } (x, I \setminus A)} \right)^{b_3} \Bigg),$$

for any $1 \le \upsilon \le n$ such that $I_{\upsilon} \subseteq A$.

Proof:

We fix $I_{\upsilon} \subseteq A$ and let $x \in I_{\upsilon}$ be such that

$$x - x_{\upsilon} \le x_{\upsilon-1} - x \tag{3.2.59}$$

So by definition of $a_{\upsilon,j}(S)_p$, we have

$$\left\| P_{\upsilon} - P_j \right\|_{L_p(I_{\upsilon})} = a_{\upsilon,j}(S)_p \left(\frac{h_{\upsilon,j}}{h_j} \right)^k .$$

Whence by virtue of (2.2.3) and (2.2.7), we have for each $\ell \in \mathbb{N}$,

$$\left\| P_{\upsilon}^{(\ell)} - P_j^{(\ell)} \right\|_{L_p(I_{\upsilon})} \le \frac{C(k,p)}{h_{\upsilon}^{\ell}} a_{\upsilon,j}(S)_p \left(\frac{h_{\upsilon,j}}{h_j} \right)^k ,$$

and for $j \neq \upsilon, \upsilon+1$, (3.2.59) and (3.2.4) imply

$$\text{dist } (x, I_j) \ge \frac{1}{2} \Delta_n(x).$$

Hence, by (3.2.56) combined with (3.2.3) and (3.2.7), we obtain

$$h_v^{\frac{1}{p}} \left\| P_\upsilon^{(r)} - P_j^{(r)} \right\|_{I_\upsilon} \left| \tilde{T}_{j,n_1}^{(q-r)}(x) \right|$$

$$\le \frac{C(k,p)}{h_\upsilon^r} a_{\upsilon,j}(S)_p \left(\frac{h_{\upsilon,j}}{h_j} \right)^k \frac{C(s)}{\left(\Delta_{n_1}(x)\right)^{q-r}} \left(\frac{\Delta_{n_1}(x)}{\Delta_{n_1}(x) + \operatorname{dist}\left(x, I_j\right)} \right)^{b_2}$$

$$\le \frac{C(k,s,p)}{h_\upsilon^r} a_{\upsilon,j}(S)_p \left(\frac{h_{\upsilon,j}}{h_j} \right)^{k+1} \frac{h_j}{h_{\upsilon,j}} \frac{1}{\left(\Delta_{n_1}(x)\right)^{q-r}} \left(\frac{\Delta_{n_1}(x)}{\Delta_{n_1}(x) + \operatorname{dist}\left(x, I_j\right)} \right)^{b_2}$$

$$\le \frac{C(k,s,p)}{h_\upsilon^r} a_{\upsilon,j}(S)_p \left(\frac{\Delta_n(x) + \operatorname{dist}\left(x, I_j\right)}{\Delta_n(x)} \right)^{2(k+1)} \frac{h_j}{h_\upsilon} \frac{1}{\left(\Delta_{n_1}(x)\right)^{q-r}} \cdot$$

$$\cdot \left(\frac{\Delta_{n_1}(x)}{\Delta_{n_1}(x) + \operatorname{dist}\left(x, I_j\right)} \right)^{q-r+1} \left(\frac{\Delta_n(x)}{\Delta_n(x) + \operatorname{dist}\left(x, I_j\right)} \right)^{b_2-q+r-1}$$

$$\le \frac{C(k,s,p)}{h_\upsilon^{r+1}} a_{\upsilon,j}(S)_p h_j \frac{1}{\left(\Delta_{n_1}(x)\right)^{q-r}} \frac{\Delta_{n_1}(x)}{\Delta_n(x)} \left(\frac{\Delta_n(x)}{\Delta_n(x) + \operatorname{dist}\left(x, I_j\right)} \right)^{b_3+1}$$

$$\le \frac{C(k,s,p)}{\left(\Delta_n(x)\right)^q} a_{\upsilon,j}(S)_p h_j \frac{n}{n_1} \left(\Delta_n(x)\right)^{b_3} \left(\frac{1}{\Delta_n(x) + \operatorname{dist}\left(x, I_j\right)} \right)^{b_3+1}, \qquad (3.2.60)$$

$$, 0 \le r \le q,$$

where we applied the inequality $\operatorname{dist}\left(x, I_j\right) \ge \frac{1}{2}\Delta_n(x)$ in the fourth inequality, the inequality

$$\frac{\Delta_{n_1}(x)}{\Delta_n(x)} \le \frac{n}{n_1}, \qquad (3.2.61)$$

and (3.2.4) in the last inequality.

So, by virtue of (3.2.54), we can represent

$$S''(x) - D''_{n_1}(x) := \left(\left(P_\upsilon(x) - P_{\upsilon+1}(x)\right)\tilde{T}_{v+1,n_1}(x)\right)'' + \sum_{\substack{I_j \subseteq A, \\ j \neq \upsilon, \upsilon+1}} \left(\left(P_\upsilon(x) - P_j(x)\right).\right.$$

$$\left. .\tilde{T}_{j,n_1}(x)\right)'' + \sum_{\substack{I_j \not\subseteq A \\ j \neq \upsilon, \upsilon+1}} \left(\left(P_\upsilon(x) - P_j(x)\right)\tilde{T}_{j,n_1}(x)\right)''$$

$$= \zeta_1(x) + \zeta_2(x) + \zeta_3(x),$$

where we write $P_{n+1} = P_n$, if $\upsilon = n$.

Now, to estimate $\zeta_1(x)$, we have, if $\upsilon = n$ then $\zeta_1(x) \equiv 0$, so that we can assume that $\upsilon < n$, and since $S \in \sum_{k.n}$, $q = 2$, $r = 0$, and $I_\upsilon \subseteq A$, then immediately it follows from the Markov type inequality (2.2.7) and the inequalities (3.2.3) and (3.2.4) that

$$\left\| P''_\upsilon - P''_j \right\|_{L_p(I_\upsilon)} \leq \frac{C(k,p)}{h_\upsilon^2} \left\| P_\upsilon - P_j \right\|_{L_p(I_\upsilon)}$$

$$\leq \frac{C(k,p)}{h_\upsilon^2} a_{\upsilon,\upsilon+1}(S)_p \left(\frac{h_{\upsilon,\upsilon+1}}{h_{\upsilon+1}}\right)^k$$

$$\leq \frac{C(k,p)}{(\Delta_n(x))^2} a_{\upsilon,\upsilon+1}(S)_p.$$

This in turn implies

$$\left\| P'_\upsilon - P'_j \right\|_{L_p(I_\upsilon)} \leq \left\| \int_{x_\upsilon}^{x} \left(P''_\upsilon - P''_j\right)(u)\,du \right\|_{L_p(I_\upsilon)}$$

$$\leq C(k,p)\left(x - x_\upsilon\right) \left\| P''_\upsilon - P''_j \right\|_{L_p(I_\upsilon)}$$

$$\leq \frac{C(k,p)}{(\Delta_n(x))^2}\ a_{\upsilon,\upsilon+1}(S)_p\ \left(x - x_\upsilon\right),$$

also

$$\| P_{\upsilon} - P_j \|_{L_p(I_{\upsilon})} \le \left\| \int_{x_{\upsilon}}^{x} (P'_{\upsilon} - P'_j)(u)\, du \right\|_{L_p(I_{\upsilon})}$$

$$\le C(k,p)(x - x_{\upsilon}) \| P'_{\upsilon} - P'_j \|_{L_p(I_{\upsilon})}$$

$$\le \frac{C(k,p)}{(\Delta_n(x))^2}\ a_{\upsilon,\upsilon+1}(S)_p\ (x - x_{\upsilon})^2 .$$

Therefore, by the above inequalities and (3.2.56) we obtain

$$h_{\upsilon}^{\frac{1}{p}} |\zeta_1(x)| \le h_{\upsilon}^{\frac{1}{p}} \left(\|(P_{\upsilon} - P_{\upsilon+1})\|_{I_{\upsilon}} \left|\tilde{T}''_{\upsilon+1,n_1}(x)\right| + 2\|(P_{\upsilon} - P_{\upsilon+1})'\|_{I_{\upsilon}} \cdot \right.$$

$$\left. \cdot \left|\tilde{T}'_{\upsilon+1,n_1}(x)\right| + \|(P_{\upsilon} - P_{\upsilon+1})''\|_{I_{\upsilon}} \left|\tilde{T}_{\upsilon+1,n_1}(x)\right| \right)$$

$$\le \frac{C(k,s,p)}{(\Delta_n(x))^2} a_{\upsilon,\upsilon+1}(S)_p \left(1 + \frac{x - x_{\upsilon}}{\Delta_n(x)} + \left(\frac{x - x_{\upsilon}}{\Delta_n(x)} \right)^2 \right).$$

$$\cdot \left(\frac{\Delta_{n_1}(x)}{\Delta_{n_1}(x) + |x - x_v|} \right)^{b_2}$$

$$\le \frac{C(k,s,p)}{(\Delta_n(x))^2} a_{\upsilon,\upsilon+1}(S)_p \left(\frac{\Delta_{n_1}(x)}{\Delta_{n_1}(x) + |x - x_v|} \right)^{b_2-2} . \qquad (3.2.62)$$

Now, if $I_{\upsilon+1} \subseteq A$, then (3.2.62) yields

$$h_{\upsilon}^{\frac{1}{p}} |\zeta_1(x)| \le \frac{C(k,s,p)}{(\Delta_n(x))^2} a_k(S,A)_p , \qquad (3.2.63)$$

if $I_{\upsilon+1} \not\subseteq A$, then (3.2.62) implies

$$h_{\upsilon}^{\frac{1}{p}} |\zeta_1(x)| \le \frac{C(k,s,p)}{(\Delta_n(x))^2} a_k(S)_p\ \frac{\Delta_{n_1}(x)}{\Delta_n(x)} \frac{\Delta_n(x)}{\Delta_{n_1}(x) + |x - x_{\upsilon}|} \left(\frac{\Delta_{n_1}(x)}{\Delta_{n_1}(x) + |x - x_{\upsilon}|} \right)^{b_2-3}$$

$$\le \frac{C(k,s,p)}{(\Delta_n(x))^2} a_k(S)_p \frac{n}{n_1} \frac{\Delta_n(x)}{|x-x_\upsilon|} \cdot \left(\frac{\Delta_n(x)}{\Delta_n(x)+|x-x_\upsilon|} \right)^{b_2-3}$$

$$\le \frac{C(k,s,p)}{(\Delta_n(x))^2} a_k(S)_p \frac{n}{n_1} \frac{\Delta_n(x)}{|x-x_\upsilon|} \cdot \left(\frac{\Delta_n(x)}{\Delta_n(x)+|x-x_\upsilon|} \right)^{b_3-1}$$

$$\le \frac{C(k,s,p)}{(\Delta_n(x))^2} a_k(S)_p \frac{n}{n_1} \frac{\Delta_n(x)}{|x-x_\upsilon|} \cdot \left(\frac{\Delta_n(x)}{|x-x_\upsilon|} \right)^{b_3-1}$$

$$= \frac{C(k,s,p)}{(\Delta_n(x))^2} a_k(S)_p \frac{n}{n_1} \left(\frac{\Delta_n(x)}{|x-x_\upsilon|} \right)^{b_3}$$

$$\le \frac{C(k,s,p)}{(\Delta_n(x))^2} a_k(S)_p \frac{n}{n_1} \left(\frac{\Delta_n(x)}{\operatorname{dist}(x, I \setminus A)} \right)^{b_3}. \qquad (3.2.64)$$

It remains only to estimate ζ_2 and ζ_3 , it follows from (3.2.60) that

$$h_\upsilon^{\frac{1}{p}} |\zeta_3(x)| = h_\upsilon^{\frac{1}{p}} \sum_{\substack{I_j \not\subseteq A \\ j \neq \upsilon, \upsilon+1}} \Big(\|(P_\upsilon - P_j)\|_{I_\upsilon} |\tilde{T}''_{j,n_1}(x)| + 2\|(P_\upsilon - P_j)'\|_{I_\upsilon} \cdot |\tilde{T}'_{j,n_1}(x)|$$

$$+ \|(P_\upsilon - P_j)''\|_{I_\upsilon} |\tilde{T}_{j,n_1}(x)| \Big)$$

$$\le \sum_{\substack{I_j \not\subseteq A \\ j \neq \upsilon, \upsilon+1}} \frac{C(k,s,p)}{(\Delta_n(x))^2} a_{\upsilon,j}(S)_p \frac{n}{n_1} (\Delta_n(x))^{b_3} h_j \left(\frac{1}{\Delta_n(x) + \operatorname{dist}(x, I_j)} \right)^{b_3+1}$$

$$\le \frac{C(k,s,p)}{(\Delta_n(x))^2} a_k(S)_p \frac{n}{n_1} (\Delta_n(x))^{b_3} \sum_{\substack{I_j \not\subseteq A \\ j \neq \upsilon, \upsilon+1}} \frac{h_j}{\Delta_n(x) + \operatorname{dist}(x, I_j))^{b_3+1}}$$

$$\le \frac{C(k,s,p)}{(\Delta_n(x))^2} a_k(S)_p \frac{n}{n_1} \left(\frac{\Delta_n(x)}{\Delta_n(x) + \operatorname{dist}(x, I \setminus A)} \right)^{b_3}. \qquad (3.2.65)$$

We have $\operatorname{dist}(x, I_\upsilon^*) := \min\{\operatorname{dist}(x, I_{\upsilon+1}), \operatorname{dist}(x, I_{\upsilon-1})\}$, then similarly from (3.2.60), we obtain

$$h_{\upsilon}^{\frac{1}{p}} |\zeta_2(x)| \le \frac{C(k,s,p)}{(\Delta_n(x))^2} a_k(S,A)_p \frac{n}{n_1}\left(\frac{\Delta_n(x)}{\Delta_n(x)+\text{dist}\,(x, I_{\upsilon}^*)}\right)^{b_3}$$

$$\le \frac{C(k,s,p)}{(\Delta_n(x))^2} a_k(S,A)_p . \qquad (3.2.66)$$

Hence the theorem follows from (3.2.63) through (3.2.66). ◊

Lemma 3.2.67.

If $S \in \sum_{k,n}$, then

$$\left\| S - D_{n_1} \right\|_p \le C(k,p) a_k(S)_p . \qquad (3.2.68)$$

Furthermore, if $S \in \sum_{k,n}(Y_s)$ and

$$S''(y_i) = 0, \quad i = 1,2,\ldots,s , \qquad (3.2.69)$$

then

$$D''_{n_1}(y_i) = 0, \quad i = 1,2,\ldots,s . \qquad (3.2.70)$$

Proof:

Recalling the definition of D_{n_1} and by virtue of (3.2.54) we have

$$\left\| S - D_{n_1} \right\|_p^p = \sum_{\ell=1}^{n} \left\| P_\ell - D_{n_1} \right\|_{L_p(I_\ell)}^p = \sum_{\ell=1}^{n} \left\| \sum_{j=1}^{n} (P_\ell - P_j) \tilde{T}_{j,n_1} \right\|_{L_p(I_\ell)}^p$$

$$\le \sum_{\ell=1}^{n}\left\|\sum_{j=1}^{n} h_\ell^{\frac{-1}{p}} C(k,p) a_{\ell,j}(S)_p \frac{n}{n_1}(\Delta_n(x))^{b_3} h_j\left(\frac{1}{\Delta_n(x)+\operatorname{dist}(x,I_j)}\right)^{b_3+1}\right\|_{L_p(I_\ell)}^p$$

$$\le C(k,p)^p\, a_k(S)_p^p\left(\frac{n}{n_1}\right)^p \sum_{\ell=1}^{n} h_\ell^{-1}\left\|\sum_{j=1}^{n} h_j\left(\frac{1}{\Delta_n(x)+\operatorname{dist}(x,I_j)}\right)^{b_3+1}\right\|_{L_p(I_\ell)}^p$$

$$= C(k,p)^p\, a_k(S)_p^p\left(\frac{n}{n_1}\right)^p \sum_{\ell=1}^{n} h_\ell^{-1}\left(\int_{I_\ell}\left|\sum_{j=1}^{n} h_j \frac{(\Delta_n(x))^{b_3}}{(\Delta_n(x)+\operatorname{dist}(x,I_j))^{b_3+1}}\right|^p\right)$$

$$\le C(k,p)^p\, a_k(S)_p^p\left(\frac{n}{n_1}\right)^p \max_{\ell=1,2,\ldots,n}\left(h_\ell^{-1}\right)\sum_{\ell=1}^{n}\left(\int_{I_\ell}\left|\frac{(\Delta_n(x))}{(\Delta_n(x)+\operatorname{dist}(x,I\setminus I_j))}\right|^{b_3 p}\right)$$

$$\le C(k,p)^p\, a_k(S)_p^p,$$

where, we applied (3.2.69) with $q=0=r$ in the first inequality, we chose b_2 in Lemma 3.2.53, such that $b_2 p \ge 2$ (so that $b_3 p \ge 2$) and we used (3.2.4) and (3.2.14) in the third inequality and used the fact that $h_i \sim n^{-1} \sim h_j$ for each $i, j = 1, 2, \ldots, n$ in the last inequality.

The second part of this lemma was proved by Leviatan and Shevchuk [50], in the following way:

We fix $1 \le i \le s$, and let υ be such that $y_i \in I_\upsilon$. Since $P_j \equiv P_\upsilon$ for all $I_j \subseteq I_\upsilon^*$, then

$$D''_{n_1}(y_i) = \sum_{j=1}^{n}\left(P_j(y_i)\tilde{T}''_{j,n_1}(y_i) + P'_j(y_i)\tilde{T}'_{j,n_1}(y_i)\right) + \sum_{I_j \not\subseteq I_\upsilon^*}^{n} P''_j(y_i)\tilde{T}_{j,n_1}(y_i)$$
$$+ P''_j(y_i)\sum_{I_j \not\subseteq I_\upsilon^*}^{n} \tilde{T}_{j,n_1}(y_i).$$

Now, by virtue of (3.2.55), the first and the second sums are zero, and since $P''_{\upsilon}(y_i) = S''(y_i) = 0$, it follows that the third term vanishes. ◊

To prove our Theorems 3.2.11 and 3.2.12 we need the following lemma.

Lemma 3.2.71. [50]

If A is a proper interval, $S \in \sum_{k,n}^{1}(Y_s)$ and (3.2.74) holds, then

$$\left| S''(x) - D''_{n_1}(x) \right| \le \frac{C_0(k,s,b)\Pi(x)}{(\Delta_n(x))^2} \Bigg(a_k(S,A) + \\ + a_k(S) \frac{n}{n_1} \left(\frac{\Delta_n(x)}{\operatorname{dist}(x, I \setminus A)} \right)^{b_3} \Bigg), \; x \in A,$$

where we recall $\Pi(x)$ from (3.2.1).

3.3 Proof of Theorem 3.1.1:

We use the mathematical induction on s - the number of convexity changes of f and the idea of flipping technique of f, which originally introduced by Beatson and Leviatan in [1].

For $s = 0$ (i.e., f is convex in I), then the theorem is valid and it was proved by DeVore, Leviatan and Hu [11].

Thus we will assume that $s \geq 1$, and it is clear that f is either concave or convex in the interval $[-1, y_1]$, and each case where will need a separate through similar construction. We will detail the construction for the case where f is concave in $[-1, y_1]$. For the sake of simplicity in written we write $\alpha = y_1$.

Now, we may assume that $\alpha \in [x_{j_0}, x_{j_0 - 1})$. Then, if $n > N_\alpha := \max\left\{\frac{50}{y_2 - \alpha}, \frac{50}{1+\alpha}\right\}$, we are assured that $x_{j_0+3} \geq -1$ and that $x_{j_0-4} \leq y_2$. Set $h := c\Delta_n(\alpha) < \frac{1}{6}h_{j_0}$, where c is chosen sufficiently small to guarantee the right inequality. We observing that this implies

$$x_{j_0+1} < \alpha + 2h < \alpha - 2h < x_{j_0-2}.$$

We are going to replace f on the interval $[\alpha - h, \alpha + h]$ in a way that will keep us near the original function (see g below) will be smoother at α. As we said above, the case $s = 0$ is known and serves as the beginning of the induction process; it has been proved by DeVore, Leviatan and Hu [11], and in this case (3.1.2) holds, for all $n \geq 2$. Thus, we proceed by induction.

To this end, we observe that either $\Delta_h^2(f, \alpha) \geq 0$ or $\Delta_h^2(f, \alpha) < 0$.

In the first case, let $\ell_1(x)$ be denote the linear function interpolating f at $\alpha - h$ and α. Then the function $\bar{f} := f - \ell_1$ satisfies

$$\bar{f}(\alpha - h) = \bar{f}(\alpha) = 0, \bar{f}(\alpha + h) \geq 0, \text{ and } \bar{f}(x) \leq 0, \ -1 \leq x < \alpha - h .$$

Hence, for $J := [x_{j_0+1}, x_{j_0-2}]$, we have,

$$\begin{aligned} 0 \leq \bar{f}(\alpha + h) &\leq \bar{f}(\alpha + h) - \bar{f}(\alpha - 2h) \\ &= \bar{f}(\alpha + h) - 3\bar{f}(\alpha) + 3\bar{f}(\alpha + h) - \bar{f}(\alpha - 2h) \\ &= \vec{\Delta}_h^3(\bar{f}, \alpha). \end{aligned}$$

Now, since $\bar{f}$ is concave in $[-1, \alpha]$ and it's convex in $[\alpha, y_2]$, then from Theorem 3.2.23, we have $\bar{f}$ is continuous on $(-1, \alpha)$ and (α, y_2), and since $\bar{f}(\alpha) = 0$ which is finite, so $\bar{f}$ is bounded on $(-1, y_2)$, then $\Delta_h^3(\bar{f}, \alpha)$ is finite on J. On the other hand, we have from the definition of ordinary modulus of smoothness that

$$\left\| \vec{\Delta}_h^3(\bar{f}, \alpha) \right\|_{L_p(J)} \leq \vec{\omega}_3(\bar{f}, h, J)_p \leq \vec{\omega}_3(f, h, J)_p \leq C(p)\omega_3(f, h, J)_p ,$$

then

$$\left| \Delta_h^3(\bar{f}, \alpha) \right| \leq |J|^{-\frac{1}{p}} \omega_3(f, h, J)_p .$$

Thus

$$0 \leq \bar{f}(\alpha + h) \leq \left| \Delta_h^3(\bar{f}, \alpha) \right| \leq |J|^{-\frac{1}{p}} \omega_3(f, h, J)_p .$$

Similarly, in the latter case, let $\ell_1(x)$ be denote the linear function interpolating f at α and $\alpha+h$. Then the function $\bar{f} := f - \ell_1$ satisfies

$$\bar{f}(\alpha)=\bar{f}(\alpha+h)=0, \bar{f}(\alpha-h)<0, \text{ and } \bar{f}(x)\ge 0, \ -1\le x<\alpha-h .$$

Hence, for $J:=[x_{j_0+1}, x_{j_0-2}]$, we have,

$$0\le -\bar{f}(\alpha-h)\le \bar{f}(\alpha+2h)+\bar{f}(\alpha-h)$$
$$=\bar{f}(\alpha+2h)-3\bar{f}(\alpha+h)+3\bar{f}(\alpha)-\bar{f}(\alpha-h)=\overleftarrow{\Delta}_h^3(\bar{f},\alpha)$$

Also

$$0\le \left|\bar{f}(\alpha-h)\right|\le\left|\overleftarrow{\Delta}_h^3(\bar{f},\alpha)\right|\le |J|^{-\frac{1}{p}}\omega_3(f,h,J)_p .$$

Thus in both cases we have,

$$\max\left\{\left|\bar{f}(\alpha-h)\right|,\left|\bar{f}(\alpha)\right|,\left|\bar{f}(\alpha+h)\right|\right\}\le |J|^{-\frac{1}{p}}\omega_3(f,h,J)_p$$

which in turn implies that the quadratic polynomial ℓ_2, interpolating f at $\alpha-h,\alpha$ and $\alpha+h$, is bounded by the same quantity on $[\alpha-h,\alpha+h]$. This means that

$$|\ell_2(x)|\le |J|^{-\frac{1}{p}}\omega_3(f,h,J)_p$$
$$\le c(2h)^{-\frac{1}{p}}\omega_3(f,h,J)_p, \quad \forall\, x\in[\alpha-h,\alpha+h],$$

so

$$\|\ell_2\|_{L_p[\alpha-h,\alpha+h]}\le c\,\omega_3(f,h,J)_p ,$$

then by applying Lemma 3.2.25, we obtain

$$\|\ell_2\|_{L_p(J)}\le C(p)\omega_3(f,h,J)_p \qquad (3.3.1)$$

At the same time applying Whitney's theorem we conclude that

$$\left\| \bar{f} - \ell_2 \right\|_{L_p(J)} \le C(p)\,\omega_3(f,h,J)_p \qquad (3.3.2)$$

Hence

$$\left\| \bar{f} \right\|_{L_p(J)} \le C(p)\,\omega_3(f,h,J)_p \qquad (3.3.3)$$

Now, since ℓ_2 is bounded on I, then

$$\left\| \ell_2 \right\|_{L_p(I_j)} \sim \left\| \ell_2 \right\|_{L_p(J)}, \ \forall\, j = 1,2,\ldots,n\ , \qquad (3.3.4)$$

and also by virtue of Theorem 3.2.23, we have

$$\omega_3(f,h,J)_p \sim \omega_3(f,h_j,I_j)_p\ , \ \forall\, j = 1,2,\ldots,n \qquad (3.3.5)$$

then by using (3.3.1), (3.3.4) and (3.3.5) we conclude that for each $j = 1,2,\ldots,n$

$$\left\| \ell_2 \right\|_{L_p(I_j)} \le C(p)\left\| \ell_2 \right\|_{L_p(J)} \le C(p)\,\omega_3(f,h,J)_p \le C(p)\,\omega_3(f,h_j,I_j)_p\,.$$

So, by applying Lemma 3.2.22, we obtain

$$\left\| \ell_2 \right\|_p \le C(p)\,\omega_3^{\varphi}(f,n^{-1})_p$$

Analogously

$$\left\| \bar{f} - \ell_2 \right\|_p \le C(p)\,\omega_3^{\varphi}(f,n^{-1})_p\ ,$$

hence

$$\left\| \bar{f} \right\|_p \le C(p)\,\omega_3^{\varphi}(f,n^{-1})_p \qquad (3.3.6)$$

Now, let

$$\hat{f}(x) := \begin{cases} -\bar{f}(x) & -1 \le x \le \alpha \\ \bar{f}(x) & \text{o.w}\ , \end{cases}$$

and

$$g(x):=\begin{cases}\hat{f}(x) & x\notin[\alpha-h,\alpha+h]\\ \max\left\{\hat{f}(x),0\right\} & x\in[\alpha-h,\alpha+h].\end{cases}$$

by virtue of(3.3.6) we have

$$\left\|\hat{f}-g\right\|_p\le C(p)\omega_3^{\varphi}\left(f,n^{-1}\right)_p \tag{3.3.7}$$

Thus, by using Lemma 3.2.21 and the inequalities (3.3.6) and (3.2.7), we obtain

$$\begin{aligned}\omega_3^{\varphi}\left(g,n^{-1}\right)_p &\le C(p)\left(\omega_3^{\varphi}\left(\hat{f}-g,n^{-1}\right)_p+\omega_3^{\varphi}\left(\hat{f},n^{-1}\right)_p\right)\\ &\le C(p)\left(\left\|\hat{f}-g\right\|_p+\left\|\hat{f}\right\|_p\right)\\ &\le C(p)\omega_3^{\varphi}\left(f,n^{-1}\right)_p\end{aligned} \tag{3.3.8}$$

It is readily that $g\in L_p(I)$, that it is convex in$[-1,y_2]$ and that it changes convexity at $\hat{Y}_{s-1}:=Y_s\setminus\{y_1\}$. If, on the other hand, f was convex in $[-1,\alpha]$, then g would be concave in $[-1,y_2]$ and change convexity at $\hat{Y}_{s-1}$. Thus in any case g had fewer convexity changes, so by induction, we may assume that for $n>\frac{A(s)}{d(s)}$, there exists an n^{th} degree polynomial q_n which is coconvex with g, and satisfies the analogues of (3.1.2). Namely (by (3.3.8))

$$\left\|g-q_n\right\|_p\le C(s-1,p)\ \omega_3^{\varphi}\left(g,\frac{1}{n}\right)_p\le C(s,p)\omega_3^{\varphi}\left(f,\frac{1}{n}\right)_p. \tag{3.3.9}$$

Note that, since $g(\alpha)=0$, we may assume that $q_n(x)=0$.

We fix $n > \max\left\{\frac{A(s-1)}{d(s-1)}, N_\alpha\right\}$ readily leads to the definition of $A(s)$. Kopotun [9] has constructed, for α, q_n sufficiently large $\mu \ge 2$, and for each n like above, two polynomials ν_n and W_n of degree $\le C(s)n$ such that the polynomial

$$P_n(x) := \int_\alpha^x \left[\left(q'_n(u) - q'_n(\alpha)\right)\nu_n(u) + q'_n(\alpha)W_n(u)\right]du ,$$

is coconvex with f, and the following inequalities are satisfied $x \in I$,

$$\nu_n(x)\,\text{sgn}\,(x-\alpha) \ge 0 ,$$

$$\nu'_n(x)q''_n(x)\left(q'_n(x) - q'_n(\alpha)\right)\text{sgn}\,(x-\alpha) \ge 0 ,$$

$$\left|\nu_n(x) - \text{sgn}\,(x-\alpha)\right| \le C(s)\psi_{j_0}^{\mu} , \tag{3.3.10}$$

$$\left|W_n(x) - \text{sgn}\,(x-\alpha)\right| \le C(s)\psi_{j_0}^{\mu} , \tag{3.3.11}$$

and

$$\left|\nu'_n(x)\right| \le C(s)h_{j_0}^{-1}\psi_{j_0}^{\mu} . \tag{3.3.12}$$

Observe that $P_n(x) := P_n(x) + \ell_1(x)$ is of same degree of P_n and it too is coconvex with f, so we conclude the induction step by proving (3.3.2) for P_n, and to this end, we begin with

$$
\begin{aligned}
\|f-P_n\|_p &= \|\bar{f}-P_n\|_p = \|\hat{f}(x)\,\mathrm{sgn}\,(x-\alpha)-P_n\|_p \\
&\le C(p)\left(\|\hat{f}-g\|_p + \|g(x)\,\mathrm{sgn}\,(x-\alpha)-P_n\|_p \right) \\
&\le C(p)\left(\omega_3^{\varphi}\left(f,\frac{1}{n}\right)_p + \|g(x)\,\mathrm{sgn}\,(x-\alpha)-P_n\|_p \right) \\
&\le C(p)\Bigg(\omega_3^{\varphi}\left(f,\frac{1}{n}\right)_p + \|g-q_n\|_p + \Big\|q_n(x)\,\mathrm{sgn}\,(x-\alpha) \\
&\qquad - \int_\alpha^x q_n'(u)\nu_n(x)\,du \Big\|_p + \Big\| |q_n'(\alpha)| \int_\alpha^x \left(\nu_n(u)-W_n(u)\right)du \Big\|_p \Bigg) \\
&\le C(s,p)\Bigg(\omega_3^{\varphi}\left(f,\frac{1}{n}\right)_p + \Big\|q_n(x)\,\mathrm{sgn}\,(x-\alpha) - \int_\alpha^x q_n'(u)\nu_n(x)\,du \Big\|_p \\
&\qquad + \Big\| |q_n'(\alpha)| \int_\alpha^x \left(\nu_n(u)-W_n(u)\right)du \Big\|_p \Bigg) \\
&=: C(s,p)\left(E_1+E_2+E_3\right),
\end{aligned}
$$

where we applied (3.3.7) and (3.3.9) in the first and last inequality respectively.

Recalling that $q_n(\alpha)=0$, integration by parts, (3.3.10) and (3.3.12) yield

$$
\begin{aligned}
E_2 &= \Big\|q_n(x)\,\mathrm{sgn}\,(x-\alpha) - \int_\alpha^x q_n'(u)\nu_n(x)\,du \Big\|_p \\
&\le \Big\| |q_n(x)|\,|\mathrm{sgn}\,(x-\alpha)-\nu_n(x)| - \int_\alpha^x |q_n(u)|\,|\nu_n'(x)|\,du \Big\|_p \\
&\le C(s)\Big\| |q_n(x)|\,\psi_{j_0}^{\mu} - \int_\alpha^x |q_n(u)|\,h_{j_0}^{-1}\psi_{j_0}^{\mu}(u)\,du \Big\|_p
\end{aligned}
$$

$$\le C(s,p)\left(\|q_n\|_p + \left\| \int_{\alpha}^{x} |q_n(u)| h_{j_0}^{-1}\psi_{j_0}^{\mu}(u)du \right\|_p\right)$$

$$=: C(s,p)\left(E_{2,1} + E_{2,2}\right).$$

To estimate E_2 , we need estimate $E_{2,1}$ and $E_{2,2}$.

By virtue of (3.3.6), (3.3.7) and (3.3.9),

$$E_{2,1} = \|q_n\|_p \le C(p)\left(\|g - q_n\|_p + \|\hat{f} - g\|_p + \|\hat{f}\|_p \right) \le C(s,p)\omega_3^{\varphi}\left(f, \frac{1}{n}\right)_p , \tag{3.3.13}$$

Now, to estimate $E_{2,2}$, we separate the cases $p \ge 1$, from the cases $0 < p < 1$, and we reall Jensen's inequality from[68], which is

$$\phi\left\{ \frac{\int_a^b f(x)p(x)dx}{\int_a^b p(x)dx} \right\} \le \frac{\int_a^b \phi(f(x))p(x)dx}{\int_a^b p(x)dx}, \tag{3.3.14}$$

where ϕ is convex in interval $d \le f(x) \le e$, that $d \le f(x) \le e$ in $a \le x \le b$, the $p(x)$ is nonnegative and $\not\equiv 0$, and all the integrals in the inequality exists.

First: For $1 \le p < \infty$, since we have $\|.\|_p$ is convex, and all integrals exist, so by applying the Jensen's inequality (3.3.14) and (3.2.14), we obtain

$$E_{2,2} := \left\| \int_{\alpha}^{x} |q_n(u)| h_{j_0}^{-1}\psi_{j_0}^{\mu}(u)du \right\|_p \le \left\| \int_{-1}^{1} |q_n(u)| h_{j_0}^{-1}\psi_{j_0}^{\mu}(u)du \right\|_p$$

$$\le \int_{-1}^{1} \| q_n \|_p h_{j_0}^{-1} \psi_{j_0}^{\mu}(u) du \quad \le C \| q_n \|_p .$$

Second: For the other case, fix $0 < p < 1$, since $q_n h_{j_0}^{-1} \psi_{j_0}^{\mu} \in M(p, \varepsilon)$, for some $\varepsilon > 0$, then by choosing (for example $q = 2$) from Theorem F, it follows that

$$\left\| \int_{\alpha}^{x} q_n(u) h_{j_0}^{-1} \psi_{j_0}^{\mu}(u) du \right\|_p \le c \left\| \int_{\alpha}^{x} q_n(u) h_{j_0}^{-1} \psi_{j_0}^{\mu}(u) du \right\|_2 ,$$

then we use the Jensen's inequality (3.3.14), to obtain

$$\left\| \int_{\alpha}^{x} |q_n(u)| h_{j_0}^{-1} \psi_{j_0}^{\mu}(u) du \right\|_2 \le \int_{-1}^{1} \| q_n \|_2 h_{j_0}^{-1} \psi_{j_0}^{\mu}(u) du \quad \le C \| q_n \|_2 .$$

Hence by using Theorem E, in new, we obtain

$$E_{2,2} := \left\| \int_{\alpha}^{x} |q_n(u)| h_{j_0}^{-1} \psi_{j_0}^{\mu}(u) du \right\|_p \le c \| q_n \|_2 \le C(p) \| q_n \|_p .$$

Thus, in each case, we have

$$E_{2,2} \le C(p) \| q_n \|_p := C(p) E_{2,1} .$$

So by virtue (3.3.13)

$$E_2 \le C(s, p)\, \omega_3^{\varphi}\left(f, \frac{1}{n} \right)_p \tag{3.3.15}$$

Finally, it remains only to estimate E_3, to do so, we notice that q_n is convex in $[-1, y_2]$, then q_n' is monotone increasing there. If $q_n'(\alpha) \ge 0$, then by mean value theorem, for some $\beta \in (\alpha, \alpha + h_{j_0})$,

$$0 \le q_n'(\alpha) \le q_n'(\beta) = \frac{q_n(\alpha + h_{j_0}) - q_n(\alpha)}{h_{j_0}} = h_{j_0}^{-1} q_n(\alpha + h_{j_0}).$$

Then, by (3.2.15), (3.3.10), (3.3.11) and (3.3.13) we have

$$E_3 = \left\| \left| q_n'(\alpha) \right| \int_\alpha^x \left(\nu_n(u) - W_n(u) \right) du \right\|_p$$

$$\leq \left(\int_{-1}^{1} \left(\left| h_{j_0}^{-1} q_n' \left(\alpha + h_{j_0} \right) \right| \int_{-1}^{1} C(s) \psi_{j_0}^{\mu}(u) du \right)^p dx \right)^{\frac{1}{p}}$$

$$\leq C(s,p) \left(\int_{-1}^{1} \left| q_n' \left(\alpha + h_{j_0} \right) \right|^p dx \right)^{\frac{1}{p}} = C(s,p) \|q_n\|_p$$

$$\leq C(s,p)\, \omega_3^{\varphi}\left(f, n^{-1}\right)_p .$$

Similarly, if $q_n'(\alpha) < 0$, then, for some $\beta \in \left(\alpha - h_{j_0}, \alpha\right)$,

$$0 \leq -q_n'(\alpha) \leq -q_n'(\beta) = \frac{q_n\left(\alpha - h_{j_0}\right) - q_n(\alpha)}{h_{j_0}} = h_{j_0}^{-1} q_n\left(\alpha - h_{j_0}\right),$$

then

$$E_3 \leq C(s,p)\, \omega_3^{\varphi}\left(f, \frac{1}{n}\right)_p .$$

This implies our assertion◊

3.4 Proof of Theorem 3.1.6:

For $s = 4$, then for each $n \geq 3$ and $A > 0$, let us we choose $0 < b < \frac{1}{2}$ from the condition

$$\frac{2 - n^2 b^2}{cn^2 b} = A ,$$

where c is a positive constant.

Put

$$f(x) := \begin{cases} -\left(x^2 - b^2\right)^2 & |x| \le b \\ \left(x^2 - b^2\right)^2 & |x| > b, \end{cases}$$

Set the points, $y_1 := -b$, $y_2 = \frac{-1}{\sqrt{3}} b$, $y_3 = \frac{1}{\sqrt{3}} b$ and $y_4 = b$, then it is clear that $f \in C^1(I) \cap \Delta^2(Y_4)$.

Let $P_n \in \Pi_n \cap \Delta^2(Y_4)$, that is, P_n'' is convex in $[-1, b]$, $\left[\frac{-1}{\sqrt{3}} b, \frac{1}{\sqrt{3}} b\right]$ and $[b, 1]$, and its concave in $\left[-b, \frac{-1}{\sqrt{3}} b\right]$ and $\left[\frac{1}{\sqrt{3}} b, b\right]$.

Set

$$Q(x) := \left(x^2 - b^2\right)^2,$$

and

$$M_n(x) := P_n(x) + Q(x).$$

Now, since $P_n''(0) \ge 0$, then

$$M_n''(0) + Q_n''(0) = P_n''(0) \ge 0 ,$$

so

$$M_n''(0) \ge -Q_n''(0) = 4b^2 \ge 0 .$$

Let us make use of S. N. Bernstein's inequality, to obtain

$$4b^2 \le M_n''(0) \le n^2 \|M_n\|,$$

hence

$$\begin{aligned} \frac{4b^2}{n^2} \le \|M_n\| &\le \|f - P_n\| + \|f - Q\| \\ &\le \|f - P_n\| + 2b^4 , \end{aligned}$$

so

$$\frac{c\,n}{2}\left\|f-P_n\right\|_p \geq \left\|f-P_n\right\| \geq \frac{4b^2}{n^2}-2b^4 .$$

On the other hand, we have

$$\begin{aligned}\omega\left(f,n^{-1}\right)_p &< 2^{\frac{1}{p}}\,\omega\left(f,n^{-1}\right)\\ &\leq 2\,\omega\left(f-Q,n^{-1}\right)\\ &\leq 2n^{-1}\left\|f'-Q'\right\|\\ &\leq 2n^{-1}\,\frac{2^3}{3\sqrt{3}}b^3\\ &< 2^4 n^{-1} 2^{-2} b^3\\ &\leq 2^2 n^{-1} b^3 .\end{aligned}$$

Thus

$$\frac{\left\|f-P_n\right\|_p}{\omega\left(f,n^{-1}\right)_p} > \frac{\frac{2}{c}n^{-1}\left(\frac{4b^2}{n^2}-2b^4\right)}{2^2 n^{-1} b^3} = \frac{2-n^2b^2}{cn^2b} = A .$$

Hence the proof is complete in this case. Now, for the other case $s>4$, we will restrict our definition of f on $\left[\frac{-1}{2},\frac{1}{2}\right]$ and we set $f(x)=f(1/2)$, for any x such that $|x|>\frac{1}{2}$, and we do the same thing to Q, in other words we will defining f as the following

$$f(x):=\begin{cases} -\left(x^2-b^2\right)^2 & |x|\leq b\\ \left(x^2-b^2\right)^2 & b<|x|\leq \frac{1}{2}\\ \left(\frac{1}{4}-b^2\right)^2 & \text{o.w},\end{cases}$$

and

$$Q(x) := \begin{cases} -\left(x^2 - b^2\right)^2 & |x| \le \frac{1}{2} \\ \left(\frac{1}{4} - b^2\right)^2 & \text{o.w} \end{cases}$$

It is clear that $f \in C(I)$ and change it's convexity at y_1, y_2, y_3 and y_4. Now we take $s-4$ arbitrary points satisfying $\frac{1}{2} < y_5 < y_6 < \ldots < y_s < 1$, and regarding f as changing convexity at these points too, hence $f \in \Delta^2(Y_s)$, and we complete proof in the similar way as above, this completes the proof of our theorem. $\Diamond$

3.5 Proof of Theorem 3.1.7 and Theorem 3.1.8:

We begin with the Proof of Theorem 3.1.8:

If $k = 1$, then Theorem 3.1.8, is trivial and there is nothing to prove, since $\sum_{k,n} \subseteq \Pi_0$. Thus we have to show our assertion for the case $k \ge 2$. To this end, we need the polynomials $\tilde{\tau}_j$'s, $j = 1, 2, \ldots, n-1$, which are defined by Leviatan and Shevchuk [50], as the following:

Given $n \in \mathbb{N}$, let $G_\upsilon := (x_{J_\upsilon}, x_{j_\upsilon})$ be denote the connected component of $O = O(n, Y_s)$. For each $j = 1, 2, \ldots, n-1$, let $\tilde{\tau}_j$ be polynomials of degree $\le c(s)n$ defined as follows.

a. if $j \in H$, then

$$\tilde{\tau}_j(x) := \tau_j(x)$$

where τ_j are from Lemma 3.2.17,

b. if $j_\upsilon = 0$ and $0 < j < J_\upsilon$, then $\tilde{\tau}_j(x) := 0$,

c. if $J_\upsilon = n$ and $j_\upsilon < j < n$, then $\tilde{\tau}_j(x) := (x - x_j)$,

For the other j's, which is $0 < j_\upsilon < j < J_\upsilon < n$. We divide the υ's into two groups. Let $n_1 := 22s(k-1)^3 n$. We say that $\upsilon \in Od$ if there exists an $\ell_\upsilon \in H(n_1, Y_s)$ such that $I_{\ell_\upsilon, n_1} \cap G_\upsilon \ne \phi$, and the interval $(x_{\ell_\upsilon, n_1}, x_{j_\upsilon, n_1})$ contains an odd number of points y_i. Note that if $\upsilon \notin Od$ then the set G_υ contains an even number, say $2m$, of points y_i, say $y_{i_0} < y_{i_0+1} \ldots < y_{i_0+2m-1}$. In this case each two consecutive points y_{i_0+2v} and y_{i_0+2v+1}, $v = 0, 1, \ldots, m-1$, must belong to the union of four consecutive intervals, say $[x_{\ell_v+2, n_1}, x_{\ell_v-2, n_1})$, hence

$$\{x \in G_\upsilon : \pi(x_{j_\upsilon}) S''(x) < 0\} \subseteq \bigcup_{v=0}^{m-1} [x_{\ell_v+2, n_1}, x_{\ell_v-2, n_1}].$$

It follows by the left-hand side of (3.2.8) that,

$$\text{meas}\ \left\{x \in G_\upsilon : \pi\left(x_{j_\upsilon}\right)S''(x) < 0\right\} \le \sum_{\nu=0}^{m-1} 4 \max_{I_{\ell,n_1} \subseteq \left(x_{J_\upsilon}, x_{j_\upsilon}\right)} \left|I_{\ell,n_1}\right|$$

$$\le 2s \max_{I_{\ell,n_1} \subseteq \left(x_{J_\upsilon}, x_{j_\upsilon}\right)} \left|I_{\ell,n_1}\right|$$

$$\le 4s \frac{n}{n_1} \max_{I_\ell \subseteq \left(x_{J_\upsilon}, x_{j_\upsilon}\right)} \left|I_\ell\right|$$

$$\le 4s \frac{n}{n_1} \frac{\left|G_\upsilon\right|}{\left(J_\upsilon - j_\upsilon\right)}$$

$$\le 4s \frac{\left|G_\upsilon\right|}{3\frac{n_1}{n}}$$

$$= 4 \frac{\left|G_\upsilon\right|}{66(k-1)^3}$$

$$\le \frac{1}{16(k-1)^3} \left|G_\upsilon\right| . \qquad (3.5.1)$$

We need the polynomials τ_{j_υ} and τ_{J_υ}, however, we note that j_υ might not be in H. Since $2j_\upsilon$ is always in $H(2n, Y_s)$, in the case $j_\upsilon \notin H$, we define $\tilde{\tau}_{j_\upsilon} := \tau_{j_\upsilon} := \tau_{2j_\upsilon,2n}$. Similarly, we always have $2J_\upsilon \in H(2n, Y_s)$, in the case

$J_\upsilon \notin H$, so we define $\tilde{\tau}_{J_\upsilon} := \tau_{J_\upsilon} := \tau_{2J_\upsilon,2n}$. Now,

d. if $0 < j_\upsilon < j < J_\upsilon < n$ and $\upsilon \notin Od$, then we let

$$\tilde{\tau}_j(x) := \tau_{j_\upsilon}(x),$$

e. if $0 < j_\upsilon < j < J_\upsilon < n$ and $\upsilon \in Od$, then we let

$$\tilde{\tau}_j(x) := \delta_j \tau_{j_\upsilon}(x) + \left(1 - \delta_j\right)\tau_{\ell_\upsilon,n_1}(x),$$

where $\delta_j = 0$ or $= 1$, is to be prescribed.

Now, we recall the piecewise linear function L that interpolates S, at the points x_j's, satisfies

$$\| S - L \|_p \le C(p)\omega_2^{\varphi}\left(S, \frac{1}{n}\right)_p , \tag{3.5.2}$$

and may be written in the following form

$$L(x) := \ell(x) + \sum_{j=1}^{n-1} [x_{j-1}, x_j, x_{j+1}; S](x_{j-1} - x_{j+1})(x - x_j)_+ ,$$

where $\ell(x)$ is a linear function and $(x - a)_+ := \begin{cases} x - a & x \ge a \\ 0 & o.w \end{cases}$.

We define the polynomial P_n of degree $\le c(s)n$, by

$$P_n(x) := \ell(x) + \sum_{j=1}^{n-1} [x_{j-1}, x_j, x_{j+1}; S](x_{j-1} - x_{j+1})\tilde{\tau}_j(x).$$

Firstly, we will show that $P_n \in \Delta^2(Y_s)$, to this end, we use the same proof, which Leviatan and Shevchuk have used

Now, we denote

$$Q_j(x) := [x_{j-1}, x_j, x_{j+1}; S](x_{j+1} - x_{j-1})\tilde{\tau}_j(x), \quad j = 1, 2, \ldots, n-1,$$

and

$$P_n(x) =: \ell(x) + A(x) + B(x) + C(x) + D(x) + E(x),$$

where

$$A(x) := \sum_{j \in H} Q_j(x) + \sum_{J_\upsilon < n} Q_{J_\upsilon}(x),$$

$$B(x) := \sum_{j=1}^{J_\upsilon - 1} Q_j(x), \quad j_\upsilon = 0,$$

$$C(x) := \sum_{j=j_\upsilon}^{n-1} Q_j(x), \quad J_\upsilon = n,$$

$$D(x) := \sum_{\upsilon \in Od} \sum_{j=j_\upsilon+1}^{J_\upsilon-1} Q_j(x),$$

$$E(x) := \sum_{\upsilon \notin Od} \sum_{j=j_\upsilon+1}^{J_\upsilon-1} Q_j(x) =: \sum_{\upsilon \notin Od} E_\upsilon(x).$$

Since $\ell''(x) \equiv 0$, so $\pi(x)\ell''(x) \equiv 0$, $\forall x \in I$.

It is important to emphasize that, we either $j_\upsilon \in H$ or $j_\upsilon = J_\upsilon$, so that indeed all $1 \le j \le n-1$ are taken care of. We have to investigate each case separately.

For case a., that is, if $j \in H$, then by definition of $\Delta^2(Y_s)$, we have, $\pi(x_j)[x_{j-1}, x_j, x_{j+1}; S] \ge 0$. Then by (3.2.18), we get

$$\pi(x)Q_j''(x) = \pi(x)[x_{j-1}, x_j, x_{j+1}; S](x_{j-1} - x_{j+1})\tau_j''(x) \ge 0,$$

similarly, for $J_\upsilon < n$, $\pi(x)Q_{J_\upsilon}''(x) \ge 0$, hence

$$\pi(x)A''(x) \ge 0, \quad \forall x \in I.$$

For the cases b. and c., we have $B(x)$ and $C(x)$ are linear functions, so $B''(x) \equiv 0$ and $C''(x) \equiv 0$, thus $\pi(x)B''(x) \equiv 0$ and $\pi(x)C''(x) \equiv 0$, $\forall x \in I$.

For the case d., that is if $\upsilon \notin Od$, then

$$\begin{aligned} E_\upsilon &= \sum_{j=j_\upsilon+1}^{J_v-1} Q_j(x) = \tau_{j_\upsilon}(x) \sum_{j=j_\upsilon+1}^{J_v-1} [x_{j-1}, x_j, x_{j+1}; S](x_{j-1} - x_{j+1}) \\ &= \tau_{j_\zeta}(x)\Big([x_{j_\upsilon}, x_{j_\upsilon+1}, x_{J_\upsilon}; S](x_{j_\upsilon} - x_{J_\upsilon}) + [x_{j_\upsilon}, x_{J_\upsilon-1}, x_{J_\upsilon}; S](x_{j_\upsilon} - x_{J_\upsilon})\Big) \\ &= \tau_{j_\zeta}(x) e_\upsilon. \end{aligned}$$

In virtue of Lemma 3.2.16, and using (3.5.1), we conclude

$$\pi(x_{j_\upsilon}) e_\upsilon \ge 0.$$

Hence, (3.2.18) implies

$$\pi(x)E''(x)=\pi(x)\pi(x_{j_v})\tau''_{j_\upsilon}(x)\pi(x_{j_v})e_\upsilon(x)\frac{1}{\pi(x_{j_\upsilon})^2}\geq 0\ ,\ \forall x\in I .$$

It remained only for the case e., which is the case $\upsilon\in Od$, in this case by definition $\upsilon\in Od$; we have an odd number of points $y_i\in(x_{\ell_\upsilon,n_1},x_{j_\upsilon})$, and this means that

$$\pi(x_{\ell_\upsilon,n_1})\pi(x_{j_v})<0 .$$

Now, Since in virtue of (3.2.18), we have

$$\tau''_{\ell_\upsilon,n_1}(x)\pi(x)\pi(x_{\ell_\upsilon,n_1})\geq 0 \text{ and } \tau''_{j_\upsilon}(x)\pi(x)\pi(x_{j_\upsilon})\geq 0\ ,\ \forall x\in I .$$

Therefore the above inequalities implies that

$$\tau''_{\ell_\upsilon,n_1}(x)\tau''_{j_\upsilon}(x)\leq 0\ ,\ \forall x\in I ,$$

this means that $\tau''_{\ell_\upsilon,n_1}(x)$ and $\tau''_{j_\upsilon}(x)$ have different signs for all $x\in I$. Hence for each $j=j_\upsilon+1,j_\upsilon+2,\ldots,J_\upsilon-1$, we may subscribe δ_j so that

$$\pi(x)Q''_j(x)\geq 0\ ,\ \forall x\in I .$$

With this choice

$$\pi(x)D''_j(x)\geq 0\ ,\ \forall x\in I .$$

Thus the above discussion yields that $\pi(x)P''_n(x)\geq 0\ ,\ \forall x\in I$.

Now, we will prove (3.1.9) and (3.1.10), to this: We have, in virtue of Lemma 3.2.17, the following inequality

$$\left|\left(x-x_j\right)-\tilde{\tau}_j(x)\right| \le c(s)h_j\psi_j^2(x), \qquad (3.5.3)$$

it is well- known that for each $j=1,2,\dots,n-1$, we have

$$\left\|[x_{j-1},x_j,x_{j+1};S]\right\|_p \le C(p)h_j^{-2}\,\omega_2\left(S,n^{-1}\right)_p, \qquad (3.5.4)$$

and since

$$\left\|[x_{j-1},x_j,x_{j+1};S]\right\|_p = 2^{\frac{1}{p}}\left|[x_{j-1},x_j,x_{j+1};S]\right|.$$

Then by using the definitions of L and P_n, and the ineq- ualityies (3.2.15), (3.5.2), (3.5.3) and (3.5.4), we obtain

$$\begin{aligned}
\|L-P_n\|_p &= \left\|\sum_{j=1}^{n-1}\left|[x_{j-1},x_j,x_{j+1};S]\right|\left(x_{j-1}-x_{j+1}\right)\left(\left(x-x_j\right)_+-\tilde{\tau}_j(x)\right)\right\|_p \\
&\le \left\|\sum_{j=1}^{n-1}C(p)h_j^{-2}\,\omega_2\left(S,n^{-1}\right)_p\,4h_j\left(C(s)h_j\psi_j^2(x)\right)\right\|_p \\
&\le C(s,p)\left\|\sum_{j=1}^{n-1}h_j^{-2}\omega_2\left(S,n^{-1}\right)_p\left(4h_j\right)\left(h_j\psi_j^2(x)\right)\right\|_p \qquad (3.5.5)\\
&= C(s,p)\left\|\sum_{j=1}^{n-1}\psi_j^2(x)\right\|_p\,\omega_2\left(S,n^{-1}\right)_p \\
&\le C(s,p)\omega_2\left(S,n^{-1}\right)_p,
\end{aligned}$$

hence

$$\|S-P_n\|_p \le C(p)\left(\|S-L\|_p+\|L-P_n\|_p\right) \le C(s,p)\omega_2\left(S,n^{-1}\right)_p.$$

This is the proof of (3.1.9).

Now on the other hand, we have

$$\left\|[x_{j-1},x_j,x_{j+1};S]\right\|_p \le C(p)\sum_{i=1}^{n-1}\left\|[x_{j-1},x_j,x_{j+1};S]\right\|_{L_p(I_i)},$$

but since S is a continuous piecewise polynomial of order k, then

$$\|[x_{j-1},x_j,x_{j+1};S]\|_p \le C(p)\sum_{i=1}^{n-1} C(k)\|[x_{i-1},x_i,x_{i+1};S]\|_{L_p(I_i)}$$
$$\le C(k,p)\sum_{i=1}^{n-1} C(k)\|[x_{i-1},x_i,x_{i+1};S]\|_{L_p(I_{i-1}\cup I_i\cup I_{j+1})}$$
$$\le C(k,p)\sum_{j=1}^{n-1} h_j^{-2}\omega_2(S,h_j,I_{j-1}\cup I_j\cup I_{j+1})_p$$
$$\le C(k,p)\max_{j=1,\dots n-1} h_j^{-2}\sum_{j=1}^{n-1}\omega_2(S,h_j,I_{j-1}\cup I_j\cup I_{j+1})_p \ .$$

So, by using Lemma 3.2.22, we obtain

$$\|[x_{j-1},x_j,x_{j+1};S]\|_p \le C(k,p)\max_{j=1,..,n-1} h_j^{-2}\omega_2(S,n^{-1})_p \,. \qquad (3.5.6)$$

Then, by the same way as we do in the proof of (3.5.5), with using (3.5.6) in place of (3.5.4), we can obtain

$$\|L-P_n\|_p \le C(s,p)\omega_2^{\varphi}(S,n^{-1})_p \ . \qquad (3.5.7)$$

Thus (3.1.10) follows, by using (3.5.2) and (3.5.7). ◊

Proof of theorem 3.1.7:

By taking a look at what we presented in the above, exactly in the proof of (3.1.9). One notices that the purpose behind the assumption that our function is piecewise polynomial was in order to apply Lemma 3.2.16, in the case $\upsilon \notin Od$; else we need not to that supposition.

Thus for any arbitrary function $f \in \Delta^2(Y_s)$, and a natural number n big enough such that each component G_υ contains an odd number of points of Y_s, in particular one point, then one may get the same result. If f alters convexity just once, apparently the requirement that each component G_υ contains an odd number of points of Y_s, especially one point holds for all $n \geq 1$. This proves Theorem. ◊

3.6 Proof of Theorem 3.1.11 and Theorem 3.1.12:

In order to prove Theorem 3.1.11, it is enough when we prove Theorem 3.1.12, this by virtue of Lemma 3.2.34, recalling that we may assume $k \geq 3$.

Now, without loss of generality we may assume that

$$a_k(S) \leq 1$$

and (3.6.1)

$$a_k(S)_p \leq 1.$$

Now, to prove our assertion, we have only to find a polynomial P_n of degree at most $c(s,p)n$ which is coconvex with S in the interval I. So by virtue of Lemma 3.2.28, we have only to show that P_n is satisfy

$$\|S - P_n\|_p \leq C(s,k,p). \tag{3.6.2}$$

We need some notations and results from [50], we fix b is too big that $b_2 p \geq 2$ and $b_3 \geq 25(s+1)$, where b_2 and b_3 was defined in Lemma 3.2.53 and 3.2.58, this makes $C_0(k,s)$ the constant in Lemma 3.2.71, so we denote $c_2 = C_0$, and we fix an integer c_3 such that

$$c_3 \geq \max\left\{\frac{8k}{c_1}, 12s\right\},$$

where c_1 is the constant from Lemma 3.2.48 .

Without any losing of generality, we may assume that n is divisible by c_3, (i.e., $n = M c_3$), where this defines M and we divided I into M intervals

$$E_q := \lfloor x_{qc_3}, x_{(q-1)c_3} \rfloor = I_{qc_3} \cup \ldots \cup I_{(q-1)c_3+1}\ ,\ q = 1,2,\ldots,M\ .$$

We write $j \in UC$ (for "*Under Control*"), if there is an $x \in I_j$ such that

$$|\, S''(x)| \leq \frac{5c_2}{(\Delta_n(x))^2}\ , \tag{3.6.3}$$

We say that $q \in G$, if E_q contains at least $2k-5$ intervals I_j with $j \in UC$, also we say $q \in G_1$, if either $q \in G$ or there is a $q^* \in G$, such that

$$E^{e}_{q+\upsilon} \cap O \neq \phi, \quad \begin{cases} \upsilon = 0,1,\ldots,q^* - q & \text{if } q^* \geq q \\ \upsilon = 0,-1,\ldots,q^* - q & \text{if } q^* \leq q, \end{cases}$$

where for any Given $A \subseteq I$, we have

$$A^e := \bigcup_{I_j \cap A \neq \phi} I_j, \quad A^{2e} := \left(A^e\right)^e \quad \text{and} \quad A^{3e} := \left(A^{2e}\right)^e\ ,$$

Note that if $q \in G_1 \setminus G$, then $\left|q - q^*\right| \leq 2s$,), thus (3.6.1), (3.6.3), Lemma 3.2.40 and Lemma 3.2.25 implies

$$\left\| (\Delta_n(x))^2 S''(x) \right\|_{E_q} \le C(k,s), \quad q \in G_1, \text{ [50]} \tag{3.6.4}$$

and

$$\left\| (\Delta_n(x))^2 S''(x) \right\|_{L_p(E_q)} \le C(k,s,p), \quad q \in G_1. \tag{3.6.5}$$

Set

$$E := \bigcup_{q \notin G_1} E_q ,$$

we decompose S into *small* part and a *big* one by setting

$$s_1(x) := \begin{cases} S''(x) & x \notin E^e \\ 0 & x \in E^e \end{cases} \quad \text{and} \quad s_2(x) := S''(x) - s_1(x).$$

and putting

$$S_1(x) := S(-1) + (x+1)S'(x) + \int_{-1}^{x} (x-u)s_1(u)du \ ,$$

$$S_2(x) := \int_{-1}^{x} (x-u)s_2(u)du \ .$$

Now, since $S''(x)$ are well defined for $x \neq x_j$, $j = 0,1,\dots,n$, so that s_1 and s_2 are well defined for $x \neq x_j$, $j = 0,1,\dots,n$, hence S_1 and S_2 are well defined everywhere and possess a second derivative again for $x \neq x_j$, $j = 0,1,\dots,n$. Thus from now and over whenever we write S''_υ we will mean $x \neq x_j, j = 0,1,\dots,n$.

It is clear that $S_1, S_2 \in \sum_{k,n}^{1}(Y_s)$. Thus

$$S_1''(x)\delta(x) \ge 0, \quad x \in I, \text{ and } \quad S_2''(x)\delta(x) \ge 0, \quad x \in I .$$

By using Lemma 3.2.39 and the inequalities (3.6.4) and (3.6.5) we obtain

$$a_k(S_1) \le C(k,s), \ [50]$$

and

$$a_k(S_1)_p \le C(k,s,p),$$

then by virtue of (3.6.1) and the fact that $S = S_1 + S_2$, we have

$$a_k(S_2) \le 1 + C(k,s) \le [1 + C(k,s)] =: c_4, \ [50] \qquad (3.6.6)$$

and

$$a_k(S_2)_p \le C(p)(1 + C(k,s,p)) \le [C(k,s,p)] =: c_5, \qquad (3.6.7)$$

where $[d]$ denote the smallest integer greater than d.

The set E is a union of disjoint intervals $F_\ell = [a_\ell, b_\ell]$, between any two of which there is an interval E_q with $q \in G_1$. We may assume that $n \ge c_3 c_6$ where $c_6 := \max\{c_4, c_5\}$, and we write $\ell \in AG$ (for "*Almost Good*"), if F_ℓ consists of no more than c_6 intervals E_q, in particular it consists of no more than $c_3 c_6$ intervals I_j. Set

$$F := \bigcup_{\ell \notin AG} F_\ell .$$

and let

$$s_4(x) := \begin{cases} S''(x) & x \in F^e \\ 0 & x \notin F^e, \end{cases}$$

and

$$s_3(x) := S''(x) - s_4(x).$$

Put

$$S_3(x) := S(-1) + (x+1)S'(x) + \int_{-1}^{x} (x-u) s_1(u) du \quad ,$$

and

$$S_4(x) := \int_{-1}^{x} (x-u) s_2(u) du \quad .$$

Then evidently

$$S_3, S_4 \in \sum_{k,n}^{1} (Y_s), \qquad (3.6.8)$$

$$S_3''(x)\delta(x) \geq 0, \quad x \in I, \qquad (3.6.9)$$

and

$$S_4''(x)\delta(x) \geq 0, \quad x \in I. \qquad (3.6.10)$$

For $\ell \in AG$, then from the definitions of S_2 and S_3 , the inequality (3.6.4) implies

$$\left| S_3''(x) \right| = \left| S_2''(x) \right| \leq \frac{c(k,s)}{(\Delta_n(x))^2} , \ x \in F_\ell .$$

So, by using Lemma 3.2.25, we obtain

$$\left| S_3''(x) \right| \leq \frac{C(k,s)}{(\Delta_n(x))^2} , \ x \in I ,$$

Hence

$$\left\| (\Delta_n(x))^2 S_3''(x) \right\|_p \leq C(k,s,p) \ .$$

Then by applying Lemma 3.2.39 we get

$a_k(S_3) \leq C(k,s)$, and $a_k(S_3) \leq C(k,s,p)$, (3.6.11)

thus, by virtue of (3.6.1), we have

$$a_k(S_4) \le [1 + C(k,s)] =: c_7 , \qquad (3.6.12)$$

and

$$a_k(S_4)_p \le [C(k,s,p)] . \qquad (3.6.13)$$

Now, in view of (3.6.8), Theorem 3.1.8 implies the existence of a polynomial $r_n(x)$ of degree $\le C(s)n$, which is coconvex with S_3 and satisfies

$$\| S_3 - r_n \|_p \le C(k,s,p) \omega_2^{\varphi}(S_3, n^{-1})_p .$$

By combining this with Lemma 3.2.28 and the inequa-lities in (3.6.11), we obtain

$$\| S_3 - r_n \|_p \le C(k,s,p). \qquad (3.6.14)$$

On the other hand Leviatan and Shevchuk [50] have constructed three polynomials Q_n ,and M_n , such that

$$\| Q_n \| \le C(k,s), \text{ and } \| M_n \| \le C(k,s),$$

and the polynomial $R_n(x) := D_{n_1}(x) + c_2 Q_n(x) + M_n(x)$, of degree $\le c(s)n \le c(s,p)n$ which is coconvex with S_4 , and n_1 is chosen to be divisible by n ,and greater than $n \max \{c_6, c_7\}$. Such that

$$\| S_4 - R_n \| \le C(k,s).$$

Hence, it is clear that the polynomial $P_n := r_n + R_n$, have the same degree of R_n , and it's coconvex with S . Thus it remains only to show P_n satisfies (3.6.2), to this end, we have

$$\begin{aligned}\|S-P_n\|_p &\le C(p)\left(\|S_3-r_n\|_p+\|S_4-R_n\|_p\right)\\ &\le C(p)\left(\left(\|S_3-r_n\|_p+\left\|S_4-D_{n_1}\right\|_p+c_2\left(\|Q_n\|_p+\|Q_n\|_p\right)\right)\right)\\ &\le C(k,s,p),\end{aligned}$$

where we used Lemma 3.2.67 and the inequalities (2.2.3), (3.6.13) and (3.6.14).

Thus we proved our assertion, for $n \ge c$, divisible by c_3. For all other n's Theorem 3.2.12 follows by the inclusion

$$\sum\nolimits_{k,n}^{1}(Y_s) \subseteq \sum\nolimits_{k,c_3 n}^{1}(Y_s). \Diamond$$

Chapter Two

Approximation by Some Piecewise Linear Functions.

In this chapter we approximate a function $f \in L_p(I)$, by some piecewise linear functions which are introduced by DeVore and Yu [15] and Zoltan [67], using a special partition of the interval I, and we shall obtain global estimates using the second order of the Ditzian – Totik modulus of smoothness.

2.1 Introduction and Main Results

Let $<\delta_k>_{k=-n}^{n}$ be a sequence such that $\delta_{-k} = \delta_k$ $(k = 1, 2, \ldots, n)$ and $n^{-2} = \delta_n < \delta_{n-1} < \cdots < \delta_1 < \delta_0 < c_0 n^{-1}$. Moreover, let $<x_k>_{k=-n}^{n}$: $-1 := x_{-n} < x_{-n+1} < \cdots < x_{n+1} < 1 := x_n$ be a partition of the interval I, and $I_k := [x_k, x_{k+1}]$, each I_k has the following properties:

(i) $c_1\delta_k < |I_k| < c_2\delta_k$, ($k = -n, -n+1, \ldots, n-2, n-1$),

(ii) For any $u \in I_k$, ($k = -n+1, \ldots, n-2$), we have

$$|I_k| \le cn^{-1}\varphi(u),$$

where the value of c may vary with each occurrence.

The existence of $<\delta_k>_{k=-n}^{n}$ and $<x_k>_{k=-n}^{n}$ is guaranteed by DeVore and Yu [15]. We are interested in such app-roximation because its significance in polynomial approximations, as we have presented in section 1.4.

First we approximate a function $f \in L_p(I)$, by the piecewise linear function $S_n f$, which is introduced by DeVore and Yu [15], and interpolate f at the points x_k ($k = -n, -n+1, \ldots, n-1, n$), defined as follows

$$(S_n f)(x) := \frac{x_{-n+1} - x + |x_{-n+1} - x|}{2(x_{-n+1} - x_{-n})} f(x_{-n}) + $$
$$+ \sum_{k=-n+1}^{n-1} \frac{x_{k+1} - x_k}{2} [x_{k-1}, x_k, x_{k+1}; |t - x|]_t f(x_k) +$$
$$+ \frac{x - x_{n-1} + |x - x_{n-1}|}{2(x_n - x_{n-1})} f(x_n) \qquad \forall x \in I,$$

where the notation $[x_{k-1}, x_k, x_{k+1}; |t - x|]_t$ means that the divided difference is applied on the variable t.

It should be noted that for any $x \in I_k$, $k = -n, -n+1, \dots, n-1, n$

$$(S_n f)(x) := \frac{x_{k+1} - x}{x_{k+1} - x_k} f(x_k) + \frac{x - x_k}{x_{k+1} - x_k} f(x_{k+1}), \qquad (2.1.1)$$

and $S_n : L_p(I) \to L_p(I)$ is a positive linear operator, that is, S_n preserves the positivity of f.

Our first achievement is the following:

Theorem 2.1.2.

For $n \ge 2$, $P_n \in \Pi_n$ and $0 < p < \infty$ we have

$$\| f - S_n f \|_p \le C(p)\, \omega_2^{\varphi}\left(f, \frac{1}{n}\right)_p .$$

In the next, we approximate $f \in L_p(I)$ by a new piecewise linear function $U_n f$, defined by Zoltan [67], as

$$\begin{aligned}(U_n f)(x) &:= \frac{x_{-n+1} - x + |x_{-n+1} - x|}{2(x_{-n+1} - x_{-n})} f(x_{-n}) + \sum_{k=-n+1}^{n-1} \frac{x_{k+1} - x_k}{2} . \\ &.[x_{k-1}, x_k, x_{k+1}; |t - x|]_t .\left(f(x_k + n^{\gamma}\varphi(x)) - f(x_k) + f(x_k - n^{\gamma}\varphi(x))\right) + \\ &+ \frac{x - x_{n-1} + |x - x_{n-1}|}{2(x_n - x_{n-1})} f(x_n) \qquad \forall x \in I,\end{aligned}$$

where $\gamma \ge 1$ such that $c_1 \delta_{-n} = c_1 n^{-2} \ge n^{-\gamma}$.

However, the well-definedness of $U_n f$ follows from the following inequalities [67]

$$x_{-n} \le x_{-n+1} - n^{-\gamma}\varphi(x_{-n+1}) \le x_{-n+1} , \qquad (2.1.3)$$

$$x_k \le x_k + n^{-\gamma}\varphi(x_k) \le x_{k+1} \quad (k = -n+1, -n+2, \dots, n-1), \qquad (2.1.4)$$

and

$$x_{k-1} \le x_k - n^{-\gamma}\varphi(x_{k-1}) \le x_k \quad (k = -n+2, -n+3, \dots, n). \qquad (2.1.5)$$

The operator $U_n : L_p(I) \to L_p(I)$ is linear which preserves the linear functions, and interpolates f at end points of the interval I. Furthermore, $U_n f$ is positive, for any positive and convex $f \in L_p(I)$.

Our second main result is:

Theorem 2.1.6.

For $f \in L_p(I), 0 < p \le \infty$ and $n \ge 2$, then

$$\| f - U_n f \|_p \le C(p)\, \omega_2^{\varphi}\left(f, \frac{1}{n}\right)_p .$$

Our next piecewise linear function is $V_n f$, which is introduced by Zoltan [67], for continuous functions, but here we define $V_n f$, for any $f \in L_p(I)$ with $p \ge 1$, and it is well known that $L_p \subset L_1$, for $p \geq 1$, then for $f \in L_p(I)$ with $p \geq 1$, the piecewise linear function $V_n f$ defined as follows

$$(V_n f)(x) := \frac{x_{k+1} - x}{x_{k+1} - x_k} \frac{1}{c_1 \delta_k} \int_{x_k}^{x_k + c_1 \delta_k} f(u)\,du + \frac{x - x_k}{x_{k+1} - x_k} \frac{1}{c_1 \delta_k} \int_{x_{k+1} - c_1 \delta_k}^{x_k} f(u)\,du \ ,$$

if $x_k < x < x_{k+1}, (k = -n, -n+1, \dots, n-1)$ and $(V_n f)(x_k) = f(x_k)$, $(k = -n, -n+1, \dots, n)$.

It follows directly from the following theorem [59], that the above definition of $V_n f$ is well – defined

Theorem 2.1.7.

A function f is lebesgue integrable if and only if $|f|$ lebesgue integrable.

It must be noticed that V_n is a linear, positive operator from $L_p(I)$ into $L_p(I)$ with $1 \leq p < \infty$, but whenever defined on $L_\infty(I)$, its codomain will be the space of all bounded measurable function, since $V_n f$ need not to be continuous in the case f does, and this is the main difference between $S_n f$ and $V_n f$. Moreover $V_n f$ inter-polate f at all of the point x_k, $(k = -n, -n+1, \dots, n)$ as $S_n f$ does, and it preserves the positivity of f. Furthermore, $V_n e_0 := e_0$ and $V_n e_1 := e_1 + O(n^{-1})$ [67], where e_0 and e_1 denote the constant and identity functions respectively.

Our next achievement is:

Theorem 2.1.8.

Let $f \in L_p(I)$ with $1 \le p < \infty$ and $n > r \ge 2$, then

$$\| f - V_n f \|_p \le C(r)\left(\omega_r^{\varphi}(f, n^{-1})_p + \frac{1}{n}\int_0^{n^{-1}} \frac{\omega_r^{\varphi}(f,t)_p}{t^2}\,dt + E_0(f)_p \right).$$

As an immediate consequence of the above theorems we have the following corollary

Corollary 2.1.9.

For any $f \in L_p(I)$ with $1 \le p < \infty$ and $n \ge 2$, we have

$$\| f - V_n f \|_p \le C\left(\frac{1}{n}\int_0^{n^{-1}} \frac{\omega_2^{\varphi}(f,t)_p}{t^2}\,dt + \| f \|_p \right).$$

The proof of the above corollary is follows from the property ($\bar{d}$) and the fact that, for $1 \le p < \infty$, we have $\omega_r^{\varphi}(f,t)_p \in L_1(I)$, and $t^{-2}\omega_r^{\varphi}(f,t)_p$ is a nonincreasing function with respect to t on $[0, n^{-1}]$, then

$t^{-2}\omega_r^{\varphi}(f,t)_p \geq n^2\omega_r^{\varphi}(f,n^{-1})_p$, $\forall\ t \in [0,n^{-1}]$, and so by integrating both sides with respect to t on $[0,n^{-1}]$, we obtain

$$\frac{1}{n}\int_0^{1/n}\frac{\omega_r^{\varphi}(f,t)_p}{t^2}\,dt \geq \omega_2^{\varphi}(f,n^{-1})_p . \Diamond$$

2.2 Auxiliary Lemmas

For the proof of our results we need the following Lemma, which proved in [13] for the case $0 < p < 1$, and in [19] for the other cases (i.e., for $1 \leq p \leq \infty$).

Lemma 2.2.1.

For $E_n(f)_p$ with $0 < p \leq \infty$ we have

$$E_n(f)_p \leq C(r,p)\omega_r^{\varphi}(f,n^{-1})_p \quad , \forall\ n \geq r .$$

We recall several well known facts about algebraic polynomials which will be used in the sequel. The first lemma is merely the equivalence of norms on a finite dimensional space and well known Markov's and Bernstein's type inequality (for example see [5], [17], [18], [19], [41] and [42]).

Lemma 2.2.2.

For any polynomial $q_k \in \Pi_k$ and any interval $J \subseteq I$

$$\|q_k\|_{L_p(J)} \le |J|^{\frac{1}{p}} \|q_k\|_{L_\infty(J)}, \; 0 < p < \infty, \qquad (2.2.3)$$

$$\|q_k\|_J \le c(k,p)|J|^{\frac{-1}{p}} \|q_k\|_{L_p(J)}, \; 0 < p < \infty, \qquad (2.2.4)$$

$$\|q'_k\|_J \le 2k^2|J|^{-1}\|q_k\|_J, \qquad (2.2.5)$$

$$\|q'_k\|_{L_p(J)} \le c(p)k^2\|q_k\|_{L_p(J)}, \; 0 < p < \infty, \qquad (2.2.6)$$

for $0 \le r \le k$, $0 < p < \infty$ and $\varphi(x) := \sqrt{(1-x^2)}$

$$\|q_k^{(r)}\|_J \le c(k,r,p)|J|^{-r-\frac{1}{p}}\|q_k\|_{L_p(J)}, \qquad (2.2.7)$$

and

$$\|\varphi^r q_k^{(r)}\|_p \le c(r,p)k^r\|q_k\|_p . \qquad (2.2.8)$$

Finally, we will prove the following useful lemmas.

Lemma 2.2.9.

If $f \in L_p(I)$, $0 < p < \infty$, $r \in \mathbb{N}_0$ and $n \ge 2$, then

for $k = -n+1, -n+2, \ldots, n-2$ we have

$$\omega_r(f, |I_k|, I_k)_p \le c(r,p)\, \omega_r^{\varphi}\left(f, \frac{1}{n}, I_k\right)_p, \qquad (2.2.10)$$

and for $k = -n, n-1$

$$\omega_r(f, |I_k|, I_k)_p \le C(r,p)\, \omega_r^{\varphi}\left(f, \frac{1}{n}\right)_p. \qquad (2.2.11)$$

Proof:

For $k=-n+1,-n+2,\dots,n-2$, we have, from the property (ii) of our partition, that $|I_k| \le cn^{-1}\varphi(x),\ \forall\ x \in I_k$, then

$$\begin{aligned}
\omega_r\left(f,|I_k|,I_k\right)_p &:= \sup_{0<h<|I_k|} \left\| \Delta_h^r(f,x) \right\|_{L_p(I_k)} \\
&\le \sup_{0<h<cn^{-1}\varphi(x)} \left\| \Delta_h^r(f,x) \right\|_{L_p(I_k)} \\
&= \sup_{0<h<cn^{-1}} \left\| \Delta_{h\varphi(x)}^r(f,x) \right\|_{L_p(I_k)} \\
&= \omega_r^{\varphi}\left(f,cn^{-1},I_k\right)_p \le c(r,p)\,\omega_r^{\varphi}\left(f,n^{-1},I_k\right)_p .
\end{aligned}$$

Since if $k=-n, n-1$, then by property (i) of $|I_k|$ and $\delta_k \le c\dfrac{1}{n^2}$ we have $|I_k| \approx \dfrac{1}{n^2}\ \ \forall\ n \ge 2$, hence

$$\begin{aligned}
\omega_r\left(f,|I_k|,I_k\right)_p &\le C(r,p)\,\omega_r\left(f,n^{-2},I_k\right)_p \\
&\le C(r,p)\,\omega_r^{\varphi}\left(f,n^{-2}\right)_p \\
&\le C(r,p)\,\omega_r^{\varphi}\left(f,n^{-1}\right)_p ,
\end{aligned}$$

where we applied (3) in the third inequality. $\Diamond$

Lemma 2.2.12.

For $n \ge 2$, $P_n \in \Pi_n$ and $1 \le p < \infty$, we have

$$\| V_n P_n \|_p \le C \| P_n \|_p , \tag{2.2.13}$$

and

$$\| P_n - V_n P_n \|_p \le C \left(\frac{1}{n} \| P_n' \|_p + \frac{1}{n^2} \| \varphi^2 P_n'' \|_p \right). \tag{2.2.14}$$

Proof:

For any $k = -n, -n+1, \ldots, n-1$, we have

$$\| V_n P_n \|_{L_p(I_k)} = \Bigg\| \frac{x_{k+1} - x}{x_{k+1} - x_k} \frac{1}{c_1 \delta_k} \int_{x_k}^{x_k + c_1 \delta_k} P_n(u) du + \frac{x - x_k}{x_{k+1} - x_k} \frac{1}{c_1 \delta_k} \int_{x_{k+1} - c_1 \delta_k}^{x_{k+1}} P_n(u) du \Bigg\|_{L_p(I_k)}$$

$$\le \Bigg\| \frac{x_{k+1} - x}{x_{k+1} - x_k} \frac{1}{c_1 \delta_k} \int_{x_k}^{x_k + c_1 \delta_k} |P_n(u)| du + \frac{x - x_k}{x_{k+1} - x_k} \frac{1}{c_1 \delta_k} \int_{x_{k+1} - c_1 \delta_k}^{x_{k+1}} |P_n(u)| du \Bigg\|_{L_p(I_k)}$$

$$\le \Bigg\| \frac{1}{c_1 \delta_k} \int_{x_k}^{x_k + c_1 \delta_k} |P_n(u)| du + \frac{1}{c_1 \delta_k} \int_{x_k}^{x_k + c_1 \delta_k} |P_n(u)| du \Bigg\|_{L_p(I_k)}$$

$$\le \frac{1}{c_1 \delta_k} \left(\Bigg\| \int_{x_k}^{x_k + c_1 \delta_k} |P_n(u)| du \Bigg\|_{L_p(I_k)} + \Bigg\| \int_{x_k}^{x_k + c_1 \delta_k} |P_n(u)| du \Bigg\|_{L_p(I_k)} \right)$$

$$= \frac{1}{c_1 \delta_k} \left(\xi_1 + \xi_2 \right).$$

Now, to estimate ξ_1 and ξ_2, we apply property (i) of our partition and the folder's inequality to obtain

$$\xi_1 \le \left\| \int_{x_k}^{x_{k+1}} |P_n(u)| du \right\|_{L_p(I_k)} \le |I_k|^{\frac{1}{p}} \| P_n \|_{L_1(I_k)}$$

$$\le |I_k|^{\frac{1}{p}+\frac{1}{q}} \| P_n \|_{L_p(I_k)}, \quad \left(\begin{array}{l} \text{where } q \text{ is the conjugate} \\ \text{exponent to } p \end{array} \right)$$

$$\le (c_2 \delta_k) \| P_n \|_{L_p(I_k)}.$$

By the same way we obtain

$$\xi_2 \le (c_2 \delta_k) \| P_n \|_{L_1(I_k)}.$$

Thus

$$\| V_n P_n \|_{L_p(I_k)} \le 2 \frac{c_2}{c_1} \| P_n \|_{L_p(I_k)}, \tag{2.2.15}$$

then

$$\| V_n P_n \|_p = \left(\sum_{k=-n}^{n-1} \| V_n P_n \|_{L_p(I_k)}^p \right)^{\frac{1}{p}}$$

$$\le \left(\sum_{k=-n}^{n-1} \left(2 \frac{c_2}{c_1} \right)^p \| P_n \|_{L_p(I_k)}^p \right)^{\frac{1}{p}}$$

$$\le \left(2 \frac{c_2}{c_1} \right) \| P_n \|_p$$

Thus the proof of (2.2.13) is complete.

Now, to the prove of (2.2.14), we need the following inequalities [67], which are direct consequences of the properties (i) and (ii), for $x \in I_k$, $(k = -n+1, -n+2, \ldots, n-2)$

$$\left| -\frac{1}{2}(x_k - x)^2 + \frac{1}{2}(x_k + c_1\delta_k - x)^2 \right| \le c_1\delta_k |I_k| \le cc_1\delta_k \frac{\varphi(x)}{n}, \tag{2.2.16}$$

and

$$\left|-\frac{1}{2}\left(x_{k+1}-x\right)^2+\frac{1}{2}\left(x_{k+1}-c_1\delta_k-x\right)^2\right|\le cc_1\delta_k\frac{\varphi(x)}{n}, \qquad (2.2.17)$$

for $x_k<u<x_{k+1}$

$$\left(x_k-u\right)^2\le c\frac{\varphi^2(u)}{n^2} \text{ and } \left(x_{k+1}-u\right)^2\le c\frac{\varphi^2(u)}{n^2}, \qquad (2.2.18)$$

for $x_k<x\le u\le x_k+c_1\delta_k<x_{k+1}$ or $x_k<x_k+c_1\delta_k\le u\le x<x_k$,

$$\left(x_k+c_1\delta_k-u\right)^2\le c\frac{\varphi^2(u)}{n^2}, \qquad (2.2.19)$$

and for $x_k<x\le u\le x_k+c_1\delta_k<x_{k+1}$ or $x_k<x_k+c_1\delta_k\le u\le x<x_k$,

$$\left(x_{k+1}-c_1\delta_k-u\right)^2\le c\frac{\varphi^2(u)}{n^2}. \qquad (2.2.20)$$

Now, by simple computation we can show, for $a<b$

$$\int_a^b\left[P_n(u)-P_n(a)\right]du\le\int_a^b(b-u)P_n'(u)du$$

and (2.2.21)

$$\int_a^b\left[P_n(u)-P_n(a)\right]du\le\int_a^b(b-u)P_n'(u)du$$

By partial integration and using (2.2.21), we obtain

$$\left|P_n(x)-(V_nP_n)(x)\right|\le$$

$$\frac{x_{k+1}-x}{x_{k+1}-x_k}\frac{1}{c_1\delta_k}\left\{\left|-\frac{1}{2}\left(x_k-u\right)^2+\frac{1}{2}\left(x_k+c_1\delta_k-u\right)^2\right|\left|P_n'(x)\right|\right.$$

$$+\frac{1}{2}\int_{x_k}^{x}(u-x_k)^2\left|P_n''(u)\right|du+\frac{1}{2}\int_{x}^{x_k+c_1\delta_k}(x_k+c_1\delta_k-u)^2\left|P_n''(u)\right|du\Bigg\}$$

$$+\frac{x-x_k}{x_{k+1}-x_k}\frac{1}{c_1\delta_k}\Bigg\{\left|-\frac{1}{2}(x_{k+1}-c_1\delta_k-u)^2+\frac{1}{2}(x_{k+1}-u)^2\right|\left|P_n'(x)\right| \qquad (2.2.22)$$

$$+\frac{1}{2}\int_{x_{k+1}-c_1\delta_k}^{x}(x_{k+1}-c_1\delta_k-u)^2\left|P_n''(u)\right|du+\frac{1}{2}\int_{x}^{x_{k+1}}(x_{k+1}-u)^2\left|P_n''(u)\right|du\Bigg\}.$$

Then by using the above inequalities (2.2.16) – (2.2.20), and (2.2.22) with the fact that,

$$\forall\ x\in I_k,\ \text{and}\quad (k=-n+1,-n+2,\dots,n-1)$$

$$0\le\frac{x_{k+1}-x}{x_{k+1}-x_k},\frac{x-x_k}{x_{k+1}-x_k}\le 1, \qquad (2.2.23)$$

we obtain

$$\left|P_n(x)-(V_nP_n)(x)\right|\le$$

$$\frac{1}{c_1\delta_k}\Bigg\{2cc_1\delta_k\frac{\varphi(x)}{n}\left|P_n'(x)\right|+\frac{1}{2}\int_{x_k}^{x}c\frac{\varphi^2(u)}{n^2}\left|P_n''(u)\right|du$$

$$+\frac{1}{2}\left|\int_{x}^{x_k+c_1\delta_k}c\frac{\varphi^2(u)}{n^2}\left|P_n''(u)\right|du\right|+\left|\frac{1}{2}\int_{x_{k+1}-c_1\delta_k}^{x}c\frac{\varphi^2(u)}{n^2}\left|P_n''(u)\right|du\right|$$

$$+\frac{1}{2}\int_{x}^{x_{k+1}}c\frac{\varphi^2(u)}{n^2}\left|P_n''(u)\right|du\Bigg\},$$

since

$$\int_{x_k}^{x}+\left|\int_{x}^{x_k+c_1\delta_k}\right|\le\begin{cases}\displaystyle\int_{x_k}^{x_k+c_1\delta_k} & \text{if } x_k\le x\le x_k+c_1\delta_k\\[2ex] \displaystyle 2\int_{x_k}^{x} & \text{if } x_k+c_1\delta_k\le x\le x_{k+1}\end{cases}\le 2\int_{x_k}^{x_{k+1}}$$

and

$$\left| \int_{x_{k+1}-c_1\delta_k}^{x} \right| + \int_{x}^{x_{k+1}} \le \begin{cases} \int_{x_{k+1}-c_1\delta_k}^{x_{k+1}} & \text{if } x_k - c_1\delta_k \le x \le x_{k+1} \\ 2\int_{x}^{x_{k+1}} & \text{if } x_k \le x \le x_{k+1} - c_1\delta_k \end{cases} \le 2\int_{x_k}^{x_{k+1}},$$

then

$$\left| P_n(x) - (V_n P_n)(x) \right| \le \frac{2c}{c_1\delta_k}\left(c_1\delta_k \frac{\varphi(x)}{n}\left|P_n'(x)\right| + \int_{x_k}^{x_{k+1}} \frac{\varphi^2(x)}{n^2}\left|P_n''(u)\right| du \right),$$

whence

$$\left\| P_n - V_n P_n \right\|_{L_p(I_k)} \le \frac{2c}{c_1\delta_k}\left(\frac{c_1\delta_k}{n}\left\| \varphi P_n' \right\|_{L_p(I_k)} + \frac{1}{n^2}\left\| \int_{x_k}^{x_{k+1}} \varphi^2(x)\left|P_n''(x)\right| dx \right\|_{L_p(I_k)} \right)$$

$$\le \frac{2c}{c_1\delta_k}\left(\frac{c_1\delta_k}{n}\left\| \varphi P_n' \right\|_{L_p(I_k)} + \frac{1}{n^2} E_1 \right).$$

To estimate E_1 we will use the similar proof, as we used to estimate ξ_1 above, we obtain

$$E_1 \le (c_2\delta_k)\left\| \varphi^2 P_n'' \right\|_{L_p(I_k)}.$$

Thus

$$\left\| P_n - V_n P_n \right\|_{L_p(I_k)} \le \frac{2c}{c_1\delta_k}\left(\frac{c_1\delta_k}{n}\left\| \varphi P_n' \right\|_{L_p(I_k)} + \frac{1}{n^2}(c_2\delta_k)\left\| \varphi^2 P_n'' \right\|_{L_p(I_k)} \right)$$

$$\le C\left(n^{-1}\left\| \varphi P_n' \right\|_{L_p(I_k)} + n^{-2}\left\| \varphi^2 P_n'' \right\|_{L_p(I_k)} \right). \qquad (2.2.24)$$

Now, for I_{-n}, we have the following inequality [67]

$$\left| P_n(x) - (V_n P_n)(x) \right| \le$$

$$\frac{x_{-n+1} - x}{x_{-n+1} - x_{-n}} \left(\frac{1}{c_1 \delta_{-n}} \int_{x_{-n}}^{x_{-n} + c_1 \delta_{-n}} \left(\int_{x_{-n}}^{u} \left| P_n'(v) \right| dv \right) + \int_{x_{-n}}^{x} \left| P_n'(u) \right| du \right)$$

$$+ \frac{x - x_{-n}}{x_{-n+1} - x_{-n}} \left(\frac{1}{c_1 \delta_{-n}} \int_{x_{-n+1} - c_1 \delta_{-n}}^{x_{-n+1}} \left(\int_{u}^{x_{-n+1}} \left| P_n'(v) \right| dv \right) + \int_{x}^{x_{-n+1}} \left| P_n'(u) \right| du \right)$$

$$\le \int_{x_{-n}}^{x_{-n+1}} \left| P_n'(u) \right| du + \frac{1}{c_1 \delta_{-n}} \left(\int_{x_{-n}}^{x_{-n} + c_1 \delta_{-n}} \left(\int_{x_{-n}}^{u} \left| P_n'(v) \right| dv \right) + \int_{x_{-n+1} - c_1 \delta_{-n}}^{x_{-n+1}} \left(\int_{u}^{x_{-n+1}} \left| P_n'(v) \right| dv \right) \right).$$

Then

$$\left\| P_n - V_n P_n \right\|_{L_p(I_{-n})} \le \left\| \int_{x_{-n}}^{x_{-n+1}} \left| P_n'(u) \right| du \right\|_{L_p(I_{-n})} + \frac{1}{c_1 \delta_{-n}} \left(\left\| \int_{x_{-n}}^{x_{-n} + c_1 \delta_{-n}} \left(\int_{x_{-n}}^{u} \left| P_n'(v) \right| dv \right) \right\|_{L_p(I_{-n})} \right.$$

$$\left. + \left\| \int_{x_{-n+1} - c_1 \delta_{-n}}^{x_{-n+1}} \left(\int_{u}^{x_{-n+1}} \left| P_n'(v) \right| dv \right) \right\|_{L_p(I_{-n})} \right)$$

$$\le E_2 + \frac{1}{c_1 \delta_{-n}} \left(E_3 + E_4 \right).$$

By using similar proof, once to E_2 and twice times to E_3 and E_4, as we estimated ξ_1, we obtain

$$E_2 \le (c_2 \delta_{-n}) \left\| P_n' \right\|_{L_p(I_{-n})},$$

$$E_3 \le (c_2 \delta_{-n})^2 \left\| P_n' \right\|_{L_p(I_{-n})},$$

and

$$E_4 \le (c_2 \delta_{-n})^2 \left\| P_n' \right\|_{L_p(I_{-n})},$$

then

$$\| P_n - V_n P_n \|_{L_p(I_{-n})} \le (c_2\delta_{-n}) \| P_n' \|_{L_p(I_{-n})} + \frac{1}{c_1\delta_{-n}} \left(2(c_2\delta_{-n})^2 \| P_n' \|_{L_p(I_{-n})} \right)$$

$$\le C \frac{1}{n^2} \| P_n' \|_{L_p(I_{-n})}. \tag{2.2.25}$$

Analogously

$$\| P_n - V_n P_n \|_{L_p(I_{n-1})} \le C \frac{1}{n^2} \| P_n' \|_{L_p(I_{n-1})}. \tag{2.2.26}$$

Hence $\forall k = -n, -n+1, \ldots, n-1$

$$\| P_n - V_n P_n \|_{L_p(I_k)} \le C \left(\frac{1}{n} \| P_n' \|_{L_p(I_k)} + \frac{1}{n^2} \| \varphi^2 P_n'' \|_{L_p(I_k)} \right), \tag{2.2.27}$$

then

$$\| P_n - V_n P_n \|_p = \left(\sum_{k=-n}^{n-1} \| P_n - V_n P_n \|_{L_p(I_k)}^p \right)^{\frac{1}{p}}$$

$$\le \left(\sum_{k=-n}^{n-1} c^p \left(\frac{1}{n} \| P_n' \|_{L_p(I_k)} + \frac{1}{n^2} \| \varphi^2 P_n'' \|_{L_p(I_k)} \right)^p \right)^{\frac{1}{p}}.$$

Now, since

$$\left(\frac{1}{n} \| P_n' \|_{L_p(I_k)} + \frac{1}{n^2} \| \varphi^2 P_n'' \|_{L_p(I_k)} \right)^p \le \left(2 \max \left(\frac{1}{n} \| P_n' \|_{L_p(I_k)}, \frac{1}{n^2} \| \varphi^2 P_n'' \|_{L_p(I_k)} \right) \right)^p$$

$$= 2^p \max \left(\frac{1}{n^p} \| P_n' \|_{L_p(I_k)}^p, \frac{1}{n^{2p}} \| \varphi^2 P_n'' \|_{L_p(I_k)}^p \right)$$

$$\le 2^p \left(\frac{1}{n^p} \| P_n' \|_{L_p(I_k)}^p + \frac{1}{n^{2p}} \| \varphi^2 P_n'' \|_{L_p(I_k)}^p \right),$$

so

$$\| P_n - V_n P_n \|_p \le \left(\sum_{k=-n}^{n-1} 2^p C^p \left(\frac{1}{n^p} \| P_n' \|_{L_p(I_k)}^p + \frac{1}{n^{2p}} \| \varphi^2 P_n'' \|_{L_p(I_k)}^p \right) \right)^{\frac{1}{p}}$$

$$\le 2C\left(\left(\sum_{k=-n}^{n-1}\frac{1}{n^p}\| P'_n \|^p_{L_p(I_k)}\right)^{\frac{1}{p}}+\left(\sum_{k=-n}^{n-1}\frac{1}{n^{2p}}\| \varphi^2 P''_n \|^p_{L_p(I_k)}\right)^{\frac{1}{p}}\right)$$

$$\le C\left(\left(\frac{1}{n^p}\| P'_n \|^p_p\right)^{\frac{1}{p}}+\left(\frac{1}{n^{2p}}\| \varphi^2 P''_n \|^p_p\right)^{\frac{1}{p}}\right)$$

$$\le C\left(\frac{1}{n}\| P'_n \|_p+\frac{1}{n^2}\| \varphi^2 P''_n \|_p\right). \qquad \Diamond$$

2.3 Proof of Theorem 2.1.2:

From (2.1.1) we have

$$(S_n f)(x)-f(x):=(x_{k+1}-x)(x-x_k)[x_k, x, x_{k+1}, f],$$
$$\forall x \in I_k \text{ and } \quad k=-n, -n+1, \dots, n-1$$

Then by using Lemma 2.2.10, we obtain

$$\| f-S_n f \|_p=\left(\sum_{k=-n}^{n-1}\| S_n f-f \|^p_{L_p(I_k)}\right)^{\frac{1}{p}}$$

$$=\left(\sum_{k=-n}^{n-1}\| (x_{k+1}-x)(x-x_k)[x_k, x, x_{k+1}; f] \|^p_{L_p(I_k)}\right)^{\frac{1}{p}}$$

$$\le\left(\sum_{k=-n}^{n-1}C(p)\,\omega_2(f, |I_k|, I_k)^p_p\right)^{\frac{1}{p}}$$

$$=C(p)\Bigg(\omega_2(f, |I_{-n}|, I_{-n})^p_p+\sum_{k=-n+1}^{n-2}\omega_2(f, |I_k|, I_k)^p_p +$$

$$+\,\omega_2(f, |I_{n-1}|, I_{n-1})^p_p\Bigg)^{\frac{1}{p}}$$

$$\le C(p)\left(C(p)\,\omega_2^{\varphi}\left(f,\frac{1}{n}\right)_p^p + \sum_{k=-n+1}^{n-2} c(p)\,\omega_2^{\varphi}\left(f,\frac{1}{n},I_k\right)_p^p\right)^{\frac{1}{p}}$$

$$\le C(p)\left(\omega_2^{\varphi}\left(f,\frac{1}{n}\right)_p^p + \omega_2^{\varphi}\left(f,\frac{1}{n},\bigcup_{k=-n+2}^{n-2} I_k\right)_p^p\right)^{\frac{1}{p}}$$

$$\le C(p)\,\omega_2^{\varphi}\left(f,\frac{1}{n}\right)_p . \Diamond$$

2.4 Proof of Theorem 2.1.6:

Form the definition of $U_n f$ and $S_n f$ for any $f \in L_p(I)$ with $0 < p < \infty$, we have

For $x \in I_{-n}$

$$(U_n f)(x) - (S_n f)(x) := \frac{x - x_{-n}}{x_{-n+1} - x_{-n}}\left(f\left(x_{-n+1} + n^{-\gamma}\varphi(x_{-n+1})\right) - 2f(x_{-n+1}) + f\left(x_{-n+1} - n^{-\gamma}\varphi(x_{-n+1})\right)\right), \quad (2.4.1)$$

for $x \in I_k$, $(k = -n+1, -n+2, \ldots, n-2)$

$$(U_n f)(x) - (S_n f)(x) := \frac{x_{k+1} - x}{x_{k+1} - x_k}\left(f\left(x_k + n^{-\gamma}\varphi(x_k)\right) - 2f(x_k) + f\left(x_k - n^{-\gamma}\varphi(x_k)\right)\right) + \quad (2.4.2)$$
$$+ \frac{x - x_k}{x_{k+1} - x_k}\left(f\left(x_{k+1} + n^{-\gamma}\varphi(x_{k+1})\right) - 2f(x_{k+1}) + f\left(x_{k+1} - n^{-\gamma}\varphi(x_{k+1})\right)\right),$$

and for $x \in I_{n-1}$

$$(U_n f)(x) - (S_n f)(x) := \frac{x_n - x}{x_n - x_{n-1}} \left(f\left(x_{n-1} + n^{-\gamma}\varphi(x_{n-1})\right) - 2f(x_{n-1}) + f\left(x_{n-1} - n^{-\gamma}\varphi(x_{n-1})\right)\right) \tag{2.4.3}$$

On the other hand, by definition of Ditzian- Totik modulus of smoothness, and using the inequalities (2.1.3), (2.1.4), (2.1.6), (2.2.23), and the above identities, we obtain

$$\| U_n f - S_n f \|_{L_p(I_{-n})} :=$$

$$\left\| \frac{x - x_{-n}}{x_{-n+1} - x_{-n}} \left| f\left(x_{-n+1} + n^{-\gamma}\varphi(x_{-n+1})\right) - 2f(x_{-n+1}) + f\left(x_{-n+1} - n^{-\gamma}\varphi(x_{-n+1})\right)\right| \right\|_{L_p(I_{-n})}$$

$$\leq \left\| f\left(x_{-n+1} + n^{-\gamma}\varphi(x_{-n+1})\right) - 2f(x_{-n+1}) + f\left(x_{-n+1} - n^{-\gamma}\varphi(x_{-n+1})\right)\right| \right\|_{L_p(I_{-n})}$$

$$\leq \left\| \Delta^2_{n^{-\gamma}\varphi(x_{-n+1})}(f, x_{-n+1}) \right\|_{L_p(I_{-n} \cup I_{-n+1})},$$

since $\forall\, x \in I_{-n} \cup I_{-n+1}$, by virtue of properties of δ_k and the property (i) of our partition, we have

$$|x_{-n+1} - x| \leq |I_{-n} \cup I_{-n+1}| = |I_{-n}| + |I_{-n+1}| \leq c_2\delta_{-n} + c_2\delta_{-n+1} \leq 2c_2c_0n^{-1}$$

Thus $|x_{-n+1} - x| = O(n^{-1})$, so $\varphi(x_{-n+1}) \approx \varphi(x)$.

Then

$$\begin{aligned} \| U_n f - S_n f \|_{L_p(I_{-n})} &\leq c(p) \left\| \Delta^2_{n^{-\gamma}\varphi(x)}(f, x) \right\|_{L_p(I_{-n} \cup I_{-n+1})} \\ &\leq c(p)\, \omega_2^{\varphi}\left(f, n^{-\gamma}, I_{-n} \cup I_{-n+1}\right)_p \\ &\leq c(p)\, \omega_2^{\varphi}\left(f, n^{-1}, I_{-n} \cup I_{-n+1}\right)_p. \end{aligned} \tag{2.4.4}$$

Using the same technique, we can obtain

$$\| U_n f - S_n f \|_{L_p(I_k)} \leq c(p)\, \omega_2^{\varphi}\left(f, n^{-1}, I_{k-1} \cup I_k \cup I_{k+1}\right)_p, \quad k = -n+1, -n+2, \dots, n-2, \tag{2.4.5}$$

and

$$\| U_n f - S_n f \|_{L_p(I_{n-1})} \le c(p)\,\omega_2^{\varphi}\left(f, n^{-1}, I_{n-2} \cup I_{n-1}\right)_p \quad . \tag{2.4.6}$$

Now, $\forall\, k = -n+1, -n+2, \dots, n-2$, we have

$$\omega_2^{\varphi}\left(f, n^{-1}, I_{k-1} \cup I_k \cup I_{k+1}\right)_p^p := \sup_{0<h\le n^{-1}} \int_{I_{k-1}\cup I_k \cup I_{k+1}} \left|\Delta^2_{h\varphi(x)}(f,x)\right|^p dx$$

$$= \sup_{0<h\le n^{-1}} \left(\int_{I_{k-1}} \left|\Delta^2_{h\varphi(x)}(f,x)\right|^p dx + \int_{I_k} \left|\Delta^2_{h\varphi(x)}(f,x)\right|^p dx + \int_{I_{k+1}} \left|\Delta^2_{h\varphi(x)}(f,x)\right|^p dx \right)$$

$$\le \sup_{0<h\le n^{-1}} \int_{I_{k-1}} \left|\Delta^2_{h\varphi(x)}(f,x)\right|^p dx + \sup_{0<h\le n^{-1}} \int_{I_k} \left|\Delta^2_{h\varphi(x)}(f,x)\right|^p dx + $$

$$+ \sup_{0<h\le n^{-1}} \int_{I_{k+1}} \left|\Delta^2_{h\varphi(x)}(f,x)\right|^p dx \Bigg)$$

$$\le \sum_{i=-1}^{1} \omega_2^{\varphi}\left(f, n^{-1}, I_{k+i}\right)_p^p \,, \tag{2.4.7}$$

hence

$$\| U_n f - S_n f \|_p = \left(\sum_{k=-n}^{n-1} \| U_n f - S_n f \|^p_{L_p(I_k)} \right)^{\frac{1}{p}}$$

$$\le c(p)\Bigg(\omega_2^{\varphi}\left(f, n^{-1}, I_{-n} \cup I_{-n+1}\right)_p^p + \omega_2^{\varphi}\left(f, n^{-1}, I_{n-2} \cup I_{n-1}\right)_p^p$$

$$+ \sum_{k=-n+1}^{n-2} \omega_2^{\varphi}\left(f, n^{-1}, I_{k-1} \cup I_k \cup I_{k+1}\right)_p^p \Bigg)^{\frac{1}{p}}$$

$$\le c(p)\left(\omega_2^{\varphi}\left(f, n^{-1}\right)_p^p + \sum_{k=-n+1}^{n-2} \sum_{i=-1}^{1} \omega_2^{\varphi}\left(f, n^{-1}, I_{k+i}\right)_p^p \right)^{\frac{1}{p}}$$

$$\le c(p)\left(\omega_2^{\varphi}\left(f, n^{-1}\right)_p^p + \sum_{k=-n+1}^{n-2} 3\, \omega_2^{\varphi}\left(f, n^{-1}, I_k\right)_p^p \right)^{\frac{1}{p}}$$

$$\leq c(p)\left(\omega_2^{\varphi}(f, n^{-1})_p^p + \omega_2^{\varphi}\left(f, n^{-1}, \bigcup_{k=-n+1}^{n-2} I_k\right)_p^p\right)^{\frac{1}{p}} \tag{2.4.8}$$
$$\leq c(p)\omega_2^{\varphi}(f, n^{-1})_p .$$

Then, by using (2.4.8) and Theorem 2.1.2, we obtain the conclusion of our theorem. ◊

2.5 Proof of Theorem 2.1.8:

Let P_n be the best n[th] degree polynomial approximation to f, then we may write $f = P_n + \sum_{i=0}^{\infty}\left(P_{2^{i+1}n} - P_{2^i n}\right)$, and recalling that for $m < n$, $\| P_n - P_m \|_p \leq 2\ E_m(f)_p$ [19]. Then by using the linear property of operator V_n, (2.2.13) and Lemma 2.2.1, we have

$$\| f - V_n f \|_p \leq \| f - P_n \| + \| v_n P_n - P_n \|_p + \left\| V_n \sum_{i=0}^{\infty}\left(P_{2^{i+1}n} - P_{2^i n}\right)\right\|_p$$
$$\leq C(r)\omega_r^{\varphi}(f, n^{-1})_p + \| v_n P_n - P_n \|_p + C\left\| \sum_{i=0}^{\infty}\left(P_{2^{i+1}n} - P_{2^i n}\right)\right\|_p \tag{2.5.1}$$
$$\leq C(r)\left(\omega_r^{\varphi}(f, n^{-1})_p + \| v_n P_n - P_n \|_p + \sum_{i=0}^{\infty}\left\| P_{2^{i+1}n} - P_{2^i n}\right\|_p\right)$$
$$\leq C(r)(\eta_1 + \eta_2 + \eta_3).$$

To estimate η_3 we have from our assumption, Lemma 2.2.1, and the fact that $L_p \subset L_1$

$$\eta_3 \le \sum_{i=0}^{\infty} 2\ E_{2^i n}(f)_p \le C(r)\sum_{i=0}^{\infty} \omega_r^{\varphi}\left(f, \frac{1}{2^i n}\right)_p$$

$$\le C(r)\frac{1}{n}\sum_{i=0}^{\infty} 2^i n\, \omega_r^{\varphi}\left(f, \frac{1}{2^i n}\right)_p \qquad (2.5.2)$$

$$\le C(r)\ \frac{1}{n}\int_0^{1/n} \frac{\omega_r^{\varphi}(f,t)_p}{t^2}\,dt\ .$$

Now, let P_0 denotes the best 0th degree of polynomial approximation to f . Then by using (2.2.24) and (2.2.8), we have $\forall k = -n+1, -n+2, \dots, n-2$

$$\| P_n - V_n P_n \|_{L_p(I_k)} \le C\left(\frac{1}{n}\| \varphi P_n' \|_{L_p(I_k)} + \frac{1}{n^2}\| \varphi^2 P_n'' \|_{L_p(I_k)}\right)$$

$$\le C\left(\frac{1}{n}\left\| \varphi (P_n - P_0)' \right\|_{L_p(I_k)} + \frac{1}{n^2}\left\| \varphi^2 (P_n - P_0)'' \right\|_{L_p(I_k)}\right)$$

$$\le C\| P_n - P_0\|_{L_p(I_k)} \le C\left(\| f - P_n\|_{L_p(I_k)} + \| f - P_0\|_{L_p(I_k)}\right), \qquad (2.5.3)$$

and for $k = -n, n-1$ then by using (2.2.25), (2.2.26) and (2.2.6) we obtain

$$\| P_n - V_n P_n \|_{L_p(I_k)} \le C\,\frac{1}{n^2}\| P_n' \|_{L_p(I_k)} = C\,\frac{1}{n^2}\left\| (P_n - P_0)' \right\|_{L_p(I_k)}$$
$$\le C\| P_n - P_0 \|_{L_p(I_k)} \le C\left(\| f - P_n \|_{L_p(I_k)} + \| f - P_0 \|_{L_p(I_k)}\right) \qquad (2.5.4)$$

Hence by virtue of (2.5.3) and (2.5.4), we have

$$\eta_2 = \left(\sum_{k=-n}^{n-1} \| V_n P_n - P_n \|_{L_p(I_k)}^p\right)^{\frac{1}{p}}$$

$$\le \left(\sum_{k=-n}^{n-1} C^p\left(\| f - P_n\|_{L_p(I_k)} + \| f - P_0\|_{L_p(I_k)}\right)^p\right)^{\frac{1}{p}}$$

$$\le C\left(\sum_{k=-n}^{n-1} 2^p\left(\|f-P_n\|_{L_p(I_k)}^p+\|f-P_0\|_{L_p(I_k)}^p\right)\right)^{\frac{1}{p}}$$

$$\le C\left(\|f-P_n\|_p^p+\|f-P_0\|_p^p\right)^{\frac{1}{p}} \le C\left(\|f-P_n\|_p+\|f-P_0\|_p\right) \tag{2.5.5}$$

$$\le C(r)\left(\omega_r^{\varphi}\left(f, n^{-1}\right)_p+E_0(f)_p\right).$$

Then, in virtue of (2.5.1), (2.5.2) and (2.5.5) we obtain the assertion of theorem. ◊

Future Works

Our strategy for future is to answer the some problems which remained unanswered, among them.

What about the estimate (3.1.5) in the cases $s = 2, 3$, that is, if $f \in L_p(I)$, with $1 \le p < \infty$ and have two or three convexity change points?While we known that Leviatan and Shevchuk in [50] were proved that (3.1.5) holds, for a general continuous function that has more than one inflection points. The same question still holds for the case $0 < p < 1$ and $s > 1$.

Can we improve Theorem 3.1.7? In other words,(is it possible to replace ω_2 by ω_2^{φ} in Theorem 3.1.7 with the constants remaining as there?), with this, we want to approximate an arbitrary function from $L_p(I)$, quasi-norm spaces with $0 < p < \infty$ that changes convexity finitely many times in the interval, by an appropriate coconvex piece-wise polynomial which in turn, by virtue of Theorem 3.1.11, will be approximated by a coconvex polynomial.

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كجزء من متطلبات نيل درجة الماجستير في علوم الرياضيات

تقدم بها

هلكورد محمد درويش

بإشراف

د. إيمان سمير بهية

ذي العقدة 1946هـ كانون الأول 2005 م

كؤماری عیَراق
وةزاروتی خویَندنی بالأ و تویَذینةوةی زانستی
زانكؤی بابـــــل
كــؤلــــــیَذی ثةروةردة
بةشـــــــی ماتـــماتیــــك

نزیك كردنةوةی شـــــیَوة ثــاریَــزران

نامةیةكی ثیَشـكةشـــكراوة بة

كؤلــــیَذی ثةروةردة، بةشــــــی ماتـــماتیــك، لة زانــكؤی بابــــــل

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لـــــــةلایةن
هةلَطورد محمد دةرویَش

بة ســـةر ثةرشــــــــتی
د. إيمان سمير بهية

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