الجمهورية العراق
وزارة التعليم عالي والبحث العلمي
جامعة بابـــــــل
كلـــية التربـــية
قســم الرياضيات

حول دالة هينون

رسالة مقدمة إلى
كلية التربية —جامعة بابل
وهي جزء من متطلبات نيل درجة ماجستير
في الرياضيات

من قبل
كاوه احمد حسن

باشراف
د.افتخارمضر طالب

كانون الثانى/٢٠٠٦

بِسْمِ أللَّهِ أَلرَّحْمَٰنِ ألرَّحِيمِ

قُلْ هَلْ يَسْتَوِى أَلَّذِينَ يعلَمُونَ وَأَلَّذِينَ لاَيَعْلَمُونَ
إِنَّمَا يَتَذَكَّرُ أُوْلُواْ أَلبَٰبِ .

الخلاصة

تعتبر النظم الدينمية من المواضع المهمة و الحديثة. درسنا احد الأنظمة الدينمية ذات البعد الثانى الغير خطى والمعروفة باسم دالة هينون والمعرفة باشكل

$$H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - by - x^2 \\ x \end{pmatrix}$$

درسنا بعض الخواص العامة لهذة الدالة مثل النقاط الثابتة و الدورية و انواعها .و لقد حصلنا على النتائج التالية:

١- اذا كان $|b|<1$ و $\frac{-(1+b)^2}{4} < a < \frac{3(1+b)^2}{4}$ فان دالة هينون تملك نقطة ثابتة جاذبة و نقطة ثابتة سرجية.

٢- اذا كان $|b|>1$ و $\frac{-(1+b)^2}{4} < a < \frac{3(1+b)^2}{4}$ فان دالة هينون تملك نقطة ثابتة نافرة و نقطة ثابتة سرجية.

٣- اذا كان $|b|>1$ و $a > \frac{3(1+b)^2}{4}$ فان دالة هينون تملك نقطتين ثابتتين سرجيتين.

و لقد برهنا ان دالة هينون لاتملك نقاط دورية عندما $a=0, -1<b<0$, ومن النتائج المهمة التى اثبتناها في تكرار معكوس دالة هينون اذا كان $b>0$ و $a > \frac{3(1+b)^2}{4}$ و كان R^2 منطقة مغلقة من $S_{a,b} = \{\begin{pmatrix} x \\ y \end{pmatrix} : |x| \le C_{a,b}, |y| \le C_{a,b}\}$ فان كل النقاط خارج المنطقة $S_{a,b}$ في R^2 اما $|x_n| \longrightarrow \infty$ او $|y_{-n}| \longrightarrow \infty$ عندما $n \longrightarrow \infty$

ON HENON MAP

A THESIS

SUBMITTED TO COLLEGE OF EDUCATION UNIVERSTY OF BABLYON IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

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JANUARY /٢٠٠٦ **BAFRANBAR**

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The Supervisor Certification

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To My Family

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Abstract

We study the dynamics of the two dimensional mapping the non linear mapping $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - by - x^2 \\ x \end{pmatrix}$.For this map. We study some general properties of this map, fixed points, periodic points and the type of fixed points till from this method ,we determine bifurcation of this map, we conclude the following:

If $|b| < 1$ and $\frac{-(1+b)^2}{4} < a < \frac{3(1+b)^2}{4}$ then Henon map has an attracting fixed point and a saddle fixed point .

If $|b| > 1$ and $\frac{-(1+b)^2}{4} < a < \frac{3(1+b)^2}{4}$ then Henon map has repelling fixed point and a saddle fixed point .

If $|b| > 1$ and $a > \frac{3(1+b)^2}{4}$ then Henon map has two saddle fixed points .

Also, we proved that there are no periodic points for this map where $a = 0, -1 < b < 0$ and there are no non wandering point where $a < \frac{-(1+b)^2}{4}$ and we proved a theorem on the iteration and inverse of Henon map :

If $S_{a,b} = \{ \begin{pmatrix} x \\ y \end{pmatrix} : |x| \le C_{a,b}, |y| \le C_{a,b} \}$ where $C_{a,b} = \frac{1 + b + \sqrt{(1+b)^2 + 4a}}{2}$, $a > \frac{-(1+b)^2}{4}$ and $b > 0$,then for all $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2 - S_{a,b}$ either $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$ or $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$.”

List of symbols

S	An open set of Euclidean space
Z	The set of Integer numbers
R	The set of real numbers
F	A map
F^{-1}	The inverse of a map F
$DF(v_0)$	The differential of F at v_0
J	Jacobian of F
O^{+}	Set of Forward Orbit
O^{-}	Set of Backward Orbit
$P_{er_n}(F)$	Set of periodic points of period n of F
$\| \ \|$	Norm
$\| \ \|$	The absolute value
GL(٢,٢)	The Set of ٢ × ٢ matrix which determinant is ± ١
$H_{a,b}$	Henon map with two parameters a and b
λ	Eigen value
M,N ,G	parameter spaces
$\langle \ \rangle$	Sequence
S^{o}	Set of interior point of a set S
$\Omega(F)$	Set of non wandering points
J(F)	Julia Set of F
K(F)	filled Julia Set of F
Λ	Non Escape set

List of Figuers

Introduction

About ٣٠ years ago the French astronomer –mathematician Michel-Henon was searched for a simple two-dimensional map possessing special properties of more complicated systems .The result was a family of maps denoted by $H_{a,b}$ given by $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1-ax^2+y \\ bx \end{pmatrix}$

where a,b are real numbers .These maps defined in above are called Henon maps [١٠].

Lorenz (١٩٦٣) has investigated a system of three first order differential equations, whose solutions tend toward a “strange attractor”. In (١٩٧٦) Henon showed that the same properties can be observed in a simple mapping of the plane defined by $x_{i+1}=y_i+1-ax_i^2$,$y=x_i$ .Numerical experiments are carried out for a=١.٤,b=٠.٣, depending on the initial point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$,the sequence of points obtained by iteration of the mapping either diverges to infinity or tends to a strange attractor [١].

In (١٩٧٧) Curry has shown that , for Henon values of the parameters, one of the fixed points has a topologically transverse homoclinic orbit , hence that there is a horseshoe embedded in the dynamics of the map [٤] .

In (١٩٧٨) Feit has shown that , for a>٠ and ٠<b<١ ,that the non wandering set Ω(H) is contained in a compact set ,and that all points outside this set escape to infinity [٤].Another form of Henon map $H_{a,b}:C^2 \longrightarrow C^2$ is defined by the rule $H_{a,b}\begin{pmatrix} z \\ w \end{pmatrix}=\begin{pmatrix} a-bw-z^2 \\ z \end{pmatrix}$.If $a,b \in$R, then $H_{a,b}$ restricts to the real

Henon map $H_{a,b} : R^2 \longrightarrow R^2$ .The importance of Henon's map for the dynamics of all polynomial diffeomorphisms was recognized by S.Friedland and J.Milnor , who proved that every quadratic polynomial diffeomorphism is conjugate to a Henon map or to an elementary maps have trivial dynamics .Henon map is normal form for all quadratic polynomial diffeomorphism with non- trivial dynamic.[١١]

In our work we used parameter space .Historically ,the first result about the parameter space of Henon maps showed the existence of two parabolic regions ,one of them is associated with maps with horseshoes,and consequently infinitely many periodic points ,and the other region corresponds to maps with no periodic points at all ; a “horseshoe” refers to a map which is hyperbolic on its nonwandering set and topologically conjugate .This result was proved in [٤] .The existence of strange attractors with in the Henon family was established in ١٩٩١ by M.Benedicks and L.Carleson .The identified set of parameters (a,b) of positive Lebesque measuer for which $H_{a,b}$ admits a strange attractor .

In this work we, care this form of Henon map $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - by - x^2 \\ x \end{pmatrix}$ [٣] .

The goal of our work is to study bifurcation of Henon map according to appearance of fixed points and periodic points ,a sudden change in the nature of the fixed points with respect to some parameter space .We introduce the iteration of $H_{0,b}$ .To determine the Henon map $H_{0,b}$, $-1 < b < 0$, has no periodic point for any period in the plane by dividing the plane to some region which are shown in chapter three .

Also we study the non wandering point for Henon map $H_{a,b}$ where $a < \frac{-(1+b)^2}{4}$

by using trapping region also we prove one theorem a bout periodic point of Henon map where $a < \frac{-(1+b)^2}{4}$ and $b > 0$,which is given in [٢].At the last section, we will prove one theorem about iteration of $H_{a,b}$ where $b > 0$ and $a > \frac{-(1+b)^2}{4}$ which is given in [٣] prove by finding some regions and prove some necessary lemma for our proof.

We clarified that the Henon map $H_{a,b}$ has saddle node bifurcation occurs in the Henon map at $a = \frac{-(1+b)^2}{4}$ and a periodic doubling bifurcation occurs at $a = \frac{3(1+b)^2}{4}$ by prove some propositions and which are exist in chapter one and shown in Remark (٢.٤.٩)

In chapter three we conclude that there exists an open set about $\begin{pmatrix}0\\0\end{pmatrix}$ in which all points tend to $\begin{pmatrix}0\\0\end{pmatrix}$ under forward iteration of H .We found these regions $Q_1, Q_{2,1}, Q_{2,2}, Q_3, Q_4$ and $R_1, R_2, R_3, R_4, R_5, R_6, R_7$, we prove there are no periodic point for Henon map in the plane .Also, we prove that the norm of iteration of Henon map $H_{0,b}$ tends to infinity for some regions shown in proposition (٣.٣.٢) and (٣.٣.٤).

In chapter four we conclude there is no non wandering point for Henon map $H_{a,b}$, where $b > 0$ and $a < \frac{-(1+b)^2}{4}$ we prove that for a closed region

$S_{a,b} = \{\begin{pmatrix}x\\y\end{pmatrix} : |x| \le C_{a,b}, |y| \le C_{a,b}\}$ if $a > \frac{-(1+b)^2}{4}$ and $b > 0$ then for all $\begin{pmatrix}x\\y\end{pmatrix} \in R^2 - S_{a,b}$ either $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$ or $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$.

CHAPTER ONE

Preliminaries

The purpose of this chapter is to introduce some definitions and theorems necessary for this research. This chapter consists of two sections. In section one,we recall some fundamental definitions and necessary theorems .In section two, we recall the definition of Henon map and some famous forms of Henon map.

١-١ Basic definitions and necessary theorems

Our goal in this section is to list the definitions and theorems that we use later in the research .We start with the general definition of a dynamical system.

<u>Definition (١.١.١)[١٣]</u>

A dynamical system is a map $K\times S \xrightarrow{\phi} S$ where S is an open set of Euclidean space and writing by $\phi\begin{pmatrix} t \\ x \end{pmatrix} = \phi_t(x)$,the map ϕ_t: $S \longrightarrow S$ satisfies

(a) $\phi_0 : S \longrightarrow S$ is the identity ;that is $\phi_0(x)=x$,for all x in S

(b) The composition $\phi_t \circ \phi_s = \phi_{t+s}$, for each t,s in K .In case K is Z the dynamical system is described to be discrete dynamical system.

In case K is real line the dynamical system is described to be continuous.

Example (١.١.٢)[١٥]

ϕ_t: R ⟶ R such that $\phi_t(x) = xe^{kt}$, t,x in R.

Example (١.١.٣)[١٥]

ϕ_t: R^٢ ⟶ R^٢ such that $\phi_t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xe^{-t} \\ (xt+y)e^{-t} \end{pmatrix}$, t in R, $\begin{pmatrix} x \\ y \end{pmatrix}$ in R^2

Definition (١.١.٤)[١٠]

Let V be a subset of R^٢, and let F: V ⟶ R^٢. Frequently such a function is called a map. The function F can always be represented in the form F(v)=$\begin{bmatrix} f(v) \\ g(v) \end{bmatrix}$ for all v in V where f and g are real –valued coordinate map of F.

Remark (١.١.٥)[١٠]

In our work, we write the map F: V ⟶ R^٢ where V $\subseteq$ R^٢, such that

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} f\begin{pmatrix} x \\ y \end{pmatrix} \\ g\begin{pmatrix} x \\ y \end{pmatrix} \end{bmatrix}; \begin{pmatrix} x \\ y \end{pmatrix} \in V.$$

Example (١.١.٦)[١٠]

If F$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ a\sin(x)+by \end{pmatrix}$ then V=R^٢, $f\begin{pmatrix} x \\ y \end{pmatrix} = y$ and $g\begin{pmatrix} x \\ y \end{pmatrix} = a\sin(x)+by$, where a, b are two fixed real numbers, x and y in R.

Definition (١.١.٧)[١٠]

The map $F:R^2 \longrightarrow R^٢$ is a **linear map** if for all v and w in $R^٢$ $F(bv+cw)=bF(v)+cF(w)$,for all real numbers b and c. Otherwise it is a **non linear** map.

Definition (١.١.٨)[٣]

A map $F: R^٢ \longrightarrow R^٢$ is **$C^١$,** if all of its first partial derivatives exist and are continuous. F is **C^∞**, if its mixed k^{th} partial derivatives exist and are continuous for all $k \in Z^+$.

Example (١.١.٩)[٧]

A map $F: R^٢ \longrightarrow R^٢$ defined by, $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + \frac{1}{4} \\ 4x - x^2 \end{pmatrix}$ is C^∞.

Definition (١.١.١٠)[٢٥]

A map $F: R^٢ \longrightarrow R^٢$ is called a **diffeomorphism** provided it is

(١) one-to-one

(٢) onto

(٣) C^∞

(٤) Its inverse $F^{-١}: R^٢ \longrightarrow R^٢$ is C^∞.

Definition (١.١.١١) [١٠]

Let V be a subset of $R^٢$, and $v_٠$ be any element in $R^٢$.Consider $F:V \longrightarrow R^٢$ be a map. Further more assume that the first partials of the coordinate maps f and g of F exist at $v_٠$.

The **differential of F** at v. is the linear map **DF(v.)** defined on R٢ by

$$\mathbf{DF(v_{\cdot})} = \begin{pmatrix} \frac{\partial f}{\partial x}(v_0) & \frac{\partial f}{\partial y}(v_0) \\ \\ \frac{\partial g}{\partial x}(v_0) & \frac{\partial g}{\partial y}(v_0) \end{pmatrix}$$

, for all v in R٢ .The determinant of DF(v.) is called the **Jacobian of F** at v_0 and is denoted by J=det DF(v_0) .

Example (١.١.١٢)[١٠]

Let F: R٢ $\longrightarrow$ R٢ be given by F$\begin{pmatrix} x \\ y \end{pmatrix}$ = $\begin{pmatrix} y \\ a\sin(x)+by \end{pmatrix}$.To find DF$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ we have for $f = y$ and $g = a\sin(x)+by$ that $\frac{\partial f}{\partial x}$ =٠, $\frac{\partial f}{\partial y}$ =١, $\frac{\partial g}{\partial x}$ = $a\cos(x)$, $\frac{\partial g}{\partial y}$ =b, so DF$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$=$\begin{pmatrix} 0 & 1 \\ a\cos(x_0) & b \end{pmatrix}$ for all $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in R٢, J=$-a\cos(x_0)$.

Definition (١.١.١٣)[١٠]

Let F: R٢ $\longrightarrow$R٢ be a map and $v_0 \in R^2$ if $0 \le |\det DF(v_0)|$ <١, then F is said to be **area-contracting at** v_0, and if $|\det DF(v_0)|$ >١, then F is said to be **area-expanding at** v_0.

Example (١.١.١٤) [١٠]

A map F in example **(١.١.١٢)** is an area-contracting at $\begin{pmatrix} \frac{\pi}{3} \\ 4 \end{pmatrix}$ if $|a|$<٢ and an area-expanding at $\begin{pmatrix} \frac{\pi}{3} \\ 4 \end{pmatrix}$ if $|a|$>٢ .

Definition (١.١.١٥)[١٠]

The forward orbit of a vector $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is the set of points $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $F\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $F^2\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ $F^3\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$,…and denoted by $O^+\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.If F a homeomorphism ,we may define the full orbit of $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $O\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$,as the set of points $F^n\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ for $n \in Z$,and the backward orbit of $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $O^-\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, as the set of points $F^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $F^{-2}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $F^{-3}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$,…,where we have $F^n\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = F^{n-1}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$.

Definition (١.١.١٦)[٢٤]

Any pair $\begin{pmatrix} p \\ q \end{pmatrix}$ for which $f\begin{pmatrix} p \\ q \end{pmatrix} = p$, $g\begin{pmatrix} p \\ q \end{pmatrix} = q$ (١.١)

is called **a fixed point** of the two dimensional dynamical system.

Example (١.١.١٧)[٧]

Let F: R^٢ $\longrightarrow$ R^٢ is given by $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y \\ x + y - 2 \end{pmatrix}$, then $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ is a unique fixed point of F .

Remark (١.١.١٨)[٢٤]

If we define two maps F, G from R^2 to R by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = x - f\begin{pmatrix} x \\ y \end{pmatrix}, \quad G\begin{pmatrix} x \\ y \end{pmatrix} = y - g\begin{pmatrix} x \\ y \end{pmatrix} \qquad (١.٢)$$

In terms of these maps, the fixed point in equation (١.١) is simply requirement that $F\begin{pmatrix} x \\ y \end{pmatrix}$ and $G\begin{pmatrix} x \\ y \end{pmatrix}$ be simultaneously zero .Fixed points may therefore be located as points of intersection of curves in the plane.

Definition (١.١.١٩)[٣]

Let F: $R^n \longrightarrow R^n$ be a map, $x_\cdot \in R^n$.The point $x_\cdot$ is a **periodic point** of **period** m if $F^m(x_\cdot) = x_\cdot$.The least positive integer m for which $F^m(x_\cdot) = x_\cdot$ is called the prime period of $x_\cdot$ .We denote the set of periodic points of F by P_{er_m} (F).

Example (١.١.٢٠)

Let F: $R^٢ \longrightarrow R^٢$ is given by $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$, then $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a periodic point of period ٢ of F .

Definition (١.١.٢١)[١٠]

Let F: $R^n \longrightarrow R^n$ be a map, $x_\cdot$ be in the domain of F. Then $x_\cdot$ is an **eventually fixed point** of F if there is a positive integer m such that $F^m(x_\cdot)$ is a fixed point of F.

Example (١.١.٢٢)

Let F: $R^٢ \longrightarrow R^٢$ is given by $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y \\ x + y \end{pmatrix}$, then $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eventually fixed point of F, $m = 1$.

Definition (١.١.٢٣)[١٩]

Let **A** be an $n \times n$ matrix **.**The real number λ is called **eigenvalue** of **A** if there exists a non zero vector X in R^n such that

$$\mathbf{AX} = \lambda \mathbf{X} \quad (١.٣)$$

Every non zero vector X satisfying (١.٣) is called **an eigen vector** of A **associated with the eigenvalue** λ .

Remark (١.١.٢٤)[١٠]

Before we show the type of fixed point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ we need to indicate distance on $R^٢$ .To this end ,let $v=\begin{pmatrix} x \\ y \end{pmatrix}$,$w=\begin{pmatrix} r \\ s \end{pmatrix}$.And let the distance $\|v-w\|$ between v and w be the distance between the corresponding points in $R^٢$,that is

$$\|v-w\| = \sqrt{(r-x)^2 + (s-y)^2} \quad (١.٤)$$

Definition (١.١.٢٥)[١٠]

Let $\begin{pmatrix} p \\ q \end{pmatrix}$ be a fixed point of F, then $\begin{pmatrix} p \\ q \end{pmatrix}$ is **attracting** fixed point if and only if there is a disk centered at $\begin{pmatrix} p \\ q \end{pmatrix}$ such that $F^{(n)}\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} p \\ q \end{pmatrix}$ for every $\begin{pmatrix} x \\ y \end{pmatrix}$ in the disk as $n \longrightarrow \infty$.By contrast $\begin{pmatrix} p \\ q \end{pmatrix}$ is **repelling** fixed point if and only if there is a disk centered at $\begin{pmatrix} p \\ q \end{pmatrix}$ such that $\left\| F\begin{pmatrix} u \\ v \end{pmatrix} - F\begin{pmatrix} p \\ q \end{pmatrix} \right\| > \left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} p \\ q \end{pmatrix} \right\|$ for every $\begin{pmatrix} u \\ v \end{pmatrix}$ in the disk for which $\begin{pmatrix} u \\ v \end{pmatrix} \neq \begin{pmatrix} p \\ q \end{pmatrix}$.

<u>Theorem (١.١.٢٦)[١٠]</u>

Let $\begin{pmatrix} p \\ q \end{pmatrix}$ be a fixed point of F. Assume that DF$\begin{pmatrix} p \\ q \end{pmatrix}$ exists, with eigen values λ_1 , λ_2, then :

(١) $\begin{pmatrix} p \\ q \end{pmatrix}$ is an attracting fixed point, if λ_1 and λ_2 are less than one in absolute value .

(٢) $\begin{pmatrix} p \\ q \end{pmatrix}$ is repelling fixed point, if λ_1 and λ_2 are greater than one in absolute value .

(٣) $\begin{pmatrix} p \\ q \end{pmatrix}$ is saddle point, if one of λ_1 , λ_2 is larger and the other is less than one in absolute value.

For proof see [١٠].

<u>Example (١.١.٢٧)[١٠]</u>

Let F$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ a\sin(x) - y \end{pmatrix}$ then

(١) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an attracting fixed point if $\frac{-1}{4} < a <$ ٠,

(٢) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a repelling fixed point if $a >$ ٢,

(٣) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is saddle fixed point if ٠ $< a <$ ٢.

<u>Definition (١.١.٢٨)[٣]</u>

Let F: $R^n \longrightarrow R^n$ be a map. The fixed point $\begin{pmatrix} p \\ q \end{pmatrix}$ is called **hyperbolic fixed point of F** if DF$\begin{pmatrix} p \\ q \end{pmatrix}$ has no eigen values on the unit circle, otherwise is said to be non hyperbolic .

For example $\begin{pmatrix}0\\0\end{pmatrix}$ in example (١.١.٢٦) is hyperbolic, if a$\neq$ ٢ and a$\neq$ ٠.

Definition (١.١.٢٩)[٢٢]

Let GL(٢,٢) be the set of all ٢$\times$٢ matrices, A=$\begin{pmatrix}a & b\\c & d\end{pmatrix}$ where a,b,c,d $\in$Z such that Det(A)= $\pm$ ١.Then if λ_1 , λ_2 are eigen values of A satisfying A$\in$GL(٢,٢) such that $|\lambda_1|$ >١> $|\lambda_2|$,then we call the matrix A, a **hyperbolic matrix .**

Definition (١.١.٣٠)[٣]

Let V,S$\subseteq R^2$,F: V$\longrightarrow$V and G:S$\longrightarrow$S be two maps .Then F and G are said to be **topologically conjugate** if there exist a homeomorphism H:V$\longrightarrow$S such that H$\circ$F=G$\circ$H .

Remark (١.١.٣١)[١٠]

The homeomorphism H is called **a topological conjugacy** .

Example (١.١.٣٢)[١٥]

Let Q$_4(x)$=٤ x(١-x) for ٠$\leq$x$\leq$١ and T(x)=$\begin{cases} 2x \quad for \quad 0 \leq x \leq \frac{1}{2} \\ 2(1-x) \quad for \quad \frac{1}{2} < x \leq 1 \end{cases}$.

Q$_4$ is topologically conjugate to T(x). Where topological conjugacy is $\sin^2 \frac{\pi}{2} x$.

Definition (١.١.٣٣)[١٠]

Let $\begin{pmatrix}p\\q\end{pmatrix}$ be a fixed point of F .The **basin of attraction** of $\begin{pmatrix}p\\q\end{pmatrix}$ consists of all $\begin{pmatrix}x\\y\end{pmatrix}$ such that F$^n\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}p\\q\end{pmatrix}$ as $n \longrightarrow \infty$.

Theorem (١.١.٣٤)[٣]

Suppose that F has an attracting fixed point at $\begin{pmatrix} p \\ q \end{pmatrix}$. Then there is an open set about $\begin{pmatrix} p \\ q \end{pmatrix}$, in which all points tend to $\begin{pmatrix} p \\ q \end{pmatrix}$ under forward iteration of F.

For proof see [٣].

١-٢ Definition and some forms of Henon map

In this section, we start by recalling a definition for Henon map, we give some forms of it, and we will study the two parameter case.

Henon map is a simple two-dimensional map and special quadratic polynomial diffeomorphism of R٢ which may be written in the form $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - by - x^2 \\ x \end{pmatrix}$

where (a,b)$\in$R$\times$(-١,١)\{٠}.

This map was introduced by French astronomer-mathematician Michel Henon in ١٩٧٦, primarily for experimental purposes .

Another form of Henon map is an analogue of the logistic equation . It is defined by the equations .

$$x_{n+1} = 1 - ax_n^{\;2} + y_n$$

$$y_{n+1} = bx_n \qquad (١.٥)$$

where a and b are real constants ; a controls the extent of the nonlinearity, while b controls the degree of dissipation .Note that ,unlike the logistic map ,the henon map is invertible ,while noninvertibility is necessary for chaos in one dimensional maps , it is not required in higher-dimensions . Generally, the sequence of points $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \ldots$ for $i = 0,1,2,\ldots$ either diverges to infinity (for x_0 large) or settles onto an attractor (for $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ near the origin). A fixed point analysis similar to the one performed earlier for the logistic equation may be carried out here to determine the behavior of the map as a map of a and b . For example the two fixed points are found to be $x_{\mp}^{*} = \frac{1}{2a}[-(1-b) \pm \sqrt{(1-b)^2 + 4a}]$ where $a \neq 0$, $y_{\mp}^{*} = bx_{\mp}^{*}$ [١٠]. The point $\begin{pmatrix} x_{-}^{*} \\ y_{-}^{*} \end{pmatrix}$ is always unstable and $\begin{pmatrix} x_{+}^{*} \\ y_{+}^{*} \end{pmatrix}$ becomes repelling for $a > \frac{3(1-b)^2}{4}$ .And for $\frac{3}{10}$ both fixed points become repelling for $a = 0.3675$ and a two-cycle is born. Now let us list some forms of Henon map from

[١],[٥],[٦],[٨],[١١],[١٥],[١٦],[١٧],[١٨] .

(١) $H_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} a - x^2 + by \\ x \end{pmatrix}$

(٢) $H_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 1 + y - ax^2 \\ bx \end{pmatrix}$

(٣) $H_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} a - y^2 - bx \\ x \end{pmatrix}$

(٤) $H_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ y^2 + b + ax \end{pmatrix}$

(٥) $H_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ a - bx - y^2 \end{pmatrix}$

(٦) $H_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 1 + y + bx^2 \\ ax \end{pmatrix}$

In our research we take the form $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} a-by-x^2 \\ x \end{pmatrix}$, $H_{a,b}$ depends on two real parameters, where $|b|<1$ .We note that there is only one nonlinear term ,so that $H_{a,b}$ is indeed one of the simplest nonlinear maps in higher dimensions .

Definition (١.٢.١) [٧],[١٠]

A bifurcation is a sudden change in the number or nature of the fixed and periodic points of the system .Fixed points may appear or disappear, change their stability, or even break a part into periodic points.

If $a \neq 0, a \geq \frac{-(1-b)^2}{4}$ Henon map in (١.٥) has two fixed points $\begin{pmatrix} x_-^* \\ y_-^* \end{pmatrix}$, $\begin{pmatrix} x_+^* \\ y_+^* \end{pmatrix}$ one of them is attracting fixed point where a is a nonzero number lying in the interval [$\frac{-(1-b)^2}{4}, \frac{3(1-b)^2}{4}$],and the second fixed point is a saddle point where $a > \frac{-(1-b)^2}{4}$, $a \neq 0$ .Furthermore, we have the following situation for a given value b in $(0,1)$.If $a < \frac{-(1-b)^2}{4}$,then (١.٥)has no fixed points, if a is nonzero number , $\frac{-(1-b)^2}{4} < a < \frac{3(1-b)^2}{4}$,then (١.٥) has two fixed points one is attractor and the other is a saddle point .Now let b be fixed in the interval $(0,1)$,and let the parameter a increase .In addition to the bifurcation at $a = \frac{-(1-b)^2}{4}$, (١.٥) has a bifurcation at $a = \frac{3(1-b)^2}{4}$,because one of the two eigen values of Henon map defined in (١.٥) descends through -1,so that $\begin{pmatrix} x_-^* \\ y_-^* \end{pmatrix}$ is transformed from an attracting fixed point to a saddle point ,we might suspect that as a passes through $\frac{3(1-b)^2}{4}$, an attracting periodic point of period m for (١.٥) would be born .

CHAPTER TWO

A characterization of the Henon map in parameter spaces

The goal of this chapter is to study type of fixed points in different parameter spaces and the existence of periodic point of Henon map with respect to parameter spaces. This chapter consists of four sections. In section one, we give some properties of Henon map. In section two we determine type of two fixed points where $|b|<1$, $\frac{-(b+1)^2}{4}<a<\frac{3(b+1)^2}{4}$.In section three, we determine type of fixed points where $|b|<1$,$\frac{-(b+1)^2}{4}<a<\frac{3(b+1)^2}{4}$. In section four, we determine type of fixed points where first $|b|>1$ and $a>\frac{3(b+1)^2}{4}$ and second $b=-1$ and $a>0$.

٢.١ Some properties of Henon map

In this section is to study some properties of Henon map. First of all, let us observe how the Henon map acts . For our illustration , we will take a rectangle .The first thing that we can observe that $T_1:\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ is contracting $T_2:\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ a-y \end{pmatrix}$ is translation.

There is a folding $T_3: \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y - x^2 \end{pmatrix}$, a rotation in y by $\frac{-\pi}{2}$

$T_4: \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ and a reflection in y, $T_5: \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Let us observe the orbit of a typical point $\begin{pmatrix} x \\ y \end{pmatrix}$ under the composition $T_5 \circ T_4 \circ T_3 \circ T_2 \circ T_1$, if T_i (x)=x_i, $i = 1,\ldots,5$,then we get x_1=x, $y_1 = b$ y, $x_2 = x_1$, $y_2 = a - y_1$, $x_3 = x_2$, $y_3 = y_2 - x_2^2$, $x_4 = y_3$, $y_4 = -x_3$ $x_5 = x_4$, $y_5 = -y_4$ then for

$T_5 \circ T_4 \circ T_3 \circ T_2 \circ T_1 \begin{pmatrix} x \\ y \end{pmatrix}$

the first coordinate is :	the second coordinate is:
$x_5 = x_4$	$y_5 = -y_4$
$= y_3$	$= x_3$
$= y_2 - x_2^2$	$= x_2$
$= a - y_1 - x_1^2$	$= x_1$
$= a - b\,y - x^2$	$= x$

therefore $T_5 \circ T_4 \circ T_3 \circ T_2 \circ T_1 \begin{pmatrix} x \\ y \end{pmatrix} = H_{a,b} \begin{pmatrix} x \\ y \end{pmatrix}$.The Henon map can be realized as a composition of contracting, translation, folding, rotation and reflection. These properties of stretching and folding are responsible for the beautiful dynamics of Henon map .

Proposition (٢.١.١)[٣],[١٢]

Let H: $R^٢ \longrightarrow R^٢$ be a Henon map and b be any fixed real number. Then

(١) $JH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = b$, $\forall$ x, y in R.

(٢) If $x^{٢}$-b≥ ٠, then the eigen values of $DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ are the real numbers

$-x \pm \sqrt{x^2 - b}$.

(٣) If b≠ ٠ , $H_{a,b}$ is one-to-one map

(٤) The Henon map $H_{a,b}$ is C^{∞}.

(٥) If b≠ ٠ then $H_{a,b}$ has an inverse.

(٦) If b≠ ٠ $H_{a,b}$ is diffeomorphism .

proof :(١) from definition (١.١.١١), $DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x & -b \\ 1 & 0 \end{pmatrix}$, then

$JH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \det \begin{pmatrix} -2x & -b \\ 1 & 0 \end{pmatrix} = b$

(٢) If λ is eigen value of $DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ then must be satisfied the characteristic equation $\lambda^2 + 2x\lambda + b = 0$ and the solutions of this equation are $\lambda_{1,2}$ where $\lambda_{1,2} = \frac{-2x \pm \sqrt{4x^2 - 4b}}{2} = -x \pm \sqrt{x^2 - b}$ which are real since $x^2 - b \geq 0$.

(٣) Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in R^{٢}$ such that $H_{a,b}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = H_{a,b}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ then

$\begin{pmatrix} a - by_1 - x_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} a - by_2 - x_2 \\ x_2 \end{pmatrix}$ then $a-by_1-x_1 = a-by_2-x_2$ and $x_1 = x_2$,

hence $by_1 = by_2$ and b≠ ٠ so $y_1 = y_2$, that is , $H_{a,b}$ one-to-one.

(٤) $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - by - x^2 \\ x \end{pmatrix}$ then $\frac{\partial f(x,y)}{\partial x}$=-٢x , $\frac{\partial f(x,y)}{\partial y}$=-b , $\frac{\partial g(x,y)}{\partial x}$=١,

$\frac{\partial g(x,y)}{\partial y}$=٠, so all first partial derivatives exist and continuous. Note that $\frac{\partial^2 f(x,y)}{\partial x^2}$=-٢, $\frac{\partial^n f(x,y)}{\partial x^n}$=٠ $\forall\ n \in$ N and $n \geq$ ٣, $\frac{\partial^n f(x,y)}{\partial y^n}$=٠, $\frac{\partial^n g(x,y)}{\partial y^n}$=٠ and $\frac{\partial^{n-1} g(x,y)}{\partial y^{n-1}}$=٠ $\forall\ n \in$ N and $n \geq$ ٢ .

We get that all its mixed k^{th} partial derivatives exist and continuous for all k. From definition (١.١.٨) $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is C^∞ .

(٥) We define the following two dimensional maps H_1, H_2 and H_3 where $H_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ by \end{pmatrix}$, $H_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$,$H_3\begin{pmatrix} a + x - y^2 \\ y \end{pmatrix}$,the Henon map is composed of H_1, H_2, H_3 and these maps are invertible maps so $H_{a,b}$ is also invertible map and $H_{a,b}^{-1} = H_1^{-1} \circ H_2^{-1} \circ H_3^{-1}$ where $H_1^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \frac{y}{b} \end{pmatrix}$,$H_2^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$, $H_3^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - a + y^2 \\ y \end{pmatrix}$,$H_1^{-1} \circ H_2^{-1} \circ H_3^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \frac{a - x - y^2}{b} \end{pmatrix}$, hence $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is

invertible where $H_{a,b}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \frac{a - x - y^2}{b} \end{pmatrix}$.

(٦) By part (٣) and part (٤) of this proposition $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is one-to-one map and C^∞.

To show it is onto let $\begin{pmatrix} z \\ w \end{pmatrix}$ be any element in R٢ then there exist x=w, y=$\frac{a-w-z^2}{b}$ in R such that $\begin{pmatrix} x \\ y \end{pmatrix}$ in R٢ and $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} z \\ w \end{pmatrix}$ hence $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is onto, and by part(٥) $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is C^{∞} and $H_{a,b}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$ is C^{∞}.Hence by definition (١.١.١٠) $H_{a,b}$ is diffeomorphism. □

Proposition (٢.١.٢)

(١) $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in$(Z\{-١,٠,١})×R, $DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is a hyperbolic matrix iff $|b|=$١.

(٢)If $|b|<$١ then $H_{a,b}$ is area-contracting and its area-expanding if $|b|>$١.

Proof (١) ($\longrightarrow$) Let $DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ be a hyperbolic matrix then in view of definition(١.١.٢٩) $DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} \in$GL(٢,٢) then det($DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$) = b =± ١, hence $|b|=$١.

($\longleftarrow$) Let $|b|=$١ then det($DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$) = b =± ١ , $DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} \in$GL(٢,٢) and by the relation between roots and coefficients $\lambda_1 \lambda_2 =\pm$ ١ so $|\lambda_2| = \frac{1}{|\lambda_1|}$ (٢.١)

and by proposition (٢.١.١) λ_1, λ_2 are two distinct real numbers and since x $\notin$ {-١,٠,١} , $|\lambda_i| \neq$ ١ , $\forall\ i =$ ١,٢ and since R is totally order set so either $|\lambda_i|>$١

or $|\lambda_i|<$١ $\forall\ i =$١,٢ , if $|\lambda_1|>$١ from (٢.١) $|\lambda_2| = \frac{1}{|\lambda_1|}$ <١and if $|\lambda_1|<$١ then

$|\lambda_2| = \frac{1}{|\lambda_1|}>$١.

(٢) If $|b|<$١ then $\left|\det(DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix})\right|<$١ and from definition (١.١.١٣) $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is area-contracting map and if $|b|>$١ then $\left|\det(DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix})\right|>$١ this implies that $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is an area-expanding map.

Proposition (٢.١.٣)[٣]

For each value of b in R, there exists b_0 in $R^- \cup \{٠\}$ such that, if $a > b_0$ the Henon map has two fixed points.

Proof: Let b$\in$R, the point $\begin{pmatrix} p \\ q \end{pmatrix}$ is a fixed point of $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ provided that $\begin{pmatrix} p \\ q \end{pmatrix}$=$H_{a,b}\begin{pmatrix} p \\ q \end{pmatrix}$=$\begin{pmatrix} a-bq-p^2 \\ p \end{pmatrix}$. Then $p=q$ and $p=a-bq-p^2$, implies that $p=a-bp-p^2$.This is equivalent to $p^2+(b+1)p-a=0$ which by the quadratic formula yields $p=\dfrac{-(b+1)\pm\sqrt{(b+1)^2+4a}}{2}$ (٢.٢)

such p exists if $(b+1)^2+4a \geq 0$, that is if a>$\dfrac{-(b+1)^2}{4}$, we take $b_0 = \dfrac{-(b+1)^2}{4}$ in $R^- \cup \{٠\}$ and since $p=q$ if a> b_0 $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ has two fixed points

$$P_+=\begin{pmatrix} p_+(a,b) \\ p_+(a,b) \end{pmatrix}=\begin{pmatrix} \frac{-(b+1)+\sqrt{(b+1)^2+4a}}{2} \\ \frac{-(b+1)+\sqrt{(b+1)^2+4a}}{2} \end{pmatrix} \qquad (٢.٣)$$

$$P_-=\begin{pmatrix} p_-(a,b) \\ p_-(a,b) \end{pmatrix}=\begin{pmatrix} \frac{-(b+1)-\sqrt{(b+1)^2+4a}}{2} \\ \frac{-(b+1)-\sqrt{(b+1)^2+4a}}{2} \end{pmatrix} \quad \square \qquad (٢.٤)$$

Proposition (٢.١.٤)

The set of fixed points of Henon map is closed .

Proof: Let A be the set of fixed points of Henon map then

A={$\begin{pmatrix} x \\ y \end{pmatrix}$:$H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$},A$\subset R^2$.To show that A is closed set , let$\begin{pmatrix} x \\ y \end{pmatrix}\in A^c$then $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}\neq\begin{pmatrix} x \\ y \end{pmatrix}$ and since A$\subset R^2$,we have two distinct elements in R^2 and R^2 is hausdorf space then there exists two disjoint open sets M,N in R^2 such that $\begin{pmatrix} x \\ y \end{pmatrix}\in$M and $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}\in$ N ,hence $\begin{pmatrix} x \\ y \end{pmatrix}\in M\cap H_{a,b}^{-1}(N)$,since N open subset in R^2 and $H_{a,b}$ is continuous map we have $H_{a,b}^{-1}(N)$ is open subset in R^2.Let $M\cap H_{a,b}^{-1}(N)$ =V, we claim that V$\subset A^c$. To show this, let $\begin{pmatrix} r \\ s \end{pmatrix}\in$V then $\begin{pmatrix} r \\ s \end{pmatrix}\in$M and $\begin{pmatrix} r \\ s \end{pmatrix}\in H_{a,b}^{-1}(N)$ so$\begin{pmatrix} r \\ s \end{pmatrix}\in$M, $H_{a,b}\begin{pmatrix} r \\ s \end{pmatrix}\in$N but since $M\cap N=\phi$,then $\begin{pmatrix} r \\ s \end{pmatrix}\neq H_{a,b}\begin{pmatrix} r \\ s \end{pmatrix}$ hence $\begin{pmatrix} r \\ s \end{pmatrix}\in A^c$.That is our claim is true. Hence for each $\begin{pmatrix} x \\ y \end{pmatrix}$in A^c we could find the open set V such that V$\subset A^c$ so A^c is open. Then A is closed set. □

Proposition (٢.١.٥)[٣]

For each value of b in R there exists a unique b_0 in $R^- \cup \{٠\}$ such that if $a = b_0$ the Henon map $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ has a unique fixed point $\begin{pmatrix} \frac{-(b+1)}{2} \\ \frac{-(b+1)}{2} \end{pmatrix}$ and if $a < b_0$, $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ has no fixed point.

Proof: Let (a,b) be any element in R^2, Clearly $b_0 = \frac{-(b+1)^2}{4} \in R^- \cup \{٠\}$. If $a = b_0$ then $(b+1)^2 + 4a = 0$, from (٢.٣), (٢.٤) $P_+ = P_- = \begin{pmatrix} \frac{-(b+1)}{2} \\ \frac{-(b+1)}{2} \end{pmatrix}$ is a unique fixed point of $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ but if $a < b_0$ we get that $(b+1)^2 + 4a < 0$ and the equation $p^2 + (b+1)p - a = 0$ has no real solution, so $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ has no fixed point in this case . □

٢.٢ A characterization of the Henon map for

$$|b|<1\ ,\ \frac{-(b+1)^2}{4}<a<\frac{3(b+1)^2}{4}$$

The goal of this section is to determine a type of fixed points of Henon map $H_{a,b}$ where $(a,b)\in$ M and M is a parameter space defined as:

$$M=\{(a,b):|b|<1,\frac{-(b+1)^2}{4}<a<\frac{3(b+1)^2}{4},a,b\in R\}\ ,\quad H_M=\{H_{a,b}:(a,b)\in M\}.$$

Proposition (٢.٢.١)[٣]

Let $H_{a,b}\in H_M$. If $p_{\pm}^{2}(a,b)-b\geq 0$ then $H_{a,b}$ has an attracting fixed point $\begin{pmatrix}p_+(a,b)\\p_+(a,b)\end{pmatrix}$, and a saddle fixed point $\begin{pmatrix}p_-(a,b)\\p_-(a,b)\end{pmatrix}$.

Proof: By theorem (١.١.٢٦) we get $\begin{pmatrix}p_+(a,b)\\p_+(a,b)\end{pmatrix}$ is attracting fixed point if the eigenvalues of $H_{a,b}\begin{pmatrix}p_+(a,b)\\p_+(a,b)\end{pmatrix}$ are less than ١ in absolute value . Let $H_{(a,b)}\begin{pmatrix}x\\y\end{pmatrix}\in H_M$ so $|b|<$١ and $\frac{-(b+1)^2}{4}<a<\frac{3(b+1)^2}{4}$ so $0<(b+1)^2+4a<4(b+1)^2$,

thus $0<\sqrt{(b+1)^2+4a}<2(b+1)$ (٢.٥)

by adding $-(b+1)$ for both sides of inequality (٢.٥) ,we get that $-(b+1)<-(b+1)+\sqrt{(b+1)^2+4a}<(b+1)$ then

$-2 < -(b+1) < -(b+1) + \sqrt{(b+1)^2 + 4a} < (b+1) < 2$, so

$-1 < \dfrac{-(b+1) + \sqrt{(b+1)^2 + 4a}}{2} < 1$, from (٢.٣), we get $-1 < p_+(a,b) <$ ١.From now for simply we refer to $p_+(a,b)$ as p_+, hence $p_+ + 1 > 0$ and $p_+ - 1 < 0$ so

$|p_+ + 1| = p_+ + 1, |p_+ - 1| = 1 - p_+$ (٢.٦)

by proposition (٢.١.١) , the eigenvalues of $DH_{a,b}\begin{pmatrix} p_+ \\ p_+ \end{pmatrix}$ are less than ١ in absolute value if $\left|-p_+ \pm \sqrt{p_+^{\ 2} - b}\right| < 1$.

Now from (٢.٣), $2\ p_+ = -(b+1) + \sqrt{(b+1)^2 + 4a}$ .Therefore $2\ p_+ > -(b+1)$

or equivalently, $2\ p_+ + 1 > -b$ and by adding $p_+^{\ 2}$ for both sides, we get that

$p_+^{\ 2} + 2\ p_+ + 1 > p_+^{\ 2} - b$, thus $(p_+ + 1)^2 > p_+^{\ 2} - b$, so $\sqrt{(p_+ + 1)^2} > \sqrt{p_+^{\ 2} - b}$

that is $|p_+ + 1| > \sqrt{p_+^{\ 2} - b}$, then by (٢.٦) $p_+ + 1 > \sqrt{p_+^{\ 2} - b}$. Hence

$-p_+ + \sqrt{p_+^{\ 2} - b} < 1$ (٢.٧)

Since $p_+ < 1$ and $\sqrt{p_+^{\ 2} - b} \geq 0$, we get $-p_+ + \sqrt{p_+^{\ 2} - b} > -1$. (٢.٨)

Now by (٢.٧) and (٢.٨) $\left|-p_+ + \sqrt{p_+^{\ 2} - b}\right| < 1$ so $|\lambda_1| < 1$, for $|\lambda_2|$ since $p_+ > -1$

and $\sqrt{p_+^{\ 2} - b} \geq 0$, we get $p_+ + \sqrt{p_+^{\ 2} - b} > -1$. (٢.٩)

Also we have $a < \frac{3(b+1)^2}{4}$, so $\sqrt{(b+1)^2+4a} < 2|b+1| = 2(b+1)$, hence

$\frac{-(b+1)+\sqrt{(b+1)^2+4a}}{2} < \frac{(b+1)}{2}$ by (٢.٣) $p_+ < \frac{(b+1)}{2}$, that is $-b < 1-2p_+$ (٢.١٠)

By adding $p_+{}^2$ for both sides of (٢.١٠) ,we get $p_+{}^2 - b < p_+{}^2 + 1 - 2p_+$,

then by (٢.٦), $\sqrt{p_+{}^2 - b} < |p_+ - 1|$ so $\sqrt{p_+{}^2 - b} < 1 - p_+$. Hence

$$p_+ + \sqrt{p_+{}^2 - b} < 1 \qquad (٢.١١)$$

Now from (٢.٨) and (٢.١١), $\left|p_+ + \sqrt{p_+{}^2 - b}\right| < 1$, so $|\lambda_2| < 1$,so by proposition (١.١.٢٦) $\begin{pmatrix} p_+ \\ p_+ \end{pmatrix}$ is attractor fixed point.

For the second part ,from (٢.٤) $p_-(a,b) = \frac{-(b+1)-\sqrt{(b+1)^2+4a}}{2}$. From now for simply we refer to $p_-(a,b)$ as p_-. Hence $2p_- = -(b+1) - \sqrt{(b+1)^2+4a}$

so $2p_- < -(b+1)$. Hence $2p_- + 1 < -b$. (٢.١٢)

By adding $p_-{}^2$ for both sides of (٢.١٢) we get $p_-{}^2 + 2p_- + 1 < p_-{}^2 - b$

so $\sqrt{(p_- + 1)^2} < \sqrt{p_-{}^2 - b}$. Hence $|p_- + 1| < \sqrt{p_-{}^2 - b}$. (٢.١٣)

Therefore $p_- \in$ R then either $p_- \geq -1$ or $p_- < -1$. If $p_- \geq -1$ from (٢.١٣)

$p_- + 1 = |p_- + 1| < \sqrt{p_-{}^2 - b}$ .Hence $-p_- + \sqrt{p_-{}^2 - b} > 1$. (٢.١٤)

If $p_- < -1$ then $-p_- > 1$,so (٢.١٤) holds, and from this

$\left|-p_- + \sqrt{p_-{}^2 - b}\right| = -p_- + \sqrt{p_-{}^2 - b} > 1$. By proposition (٢.١.١), $|\lambda_1| > 1$, now from (٢.١), we get $|\lambda_{1_2}\lambda_2| = |b| < 1$ then $|\lambda_2| < \frac{1}{|\lambda_1|} < 1$ and by theorem (١.١.٢٦) $\begin{pmatrix} p_- \\ p_- \end{pmatrix}$ is a saddle fixed point . □

Proposition (٢.٢.٢)

There are no point of period two for $H_{a,b}\begin{pmatrix}x\\y\end{pmatrix} \in H_M$.

Proof: Suppose that there exists a periodic point of period two for $H_{a,b} \in H_M$ then from definition (١.١.١٩) we have $H^2_{a,b}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$, so

$\begin{pmatrix}a-bx-(a-by-x^2)^2\\a-by-x^2\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$. Then $a-bx-(a-by-x^2)^2=x$

$$a-by-x^2=y \qquad (٢.١٥)$$

so

$$a-bx-y^2=x$$

$$x=\frac{a-y^2}{b+1} \qquad (٢.١٦)$$

substituting for x , $a-by-(\frac{a-y^2}{b+1})^2=y$, we get

$(b+1)^2\ a-(b+1)^2by-(a-y^2)^2=(b+1)^2\ y$ so $(a-y^2)^2+(b+1)^3\ y-a(b+1)^2=0$.

hence $y^4-2ay^2+(b+1)^3y+(a^2-a(b+1)^2\ )=0$. (٢.١٧)

Since the equation $y^2+(b+1)y-a$ corresponds to the fixed points, the polynomial $p(y)=y^2+(b+1)y-a$ is factor polynomial of $q(y)=y^4-2ay^2+(b+1)^3y+(a^2-a(b+1)^2$ and by factor theorem there exists a polynomial $h(y)$ such that $q(y)=h(y)\ p(y)$ and periodic points are roots of the equation $h(y)=0$ .We can find $h(y)=y^2-(b+1)y+((b+1)^2-a)=0$, where $(a,b)\in M$, and the quadratic equation $y^2-(b+1)y+((b+1)^2-a)=0$ has a solution, for $(b+1)^2-4((b+1)^2-a)\geq 0$ .That is $-3(b+1)^2+4a\geq 0$,hence $a\geq\frac{3(b+1)^2}{4}$ which is contradiction □

٢-٣ A characterization of the Henon map for $|b|>1$ and

$$\frac{-(b+1)^2}{4}<a<\frac{3(b+1)^2}{4}$$

The goal of this section is to study a type of fixed points of Henon map $H_{a,b}$,where (a,b) in N, N is a parameter space defined as:

N={ $(a,b):|b|>1, \frac{-(b+1)^2}{4}<a<\frac{3(b+1)^2}{4}, a,b\in R$ } , we divide N into two disjoint spaces $N^+\{(a,b)\in N: b>1\}$, $N^-=\{(a,b)\in N: b<-1\}$, and we denote $H_{N^+}=\{H_{a,b}:(a,b)\in N^+\}$, $H_{N^-}=\{H_{a,b}:(a,b)\in N^-\}$.

Proposition (٢.٣.١)

Let $H_{a,b}\in H_{N^+}$. If $p_\pm^2(a,b)-b\geq 0$ then $H_{a,b}$ has a saddle fixed point $\begin{pmatrix} p_-(a,b)\\ p_-(a,b)\end{pmatrix}$, and a repelling fixed point $\begin{pmatrix} p_+(a,b)\\ p_+(a,b)\end{pmatrix}$.

p**roof:** Let $H_{a,b}\begin{pmatrix} x\\ y\end{pmatrix}$ be any element in H_{N^+}. Then $(a,b)\in N^+$ hence $b>1, \frac{-(b+1)^2}{4}<a<\frac{3(b+1)^2}{4}$. From (٢.٤) $2p_-=-(b+1)-\sqrt{(b+1)^2+4a}$ .Therefore $2p_-<-(b+1)<-2$ .We note that $-p_->1$ and since $\sqrt{p_-^2-b}\geq 0$, we get

$$\left|-p_-+\sqrt{p_-^2-b}\right|=-p_-+\sqrt{p_-^2-b}>1. \qquad (٢.١٨)$$

On the other hand, $2p_-<-(b+1)<(b+1)$,or equivalently $-2p_-+1>-b$ by

adding p_-^2 for both sides, we get , $p_-^2-2p_-+1>p_-^2-b$, thus

$(p_- -1)^2 > p_-^{\,2} - b$ or $\sqrt{(p_- -1)^2} > \sqrt{p_-^{\,2} - b}$, hence $|p_- -1| > \sqrt{p_-^{\,2} - b}$. (٢.١٩)

Since, $p_- < -1$, hence $|p_- -1| = 1 - p_-$,thus $1 - p_- > \sqrt{p_-^{\,2} - b}$.

Hence $p_- + \sqrt{p_-^{\,2} - b} < 1$ (٢.٢٠)

In the same way, since $2\,p_- < -b-1$,or equivalently $2p_- + 1 < -b$ by adding $p_-^{\,2}$ for both sides, we get $p_-^{\,2} + 2\,p_- + 1 < p_-^{\,2} - b$, or $(p_- + 1)^2 < p_-^{\,2} - b$,

hence $\sqrt{(p_- +1)^2} < \sqrt{p_-^{\,2} - b}$. That is $|p_- + 1| < \sqrt{p_-^{\,2} - b}$, since $p_- < -1$, we get

$-p_- - 1 < \sqrt{p_-^{\,2} - b}$, so $-1 < p_- + \sqrt{p_-^{\,2} - b}$. (٢.٢١)

Now from (٢.٢٠) and (٢.٢١), we get that $\left|p_- + \sqrt{p_-^{\,2} - b}\right| < 1$, hence $|\lambda_2| < 1$ and by (٢.١٨) $|\lambda_1| > 1$, in view of theorem (١.١.٢٦) $\begin{pmatrix} p_- \\ p_- \end{pmatrix}$ is a saddle fixed point, for the second part since $b > 1$, under the given condition must be $p_+ > 1$ or $p_+ < -1$.

Case ١: If $p_+ > 1$ then $|p_+ - 1| = p_+ - 1$ (٢.٢٢)

we have $a < \dfrac{3(b+1)^2}{4}$, so $-(b+1) + \sqrt{4a + (b+1)^2} < (b+1)$,by (٢.٣) we get $2p_+ < (b+1)$. That is $-b < -2p_+ + 1$ (٢.٢٣)

By adding $p_+^{\,2}$ for both sides of (٢.٢٣) ,we get that $p_+^{\,2} - b < p_+^{\,2} - 2\,p_+ + 1$ or $\sqrt{p_+^{\,2} - b} < \sqrt{(p_+ - 1)^2}$ then by (٢.٢١) we have $\sqrt{p_+^{\,2} - b} < |p_+ - 1| = p_+ - 1$. Hence $p_+ - \sqrt{p_+^{\,2} - b} > 1$.

(٢.٢٤) Since $\left|-p_+ + \sqrt{p_+^{\;2} - b}\right| = \left|p_+ - \sqrt{p_+^{\;2} - b}\right|$ by (٢.٢٤), we get $\left|-p_+ + \sqrt{p_+^{\;2} - b}\right| > 1$,

hence $|\lambda_1| > 1$ since $p_+ > 1$, clearly $p_+ + \sqrt{p_+^{\;2} - b} > 1$ so $\left|p_+ + \sqrt{p_+^{\;2} - b}\right| = |\lambda_2| > 1$ by theorem (١.١.٢٦), $\begin{pmatrix} p_+ \\ p_+ \end{pmatrix}$ is a repelling fixed point.

Case ٢: If $p_+ < -1$, since $\sqrt{p_+^{\;2} - b} \geq 0$ then $p_+ - \sqrt{p_+^{\;2} - b} < -1$, hence

$$\left|p_+ - \sqrt{p_+^{\;2} - b}\right| = \left|-p_+ + \sqrt{p_+^{\;2} - b}\right| > 1 \qquad (٢.٢٥)$$

so $|\lambda_1| > 1$, since $\sqrt{(b+1)^2 + 4a} > 0$ we have $-(b+1) + \sqrt{(b+1)^2 + 4a} > -(b+1)$

, by (٢.٣) we get that $p_+ > \frac{-(b+1)}{2}$. That is $2p_+ + 1 > -b$. (٢.٢٦)

By adding $p_+^{\;2}$ for both sides of (٢.٢٦), we get that $p_+^{\;2} - b < p_+^{\;2} + 2\,p_+ + 1$,

thus $\sqrt{p_+^{\;2} - b} < \sqrt{(p_+ + 1)^2}$,since $p_+ < -1$ we have $\sqrt{p_+^{\;2} - b} < |p_+ + 1| = -p_+ - 1$,

hence $p_+ + \sqrt{p_+^{\;2} - b} < -1$. (٢.٢٧)

so $\left|p_+ + \sqrt{p_+^{\;2} - b}\right| = |\lambda_2| > 1$,by theorem (١.١.٢٦) $\begin{pmatrix} p_+ \\ p_+ \end{pmatrix}$ is a repelling fixed point. □

Proposition (٢.٣.٢)

If $H_{a,b} \in H_N^-$ then $H_{a,b}$ has a saddle fixed point, $\begin{pmatrix} p_+(a,b) \\ p_+(a,b) \end{pmatrix}$, and a repelling fixed point $\begin{pmatrix} p_-(a,b) \\ p_-(a,b) \end{pmatrix}$.

Proof: Let $H_{a,b} \in H_N^-$, then we have $(a,b) \in N^-$ and from (٢.٣) we have $2\ p_+ = -(b+1) + \sqrt{(b+1)^2 + 4a}$, hence $2\ p_+ > -(b+1)$. (٢.٢٨)

Or equivalently , $2\ p_+ + 1 > -b$. By adding $p_+^{\ 2}$ for both sides , we get that

$p_+^{\ 2} + 2\ p_+ + 1 > p_+^{\ 2} - b$, thus $\sqrt{(p_+ + 1)^2} > \sqrt{p_+^{\ 2} - b}$, hence

$|p_+ + 1| > \sqrt{p_+^{\ 2} - b}$. (٢.٢٩)

Now since $(a,b) \in N^-$,we have $-(b+1) > 0$ .From (٢.٢٨) $p_+ + 1 > 0$ then (٢.٢٩) becomes $p_+ + 1 > \sqrt{p_+^{\ 2} - b}$, hence $-p_+ + \sqrt{p_+^{\ 2} - b} < 1$. (٢.٣٠)

On the other hand, since $-b > 0$, we have $\sqrt{p_+^{\ 2} - b} > \sqrt{p_+^{\ 2}} = |p_+|$, hence

$-p_+ + \sqrt{p_+^{\ 2} - b} > -p_+ + |p_+| = 0$, so we get $\left|-p_+ + \sqrt{p_+^{\ 2} - b}\right| = -p_+ + \sqrt{p_+^{\ 2} - b}$

from (٢.٣٠) and proposition (٢.١.١) $\left|-p_+ + \sqrt{p_+^{\ 2} - b}\right| = |\lambda_1| < 1$, for λ_2 since $p_+ > 0$,

we get that $\left|p_+ + \sqrt{p_+^{\ 2} - b}\right| = p_+ + \sqrt{p_+^{\ 2} - b}$. (٢.٣١)

On the other hand , since $-b > 1$, $p_+^{\ 2} > 0$, we have $\sqrt{p_+^{\ 2} - b} > 1$,then from (٢.٢٨) we can say that , $p_+ + \sqrt{p_+^{\ 2} - b} > 1$. (٢.٣٢)

By (٢.٣١) , (٢.٣٢) and proposition (٢.١.١) we get $\left|p_+ + \sqrt{p_+^{\,2} - b}\right| = |\lambda_2| > 1$, hence by theorem (١.١.٢٦) $\begin{pmatrix} p_+ \\ p_+ \end{pmatrix}$ is a saddle fixed point. To show that $\begin{pmatrix} p_- \\ p_- \end{pmatrix}$ is a repelling fixed point, from (٢.٤) we have $2\ p_- = -(b+1) - \sqrt{(b+1)^2 + 4a}$

then $2\ p_- < -(b+1)$ (٢.٣٣)

as the same as (٢.٢١), we get that $|p_- + 1| < \sqrt{p_-^{\,2} - b}$, since $(a,b) \in N^-$ we have $a < \frac{3(b+1)^2}{4}$ and $b < -1$ thus $\sqrt{(b+1)^2 + 4a} < 4(b+1)^2$ and $b+1 < 0$ **,** hence

$\sqrt{(b+1)^2 + 4a} < 2|b+1| = -2(b+1)$, so $-(b+1) - \sqrt{(b+1)^2 + 4a} > (b+1)$, then from

(٢.٤), we get $2p_- > (b+1)$, from (٢.٣٣) we get that $(b+1) < 2p_- < -(b+1)$

then we have two cases :

Case١: If $(b+1) < 2p_- < 0$, then $-b > -2p_- + 1$**,** thus $|p_- - 1| < \sqrt{p_-^{\,2} - b}$

hence $1 - p_- < \sqrt{p_-^{\,2} - b}$,that is $p_- + \sqrt{p_-^{\,2} - b} > 1$. (٢.٣٤)

then by(٢.٣٤) and proposition(٢.١.١) $\left|p_- + \sqrt{p_-^{\,2} - b}\right| = |\lambda_2| > 1$.

(٢.٣٥) For λ_1 , since $p_- < 0$ and $b < -1$**,** we have $\sqrt{p_-^{\,2} - b} > 1$, so $-p_- + \sqrt{p_-^{\,2} - b} > 1$, then by proposition (٢.١.١) $|\lambda_1| = \left|-p_- + \sqrt{p_-^{\,2} - b}\right| > 1$.

(٢.٣٦) Now by (٢.٣٤),(٢.٣٥) and theorem(١.١.٢٦) $\begin{pmatrix} p_- \\ p_- \end{pmatrix}$ is a repelling fixed point

Case ٢: If $0<2p_-<-(b+1)$, in the same way as (٢.١٣), $|p_-+1|<\sqrt{p_-^{\,2}-b}$, thus $p_-+1<\sqrt{p_-^{\,2}-b}$, hence $-p_-+\sqrt{p_-^{\,2}-b}>1$, hence by proposition (٢.١.١) $|\lambda_1|=\left|-p_-+\sqrt{p_-^{\,2}-b}\right|>1$ .Since $p_->0$, $\sqrt{p_-^{\,2}-b}>1$,we get that $p_-+\sqrt{p_-^{\,2}-b}>1$. By proposition (٢.١.١), $|\lambda_2|=\left|p_-+\sqrt{p_-^{\,2}-b}\right|>1$, hence by theorem (١.١.٢٦) $\begin{pmatrix}p_-\\p_-\end{pmatrix}$ is a repelling fixed point. □

٢-٤ A characterization of the Henon map for $|b|>1$, $a>\frac{3(b+1)^2}{4}$

Our goal of this section is to determine a type of fixed points of Henon map $H_{a,b}$, where (a,b) in G , G=$\{(a,b):|b|>1, a>\frac{3(b+1)^2}{4}, a,b\in R\}$,we divide

G into G_+, G_- where ,$G^+=\{(a,b)\in G: b>1\}$,$G^-=\{(a,b)\in G: b<-1\}$.

We have $H_{G^+}=\{H_{a,b}:(a,b)\in G^+\}$, $H_{G^-}=\{H_{a,b}:(a,b)\in G^-\}$.

Proposition (٢.٤.١)

Let $H_{a,b}\in H_{G^+}$, if $p_\pm^{\,2}(a,b)-b\geq 0$ then $H_{a,b}$ has two saddle fixed points .

Proof: Let $H_{a,b}\begin{pmatrix}x\\y\end{pmatrix}\in H_{G^+}$, then $(a,b)\in G^+$. From **(**٢.٣), we have

$2p_+=-(b+1)+\sqrt{(b+1)^2+4a}$, thus $2p_+>-(b+1)$. Hence $2p_++1>-b$.

By adding $p_+^{\,2}$ for both sides, we get that $p_+^{\,2}+2p_++1>p_+^{\,2}-b$, so $\sqrt{(p_++1)^2}>\sqrt{p_+^{\,2}-b}$ .Hence $|p_++1|>\sqrt{p_+^{\,2}-b}$.

(٢.٣٧) Now since $(a,b)\in G^+$ we have $a>\frac{3(b+1)^2}{4}$, thus

$-(b+1)+\sqrt{(b+1)^2+4a}>(b+1)$. By (٢.٣) $2p_+>(b+1)$, hence $p_+>1$, from

(٢.٣٧), $-p_+ + \sqrt{p_+{}^2-b}<1$. (٢.٣٨)

Since $2p_+>(b+1)$, we have $-b>-2p_+ +1$, by adding $p_+{}^2$ for both sides , we get that $p_+{}^2-2p_+ +1<p_+{}^2-b$, thus $\sqrt{(p_+-1)^2} < \sqrt{p_+{}^2-b}$, by definition of absolute value $|p_+-1|<\sqrt{p_+{}^2-b}$,since $p_+>1$ we get $p_+-1<\sqrt{p_+{}^2-b}$.

Hence $-p_+ + \sqrt{p_+{}^2-b}>-1$. (٢.٣٩)

So from (٢.٣٨),(٢.٣٩), we have $\left|-p_+ + \sqrt{p_+{}^2-b}\right|<1$ and by proposition (٢.١.١)

$|\lambda_1|<1$, for λ_2 from (٢.١) $|\lambda_{1_2}\lambda_2|=|b|>1$, then $|\lambda_2|>\frac{1}{|\lambda_1|}>1$, then by theorem (١.١.٢٦) $\begin{pmatrix} p_+ \\ p_+ \end{pmatrix}$ is a saddle fixed point. To show that $\begin{pmatrix} p_- \\ p_- \end{pmatrix}$ is a saddle fixed point. From (٢.٤), we have $2\ p_- = -(b+1)-\sqrt{(b+1)^2+4a}$. It is clearly that

$2\ p_- < -(b+1) < (b+1)$ and $-b < -2p_- +1$, so $|p_- -1| > \sqrt{p_-{}^2-b}$. (٢.٤٠)

Since $2\ p_- < -(b+1) < -2$, we get that $1-p_- > \sqrt{p_-{}^2-b}$ hence we have

$p_- + \sqrt{p_-{}^2-b}<1$.

(٢.٤١)

On the other hand, since $2\ p_- < -(b+1)$, in the same way as (٢.٢١) ,we get

$-1 < p_- + \sqrt{p_-^2 - b}$ (٢.٤٢)

now from (٢.٤١) and (٢.٤٢), we get $\left|p_- + \sqrt{p_-^2 - b}\right| < 1$. Hence $|\lambda_2| < 1$ for λ_1 from (٢.١), $|\lambda_1 \lambda_2| = |b| > 1$, then $|\lambda_2| > \frac{1}{|\lambda_1|} > 1$ and by theorem (١.١.٢٦) $\begin{pmatrix} p_- \\ p_- \end{pmatrix}$ is a saddle fixed point. □

Proposition (٢.٤.٢)

If $H_{a,b} \in H_{G^-}$,then $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ has two saddle fixed points.

Proof: Let $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} \in H_{G^-}$, by the same way in (٢.١٣), we can show that

$|p_- + 1| < \sqrt{p_-^2 - b}$. (٢.٤٣)

Since $a > \frac{3(b+1)^2}{4}$, so $\sqrt{(b+1)^2 + 4a} > 2|b+1| = -2(b+1)$.By (٢.٣) $2p_+ < (b+1)$ in the same way as (٢.١٩), we can show that $|p_- - 1| > \sqrt{p_-^2 - b}$. (٢.٤٤)

Since $2p_- < (b+1)$ and $(b+1) < 0$, we have $|p_- - 1| = 1 - p_-$, hence from (٢.٤٤)

$p_- + \sqrt{p_-^2 - b} < 1$. (٢.٤٥)

Now since $p_- \in$ R, we have two cases

Case١: If $p_- \le -1$, then from (٢.٤٣) $p_- + \sqrt{p_-^2 - b} > -1$

Case٢: If $p_- > -1$, then clearly $p_- + \sqrt{p_-^{\,2} - b} > -1$. (٢.٤٦)

Hence from (٢.٤٥), (٢.٤٦) $\left| p_- + \sqrt{p_-^{\,2} - b} \right| = |\lambda_2| < 1$. (٢.٤٧)

For λ_1 from (٢.١), $|\lambda_{1_2} \lambda_2| = |b| > 1$ then $|\lambda_2| > \dfrac{1}{|\lambda_1|} > 1$ and by theorem (١.١.٢٦)

$\begin{pmatrix} p_- \\ p_- \end{pmatrix}$ is a saddle fixed point. To show that $\begin{pmatrix} p_+ \\ p_+ \end{pmatrix}$ is a saddle fixed point, by the

same way in (٢.٢٩) we get $|p_+ + 1| > \sqrt{p_+^{\,2} - b}$. Since $2p_+ > -(b+1) > 0$, we have $-p_+ + \sqrt{p_+^{\,2} - b} < 1$, on the other hand $2p_+ > -(b+1) > (b+1)$ then by the same way in (٢.٣٩) $|p_+ - 1| < \sqrt{p_+^{\,2} - b}$. (٢.٤٨)

Now if $p_+ < 1$ then $-p_+ + \sqrt{p_+^{\,2} - b} > -1$ and if $p_+ \geq 1$ by (٢.٤٨), we have $-p_+ + \sqrt{p_+^{\,2} - b} > -1$, hence $\left| -p_+ + \sqrt{p_+^{\,2} - b} \right| = |\lambda_1| < 1$. (٢.٤٩)

Since $|\lambda_2| > \dfrac{1}{|\lambda_1|}$ we have $|\lambda_2| > 1$ the by theorem (١.١.٢٦) $\begin{pmatrix} p_- \\ p_- \end{pmatrix}$ is a saddle

fixed point. □

Proposition (٢.٤.٣)

If $(a,b) \in$ G then the Henon H$_{a,b}$ map has two periodic points of period two .

Proof: Let $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ be two different vectors such that $H_{a,b}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$

and $H_{a,b}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ thus $\begin{pmatrix} a - bp_2 - p_1^2 \\ p_1 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \begin{pmatrix} a - bq_2 - q_1^2 \\ q_1 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ then

$a - bp_2 - p_1^2 = q_1$, $a - bq_2 - q_1^2 = q_2$ and

$$p_1 = q_2 \;,\; q_1 = p_2 \qquad (٢.٥٠)$$

equivalently $q_1 - q_2 = b(q_2 - p_2) + (q_1^2 - p_1^2)$.

$$p_2 - p_1 = b(p_1 - p_2) + (p_2^2 - p_1^2) \;. \qquad (٢.٥١)$$

Since $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ are different $p_2 - p_1 \neq 0$, hence from (٢.٥١), we get that $p_1 + p_2 = b + 1$. From (٢.٥٠) $p_2 = b + 1 - p_1 = a - bp_2 - p_1^2 = a - b(b + 1 - p_1) - p_1^2$

thus $p_1^2 + b(b+1-p_1) + b + 1 - p_1 - a = 0$, so $p_1^2 + (b+1)(b+1-p_1) - a = 0$ (٢.٥٢)

since $(a,b) \in G$, thus (٢.٥٢) has two real solutions

$$p_{1\,+} = \frac{(b+1) + \sqrt{4a - 3(b+1)^2}}{2}$$

$$p_{1\,-} = \frac{(b+1) - \sqrt{4a - 3(b+1)^2}}{2} \qquad (٢.٥٣)$$

and $p_{2\,\pm} = (b+1) - p_{1\pm} = \dfrac{(b+1) \pm \sqrt{4a - 3(b+1)^2}}{2}$, hence we get that

$$\left(\begin{array}{c}\frac{(b+1)+\sqrt{4a-3(b+1)^2}}{2}\\ \frac{(b+1)-\sqrt{4a-3(b+1)^2}}{2}\end{array}\right),\left(\begin{array}{c}\frac{(b+1)-\sqrt{4a-3(b+1)^2}}{2}\\ \frac{(b+1)+\sqrt{4a-3(b+1)^2}}{2}\end{array}\right)$$ are two periodic point

for Henon map of period two. □

Proposition (٢.٤.٤)[٢]

If $a=\frac{3(b+1)^2}{4}$ and $b>-1$, then one of the eigen values of $DH_{a,b}$ at $P_+(a,b)$ is -1 .

Proof: If $a=\frac{3(b+1)^2}{4}$, then $\sqrt{(b+1)^2+4a}=2|b+1|=+2(b+1)$ and from (١.١.٤)

$$p_+=\frac{-(b+1)+\sqrt{(b+1)^2+4a}}{2}=\frac{(b+1)}{2} \quad . \qquad (٢.٥٤)$$

Now by proposition (٢.١.١) and (٢.٥٤), we get that the eigen values of $DH_{a,b}$ at $P_+(a,b)$ are $-p_+(a.b)\pm\sqrt{((p_+(a,b))^2-b}$

$$=-\frac{(b+1)}{2}\pm\sqrt{\frac{(b+1)^2}{4}-b}=-\frac{(b+1)}{2}\pm\frac{|b-1|}{2}=-1 \text{ or } -b \text{ . } □$$

Proposition (٢.٤.٥)[٢]

If $a=\frac{3(b+1)^2}{4}$ and $b<-1$, then one of the eigen values of $DH_{a,b}$ at $P_-(a,b)$ is -1 .

Proof: similar to proposition (٢.٤.٤) .

Proposition (٢.٤.٦)

If $b=-1, a>0$ then $H_{a,b}$ has two saddle fixed points .

Proof : Since $b=-1$ and $a>0$,then from (٢.٣) and (٢.٤) the fixed points are

$$P_+=\begin{pmatrix} p_+ \\ p_+ \end{pmatrix}=\begin{pmatrix} \sqrt{a} \\ \sqrt{a} \end{pmatrix} \qquad (٢.٥٥)$$

$$P_-=\begin{pmatrix} p_-(a,b) \\ p_-(a,b) \end{pmatrix}=\begin{pmatrix} -\sqrt{a} \\ -\sqrt{a} \end{pmatrix} \quad . \qquad (٢.٥٦)$$

Then by proposition (٢.١.١), $\lambda_1=-\sqrt{a}+\sqrt{a+1}$ .Now suppose that $|\lambda_1|\geq 1$, then $\left|-\sqrt{a}+\sqrt{a+1}\right|\geq 1$, since $\sqrt{a+1}>\sqrt{a} \quad \forall a>0$,we have $\sqrt{a+1}-\sqrt{a}\geq 1$ so $(a+1)\geq 1+2\sqrt{a}+a$, that is $\sqrt{a}\leq 0$ .But this is contradiction hence $|\lambda_1|<1$. Since $\lambda_1\lambda_2=b=-1$, we have $|\lambda_2|=\frac{1}{|\lambda_1|}$ but $|\lambda_1|<1$, hence $|\lambda_2|>1$. By theorem (١.١.٢٦),P_+ is a saddle fixed point. To show that P_- is a saddle fixed point .We suppose that $|\lambda_2|\geq 1$,then $\left|\sqrt{a}-\sqrt{a+1}\right|\geq 1$,so $\sqrt{a+1}-\sqrt{a}\geq 1$ thus $(a+1)\geq 1+2\sqrt{a}+a$ hence, $\sqrt{a}<0$ but this is contradiction, so must be $|\lambda_2|<1$,since $|\lambda_1|>\frac{1}{|\lambda_2|}>1$, so by theorem (١.١.٢٦), $\begin{pmatrix} -\sqrt{a} \\ -\sqrt{a} \end{pmatrix}$ is a saddle fixed point. □

Proposition (٢.٤.٧)

For the Henon map $H_{a,b}$, if $b=-1$ then there are no periodic points of period two.

Proof: From the proof of proposition (٢.٢.٢), to determine a point $\begin{pmatrix} x \\ y \end{pmatrix}$ to be of period two, it must satisfy the equation $x = \frac{a - y^2}{b+1}$ whose solution is undefined for $b = -1$. □

Proposition (٢.٤.٨)

If $b = -1, a > 0$, then there are no eventually fixed point of $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$, other than fixed points .

Proof: by proposition (٢.١.٣), we have only two fixed points P_+, P_- and clearly they are eventually fixed points . To show that there is no eventually fixed point except $\begin{pmatrix} \sqrt{a} \\ \sqrt{a} \end{pmatrix}, \begin{pmatrix} -\sqrt{a} \\ -\sqrt{a} \end{pmatrix}$. For $\begin{pmatrix} \sqrt{a} \\ \sqrt{a} \end{pmatrix}$ we suppose that there exists an eventually fixed point $\begin{pmatrix} x \\ y \end{pmatrix}$ for $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} \sqrt{a} \\ \sqrt{a} \end{pmatrix}$ then by definition (١.١.٢١) there exists a positive integer number n such that $H^n_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is a fixed point of $H_{a,b}$. Since we have only two fixed points, so there is $n \in Z^+$ such that $H^n_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{a} \\ \sqrt{a} \end{pmatrix}$. If $n = 1$, then $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a + y - x^2 \\ x \end{pmatrix} = \begin{pmatrix} \sqrt{a} \\ \sqrt{a} \end{pmatrix}$ so $a + y - x^2 = \sqrt{a}$ and $x = \sqrt{a}$ so $a + y - a = \sqrt{a}$. That is $y = \sqrt{a}$, hence $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{a} \\ \sqrt{a} \end{pmatrix}$ which is contradiction. If $n=2$ then $H^2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a+x-(a+y-x^2)^2\\a+y-x^2\end{pmatrix}=\begin{pmatrix}\sqrt{a}\\\sqrt{a}\end{pmatrix}$ thus

$a+x-(a+y-x^2)^2=\sqrt{a}\,,a+y-x^2=\sqrt{a}\,,$ so $x=\sqrt{a}\,,a+y-x^2=\sqrt{a}\,,$ that is $x=\sqrt{a},y=\sqrt{a}$ hence $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\sqrt{a}\\\sqrt{a}\end{pmatrix}$ but this is contradiction .Now suppose that it is true for $k-1$ .We must show that it is true for k thus we have ,if $H^{k-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\sqrt{a}\\\sqrt{a}\end{pmatrix}$then $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\sqrt{a}\\\sqrt{a}\end{pmatrix}$, we must show that it is true for k that is

if $H^{k}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\sqrt{a}\\\sqrt{a}\end{pmatrix}$then $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\sqrt{a}\\\sqrt{a}\end{pmatrix}$.

Now $H(H^{k-1}\begin{pmatrix}x\\y\end{pmatrix})=H\begin{pmatrix}x_{k-1}\\y_{k-1}\end{pmatrix}=\begin{pmatrix}a+y_{k-1}-x^2{}_{k-1}\\x_{k-1}\end{pmatrix}$,$H(H^{k-1}\begin{pmatrix}x\\y\end{pmatrix})=\begin{pmatrix}\sqrt{a}\\\sqrt{a}\end{pmatrix}$ then

$$a+y_{k-1}-x^2{}_{k-1}=\sqrt{a}\,,x_{k-1}=\sqrt{a} \qquad (٢.٥٧)$$

we put $x_{k-1}=\sqrt{a}$ in (٢.٥٧) we get $y_{k-1}=\sqrt{a}$, hence $H^{k-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\sqrt{a}\\\sqrt{a}\end{pmatrix}$ by our supposition $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\sqrt{a}\\\sqrt{a}\end{pmatrix}$, so it is true $\forall\ n\in Z^+$.In the same way, we can show that if $H^n\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-\sqrt{a}\\-\sqrt{a}\end{pmatrix}$, then $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-\sqrt{a}\\-\sqrt{a}\end{pmatrix}$, that means there are no eventually fixed points other than $\begin{pmatrix}\sqrt{a}\\\sqrt{a}\end{pmatrix}$, $\begin{pmatrix}-\sqrt{a}\\-\sqrt{a}\end{pmatrix}$which are fixed points. □

Remark (٢.٤.٩)

From some theorems in this chapter we can deal with bifurcation point of Henon map. Thus we have the following situation for a given value b in $(-1,1)$ .From proposition (٢.١.٣) if $a > \frac{-(1+b)^2}{4}$, then Henon map $H_{a,b}$ has two fixed points .From proposition (٢.١.٥) if $a = \frac{-(1+b)^2}{4}$, then $H_{a,b}$ has a unique fixed point and if $a < \frac{-(1+b)^2}{4}$, then $H_{a,b}$ has no fixed points .Thus sudden change in fixed point behavior is bifurcation .Thus particular example (with the sudden appearance and then splitting of a fixed point)is called a saddle node bifurcation .

The Henon map $H_{a,b}$ has a bifurcation at $a = \frac{3(1+b)^2}{4}$, because from proposition (٢.٤.٤) and proposition (٢.٤.٥) one of the two eigen values of $H_{a,b}$ is -1,from proposition (٢.٢.٢) if $a < \frac{3(1+b)^2}{4}$ there are no periodic points of period two and from proposition (٢.٤.٣) if $a > \frac{3(1+b)^2}{4}$ there are two periodic points of period two. Furthermore from proposition (٢.٣.٢) if $a < \frac{3(1+b)^2}{4}$ $H_{a,b}$ has a repelling fixed point $\begin{pmatrix} p_- \\ p_- \end{pmatrix}$, from proposition (٢.٤.٢) if $a > \frac{3(1+b)^2}{4}$ $H_{a,b}$ has a saddle fixed point $\begin{pmatrix} p_- \\ p_- \end{pmatrix}$.

CHAPTER THREE

The dynamics of $H_{0,b}$, $|b|<1$

In this chapter, we introduce forward and backward iteration of Henon map $H_{a,b}$, where $a=0$,-١< $b<0$ .In section one ,we study the type of fixed point of Henon map $H_{0,b}$ with a basin of attraction of one of fixed points .In section two we study non existence of periodic point of Henon map in R٢ by finding some region in R٢ .At the last we introduce forward and backward iteration of Henon map $H_{a,b}$, where $a=0$, $b>0$.

٣-١ Type of fixed points with basin of attraction

Our goal of this section is to determine type of fixed points of $H_{0,b}$ with the basin of attraction of a fixed point $\begin{pmatrix}0\\0\end{pmatrix}$. We will use maximum norm, where the maximum norm of $\begin{pmatrix}x\\y\end{pmatrix}=\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|$=max{ $|x|$, $|y|$ }.

Proposition (٣.١.١)

If $-1<b<0$, then $H_{0,b}$ has attracting fixed point $\begin{pmatrix}0\\0\end{pmatrix}$ and a saddle fixed point $\begin{pmatrix}-(b+1)\\-(b+1)\end{pmatrix}$.

Proof: Since $a=0$ and $-1<b<0$. By proposition (٢.١.٣) $P_+=\begin{pmatrix}0\\0\end{pmatrix}$, $P_-=\begin{pmatrix}-(b+1)\\-(b+1)\end{pmatrix}$ are fixed points for $H_{0,b}$.By proposition (٢.١.١) $\lambda_{1,2}=\pm\sqrt{-b}$ are eigen values of $DH_{0,b}\begin{pmatrix}x\\y\end{pmatrix}$ at P_+ . Since $-1<b<0$ then $0<\sqrt{-b}<1$ and $|\lambda_1|=|\lambda_2|<1$, so $\begin{pmatrix}0\\0\end{pmatrix}$ is attracting fixed point. To show that P_- is a saddle fixed point . From proposition (٢.١.١) , we have two eigen values,

$$\lambda_1=-(b+1)+\sqrt{(b+1)^2-b}$$

$$\lambda_2=-(b+1)-\sqrt{(b+1)^2-b} \quad . \tag{٣.١}$$

Since $b^2+b+1<(b+2)^2$ thus $\sqrt{b^2+b+1}<\sqrt{(b+2)^2}=b+2$ hence

$$-(b+1)+\sqrt{b^2+b+1}<1 \quad . \tag{٣.٢}$$

On the other hand, since $\sqrt{b^2+b+1}=\sqrt{(b+1)^2-b}>\sqrt{(b+1)^2}=|b+1|=b+1$

so $-(b+1)+\sqrt{b^2+b+1}>0$, hence by (٣.٢) $\left|-(b+1)+\sqrt{(b+1)^2-b}\right|=|\lambda_2|<1$. (٣.٣)

For λ_1 since $-1<b<0$ we have $\sqrt{b^2+b+1}>\sqrt{b^2}=|b|=-b$. (٣.٤)

By adding $(b+1)$ for both sides of (٣.٤), we get $(b+1)+\sqrt{b^2+b+1}>1$

then $\left|(b+1)+\sqrt{(b+1)^2-b}\right|=|\lambda_1|>1$. (٣.٥)

Hence from (٣.٤), (٣.٥) and theorem (١.١.٢٦), P_- is a saddle fixed point. □

Remark (٣.١.٢)

In view of theorem (١.١.٣٤), there exists an open set about $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in which all points tend to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ under forward iteration of $H_{0,b}$.The next theorem shows this open set.

Theorem (٣.١.٣)

Suppose S= $\{\begin{pmatrix} x \\ y \end{pmatrix} : x, y \in R, |x| \le 1-|b|, |y| \le 1-|b|, |b| < 1\}$ then for all $\begin{pmatrix} x \\ y \end{pmatrix} \in S^\circ$

$H^n_{0,b}\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $n \longrightarrow \infty$.

Proof: We claim that $H_{0,b}(S^\circ) \subset S^\circ$, to show this , let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ be any element in $H_{0,b}(S^\circ)$ then there is $\begin{pmatrix} x \\ y \end{pmatrix}$ in S° such that $H_{0,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -by - x^2 \\ x \end{pmatrix}$ .From this

$$|y_1| = |x| < 1-|b| \qquad (٣.٦)$$

and $|x_1| = |by + x^2| \le |b||y| + |x|^2 < |b|\,(1-|b|) + (1-|b|)^2$

$$= |b| - |b|^2 + 1 - 2|b| + |b|^2 = 1-|b| \; . \qquad (٣.٧)$$

Now from (٣.٦), (٣.٧) $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in S^\circ$.Consider the maximum norm

$\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|$=max$\{|x|, |y|\}$. Let $r = \left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|$, so $|x| \le r$, $|y| \le r$ and since $|x| \le 1-|b|$,

$|y| \le 1-|b|$, then $r < 1-|b|$. We denote $H_{0,b}{}^{n}\begin{pmatrix} x \\ y \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix}$ in S° ,by $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$,then

$$|y_2| = |x_1| = |by + x^2| \le |b||y| + |x|^2 \le |b|\ r + r^2 = r(|b|+r) \qquad (٣.٨)$$

$$|x_2| = |by_1 + x_1{}^2| \le |b|\ r + r^2(|b|+r)^2 \ . \qquad (٣.٩)$$

Now since $0 < |b| + r < 1$, we have $0 < r^2(|b|+r) < r^2$, then

$|b|\ r + r^2(|b|+r)^2 < |b|\ r + r^2 = r(|b|+r)$, hence by (٣.٨), (٣.٩)

$$|x_2| < r(|b|+r) \ . \qquad (٣.١٠)$$

Now we claim that $|x_{2n}| < r(|b|+r)^{n}$, $|y_{2n}| < r(|b|+r)^{n}$. (٣.١١)

We prove it by using mathematical induction from (٣.٨), (٣.١٠),it is true for $n=1$, suppose it is true for k then $|x_{2k}| < r(|b|+r)^{k}$,$|y_{2k}| < r(|b|+r)^{k}$

to show that it is true for $k+1$, $y_{2(k+1)} = -by_{2(k+1)-2} - x^2{}_{2(k+1)-2}$

$$= -by_{2k} - x^2{}_{2k} \ .$$

$$|y_{2(k+1)}| = |by_{2k} + x_{2k}{}^2| \le |b||y_{2k}| + |x_{2k}|^2 \le |b|\ r(|b|+r)^{k} + r^2(|b|+r)^{2k} \ . \qquad (٣.١٢)$$

Now $0 < (|b|+r)^{k} < 1$, so $r^2(|b|+r)^{k} + |b|\ r < |b|\ r + r^2 = r(|b|+r)$

Hence $r^2(|b|+r)^{2k} + |b|\ r(|b|+r)^{k} < r(|b|+r)^{k+1}$. (٣.١٣)

Hence from (٣.١٢),we get $|y_{2(k+1)}| < r(|b|+r)^{k+1}$. (٣.١٤)

Also we have $x_{2(k+1)} = -by_{2(k+1)-1} - x^2_{2(k+1)-1}$

$$= -by_{2k+1} - x^2{}_{2k+1}$$

$$= -bx_{2k} - y^2{}_{2(k+1)} \ , \quad \text{since } x_n = y_{n+1}$$

so $|x_{2(k+1)}| = |bx_{2k} + y^2{}_{2(k+1)}| < |b||x_{2k}| + |y_{2(k+1)}|^2$

$$< |b|\ r(|b|+r)^{k} + r^2(|b|+r)^{2k}$$

hence from (٣.١٣) we get that $|x_{2(k+1)}| < r(|b|+r)^{k+1}$, hence it is true for $k+1$, Now $\left\| H_{0,b}{}^{2n}\begin{pmatrix} x \\ y \end{pmatrix} \right\| = \left\| \begin{pmatrix} x_{2n} \\ y_{2n} \end{pmatrix} \right\|$ =max{ $|x_{2n}|, |y_{2n}|$ },if $\left\| H_{0,b}{}^{2n}\begin{pmatrix} x \\ y \end{pmatrix} \right\| = |x_{2n}|$ then

$\lim\limits_{n \longrightarrow \infty} \left\| H_{0,b}{}^{2n}\begin{pmatrix} x \\ y \end{pmatrix} \right\| = \lim\limits_{n \longrightarrow \infty} |x_{2n}| = \lim\limits_{n \longrightarrow \infty} r(|b|+r)^{n} = 0$, so by definition of maximum

norm of $\left\| H_{0,b}{}^{2n}\begin{pmatrix} x \\ y \end{pmatrix} \right\|$. $\lim\limits_{n \longrightarrow \infty} |y_{2n}| = 0$ so $H_{0,b}{}^{2n}\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $n \longrightarrow \infty$. (٣.١٥)

If $\left\| H_{0,b}{}^{2n}\begin{pmatrix} x \\ y \end{pmatrix} \right\| = |y_{2n}|$, as the same as (٣.١٣), $H_{0,b}{}^{2n}\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $n \longrightarrow \infty$ and

since $H_{0,b}(S^{\circ}) \subset S^{\circ}$ then $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in S^{\circ}$, we have $H_{0,b}\begin{pmatrix} x \\ y \end{pmatrix} \in S^{\circ}$ by (٣.١٥), so

$H_{0,b}{}^{2n}(H_{0,b}\begin{pmatrix} x \\ y \end{pmatrix}) \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $n \longrightarrow \infty$, so $H_{0,b}{}^{2n+1}\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $n \longrightarrow \infty$,

hence $H_{0,b}{}^{n}\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $n \longrightarrow \infty, \forall \begin{pmatrix} x \\ y \end{pmatrix} \in S^{\circ}$. □

٣-٢ The Periodic Points for Henon Map Where $a=0, -1<b<0$

In this section we will show that Henon map $H_{0,b}, -1<b<0$ has no periodic points other than fixed points in the plane. To prove this, we divide the proof to fifteen lemmas, thus we find some regions such that the union of all regions covers the plane. We prove that there is no periodic point in each of them until we get the main purpose. The regions are the following $Q_1, Q_{2,1}, Q_{2,2}, Q_3, Q_4$ and in Q_3 define the following regions $R_1, R_2, R_3, R_4, R_5, R_6, R_7$.

$$Q_1 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x \geq 0, y \geq 0 \}.$$

$$Q_{2,1} = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x < 0, y \geq 0, x \geq -y^2 \}.$$

$$Q_{2,2} = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x < 0, y > 0, x < -y^2 \}.$$

$$Q_4 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x \geq 0, y < 0 \}.$$

$$R_1 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x \geq -y^2 \}.$$

$$R_2 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x \leq b\sqrt{-y} - y^2 \}$$

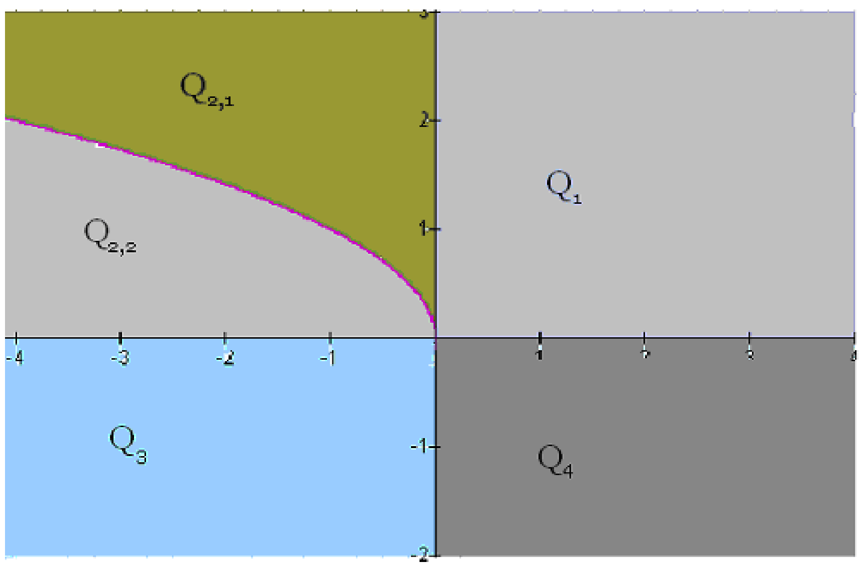
$$R_3 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : y < \frac{-x-x^2}{b}, -y^2 - by < x < -y^2, x < y \}.$$

$$R_4 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : y < \frac{-x-x^2}{b}, b\sqrt{-y} - y^2 < x \leq -by - y^2 \}$$

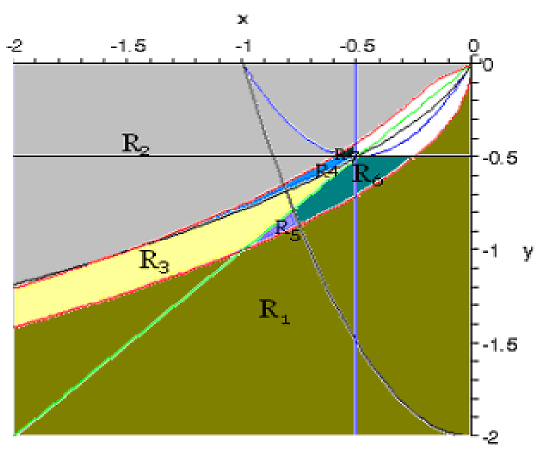
$$R_5 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : y < \frac{-x-x^2}{b}, -y^2 - by < x < -y^2, x \geq y, by + x^2 > 1 \}.$$

$$R_6 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : y < -1-b, x < -y^2, x \geq y, by + x^2 \leq 1, x > -y^2 - by, x > b\sqrt{-y} - y^2 \}.$$

$$R_7 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x < -1-b \, , y > -1-b \, , y > x \, , y \geq \frac{-x-x^2}{b} \quad , -y^2 - by > x > b\sqrt{-y} - y^2 \} .$$



Fig(1):Region Q_1 ,$Q_{2,1}$,$Q_{2,2}$,Q_3 ,Q_4



Fig(2):Region R_1 ,R_2,R_3,R_4,R_5,R_6,R_7

Lemma (٣.٢.١)

$H_{0,b}$ is topologically conjugate to $H^{-1}_{0.b^{-1}}$

Proof: Let G: $R^{٢} \longrightarrow R^{٢}$ be a map defined by $G\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} by \\ bx \end{pmatrix}$. Clearly G is continuous and bijective , G^{-1} is continuous, so G is homeomorphism and

$$G \circ H^{-1}_{0.b^{-1}} \begin{pmatrix} x \\ y \end{pmatrix} = G\begin{pmatrix} y \\ \dfrac{-x-y^2}{b^{-1}} \end{pmatrix} = \begin{pmatrix} b\dfrac{-x-y^2}{b^{-1}} \\ by \end{pmatrix} = \begin{pmatrix} \dfrac{-x-y^2}{b^{-2}} \\ by \end{pmatrix} \qquad (٣.١٦)$$

$$H_{0,b} \circ G\begin{pmatrix} x \\ y \end{pmatrix} = H_{0,b}\begin{pmatrix} by \\ bx \end{pmatrix} = \begin{pmatrix} -b(bx)-(by)^2 \\ by \end{pmatrix} = \begin{pmatrix} b^2(-x-y^2) \\ by \end{pmatrix} = \begin{pmatrix} \dfrac{-x-y^2}{b^{-2}} \\ by \end{pmatrix}. \qquad (٣.١٧)$$

From (٣.١٦) and (٣.١٧), $G \circ H^{-1}_{0.b^{-1}} = H_{0,b} \circ G$. Hence by definition **(١.١.٣٠),** $H_{0,b}$ is topologically conjugate to $H^{-1}_{0.b^{-1}}$. □

Lemma (٣.٢.٢)

There are no periodic points in Q_1 other than fixed points.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_1$, we define norm $\left\| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\|$ by $\left\| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| = \sqrt{x_0^{\,2} + y_0^{\,2}}$

From now for simply we refer to $H_{0,b}$ as H and $H^{-1}_{0.b^{-1}}$ as H^{-1}

$$H^{-n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{-(n+1)} \\ y_{-(n+1)} \end{pmatrix} = \begin{pmatrix} y_{-n} \\ \dfrac{-1}{b}(x_{-n} + y_{-n}^{\,2}) \end{pmatrix}. \qquad (٣.١٨)$$

We have $-1 < b < 0$, $x_{-1} = y_0$, $y_{-1} = \dfrac{-1}{b}(x_0 + y_0)$ then $y_{-1} = \dfrac{-1}{b}(x_0 + y_0) > \dfrac{-1}{b}x_0 > x_0$

$y_{-1} > x_0$ and since y_{-1}, x_0 are non negative $y_{-1}^{\,2} > x_0^{\,2}$.That is $y_{-1}^{\,2} + y_0^{\,2} > x_0^{\,2} + y_0^{\,2}$

thus $y_{-1}^{\,2} + x_{-1}^{\,2} > x_0^{\,2} + y_0^{\,2}$ so $\sqrt{y_{-1}^{\,2} + x_{-1}^{\,2}} > \sqrt{x_0^{\,2} + y_0^{\,2}}$, hence $\left\|\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix}\right\| > \left\|\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right\|$, that is $\left\|H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right\| > \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. (٣.١٩)

Since $x_{-1} = y_0 \geq 0$, $y_{-1} = \frac{-1}{b}(x_0 + y_0) \geq 0$,we have $H^{-1}(Q_1) \subset Q_1$, then (٣.١٩) is true for all $n \in Z^+$,hence $\left\|H^{-n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right\| > \left\|H^{-n+1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right\| \quad \forall n \in Z^+$, (٣.٢٠)

so the norms of the points in the orbit are strictly increasing. Hence there are no periodic point in the region Q_1. □

Lemma (٣.٢.٣)

There are no periodic points in $Q_{2,1}$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,1}$, since $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ \frac{-1}{b}(x_0 + y_0{}^2) \end{pmatrix}$, $y_0 > 0$ and $x_0 \geq -y_0{}^2$, hence $x_{-1} > 0$, $\frac{-1}{b}(x_0 + y_0{}^2) \geq 0$, so $x_{-1} > 0$, $y_{-1} \geq 0$ $\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} \in Q_1$, that means $Q_{2,1}$ maps into Q_1 then by lemma (٣.٢.٢) there are no periodic point for H in $Q_{2,1}$. □

Lemma (٣.٢.٤)

There are no periodic points of even period in $Q_{2,2}$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,2}$ .From (٣.١٨),since $y_{-n} = \frac{-1}{b}(x_{-(n-1)} + y_{-(n-1)}{}^2)$, $x_{-n} = y_{-(n-1)}$. Since $x_0 < 0, y_0 \geq 0$,so $x_{-1} = y_0 > 0$, $y_{-1} = \frac{-1}{b}(x_0 + y_0{}^2) < 0$, hence $x_{-2} = y_{-1} < 0$, $y_{-2} = \frac{-1}{b}(x_{-1} + y_{-1}{}^2) > \frac{-1}{b}x_{-1}$. Since $-1 < b < 0$, $\frac{-1}{b}x_{-1} > x_{-1} = y_0 > 0$, hence $y_{-2} > y_0 > 0$ .In general $y_{-n} > \frac{-1}{b} x_{-(n-1)} > x_{-(n-1)} = y_{-(n-2)}$, so $\langle y_{-2n} \rangle$ is strictly increasing sequence ,$H^{-2n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in $Q_{2,1} \cup Q_{2,2}$ then we have two cases.

Case ١: If $H^{-2n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,1}$, then by lemma (٣.٢.٣) there are no periodic points of even period for H.

Case ٢: If $H^{-2n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,2}$ and since $\langle y_{-2n} \rangle$ is strictly increasing sequence so there are no periodic points of even period for H. □

Lemma (٣.٢.٥)

There are no periodic points of odd period in Q_4.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_4$ since $x_{-1} = y_0 < 0$, $y_{-1} = \frac{-1}{b}(x_0 + y_0{}^2)$, $x_0 > 0$ so $y_{-1} = \frac{-1}{b}(x_0 + y_0{}^2) > \frac{-1}{b} x_0 > x_0$, hence $y_{-1} > 0$,from (٣.١٨) $x_{-(n+1)} = y_{-n}$, we have $x_{-2} > 0$,then $y_{-3} = \frac{-1}{b}(x_{-2} + y_{-2}{}^2) > \frac{-1}{b} x_{-2} > x_{-2} = y_{-1} > 0$,hence $x_{-4} > 0$ by the same way $y_{-5} = \frac{-1}{b}(x_{-4} + y_{-4}{}^2) > \frac{-1}{b} x_{-4} > x_{-4} = y_{-3} > 0$, hence

$$y_{-(2n+1)} > y_{-(2n-1)} > \ldots > y_{-5} > y_{-3} > y_{-1} > 0 \quad . \qquad (٣.٢١)$$

Either $x_{-2n-1} > 0$ or $x_{-2n-1} < 0$ so $H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_1$ or $H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_2$

Case ١: If $H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_1$, by lemma (٣.٢.٢), there are no periodic point of odd period

Case ٢: If H $^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_2$ since $\langle y_{-2n-1} \rangle$ is strictly increasing sequence there are no periodic points of odd period . □

Lemma (٣.٢.٦)

There are no periodic points of even period in Q_4.

Proof: Suppose there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in Q_4 such that $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is a periodic point of even period. That is there is $2n$ in Z^+, H $^{-2n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. (٣.٢٢)

Since $x_{-n} = y_{-n-1}, y_{-n} = \frac{-1}{b}(x_{-(n-1)} + y^2_{-(n-1)})$,then either H $^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,1}$ or H $^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_1$ for all $n \in N$ .If H $^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,1}$,from (٣.٢٢) H $^{-2n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ so H $^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$= H $^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, hence H $^{-2n}$(H $^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$)= H $^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, H $^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,1}$ so $Q_{2,1}$ has a periodic point which is contradiction .If H $^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,2}$, in the same way, H $^{-2n}$(H $^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$)= H $^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, H $^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_1$,that means there is a periodic point of even period in Q_1 which is contradiction ,so there are no periodic points of even period in Q_4. □

Lemma (٣.٢.٧)

There are no periodic points of odd period in $Q_{2,2}$.

Proof**:** Suppose that there is a periodic point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ of odd period m. That is there exists at least positive integer n such that $m = 2n+1$, and

$H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, since $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,2}$, we have $x_0 < -y_0^2$, $x_{-1} = y_0 \geq 0$, also

$y_{-1} = \frac{-1}{b}(x_0 + y_0^2) < 0$, hence $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_4$, that is $H^{-1}(H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}) = H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$,

also $H^{-2n-1}(H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}) = H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ hence $\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix}$ is a periodic point of odd period in Q_4 which is contradiction by lemma (٣.٢.٥),hence there are no periodic points of odd period in $Q_{2,2}$.

Lemma (٣.٢.٨)

There are no periodic points in R_1.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be any element in R_1, then $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_3, x_0 \geq -y_0^2$, that is $y_0 < 0$

$x_0 < 0$, $x_0 \geq -y_0^2$, so $x_{-1} = y_0 < 0$, $y_{-1} = \frac{-1}{b}(x_0 + y_0^2) \geq 0$, hence $\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} \in Q_2$. That is

$H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_2$ that means R_1 maps into Q_2, so by lemma (٣.٢.٤) and lemma (٣.٢.٥) there are no periodic points in R_1. □

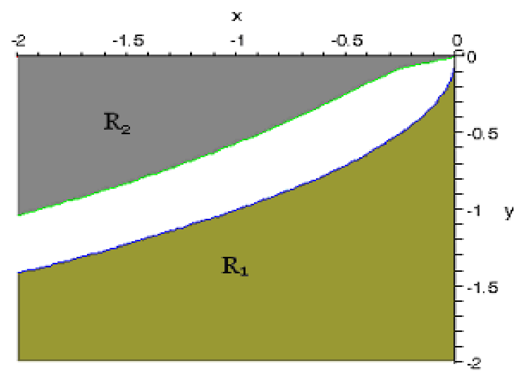
Lemma (٣.٢.٩)

There are no periodic points in R_2.

Proof : Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_2$ then $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_3, x_0 \leq b\sqrt{-y_0} - y_0^2$, that is $y_0 < 0, x_0 < 0$

$x_0 \leq b\sqrt{-y_0} - y_0^2$. Thus $x_0 + y_0^2 \leq b\sqrt{-y_0}$ so $\frac{-1}{b}(x_0 + y_0^2) \leq -\sqrt{-y_0}$, that is

$y_{-1} \le -\sqrt{-y_0}$ .Hence $y_{-1}^{2} \ge -y_0$, that is $y_{-1}^{2} \ge -x_1$ so $-y_{-1}^{2} \le x_1$, hence $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_1$.Hence R_2 maps into R_1, by lemma (٣.٢.٨) ,there are no periodic point in R_2 . □



Fig(3):The region R1,R2 where b=-0.5

Lemma (٣.٢.١٠)

There are no periodic point in R_3.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_3$, so $y_0 < \dfrac{-x_0 - x_0^{2}}{b}, -y_0^{2} - by_0 < x_0 < -y_0^{2}, x_0 < y_0$,we have $x_1 = -by_0 - x_0^{2}$, $y_1 = x_0$, since $-x_0 < y_0^{2} + by_0$, $x_0^{2} > y_0^{2}$, we get that $y_0^{2} + by_0 < x_0^{2} + by_0$, so $-x_0 < x_0^{2} + by_0$, that is , $x_0 > x_1$,since $x_0 = y_1$, we have

$y_1 > x_1$. (٣.٢٣)

Suppose that $-x_1 \ge y_1^{2} + by_1$, that is , $-x_1 \ge x_0^{2} + bx_0, bx_0 > by_0$, so

$-x_1 > x_0^{2} + by_0 = -x_1$ which is contradiction, hence $x_1 > -y_1^{2} - by_1$. (٣.٢٤)

Clearly, since $-by_0 < 0$, then $-by_0 - x_0^{2} < -x_0^{2}$,hence $x_1 < -y_1^{2}$. (٣.٢٥)

To show that $y_1 < \frac{-x_1 - x_1^{\,2}}{b}$, since $x_0 < y_0, b < 0$, we have

$-x_0^{\,2} - by_0 > -x_0^{\,2} - bx_0$,so $x_1 > -x_0^{\,2} - bx_0 = -x_0^{\,2} - by_1$. (٣.٢٦)

From $y_0 < \frac{-x_0 - x_0^{\,2}}{b}$, we have $by_0 > -x_0 - x_0^{\,2}$,hence $x_0 > x_1$,since $x_0 < 0$

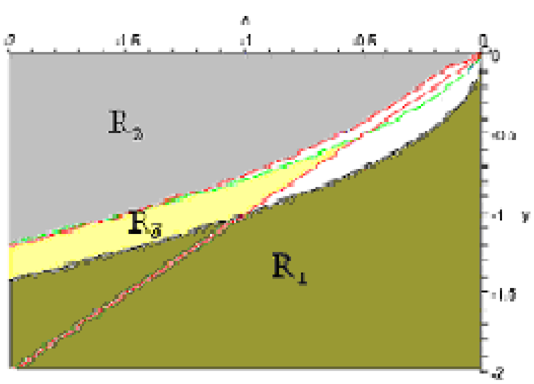
$x_0^{\,2} < x_1^{\,2}$,thus $-x_0^{\,2} - by_1 > -x_1^{\,2} - by_1$ then from (٣.٢٦),we get that

$x_1 > -x_1^{\,2} - by_1$,that is $by_1 > -x_1 - x_1^{\,2}$,hence $y_1 < \frac{-x_1 - x_1^{\,2}}{b}$. (٣.٢٧)

Now from (٣.٢٣), (٣.٢٤), (٣.٢٥)and (٣.٢٧), we can say that $H(R_3) \subset R_3$,so $H^2(R_3) \subset R_3$ and so on $H^n(R_3) \subset H^{n-1}(R_3) \subset R_3$,where n is positive integer hence ,

$x_n < y_n, y_n < \frac{-x_n - x_n^{\,2}}{b}$,that is $by_n > -x_n - x_n^{\,2}$,so $x_n > -x_n^{\,2} - by_n = x_{n+1}$

hence $\langle x_n \rangle$ is strictly decreasing sequence in R_3, so there are no periodic points in R_3.□



Fig(٤):The region R3 where b=-0.5

Lemma (٣.٢.١١)

There are no periodic point in R_4.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be any element in R_4 , then $y_0 < \dfrac{-x_0 - x_0^2}{b}$ (٣.٢٨)

and $b\sqrt{-y_0} - y_0^2 < x_0 \le -by_0 - y_0^2$. (٣.٢٩)

By (٣.٢٨) $by_0 > -x_0 - x_0^2$ or $x_0 > -by_0 - x_0^2 = x_1$, from (٣.١٨) $y_1 = x_0$,so

$y_1 > x_1$. (٣.٣٠)

Since $x_1 = -by_0 - x_0^2, -by_0 < 0$, we have $x_1 < -y_1^2$. (٣.٣١)

Now ,to show that $x_1 > -y_1^2 - by_1$, since $by_0 > -x_0 - x_0^2, -x_0 \ge by_0 + y_0^2$,we get $by_0 - x_0 > -x_0 - x_0^2 + y_0^2 + by_0$, so $-x_0^2 + y_0^2 < 0$, that is , $\sqrt{x_0^2} > \sqrt{y_0^2}$ hence $|x_0| > |y_0|$,since $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_3$, $|x_0| = -x_0$, $|y_0| = -y_0$,thus $x_0 < y_0$, so $y_1 < y_0$,since $b < 0$,we have $x_0^2 + by_0 < x_0^2 + by_1$, hence, $-x_1 < x_0^2 + by_1$. Now since $x_0 = y_1$, we get that $-x_1 < y_1^2 + by_1$ that is , $x_1 > -y_1^2 - by_1$. (٣.٣٢)

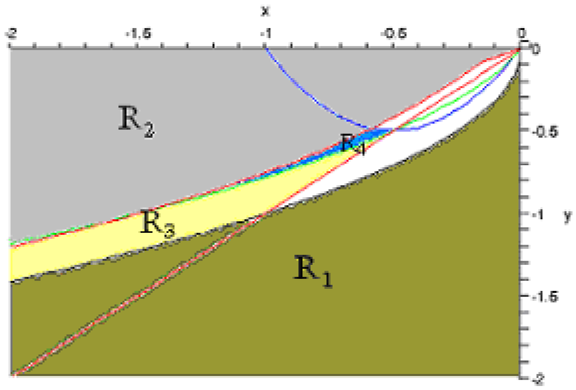
On the other hand , $x_0 < y_0$,thus , $-x_0^2 - by_0 > -x_0^2 - bx_0$,hence we get that

$x_1 > -x_0^2 - by_1$. (٣.٣٣)

From $by_0 > -x_0 - x_0^2$,then $x_0 > -x_0^2 - by_0$, so $x_0 > x_1$, x_0, x_1 are negative ,so $x_0^2 < x_1^2$, thus $-x_0^2 - by_1 > -x_1^2 - by_1$,hence from (٣.٣٣) $x_1 > -x_1^2 - by_1$, that is

$y_1 < \dfrac{-x_1 - x_1^2}{b}$. (٣.٣٤)

Now from (٣.٣٠), (٣.٣١), (٣.٣٢) and (٣.٣٤), we get that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in R_3$,so R_4 maps into R_3 ,then by lemma (٣.٢.١٠) there are no periodic points in R_4. □



Fig(5):The region R_4 where b=-0.5

Lemma (٣.٢.١٢)

There are no periodic points in R_5.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_5$, so $y_0 < \frac{-x_0 - x_0^2}{b}$, $-y_0^2 - by_0 < x_0 < -y_0^2$, $x_0 > y_0$

$by_0 + x_0^2 > 1$,we have $x_1 = -by_0 - x_0^2, y_1 = x_0$.

We claim that R_5 maps into R_4, since $x_0 \geq y_0$,we have $-x_0^2 - bx_0 \geq -x_0^2 - by_0$

also since $-x_0^2 - by_0 = x_1\ x_0 = y_1$, we have $x_1 \leq -y_1^2 - by_1$. (٣.٣٥)

On the other hand, since we have $x_0 < -y_0^2$,then $\sqrt{-x_0} > \sqrt{y_0^2} = |y_0| = -y_0$ so,

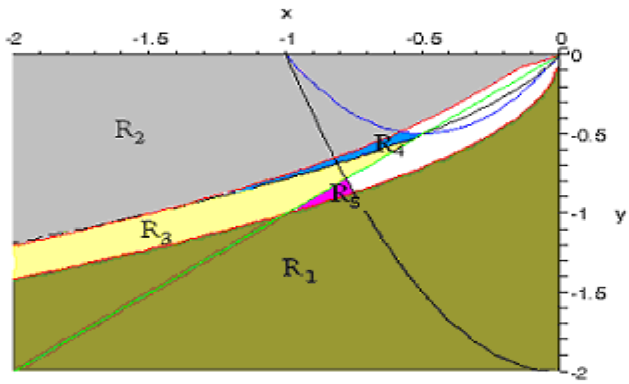
$by_0 < -b\sqrt{-x_0}$, hence $by_0 + x_0^2 < x_0^2 - b\sqrt{-x_0}$,thus $-x_1 < y_1^2 - b\sqrt{-y_1}$. (٣.٣٦)

To show that $y_1 < \frac{-x_1 - x_1^2}{b}$,since $by_0 + x_0^2 > 1$, we have $x_1 < -1$, so $x_1^2 > -x_1$.

Since $bx_0 > 0$,we get $bx_0 > -x_1 - x_1^2$,that is $x_0 < \frac{-x_1 - x_1^2}{b}$ but $y_1 = x_0$, so

$$y_1 < \frac{-x_1 - x_1^2}{b} \quad . \tag{٣.٣٧}$$

Now from (٣.٣٥), (٣.٣٦)and (٣.٣٧) ,our claim is true, so by lemma (٣.٢.١١) there are no periodic points in R_5. □



Fig(6):The region R_5 where b=-0.5

Lemma (٣.٢.١٣)

The region R_6 maps into $R_2 \cup R_7$ under backward iteration .

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be any element in R_6.We must show that $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_2 \cup R_7$.

From (٣.١٨) $x_{-1} = y_0, y_{-1} = \frac{-x_0 - y_0^2}{b}$, since $y_0 < -1-b$, we have

$$x_{-1} < -1-b \quad . \tag{٣.٣٨}$$

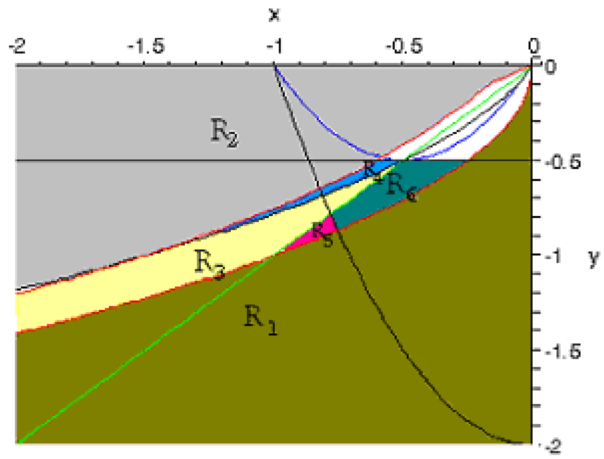
Since $x_0 > -y_0^2 - by_0$, we have $y_0 < \frac{-x_0 - y_0^2}{b}$,that is $x_{-1} < y_{-1}$. (٣.٣٩)

Also, since $x_0 > b\sqrt{-y_0} - y_0^2$,we have $\frac{x_0 + y_0^2}{b} < \sqrt{-y_0}$,so $-y_{-1} < \sqrt{-y_0}$

thus, $y_0 < -y_{-1}^2$,hence $x_{-1} < -y_{-1}^2$. (٣.٤٠)

From $x_0 \geq y_0$,we have $\frac{-x_0 - y_0{}^2}{b} \geq \frac{-y_0 - y_0{}^2}{b}$,hence $y_{-1} \geq \frac{-x_{-1} - x_{-1}{}^2}{b}$. (٣.٤١)

Now from (٣.٣٨), (٣.٣٩),(٣.٤٠)and (٣.٤١) $\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} \in R_2 \cup R_7$,that is $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in $R_2 \cup R_7$. □



Fig(٧):The region R_6 where b=-0.5

Lemma (٣.٢.١٤)

There are no periodic points in R_7.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be any element in R_7, since $y_0 \geq \frac{-x_0 - x_0{}^2}{b}$, the graph of the map $y = \frac{-x - x^2}{b}$ intersects x-axis at point (-١,٠),thus $x_0 > -1$,so $b\frac{-x_0 - y_0{}^2}{b} + y_0{}^2 < 1$, that is $by_{-1} + x_{-1}{}^2 < 1$. (٣.٤٢)

Since $-x_0 > 1+b$, $y_0 > -1-b$, we have $-y_0{}^2 - x_0 > (1+b)-(1+b)^2 = b(-1-b)$, hence $\frac{-x_0-y_0^{\ 2}}{b}<-1-b$, $y_{-1}<-1-b$. (٣.٤٣)

On the other hand , $x_0<-y_0^{\ 2}-by_0$, thus $y_0>\frac{-x_0-y_0^{\ 2}}{b}$, so $x_{-1}>y_{-1}$. (٣.٤٤)

Also , since $x_0>b\sqrt{-y_0}-y_0^{\ 2}$,we get $\frac{x_0+y_0^{\ 2}}{b}<\sqrt{-y_0}$,thus $-y_{-1}<\sqrt{-y_0}$

so $y_{-1}^{\ 2}<-y_0$,that is $x_{-1}<-y_{-1}^{\ 2}$. (٣.٤٥)

Hence, from (٣.٤٢), (٣.٤٣), (٣.٤٤) and (٣.٤٥) R_7 maps into R_6, by lemma (٣.٢.١٣) R_6 maps into $R_2\cup R_7$, so there is no point $\begin{pmatrix}x_0\\ y_0\end{pmatrix}$ in R_7 such that

$H^{-2n-1}\begin{pmatrix}x_0\\ y_0\end{pmatrix}\in R_7$.That is, there is no point $\begin{pmatrix}x_0\\ y_0\end{pmatrix}$ in R_7 such that $H^{-2n-1}\begin{pmatrix}x_0\\ y_0\end{pmatrix}=\begin{pmatrix}x_0\\ y_0\end{pmatrix}$

hence there are no periodic points of odd period in R_7.To show that there are no periodic points of even period in R_7.

Let P={$\begin{pmatrix}x_0\\ y_0\end{pmatrix}\in R_7$: $H^{-2n-1}\begin{pmatrix}x_0\\ y_0\end{pmatrix}\in R_6$, $H^{-2n}\begin{pmatrix}x_0\\ y_0\end{pmatrix}\in R_7$} $\forall n\in N$,if $\begin{pmatrix}x_0\\ y_0\end{pmatrix}\in$ P ,we claim that the sequence $\langle y_{-2n}\rangle$ is strictly increasing ,since $y_0>-1-b$,we have

$y_0(-1-b)<(1+b)^2$. (٣.٤٦)

Also $y_0^{\ 2}<(1+b)^2$, $-x_0>1+b$,so $-x_0-y_0^{\ 2}>-b(1+b)$, hence $\frac{-x_0-y_0^{\ 2}}{b}<-1-b$

that is , $(\frac{-x_0-y_0^{\ 2}}{b})^2>(1+b)^2$. (٣.٤٧)

Now from (٣.٤٦) , (٣.٤٧), we get that $(\frac{-x_0-y_0^{\ 2}}{b})^2>y_0(-1-b)=-y_0-by_0$

so , $by_0>-y_0-y_{-1}^{\ 2}$, that is $y_0<\frac{-x_{-1}-y_{-1}^{\ 2}}{b}=y_{-2}$.In general, since $\begin{pmatrix}x_{-2n}\\ y_{-2n}\end{pmatrix}\in R_7$

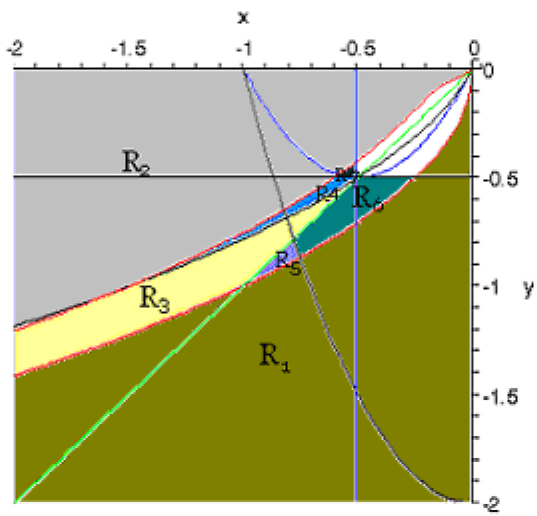
we have, $y_{-2n}>-1-b$, $x_{-2n}<-1-b$, so $y_{-2n}(-1-b)<(1+b)^2$, $y_{-2n}^{\ 2}<(1+b)^2$.

Hence $-x_{-2n}-y_{-2n}^{\ 2}>-b(1+b)$, that is $(\frac{-x_{-2n}-y_{-2n}^{\ 2}}{b})^2>(1+b)^2>(-1-b)y_{-2n}$.

So $y_{-2n-1}^{\ 2}>(-1-b)y_{-2n}=-y_{-2n}-by_{-2n}=-x_{-2n-1}-by_{-2n}$, that is , we get that

$by_{-2n}>-x_{-2n-1}-y_{-2n-1}^{\ 2}$, hence $y_{-2n}<\frac{-x_{-2n-1}-y_{-2n-1}^{\ 2}}{b}=y_{-2n-2}$, for all $n\geq 1$.

So, $\langle y_{-2n}\rangle$ is strictly increasing sequence in R_7, so there are no periodic points of even period in P. □



Fig(٤):The region R_7 where b=-0.5

Lemma (٣.٢.١٥)

There are no periodic points in R_6.

Proof: by lemma (٣.٢.١٣), R_6 maps into $R_2\cup R_7$, there are no periodic points in $R_2\cup R_7$, so there are no periodic points in R_6. □

Theorem (٣.٢.١٥)

There are no periodic points of $H_{0,b}$ in $R^{٢}$ other than fixed points.

Proof: If $x_0 > -1-b, y_0 = -1-b$,we have $-x_0^{\ 2} > -(1+b)^2$, $-by_0 = b+b^2$, hence $-x_0^{\ 2} - by_0 > b+b^2 \ -1-b^2-2b = -1-b$, so $x_1 > -1-b$, $y_1 = x_0 > -1-b$,all points $\begin{pmatrix} x \\ y \end{pmatrix}$ in R_3 satisfiy the equation $y < \frac{-x^2}{b}$,if $y_0 > -1-b, x_0 = -1-b$,we get that $-by_0 > b+b^2$, $x^2 = (1+b)^2$ so $-x^2 - by > -1-b$, that is $x_1 > -1-b$, $y_1 = x_0 = -1-b$

Hence from theorem (٣.١.٣) ,and previous lemmas (٣.٢.١),(٣.٢.٢),…,(٣.٢.١٤), ,(٣.٢.١٥) , since a periodic point under forward iteration implies one under backwards iteration, and viceversa . □

٣-٣ Iteration of Henon Map ($H_{0,b}$,b>٠)

The main purpose of this section is to give the proof of two propositions which ensures that orbits of every point in the two regions are infinite under forward iteration of Henon map $H_{0,b}$,which are W and V where

W=$\{\begin{pmatrix} x \\ y \end{pmatrix}: , x \le 0, y < -1-b, x \ge y\}$,V=$\{\begin{pmatrix} x \\ y \end{pmatrix}: , x < \frac{-|y+1+b|+y-1-b}{2}, y \ne 1-b\}$

where b is a positive real number .

Lemma (٣.٣.١)

Suppose W= $\{\begin{pmatrix} x \\ y \end{pmatrix}: , x \le 0, y < 1-b, x \ge y\}$, then there is no fixed point for $H_{0,b}^{-1}$ in W.

Proof : Suppose there is a fixed point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ for $H_{0,b}^{-1}$ in W, then this fixed point satisfies two equations $x_0 = y_0$, $y_0 = \frac{-x_0 - y_0^2}{b}$, thus $x_0(x_0 + (b+1)) = 0$.

Either $x_0 = 0$ or $x_0 = -1-b$, hence $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -1-b \\ -1-b \end{pmatrix}$

which is contradiction . □

Proposition (٣.٣.٢)

For $\begin{pmatrix} x \\ y \end{pmatrix} \in$ W, $\left\| H_{0,b}^{-n}\begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in$ W, then $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ \frac{-x_0 - y_0^{2}}{b} \end{pmatrix}$, since $x_0 \geq y_0$,we get that $\frac{-x_0 - y_0^{2}}{b} \leq \frac{-y_0 - y_0^{2}}{b}$. (٣.٤٨)

On the other hand, $y_0 < -1-b$, so $by_0 > -y_0^{2} - y_0$,hence $y_0 > \frac{-y_0 - y_0^{2}}{b}$ .From (٣.٤٨),we get that $y_0 > \frac{-x_0 - y_0^{2}}{b}$, that is $y_0 > y_{-1}$, $y_{-1} < -1-b$,so $x_{-1} > y_{-1}$.Now $y_{-1} < -1-b$. As above $by_{-1} > -y_{-1}^{2} - y_{-1}$, that is $y_{-1} > \frac{-y_{-1} - y_{-1}^{2}}{b}$.

Since $x_{-1} > y_{-1}$,we have $\frac{-x_{-1} - y_{-1}^{2}}{b} < \frac{-y_{-1} - y_{-1}^{2}}{b}$,that is $y_{-1} > y_{-2}$,we suppose that $y_{-(k-1)} > y_{-k}$, $y_{-k} < -1-b$. To show that $y_{-k} > y_{-(k+1)}$, since $y_{-k} < -1-b$ so $by_{-k} > -y_{-k}^{2} - y_{-k}$,hence $y_{-k} > \frac{-y_{-k} - y_{-k}^{2}}{b}$. (٣.٤٩)

Since $y_{-(k-1)} = x_{-k}$,by our supposition $y_{-k} < x_{-k}$,then $\frac{-x_{-k} - y_{-k}^{2}}{b} < \frac{-y_{-k} - y_{-k}^{2}}{b}$. From (٣.٤٩), we get $y_{-(k+1)} < y_{-k}$, so $\langle y_{-n} \rangle$ is strictly decreasing sequence ,that is $\langle y_{-n} \rangle$ is monotonic sequence , $\langle y_{-n} \rangle$ is bounded above by a negative real number M, if possible $\langle y_{-n} \rangle$ is bounded below then $\langle y_{-n} \rangle$ is convergent, $x_{-(n+1)} = y_{-n}$,that is $\langle x_{-n} \rangle$, also convergent so there is a fixed point for $H_{0,b}^{-1}$ in W which is contradiction to lemma (٣.٣.١). Hence $\langle y_{-n} \rangle$ is not bounded below, $\langle y_{-n} \rangle$ is strictly decreasing sequence $\lim_{n \to \infty} y_{-n} \longrightarrow -\infty$ and $\lim_{n \to \infty} x_{-n} \longrightarrow -\infty$, so $\lim_{n \to \infty} \left\| H_{0,b}^{-n}\begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$. □

Corollary (٣.٣.٣)

There are no periodic points for $H_{0,b}^{-1}$ in W.

Proof: Directly from Proposition (٣.٣.٢).

Proposition (٣.٣.٤)

Let V=$\{\begin{pmatrix} x \\ y \end{pmatrix} : x < \frac{-|y+1+b|+y-1-b}{2}, y \neq 1-b\}$ be an open region in Q_3, then for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in V , $\left\| H_{0,b}^{\ n} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be any element in V, since $y_0 \neq 1-b$,either $y_0 > 1-b$ or $y_0 < -1-b$ .If $y_0 > -1-b$, then $x_0 < -1-b < y_0$. (٣.٥٠)

If $y_0 < -1-b$, then $x_0 < y_0 < -1-b$. (٣.٥١)

So from (٣.٥٠), (٣.٥١), we get $x_0 < y_0$ and $x_0 < -1-b$. (٣.٥٢)

Now $x_1 - x_0 = -x_0^{\ 2} - by_0 - x_0 < -x_0^{\ 2} - (b+1)x_0$. (٣.٥٣)

Also $x_0 < -1-b$, so $x_0^{\ 2} > -x_0 - bx_0$, that is $-x_0^{\ 2} - (b+1)x_0 < 0$. Hence from (٣.٥٣) ,we get that $x_1 < x_0$,since $y_1 = x_0$,we have $x_1 < y_1$, $x_1 < -1-b$,hence

$x_2 - x_1 = -x_1^{\ 2} - by_1 - x_1 < -x_1^{\ 2} - (b+1)x_1$. (٣.٥٤)

Also $x_1 < -1-b$, so $x_1^{\ 2} > -x_1 - bx_1$, that is $-x_1^{\ 2} - (b+1)x_1 < 0$. Hence from (٣.٥٤) we get that $x_2 < x_1$,by the same way we continue ,we get for $n \in Z^+$ $x_n < x_{n-1} < ... < x_3 < x_2 < x_1 < x_0$ so $\langle x_n \rangle$ is strictly negative decreasing sequence so it is a monotonic sequence , if possible $\langle x_n \rangle$ is bounded below then $\langle x_n \rangle$ is convergent , $y_{n+1} = x_n$ that is $\langle y_n \rangle$ also convergent , H is continuous, so there exits a fixed point for H $_{0,b}$ in V which is contradiction .Hence $\langle x_n \rangle$ not bounded below that is $\langle x_n \rangle \longrightarrow -\infty$ as $n \longrightarrow \infty$ and $\langle y_n \rangle \longrightarrow -\infty$ as $n \longrightarrow \infty$ so $\underset{n \longrightarrow \infty}{Lim} \left\| H_{0,b}{}^n \begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$. □

<u>Corollary (٣.٣.٥)</u>

There are no periodic points for H $_{0,b}$ in V .

Proof: Directly from Proposition (٣.٣.٣).

CHAPTER FOUR

The Non Wandering Point and Periodic Point of Henon map

This chapter consists of two sections. In section one , we study non wandering point of Henon map. In section two, we study periodic point and non escape set of Henon map .

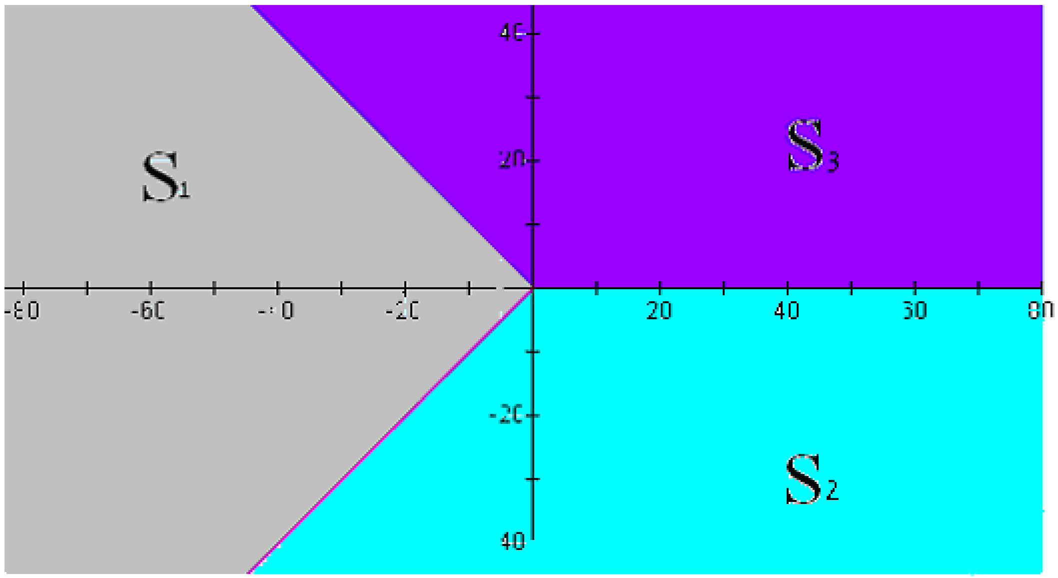
٤-١ The Non Wandering Point of Henon Map Where $b > 0$ and $a < -\frac{(1+b)^2}{4}$.

The main goal of this section is to prove two theorems on the henon map $H_{a,b}, b > 0$. To prove these theorems , we divide the plane into three regions such that the union of this three regions covers the plane. we suppose $a_0 = \frac{-(1+b)^2}{4}$, prove some lemmas with respect to parameters a,b and regions till we get the main purpose ,the regions are the following :

$$S_1 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in R, x \leq -|y| \}$$

$$S_2 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in R, x \geq -|y|, y \leq 0 \}$$

$$S_3 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in R, x \geq -|y|, y > 0 \}$$



Fig(9):Region S_1 ,S_2 ,S_3

Definition (٤.١.١)[٣]

Let F: $R^n \longrightarrow R^n$ be a map .A closed region Q $\subset R^n$ is trapping region for F if F(Q) is contained in the interior of Q.

Lemma (٤.١.٢)

Let $a < a_0$, $b > 0$ .Then S_1 is trapping region for $H_{a,b}$.

Proof: clearly S_1 is closed region .To show that $H_{a,b}(S_1) \subset S_1^{\,0}$.

Let $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{a,b}(S_1)$, then there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in S_1, such that $H_{a,b}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ so

$x = a - by_0 - x_0^{\,2}$, $y = x_0$, since $x_0 \in R$ we have two cases :

case١: $y_0 \geq 0$ then $|y_0| = y_0$, $x_0 \leq -y_0$, since $b > 0$, we have $by_0 \leq -bx_0$. (٤.١)

Thus $a + by_0 - x_0^{\,2} \leq a - bx_0 - x_0^{\,2}$. (٤.٢)

Since $-by_0$ is a negative real number we have $-by_0 < by_0$ hence

$a-by_0-x_0^{2} < a+by_0-x_0^{2}$. (٤.٣)

Now from (٤.٢) and (٤.٣) we get $x < a-bx_0-x_0^{2}$. (٤.٤)

Since $a < a_0$ from (٤.٤), we get $x < -\frac{(1+|b|)^2}{4} - bx_0 - x_0^{2}$

$$= -[\frac{(1+|b|)^2}{4} + (1+|b|)x_0 + x_0^{2}] + x_0$$

$$= -[\frac{1+|b|}{2} + x_0]^2 + x_0 .$$

So $x < x_0$, that is $x < y$. On the other hand $x_0 < 0$, so $|y| = -y$, $x < -|y|$, hence

$\begin{pmatrix} x \\ y \end{pmatrix} \in S_1^{0}$,that is $H_{a,b}(S_1) \subset S_1^{0}$.

Case٢: $y_0 < 0$ then $|y_0| = -y_0$,so $x_0 \le y_0$, since $b > 0$, we have $-by_0 \le -bx_0$ (٤.٥)

thus $a-by_0-x_0^{2} \le a-bx_0-x_0^{2}$. (٤.٦)

Now as the same as above case , we get $x < x_0$, $x_0 \le y_0 < 0$ so $|y| = -y$ thus

$x < -|y|$, hence $\begin{pmatrix} x \\ y \end{pmatrix} \in S_1^{0}$, that is $H_{a,b}(S_1) \subset S_1^{0}$. □

Lemma (٤.١.٣)

Let $a < a_0$, $b > 0$ .Then S_2 is trapping region for $H_{a,b}^{-1}$.

Proof: clearly, S_2 is closed region. To show that $H_{a,b}^{-1}(S_2) \subset S_2^{0}$, let

$\begin{pmatrix} x \\ y \end{pmatrix} \in H_{a,b}^{-1}(S_2)$,then there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in S_2, such that $H_{a,b}^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, so

$x = y_0$, $y = \frac{a-x_0-y_0^{2}}{b}$, since $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_2$ we have $x_0 \ge -|y_0|$ and $y_0 \le 0$ that is

$x_0 \ge y_0$. (٤.٧)

Now since $b > 0, a < a_0$ from (٤.٧) we have :

$$by = a - x_0 - y_0{}^2 < -\frac{(1+b)^2}{4} - x_0 - y_0{}^2$$

$$< -\frac{(1+b)^2}{4} - y_0 - y_0{}^2$$

$$= -[\frac{(1+b)^2}{4} + y_0 + by_0 - by_0 + y_0{}^2]$$

$$= -[\frac{(1+b)^2}{4} + (1+b)y_0 + y_0{}^2] + by_0$$

$$= -[\frac{(1+b)}{2} + y_0]^2 + by_0 .$$

Hence $by < by_0$,since $b > 0$, we get $y < y_0$,that is $y < x$, since $x = y_0 \le 0$, we get $|y| = -y, y < x$. Hence $-|y| < x$, so $\begin{pmatrix} x \\ y \end{pmatrix} \in S_2{}^0$, hence $H_{a,b}{}^{-1}(S_2) \subset S_2{}^0$. □

Proposition (٤.١.٤)[٤]

Let $a < a_0$, $b > 0$.

(i) $H_{a,b}(S_1 \cup S_3) \subset S_1{}^0$

(ii) $\langle x_n \rangle$ is strictly decreasing sequence along $H_{a,b}$ -orbits and $\left\| H_{a,b}{}^n \begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $n \longrightarrow \infty$, for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_1.

(iii) $H_{a,b}{}^{-1}(S_2 \cup S_3) \subset S_2{}^0$.

(iv) $\langle |y_{-n}| \rangle$ is strictly increasing sequence along $H_{a,b}{}^{-1}$ -orbits and $\left\| H_{a,b}{}^{-n} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $n \longrightarrow \infty$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_2.

proof:(i) Let $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{a,b}(S_1 \cup S_3)$, so there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in $S_1 \cup S_3$, such that $x = a - by_0 - x_0{}^2$, $y = x_0$,since $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_1 \cup S_3$, we have $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_1$ or $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_3$,if $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_1$. From lemma (٤.١.٢), $\begin{pmatrix} x \\ y \end{pmatrix} \in S_1{}^0$,in the case $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_3$ we have $x_0 > -|y_0|$ and $y_0 \geq 0, x_0 \in R$.

case١: If $x_0 \leq 0$, then $|y| = -y$ and $-x_0$ is positive, so $-x_0 \geq x_0 > -y_0$. Since $b > 0$, we get $-bx_0 > -by_0$,hence $x = a - by_0 - x_0{}^2 < -\frac{(1+b)^2}{4} - by_0 - x_0{}^2$ thus $x < -[\frac{(1+b)^2}{4} + (1+b)x_0 + x_0{}^2] + x_0$, that is $x < -[\frac{(1+b)}{2} + x_0]^2 + x_0$ so $x < x_0$.

Since $x_0 = y$, we get that $x < -|y|$ so $\begin{pmatrix} x \\ y \end{pmatrix} \in S_1{}^0$.

Case ٢: If $x_0 > 0$, then $|y| = y, x_0 > -y_0$, since $b > 0$, we get $by_0 > -bx_0$. (٤.٨)

Since $y_0 > 0, a < 0$,we get $x = a - by_0 - x_0{}^2 < 0$ so $|x| = -x$. (٤.٩)

On the other hand $x = a - by_0 - x_0{}^2 < -\frac{(1+b)^2}{4} - by_0 - x_0{}^2 < 0$, hence we get that

$$|x| > \left|\frac{(1+b)^2}{4} + by + x_0{}^2\right| = \frac{(1+b)^2}{4} + by_0 + x_0{}^2 \quad . \qquad (٤.١٠)$$

Now from (٤.٨), (٤.١٠) ,we get that $|x| > \frac{(1+b)^2}{4} - bx_0 + x_0{}^2 + x_0 - x_0$ thus

$$|x| > (\frac{(1+b)}{2} - x_0)^2 + x_0 \quad . \qquad (٤.١١)$$

Now from (٤.٩),(٤.١١),we get $-x > x_0$,that is $-x > y = |y|$ so $x < -|y|$, hence $\begin{pmatrix} x \\ y \end{pmatrix} \in S_1{}^0$ that is $H_{a,b}(S_1 \cup S_3) \subset S_1{}^0$.

(ii) From lemma (٤.١.٢) S_1 is trapping region for $H_{a,b}$, that is $H_{a,b}(S_1) \subset S_1{}^0$ and $x_1 < x_0$ and this is true for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_1, so it is true for $H_{a,b}(S_1)$,that is we have $H_{a,b}(H_{a,b}(S_1)) \subset (H_{a,b}(S_1))^0$ and $(H_{a,b}(S_1))^0 \subset (S_1{}^0)^0 = S_1{}^0$, so $H_{a,b}{}^2(S_1) \subset S_1{}^0$, and $x_2 < x_1 < x_0$ and by the same way, we get for $n \in N$ $H_{a,b}{}^n(S_1) \subset S_1{}^0$ and $x_n < \ldots < x_2 < x_1 < x_0$,that is $\langle x_n \rangle$ is strictly decreasing sequence .

For the second part , since $x_0 < 0$ a sequence $\langle x_n \rangle$ is bounded above by zero, if possible $\langle x_n \rangle$ is bounded below then it is bounded and its monotone so it is convergent , $y_{n+1} = x_n$ that is $\langle y_n \rangle$ also convergent , $H_{a,b}$ is continuous, so there exits a fixed point for $H_{a,b}$ in S_1 which is contradiction to proposition (٢.١.٥).Hence $\langle x_n \rangle$ not bounded below that is $\langle x_n \rangle \longrightarrow -\infty$ as $n \longrightarrow \infty$ and the sequence $\langle y_n \rangle \longrightarrow -\infty$ as $n \longrightarrow \infty$ that is $\left\| H_{a,b}{}^n \begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $n \longrightarrow \infty$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_1.

(iii) Let $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{a,b}{}^{-1}(S_2 \cup S_3)$,so there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in $S_2 \cup S_3$, such that $x = y_0, y = \dfrac{a - x_0 - y_0{}^2}{b}$,since $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_1 \cup S_3$, we have $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_2$ or $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_3$.

If $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_2$ then $H_{a,b}{}^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in H_{a,b}{}^{-1}(S_2)$, from lemma (٤.١.٣) S_2 is trapping region for $H_{a,b}$,so $\begin{pmatrix} x \\ y \end{pmatrix} \in S_2{}^0$,in the case $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_3$, we have $x_0 > -|y_0|$ and $y_0 \geq 0$ that is $x_0 > -y_0$. (٤.١٢)

Now since $b > 0, a < a_0$ from (٤.١٢) we have :

$$by = a - x_0 - y_0{}^2 < -\frac{(1+b)^2}{4} - x_0 - y_0{}^2$$

$$< -\frac{(1+b)^2}{4} + y_0 - y_0{}^2$$

$$= -[\frac{(1+b)^2}{4} - y_0 - by_0 + by_0 + y_0{}^2]$$

$$= -[\frac{(1+b)^2}{4} - (1+b)y_0 + y_0{}^2] - by_0$$

$$= -[\frac{(1+b)}{2} - y_0]^2 - by_0$$

hence $by < -by_0$ and $b > 0$,dividing both sides by b , we get $y < -y_0$,that is $y < -x$. Since $x = y_0 > 0$, we have $-x$ is a negative real number, so $y < x$.

Now since $y < -y_0$, we have $y < 0$ thus $|y| = -y$ hence $-|y| < x$, so $\begin{pmatrix} x \\ y \end{pmatrix} \in S_2{}^0$.

Hence $H_{a,b}{}^{-1}(S_2 \cup S_3) \subset S_2{}^0$.

(iv)By lemma(٤.١.٣), S_2 is trapping region for $H_{a,b}{}^{-1}$ that is $H_{a,b}{}^{-1}(S_2) \subset S_2{}^0$ and $y_{-1} < y_0 < 0$, $\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} \in S_2{}^0$, which is a subset of S_2, so it is true for $S_2{}^0$ that is $H_{a,b}{}^{-1}(S_2{}^0) \subset S_2{}^0$, so $y_{-2} < y_{-1} < y_0 < 0$ and $H_{a,b}{}^{-2}(S_2) \subset H_{a,b}{}^{-1}(S_2{}^0)$ and $H_{a,b}{}^{-3}(S_2{}^0) \subset S_2{}^0$ so we get that $y_{-3} < y_{-2} < y_{-1} < y_0 < 0$, by the same way we get that for $n \in N$, $H_{a,b}{}^{-n}(S_2) \subset S_2{}^0$, so we have $\dots y_{-n} < \dots < y_{-3} < y_{-2} < y_{-1} < y_0 < 0$, that is $\langle |y_{-n}| \rangle$ is strictly increasing sequence .

For the second part a sequence $\langle y_{-n} \rangle$ bounded above by zero ,if possible $\langle y_{-n} \rangle$ is bounded below then it is bounded and its monotone,so it is convergent ,since $x_{-(n+1)} = y_{-n}$,that is $\langle x_{-n} \rangle$ also convergent , $H_{a,b}$ is continuous so there exits a fixed point for $H_{a,b}$ in S_2, which is contradiction, hence $\langle y_{-n} \rangle$ is not bounded below that is $\langle y_{-n}\rangle \longrightarrow -\infty$ as $n \longrightarrow \infty$ and $\langle x_{-n}\rangle \longrightarrow -\infty$ as $n \longrightarrow \infty$, that is $\left\| H_{a,b}{}^{-n}\begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $n \longrightarrow \infty$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_2. □

Remark: From Proposition (٤.١.٤) we note that $S_1 \cup S_3$ is trapping region for $H_{a,b}$ and $S_2 \cup S_3$ is trapping region for $H_{a,b}{}^{-1}$.

Theorem (٤.١.٥)

Let $a < a_0(b)$, $b > 0$,the Henon map $H_{a,b}$ has no periodic point for any period, that is $P_{er_n}(H_{a,b}) = \phi$.

Proof: Suppose that there exists a periodic point $\begin{pmatrix} x \\ y \end{pmatrix}$ for $H_{a,b}$ of period n, where $n \in N$ then must be in R٢ and from the partition with respect to S_1, S_2 and S_3.Since $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$ then either $\begin{pmatrix} x \\ y \end{pmatrix}$ in $S_1 \cup S_2$ or S_3. If $\begin{pmatrix} x \\ y \end{pmatrix} \in S_1 \cup S_2$, then by lemma(٤.١.٤)(ii) $\langle x_n\rangle$ is strictly decreasing sequence along $H_{a,b}$ -orbits in S_1 and $\left\| H_{a,b}{}^{n}\begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $n \longrightarrow \infty$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_1 and from lemma(٤.١.٤)(iv) $\langle |y_{-n}|\rangle$ is strictly increasing sequence along $H_{a,b}{}^{-1}$ orbits in S_2 and $\left\| H_{a,b}{}^{-n}\begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $n \longrightarrow \infty$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_2 that means there is no finite orbit for any point in $S_1 \cup S_2$, so $\begin{pmatrix} x \\ y \end{pmatrix}$ has no finite orbit which is contradiction .

If $\begin{pmatrix} x \\ y \end{pmatrix} \in S_3$, by lemma (٤.١.٤)(i) $H_{a,b}(S_3) \subset S_1{}^0$ and $S_1{}^0 \subset S_1$, thus S_3 maps into S_1 and by our supposition $\begin{pmatrix} x \\ y \end{pmatrix}$ is periodic, so $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is a periodic point in S_1 which is contradiction . So there is no periodic point for $H_{a,b}$ in

$S_1 \cup S_2 \cup S_3$ and since $S_1 \cup S_2 \cup S_3 = R^٢$. We get $H_{a,b}$ has no periodic point for any period .□

Definition (٤.١.٦)[٣],[٢٥]

Let F: $R^٢ \longrightarrow R^٢$ be a map. A point $\begin{pmatrix} p \\ q \end{pmatrix}$ is called **non wandering** provided that for every neighborhood U of $\begin{pmatrix} p \\ q \end{pmatrix}$, there is an integer $n > 0$ such that $F^n(U) \cap U \neq \phi$ .Thus ,there is a point $\begin{pmatrix} r \\ s \end{pmatrix} \in U$ with $F^n \begin{pmatrix} r \\ s \end{pmatrix} \in U$. The set of all non wandering points for F is called the **non wandering set** and is denoted by Ω(F).

Definition (٤.١.٧)[٢٣]

Let F: $R^٢ \longrightarrow R^٢$ be a map. Let K^+(F) denotes the set of points in $R^٢$ with bounded forward orbits .Let K^-(F) denotes the set of points in $R^٢$ with bounded backward orbits then the set K(F)= $K^+(F) \cap K^-(F)$ is called filled Julia set .

Definition (٤.١.٨)[٢٣]

Let F: $R^٢ \longrightarrow R^٢$ be a map. Let $J^\pm(F) = \partial K^\pm(F)$,and J(F)= $J^+(F) \cap J^-(F)$ then $J^\pm$(F) is called the forward /backward Julia set and J(F) is the Julia set of F.

Theorem (٤.١.٩)[٢٣]

Let F be a hyperbolic regular polynomial automorphism of C^n with $|\det DF| \le 1$ .Then Ω(F) = J(F)$\cup \{ \alpha_1, \alpha_2, \ldots, \alpha_m \}$, where the α_i are the attracting periodic points of F.

Theorem (٤.١.١٠)

Let $a < a_0(b)$, $b > 0$, the Henon map $H_{a,b}$ has no non wandering point , that is $\Omega(H_{a,b}) = \phi$.

Proof: since $b < 1$ and from definition (١.١.١١) $|\det DF| = \left|\det\begin{pmatrix} -2x & -b \\ 1 & 0 \end{pmatrix}\right| = |b| < 1$,

for $\begin{pmatrix} x \\ y \end{pmatrix} \in R^{٢}$ we have three cases :

case ١ : If $\begin{pmatrix} x \\ y \end{pmatrix} \in S_1$, from lemma (٤.١.٤) part (ii) , $\langle x_n \rangle$ is strictly decreasing sequence along $H_{a,b}$-orbits and $\langle x_n \rangle$ is not bounded below that is $\langle x_n \rangle \longrightarrow -\infty$ as $n \longrightarrow \infty$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_1 ,that is there is no element in S_1 such that has bounded forward orbits ,so by definition (٤.١.٧) $K^+(H_{a,b}) = \phi$ and by definition (٤.١.٨) $J^+(H_{a,b}) = \partial(\phi)$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_1,since $\partial(\phi) = \phi$,we have $J^+(H_{a,b}) = \phi$ hence by definition (٤.١.٨) $J(H_{a,b}) = \phi \cap J^-(H_{a,b}) = \phi$.

Case ٢:If $\begin{pmatrix} x \\ y \end{pmatrix} \in S_2$, from lemma (٤.١.٤) part (iv) the sequence $\langle y_{-n} \rangle$ is strictly decreasing sequence along $H_{a,b}{}^{-1}$-orbit and it is not bounded below , that is $\langle y_{-n} \rangle \longrightarrow -\infty$ as $n \longrightarrow \infty$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_2 ,that is there is no element in S_2 such that has bounded backward orbit so by definition (٤.١.٧) $K^-(H_{a,b}) = \phi$ and by definition (٤.١.٨) $J^-(H_{a,b}) = \partial(\phi)$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_2 ,we have $\partial(\phi) = \phi$ so $J^-(H_{a,b}) = \phi$, hence by definition (٤.١.٨) we get $J(H_{a,b}) = J^+(H_{a,b}) \cap \phi = \phi$.

Case ٣: If $\begin{pmatrix} x \\ y \end{pmatrix} \in S_3$,by lemma (٤.١.٤) part (i) $H_{a,b}(S_3) \subset S_1^{\,0}$,hence $H_{a,b}(S_3) \cap S_3 \subset S_3 \cap S_1^{\,0}$, since $S_3 \cap S_1^{\,0} = \phi$, we have $H_{a,b}(S_3) \cap S_3 = \phi$ $H_{a,b}(S_3) \subset S_1^{\,0}$, so S_3 maps into S_1, hence by case ١ ,we have $J^+(H_{a,b}) = \phi$. So from definition (٤.١.٨), $J(H_{a,b}) = \phi \cap J^+(H_{a,b}) = \phi$ .Now for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in R^٢, we get $J(H_{a,b}) = \phi$,by theorem (٤.١.٥), $H_{a,b}$ has no periodic point for any period ,hence by theorem (٤.١.٩) $\Omega(H_{a,b}) = \phi$. □

٤-٢ Iteration with Periodic Point of Henon Map where $a > \frac{-(1+|b|)^2}{4}$

The main purpose of this section is to prove one theorem on Henon map $H_{a,b}$ where $a > \frac{-(1+|b|)^2}{4}$. To prove this theorem, we need to prove some lemmas. To state and prove our lemmas, we fix b and define two crucial $a-$values

١- $a_0(b) = \frac{-(1+|b|)^2}{4}$

٢- $a_1(b) = 2(1+|b|)^2$

and, for any particular $a-$value ,we define C=$C_{a,b}$ by

$C_{a,b} = \frac{1+|b|+\sqrt{(1+|b|)^2+4a}}{2}$, we will go to prove necessary lemmas.

<u>Lemma (٤.٢.١)[٤]</u>

(i) C is positive real and the larger root of $C^2-(1+|b|)C-a=0$ if and only if $a_0 \le a$.

(ii) $a-|b|C > C$ if and only if $a > a_1$

proof: (i) Since $a_0 \le a$, we have $(1+|b|)^2+4a \ge 0$,so C is positive real number, if $C^2-(1+|b|)C-a=0$,since $(1+|b|)^2+4a \ge 0$ .We have two distinct real roots which are $\frac{1+|b|+\sqrt{(1+|b|)^2+4a}}{2}$ and $\frac{1+|b|-\sqrt{(1+|b|)^2+4a}}{2}$,the first is positive and the second may be negative so C is larger root .

(ii) If $a-|b|C>C$, this implies that $a>(1+|b|)C$ (٤.١٣)

We have $2C-(1+|b|)=\sqrt{(1+|b|)^2+4a}$. (٤.١٤)

Now let $(1+|b|)=\beta$, then $2C-\beta=\sqrt{\beta^2+4a}$, that is $(2C-\beta)^2=\beta^2+4a$, so $C^2-\beta C=a$ .We put the value of a in (٤.١٣) ,we get that $C^2-\beta C>\beta C$ (٤.١٥) so $C^2>2\beta C$,since C is positive we have $C>2\beta$, that is $C>2(1+|b|)$. (٤.١٦)

Now by (٤.١٦) $(1+|b|)+\sqrt{(1+|b|)^2+4a}>4(1+|b|)$,that is $4a+(1+|b|)^2>9(1+|b|)^2$, hence $a>2(1+|b|)^2$.

By the same way, we can prove that if $a>a_1$ then $a-|b|C>C$. □

Lemma (٤.٢.٢)[٤]

(i) The image under $H_{a,b}$ of the horizontal strip $|y_0|\le\gamma$ is the region bounded by the two parabolas $a-|b|\gamma-{y_1}^2\le x_1\le a+|b|\gamma-{y_1}^2$ and the image under $H_{a,b}$ of the vertical strip $|x_0|\le\gamma$ is the horizontal strip $|y_1|\le\gamma$, where γ is positive real number .

(ii) The inverse image of the vertical strip $|x_0|\le\gamma$ is the region bounded by two parabolas $-\gamma+a-{x_{-1}}^2\le by_{-1}\le a+\gamma-{x_{-1}}^2$ and the image of the horizontal strip $|y_0|\le\gamma$ is the vertical strip $|x_{-1}|\le\gamma$.

Proof: (i) we have $|y_0|\le\gamma$, so $-\gamma\le y_0\le\gamma$, if $-1<b<0$ then $b\gamma\le -by_0\le -b\gamma$ and $|b|=-b$, hence $-|b|\gamma\le -by_0\le|b|\gamma$. (٤.١٧)

If $0 < b < 1$,then $b\gamma \geq -by_0 \geq -b\gamma$, $|b| = b$ hence $-|b|\gamma \leq -by_0 \leq |b|\gamma$. (٤.١٨)

Now in all case $-|b|\gamma \leq -by_0 \leq |b|\gamma$,so $a-|b|\gamma \leq a-by_0 \leq a+|b|\gamma$,since $x_0 = y_1$

$$a-|b|\gamma - y_1^{\,2} \leq x_1 \leq a+|b|\gamma - y_1^{\,2} \quad . \qquad (٤.١٩)$$

Also if $|x_0| \leq \gamma$, since $x_0 = y_1$,we have $|y_1| \leq \gamma$ that means the image under $H_{a,b}$ of the vertical strip $|x_0| \leq \gamma$ is the horizontal strip $|y_1| \leq \gamma$.

(ii) We have $|x_0| \leq \gamma$ so $\gamma \leq -x_0 \leq -\gamma$ so $a-\gamma - x_{-1}^{\,2} \leq by_{-1} \leq a+\gamma - x_{-1}^{\,2}, y_0 = x_{-1}$, so $|y_0| = |x_{-1}|$,the image under $H_{a,b}^{\ -1}$ of the horizontal strip $|y_0| \leq \gamma$ is the vertical strip $|x_{-1}| \leq \gamma$. □

Lemma (٤.٢.٣)[٢]

Let $P\begin{pmatrix} a \\ b \\ \gamma \end{pmatrix} = \{ \begin{pmatrix} x \\ y \end{pmatrix} : a-|b|\gamma - y^2 \leq x \leq a+|b|\gamma - y^2 \}$ $S_h(\alpha, \beta) = R \times [\alpha, \beta]$ and $S_v(\alpha, \beta) = [\alpha, \beta] \times R, \gamma \in R$ and then for the Henon map $H_{a,b}$ the following are hold :

(i) $H_{a,b}(S_v(-\gamma, \gamma)) = S_h(-\gamma, \gamma)$.

(ii) $H_{a,b}(S_h(-\gamma, \gamma)) \subset P\begin{pmatrix} a \\ b \\ \gamma \end{pmatrix}$.

(iii) $H_{a,b}^{\ -1}(P\begin{pmatrix} a \\ b \\ \gamma \end{pmatrix}) \subset S_h(-\gamma, \gamma)$.

Proof: (i) Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{H}_{a,b}(\mathrm{S}_v(-\gamma,\gamma))$, so there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathrm{S}_v(-\gamma,\gamma)$ such that $\mathrm{H}_{a,b}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, that is $x = a - by_0 - x_0^{\ 2}$, $y = x_0$, since $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in [-\gamma,\gamma] \times \mathrm{R}$ we have $x_0 \in [-\gamma,\gamma]$, hence $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{R} \times [-\gamma,\gamma]$, that is $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{S}_h(-\gamma,\gamma)$.

Conversely: Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{S}_h(-\gamma,\gamma)$, then $x \in \mathrm{R}$ and $y \in [-\gamma,\gamma]$,so $\frac{a - x - y^2}{b} \in \mathrm{R}$,thus $\begin{pmatrix} y \\ \frac{a - x - y^2}{b} \end{pmatrix} \in [-\gamma,\gamma] \times \mathrm{R}$, since $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{H}_{a,b}\begin{pmatrix} y \\ \frac{a - x - y^2}{b} \end{pmatrix}$,we have $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{H}_{a,b}(\mathrm{S}_v(-\gamma,\gamma))$.

(ii) Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{H}_{a,b}(\mathrm{S}_h(-\gamma,\gamma))$,so there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathrm{S}_h(-\gamma,\gamma)$ such that $\mathrm{H}_{a,b}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathrm{R} \times [-\gamma,\gamma]$,so we get that $|y_0| \le \gamma$ and by lemma (٤.٢.٢) Part (i) $a - |b|\gamma - y_1^{\ 2} \le x_1 \le a + |b|\gamma - y_1^{\ 2}$ so $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{P}\begin{pmatrix} a \\ b \\ \gamma \end{pmatrix}$.

(iii) Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{H}_{a,b}^{\ -1}(\mathrm{P}\begin{pmatrix} a \\ b \\ \gamma \end{pmatrix})$,so there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathrm{P}\begin{pmatrix} a \\ b \\ \gamma \end{pmatrix}$ such that $\mathrm{H}_{a,b}^{\ -1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, hence $a - |b|\gamma - y_0^{\ 2} \le x_0 \le a + |b|\gamma - y_0^{\ 2}$. Clearly $x \in \mathrm{R}$.To show that $y \in [-\gamma,\gamma]$, suppose that $y > \gamma$ then $\frac{a - x_0 - y_0^{\ 2}}{b} > \gamma$ (٤.٢٠)

Now if $b > 0$, then from (٤.٢٠) $a - x_0 - y_0^{\ 2} > |b|\gamma$, so $x_0 < a - |b|\gamma - y_0^{\ 2}$ which is contradiction ,if $b < 0$ then from (٤.٢٠) $a - x_0 - y_0^{\ 2} < b\gamma$, so $x_0 > a + |b|\gamma - y_0^{\ 2}$ which is contradiction .

To show that $y \ge -\gamma$, if $b > 0$ then $x_0 \le a + |b|\gamma - y_0^{\ 2}$ so $-b\gamma \le a - x_0 - y_0^{\ 2}$ thus $-\gamma \le \frac{a - x_0 - y_0^{\ 2}}{b} = y$, that is $y \ge -\gamma$,if $b < 0$ then $|b| = -b$, hence $x_0 \ge a + b\gamma - y_0^{\ 2}$

so $-b\gamma \geq a - x_0 - y_0{}^2$ thus $-\gamma \leq \frac{a - x_0 - y_0{}^2}{b} = y$, that is $y \geq -\gamma$ hence $y \in [-\gamma, \gamma]$

$\begin{pmatrix} x \\ y \end{pmatrix} \in S_h(-\gamma, \gamma)$. □

Proposition (٤.٢.٤)[٢]

Let $S_{a,b} = \{\begin{pmatrix} x \\ y \end{pmatrix} : |x| \leq C_{a,b}, |y| \leq C_{a,b}\}$ be a closed region in R^2, for Henon map $H_{a,b}$, if $b \neq 0$ then $H_{a,b}(S_{a,b}) = P\begin{pmatrix} a \\ b \\ C \end{pmatrix} \cap S_h(-C, C)$.

Proof: From definition of $S_{a,b}$, we have $S_{a,b} = S_h(-C,C) \cap S_v(-C,C)$.

So $H_{a,b}(S_{a,b}) = H_{a,b}(S_h(-C,C)) \cap H_{a,b}(S_v(-C,C))$. (٤.٢١)

Now by lemma (٤.٢.٣)(ii) $H_{a,b}(S_h(-C,C)) \subset P\begin{pmatrix} a \\ b \\ C \end{pmatrix}$. (٤.٢٢)

By lemma (٤.٢.٣)(iii),since $H_{a,b}$ is diffeomorphism , we get that

$P\begin{pmatrix} a \\ b \\ C \end{pmatrix} \subset H_{a,b}(S_h(-C,C))$. (٤.٢٣)

Hence from (٤.٢٢) and (٤.٢٣), we get that $H_{a,b}(S_h(-C,C)) = P\begin{pmatrix} a \\ b \\ C \end{pmatrix}$ (٤.٢٤)

From lemma (٤.٢.٣)(i) we have $H_{a,b}(S_v(-C,C)) = S_h(-C,C)$. (٤.٢٥)

Now by (٤.٢١), (٤.٢٤) and (٤.٢٥) $H_{a,b}(S_{a,b}) = P\begin{pmatrix} a \\ b \\ C \end{pmatrix} \cap S_h(-C,C)$. □

Remark (٤.٢.٥)

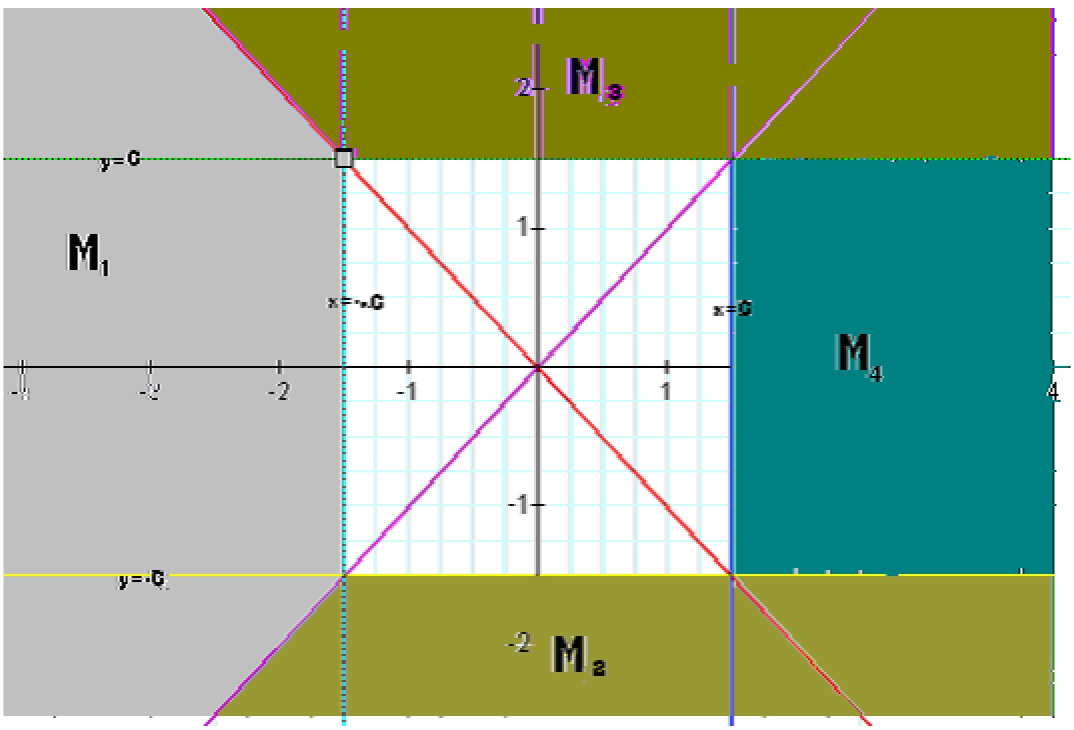
We define some region and proof one theorem on Henon map for $a > a_0$, the regions are M_1, M_2, M_3 and M_4 where :

$$M_1 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x < \frac{-\left|C - |y|\right| - C - |y|}{2} \}.$$

$$M_2 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x \geq y, y < -C \}.$$

$$M_3 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x \geq -y, y > C \}.$$

$$M_4 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x > C, |y| < C \}.$$



Fig(10)Region M_1,M_2,M_3,M_4

<u>Theorem (٤.٢.٦)[٣]</u>

Suppose $S_{a,b} = \{ \begin{pmatrix} x \\ y \end{pmatrix} : |x| \le C_{a,b}, |y| \le C_{a,b} \}$ is a closed region in R^2 .If $a > a_0(b)$ and $b > 0$,then for all $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2 - S_{a,b}$ either $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$ or $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Lemma (٤.٢.٧)

Let $H_{a,b}$ be a Henon map, $\begin{pmatrix} x \\ y \end{pmatrix} \in M_1$ and $b > 0$ .Then the sequence $\langle x_n \rangle$ is strictly decreasing sequence and $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Proof: If $|y_0| \ne C$ then we have two cases:

Case ١: $|y_0| < C$, hence $\left| C - |y_0| \right| = C - |y_0|$, so $x_0 < \dfrac{-(C - |y_0|) - C - |y_0|}{2} = -C$ thus

$x_0 < -C < -|y_0|$. (٤.٢٦)

Now $x_1 - x_0 = a - by_0 - x_0^2 - x_0$, so $x_1 - x_0 \le a + |b| y_0 - x_0^2 - x_0$. (٤.٢٧)

Since $x_0 < -|y_0|$,we have $|y_0| < -x_0$, so by (٤.٢٧) $x_1 - x_0 < a - |b| x_0 - x_0^2 - x_0$ (٤.٢٨)

$a - |b| x_0 - x_0^2 - x_0 = a - (|b| + 1) x_0 - x_0^2$ but this equation has two roots which are

$x_0^{\mp} = \dfrac{-(1 + |b|) \mp \sqrt{(1 + |b|)^2 + 4a}}{2}$, one of them is $-C$, for any value less than $-C$

$a - (|b| + 1) x_0 - x_0^2 < 0$. From (٤.٢٦), we have $x_0 < -C$, so $a - (|b| + 1) x_0 - x_0^2 < 0$, hence by (٤.٢٨) $x_1 - x_0 < 0$, that is $x_1 < x_0$.

Case ٢: $|y_0| > C$, hence $\left| C - |y_0| \right| = |y_0| - C$, so $x_0 < \dfrac{-(|y_0| - C) - C - |y_0|}{2} = -|y_0|$

and $|y_0| > C$, thus $x_0 < -|y_0| < -C$. (٤.٢٩)

Now $x_1 - x_0 = a - by_0 - {x_0}^2 - x_0 \le a + |b||y_0| - {x_0}^2 - x_0$

$$< a - |b|x_0 - {x_0}^2 - x_0 = a - (1+|b|)x_0 - {x_0}^2 \quad . \qquad (٤.٣٠)$$

As case ١, $a - (1+|b|)x_0 - {x_0}^2 = 0$ has two roots one of them is $-C$ .From (٤.٢٩) , $x_0 < -C$, so $a - (1+|b|)x_0 - {x_0}^2 < 0$, hence by (٤.٣٠) $x_1 < x_0$.

Now since $x_0 < \dfrac{-(C-|y_0|) - C - |y_0|}{2}$,we have $x_0 < -C$, that is $y_1 < -C$ so $|y_1| > C$.

Hence $|y_1| - C = |C - |y_1||$. (٤.٣١)

On the other hand $x_1 < x_0$ and x_0 is a negative real number so $x_1 < -|x_0| = -|y_1|$.

From (٤.٣١), we get $x_1 < \dfrac{-(C-|y_1|) - C - |y_1|}{2}$. As above we get that

$$x_2 < x_1 < \frac{-(C-|y_1|) - C - |y_1|}{2} \quad . \qquad (٤.٣٢)$$

Now $|y_1| > C$, from (٤.٣٢) ,we have $x_1 = y_2 < -C$ so $|y_2| - C = |C - |y_2||$. (٤.٣٣)

Also $x_2 < x_1$ and x_1 is a negative real number so $x_2 < -|x_1| = -|y_2|$.

From (٤.٣٣) we get $x_2 < \dfrac{-(C-|y_2|) - C - |y_2|}{2}$ .As (٤.٣٠), we get $x_3 < x_2$. continuing in this procedure, we get that for a positive integer n $x_n < x_{n-1} < \ldots < x_2 < x_1 < x_0$, that is $\langle x_n \rangle$ is strictly decreasing sequence .

For the second part ,if possible $\langle x_n \rangle$ is bounded, since it is monotone, we get $\langle x_n \rangle$ convergent , since $x_n = y_{n+1}$, that is $\langle x_n \rangle$, also convergent , $H_{a,b}$ is continuous. So there exits a fixed point for $H_{a,b}$ in M_1 which is contradiction, hence $\langle x_n \rangle$ is not bounded ,that is $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$.□

Remark (٤.٢.٨)[٨]

In theorem (٤.٢.٧) the equality holds only for $x_0 = -C, y_0 = \mp C$.

Proof: If $b \ge 0$, then $x_1 - x_0 = a - |b|y_0 - {x_0}^2 - x_0$ if $x_0 = -C$, $y_0 = -C$, then $x_1 - x_0 = a + (1+|b|)C - C^2$, by lemma (٤.٢.١) $x_1 - x_0 = 0$, that is $x_1 = x_0$.

If $b < 0$, then $x_1 - x_0 = a + |b|y_0 - {x_0}^2 - x_0$ if $x_0 = -C$, $y_0 = C$, then

$x_1 - x_0 = a + (1+|b|)C - C^2$, by lemma (٤.٢.١) $x_1 - x_0 = 0$, that is $x_1 = x_0$. □

Lemma (٤.٢.٩)

Let $H_{a,b}^{-1}$ be the inverse of Henon map, suppose $\begin{pmatrix} x \\ y \end{pmatrix} \in M_2$ and $b > 0$. Then the sequence $\langle y_{-n} \rangle$ is strictly decreasing sequence and $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in M_2$, from remark (٤.٢.٥) $x_0 \geq y_0$, $y_0 < -C$, consider

$$b(y_{-1} - y_0) = by_{-1} - by_0 = a - x_0 - y_0^2 - by_0 . \quad (٤.٣٤)$$

Since $x_0 \geq y_0$, we have $a - x_0 - y_0^2 - by_0 \leq a - y_0 - y_0^2 - by_0$

$$= a - (1+b)y_0 - y_0^2 . \quad (٤.٣٥)$$

From (٤.٣٤) and (٤.٣٥), we get $b(y_{-1} - y_0) \leq a - (1+b)y_0 - y_0^2$. (٤.٣٦)

Now the quadratic equation $a - (1+b)y_0 - y_0^2 = 0$ has a negative root $y_0^- = -C$ if we take another value $y = y_0^*$ such that $y_0^* < -C$,

becomes $a - (1+b)y - y^2 < 0$, and we have $y_0 < -C$, so $a - (1+b)y_0 - y_0^2 < 0$.
From (٤.٣٦), we get that $b(y_{-1} - y_0) < 0$, since $b > 0$, we get $y_{-1} < y_0 < -C$ and since $x_{-1} = y_0$, we have $y_{-1} < x_{-1}$ that is $\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} \in M_2$.

Now if possible $y_{-k} < y_{-(k-1)} < \ldots < y_{-1} < y_0$, since $y_0 < -C$, we have $y_{-k} < -C$.
Also $x_{-k} = y_{-(k-1)}$, we have $y_{-k} < x_{-k}$. (٤.٣٧)

$$b(y_{-(k+1)} - y_{-k}) \leq a - x_{-k} - y_{-k}^2 - by_{-k} < a - y_{-k} - y_{-k}^2 - by_{-k}$$

$$= a - (1+b)y_{-k} - y_{-k}^2 . \quad (٤.٣٨)$$

Now the quadratic equation $a - (1+b)y_{-k} - y_{-k}^2 = 0$ has a negative root $y_{-k}^- = -C$ if we take another value $y = y_{-k}^*$ such that $y_{-k}^* < -C$ becomes $a - (1+b)y - y^2 < 0$ and we have $y_{-k} < -C$, so $a - (1+b)y_{-k} - y_{-k}^2 < 0$, so from (٤.٣٨) we get that $b(y_{-(k+1)} - y_{-k}) < 0$, since $b > 0$, we get $y_{-(k+1)} < y_{-k}$, so we get that $y_{-n} < y_{-(n-1)} < \ldots < y_{-1} < y_0$, that is $\langle y_{-n} \rangle$ is strictly decreasing sequence .

For the second part ,if possible $\langle y_{-n}\rangle$ is bounded, since it is monotone we get $\langle y_{-n}\rangle$ convergent ,since $x_{-(n+1)} = y_{-n}$,that is $\langle x_{-n}\rangle$ also convergent , $H_{a,b}$ is continuous so there exits a fixed point for $H_{a,b}$ in M_2 which is contradiction hence $\langle y_{-n}\rangle$ not bounded that is $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$.□

Lemma(٤.٢.١٠)

The region M_3 maps into M_2 under backward iteration of Henon map $H_{a,b}$, provided $a > a_0, b > 0$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in M_3$, hence by remark (٤.٢.٥) we have $x_0 \geq -y_0$ and $y_0 > C_{a,b}$

so $b(y_{-1} - y_0) = by_{-1} - by_0 = a - x_0 - y_0^2 - by_0$. (٤.٣٩)

Since $x_0 \geq -y_0$ and $y_0 > 0$ we have $-by_0 < by_0$ so by (٤.٣٩), we have

$b(y_{-1} - y_0) \leq a + y_0 - y_0^2 + by_0 = a + (1+b)y_0 - y_0^2$. (٤.٤٠)

Now by lemma (٤.٢.١)(i) the quadratic equation $y_0^2 - (1+b)y_0 - a = 0$ has a positive real root $y_0^+ = C$,for any value $y_0^* > C$ this quadratic equation is negative and we have $y_0 > C$ so from (٤.٤٠) $b(y_{-1} - y_0) < 0$,since $b > 0$,we get $y_{-1} < y_0$ and since $x_{-1} = y_0$,we have $y_{-1} < x_{-1}$.

Now we have to show that $y_{-1} < -C$ since $y_{-1} + C = \dfrac{a - x_0 - y_0^2}{b} + C$, $y_0 > 0$, we have $b(y_{-1} + C) = a - x_0 - y_0^2 + bC < a + y_0 - y_0^2 + by_0$

$= a + (1+b)y - y^2$. (٤.٤١)

As above and since $y_0 > C$, we have $a + (1+b)y - y^2 < 0$, hence by (٤.٤١) we get that $b(y_{-1} + C) < 0$,since $b > 0$, we get $y_{-1} < -C$,hence $\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} \in M_2$.□

Lemma (٤.٢.١١)

The region M_4 maps into M_1 under forward iteration of the Henon map $H_{a,b}$, provided $a > a_0, b > 0$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in M_4$ hence from remark (٤.٢.٥), we have $|y_0| < C$ and $x_0 > C$

from definition of Henon map $x_1 + C = a - by_0 - x_0^2 + C$. (٤.٤٢)

Since $x_0 > C$, we have $x_0^2 > C^2$. Furthermore $|y_0| < C$ so $-by_0 < bC$. (٤.٤٣)

Now from (٤.٤٢) and (٤.٤٣) we get that

$x_1 + C < a + bC - C^2 + C = a + (1+b)C - C^2$ (٤.٤٤)

From lemma (٤.٢.١)(i) , C is a positive real root $y^2 - (1+b)y - a = 0$

so $a + (1+b)C - C^2 = 0$, from (٤.٤٤), we get $x_1 + C < 0$, that is $x_1 < -C$,since

$y_1 = x_0, x_0 > C$ and we have $x_1 < -C$, we get $y_1 > x_1$. (٤.٤٥)

Now $x_0 + x_1 = x_0 + a - by_0 - x_0^2$. (٤.٤٦)

Since $x_0 > C > |y_0|$,so if $y_0 \geq 0$ then $x_0 > y_0$,hence $bx_0 > by_0 > -by_0$.

If $y_0 < 0$ then $x_0 > -y_0$ that is $bx_0 > -by_0$.

From (٤.٤٦), we get $x_0 + x_1 < x_0 + a + bx_0 - x_0^2 = a + (1+b)x_0 - x_0^2$. (٤.٤٧)

As above C is a positive real root , $x^2 - (1+b)x - a = 0$,for $x_0 > C$

$a + (1+b)x_0 - x_0^2 < 0$, from (٤.٤٧) ,we get that, $x_0 + x_1 < 0$ that is $x_1 < -x_0 = -y_1$.

Now we have $x_1 < y_1 < -x_1$ so $x_1 < -|y_1|$, $x_0 = |y_1| > C$,which becomes $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in M_1$.□

<u>proof of theorem (٤.٢.٦)</u>

Let $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2 - S_{a,b}$, by remark (٤.٢.٥) $\begin{pmatrix} x \\ y \end{pmatrix} \in \bigcup_{i=1}^{4} M_i$ so we have the following cases :

Case I: If $\begin{pmatrix} x \\ y \end{pmatrix} \in M_1$, by lemma (٤.٢.٧) the sequence $\langle x_n \rangle$ is strictly decreasing sequence and $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Case II: If $\begin{pmatrix} x \\ y \end{pmatrix} \in M_2$, by lemma (٤.٢.٩) the sequence $\langle y_{-n} \rangle$ is strictly decreasing sequence and $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Case III: If $\begin{pmatrix} x \\ y \end{pmatrix} \in M_2$ by lemma (٤.٢.١٠) $\begin{pmatrix} x \\ y \end{pmatrix}$ maps into M_3 under backward iteration of Henon map $H_{a,b}$ .So from case II the sequence $\langle y_{-n} \rangle$ is strictly decreasing sequence and $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Case IV: If $\begin{pmatrix} x \\ y \end{pmatrix} \in M_4$ by lemma (٤.٢.١١) $\begin{pmatrix} x \\ y \end{pmatrix}$ maps into M_1 under forward iteration of the Henon map $H_{a,b}$ so from case I the sequence $\langle x_n \rangle$ is strictly decreasing sequence and $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$. □

Corollary (٤.٢.١٢)[٢]

If $a > a_0(b)$ then $P_{er_n}(H_{a,b}) \subset S_{a,b}$.

Proof: Let $\begin{pmatrix} x \\ y \end{pmatrix} \in$ Per($H_{a,b}$),if $\begin{pmatrix} x \\ y \end{pmatrix} \notin S_{a,b}$,then $\begin{pmatrix} x \\ y \end{pmatrix} \in (S_{a,b})^C$,that is $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$ - $S_{a,b}$, by theorem (٤.٢.٥) either $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$, or $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$, $x_n = y_{n+1}$, $x_{-(n+1)} = y_{-n}$, so there is no finite orbit for $H_{a,b}$ of $\begin{pmatrix} x \\ y \end{pmatrix}$ which is contradiction .So $\begin{pmatrix} x \\ y \end{pmatrix} \in S_{a,b}$,that is $P_{er_n}(H_{a,b}) \subset S_{a,b}$.□

Definition (٤.٢.١٣)[٢]

For $a > a_0(b)$ we define non-escape set of $H_{a,b}$ with respect to a and b by

$$\Lambda\begin{pmatrix} a \\ b \end{pmatrix} = \{\begin{pmatrix} x \\ y \end{pmatrix} \in R^2 : \lim_{n \longrightarrow \pm\infty} \left\| H_{a,b}{}^n \begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty \}^C .$$

Corollary (٤.٢.١٤)[٢]

For the Henon map $H_{a,b}$, $\Lambda\begin{pmatrix} a \\ b \end{pmatrix} = \bigcap_{m \in Z} H_{a,b}{}^m (S_{a,b})$.

Proof: Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \Lambda\begin{pmatrix} a \\ b \end{pmatrix}$. To show that $\begin{pmatrix} x \\ y \end{pmatrix} \in \bigcap_{m \in Z} H_{a,b}{}^m (S_{a,b})$, if we suppose that $\begin{pmatrix} x \\ y \end{pmatrix} \notin \bigcap_{m \in Z} H_{a,b}{}^m (S_{a,b})$, then there exists $k \in Z$, such that $\begin{pmatrix} x \\ y \end{pmatrix} \notin H_{a,b}{}^k (S_{a,b})$, that means there exists $r \in Z$ such that $H_{a,b}{}^r (S_{a,b}) \notin S_{a,b}$, that is $H_{a,b}{}^r (S_{a,b}) \in (S_{a,b})^C$, so by theorem (٤.٢.٥) $\left\| H_{a,b}{}^p \begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $p \longrightarrow \infty$ so $\begin{pmatrix} x \\ y \end{pmatrix} \in (\Lambda\begin{pmatrix} a \\ b \end{pmatrix})^C$, that is

$\begin{pmatrix} x \\ y \end{pmatrix} \notin \Lambda\begin{pmatrix} a \\ b \end{pmatrix}$

which is contradiction hence $\begin{pmatrix} x \\ y \end{pmatrix} \in \bigcap_{m \in Z} H_{a,b}{}^m (S_{a,b})$.

To show that $\bigcap_{m \in Z} H_{a,b}{}^m (S_{a,b}) \subset \Lambda\begin{pmatrix} a \\ b \end{pmatrix}$, let $\begin{pmatrix} x \\ y \end{pmatrix} \in \bigcap_{m \in Z} H_{a,b}{}^m (S_{a,b})$ so $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{a,b}{}^m (S_{a,b})$ for all m in Z, hence $H_{a,b}{}^n \begin{pmatrix} x \\ y \end{pmatrix} \in S_{a,b}$ for all $n \in Z$, that means if n is very large

$H_{a,b}{}^{n}\begin{pmatrix} x \\ y \end{pmatrix} \in S_{a,b}$, where $n \longrightarrow \infty$ or $n \longrightarrow -\infty$ then $\underset{n \longrightarrow \pm\infty}{Lim} H_{a,b}{}^{n}\begin{pmatrix} x \\ y \end{pmatrix} \in S_{a,b}$ so $\underset{n \longrightarrow \pm\infty}{Lim} \left\| H_{a,b}{}^{n}\begin{pmatrix} x \\ y \end{pmatrix} \right\|$ is real number that is $\begin{pmatrix} x \\ y \end{pmatrix} \in \Lambda\begin{pmatrix} a \\ b \end{pmatrix}$ so $\bigcap_{m \in Z} H_{a,b}{}^{m}(S_{a,b}) \subset \Lambda\begin{pmatrix} a \\ b \end{pmatrix}$, that is $\Lambda\begin{pmatrix} a \\ b \end{pmatrix} = \bigcap_{m \in Z} H_{a,b}{}^{m}(S_{a,b})$.□

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