جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل/ كلية التربية

الصيغة التقاربية لكل من توزيعي كاما وبيتا اللامركزيان

رسالة مقدمة الى

قسم الرياضيات- كلية التربية- جامعة بابل

وهي جزء من متطلبات نيل درجة الماجستير في علوم الرياضيات

من قبل

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٢٠٠٦م أيار   ١٤٢٧هـ ربيع الثاني

Republic of Iraq

Ministry of Higher Education

And Scientific Research

Babylon University

College of Education

An Asymptotic Expansion for each of the Two Non-Central Distributions Gamma and Beta

A thesis

Submitted to the Department of Mathematics- College of Education- University of Babylon in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

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Supervised by

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May ٢٠٠٦

بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

﴿وَمَا جَعَلَهُ اللهُ إِلاَ بُشَرى لَكمُ وَلِتَطمَئِنَّ قُلُوبُكم بهِ وَمَا النَصرُ إِلاَ مِن عِندِ اللهِ العزِيزِ الحَكِيمِ﴾

صدق الله العلي العظيم

سورة آل عمران آية (١٢٦)

الاهداء

الى صاحب الفضل الاول   ابي

الى منبع الحنان والطيبة   امي

الى اصحاب التشجيع وشد الازر   اخواني واخواتي

الى رفيقة عمـري   زوجتي

الى احباء قلبي   اولادي

الى كل معلم علمني، وكل مدرس درسني، وكل استاذ فهمني اهدي جهدي المتواضع هذا.

محمد

الخـلاصـة

في هذه الرسالة تم تناول العديد من المفاهيم والادوات الاساسية والرئيسية في التحليل التقاربي، وذلك بتوظيف بعض المفاهيم في التحليل الرياضي. وتقديم توطئة ويتسن وبرهانها لكونها الخطوة الاولى في تقاربية التكاملات، وكما تم استعراض بعض التوزيعات الاحصائية الخاصة كالتوزيعات اللامركزية وبعض التوزيعات المركزية كحالة خاصة منها عندما يكون البارميتر اللامركزي مساوياً للصفر. كما تم اشتقاق صيغة سترلنج التقاربية باستخدام طريقة لابلاس التقاربية، وكذلك تم اشتقاق الصيغة التقاربية لكل من توزيعي كاما وبيتا اللامركزيين مع بعض الاستنتاجات والتوصيات.

Abstract

In this thesis, we have many concepts and basic tools in the asymptotic analysis based on some concepts in mathematical analysis. We introduce Watson's Lemma and its proof to be the first step in asymptotic the integrations, so we review some special statistical distributions such as the non-central distributions and central distributions as special case of it when the non central parameter is equal to zero. Also we derive the asymptotic stirling formula by using the asymptotic Laplace method, and, we derive an asymptotic expansion for each of the two non central distributions Gamma and Beta with some conclusions and recommendations.

Acknowledgement

Praise is to Allah for helping me to complete this thesis.

I would like to express my gratitude to my supervisor Dr. Kareema Abdul-Kaddum Alkafagi for her advice, guidance and encouragement throughout the preparation of this work.

Also, I'm indebted to the dean of the college of education, Babylon University, Assist. Prof. Dr. Lua'y Abdul- Hani Al-Suwidi and to all the staff of the department of the Mathematics especial the head of department of mathematics, Dr. Ifticharr Mudhar Talib, for their cooperation.

Thanks are due to the staff of the central library in Babylon university.

Also I thank all those helped me very much in the fulfillment of this work.

My thanks are due to my family for their spiritual support during the period of writing this thesis.

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Examining Committee Certification

We certify that we have read this thesis entitled "**An Asymptotic Expansion for each of the Two Non-Central Distributions Gamma and Beta**" and, as an Examining Committee, we examined the student in its content, and what is related to it, and that in our opinion it is adequate with standing as a thesis for the degree of Master of Science in Mathematics.

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SUPERVISOR CERTIFICATION

I certify that this thesis was prepared under my supervision at the Department of Mathematics/ College of Education/ University of Babylon, as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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RECOMMENDATION OF THE HEAD OF THE DEPARTMENT

In view of the available recommendations, I forward this research for debate by the Examining Committee.

Signature:

Name: ***Assist. Prof. Dr*****. Iftichar Mudhar Talib**

Date: / / ٢٠٠٦

List of Symbols

Symbol	***Description***
P	probability
$\{X_n\}$	Sequence of random variables
$F(x)$	Cumulative distribution function (c.d.f
$w.p.$ '	Converge with probability one
$a.s$	Converge almost sure
$m.s.e$	Mean square error
L	Converge in law
d	Converge in distribution
$(WLLN)$	Weak law of large numbers
$(SLLN)$	Strong law of large numbers
$f(x)$	Probability density function (p.d.f)
$p(x)$	Probability mass function (c.d.f)
$\{\phi_n\}$	Asymptotic sequence
E	Expected value

REVIEW OF LITERATURE

There are many researchers who worked in our field at the end of twenty century and beginning of twenty one century:

Stieltjes, T. J. see [١] has several results on asymptotic expansions, including examples on asymptotic of special functions and discussions of the remainder terms in the expansions, almost ١٥٠ years ago.

In ١٩٩٥, Temme, N. M [١] studied the uniform asymptotic expansions of integrals by using examples of Stieltjies work on asymptotic of special functions.

In the same year, Cohen, J.K. [٢] studied the symbolic integration and asymptotic expansions.

In ١٩٩٦, Temme, N. M [٣] derived uniform asymptotic for the incomplete gamma functions starting from negative values of the parameters.

In ١٩٩٧, Temme, N. M. [٤] studied the numerical algorithms for uniform Airy- type asymptotic expansions.

In ١٩٩٨, Dunster, T. M, Paris, R.B., and Cang, S. [٥] are studied on the high-order coefficients in the uniform asymptotic expansion for the incomplete gamma function.

In ٢٠٠٠, Wolfgang, B. [٦] derived a special case of the asymptotic expansion for a ratio of products of gamma functions. He generalized a formula which was stated by Dingle in ١٩٧٣. [٧], first proved by Paris in ١٩٩٢. [٨] and recently reconsidered by Oliver in ١٩٩٥. [٩]

In ٢٠٠١ Gotze, F., and Tikhomirov, A. N. [١٠] studied the asymptotic expansions in non central limit theorems for quadratic forms.

In ٢٠٠٣ Lieberman, O., Rousseau, J. and Zucker, D.M. [١١] are studied the valid asymptotic expansions for the maximum likelihood estimator of the parameter of a stationary, Gaussian, strongly dependant process.

In the same year, Wolfgang, B. [١٢] derived general case of the asymptotic expansion for a ratio of products of gamma functions.

In ٢٠٠٤, Dunster, T. M. [١٣] derived uniform asymptotic expansions for incomplete Rieman zeta functions.

In the same year, Simon, J, A, Malham. [١٤] studied in the asymptotic analysis and properties of asymptotic expansions.

In ٢٠٠٥, Daalhuis, A. B. [١٥] studied the uniform asymptotic expansions for hypogeometric functions with large parameters.

INTRODUCTION

Asymptotic theory or large sample theory studies the behaviour of random variables when the size of the sample tends to infinity. It is involves generalizing the usual notions of convergence for real sequences to allow for random variables and it is important to emphasis that limiting distributions obtained by central limit theorem (CLT) all involve unknown parameters which we seek to estimate.

Asymptotic expansions of functions are useful in statistics in three main ways.

Firstly, conventional asymptotic expansions of special functions are useful for approximate computation of integrals arising in statistical calculations.

Second, asymptotic expansions of density or distribution functions estimators or test statistics can be used to give approximate confidence limits for a parameter of interest or p-values for an hypothesis tests.

Third, asymptotic expansions for distributions of estimators or test statistics may be used to investigate properties such as the efficiency of an estimators or the power of a test.

In this thesis, we deal with first useful indirectly from through study and derive an asymptotic expansions for each of the two non-central distributions gamma and beta, its special functions are usually defined in terms of integrals. The present study is of three chapters.

Chapter one provides many basic concepts and results of asymptotic analysis.

Chapter two gives some central and non- central distributions.

Chapter three contains derivation an asymptotic expansions for each of the two non-central distributions gamma and beta.

The purpose of this chapter is to introduce many basic concepts, definitions and results from asymptotic analysis. We give two sections, section one includes some basic concepts about asymptotic theory and section two contains many definitions from asymptotic expansions which find use in later chapters.

١.١ *Some Definitions and Concepts in Asymptotic Theory*

١.١.١ *Asymptotic Theory [١٦]*

Asymptotic theory or large sample theory studies the behaviour of random variables when the size of the sample tends to infinity.

١.١.٢ *Modes of Convergence [١٧] [١٨] [١٩]*

(a) Convergence in Probability:

A sequence of random variables $\{X_n\}=(X_1,X_2,...,X_n)$ is said to converges in probability to a random variable *X* (or weakly convergent) if for every $\varepsilon>0$, $\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon)=0$,

or equivalently, $\lim_{n\to\infty} P(|X_n - X| < \varepsilon)=1$, where *ε* is arbitrarily small

It is common to write $X_n \xrightarrow{P} X$ or $P \lim_{n\to\infty} X_n = X$.

Remark

We say that an estimator $\hat{\theta}_n$ of a parameter ϑ is weak consistent if

$$P\lim_{n\to\infty} \theta_n^{\wedge} = \theta.$$

Example

Suppose X_n is normally distributed with mean $\mu_n = \mu + \frac{k}{n}$ and the variance $\sigma_n^2 = \frac{\sigma^2}{n}$. To show that $\{X_n\}$ converges in probability to μ, a fixed constant. Here we show the convergence X_n to μ by obtaining $P(|X_n - \mu| < \varepsilon)$ directly.

Since $X_n - \mu \sim N\left(\frac{k}{n}, \frac{\sigma^2}{n}\right)$ it is easily seen that

$$P(|X_n - \mu| < \varepsilon) = F\left(\frac{\varepsilon - \frac{k}{n}}{\frac{\sigma}{\sqrt{n}}}\right) - F\left(\frac{-\varepsilon - \frac{k}{n}}{\frac{\sigma}{\sqrt{n}}}\right),$$

Where $F(.)$ represents the cumulative distribution function of a standard normal variate. But

$$\lim_{n\to\infty} F\left(\frac{\varepsilon - \frac{k}{n}}{\frac{\sigma}{\sqrt{n}}}\right) = \lim_{n\to\infty} F\left(\frac{\varepsilon\sqrt{n}}{\sigma}\right) = 1 \qquad \text{for any } \varepsilon > ۰$$

and

$$\lim_{n\to\infty} F\left(\frac{-\varepsilon - \frac{k}{n}}{\frac{\sigma}{\sqrt{n}}}\right) = \lim_{n\to\infty} F\left(\frac{-\varepsilon\sqrt{n}}{\sigma}\right) = 0 \qquad \text{for any } \varepsilon > ۰$$

Therefore

$$\lim_{n\to\infty} P(|X_n - \mu| < \varepsilon) = 1,$$

As required

For a given value of ε, the rate of convergence of X_n to μ clear depends on k, σ and the shape of the distribution function $F(.)$. The larger value of σ, the slower will be the rate of convergence of $F(.)$ to μ.

(b) Convergence with probability one:

A sequence of random variables $\{X_n\} = (X_1, X_2, ..., X_n)$ is said to converge with probability one (or strongly almost surely, almost every where) to *X* if

$P(\lim_{n\to\infty} X_n = X) = 1$, this is written $X_n \xrightarrow{w.P.1} X$, $n \to \infty$.

Or equivalently, $X_n \xrightarrow{a.s} X$ if and only if for every $\varepsilon > 0$

$P(|X_m - X| < \varepsilon$ for all $m \geq n) \to 1$ as $n \to \infty$.

Remark

We say that an estimator $\hat{\theta}_n$ of a parameter θ is strongly consistent if $\hat{\theta}_n \xrightarrow{a.s} \theta$.

(c) Convergence in r^{th} mean

For a real number *r*>٠, the sequence $\{X_n\}$ convergence in the r^{th} mean to *X*, if $\lim_{n\to\infty} E\,|\,X_n - X\,|^r = 0$.

This is written as $X_n \xrightarrow{r^{th}} X$, whenever, $n \to \infty$.

If *r*=٢, then a sequence $\{X_n\}$ is said to converge to *X* in mean square error (MSE) if $\lim_{n\to\infty} E(X_n - X)^2 = 0$, written as $X_n \xrightarrow{m.s} X$.

Remark

We say that an estimator $\overset{\wedge}{\theta_n}$ is of a parameter ϑ is r^{th} mean consistent if $\overset{\wedge}{\theta_n} \xrightarrow{r^{th}} \theta$

(d) Convergence in Distribution

A sequence $\{X_n\}$ is said to converge to *X* in distribution (or in law) if the distribution function $F_n(x)$ of X_n converges to the distribution function $F(x)$ of *X* at every continuity point *x* of *F*.

We write

$$\lim_{n\to\infty} F_{X_n}(x) - F_X(x) = 0 \text{ or } X_n \xrightarrow{L} X,\ X_n \xrightarrow{d} X$$

Example:

Let $\{X_1,....,X_n\}$ be a sequence of random variables having an exponential distribution with $\lambda = 1 + 1/n$.

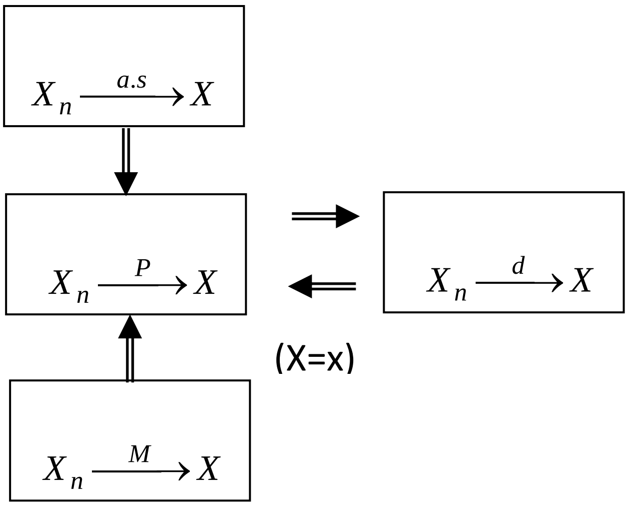
$$F_n(x) = 1 - e^{-\left(1+\frac{1}{n}\right)x} \qquad x > ٠,$$

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} [1 - e^{-\left(1+\frac{1}{n}\right)x}] = 1 - e^{-x}.$$

Which is an exponential distribution with λ =١.

١.١.٣ *Relationships among modes of convergence*

Let us summarize relationships among the modes of convergence in the following diagram:



١.١.٤ *The Law of Large Numbers (LLN) [٢٠] [٢١]*

Let $X_1, X_2,...$ be independent and identically distributed (i.i.d) random variables, and let

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \ .$$

(a) If $E|x| \leq \infty$, then $\overline{X}_n \xrightarrow{p} \mu = E(x)$. This is the weak law of large numbers (WLLN).

(b) $\overline{X}_n \xrightarrow{a.s} \mu$ if and only if $E|x| < \infty$ and $\mu = E(x)$. This is the strong law of large numbers (SLLN).

١.١.٥ *The Central Limit Theorem (CLT) [٢٢]*

Let $X_1, X_2, ... X_n$ are (i.i.d) random variables with mean $\mu = E(x_i)$ and variance $\sigma^2 = v(x_i) < \infty$. Then

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1) \ .$$

We sometimes say that if $\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$, then asymptotically $\overline{X}_n \sim N[\mu, \frac{\sigma^2}{n}]$ or $\overline{X}_n \xrightarrow{a} N[\mu, \frac{\sigma^2}{n}]$.

١.٢ *Asymptotic Expansions*

Asymptotic expansions are used in many areas of mathematical analysis and it is explained by the following concepts.

١.٢.١ *Asymptotic Estimates*

The symbols *O*, *o* and ~, were first used by E. Landau and P. Du Bois Reymond in [١٤] and are defined as follows.

١.٢.١.١ *Definition [١٤] [٢٣]*

Let *M* a subset of real or complex members with a limit point *a*, and let *f*, *g*: *M*→*R* (or *f*, *g*: *M*→*C*) be some functions on *M*. The following are asymptotic estimates.

(i) ***Asymptotically Bounded:*** $f(x) = O(g(x))$ as *x*→*a*, *x*∈*M*, means that: there exist constants $k \geq 0$ and δ>٠ such that, for $0 < |x - a| < \delta$, we have $|f(x)| \leq k|g(x)|$, (i.e $f(x)$ is asymptotically bounded by $g(x)$ in magnitude as $x \to a$) and we say that *f*(*x*) is of order 'big *O*' of $g(x)$. Hence provided $g(x)$ is not zero in a neighbourhood of *a*; except possibly at *a*, then *f* / *g* is bounded.

(ii) ***Asymptotically Smaller:*** $f(x) = o(g(x))$ as *x* → *a*, *x* ∈*M*, means that: for all ε>٠, there exists $\delta > 0$ such that, for $0 < |x - a| < \delta$, we have $|f(x)| \leq \varepsilon |g(x)|$; or equivalently $\dfrac{f(x)}{g(x)} \to 0$ as *x* → *a*, *x*∈*M*. Provided $g(x)$ is not zero in a neighbourhood of *a*, except possibly at *a*. We say that $f(x)$ is asymptotically smaller than $g(x)$ or $f(x)$ is of order 'little o' of $g(x)$ as *x* → *a*.

(iii) ***Asymptotically Equal:*** $f(x) \sim g(x)$ as *x* → *a*, *x*∈*M* means that:

$f(x) = g(x) + o\,(g(x))$, as *x* → *a* ;

or equivalently

$$\frac{f(x)}{g(x)} \to 1 \text{ as } x \to a,\ x \in M,$$

and we say that $f(x)$ asymptotically equivalent/ equal to $g(x)$ in this limit (or more colloquially, $f(x)$, 'goes like' $g(x)$ as $x \to a$).

Examples [١٤] [٢٤]

١- $f(x)=O(١)$ as $x \to x_\circ$ means $f(x)$ is bounded close to $x_\circ$.

٢- $f(x)=o(١) \Rightarrow f(x) \to ٠$ as $x \to x_\circ$.

٣- If $f(x)=5x^2+x+3$, then $f(x)=o(x^3)$, $f(x)=O(x^2)$ and $f(x) \sim 5x^2$ as $x \to \infty$; but

$f(x) \sim 3$ as $x \to 0$ and $f(x)=o(1/x)$ as $x \to 0$

٤- $x! \sim \sqrt{2\pi x}e^{-x}x^x$, $(x \to \infty)$ (see ٣.١), page (٤٠)

٥- $\ln x = o(x^{-\alpha})$, $(x \to o^+)$, $\alpha > 0$; but

$\ln x = o(x^{\alpha})$, $(x \to \infty)$, $\alpha > 0$

Remarks [١٤]

١- In definition ١.٢.١.١, the function $g(x)$ is often called a *gauge function* because it is the function against which the behaviour of $f(x)$ is gauged.

٢- Note that *O*- order is more informative than *o*- order about the behaviour of the function concerned as $x \to a$. For example, $\sin x = x + o(x^2)$ as $x \to 0$ tells us that $\sin x - x \to 0$ faster than x^2, however, $\sin x = x + O(x^3)$, tells us specifically that $\sin x - x \to 0$ like x^3.

٣- This notation is also easily adaptable to function of a discrete variable such as sequence of real numbers. For example, if $X_n = 3n^2 - 7n + 8$ then $X_n = o(n^3)$, $X_n = O(n^2)$ and $X_n \sim 3n^2$ as $n \to \infty$.

٤- Often the alternation $f(x) << g(x)$ as $x \to a$. is used in place of $f(x) = o(g(x))$ as $x \to a$.

١.٢.١.٢ *Some Properties of the Asymptotic Estimates [٢٤]*

These properties hold as $x \to a$, $x \in M$

١- $o(f(x)) + o(f(x)) = o(f(x))$

٢- $o(f(x))o(g(x)) = o(f(x)g(x))$

٣- $o(o\ f(x)) = o(f(x))$

٤- $O(f(x)) + O(f(x)) = O(f(x))$

٥- $O(f(x))O(g(x)) = O(f(x)g(x))$

٦- $O(O\ f(x)) = O(f(x))$

٧- $o(f(x)) + O(f(x)) = O(f(x))$

٨- $o(f(x))O(g(x)) = o(f(x)g(x))$

٩- $O(o(f(x))) = o(f(x))$

١٠- $o(O(f(x))) = o(f(x))$

١.٢.٢ *Asymptotic Sequences*

١.٢.٢.١ *Definition [١٤] [٢٥]*

Let $\phi_n : M \to R$, $n \in N$, and *a* be a limit point of *M*, and let $\phi_n(x) \neq 0$ in a neighbourhood U_n of *a*. Then the sequence $\{\phi_n\}$ is called asymptotic sequence at *x* → *a*, $x \in M$ if $\forall n \in N$, we have $\phi_{n+1}(x) = o(\phi_n(x))$, (*x* → *a*, $x \in M$). ... (١.١)

Examples (power asymptotic sequences)

١- $\{(x-a)^n\}$ as $x \to a$

٢- $\{(x^{-n})\}$ as $x \to \infty$

٣- $\{(\sin x)^n\}$ as $x \to 0$

١.٢.٢.٢ *Some Properties of the Asymptotic Sequences [٢٤]*

١- Any subsequence of an asymptotic sequence is also asymptotic sequence.

٢- Let $f(x) \neq 0$ for $x \in M$ in some neighbourhood of *a* and $\{\phi_n\}$ be an asymptotic sequence at $x \to a$, $x \in M$. Then the sequence $\{f(x)\phi_n(x)\}$ is an asymptotic sequence as $x \to a$, $x \in M$.

٣- Let $\{\phi_n(x)\}$, $\{\psi_n(x)\}$ be asymptotic sequences as $x \to a$, $x \in M$. Then the sequence $\{\phi_n(x)\psi_n(x)\}$ is an asymptotic sequence as $x \to a$, $x \in M$.

١.٢.٣ *Asymptotic Series*

١.٢.٣.١ *Definition [١٤] [٢٥]*

Let $f : M \to R$ and *a* be a limit point of *M*, and let $\{\phi_n\}$ be an asymptotic sequence as *x* → *a*, $x \in M$. We say that the function $f(x)$ is expanded in an asymptotic series,

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x), \ (x \to a,\ x \in M), \qquad \dots (١.٢)$$

where a_n are constants, if for each *N* ≥ ٠ we have

$$f(x) = \sum_{n=0}^{N} a_n \phi_n(x) + o(\phi_N(x)) \text{ as } x \to a, \qquad \dots (١.٣)$$

or $$R_N(x) \equiv f(x) - \sum_{n=0}^{N} a_n \phi_n(x) = o(\phi_N(x)), \ (x \to a,\ x \in M).$$

Or equivalently

$$f(x)=\sum_{n=1}^{N-1} a_n\phi_n(x)+O(\phi_N(x)) \text{ as } x \to a. \qquad \text{... (١.٤)}$$

i.e. the error is of the order of the first term neglected in the expansion.

This series is called asymptotic expansion of the function f with respect to the asymptotic sequence $\{\phi_n\}$. $R_N(x)$ is called the rest term (remainder term).

Remarks [٢٤]

١- The condition $R_N(x)=o(\phi_N(x))$ means, in particular, that $\lim_{x\to a} R_N(x)=0$ for any fixed *N*.

٢- Asymptotic series could diverge. This happens if $\lim_{N\to\infty} R_N(x)\neq 0$ for some fixed *x*.

٣- There are three possibilities

(a) asymptotic series converges to $f(x)$;

(b) asymptotic series converges to a function $g(x)\neq f(x)$;

(c) asymptotic series diverges;

١.٢.٣.٢ *Some Properties of the Asymptotic Series (Asymptotic Expansions) [١٤] [٢٤]*

In this article we introduce some important properties of the asymptotic expansions.

(١) Uniqueness. For a given asymptotic sequence $\{\phi_n(x)\}$, the asymptotic expansion of $f(x)$ is unique; i.e the a_n are uniquely determined for *n*=١, ٢, ..., *N* as follows

$$a_1 = \lim_{x\to a} \frac{f(x)}{\phi_1(x)};$$

$$a_2 = \lim_{x\to a} \frac{f(x) - a_1\phi_1(x)}{\phi_2(x)};$$

.

.

.

$$a_N = \lim_{x\to a} \frac{f(x) - \sum_{n=1}^{N-1} a_n\phi_n(x)}{\phi_N(x)} = \frac{O(\phi_N(x))}{\phi_N(x)}. \qquad \text{... (١.٥)}$$

٢- Non- Uniqueness (for a given function). A given function $f(x)$ may have many different asymptotic expansions. For example as $x \to 0$,

$$\tan x \sim x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots$$

$$\tan x \sim \sin x + \frac{1}{2} + (\sin x)^3 + \frac{3}{8}(\sin x)^5 + \ldots$$

$$\tan x \sim x\cosh(\sqrt{2/3x}) + \frac{31}{270}(x\cosh(\sqrt{2/3x}))^5 + \ldots$$

٣- **Sub dominance.** Two different functions can have the same asymptotic expansions. For example $f(x)=e^{x}$ and $g(x)=e^{x}+e^{-1/x}$ have the same asymptotic expansion with respect to the asymptotic sequence $\{X^{n}\}$:

$$e^{x} \sim e^{x}+e^{-1/x} \sim \sum_{n=0}^{\infty} \frac{x^{n}}{n!}, \quad x \to o^{+},$$

such a function $g(x)$ is said to be subdominant to the asymptotic power series.

٤- **Equaling Coefficients:** If we write

$$\sum_{n=0}^{\infty} a_{n}.(x-a)^{n} \sim \sum_{n=0}^{\infty} b_{n}(x-a)^{n} \text{ as } x \to a. \qquad \text{... (١.٦)}$$

We mean that the class of functions to which $\sum_{n=0}^{\infty} a_{n}(x-a)^{n}$ and $\sum_{n=0}^{\infty} b_{n}(x-a)^{n}$ are asymptotic as $x \to a$ are the same. And,

uniqueness of asymptotic expansions means that $a_{n}=b_{n}$ for all *n* i.e. we may equal coefficients of like powers of $x-a$ in (١.٦).

٥- **Arithmetical Operation**

Suppose $f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x)$ and $g(x) \sim \sum_{n=0}^{\infty} b_n \phi_n(x)$ as $x \to a$, then

$\alpha f(x) + \beta g(x) \sim \sum_{n=0}^{\infty} (a_n + b_n)\phi_n(x)$ as $x \to a$, where α and β are constants. Asymptotic expansions can also be multiplied and divided-perhaps based on an enlarged asymptotic sequence. (which we will need to be able to order). In particular for asymptotic power series, when $\phi_n(x) = (x-a)^n$, these operations are straightforward:

$f(x)g(x) \sim \sum_{n=0}^{\infty} C_n (x-a)^n$, as $x \to a$, where $C_n = \sum_{m=0}^{n} a_n b_{n-m}$, and if $b_\circ \neq 0$, $d_\circ = a_\circ / b_\circ$, then

$$\frac{f(x)}{g(x)} \sim \sum_{n=0}^{\infty} d_n (x-a)^n \text{, as } x \to a$$

where $d_n = \left(a_n - \sum_{m=0}^{n-1} d_m b_{n-m} \right) \Big/ b_\circ$.

٦- **Integration**. An asymptotic power series can be integrated term by term (if $f(x)$ is integrable near *x*=*a*) resulting in the correct asymptotic expansions for the integral. Hence, if

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x-a)^n \text{ as } x \to a \text{, then}$$

$$\int_a^x f(t)dt \sim \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1} \text{ as } x \to a.$$

٧- **Differentiation.** Asymptotic expansions can not in general be differentiated term by term. The problem with differentiation is connected with sub dominance. For instance the two functions

$$f(x) \text{ and } g(x) = f(x) + e^{-\frac{1}{(x-a)^2}} \sin\left(e^{\frac{1}{(x-a)^2}}\right)$$

differ by a subdominant function and thus have the same asymptotic power series expansion as $x \to a$. However $f'(x)$ and $g'(x)$ where

$$g'(x) = f'(x) - 2(x-a)^{-3} \cos\left(e^{\frac{1}{(x-a)^2}}\right) + 2(x-a)^{-3} e^{-\frac{1}{(x-a)^2}} \sin\left(e^{\frac{1}{(x-a)^2}}\right)$$

do not have the same asymptotic series expansion as $x \to a$.

١.٢.٤ *Asymptotic expansions of integrals [١] [١٤]*

Integral representations. When modeling many physical phenomena, it is often useful to know the asymptotic behaviour of integrals of the form

$$I(x) = \int_{a(x)}^{b(x)} f(x,t)dt, \text{ as } x \to a \qquad \dots (١.٧)$$

(or more generally, the integrals of a complex functions along a contour). For example, many functions have integral representations:

the error function, incomplete gamma function, incomplete beta function, and many other special functions such as the Bessel, Airy and hyper geometric functions have integral representations as they are solutions of various classes of differential equations. Also, if we use Laplace, Fourier or Hankel transformations to solve differential equations. We are often left with an integral representation of the solution.

Example (the Exponential Integral) [١٤]

This nicely demonstrates the difference between convergent and asymptotic series. Consider the exponential integral function defined for $x > 0$ by

$$Ei(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt .$$

Let us look for an analytical approximation to $Ei(x)$ for $x \geq 1$. Repeatedly integrating by parts gives

$$Ei(x) = \left[-\frac{e^{-t}}{t}\right]_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2}dt$$

$$= \frac{e^{-x}}{x} + \left[\frac{e^{-t}}{t^2}\right]_x^\infty - 2\int_x^\infty \frac{e^{-t}}{t^3}dt$$

.

.

.

$$Ei(x) = e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} + \ldots + (-1)^{N-1}\frac{(N-1)!}{x^N}\right)$$

$$+ (-1)^N N! \int_x^\infty \frac{e^{-t}}{t^{N+1}}\,dt \qquad \text{... (١.٨)}$$

$$Ei(x) = S_N(x) + R_N(x).$$

Here we set $S_N(x)$ to be the partial sum of the first N terms, and $R_N(x)$ to be the remainder after *N* terms.

The series for which $S_N(x)$ is the partial sum is divergent for any fixed *x*; notice that for large *N* the magnitude of the N^{th} term increases as *N* increases! of course $R_N(x)$ is also unbounded as $N \to \infty$, since $S_N(x)+R_N(x)$ must be bounded because $Ei(x)$ is defined (and bounded) for all $x>0$.

Suppose we consider *N* fixed and let *x* become large:

$|R_N(x)| < \frac{N!}{x^{N+1}} \int_x^{\infty} e^{-t} dt = \frac{N!}{x^{N+1}} e^{-x} \to 0$ as $x \to \infty$. Note that the ratio of $R_N(x)$ to the last term in $S_N(x)$ is

$$\left| \frac{R_N(x)}{(N-1)! e^{-x} x^{-N}} \right| < \frac{N! e^{-x} x^{-(N+1)}}{(N-1)! e^{-x} x^{-N}} = \frac{N}{x} \to 0 \text{ as } x \to \infty, \quad \ldots (١.٩)$$

thus $Ei(x) = S_N(x) + o$ (N^{th} term in $S_N(x)$) as $x \to \infty$, in particular, if *x* is sufficiently large and *N* fixed, $S_N(x)$ gives a good approximation to $Ei(x)$; the accuracy of the approximation increases as *x* increases for *N* fixed. In fact, as we shall see, this means we can write

$$Ei(x) \sim e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} + \ldots\right) \text{ as } x \to \infty. \quad \ldots (١.١٠)$$

Note [١٤]

For x sufficiently large, the terms in $S_N(x)$ will successively decrease initially. However at some value $N = N_*(x)$, the terms in $S_N(x)$ for $N > N_*$ will start to increase successively for a given *x* (however large) because the *N*th term, $(-1)^{N-1}e^{-x}\frac{(N-1)!}{x^N}$, is unbounded as $N \to \infty$.

Hence for a given *x*, there is an optimal value $N = N_*(x)$ for which the greatest accuracy is obtained. Our estimate equation (١.٨) suggests we should take N_* to be the largest integral part of the given *x*.

١.٢.٤.١ *Laplace Integrals [١] [١٤]*

A Laplace integral has the form

$$I(x) = \int_a^b f(t)\ e^{x\phi(t)}dt, \qquad \text{...(١.١١)}$$

where we assume $x > 0$. Typically *x* is a large parameter and we are interested in the asymptotic behaviour of $I(x)$ as $x \to +\infty$. Note that we can write $I(x)$ in the form

$$I(x) = \frac{1}{x}\int_a^b \frac{f(t)}{\phi'(t)}\frac{d}{dt}(\ e^{x\phi(t)})dt.$$

Integration by parts gives

$$I(x) = \underbrace{\left[\frac{1}{x}\frac{f(t)}{\phi'(t)}e^{x\phi(t)}\right]_a^b}_{\text{boundary term}} - \underbrace{\frac{1}{x}\int_a^b \frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right)e^{x\phi(t)}\ dt}_{\text{integral term}} \qquad \text{... (١.١٢)}$$

If the integral term is asymptotically smaller than the boundary term, i.e.

Integral term= *o* (boundary term) as $x \to +\infty$,

then $I(x) \sim \underbrace{\left[\frac{1}{x} \frac{f(t)}{\phi'(t)} e^{x\phi(t)} \right]_a^b}_{\text{boundary term}}$ as $x \to +\infty$

i.e. $I(x) \sim \frac{1}{x} \frac{f(b)}{\phi'(b)} e^{x\phi(b)} - \frac{1}{x} \frac{f(a)}{\phi'(a)} e^{x\phi(a)}$ as $x \to +\infty$ ...(١.١٣)

and we have a useful asymptotic approximation for $I(x)$ as $x \to +\infty$. In general, this will in fact be the case, i.e. formula (١.١٣) is valid, if $\phi(t)$, $\phi'(t)$ and $f(t)$ are continuous functions (possibly complex) and one of the following three conditions is satisfied:

(i) $\phi'(t) \neq 0$ for $a \le t \le b$ and either $f(a) \neq 0$ or $f(b) \neq 0$. These conditions ensure that the integral term in equation (١.١٢) is bounded and is asymptotically smaller than the boundary term.

(ii) $\operatorname{Re} \phi(t) \le \operatorname{Re} \phi(b)$ for $a \le t \le b$, $\operatorname{Re} \phi'(b) \neq 0$ and $f(b) \neq 0$. These conditions do not ensure that the integral term in equation (١.١٢) is bounded, but by Laplace’s method (see ١.٢.٤.٢) they ensure

$$I(x) \sim \frac{1}{x} \frac{f(b)}{\phi'(b)} e^{x\phi(b)} \text{ as } x \to +\infty$$

(iii) $\operatorname{Re} \phi(t) \le \operatorname{Re} \phi(a)$ for $a \le t \le b$, $\operatorname{Re} \phi'(a) \neq 0$ and $f(a) \neq 0$. Similar to the last case, the integral term in equation (١.١٢) is not necessarily bounded, but by Laplace's method we can ensure

$$I(x) \sim \frac{1}{x}\frac{f(a)}{\phi'(a)}e^{x\phi(a)} \text{ as } x \to +\infty$$

Further, if any one of these conditions is met then we may also continue to integrate by parts to generate further terms in the asymptotic expansion of $I(x)$; each integration by parts generates a new factor of $1/x$.

١.٢.٤.٢ *Laplace's Method [١] [١٤] [٢٤]*

We have seen that for Laplace integrals, integration by parts fails, for example, when $\phi'(t)$ has a zero somewhere in $a \le t \le b$. Laplace's method is a general technique that allows us to generate an asymptotic expansion for Laplace integrals for large *x* in such cases. Recall $I(x) = \int_a^b f(t)\ e^{x\phi(t)}dt$,

where we now suppose $f(t)$ and $\phi(t)$ are real, continuous functions.

Basic Idea

If $\phi(t)$ has a global maximum at *t*=*c* with $a \le c \le b$ and if $f(c) \ne 0$, then it is only neighbourhood of *t*=*c* that contributes to the full asymptotic expansion of $I(x)$ as $x \to +\infty$.

Procedure

Step ١. We may approximate $I(x)$ by $I(x;\varepsilon)$ where

$$I(x;\varepsilon)=\begin{cases}\int_{c-\varepsilon}^{c+\varepsilon} f(t)\ e^{x\phi(t)}\,dt, & \text{if} \quad a<c<b\\ \int_{a}^{a+\varepsilon} f(t)\ e^{x\phi(t)}\,dt, & \text{if} \quad c=a\\ \int_{b-\varepsilon}^{b} f(t)\ e^{x\phi(t)}\,dt, & \text{if} \quad c=b\end{cases} \quad \dots (١.١٤)$$

where $\varepsilon>0$ is arbitrary.

Step ٢. Now $\varepsilon>0$ can be chosen small enough so that (now we're confined to the narrow region surrounding *t*=*c*) it is valid to replace $\phi(t)$ by the first few terms in its Taylor or asymptotic series expansion.

(i) If $\phi'(c)=0$ with $a\le c\le b$ and $\phi''(c)\ne 0$, approximate $\phi(t)$ by

$$\phi(t)\approx\phi(c)+\frac{1}{2}\phi''(c)(t-c)^2.$$

(ii) If *c*=*a* or *c*=*b* and $\phi'(c)\ne 0$, approximate $\phi(t)$ by

$$\phi(t)\approx\phi(c)+\phi'(c)(t-c).$$

In either case approximate $f(t)$ by the leading order term in its series expansion about *t*=*c*,

$$f(t)\approx f(c)\ne 0 \quad \dots (١.١٥)$$

Step ٣. Having substitute the approximations for ϕ and f indicated above, we now extend the endpoints of integration to infinity, in order to evaluate the resulting integrals:

(i) If $\phi'(c)=0$ with $a<c<b$, we must have $\phi''(c)<0$ (t=c is a maximum) and so as $x\to+\infty$,

$$I(x;\varepsilon)\sim\int_{c-\varepsilon}^{c+\varepsilon} f(c)\ e^{x(\phi(c)+\frac{1}{2}\phi''(c)(t-c)^2)}\,dt$$

$$\sim f(c)\ e^{x\phi(c)}\int_{-\infty}^{\infty} e^{x\frac{\phi''(c)}{2}(t-c)^2}\,dt$$

$$=\frac{\sqrt{2}f(c)\ e^{x\phi(c)}}{\sqrt{-x\phi''(c)}}\int_{-\infty}^{\infty} e^{-s^2}\,ds, \qquad \text{... (١.١٦)}$$

where in the last step we made the substitution

$$s^2=-x\frac{\phi''(c)}{2}(t-c)^2.$$

Since $\int_{-\infty}^{\infty} e^{-s^2}\,ds=\sqrt{\pi}$,

We get $I(x)\sim\dfrac{\sqrt{2\pi}\,f(c)\ e^{x\phi(c)}}{\sqrt{-x\phi''(c)}}$ as $x\to+\infty$ ... (١.١٧)

If c=a or c=b. then the leading order behaviour for $I(x)$ is the same as that in formula (١.١٧) except multiplied by a factor $\frac{1}{2}$ when we extend the limits of integration, we only do so in one direction, so that the integral in formula (١.١٦) only extends over a semi-infinite range.

(ii) If *c*=*a* and $\phi'(c) \neq 0$, we must have $\phi'(c) < 0$, and as $x \to +\infty$,

$$I(x;\varepsilon) \sim \int_a^{a+\varepsilon} f(a)\ e^{x(\phi(a)+\frac{1}{2}\phi'(a)(t-a))}\ dt$$

$$\sim f(a)\ e^{x\phi(a)} \int_0^{\infty} e^{x\phi'(c)(t-c)}\ dt,$$

which implies that $I(x) \sim -\dfrac{f(a)\ e^{x\phi(a)}}{x\phi'(a)}$.

A similar argument for the case *c*=*b*, for which $\phi'(b) > 0$, gives

$$I(x) \sim \frac{f(b)\ e^{x\phi(b)}}{x\phi'(b)} \text{ as } x \to +\infty.$$

١.٢.٤.٣ Watson's Lemma [١] [٢٦]

The first step in the asymptotic of integrals is Wastson's lemma. Consider the following example of a Laplace integral

$$I(x) = \int_0^b f(t)\ e^{-xt}\ dt \quad (b > 0) \qquad \text{... (١.١٨)}$$

Suppose $f(t)$ is continuous on [٠, *b*] and has the asymptotic expansion

$$f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{\beta n} \quad \text{as} \quad t \to 0^+ \qquad \text{... (١.١٩)}$$

We assume that $\alpha > -1$ and $\beta > 0$ so that the integral is bounded near t=٠; if b=∞, we also require that $f(t) = o(e^{ct})$ as $t \to +\infty$ for some c>٠, to guarantee the integral is bounded for large t. Then

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}} \quad \text{as} \quad x \to +\infty \qquad \text{... (١.٢٠)}$$

Proof

Step١: Replace $I(x)$ by $I(x,\varepsilon)$ where

$$I(x,\varepsilon) = \int_0^{\varepsilon} f(t) e^{-xt} dt . \qquad \text{... (١.٢١)}$$

This approximation only introduces exponentially small errors for any $\varepsilon > 0$.

Step ٢: we can now choose $\varepsilon > 0$ small enough so that the first *N* terms in the asymptotic series for $f(t)$ are a good approximation to $f(t)$, i.e.

$$\left| f(t) - t^{\alpha} \sum_{n=0}^{N} a_n t^{\beta n} \right| \le k t^{\alpha + \beta(N+1)}, \qquad \text{... (١.٢٢)}$$

for $0 \le t \le \varepsilon$ and some constant k>٠. Substituting the first *N* terms in the series for $f(t)$ in to (١.٢١) we have by using (١.٢٢), that

$$\left| I(x;\varepsilon) - \sum_{n=0}^{N} a_n \int_0^{\varepsilon} t^{\alpha+\beta n} e^{-xt} dt \right| = \left| \int_0^{\varepsilon} \left(f(t) - t^{\alpha} \sum_{n=0}^{N} a_n t^{\beta n} \right) e^{-xt} dt \right|$$

$$\leq \int_0^{\varepsilon} \left| f(t) - t^{\alpha} \sum_{n=0}^{N} a_n t^{\beta n} \right| e^{-xt} dt$$

$$\leq k \int_0^{\varepsilon} t^{\alpha+\beta(N+1)} e^{-xt} dt$$

$$\leq k \int_0^{\infty} t^{\alpha+\beta(N+1)} e^{-xt} dt .$$

By using the identity

$$\int_0^{\infty} t^m e^{-xt} dt \equiv \frac{\Gamma(m+1)}{x^{m+1}} \qquad \text{... (١.٢٣)}$$

We have just established that

$$\left| I(x,\varepsilon) - \sum_{n=0}^{N} a_n \int_0^{\varepsilon} t^{\alpha+\beta n} e^{-xt} dt \right| \leq k \frac{\Gamma(\alpha+\beta+\beta N+1)}{x^{\alpha+\beta+\beta N+1}} .$$

Step ٣: Extending the range of integration to [٠, ∞) and using the identity (١.٢٣) again, we get that

$$I(x) = \sum_{n=0}^{N} a_n \frac{\Gamma(\alpha+\beta n+1)}{X^{\alpha+\beta n+1}} + o\left(\frac{1}{X^{\alpha+\beta N+1}} \right) \quad \text{as } n \to +\infty \quad \text{...(١.٢٤)}$$

Since this time for every *N*, we have proved (١.٢٠) and thus the Lemma.

Example [٢٦]

To apply Watson's lemma to the modified Bessel function

$$k_{\circ}(x) = \int_1^{\infty} (s^2 - 1)^{-\frac{1}{2}} e^{-xs} ds,$$

We first substitute $s = t + 1$, so the lower endpoint of the integration is t= ,

$$k_\circ(x) = e^{-x} \int_0^\infty (t^2 + 2t)^{-\frac{1}{2}} e^{-xt} dt .$$

For $|t| < 2$, the binomial theorem implies

$$(t^2 + 2t)^{-\frac{1}{2}} = (2t)^{-\frac{1}{2}} (1 + \frac{t}{2})^{-\frac{1}{2}} = (2t)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \left(-\frac{t}{2} \right)^n \frac{\Gamma\left(n + \frac{1}{2}\right)}{n! \Gamma\left(\frac{1}{2}\right)} .$$

Watson's lemma, then immediately tells us that

$$k_\circ(x) \sim e^{-x} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\Gamma\left(n + \frac{1}{2}\right)\right)^2}{2^{n+\frac{1}{2}} n! \Gamma\left(\frac{1}{2}\right) x^{n+\frac{1}{2}}} \text{ as } n \to +\infty .$$

Example:

To find the asymptotic expansions of the integral

$$\int_0^\pi e^{-xt} t^{-1} \sin t \, dt \quad \text{for large } x$$

Since $\quad t^{-1} \sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$

$$t^{-1}\sin t \sim \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n+1)!}$$

Then by using Watson's lemma, we get

$$\int_0^{\pi} e^{-xt}t^{-1}\sin t\, dt \sim \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(2n+1)}{x^{(2n+1)}(2n+1)!}$$

$$\sim \sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{x^{2n+1}(2n+1)!}$$

$$\sim 1/x - 1/3x^3 + 1/5x^5 \ldots \text{ For } x \to \infty .$$

In this chapter, we explain the importance of normal distribution and the definition of Poisson distribution in section one, while section two contains a review for some central distributions, and in section three speaks about non central distributions.

٢. ١ *Normal Distribution [١٩] [٢٧]*

The normal distribution or, as it is often called the Gaussian distribution is the most important distribution in statistical theory and application and clear importance from central limit theorem (CLT) which proves that most of the probability distributions (discrete or continuous) converge from the normal distribution when sample size is very large. Also, sampling distributions base their derivation for the normal distribution.

A continuous random variable X has a normal distribution if its probability density function (p.d.f) is given by

$$f(x;\mu,\sigma^2)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \begin{array}{c}-\infty<x<\infty\\ -\infty<\mu<\infty,\sigma>0\end{array} \qquad \text{... (٢.١)}$$

where μ is a location parameter, equal to the mean and σ the standard deviation are real constants we write $X \sim N(\mu,\sigma^2)$. The cumulative distribution function (c.d.f) or simply (the distribution function) is given by

$$\text{... (٢.٢)} \qquad .F(X\mid\mu,\sigma^2)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}dt$$

Notes

١- If $X \sim N(\mu,\sigma^2)$, then $Z=\frac{X-\mu}{\sigma}$ is a standard normal variate with mean ٠ and variance ١. we write $Z \sim N(0,1)$.

٢- The p.d.f of the standard normal variable Z is given by

$$.\ f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2} \qquad -\infty<z<\infty$$

And the corresponding distribution function is given by $\phi(z)=p(Z\le z)$

$$.\phi(z)=\int_{-\infty}^{z} f(t)dt=\int_{-\infty}^{z}\frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}dt$$

٣- $$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\ e^{-\frac{Z^2}{2}}dz=1$$

(ii) Poisson Distribution [٢٨]

A discrete random variable X has a Poisson distribution if its Mass probability function (m. p.f) given by

x=٠, ١, ... , λ> ٠ ... (٢.٣) $$f(x;\lambda)=P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}$$

(λ is a parameter distribution) We write $X \sim Po(\lambda)$.

The distribution function is given by

... (٢.٤) $$,F(X\mid\lambda)=P_r(X\le x)=\sum_{r=0}^{x}\frac{e^{-\lambda}\lambda^r}{r!}$$

and, $$\sum_{r=0}^{\infty}\frac{e^{-\lambda}\lambda^r}{r!}=1$$

٢.٢ Some Central Distributions

In this section, we explain the some central distributions as follows:

(i) Gamma Distribution [١] [٢٩] [٣٠]

A continuous random variable X, taking all real values in the range $(0,\infty)$ is said to have a gamma distribution with parameters *α* and *β* if its probability density function is given by

$$f(x;\alpha,\beta)=\frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}} \qquad x>0,\alpha,\beta>0 \qquad \text{... (٢.٥)}$$

where *α*, *β* are constants. We write $X \sim G(\alpha,\beta)$.

The Gamma function $\Gamma(\alpha)$ is defined as

$$\Gamma(\alpha)=\int_0^{\infty} x^{\alpha-1}e^{-x}dx$$

which converges for all $\alpha>0$ and diverges if $\alpha\le 0$ and has the following properties:

(١) If *α* is a positive integer, then

$$\Gamma(\alpha+1)=\alpha!=\int_0^{\infty} x^{\alpha}e^{-x}dx.$$

For this reason $\Gamma(\alpha)$ is sometimes called the factorial function.

(٢) for any positive real number

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha).$$

(٣) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

The distribution function is given by,

$$F(X\mid\alpha)=\frac{1}{\Gamma(\alpha)}\int_0^{x} t^{\alpha-1}e^{-t}dt \qquad \text{... (٢.٦)}$$

The incomplete gamma functions definitions are

$$, \qquad \Gamma(\alpha, x) = \int_x^{\infty} t^{\alpha-1}.e^{-t}dt \,.\, \gamma(\alpha, x) = \int_0^x t^{\alpha-1}e^{-t}dt$$

It is easy to check that, $\gamma(\alpha,0) = 0,\ \gamma(\alpha,\infty) = \Gamma(\alpha),$ $\Gamma(\alpha,0) = \Gamma(\alpha), \Gamma(\alpha,\infty) = 0$ and obviously $\Gamma(\alpha) = \gamma(\alpha, x) + \Gamma(\alpha, x)$ **.**

The incomplete gamma functions ratio definitions are

$, Q(\alpha, x) = \dfrac{\Gamma(\alpha, x)}{\Gamma(\alpha)}$ and clear $p(\alpha, x) + Q(\alpha, x) = 1$. $p(\alpha, x) = \dfrac{\gamma(\alpha, x)}{\Gamma(\alpha)}$

Notes:

(١) If $\alpha = 1$, the (p.d.f) (٢.٥) becomes $f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}$ it is common the exponential distribution is a special case of the gamma distribution.

We write $X \sim Exp(\beta)$.

The distribution function is given by

$$\cdot < x < \infty.\ F(X \mid \beta) = \int_0^x \frac{1}{\beta} e^{-\frac{t}{\beta}} dt = 1 - e^{-\frac{x}{\beta}}$$

(٢) Another special case gives if $\alpha = \frac{n}{2}, \beta = 2$ then p.d.f (٢.٥) become

$$f(x) = \frac{1}{\Gamma\left(\frac{n}{2}\right) \boldsymbol{2}^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-x/2}$$

It is common the central chi-square distribution with *n* degree of freedom (n.d.f) (positive integer), we write $\chi^2_{(n)}$, and the distribution function is given by

$$. \qquad 0<x<\infty \; F(X\mid n)=\frac{1}{\Gamma\left(\frac{n}{2}\right)\,\mathbf{2}^{\frac{n}{2}}}\int_0^x t^{\frac{n}{2}-1}e^{-t/2}dt$$

(ii) Beta Distribution [٣١] [٣٢]

A continuous random variable X, taking all real values in the range $(0,\infty)$ is said to have a beta distribution with parameter *a* and *b* if its probability density function is given by

$$\ldots\ (٢.٧)\ f(x;a,b)=\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1} \quad 0<x<1,\ \ a,b>0$$

where *a*, *b* are constants, we write *X~B* (*a*,*b*). The Beta function B(*a*,*b*) is defined as

which converges if *a* >٠, and *b*>٠ $B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$

The Beta function is related to the gamma function according to the following formula:

$$.\,B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

The distribution function is given by

٠<t<١, *a*, *b*>٠ ... (٢.٨) $F(X \mid a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt$

$$= \frac{B_X(a,b)}{B(a,b)} = I_X(a,b) .$$

It is common incomplete Beta function.

Notes

(١) The Beta distribution reduces to the uniform distribution over (٠,١) if $a = b = 1$.

(٢) The Beta function is symmetric $B(a,b) = B(b,a)$.

(٣) $B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$

(٤) If X_1 and X_2 are independent random variables, X_1 has a gamma distribution, and X_2 has a gamma distribution then $X = \frac{X_1}{X_1 + X_2}$ has a beta distribution.

(iii) Student's t-Distribution [٣٢] [٣٣]

If *X* and *Y* are independent random variables and *X* has a standard normal distribution and *Y* has a chi-square distribution with *n* degrees of freedom, then the variable $t = \frac{X}{\sqrt{Y/n}}$ has a *t*-distribution if its probability density function is given by

$$f(t;n)=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}$$

$$=\frac{\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}}{\sqrt{n}\,B\left(\frac{1}{2},\frac{n}{2}\right)} \qquad -\infty < x < \infty$$

, ... (٢.٩)

where *n* is a parameter (positive integer), we write $t \sim t_n$

The distribution function is given by

... (٢.١٠)

$$F(t\,|\,n)=\frac{1}{\sqrt{n}\,B\left(\frac{1}{2},\frac{n}{2}\right)}\int_{-\infty}^{t}\left(1+\frac{u^2}{n}\right)^{-\frac{n+1}{2}}du$$

(iv) F-Distribution [٣٢][٣٣]

If X_1 and X_2 are independent random variables and X_1 has chi-square distribution with *m* degrees of freedom and X_2 has a chi-square distribution with *n* degrees of freedom then the variable $F=\frac{X_1/m}{X_2/n}$ is said to have *F*-distribution with *m*, *n* degrees of freedom if its probability density function is given by

$$f(F;m,n)=\frac{m^{\frac{m}{2}}n^{\frac{n}{2}}\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\cdot\frac{F^{\frac{m}{2}-1}}{(mF+n)^{\frac{m+n}{2}}},\qquad F\in R$$

$$=\frac{m^{\frac{m}{2}}n^{\frac{n}{2}}}{B\left(\frac{m}{2},\frac{n}{2}\right)}\cdot\frac{F^{\frac{m}{2}-1}}{(mF+n)^{\frac{m+n}{2}}},\qquad \text{... (٢.١١)}$$

where *m*, *n* are parameters (positive integer), we write $F \sim Fm,n$.

The distribution function is given by

$$.F(F\mid m,n)=\frac{m^{\frac{m}{2}}n^{\frac{n}{2}}}{B\left(\frac{m}{2},\frac{n}{2}\right)}\int_0^F\frac{u^{\frac{m}{2}-1}}{(mu+n)^{\frac{m+n}{2}}}du \qquad \text{... (٢.١٢)}$$

٢.٣ *Non-central Distributions*

In this section, we discuss the non-central distributions as follows:

(i) *Chi-square Distribution [٣٢] [٣٤] [٣٥]*

If $X_1, X_2, \ldots, X_n$ are independently distributed and X_i is $N(\mu_i, 1)$ then the random variable $U=\sum_{i=1}^{n}X_i^2$ is called a non-central chi-square variable with *n* degree of freedom. We call $\delta=\left(\sum_{i=1}^{n}\mu_i^2\right)^{1/2}$ the non centrality parameter of the distribution if its probability density function is given by

, ... (٢.١٣) $f(U;n,\lambda)=\sum_{r=0}^{\infty} p_0\left(\frac{\lambda}{2}\right) f(U;n+2r) \qquad U\geq 0,\lambda\geq 0$

, and $=\frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{1}{2}\right)} U^{\frac{n}{2}-1} e^{-\frac{1}{2}(U+\lambda)} \sum_{r=0}^{\infty} \frac{(\lambda U)^r \Gamma\left(\frac{1}{2}+r\right)}{(2r)!\Gamma\left(\frac{n}{2}+r\right)}$

is Poisson probability mass function $\lambda=\delta^2$ is the $p_0\left(\frac{\lambda}{2}\right)=\frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^r}{r!}$ non-central parameter and $f(U;n+2r)$ is a central chi-square with $(n+2r)$ degree of freedom (positive integer).

The distribution function is given by

$$.F(U\mid n,\lambda)=\frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{1}{2}\right)} e^{-\frac{\lambda}{2}} \sum_{r=0}^{\infty} \frac{\lambda^r}{(2r)!} \frac{\Gamma\left(\frac{1}{2}+r\right)}{\Gamma\left(\frac{n}{2}+r\right)} \int_0^U t^{\frac{n}{2}+r-1} e^{-\frac{t}{2}} dt$$

... (٢.١٤) $$F(U\mid n,\lambda)=e^{-\frac{\lambda}{2}} \sum_{r=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^r}{r!} p\left(\frac{n}{2}+r,\frac{U}{2}\right)$$

Where P(.,.) is incomplete gamma function ratio.

Notes

(١) The ordinary or central chi-square distribution is a special case of the noncentral distribution when the noncentral parameter δ= ٠. The same goes for the *F*-and *t*- distributions.

(٢) The noncentrality parameter is usually defined differently, some authors use $\lambda = \delta^2$, while others use $\lambda = \frac{1}{2}\delta^2$, both using the same symbol δ.

(٣) We use the symbol $\chi'^2_{n,\lambda}$ for a noncentral chi-square variable with n degrees of freedom and non centrality parameter λ, and the symbol χ^2_n for $\chi^2_{n,0}$.

(٤) If X_1 and X_2 be independent random variables and X_1 has a χ'^2_{m,δ_1} and X_2 has a χ^2_{n,δ_2}. Then $(X_1 + X_2)$ have a $\chi'^2_{r,\delta}$ with *r=m+n*, $\delta = \left(\delta_1^2 + \delta_2^2\right)^{1/2}$.

(ii) F-Distribution [٣٣] [٣٤] [٣٦]

If X_1 and X_2 are independent random variables and X_1 is anon-central chi-square distribution with *m* degrees of freedom and noncentrality parameter λ and X_2 is a central chi-square distribution with *n* degrees of freedom then the variable

is said to have a non-central *F*-Distribution with *m, n* $F' = \frac{X_1/m}{X_2/n}$

degrees of freedom (positive integers) and non-central parameter $\lambda \geq 0$, if its probability density function is given by

$$f(F'; m, n, \lambda) = e^{-\frac{\lambda}{2}} \sum_{r=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^r}{r!} \frac{\Gamma\left(\frac{m+n}{2} + r\right)}{\Gamma\left(\frac{m}{2} + r\right)\Gamma\left(\frac{n}{2}\right)}$$

$$\left(\frac{m}{n}\right)^{\frac{m}{2}+r}\frac{(F')^{\frac{m}{2}-1+r}}{\left(1+\frac{mF'}{n}\right)^{\frac{1}{2}(m+n)+r}}. \qquad \text{... (٢.١٥)}$$

And we write $F' \sim F_{m,n,\lambda}$.

The distribution function is given by

$$F(F' \mid m,n,\lambda)=\int_0^{F'} U^k f(F';m,n,\lambda)dU.$$

$$=e^{-\frac{\lambda}{2}}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{\lambda}{2}\right)^r\frac{B_q\left(\frac{m}{2}+r,\frac{n}{2}\right)}{B\left(\frac{m}{2}+r,\frac{n}{2}\right)},$$

and $$F(F' \mid m,n,\lambda)=e^{-\frac{\lambda}{2}}\sum_{r=0}^{\infty}\frac{\left(\frac{\lambda}{2}\right)^r}{r!}I_q\left(\frac{m}{2}+r,\frac{n}{2}\right), \qquad \text{... (٢.١٦)}$$

where $q=\dfrac{\frac{mF'}{n}}{1+\frac{mF'}{n}}$, $I_q(.,.)$ is incomplete Beta function.

(iii) t-distribution [٣٣] [٣٤] [٣٦]

If X and Y are independent random variables and X is a normal distribution with mean δ and variance ١ and Y is a central chi-square with n degrees of freedom then the variable $t'=\dfrac{X}{\sqrt{Y/n}}$ has a non-central t-distribution with *n* degrees of freedom (positive integer) and non- central parameter *δ* (real) if its probability density function is given by

$$f(t',n,\delta)=\frac{e^{-\frac{\delta^2}{2}}}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\sum_{r=0}^{\infty}\frac{(t'\delta)^r}{r!\,n^{\frac{r}{2}}}$$

$$\left(1+\frac{t'^2}{n}\right)^{-\frac{n+r+1}{2}}2^{\frac{r}{2}}\Gamma\left(\frac{n+r+1}{2}\right) \quad \ldots (٢.١٧)$$

And we write $t' \sim t_{n,\delta}$.

The distribution function is given by

$$F(t'\,|\,n,\delta)=\frac{e^{-\frac{\delta^2}{2}}}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\sum_{r=0}^{\infty}\frac{\delta^r}{r!\,n^{\frac{r}{2}}}2^{\frac{r}{2}}\Gamma\left(\frac{n+r+1}{2}\right)\int_{-\infty}^{t}\frac{u^r}{\left(1+\frac{u^2}{n}\right)^{\frac{n+r+1}{2}}}du$$

$$F(t'\,|\,n,\delta)=\frac{e^{-\frac{\delta^2}{2}}}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{\delta^2}{r!}2^{\frac{r}{2}-1}\Gamma\left(\frac{r+1}{2}\right)\left\{S_1+S_2 I_q\left(\frac{r+1}{2},\frac{n}{2}\right)\right\} \quad ..(٢.١٨)$$

Where S_1 and S_2 are signs differing between cases with positive or negative *t* as well as odd or even *r* in the summation. The sign S_1 is -١ if *r* is odd and +١ if it is even while S_2 is +١ unless *t*<٠ and *r* is even in which case it is -١.

(iv) Gamma Distribution [٣٣][٣٧]

If Y has a non-central chi-square distribution with ٢α degrees of freedom and non-centrality parameter ٢λ, then the random variable $X=\frac{Y}{2}$ has a non-central gamma distribution with parameter α and a non-centrality parameter λ, if its probability density function is given by

$$f(x;\alpha,\lambda)=\sum_{r=0}^{\infty} e^{-\frac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^r}{r!} \frac{x^{\alpha+r-1}e^{-x}}{\Gamma(\alpha+r)} \quad , \quad \alpha>0,\ \lambda\geq 0. \quad \text{... (٢.١٩)}$$

The distribution function is given by

$$F(X\mid\alpha,\lambda)=\sum_{r=0}^{\infty} e^{-\frac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^r}{r!\Gamma(\alpha+r)} \int_0^x t^{\alpha+r-1}.e^{-t}dt$$

$$=\sum_{r=0}^{\infty} e^{-\frac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^r}{r!\Gamma(\alpha+r)} \gamma(\alpha+r,x),$$

and $$F(X\mid\alpha,\lambda)=\sum_{r=0}^{\infty} e^{-\frac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^r}{r!} p(\alpha+r,x), \quad \text{... (٢.٢٠)}$$

where $\gamma(\alpha+r,x)$ is incomplete gamma function,

and $p(\alpha+r,x)$ is incomplete gamma function ratio.

(v) Beta Distribution [٣٣] [٣٧]

If X_1 and X_2 be independent random variables and X_1 is non-central gamma distribution with parameter *a* and non-centrality parameter λ, and X_2 is a central gamma distribution with parameter b, then the random variable $X = \frac{X_1}{X_1 + X_2}$ has a non-central beta distribution with parameters (*a*, *b*) and non centrality parameter λ. if its probability density function is given by

$$a,b > 0, \lambda \geq 0. \quad ..(٢.٢١) \quad f(x;a,b,\lambda) = \sum_{r=0}^{\infty} \frac{e^{-\frac{\lambda}{2}} \left(\frac{\lambda}{2}\right)^r}{r!} \frac{x^{a+r-1}(1-x)^{b-1}}{B(a+r,b)}$$

The distribution function is given by

$$F(X \mid a,b,\lambda) = \int_0^x \sum_{r=0}^{\infty} e^{-\frac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^r}{r!} \frac{t^{a+r-1}(1-t)^{b-1}}{B(a+r,b)} dt$$

$$= \sum_{r=0}^{\infty} e^{-\frac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^r}{r!B(a+r,b)} \int_0^x t^{a+r-1}.(1-t)^{b-1} dt$$

$$... (٢.٢٢) \qquad ,F(X \mid a,b,\lambda) = \sum_{r=0}^{\infty} e^{-\frac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^r}{r!} I_x(a+r,b)$$

where $I_x(.,.)$ is incomplete Beta function.

In this chapter, we explain stirling formula and its proof in section one, while we derive the asymptotic expansions for each the two non-central distributions gamma and beta in section two and three, respectively, conclusions and recommendations are mentioned at section four.

٣.١ *Asymptotic Formula for* $\Gamma(x)$ *(Stirling's formula)*

If *x* is large, the computational difficulties inherent in a calculation of $\Gamma(x)$ are apparent. A useful result in such case is supplied by the relation

$$\cdot<\vartheta<١ . \Gamma(x+1) = \sqrt{2\pi x}\ x^x e^{-x} e^{\theta/12(x+1)} \quad \dots (٣.١)$$

For most practical purposes the last factor, which is very close to ١ for large *x* can be omitted. If *x* is an integer, we can write

$$x! \sim \sqrt{2\pi x} x^x e^{-x} \quad \dots (٣.٢)$$

This sometimes called stirling's factorial approximation or asymptotic formula for *x*!

Proof

By using Laplace's method, (see ١.٢.٤.٢) we shall try to find the leading order behaviour of the (complete) gamma function.

$$\text{as } x \to +\infty . \Gamma(x+1) = \int_0^\infty e^{-t} t^x dt \equiv \int_0^\infty e^{-t + x\ln t} dt$$

First we try to convert it more readily to the standard Laplace integral form by making the substitution *t=xr* (this really has the effect of creating the maximum of ϕ away from the origin),

$$. \Gamma(x+1) = \int_0^\infty e^{-xr + x\ln x + x\ln r} x dr = x^{x+1} \int_0^\infty e^{x(-r+\ln r)} dr$$

Hence *f*(*r*)=١ and $\phi(r) = -r + \ln r$.

Since $\phi'(r) = -1 + \frac{1}{r}$ and $\phi''(r) = -\frac{1}{r^2}$.

For all *r*>٠, we conclude that ϕ has a local (and global) maximum at *r*=١, and $\phi''(r) < 0$. Hence after collapsing the range of integration to a narrow region surrounding *r*=١, we approximate

$$.\phi(r) \approx \phi(1) + \frac{\phi''(1)}{2}(r-1)^2 = -1 - \frac{1}{2}(r-1)^2$$

Subsequently extending the range of integration out to infinity again we see that

$$.\Gamma(x+1) \sim x^{x+1} \int_{-\infty}^{\infty} e^{-x} e^{-\frac{x}{2}(r-1)^2} dr$$

Making the substitution $S^2 = \frac{x}{2}(r-1)^2$ and by using the identity

, then reveals that $\int_{-\infty}^{\infty} e^{-S^2} ds = \sqrt{\pi}$

as $x \to +\infty$. $\Gamma(x+1) = x! \sim \sqrt{2\pi x}\, x^x e^{-x}$

When $x \in N$, this is stirling's formula for the asymptotic of the factorial function for large integers.

٣.٢ *Asymptotic Expansion for the non-Central Gamma Distribution*

We know that

$$..(٣.٣)\, F(X \mid \alpha,\lambda) = \sum_{r=0}^{\infty} e^{-\frac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^r}{r!\Gamma(\alpha+r)} \int_0^x t^{\alpha+r-1} e^{-t} dt \quad \alpha > 0, \lambda \geq 0$$

Is the cumulative distribution function (c.d.f) of non-central gamma distribution.

We now take the integral term in (٣.٣) called (incomplete gamma function $\gamma(\alpha+r,x)$). To find an asymptotic expansion of the $\gamma(\alpha+r,x)$ as $x \to \infty$ by expanding the integrand in powers of *t* and integrating term by term, we have

$$\gamma(\alpha+r,x) = \int_0^x t^{\alpha+r-1} e^{-t} dt$$

$$= \int_0^x t^{\alpha+r-1} \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right) dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{\alpha+r+n-1} dt$$

$$\sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+r+n}}{n!(\alpha+r+n)}.$$

Using the ratio test, we find that the series converges for all *x*. However, the series is only useful for small *x*. (for large values of *x*, it converges very slowly which in practical terms is not very useful).

To deal with $\gamma(\alpha+r,x)$ for *x* large, we proceed indirectly with $\Gamma(\alpha+r,x)$, write

$$\gamma(\alpha+r,x)=\Gamma(\alpha+r)-\Gamma(\alpha+r,x) \quad \text{... (٣.٤)}$$

$$.\gamma(\alpha+r,x)=\int_0^\infty e^{-t}t^{\alpha+r-1}dt-\int_x^\infty e^{-t}t^{\alpha+r-1}dt$$

We now take the incomplete gamma function

$$.\Gamma(\alpha+r,x)=\int_x^\infty e^{-t}t^{\alpha+r-1}dt$$

We shall develop the integral by repeating the integration by parts:

$$\Gamma(\alpha+r,x)=[-e^{-t}t^{\alpha+r-1}]_x^\infty+(\alpha+r-1)\int_x^\infty e^{-t}t^{\alpha+r-1}dt$$

$$=e^{-x}x^{\alpha+r-1}+(\alpha+r-1)e^{-x}x^{\alpha+r-2}$$

$$+(\alpha+r-1)(\alpha+r-2)\int_x^\infty e^{-t}t^{\alpha+r-3}dt\,.$$

$$\vdots$$

$$\Gamma(\alpha+r,x)=e^{-x}[x^{\alpha+r-1}+(\alpha+r-1)x^{\alpha+r-2}+\ldots$$

$$+(\alpha+r-1)\ldots(\alpha+r-N+1)x^{\alpha+r-N}]$$

$$\underbrace{+(\alpha+r-1)(\alpha+r-2)\ldots(\alpha+r-N)\int_x^\infty e^{-t}t^{\alpha+r-N-1}dt}_{\text{remainder term } R_N}$$

The integrands of successive integrals are becoming smaller in the regin of integration $1\le x<t$. However, this series generated by integration by parts does not convergent for fixed finite *x*. (by using the ratio test). But for fixed N, the error committed by omitting $R_N(\alpha+r,x)$

is small for large *x*, that is, as $x \to \infty$, $R_N(\alpha + r, x) \to 0$ for fixed *N*. In order to prove this, we estimate R_N as follows:

Assume $N > \alpha + r - 1$

$$\left| \int_x^\infty e^{-t} t^{\alpha+r-N-1} dt \right| \le \int_x^\infty \left| e^{-t} t^{\alpha+r-N-1} \right| dt$$

$$\le x^{\alpha+r-N-1} \int_x^\infty e^{-t} dt$$

$$= x^{\alpha+r-N-1} e^{-x}$$

$$= o(x^{\alpha+r-N} e^{-x}),$$

as $x \to \infty$.

Thus, for sufficiently large *x*, the remainder term R_N will be small and only a few terms in the series are needed to give a reasonable approximation to $\Gamma(\alpha + r, x)$. Series sums like that are called asymptotic expansions, and are written as:

$$\Gamma(\alpha + r, x) \sim e^{-x} x^{\alpha+r} \left(\frac{1}{x} + \frac{(\alpha + r - 1)}{x^2} + \frac{(\alpha + r - 1)(\alpha + r - 2)}{x^3} + \dots \right)$$

By substitution in equation (٣.٤), we get

$$\gamma(\alpha + r, x) \sim \Gamma(\alpha + r) - e^{-x} x^{\alpha+r}$$

as $x \to \infty$ ... (٣.٥) $\left(\frac{1}{x} + \frac{(\alpha + r - 1)}{x^2} + \frac{(\alpha + r - 1)(\alpha + r - 2)}{x^3} + \dots \right)$

although the series is divergent.

Now, when we substitute the formula (٣.٥) in equation (٣.٣), we get

$$F(X \mid \alpha,\lambda) \sim \sum_{r=0}^{\infty} \frac{e^{\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^{r}}{r!} - \sum_{r=0}^{\infty} \frac{e^{-\left(\frac{\lambda}{2}+x\right)}\left(\frac{\lambda}{2}\right)^{r} x^{\alpha+r}}{r!\Gamma(\alpha+r)}$$

$$\left(\frac{1}{x} + \frac{(\alpha+r-1)}{x^2} + \frac{(\alpha+r-1)(\alpha+r-2)}{x^3} + \dots\right).$$

Then by using the identity $\sum_{r=0}^{\infty} e^{\lambda} \frac{\lambda^r}{r!} = 1$, we have

$$F(X \mid \alpha,\lambda) \sim 1 - \sum_{r=0}^{\infty} \frac{e^{-\left(\frac{\lambda}{2}+x\right)}\left(\frac{\lambda}{2}\right)^{r} x^{\alpha+r}}{r!\Gamma(\alpha+r)} \left(\frac{1}{x} + \frac{(\alpha+r-1)}{x^2} +\right.$$

$$\left.\frac{(\alpha+r-1)(\alpha+r-2)}{x^3} + \dots\right). \text{ as } x \to \infty \qquad \text{... (٣.٦)}$$

This result gives an asymptotic expansion for the non-central gamma distribution.

٣.٣ *Asymptotic Expansion for the non Central Beta Distribution*

We know that $F(X \mid a,b,\lambda) = \sum_{r=0}^{\infty} e^{-\frac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^{r}}{r!} I_x(a+r,b)$

$$a,b > 0, \lambda \geq 0 \qquad \text{... (٣.٧)}$$

$$\text{... (٣.٨)} = \sum_{r=0}^{\infty} e^{-\frac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^{r}}{r!} \frac{\Gamma(a+r+b)}{\Gamma(a+r)\Gamma(b)} \int_0^x t^{a+r-1}(1-t)^{b-1} dt$$

Is the cumulative distribution function (c.d.f) of non-central Beta distribution.

We derive an asymptotic expansion of $F(X \mid a,b,\lambda)$, through two stages:

***First Stage*:**

We shall derive the asymptotic expansion of $\dfrac{\Gamma(a+r)}{\Gamma(a+r+b)}$ where *a*, *b*>٠, $a \geq b$ and $r \in N$ .we start from the Beta function $B(a+r,b)$, which has the formula.

$$B(a+r,b)=\frac{\Gamma(a+r)\Gamma(b)}{\Gamma(a+r+b)}=\int_0^1 t^{a+r-1}(1-t)^{b-1}dt \quad \text{... (٣.٩)}$$

Then by using the substitution $t=e^{-u}$ and $dt=-e^{-u}du$ we obtain

$$B(a+r,b)=\int_0^{\infty} e^{-(a+r)u}(1-e^{-u})^{b-1}du \quad \text{... (٣.١٠)}$$

And using the fact that $(1-e^{-u})=e^{-u/2}2\sinh\frac{u}{2}$ we have

$$B(a+r,b)=\int_0^{\infty} e^{-ku}u^{b-1}\left(\sinh\left(\frac{u}{2}\right)\Big/\left(\frac{u}{2}\right)\right)^{b-1}du \quad \text{... (٣.١١)}$$

where $k=(a+r)+\dfrac{(b-1)}{2}$.

Now expand $\left[\sinh\left(\frac{u}{2}\right)\Big/\left(\frac{u}{2}\right)\right]^{b-1}$ in powers of u^2 as $u \to \infty$

$$\left[\sinh\left(\frac{u}{2}\right)\Big/\left(\frac{u}{2}\right)\right]^{b-1}=\left[\sum_{n=0}^{\infty}\frac{u^{2n}}{(2n+1)!2^{2n}}\right]^{b-1}$$

$$=\left(\sum_{n=0}^{\infty}h_n u^{2n}\right)^{b-1}\sim\sum_{n=0}^{\infty}C_n u^{2n}, \quad \text{... (٣.١٢)}$$

Let $h_n=\left[\sum_{n=0}^{\infty}\frac{1}{(2n+1)!2^{2n}}\right]^{b-1}$

The last quality follows by Didonate and Morris [٣٨].

Where C_n are the expansion Coefficients of $\left[\sinh\left(\frac{u}{2}\right)\Big/\left(\frac{u}{2}\right)\right]^{b-1}$, and which can be expressed of the generalized Bernoulli polynomials [٣٩],

$$C_n=B_{2n}^{1-b}\frac{\left(\frac{1-b}{2}\right)}{(2n)!}.$$

By substitution equation (٣.١٢) in (٣.١١) we get

$$du\,B(a+r,b)\sim\int_0^{\infty}e^{-ku}u^{b-1}\sum_{n=0}^{\infty}C_n u^{2n}$$

$$.\,B(a+r,b)\sim\sum_{n=0}^{\infty}C_n\int_0^{\infty}e^{-ku}u^{b+2n-1}du$$

And using Watson's Lemma (see ١.٢.٤.٣) we obtain the asymptotic expansion

$$.\frac{\Gamma(a+r)}{\Gamma(a+r+b)}\sim\frac{1}{k^b}\sum_{n=0}^{\infty}C_n\frac{\Gamma(b+2n)}{\Gamma(b)}\left(\frac{1}{k}\right)^{2n} \quad \text{... (٣.١٣)}$$

Second Stage:

In this stage we derive the asymptotic expansion of $I_x(a+r,b)$,

$$I_x(a+r,b)=\frac{\Gamma(a+r+b)}{\Gamma(a+r)\Gamma(b)}\int_0^x t^{a+r-1}(1-t)^{b-1}dt \quad \text{... (٣.١٤)}$$

when $a,b>0$, $a\geq b, 0<x<1$ and $r\in N$, and then transform the expression for it as the same in equation (٣.٩) and changing integration terms, to obtain

$$I_x(a+r,b)=\frac{\Gamma(a+r+b)}{\Gamma(a+r)\Gamma(b)}\int_{-\log x}^{\infty} e^{-ku}u^{b-1}$$

$$\left(\sinh\left(\frac{u}{2}\right)\Big/\left(\frac{u}{2}\right)\right)^{b-1}du, \quad \text{... (٣.١٥)}$$

where as before $k=(a+r)+\left(\frac{b-1}{2}\right)$, by using (٣.١٢) we have

$$I_x(a+r,b)\sim\frac{\Gamma(a+r+b)}{\Gamma(a+r)\Gamma(b)}\sum_{n=0}^{\infty}C_n\int_{-\log x}^{\infty}e^{-ku}u^{2n+b-1}du \quad \text{... (٣.١٦)}$$

Let *w=ku*, then $u=\frac{1}{k}w$ and $du=\frac{1}{k}dw$, so from (٣.١٦) we have

$$I_x(a+r,b)\sim\frac{\Gamma(a+r+b)}{\Gamma(a+r)\Gamma(b)}\sum_{n=0}^{\infty}C_n\int_{-k\log x}^{\infty}e^{-w}\left(\frac{1}{k}w\right)^{2n+b-1}\frac{1}{k}dw$$

$$=\frac{\Gamma(a+r+b)}{\Gamma(a+r)\Gamma(b)}\sum_{n=0}^{\infty}C_n\frac{\Gamma(b+2n)}{K^{2n+b}}\frac{1}{\Gamma(b+2n)}\int_{-k\log x}^{\infty}e^{-w}w^{2n+b-1}dw$$

$$\sim \frac{\Gamma(a+r+b)}{\Gamma(a+r)\Gamma(b)k^{b}} \sum_{n=0}^{\infty} \frac{C_n}{k^{2n}} \Gamma(b+2n) Q(b+2n,-k\log x), \quad \text{... (٣.١٧)}$$

where $Q(.,.)$ is incomplete gamma function ratio. We can proceed by using the recurrence relations for $Q(.,.)$ to express $Q(b+2n,-k\log x)$ in terms of $Q(b,-k\log x)$. This gives

$$\text{... (٣.١٨)} \qquad I_x(a+r,b) \sim Q(b,-k\log x) + R(a+r,b,x)$$

Where we have use the formula (٣.١٣) to cancel out the factors multiplying *Q*, the other term $R(a+r,b,x)$ is a double summation over *n* and the ٢*n* residual terms obtained by expressing $Q(b+2n,-k\log x)$ in terms of $Q(b,-k\log x)$.

To obtain the asymptotic expansion we require to reordering this sum.

First we write (٣.١٥) in the form

$$I_x(a+r,b) = \frac{\Gamma(a+r+b)}{\Gamma(a+r)\Gamma(b)} \left[\int_{-\log x}^{\infty} e^{-ku} \right.$$

$$\left. \left([2\sinh(u/2)]^{b-1} - u^{b-1} \right) du + \int_{-\log x}^{\infty} e^{-ku} u^{b-1} du \right] \quad \text{...(٣.١٩)}$$

Integrate the first integral by parts twice as follows:

$$\int_{-\log x}^{\infty} e^{-ku} \left([2\sinh(u/2)]^{b-1} - u^{b-1} \right) du =$$

$$\frac{1}{k^2} \int_{-\log x}^{\infty} e^{-ku} \frac{d^2}{du^2} \left([2\sinh(u/2)]^{b-1} - u^{b-1} \right) du$$

$$+\frac{x^k}{k}\Big[[2\sinh(u/2)]^{b-1} - u^{b-1} + \frac{1}{k}\frac{d}{du}$$

$$\Big([2\sinh(u/2)]^{b-1} - u^{b-1}\Big)\Big]\Big|_{u=-\log x} \qquad \ldots (٣.٢٠)$$

In the integral in (٣.٢٠) we now subtract the second term $(C_1 u^{b+1})$ in expansion of $[2\sinh(u/2)]^{b-1}$ and add a corresponding integral so that the integral in (٣.٢٠) becomes

$$\int_{-\log x}^{\infty} e^{-ku}\frac{d^2}{du^2}([2\sinh(u/2)]^{b-1} - u^{b-1} - C_1 u^{b+1})du +$$

$$\ldots (٣.٢١) \qquad , \frac{\Gamma(b+2)}{\Gamma(b)} C_1 \int_{-\log x}^{\infty} e^{-ku} u^{b-1} du$$

and producing two further integrated terms evaluated at $u = -\log x$.

Again Integral the integral term in (٣.٢٠) by parts twices and subtract a third term $(C_2 u^{b+3})$ from the expansion of $[2\sinh(u/2)]^{b-1}$ and add a corresponding integral on separately. This procedure is continued indefinitely. The separate integrals stating from the ones on the right of (٣.١٩) and (٣.٢١) add together to give $Q(b, -k\log x)$ as in (٣.١٨) so that

$$I_x(a+r,b) \sim Q(b, -k\log x)$$

$$+\frac{\Gamma(a+r+b)}{\Gamma(a+r)\Gamma(b)} x^k \sum_{n=0}^{\infty}\frac{H_n(b,x)}{k^{n+1}}, \qquad \ldots (٣.٢٢)$$

where $H_n(b,x) = \frac{d^n}{du^n}\left([2\sinh(u/2)]^{b-1} - \sum_{m=0}^{n/2} C_m u^{2m+b-1}\right)\Bigg|_{u=-\log x}$

$$= \frac{d^n}{du^n}\left(\sum_{m=\frac{n}{2}+1}^{\infty} C_m u^{2m+b-1}\right)\Bigg|_{u=-\log x} ,$$

and *n*/٢ in the summation is to be interpreted as largest integer $\leq n/2$ as in integer division. The quantities H_n satisfy the simple recurrence formula

$$,H_{2n+1} = \frac{d}{du} H_{2n}$$

$$. \quad ... (٣.٢٣)\ H_{2n} = \frac{d}{du} H_{2n-1} - C_n u^{b-1} \frac{\Gamma(2n+b)}{\Gamma(b)}$$

We can express $H_n(b,x)$ directly in terms of *b* and *x*, for example, $H_{\circ}(b,x) = (1/\sqrt{x} - \sqrt{x})^{b-1} - (-\log x)^{b-1}$.

However, for *x* close to ١, evaluation of H_n in this way can lead to large rounding errors on subtraction, and so $H_n(b,x)$ is better evaluated from its power series expansion in *u*.

Now, when we substitute the formula (٣.٢٢) in equation (٣.٨), we get

$$F(X \mid a,b,\lambda) \sim \sum_{r=0}^{\infty} \frac{e^{-\frac{\lambda}{2}\left(\frac{\lambda}{2}\right)^r}}{r!} [Q(b,-k\log x)$$

$$+\frac{\Gamma(a+r+b)}{\Gamma(a+r)\Gamma(b)}x^k\sum_{n=0}^{\infty}\frac{H_n(b,x)}{k^{n+1}}]$$

$$\sim\sum_{r=0}^{\infty}\frac{e^{-\frac{\lambda}{2}\left(\frac{\lambda}{2}\right)^r}}{r!}Q(b,-k\log x)+\sum_{n=0}^{\infty}\frac{\Gamma(a+r+b)}{\Gamma(a+r)\Gamma(b)}$$

$$\frac{e^{-\frac{\lambda}{2}\left(\frac{\lambda}{2}\right)^r}}{r!}\sum_{n=0}^{\infty}\frac{H_n(b,x)}{k^{n+1}}.$$

By using the identity $\sum_{r=0}^{\infty}\frac{e^{-\lambda}\lambda^r}{r!}=1$ we get

$$F(X\mid a,b,\lambda)\sim Q(b,-k\log x)+\frac{1}{\Gamma(b)}$$

$$\sum_{r=0}^{\infty}\frac{\Gamma(a+r+b)e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^r x^k}{\Gamma(a+r)r!}\sum_{n=0}^{\infty}\frac{H_n(b,x)}{k^{n+1}}. \quad \text{... (٣.٢٤)}$$

which is an asymptotic expansion for the non-central beta distribution.

٣.٤ *Conclusions and Recommendations*

(i) Conclusions

From our work in this chapter we conclude that:

١- Asymptotic expansions are not necessary convergent.

٢- Asymptotic expansions are very useful for determining the local behaviour of functions, and study some its properties.

٣- Asymptotic expansions which are converges tell us the number of terms we need to get an accurate estimate of functions when *x* is large and also it tells us about the long-range behaviour of functions.

٤- Finding Asymptotic Expansions is complicated some how, in turn its coefficients become more complex.

(ii) Recommendations

Since we work in this field we recommend to:

١- find doubly Non-central two distributions gamma and beta and then derive asymptotic expansions for it.

٢- find multivariate non-central two distributions gamma and beta and then derive asymptotic expansions for it.

٣- Studying the degree of convergence or divergence in the optimal solutions.

٤- Using Simulaty to study the behaviour of functions.

REFERENCES

[١] Temme, N.M. (١٩٩٥). "Uniform Asymptotic Expansions of Integrals: A Selection of Problems",

http://citesseer.١st.psu.edu/temme٩٥uniform.html.

[٢] Cohen, J.K. (١٩٩٥). "Symbolic Integration and Asymptotic Expansions"

http://www.mathematical.journal.com/issue/v٥١٤/columns/cohen/٥٢.cohe.mj.pdf.

[٣] Temme, N. M. (١٩٩٦). "Uniform Asymptotic for the Incomplete Gamma Functions Starting from Negative values of the parameters.

http://www.db.cwi.nl/rapporten/abstract.pdf?abstractn١=١١١

[٤] Temme, N.M. (١٩٩٧). "Numerical Algorithms for Uniform Airy-type Asymptotic expansions.

http://www.citesseer.ist.psu.edu/temme٩٧numerical.html.

[٥] Dunster, T.M., paris, R.B., and Cang, S. (١٩٩٨). "On the High-order Coefficients in the Uniform Asymptotic Expansion for the Incomplete gamma function.

http://www.roham.sdsu.edu/~asymptoticexpansion.

[٦] Wolfgang, B. (٢٠٠٠). "An Asymptotic Expansions for a Ratio of Products of Gamma Functions", physikalisches Institute, Universitat Heidelberg, philosophenweg ١٢. ٦٩١٢٠ Heidelberg, Germany.

http://www.univie.ac.at/EmIS/journals/IJMMS/volume.٢٤/SO١٦١١٧/٢٠٠٠١٠٣١٠.pdf.

[٧] Dingle, R. B. (١٩٧٣). "Asymptotic Expansions: Their Derivation and Interpretation" (Academic press, London).

[٨] Paris, R. B. (١٩٩٢). "Smoothing of the stokes phenomenon using Mellinbarnes integrals", J. Comput Appl. Math. ٤١, ١١٧-١٣٣.

[٩] Oliver, F. W. J. (١٩٩٥). "On An Asymptotic Expansion of a Ratio of Gamma Functions", Proc. Royal Irish Acad. A٩٥-٥-٩.

[١٠] Gotze, F., and TiKhamirov, A. N. (٢٠٠١) "Asymptotic Expansions in Non- Central Limit Theorems for Quadratic Forms", university of Bielefeld.

http://www.mathmatik.uni-bielefeld.de.

[١١] Lieberman, O., Rousseau, J., and Zucker, D. M. (٢٠٠٣) "Valid Asymptotic Expansions For the Maximum Liklihood Estimator of the Parameter of a Stationary, Gaussian Strongly Dependent Process.

http://www.nhn.ou.edu/~٢٠٠٣,vol.٣١No.٢,٥٨٦-٦١٢,pdf.

[١٢] Wolfgang, B. (٢٠٠٣) "An Asymptotic Expansions For a Ratio of Products of Gamma Functions", Physikalisches Institute, Universitat Heidelberg, Philosophenweg ١٢. ٦٩١٢٠ Heidelberg, Germany.

http://www.uniric.ac.at/EmIS/journals/IJMMS/volume.٢٤/So١٦١١٧/٢٠٠٣١٠٣١٠.pdf.

[١٣] Dunster, T. M. (٢٠٠٤). "Uniform Asymptotic Expansions for Incomplete Riemam Zeta Functions", Department of Mathematics and Statistics, State University, U.S.A.

http://www.rohan.sdsu.edu/~dunster publications.

[١٤] Simon, J., A.M. (٢٠٠٤). "An Introduction in Asymptotic Analysis".

http://www.ma.hw.ac.uk/~simonm/ae/ae.pdf.

[١٥] Old Daalhuis, A.B. (٢٠٠٥). "Uniform Asymptotic Expansions for Hyper Geometric Functions with Large Parameters.

http://www.math.umn.edu/~focm/c-/Daolhuis.pdf.

[١٦] Silvio, R. (٢٠٠٣). "Asymptotic Theory", Applied Econometric ١, Handout ٣, pp. ١-١٠, University of Western Ontario, London Canada.

http://www.halweb-uc٣m.es/esp/personal/Rendon/econometric١/ps.pdf/.

[١٧] Bunzel, H. "Asymptotic Theory-Handout≠١",

http://www.econ.iastate.edu/classes/econ.٦٧١/Bnzel/Asymptotic.pdf.

[١٨] Wikipedia, (٢٠٠٥) The Free Encyclopedia, "Convergence of Random Variables",

http://en.wikipedia.org/wiki/convergence-of-random-variables.

[١٩] Fahady, K.S. and Shamoon, p. J. (١٩٩٠). "Probability", Ministry of Higher Education and Scientific Research Iraq.

[٢٠] "Chapter ٢ Some Basic Sample Theory". http://www.stat.washington/edu/jaw/COURSES/٥٨٠s/٥٨١/Lectnotes/ch٢/pdf.

[٢١] Wikipedia, (٢٠٠٥). The Free Encyclopedia "Law of Large Numbers".

http://en.wikipedia.org/wiki/law of large numbers.

[٢٢] Boswijk, H.P. "Asymptotic Theory for Integrated Processes",

http://www١.fee.uva.nl/ke/boswijk/book.pdf.

[٢٣] "Chapter ٣. Asymptotic Methods".

http://www.math.ntu.edu.tw/~hehen/teaching/large sample/notes/notprobability.pdf.

[٢٤] Götz, T. "Asymptotic analysis", Technom Athematics Group, University of Kaiserslaut Germany.

http://www.mathematik,unis-k١.de/~goetz/Dounload/asyan٠٥.pdf#search="Expannsion.pdf٪٢٠asymptotic٪٢٠analysis.

[٢٥] Abramowitz, M.and stegun, I.A. (١٩٧٢) "Handbook of Mathematical Function", over, New York.

http://www.convertit.com/go/convertit-reference/ams٥٥.asp?Res=٢٠٠&pag=١٠٣٢.

[٢٦] Arizona, "Principles and Methods of Applied Mathematics", Math٥٨٣B Elementary Asymptotics Watson Lemma, pp.١ ٥.

http://www.cwp.mines.edu/~norm/Ma٥٠٦/part٧/node٥.html.

[٢٧] Young, d.H. and Al-Saadi, S.D. (١٩٨٣). "Statistical Theory and Methods", Ministry of Higher Education and Scientific Research. Iraq.

[٢٨] Hog, R.V. and Craig, A. T. (١٩٧٨). "Introduction to Mathematical Statistics" Macmillan Publishing Co., Inc. New York. Fourth Edition.

[٢٩] Stephenson, G. "Mathematical Methods for Science Students", Empirical College, London, Second Edition.

[٣٠] Jensen, P. A. and Bard J. F. (٢٠٠١) "Probability Models. S٣ Continuous Random Variables",

http://www.me.utexas.ed/~jensen/ORMM/supplement/models/probmodel/continuous/pdf.

[٣١] Gohari, P. (٢٠٠٢). "Chapter ٣: Random Variables and Probability Distribution", Fall,

http://www.ece.concordia-ca/~gohan/engr٣٧١/٣.pdf.

[٣٢] Mood, A. M., Graybill, F. A. and Boes, D. C., (١٩٧٤). "Introduction to the Theory of Statistics", ٣d ed., Mc Graw-Hill, Inc., New York.

[٣٣] Walkk, C. (٢٠٠١). "Hand-Book on Statistical Distribution for Experimentalist Lists, University of Stockholm.

http://www.physto.se/~walck/suf٩٦٠١.pdf.

[٣٤] Henry, S. (١٩٥٩). "The Analysis of Variance". John Wiley and sons, Inc. New York.

[٣٥] "Chi-Square Distribution"

http://www.rikglossary.com/link/chi-squared_distribution.htm.

[٣٦] "Non-Central T and F-Distributions.",

http://www.mcs.une.edu.au/~stat٣٥٤/notes/note٤٨.html.

[٣٧] "Non- Central Distributions Gamma and Beta".

http://www.math.ucla.edu/~xhn/pca-fa.pdf.

[٣٨] Didonato, A. R. and Morris, A. H. (١٩٩٢)" Significant Digit Computation of the Incomplete Beta Function Ratio, A C M Trans. Math. Software ١٨, pp. ٣٦٠-٣٧٣.

[٣٩] Luke, Y. L. (١٩٦٩). "The Special Functions and Their Approximations", Vol.I, Academic Press, New York, MR ٣٩.٣٠٣٩.