جمهورية العراق
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كلية التربية
قسم الرياضيات

حول التراص الفوقي في الفضاءات ثنائية وثلاثية التبولوجيا

رسالة مقدمة إلى
كلية التربية- قسم الرياضيات -جامعة بابل
كاستكمال جزئي لمتطلبات نيل درجة الماجستير في علوم الرياضيات

تقدم بها
إحسان جبار كاظم الفتلاوي

بإشراف

أ.م. د لؤي عبد ألهاني السويدي

محرم ١٤٢٧ هـ شباط ٢٠٠٦ م

بسم الله الرحمن الرحيم

الله نور السموات و الأرض مثل نوره كمشكوة فيها
مصباح المصباح في زجاجة الزجاجة كأنها
كوكب دري يوقد من شجرة مباركة زيتونة لا شرقية
و لا غربية يكاد زيتها يضيء و لو لم تمسسه نار نور
على نور يهدي الله لنوره من يشاء و يضرب الله الأمثال
للناس و الله بكل شيء عليم

صدق الله العلي العظيم

النور (٣٥)

الإهداء

إلى وطني الجريح إخلاصا ....

إلى أرواح شهداء العراق الأبرار تخليدا.......

إلى أمي و أبي إحسانا.......

إلى إخوتي و أخواتي اعتزازا.......

إلى أصدقائي و زملائي تقديرا.....

اهدي ثمرة جهدي المتواضع.......

إحسان

المستخلص

الهدف من هذا العمل هو تعميم مفهوم التراص الفوقي في الفضاءات ثنائية وثلاثية التبولوجيا أدناه بعض النتائج الرئيسية التي تم الحصول عليها:

١. كل الفضاءات ثنائية التبولوجيا (X,τ,μ) من النوع $m(\tau\text{-}\mu)$ المرصوصة فوقيا بالنسبة إلى μ تكون $m(\tau\text{-}\mu)$ شبه المرصوصة فوقيا بالنسبة إلى μ .

٢. كل الفضاءات ثنائية تبولوجيا (X,τ,μ) من النوع $m(\tau\text{-}\mu)$ شبه المرصوصة فوقيا بالنسبة إلى μ تكون$m(\tau\text{-}\mu)$ -a- المرصوصة فوقيا بالنسبة إلى μ .

٣. اذا كان (X,τ,μ) فضاءاً ثنائي التبولوجي من النوع $m(\tau\text{-}\mu)$ المرصوص فوقيا بالنسبة إلى μ وثنائي الهاوزدورف بحيث ان كل مجموعة مغلقة بالنسبة الى τ في (X,τ,μ) لها اساس اقل من او يساوي m فان (X,τ,μ) يكون فضاءاً من النوع $m(\tau,\mu,\mu)$-منتظما.

٤. اذا كان (X,τ,μ) فضاءاً ثنائي التبولوجي من النوع $m(\tau\text{-}\mu)$ المرصوص فوقيا بالنسبة إلى μ وثنائي الهاوزدورف بحيث ان كل مجموعة مغلقة بالنسبة الى τ في (X,τ,μ) لها اساس اقل من او يساويm فان (X,τ,μ) يكون فضاءاً من النوع $m(\tau,\mu,\mu)$-متسقا.

٥. ليكن (X,τ,μ) فضاء ثنائي التبولوجيا و ليكن (Y,τ_Y,μ_Y) فضاءاً جزئيا من (X,τ,μ) مغلقاً بالنسبة الى τ . اذا كان (X,τ,μ) من النوع $m(\tau\text{-}\mu)$ المرصوص فوقيا بالنسبة إلى μ ,فان (Y,τ_Y,μ_Y) يكون من النوع $m(\tau_Y\text{-}\mu_Y)$ المرصوص فوقيا بالنسبة إلى μ.

٦. ليكن (X,τ,μ) فضاء ثنائي التبولوجيا وليكن $\chi = \{Xi : Xi \in \tau \cap \mu, i \in I\}$ تجزئة للمجموعة X يكون الفضاء (X,τ,μ) من النوع $m(\tau\text{-}\mu)$ المرصوص فوقيا بالنسبة إلى μ اذا و فقط اذا كان (X_i,τ_i,μ_i) من النوع $m(\tau_i\text{-}\mu_i)$ المرصوص فوقيا بالنسبة إلى μ_i لكل i.

٧. ليكن (X,τ,μ) فضاءاً ثنائي التبولوجيا من النوع $m(\tau\text{-}\mu)$ المرصوص فوقيا بالنسبة إلى μ و ليكن (Y,τ_Y,μ_Y) فضاء جزئيا من (X,τ,μ).اذا كان Y مجموعة من النوع $F\sigma$ بالنسبة الى τ فان (Y,τ_Y,μ_Y) يكون شبه مرصوص فوقيا بالنسبة الى μ_Y.

٨ . ليكن (X,τ,μ) فضاءاً ثنائي التبولوجيا وليكن (Y,τ_Y,μ_Y) فضاءاً جزئيا من (X,τ,μ) مغلقاً بالنسبة الى τ .اذا كان (X,τ,μ) من النوع $m(\tau\text{-}\mu)$ - a - المرصوص فوقيا بالنسبة إلى μ فان (Y,τ_Y,μ_Y) يكون من النوع $m(\tau\text{-}\mu)$ - a - المرصوص فوقيا بالنسبة إلى μ_Y .

٩ . ليكن (X,τ,μ) فضاءاً ثنائي التبولوجيا وليكن $\chi = \{Xi : Xi \in \tau \cap \mu, i \in I\}$ تجزئة للمجموعة X يكون الفضاء (X,τ,μ) من النوع $m(\tau\text{-}\mu)$ - a - المرصوص فوقيا بالنسبة إلى μ اذا و فقط اذا كان (X_i,τ_i,μ_i) من النوع $m(\tau_i\text{-}\mu_i)$ - a -المرصوص فوقيا بالنسبة إلى μ_i لكل i.

١٠. اذا كانت كل مجموعة مفتوحة بالنسبة الى τ في الفضاء ثنائي التبولوجي من النوع $m(\tau\text{-}\mu)$ المرصوص فوقيا بالنسبة إلى μ وهي من النوع $m(\tau\text{-}\mu)$- المرصوصة فوقيا بالنسبة إلى μ فان جميع الفضاءات الجزئية (Y,τ_Y,μ_Y) تكون من النوع $m(\tau_Y\text{-}\mu_Y)$ مرصوصة فوقيا بالنسبة إلى μ_Y.

١١. اذا كانت f تطبيق شامل من النوع $(\mu,\tau`)$- مغلقاً, $(\mu,\mu`)$ - مستمراً من الفضاء ثنائي التبولوجي (X,τ,μ) الى الفضاء ثنائي التبولوجي $(Y,\tau`, \mu`)$ من النوع $m(\tau`\text{-} \mu`)$ المرصوص فوقيا بالنسبة الى$\mu`$ بحيث ان $Z = f^{-1}(y), \forall y \in Y$ من النوع $m(\tau\text{-}\mu)$ - مرصوصة فان (X,τ,μ) يكون من النوع $m(\tau\text{-}\mu)$ المرصوص فوقيا بالنسبة الى μ .

١٢. اذا كانت f تطبيق شامل من النوع $(\mu,\tau`)$- مغلقاً, $(\mu,\mu`)$ - مستمراً من الفضاء ثنائي التبولوجي (X,τ,μ) الى الفضاء ثنائي التبولوجي $(Y,\tau`, \mu`)$ من النوع $m(\tau`\text{-} \mu`)$ شبه

المرصـوص فوقيـا بالنسـبة الـى μ` بحيـث ان $Z = f^{-1}(y), \forall y \in Y$ مـن النـوع m(τ-μ) - مرصوصـة فـان (X,τ,μ) يكـون من النـوع m(τ- μ) شبه المرصوص فوقيا بالنسبة الى μ .

١٣.ليكن (X,τ,μ) ثنائي هاوزدورف فالعبارات التالية متكافئة.

(i) (X,τ,μ) RR- ثنائي التراص الفوقي.

(ii) (X,τ,μ) β- ثنائي الانتظام و α – متراص فوقيا.

(iii) (X,τ,μ) β- ثنائي الانتظام و β- ثنائي – متراص فوقيا.

(iv) (X,τ,μ) β- ثنائي الانتظام وثنائي - α – متراص فوقيا.

(v) (X,τ,μ) β- ثنائي الانتظام وثنائي – متراص فوقيا.

١٤. اذا كان (X,τ,μ,ρ) فضاءاً ثلاثي التبولوجي من النوع m(τ-μ) المرصوص فوقيا بالنسبة إلى ρ وكان (X,τ,ρ) ثنائي الهاوزدورف بحيث ان كل مجموعة مغلقة بالنسبة الى τ في (X,τ,μ,ρ) لها اساس اقل من او يساوي m فان (X,τ,μ,ρ) يكون فضاءاً من النوع m(τ,ρ,μ)-منتظما.

١٥. اذا كان (X,τ,μ,ρ) فضاءاً ثلاثي التبولوجي من النوع m(τ-μ) المرصوص فوقيا بالنسبة إلى ρ وكان (X,τ,ρ) ثنائي الهاوزدورف بحيث ان كل مجموعة مغلقة بالنسبة الى τ في (X,τ,μ,ρ) لها اساس اقل من او يساويm فان (X,τ,μ,ρ) يكون فضاءاً من النوع m(τ,μ, ρ)-متسقا.

١٦. ليكن (X,τ,μ,ρ) فضاءاً ثلاثي التبولوجيا وليكن (Y,τ_Y,μ_Y,ρ_Y) فضاءاً جزئيا من (X,τ,μ,ρ) مغلقاً بالنسبة الى τ . اذا كان (X,τ,μ,ρ) من النوع m(τ-μ) المرصوص فوقيا بالنسبة إلى ρ ,فان (Y,τ_Y,μ_Y,ρ_Y) يكون من النوع $m(\tau_Y-\mu_Y)$ المرصوص فوقيا بالنسبة إلى ρ_Y .

١٧. لـيكن (X,τ,μ,ρ) فضــاء ثلاثــي التبولوجيــا ولــيكن $\chi = \{Xi : Xi \in \tau \cap \mu, i \in I\}$ تجزئـة للمجموعة X يكون الفضاء (X,τ,μ,ρ) من النوع m(τ-μ) المرصوص فوقيا بالنسبة إلى

ρ اذا و فقط اذا كان $(X_i,\tau_i,\mu_i,\rho_i)$ من النوع $m(\tau_i-\mu_i)$ المرصوص فوقيا بالنسبة إلى ρ_i لكل i.

١٨. اذا كانت كل مجموعة مفتوحة بالنسبة الى τ في الفضاء ثلاثي التبولوجي من النوع $m(\tau-\mu)$- المرصوص فوقيا بالنسبة إلى ρ وهي من النوع $m(\tau-\mu)$- المرصوصة فوقيا بالنسبة إلى ρ فان جميع الفضاءات الجزئية (Y,τ_Y,μ_Y,ρ_Y) تكون من النوع $m(\tau_Y-\mu_Y)$ مرصوصة فوقيا بالنسبة إلى ρ_Y.

١٩. ليكن (X,τ,μ,ρ) فضاءاً ثلاثي التبولوجيا من النوع $m(\tau-\mu)$ المرصوص فوقيا بالنسبة إلى ρ و ليكن (Y,τ_Y,μ_Y,ρ_Y) فضاء جزئيا من (X,τ,μ,ρ) اذا كان Y مجموعة من النوع $F\sigma$ بالنسبة الى τ فان (Y,τ_Y,μ_Y,ρ_Y) يكون شبه مرصوص فوقيا بالنسبة الى ρ_Y.

٢٠.اذا كانت f تطبيق شامل من النوع $(\mu,\tau`)$- مغلقاً ,$(\mu,\mu`)$ - مستمراً و$(\rho,\rho`)$ مستمراً من الفضاء ثلاثي التبولوجي $(X,\tau,\mu,\rho)$ الى الفضاء ثلاثي التبولوجي $(Y,\tau`, \mu`, \rho`)$ من النوع $m(\tau`- \mu`)$ المرصوص فوقيا بالنسبة الى$\rho`$ بحيث ان $Z = f^{-1}(y), \forall y \in Y$ من النوع $m(\tau-\mu)$ - مرصوصة فان (X,τ,μ,ρ) يكون من النوع $m(\tau- \mu)$ المرصوص فوقيا بالنسبة الىρ.

٢١. اذا كانت f تطبيق شامل من النوع $(\mu,\tau`)$- مغلقاً ,$(\mu,\mu`)$ - مستمراً و$(\rho,\rho`)$ مستمراً من الفضاء ثلاثي التبولوجي $(X,\tau,\mu,\rho)$ الى الفضاء ثلاثي التبولوجي $(Y,\tau`, \mu`, \rho`)$ من النوع $m(\tau`- \mu`)$ شبه المرصوص فوقيا بالنسبة الى$\rho`$ بحيث ان $Z = f^{-1}(y), \forall y \in Y$ من النوع $m(\tau-\mu)$ - مرصوصة فان (X,τ,μ,ρ) يكون من النوع $m(\tau- \mu)$ شبه المرصوص فوقيا بالنسبة الىρ.

Republic of Iraq
Ministry of Higher Education
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University of Babylon
College of Education
Department of Mathematics

On Paracompactness in Bitopological Spaces and Tritopological Spaces

A thesis
Submitted to the department of mathematics of the college of Education of Babylon University as a partial fulfilment of the requirements for the degree of Master of Science in mathematics.

By

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Supervised by

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// A knowledgement

I am deeply indebted to my supervisor

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Special thanks are due to all my friends in and out the mathematics department for there moral support.

To :my wounded country faithfully
Iraqi sacrificial spirits immortality
My mother and father kidness
My brothers and sisters dearly
My friends considerly
I present my humble efforts.

Ihsan

We certify that this thesis was prepared under my supervision at the department of mathematics in the university of Babylon as partial fulfilment of the requirements needed toward the degree of master of science in mathematics.

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We certify that we have read the thesis entilled (One Para-compactness in Bitopological Spaces and Tritopological Spaces) and as an examining committee , examined the thesis in its content and what is related to it and that in our opinion it is adequate with (   ) attending as a thesis for the degree of master of science in mathematics .

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Contents

List of Symbols

Symbol	Meaning
τ-open	The open set with respect to topology τ
τ-closed	The closed set with respect to topology τ
nhd	Neighborhood
τ-nhd	Neighborhood with respect to topology τ
w.r.t	With respect to
m	Infinite cardinality inal number
cl τ(A)	The closer of a set A with respect to topology τ
$\subset$	subset
$\not\subset$	Not subset
$\|A\|$	Cardinality inality of A
ϕ	Empty set
A/B	The different between the set A and the set B

Abstract

The aim of this work is to generalize paracompactness in bitopological spaces and tritopological spaces. We state below some of the main results that are obtained in this work

١. Every m(τ-μ) paracompact w.r.t. μ bitopological space (X,τ,μ) is m(τ-μ) semiparacompact w.r.t. μ .

٢. Every m(τ-μ) semiparacompact w.r.t. μ bitopological space (X,τ,μ) is m(τ-μ) – a –paracompact w.r.t. μ .

٣. If (X,τ,μ)be an m(τ-μ) paracompact w.r.t. μ , and pairwise Hausdorff space such that every τ-closed set in (X,τ,μ) having cardinality $\leq m$, then (X,τ,μ,) is m(τ,μ,μ) – regular space.

٤. If (X,τ,μ)be an m(τ-μ) paracompact w.r.t. μ , and pairwise Hausdorff space such that every τ-closed set in (X,τ,μ) having cardinality $\leq m$, then (X,τ,μ) is m(τ,μ,μ) – normal space.

٥. Let (X,τ,μ) be a bitopological space and let (Y, τ_y,μ_y) be a τ-closed subspace of (X,τ,μ) . If (X,τ,μ) is m(τ-μ)paracompact w.r.t μ , then (Y, τ_Y,μ_Y) is m(τ_Y-μ_Y)paracompact w.r.t μ_Y.

٦. Let (X,τ,μ) be a bitoplogical space , let $\chi = \{Xi : Xi \in \tau \cap \mu, i \in I\}$ be a partition of X .The space (X,τ,μ)is m(τ-μ) paracompact w.r.t. μ iff (Xi,τi,μi) is m(τi-μi) paracompact w.r.t. μi for every i .

٧. Let (X,τ,μ) be m(τ-μ) paracompact w.r.t. μ bitopological space , and let (Y, τ_Y,μ_Y) be a subspace of (X,τ,μ) .If Y is F μ-set relative to τ , then (Y, τ_Y,μ_Y)) is m(τ_Y-μ_Y) semi paracompact w.r.t. μ .

٨. Let (X,τ,μ) be a bitopological space and let (Y, τ_Y,μ_Y) be a τ-closed subspace of (X,τ,μ). If (X,τ,μ)is m(τ-μ) –a-paracompact w.r.t. μ , then (Y, τ_Y,μ_Y) is m(τ_Y-μ_Y)-a- paracompact w.r.t. μ_Y.

٩. Let (X,τ,μ) be a bitoplogical space , let $\chi = \{Xi : Xi \in \tau \cap \mu, i \in I\}$ be a partition of X .The space (X,τ,μ)is m(τ-μ) –a-paracompact w.r.t μ iff the space (X_i,τ_i,μ_i) is m(τ_i-μ_i)-a- paracompact w.r.t μ_i for every i .

١٠. If each τ-open set in an m(τ-μ) paracompact w .r .t. μ bitopological space (X,τ,μ) is m(τ-μ) paracompact w .r .t. μ , then every subspace (Y,τ_Y,μ_Y) is m(τ_Y-μ_Y) paracompact w.r.t. μ_Y .

١١.If *f* is (μ-τ`) closed , (μ-μ`) continuous mapping of a bitopological space (X,τ,μ) onto an m(τ`-μ`) paracompact w. r. t μ` bitopological space (Y,τ`,μ`) such that $Z = f^{-1}(y), \forall y \in Y$ is m(τ-μ) compact , then (X,τ,μ) is m(τ-μ) paracompact w. r.t. μ

١٢. If *f* is a (μ-τ`)closed, (μ-μ`) continuous mapping of a bitopological space (X,τ,μ) onto an m (τ-μ) semiparacompact w.r.tμ` bitopological space (Y,τ`,μ`) such that

$Z = f^{-1}(y), \forall y \in Y$ is m(τ-μ) compact . Then (X,τ,μ) is m(τ-μ) semiparacompact w. r. t μ

١٣. Let (X,τ,μ) be a pairwise Hausdorff bitopological space . Then the following are equivalent :

(i) (X,τ,μ) is RR-pairwise paracompact

(ii) (X,τ,μ) is β- pairwise regular , and α-pairwise paracompact

(iii) (X,τ,μ) β- pairwise regular , and β-pairwise paracompact

(iv) (X,τ,μ) is β- pairwise regular , and bi-a- paracompact

(v) (X,τ,μ) is β- pairwise regular , and bi- paracompact

١٤ .If (X,τ,μ)be an m(τ-μ)paracompact w.r.t ρ, and (X,τ,ρ)is pairwise Hausdorff such that every τ-closed set in (X,τ,μ,ρ) having cardinality $\leq m$,then (X,τ,μ,ρ) i s m (τ,ρ,μ,)-regular.

١٥. If (X,τ,μ,ρ) be an m(τ-μ)paracompact w.r.t. ρ , and (X,τ,ρ) is pairwise Hausdorff space such that every τ-closed set in (X, τ ,μ, ρ) having cardinality ≤ m , then (X,τ,μ,ρ) is m(τ,μ,ρ)– normal .

١٦. Let (X,τ ,μ, ρ) be a tritopological space and let(Y,τ_Y ,μ_Y,ρ_Y) be a τ- closed subspace of (X ,τ, μ,ρ).If (X ,τ, μ,ρ)be an m(τ-μ) paracompact.w .r .t ρ , then (Y,τ_Y ,μ_Y,ρ_Y) is an $m(\tau_Y -\mu_Y)$ paracompact.w .r .t ρ_Y .

١٧.Let (X,τ,μ,ρ) be a tritopological space , let $\chi = \{ X_i : X_i \in \tau \cap \mu \cap \rho, i \in I \}$ be a partition of X . The space

(X, τ ,μ, ρ) is m(τ -μ) paracompact.w .r .t ρ iff (X_i , τ_i ,μ_i,ρ_i) is m(τ_i -μ_i) paracompact.w .r .t ρ for every i∈I.

١٨. If each τ-open set in an m(τ-μ) paracompact w .r .t ρ tritopological space (X,τ,μ,ρ) is m(τ-μ) paracompact w .r .t ρ , then every subspace (Y, τ_Y, μ_Y, ρ_Y)is m(τ_Y-μ_Y) paracompact w .r .t ρ_Y .

١٩. Let (X,τ,μ,ρ) be m (τ-μ) paracompact w.r.t.ρ tritopological space, and let (Y,τ_Y,μ_Y,ρ_Y) be asubspace of (X,τ,μ,ρ). If Y is τ-Fσ-set relative to τ ,then (Y,τ_Y,μ_Y,ρ_Y) is m(τ_Y-μ_Y) semiparacompact w.r.t ρ_y.

٢٠. If *f* is (μ-τ`)closed , (μ-μ`) continuous and (ρ-ρ`) continuous mapping of a tritopological space (X,τ,μ,ρ) on to an m(τ`-μ`) paracompact w. r. t ρ` tritopological space (Y,τ`,μ`,ρ`) such that $Z = f^{-1}(y), \forall y \in Y$ is m(τ,μ) compact , then (X,τ,μ,ρ) is m(τ-μ) paracompact w. r. t ρ .

٢١. If *f* is a (μ-τ`)closed, (μ-μ`) continuous and (ρ,ρ`) continuous mapping of atritpoiogical space (X,τ,μ,ρ) onto an is m (τ-μ) semiparacompact w. r. t ρ` tritpoiogical space(Y,τ`,μ`,ρ`) such that $Z = f^{-1}(y), \forall y \in Y$ is m(τ-μ) compact , then(X,τ,μ,ρ) onto an is m (τ-μ) semiparacompact w. r. t ρ.

Introduction

A principal goal of this work is to study the concept of paracompactness in bitoplogical spaces , and tritopological space .

Paracompact spaces were first introduced by Die Donne in ١٩٤٤ as a natural generalization of compact spaces still retaining enough structure to enjoy many of the properties of compact spaces , yet sufficiently general to include a much winder class spaces . The notion of Para compactness gained stature with the proof , by A.H. Stone , that every metric space is paracompact and the subsequent use of this result in the solutions of the general metrezation problem by Bing , Nagata and Smirnov . the central role played by Para compactness or paracompact– like properties , in some of the current areas of intensive investigation in topology ensure it a permanent place alongside metrizability and compactness among the most important concept in general topology [٢٢] .

In the second half of twentieth century some papers were introduced on bitopological spaces and tritopological (٣-topological) spaces . we display some of them below:

Kelly , J.C. (١٩٦٣) [٥] introduced the idea of bitopological spaces .

Singal , M.K. , and Singal , A.R. , (١٩٧٠) [١٤] they introduced some more separation axioms these consider with bitopological spaces .

Reilly , I.L. (١٩٧٢) [٤] presented some properties that’s deal with separation axioms on bitopological spaces .

Maheshwari , S.N. and Prasad R. (١٩٧٥) [٢١] . They introduced some new separation axioms and studied some of their basic properties . the implications of those new separation axioms among themselves and with the well known axioms to ,T , were obtained .

Valeriu , P. (١٩٧٧) [١٩] , introduced some properties of bitopological semi separation space .

Mirevic , M. (١٩٨٦) [١٧] , studied the separation axioms in bitopological space .

Mukherjee , M.N. , and Ganguly , S. (١٩٨٧) [١٨] , studied characterized almost continuous multifunctions in bitopological spaces . Also generalized the idea of almost continuity as studied in (Bose and sinha (١٩٨٢) . such multifunctions had been investigated in relation to the extended concept of lower and upper semi – continuous multifunctions introduced for bitopological spaces .

Arya , S.P. , and Nour , T.M. (١٩٨٨) [٢٢] introduced some separation axioms that's consider with bitopological spaces and some theorems .

Jelis , M. (١٩٨٩) [١٢] introduced some T_i pairwise continuous functions and studied bitopological separation axioms .

Al – Swidi , L. and Shaker A.Y. (١٩٩٣) [١٠] , studied a semi compactness in bitopological spaces .

Jelic , M. (١٩٩٤) [١١] , studied the nearly PT_i – continuous mapping and the relations among them .

Bosi , G. (١٩٩٧) [٣] , he was present a separation theorem in pairwise normally preordered by topological spaces , which slightly generalize both a well known separation theorem by Nachbin in normally preordered topological spaces , and a separation theorem by Kelly in pairwise normal topological spaces . based on this result , he gave necessary and sufficient conditions for the existence of semi continuous order – preserving functions on such spaces . further , he discussed the existence of upper semi continuous order preserving functions on preordered topological spaces .

Kovar , M.M. (١٩٩٨) [١٦] modified the concept of θ – regularity for spaces with two and three topologies . The new morgeneral property that was given is fully preserved by sums and products .

Using some bitopological reductions of this property , Michel's theorem for several variants of bitopological Para compactness was proved . also in (١٩٩٩) [١٣] he slightly generalize some Michel's constructions and characterize RR- pairwise Para compactness in terms of bitopological θ – regularity , and some other weaker modifications of pairwise Para compactness and he conclude that the bitopological instability of RR – pairwise Para compactness in presence of pairwise hausedroff separation axioms is caused by a bitopological property which is much weaker and mor local than RR – pairwise compactness .

Tallafaha , A., AL-Bsoul , and For a , A – (١٩٩٩) [٢] introduced the concept of being countable dense homogeneous bitopological spaces and they defined several kinds of that concept . They gave some results concerning those bitopological spaces satisfying the axioms P –$T_{٠}$, and P – $T_{١}$ spaces .

Al – Swidi , L. (٢٠٠٠) [٩] defined , new space depend on three topologies which called tritopological spaces . He gave some characterictions of those spaces and defined some separation axioms as regular normal , triwise regular and completely normal relative to tritopological space also he studied the compactness of this spaces .

This thesis have three chapters :

In chapter one we study subspaces , covers , and some separation axioms .

In chapter two , we study the paracompactness in bitopological spaces .

In chapter three we study the paracompactness in tritopological spaces .

CHAPTER ONE

Elementeries and study

§ (١-١)Subspaces.

In this section we state subspaces in topological spaces , and give the definition of subspaces of bitopological and tritopological spaces.

(١.١.١)Definition.[٦]

Let (X , τ) be a topological space and let Y be a subset of X . The τ-relative topology for Y is the collection τ_Y given by :

$$\tau_Y = \{G \cap Y : G \in \tau\} .$$

The topological space (Y , τ_Y) is called subspace of (X , τ).

If Y is τ-open subset of X, then the subspace (Y , τ_Y) is called τ-open subspace of (X , τ),and other adjectives applying to subset of X apply similarly to subspace.

(١.١.٢)Theorem. [٦]

Let (Y , τ_Y) be a subspace of (X , τ) . Then

(i) a subset A of Y is closed in Y iff there exists a set F closed in X such that A=F∩Y .

(ii) a subset M of Y is a τ_Y-nhd of a point y in Y iff there exists a τ-nhd N of y such that M=N∩Y.

(١.١.٣)Theorem.[٦]

Let (Y , τ_Y) be a subspace of (X , τ). Then every open (closed) subset B of Y is open (closed) in X iff Y is open (closed) in X .

(١.١.٤)Definition.[٦]

A subset F of a topological space (X , τ) is called F_σ set if it is a countable union of $\tau-$closed sets . We will denote to such set by $\tau - F_\sigma$.

(١.١.٥)Remark.[٦]

Let $\{\tau_\lambda : \lambda \in \Delta\}$ be any collection of topologies on a set X. A topology $\underset{\lambda\in\Delta}{\vee}\tau_\lambda$ having $\bigcup_{\lambda\in\Delta}\tau_\lambda$ as a subbasis is the smallest of the topologies on X larger than τ_λ for every λ .

(١.١.٦) Definition.

Let (X , τ ,μ) be a bitopological space , and let Y be a subset of X . The collections

$\tau_Y = \{G\cap Y : G\in\tau\}$, and

$\mu_Y = \{H\cap Y : H\in \mu\}$

are two topologies for Y , called the relative topologies for Y .

Then (Y,τ_Y ,μ_Y) is called a subspace of (X , τ ,μ) .

(١.١.٧)Definition[٩]

Let (X , τ , μ , ρ)be a tritopological space , and let Y be a subset of X . The collections

$\tau_Y=\{G\cap Y : G\in\tau\}$

$\mu_Y=\{H\cap Y : H\in \mu\}$, and

$\rho_Y=\{F\cap Y : F\in \rho\}$

are three topologies for Y, called the relative topologies for Y .

Then$(Y, \tau_Y , \mu_Y, \rho_Y)$ is called subspace of (X,τ, μ,ρ).

§(١-٢)Covering Properties.

In this section ,we study the covering properties in topological spaces , bitopological space , and tritopological spaces.

(١.٢.١)Definition [٢٠].

For an infinite cardinal number m , if a set A consisting of at most m elements we say that A having cardinality $\leq$ m (or with cardinality $\leq$ m) , and denoted by $|A| \leq$ m . If the collection $U = \{U_\lambda : \lambda \in \Delta\}$ consisting of at most m members then we say that U having cardinality $\leq$ m (or with cardinality $\leq$ m) . Some times this collection denoted by $|U| \leq$ m or $|\Delta| \leq$ m.

(١.٢.٢)Definition [٦].

A cover or (covering) of a space (X , τ) is a collection $U = \{U_\lambda : \lambda \in \Delta\}$ of subsets of X whose union is all of X . A subcover of a cover U is a subcollection of U which is a cover . An open cover of X is a cover consisting of open sets , and other adjectives appling to subsets of X apply similarly to covers .

(١.٢.٣)Definition [٦].

Let (X,τ) be a topological space .Let $U=\{U_\lambda : \lambda \in \Delta\}$ and $V=\{V_\gamma : \gamma \in \Gamma\}$ be two coverings of X , V is said to be refine (or be a refinement of) U , if for each V_γ there exists some U_λ with $V_\gamma \subset U_\lambda$.

If $W=\{W_\delta : \delta \in \Omega\}$ refine two covers U, V of X, then it is called common refinement [٤] .

(١.٢.٤)Definition[٦]

A family $U=\{ U_\lambda : \lambda \in \Delta\}$of sets in a space (X,τ) is called locally finite, if each point of X has a nhd V such that $V \cap U_\lambda \neq \phi$ for at most finitely many indices λ. In other word $V \cap U_\lambda = \phi$ for all but finitely many λ.

(١.٢.٥)Definition[٦]

A family U of set in a space(X,τ) is called σ-locally finite if $U=\bigcup_{n=1}^{\infty}U_n$ where each U_n is locally finite collection in X.

(١.٢.٦)Lemma.

Let (X , τ) be a topological space .Let $U=\{U_\lambda , \lambda \in \Delta\}$ be a cover of X, and let $V=\{V_\gamma : \gamma \in \Gamma\}$ be a refinement of U. If $W=\{W_\alpha : \alpha \in \Omega\}$ refines V , then W is also refines U.

(١.٢.٧) Remark [٢١]

Every locally finite systems of sets is σ-locally finite .

(١.٢.٨) Lemma[١]

Let (Y, τ_Y) be a subspace of (X, τ) . If a system $V=\{V_\gamma : \gamma \in \Gamma\}$ of sets is (σ-) locally finite with respect to τ, then so is $\{V_\gamma \cap Y : \gamma \in \Gamma\}$ w.r.t. τ_Y.

(١.٢.٩) Lemma[٢١]

(i) If $U=\{U_\lambda : \lambda \in \Delta\}$is locally finite system of sets in (X, τ) Then any sub collection of U is locally finite.

(ii) If $U=\{U_\lambda : \lambda \in \Delta\}$ is locally finite system of sets in (X, τ) then so is $\{Cl_\tau (U_\lambda) : \lambda \in \Delta\}$. And $\bigcup_{\lambda\in\Delta} Cl_\tau(U_\lambda)=Cl_\tau\left(\bigcup_{\lambda\in\Delta} U_\lambda\right)$

In particular the union of a locally finite collection of closed sets is closed.

(iii) The union of a finite number of locally finites systems of sets is locally finite.

(١.٢.١٠) Lemma[٦]

Let (X, τ) be a topological space. Let U={U_λ : λ Є Δ} and V= {V_γ : γ Є Γ } be two covering of X and E = { $U_\lambda \cap V_\gamma$: (λ, γ) Є Δ x Γ}. Then

(i) E is covering of X refines both U and V. Furthermore if, both U and are locally finite so also is E.

(ii) Any common refinement of U and V is also refinement of E.

(iii) If E be any refinement of V, and V refines U, then E refines U .

(١.٢.١١)Definition.

A topological space (X , τ)is said to be

(i) m-paracompact[٢٠], if for every open cover of X with cardinality ≤m has a locally finite open refinement.

(ii) paracompact[٨], if for every open cover of X has a locally finite open refinement.

(iii) (m-) semiparacompact , if for every open cover of X (with cardinality ≤m) has a σ-locally finite open refinement .

(iv) (m-) a-paracompact[١] if for every open cover of X(with cardinality ≤m) has a a locally finite refinement not necessary either open or closed.

(v) (m-) z- paracompact[١] if for every open cover of X(with cardinality ≤m) has a a locally finite closed refinement.

(١.٢.١٢) Definition[٧]

A set equipped with two non identical topologies is called a bitopological spaces , and it denoted by (X, τ ,μ) where τ and μ are two topologies on X .

(١.٢.١٣) Definition

Let (X, τ ,μ) be a bitopological space, and let U = { U_λ : λ Є Δ} be a τ-open cover of X. A μ – open refinement

V= {V_γ : γ Є Γ } of U is called locally finite with respect to the topology μ if for every point x of X has a μ- nhd N such that $N \cap V_\gamma \neq \phi$ for at most finitely many γ . In other word $N \cap V_\gamma = \phi$ for all but finitely many γ.

(١.٢.١٤) Definition

Let (X, τ, μ) be a bitopological space and let U={U_λ : λ ϵ Δ}be a τ -open cover of X.A μ- open refinement V= {V_γ : γ Є Γ } of U is called σ- locally finite with respect to the topology μ if $V = \bigcup_{n=1}^{\infty} V_n$ where every V_n is locally finite. w.r.t.μ

(١.٢.١٥)Remark

Let (X, τ ,μ) be a bitopological space, and let U={U_λ:λ ϵΔ} be a τ-open cover of X.If V= {V_γ : γ Є Γ } is μ-open refinement of U which is locally finite w.r.t.μ , then V is σ-locally finite .w.r.t.μ.

(١.٢.١٦) Definition[٩]

A set equipped with three non identical topologies is called a tritopological spaces , and it denoted by (X, τ , μ , ρ) where τ , μ and ρ are three topologies on X .

(١.٢.١٧) Definition

Let (X, τ , μ , ρ)be a tritopological space and let U={U_λ:λ ϵΔ} be a τ-open cover of X . A μ-open refinement V= {Vγ : γ Є Γ } of U is called locally finite with respect to the topology ρ if for every point x of X has ρ-nhd N such that $N \cap V_\gamma \neq \phi$ for at most finitely many γ. In other word $N \cap V_\gamma = \phi$ for all but finitely many γ.

(١.٢.١٨) Definition

Let (X, τ , μ , ρ) be a tritopogical space, and let U={U_λ :λ ϵ Δ}be a τ -open cover of X . A μ-open refinement V ={V_γ : γ ϵ Γ }of U is called σ-locally finite with respect to the topology ρ if $V = \bigcup_{n=1}^{\infty} V_n$, where every V_n is locally finite. w.r.t.ρ.

(١.٢.١٩) Remark

Let (X, τ , μ , ρ) be a tritopological space and letU= {U_λ : λ Є Δ }be a τ-open cover of X .If V ={ V_γ : γ Є Γ} is μ-open refinement of U which is locally finite .w.r.t.ρ , then V is σ- locally finite .w.r.t.ρ.

(١.٢.٢٠) Lemma

Let(X, τ , μ , ρ) be a tritopological space, Let U= {U_λ : λ ϵ Δ } be a τ -open cover of X . If V={ V_γ : γ ϵ Γ }and W= {W_δ :δ Є Ω} be two μ-open refinement of U, E = { $V_\gamma \cap W_\delta$:(γ, δ) Є Γ x Ω } .Then ,

(i) The collection E is μ- open cover of X refine both V and W consequently of U.

(ii) If both V and W are locally finite w.r.t.ρ so also is E.

(iii) Any common μ- open refinement of V and W is also μ –open refinement of E.

Proof:

(i) First we need to prove that E is a μ-open cover of X . Given x Є X, since V and W be two μ-open cover of X , then there exists some γ , δ such that x Є V_γ and x Є W_δ . Thus x belongs to some $V_\gamma \cap W_\delta$; therefore E is a cover of X . Since V_γ and W_δ , are μ-open sets for every γ, δ so also is $V_\gamma \cap W_\delta$ (γ , δ) Є Γ x Ω. Hence E is a μ-open cover of X .

Now to show E refines both V and W. Let C Є E . Then there exists V_γ and W_δ such that C= $V_\gamma \cap W_\delta$. Since C $\subset V_\gamma$ and C $\subset W_\delta$, then E refines both V and W. Since fore every C in E there is V_γ such that C $\subset V_\gamma$, and since V refine U, there is U_λ in U such that $V_\gamma \subset U_\lambda$; hence we have for every C in E , there is U_λ in U such that C $\subset U_\lambda$. Therefore E refines U.

(ii) Let x Є X. Since V = { V_γ : γ Є Γ } is locally finite . w.r.t.ρ, then x has ρ-nhd N such that N∩ $V_\gamma = \phi$ for all but finitely many γ .

And since W= { W_δ : δ Є Ω } is locally finite. w.r.t.ρ , then x has p- nhd M such that M ∩ $W_\delta = \phi$ for all but finitely many δ.

Hence (N ∩ V_γ) ∩ (M ∩ W_δ) = ϕ for all but finitely many γ, δ.Then (N∩M) ∩ ($V_\gamma \cap W_\delta$) = ϕ for all but finitely money γ, δ . But N∩M is ρ-nhd of x, and $V_\gamma \cap W_\delta$ is μ- open set belong to E . Thus, we have for every point x in X there is ρ-nhd intersect at most finitely many member of E ;Hence E is locally finite. w. r. t.ρ.

(iii) Let D = {D_α :α Є Ω } be a μ-open cover of X refining both V and W. Then for every D_α there exists V_γ and W_δ such that $D_\alpha \subset V_\gamma$ and $D_\alpha \subset W_\delta$,so that $D_\alpha \subset V_\gamma \cap W_\delta$, i.e. for every D_α in D there is $V_\gamma \cap W_\delta$ in E , such that $D_\alpha \subset V_\gamma \cap W_\delta$. Hence D is μ-open refinement of E .

§(١-٣) Some Separation Axioms

In this section we study regularity and normality in bitopological spaces , and tritopolgical spaces.

(١.٣.١) Definition [٧]

A topological space (X, τ) is said to be regular if for any closed subset A of X and any point x of X which is not in A, there are open sets U and V such that x Є U, A ⊂ V, and U ∩V=ϕ.

(١.٣.٢) Theorem [٧]

A topological space (X, τ) is regular if given any x Є X and any open set U containing x there is open set V containing x such that Cl_τ (V) ⊂ U.

(١.٣.٣)Theorem

Let(X , τ) be a regular space,then the following are equivalent

(i) (X , τ) is paracompact.

(ii) (X , τ) is semiparacompact.

(iii) (X , τ) is a- paracompact.

(iv) (X , τ) is z- paracompact.

For the proof see [٢٤] .

(١.٣.٤) Definition [٥]

A bitopological space (X, τ ,μ) is called pairwise Hausdorff if for every two distinict points x and y of X, there exist τ -open set U and τ -open set V such that x Є U, y Є V and U ∩V= ϕ.

(١.٣.٥) Definition

A bitopological space (X, τ ,μ) is called (m)(τ, μ , μ)- regular if for every point x in X and every τ -closed set A ($|A| \leq m$) such that x Є A, there exist two μ- open sets U, and V such that

x Є U, A ⊂ V, and U ∩V= ϕ.

Clearly every (τ,μ, μ)-regular space is m(τ,μ,μ)-regular space .

(١.٣.٦) Theorem

A bitopological space (X, τ ,μ) is (τ, μ, μ) - regular iff for every point x of X and every τ -open set U containing x there exists a μ-open set V containing x such that Cl_μ (V) ⊂ U .

Proof

Suppose that (X, τ, μ) is (τ, μ , μ) – regular and suppose that the point x and the τ -open set U containing x are given . Let B=X/U. The B is τ –closed set and x ∉ B. By hypothesis there exist two μ - open sets V and W such that x Є V, B ⊂ W , and V∩W=ϕ.

Then V ⊂X /W

⇒Cl_μ (V) ⊂ Cl_μ (X/W) = X/W

⇒ Cl_μ (V) ∩ W =ϕ

⇒ Cl_μ (V) ∩ B =ϕ [since B ⊂ W]

Therefore Cl_μ (V) ⊂X/B= U as desired.

Conversely suppose the point x and the τ-closed set B not containing x are given. Let U= X/B, then U is τ-open set containing x. By hypothesis there exist a μ- open set V containg x such that Cl_μ (V)⊂U. Since Cl_μ (V) ⊂ X/ B then B ⊂ X / Cl_μ (V) ⊂ U , and X/ Cl_μ (V) is a μ-open set.

Evidently V ∩(X/ Cl_μ (V)) = ϕ . Hence we have μ-open set V and μ-open set X / Cl_μ (V) such that x Є V,B ⊂ X/ Cl_μ (V) and V∩(X/ Cl_μ (V)) = ϕ

(١.٣.٧)Definition

Given a property Q , a bitopological space (X,τ,μ) is called bi-Q if both (X,τ) and (X,μ) are Q .

(١.٣.٨) Definition.

The bitopological space (X,τ,μ) is called

(i) α- pairwise (m-)regular if (X,τ,μ) is (m-)(τ,τ,μ) regular and (m-)(μ,μ.τ)–regular.

(ii) β-pairwise (m-)regular if (X,τ,μ) is (m-)(τ ,μ, τ)- regular and (m-)(μ,τ,μ)– regular.

(iii)γ-pairwise (m-)regular if (X,τ,μ) is (m-)(τ, μ,μ)–regular and (m-)(μ,τ,τ)–regular.

(ir) δ-pairwise (m-)regular if (X,τ,μ) is (m-)(τvμ,μ , τvμ)- regular and (m-)(τvμ,τ ,τvμ)- regular

(١.٣.٩) Proposition.

Every β-pairwise regular bitopological space is α- pairwise regular.

Proof :

Let (X,τ,μ) be a β-pairwise regular bitopological space . To show (X,τ,μ) is α- pairwise regular . We need first to show (X,τ,μ) is (τ,τ,μ) –regular . Let x Є X and let U be a τ-open set containing x . Since (X,τ,μ) is (τ,μ,τ) – regular then there exists μ- open set V containing x such that $Cl_\tau(V) \subset U$. Also since (X,τ,μ) is (μ,τ,μ) – regular then there exists τ- open set G containing x such that $Cl_\mu(G) \subset V$.
Hence $Cl_\mu(G) \subset V \subset Cl_\tau(V) \subset U$. Hence for every point x in X and every τ-open set U containing x there exists τ-open set G containing x such that $Cl_\mu(G) \subset U$. Thus(X,τ,μ) is (τ,τ,μ) – regular. [by Theorem (١.٣.٦)] .

Now to show (X,τ,μ) is (μ,μ,τ) –regular . Let x Є X , and let U be a μ-open set containing x .Since (X,τ,μ) is (μ,τ,μ) –regular then there exist a τ- open set V containing x such that $Cl_\mu(V) \subset U$. Also since(X,τ,μ) is (τ,μ,τ) – regular, then there exist μ-open set G containing x such that $Cl_\tau(G) \subset V$. Hence $Cl_\tau(G) \subset V \subset Cl_\mu(V) \subset U$. Hence is for every point x in X and every μ-open set U containing x there exist μ-open set G containing x such that $Cl_\tau(G) \subset U$. Thus (X,τ,μ) is (μ,μ,τ) –regular by Theorem (١.٣.٦) .
And consequently (X,τ,μ) is α- pairwise regular by Definition (١.٣.٨) (i)

(١.٣.١٠) Proposition.

A bitopological space (X,τ,μ) is β- pairwise regular iff it is γ- pairwise regular.

Proof:

Suppose that (X,τ,μ) is a β- pairwise regular space . Let x Є X let U be a τ-open set containing x.There exists a μ-open set V containing x such that Cl_τ (V) ⊂U. Similarly there exists a τ-open set W containing x such that Cl_μ (W) ⊂V . And finally , there exists a μ-open set A containing x such that Cl_τ (A) ⊂W. Hence A ⊂ Cl_τ (A)⊂ W ⊂ Cl_μ (W) ⊂ V ⊂ Cl_τ (V)⊂U. Then Cl_μ (A)⊂U. Thus (X,τ, μ) is (τ,μ,μ) –regularly.

Reversing the role of τ and μ we obtain that (X,τ,μ) is also (μ,τ,τ)- regular which implies that (X,τ,μ) is γ- pairwise regular .

Conversely, let (X,τ,μ) is γ- pairwise regular . Let x Є X let U be a τ-open set containing x.Then there exist μ-open set V containing x such that Cl_μ(V) ⊂U . Similarly there exist τ-open set W containing x Such that Cl_τ (W)⊂V .And finally there exists μ-open set A containing x such that Cl_μ (A) ⊂ W. Hence A ⊂ Cl_μ (A)⊂W⊂ Cl_τ (W)⊂V ⊂ Cl_μ (V)⊂U. Then Cl_τ(A)⊂U . Thus (X,τ,μ) is (τ,μ,τ) – regular.

Reversing the role of τ and μ we obtain that (X,τ,μ) is also (μ,τ,μ) –regular which implies that (X,τ,μ) is β- pairwise regular .

(١.٣.١١) Corollary.

Every γ- pairwise regular bitopological is α- pairwise regular.

Proof :

Since every γ- pairwise regular bitopological space is β- pairwise regular by Proposition (١.٣.١٠) and every β- pairwise regular is α-pairwise regular by Proposition (١.٣.٩) then the result follows.

(١.٣.١٢)Proposition

If (X,τ,μ) is β - pairwise regular then it is bi-regular.

Proof:

Suppose (X,τ,μ) is β-pairwise regular.To show(X,τ) is regular, let $x \in X$, and let U be a τ- open set containing x. Since (X,τ,μ) is (τ,μ,τ) –regular then by Theorem (١.٣.٦) there exists μ-open set V containing x such that $Cl_\tau(V) \subset U$.Also since (X,τ,μ) is (μ,τ,μ)- regular then there exists a τ- open set G containing x such that $Cl_\mu(G) \subset V$.Since $G \subset Cl_\mu(G) \subset V$, then $Cl_\tau(G) \subset Cl_\tau(Cl_\mu(G) \subset Cl_\tau(V) \subset U$.Hence for every $x \in X$, and every τ- open set U containing x, there is τ-open set G containing x such that $Cl_\tau(G) \subset U$. Thus (X,τ) is regular by Theorem (١.٣.٢)……………(١)

Now to show that (X,μ) is regular. Let x in X and let U be a μ- open set containing x. Since (X,τ,μ) is(μ,τ,μ) – regular, then by Theorem(١.٣.٦) there exists a τ-open set V containing x such that $Cl_\mu(V) \subset U$. And since (X,τ,μ) is (τ,μ,τ) –regular then by Theorem (١.٣.٦) there exist μ- open set G containing x such that $Cl_\tau(G) \subset V$ Since $G \subset Cl_\tau(G) \subset V$, then $Cl_\mu(G) \subset Cl_\mu(Cl\,\tau(G)) \subset Cl_\mu(V) \subset U$.Hence for every point x and every μ- open set U containing x there exists a μ- open set G containing x such that $Cl_\mu(G) \subset U$. Thus (X,μ) is regular by Theorem (١.٣.٢)……………..(٢)

From (١) and (٢) the result follows.

Note

By using Proposition (١.٣.١٠) and Proposition (١.٣.١٢) we can show that the γ-pairwise regular space is bi-regular .Although we can give the independing proof as shown in the following Proposition.

(١.٣.١٣) Proposition.

If (X,τ, μ) is γ-pairwise regular, then it is bi-regular.

Proof:

Suppose (X,τ, μ) is γ-pairwise regular space to show (X,τ) is regular space . Let $x \in X$ and let U be a τ-open set containing x .Since (X,τ, μ) is (τ,μ,μ) – regular then there exists a μ-open set V containing x such that $Cl_\mu (V) \subset U$.And since (X,τ,μ) is (μ,τ,τ) – regular the there exist τ-open set G containing x such that $Cl_\tau (G) \subset V$. Hence $Cl_\tau (G) \subset V \subset Cl_\mu (V) \subset U$. Hence we have for every x in X, and every τ-open set U containing x ,there exists τ-open set G containing x such that $Cl_\tau (G) \subset U$.Thus (X,τ) is regular space by Theorem (١.٣.٢)……..(١)

Now ,to show (X,μ) is regular space Let $x \in X$, and let U be a μ- open set containing x , Since (X,τ,μ) is (μ,τ,τ) –regular then there exists a τ-open set V containing x , such that $Cl_\tau (V) \subset U$.And since (X,τ,μ) is (τ,μ,μ) –regular then there exist a μ-open set G containing x , such that $Cl_\mu (G) \subset V$. Hence $Cl_\mu (G) \subset V \subset Cl_\tau (V) \subset U$. Thus (X,μ) is regular space by Theorem (١.٣.٢).........(٢). From (١) and (٢) the result follows.

(١.٣.١٤) Proposition.

Let (X,τ, μ) be a β- pairwise regular bitopological space and let $U=\{U \lambda : \lambda \in \Delta\}$ be a collection of subsets of X. Then U is locally finite w.r.t τ iff it is locally finite . w.r.t μ .

Proof:

Suppose that U={U λ : λ Є Δ} is locally finite .w.r.t τ.Then, each x Є X has τ-open nhd V_x such that $V_x \cap U_\lambda = \phi$ for all but finitely many λ. Since (X,τ ,μ) is β- pairwise regular , then (X,τ,μ) is (τ,μ,τ) – regular. Therefore the point x has μ- open nhd N_x such that $Cl_\tau (N_x) \subset V_x$. Hence $Cl_\tau (N_x) \cap U_\lambda = \phi$ for all but finitely many λ. Consequently $N_x \cap U_\lambda = \phi$ for all but finitely λ . Thus U is locally finite w.r.t. μ.

Conversely Suppose U= {U_λ : λ Є Δ} is locally finite . w. r.t μ, then each x in X has μ- open nhd G_x such that $G_x \cap U_\lambda = \phi$ for all but finitely many λ . since (X,τ,μ) is β- pairwise regular then (X,τ,μ) is (μ,τ,μ) –regular. Therefore x has τ- open nhd H_x such that $Cl_\mu (H_x) \subset G_x$. Hence $Cl_\mu (H_x) \cap U_\lambda = \phi$ for all but finitely many λ .Consequently $H_x \cap U_\lambda = \phi$ for all but finitely many λ. Thus U is locally finite .w.r.t τ.

(١.٣.١٥) Definition.

A bitopological space (X,τ,μ) is called (m-)(τ,μ,μ) –normal if for every pair disjoint τ-closed sets A,B of X, $(|A| \leq m, |B| \leq m)$ there exist μ-open sets U,V such that $A \subset U$, $B \subset V$, and $U \cap V = \phi$.

Clearly that every (τ,μ,μ) –normal space is m(τ,μ,μ) –normal.

(١.٣.١٦)Theorem.

A bitoplogical space (X,τ,μ) is (τ,μ,μ) –normal iff for every τ- closed sets and τ-open set U with $A \subset U$, there exists a μ-open set V containing A such that $Cl_\mu (V) \subset U$.

Proof :

Suppose that (X,τ,μ) is (τ,μ,μ) -normal , and suppose that the τ- closed set A and the τ-open set U are given with $A \subset U$. Let B=X /U, then B is τ-closed. By hypothesis, there exist μ-open set Vand W containing A and B respectively , and $V \cap W = \phi$ so that $V \subset X/W$.

Then $Cl_\mu (V) \subset Cl_\mu (X/W) = X/W$ .Also $X/U \subset W$ then $X/W \subset U$.

Thus $Cl_\mu (V) \subset U$.Conversely , let A and B be τ-closed subset of X such that $A \cap B = \phi$. Let U=X/B the set U is τ-open so that $A \subset U$ .By hypothesis there exists a μ-open set V containing A such that $Cl_\mu (V) \subset U$. Then $X/U \subset X / Cl_\mu (V) \Rightarrow X/(X/B) \subset X/Cl_\mu (V) \Rightarrow B \subset X / Cl_\mu (V)$ Evidentially $V \cap (X/Cl_\mu(V)) = \phi$. Thus V and $X/Cl_\mu (V)$ are disjoint μ-open sets containing A and B respectively. Therefore (X,τ,μ) is (τ,μ,μ) –normal.

(١.٣.١٧) Definintion

A bitopological space (X,τ,μ) is called pairwise (m-)normal , if (X,τ,μ) is (m-) (τ,μ,μ) – normal and (m-) (μ,τ,τ) – normal.

(١.٣.١٨) Definintion

A tritopolgical space (X,τ,μ,ρ) is called[٩] (m-)(τ,μ,ρ) – regular if for every point x in X and every τ-closed set A , $(|A| \leq m), x \notin A$, there exists a μ -open set U and a ρ-open set V such that $x \in U$. $A \subset V$, and $U \cap V = \phi$.

In [٩] Al- Swidi state and proved the following theorem .

(١.٣.١٩) Theorem.

A tritopological space (X,τ,μ ρ) is (τ,μ,ρ) –regular iff for every point x in X and every τ-open set U containing x there exists a μ-open set V containing x such that $Cl_\rho(V) \subset U$.

(١.٣.٢٠) Definition

A tritopological space (X,τ,μ,ρ) is called [٩] (m-)(τ,μ,ρ)– normal if for every pair of disjoin τ-closed sets A,B of X $(|A| \le m, |B| \le m)$ there exists μ-open set U and ρ- open set V such that A⊂U, B⊂V. And U∩V=ϕ.

In [٩] Al- Swidi state the following theorem .

(١.٣.٢١) Theorem.

A tritopological space (X,τ,μ,ρ) is (τ,μ,ρ) – normal iff for every τ- closed set A and τ-open set U with A⊂U, there exists a μ-open set V containing A such that $Cl_\rho(V) \subset U$.

CHAPTER Two

Paracompactness
in
Bitopological spaces

§ (٢-١)Some Definitions and results

In this section we give the definitions of some types of Paracompact topological spaces, and introduce definitions of some types of Para compactness in bitopological spaces. Also we give some relations among these types.

(٢.١.١)Definition.

A bitopological space (X , τ ,μ) is called (m-) (τ - μ) compact if for every τ-open cover $U = \{U_\lambda : \lambda \in \Delta\}$ of X (with cardinality ≤ m) has a μ-open finite subcover .

(٢.١.٢) Definition

A bitopological space (X,τ,μ) is called (m-) (τ-μ) paracompact w.r.t μ ,if for every τ-open cover $U = \{U_\lambda : \lambda \in \Delta\}$ of X (with cardinality ≤ m) has a μ-open refinement $V = \{V_\gamma : \gamma \in \Gamma\}$ which is locally finite w .r .t μ.

(٢.١.٣) Proposition.

Every (τ -μ)paracompact w .r .t. μ bitopological space (X, τ ,μ) is m (τ-μ)paracompact w .r .t μ .

Proof

Let (X, τ, μ) be a (τ -μ)paracompact w .r .t. μ. Let $U = \{U_\lambda : \lambda \in \Delta\}$ is τ-open cover of X with cardinality ≤m . By hypothesis U has a μ-open refinement $V = \{V_\gamma : \gamma \in \Gamma\}$ which is locally finite w .r .t μ. Hence (X, τ ,μ) is m(τ-μ)paracompact w.r.t μ .

(٢.١.٤) Definition.

A bitopological space (X , τ ,μ) is called (m-) (τ - μ) semiparacompact w .r .t μ , if for every τ-open cover $U = \{U_\lambda : \lambda \in \Delta\}$ of X (with cardinality ≤ m) has a μ -open refinement $V = \{ V_\gamma | \gamma \in \Gamma \}$ which is σ-locally finite. w.r.t μ .

(٢.١.٥) Proposition.

Every (τ -μ)semiparacompact w .r .t. μ bitopological space (X , τ ,μ) is m(τ -μ) semiparacompact w .r .t μ .

This proof uses exactly the same proof of the Proposition (٢.١.٣) one just replaces the locally finite by σ-locally finite.

(٢.١.٦) Theorem

Every m(τ-μ)paracompact w .r .t. μ bitopological space (X , τ, μ) is m(τ -μ)semiparacompact w .r .t μ .

Proof

Let (X, τ , μ) be an m(τ-μ)paracompact w .r .t μ. Let $U = \{U_\lambda : \lambda \in \Delta\}$ is τ-open cover of X with cardinality ≤ m .By hypothesis there exist a μ-open refinement $V = \{V_\gamma : \gamma \in \Gamma\}$ of U which is locally finite w.r.t.μ. By Remark (١.٢.٧) ,V is σ-locally finite. w . r .t. μ. Hence the space (X, τ ,μ) is m(τ-μ) semiparacompact w .r. t. μ by Definition (٢.١.٤) .

(٢.١.٧) Corollary

Every (τ-μ)paracompact w .r .t. μ bitopological space (X , τ ,μ) is (τ-μ) semiparacompact w .r .t. μ .

Proof

Let (X, τ,μ) be a (τ -μ)paracompact w .r .t μ. Let $U = \{U_\lambda : \lambda \in \Delta\}$be a τ-open cover of X .By hypothesis there exist a μ-open refinement $V = \{V_\gamma : \gamma \in \Gamma\}$ which is locally finite w .r .t μ. By Remark (١.٢.٧) ,V is σ-locally finite w .r .t μ .Hence the space (X, τ, μ) is (τ-μ)semipracompact w .r. t. μ.

(٢.١.٨) Corollary

Every (τ-μ)paracompact w .r .t. μ bitopological space (X , τ ,μ) is m(τ -μ)semiparacompact w .r .t μ .

Proof

The proof is immediately follows from Proposition (٢.١.٣) and Theorem (٢.١.٦)

(٢.١.٩) Definition.

A bitopological space (X,τ ,μ) is called (m) (τ - μ) -a-paracompact w .r .t μ ,if for every τ - open cover $U = \{U_\lambda : \lambda \in \Delta\}$ of X (with cardinality ≤ m) has a refinement $V = \{V_\gamma : \gamma \in \Gamma\}$ of U not necessary either μ-open or μ-closed which is locally finite . w .r .t. μ .

(٢.١.١٠) Proposition.

Every (τ-μ)-a-paracompact w .r .t. μ bitopological space (X , τ ,μ) is m (τ -μ) -a-paracompact w .r .t μ .

Proof

Let (X, τ,μ) be a (τ -μ) -a- paracompact w .r .t.μ. Let $U = \{U_\lambda : \lambda \in \Delta\}$be a τ -open cover of X with cardinality ≤ m .By hypothesis U has a refinement $V = \{ V_\gamma | \gamma \in \Gamma \}$not necessary either μ -open or μ -closed which is locally finite .w .r .t μ. Hence (X, τ, μ) is m(τ-μ)-a-paracompact w .r. t μ.

(٢.١.١١)Theorem.

Every m(τ -μ)semiparacompact w .r .t.μ bitopological space (X , τ ,μ) is m (τ -μ) -a-paracompact w .r .t. μ .

Proof

Suppose that (X, τ ,μ) be m (τ -μ)semiparacompact w .r .t.μ. Let $U = \{U_\lambda : \lambda \in \Delta\}$be a τ -open cover of X with cardinality ≤m ,then U has μ-open refinement V of U which is σ-locally finite w .r .t. μ , i.e $V = \bigcup_{n=1}^{\infty} V_n$ where each V_n is μ-open collection which is locally finite w.r.t.μ , say $V_n = \{V_{n\beta} : \beta \in B\}$.For each n, let $W_n = \bigcup_{\beta} V_{n\beta}$, then W_n is μ - open set.

Since $X = \bigcup_{\beta}\left(\bigcup_{n=1}^{\infty} V_{n\beta}\right) = \bigcup_{n=1}^{\infty}\left(\bigcup_{\beta} V_{n\beta}\right) = \bigcup_{n=1}^{\infty} W_n$ Then the collection $W = \{W_n | n \in \mathbb{N}\}$ is μ-open cover of X.

Define :

$$A_i = W_i / \bigcup_{j \langle i} W_j \text{ , i=١,٢,…}$$

then $A = \{A_n : n \in IN\}$ be a collection of sets they are not necessary either μ-open or μ-closed.We claim that A is

(i) cover of X.

(ii) refinement of W.

(iii) locally finite w .r .t. μ .

proof of (i)

Let x Є X ,Then xЄ W_n for some n. Let n(x)is the first i for which xЄ W_i then x Є $A_{n(x)}$; hence A covers X.

proof of (ii)

Let A_n Є A then there exsist W_n ЄW such that $An = Wn / \bigcup_{m \langle n} W_m$. Hence $A_n \subset W_n$ .There fore A refines W.

proof of (iii)

Let x Є X $\Rightarrow$ x Є W_n(x) where n(x)is the first index , then W_n(x) is μ-open nhd of x. Since $W_{n(x)} \cap A_i = \phi$ for $i \rangle n(x)$, therefore A is locally finite w.r.t.μ .

Now , $\Pi = \{A_n \cap V_{n\beta} | (n, \beta) \in IN x \beta \}$ is refinement of U which is locally finite w.r.t. μ by Lemma (١.٢.١٠) and Π is a collection of sets not necessary either μ-open or μ-closed . Hence (X,τ,μ) be m(τ-μ)-a- paracompact w.r.t.μ by Definition (٢.١.٩) .

In a same way we can proof the following corollary.

(٢.١.١٢)Corollary .

Every (τ-μ)semiparacompact w .r .t. μ bitopological space (X,τ,μ) is (τ-μ)-a-paracompact w.r.t μ .

(٢.١.١٣)Corollary

Every (τ-μ)semiparacompact w .r .t. μ bitopological space (X, τ ,μ) is m (τ-μ) -a-paracompact w .r .t μ .

Proof

The proof follows immediately from Proposition (٢.١.٥) , and Theorem (٢.١.١١)

(٢.١.١٤)Corollary

Every m (τ-μ)paracompact w .r .t. μ bitopological space (X, τ ,μ) is m (τ-μ)- a-paracompact w .r .t. μ .

Proof

The proof follow immediately from Theorem (٢.١.٦) and Theorem (٢.١.١١)

(٢.١.١٥)Corollary

Every (τ-μ)paracompact w.r.t. μ bitopological space (X, τ ,μ) is (τ-μ)- a-paracompact w.r.t. μ .

Proof

The proof follow immediately from Proposition (٢.١.٧) and Corollary (٢.١.١٢) .

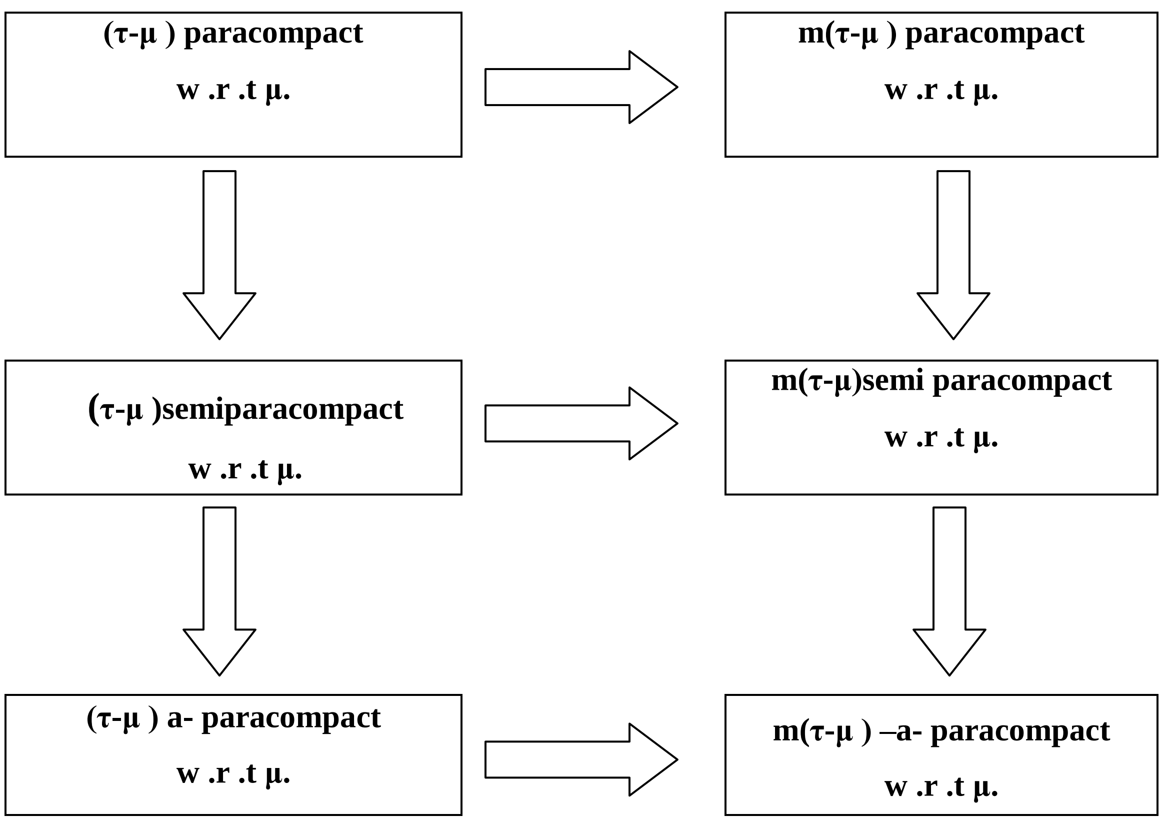
(٢.١.١٦)Corollary

Every (τ-μ) paracompact w .r .t .μ bitopological space (X, τ ,μ) is m(τ-μ) -a-paracompact w .r .t .μ .

Proof

The proof follow immediately from Corollary (٢.١.١٥) and Proposition (٢.١.١٠).

The following diagram show the relation a among the spaces which have been studied above



§ (٢-٢) Paracompactness and Subspaces .

In this section we give some theorems that related the paracompactness with separation axioms in bitopological spaces. Also we give some theorems concerning the subspaces of paracompact bitopological spaces .

(٢.٢.١) Theorem

If (X, τ ,μ) be an m(τ-μ)paracompact w .r .t μ. and pairwise Hausdroff space such that every τ- closed set in (X, τ ,μ) having cardinality ≤ m , then (X, τ ,μ) is m(τ ,μ, μ)-regular space .

Proof

Suppose that. (X, τ ,μ) be an m(τ-μ) paracompact w .r .t. μ , and A be a τ- closed set in (X, τ ,μ) having cardinality ≤ m, and x Є X / A . Since (X, τ ,μ) is pairwise Hausdorff, then for each yЄA, we can find τ–open set V_y and μ-open set U_y, such that $x \in U_y$, $, y \in V_y$ and $U_y \cap V_y = \phi$ the collection $\Pi = \{V_y : y \in A\} \cup \{X / A\}$ form a τ – open cover of X having cardinality ≤ m. and **Π** has μ-open refinement $W = \{W_\gamma : \gamma \varepsilon \Gamma\}$ which is locally finite w.r.t. μ.

Set

$$V = \bigcup_{\gamma \varepsilon \Gamma} \{W_\gamma : W_\gamma \cap A \neq \phi\}$$

then V is μ-open set containing A.

For let $p \in A \Rightarrow p \notin W_\gamma \forall \gamma \in \Gamma$ such that $W_\gamma \cap A = \phi$

$\Rightarrow P \in W_\gamma$, for some $\gamma \in \Gamma$ such that $W_\gamma \cap A \neq \phi$

$\Rightarrow p \in \bigcup_{\gamma \in \Gamma} \{W_\gamma, W \cap A \neq \phi\}$

Hence $A \subset V$. Since the μ- open cover W is locally finite. w.r.t. μ,then x has a μ-nhd U* which meet only finitely many $W\gamma_١,\ldots,W\gamma_n$ .If some $W\gamma_i$, i=١,٢,…n meets A i.e $W_{\gamma i} \cap A \neq \phi$, then $W_{\gamma i} \subset X / A$ is impossible thus there exists $W_{\gamma i}$ .such that $W_{\gamma i} \subset V_{yi}$.

Set

$$U = U^* \cap \left(\bigcap_{i=1}^{n} U_{yi} \right),$$

then x Є U and U is μ- open set. Finelly, we need to show $U \cap V = \phi$. Assume contrary that $U \cap V \neq \phi$,then there exist at least one point p such that $p \in U \cap V$

$\Rightarrow p \in U \wedge p \in V$

$\Rightarrow p \in U_{yi}, \forall_i, i = 1,2\ldots.n$

$\Rightarrow \exists V_{yi} \ni p \in V_{yi}$

$\Rightarrow p \in U_{yi} \wedge p \in V_{yi}$

$\Rightarrow p \in U_{yi} \cap V_{yi}$

Hence $U_{yi} \cap V_{yi} \neq \phi$, but that contradiction .

Thus $U \cap V = \phi$.Therefore the bitopological space (X,τ,μ) is m(τ,μ,μ)- regular

(٢.٢.٢) Corollary

If (X, τ ,μ) be a (τ-μ) paracompact w .r .t. μ ,and pairwise Hausdroff then (X, τ ,μ) is (τ,μ,μ) –regular.

The proof uses exactly the same as above theorem with omitting the condition "every τ- closed set having cardinality ≤ m".

(٢.٢.٣) Theorem

If (X, τ,μ) be an m (τ -μ)paracompact w .r .t μ , and pairwise Hausdorff space, such that every τ-closed set in (X, τ,μ) having cardinality ≤ m , then (X, τ,μ) is m(τ,μ,μ)-normal .

proof

Suppose that (X, τ ,μ) be an m(τ -μ) paracompact w .r .t μ .Let A , and B be disjoint τ- closed set in (X, τ ,μ)such that they having cardinality ≤ m. Since (X, τ ,μ) is pairwise Hausdorff, then for each x∈A,y∈B we can find τ–open set U_x and μ-open set V_x , such that $x \in U_x, y \in V_x$, and $U_x \cap V_x = \phi$. Then $\prod = \{U_x : x \in A\} \cup \{X / A\}$ form a τ-open cover of X having cardinality ≤ m. Then **Π** has μ-open refinement $W = \{W_\gamma : \gamma \in \Gamma\}$ which is locally finite w.r.t μ.

Set

$$U = \bigcup_{\gamma \in \Gamma} \{W_\gamma, W_\gamma \cap A \neq \phi\}.$$

Then U is μ-open set contains A. For, let $p \in A \Rightarrow p \notin W_\gamma, \forall \gamma \in \Gamma$ such that $W_\gamma \cap A = \phi$ $\Rightarrow P \in W_\gamma$, for some Γ such that $W_\gamma \cap A \neq \phi$ $\Rightarrow p \in \bigcup_{\gamma \in \Gamma} \{W_\gamma, W_\gamma \cap A \neq \phi\} \Rightarrow p \in U$. Hence $A \subset U$.

For each y∈B ,we can find μ-open nhd H_y of y which meets only finitely many W_γ, say $W_{\gamma 1(y)},\ldots, W_{\gamma n(y)}$ (the value of n also depending on y) . Each $W_{\gamma i(y)}$ which meets A i.e $W_{\gamma i}\cap A\neq\phi$,then $W_{\gamma i}\subset X/A$ is impossible . Thus there exists $U_{x\,i}$ such that $W_{\gamma i(y)}\subset U_{x\,i}$ for x_i ∈ A.

Set $$G_y = H_y \cap \left(\bigcap_{i=1}^{n} V_{xi} \right).$$

Then G_y is μ-open set which contains y but does not meet U

Let $V=\bigcup_{y\in B} G_y$ .Then V is μ-open set, and $B\subset V$ .For , let $p\in B \Rightarrow p\in G_y$ for some $y\in B.\Rightarrow p\in \underset{y\in B}{U} G_y$ i.e $p\in V$ .Finelly, we need to show $U\cap V=\phi$.

Assume contrary that $U\cap V\neq\phi$ then there exists at least one point p such that $p\in U\cap V$

$\Rightarrow p\in U \wedge p\in V$

$\Rightarrow p\in V_{xi}, \forall_i, i=1,2....n$

$\Rightarrow \exists U_{xi} \ni p\in U_{xi}$

$\Rightarrow p\in V_{xi} \wedge p\in U_{xi}$

$\Rightarrow p\in V_{xi}\cap U_{xi} \Rightarrow V_{xi}\cap U_{xi}\neq\phi$ but that contradiction .

Thus $U\cap V=\phi$. Therfore (X, τ ,μ) is m(τ ,μ,μ)-normal.

(٢.٢.٤) Corollary.

If (X, τ,μ) be a (τ -μ) paracompact w .r .t. μ ,and pairwise Hausdorff space then it is (τ, μ, μ) –normal .

proof

The proof uses exactly the same as above theorem with omitting the condition "every τ- closed set having cardinality ≤ m".

(٢.٢.٥)Theorem

Let (X, τ ,μ) be a bitopological space and let (Y, τ_Y ,μ_Y) be a τ- closed subspace of (X ,τ, μ).If (X ,τ, μ) is m(τ-μ) paracompact w .r .t .μ , then (Y, τ_Y ,μ_Y) is m(τ_Y-μ_Y) paracompact w .r .t. μ_Y .

Proof

Suppose that (Y,τ_Y,μ_Y) be a τ- closed subspace of m(τ-μ) paracompact w .r .t.μ space (X,τ ,μ).To show that (Y,τ_Y,μ_Y) is m(τ_Y-μ_Y)paracompact w .r .t μ_Y.

Let $U=\{U_\lambda : \lambda \in \Delta\}$ be a τ_Y- open cover of Y with cardinality $\leq$ m.

Since U_λ is τ_Y-open subset of Y, there is τ- open subset V_λ of X such that each $U_\lambda = V_\lambda \bigcap Y$. The collection.

$$\Pi = \{V_\lambda : \lambda \in \Delta\} \bigcup \{X/Y\}$$

form a τ-open cover of`` X with cardinality $\leq$ m. For , let $x \in X$ .If $x \notin Y$,then $x \in X/Y \Rightarrow x \in \bigcup_{\lambda \varepsilon \Delta} V_\lambda \bigcup (X/Y)$.. If $x \in Y \Rightarrow x \in \bigcup_{\lambda \varepsilon \Delta} U_\lambda$

$\Rightarrow x \in \bigcup_{\lambda \varepsilon \Delta} (V_\lambda \bigcap Y) = \bigcup_{\lambda \varepsilon \Delta} V_\lambda \bigcap Y \Rightarrow x \in \bigcup_{\lambda \in \Delta} V_\lambda$.

$\Rightarrow x \in \bigcup_{\lambda \varepsilon \Delta} V_\lambda \bigcup (X/Y)$.Thus Π is τ-open cover of X. Since $|\Delta| \leq$m, then Π having cardinality$\leq$m.By hypothesis Π has μ-open refinement $W=\{W_\gamma : \gamma \in \Gamma\}$ which is locally finite w.r.t. μ.

Now, let A $=\{W_\gamma \cap Y | \gamma \in \Gamma\}$,then A is a collection of μ_Y-open subset of Y. We claim that A is

(i) cover Y
(ii) refine U
(iii) locally finite w.r.t.μ.

proof of (i).

Let $y \in Y \Rightarrow y \in X$.Since W covers X, then $y \in \bigcup_{\gamma \in \Gamma} W_\gamma \Rightarrow y \in W_\gamma$ for some $\gamma \Rightarrow y \in W_\gamma \cap Y$ for some $\gamma \Rightarrow x \in \bigcup_{\gamma \in \Gamma} (W_\gamma \cap Y)$, hence A covers Y.

proof of (ii)

Since W refines $\prod$,then $\forall W_\gamma \in W$,there exists $V_\lambda \in \prod$ such that $W_\gamma \subset V_\lambda \Rightarrow W_\gamma \cap Y \subset V_\lambda \cap Y$.But $U_\lambda = V_\lambda \cap Y$; Hence $W_\gamma \cap Y \subset U_\lambda$.Hence for every member of A there exists member of U containing it. Thus A refines U

Proof of (iii)

A is locally finite.w.r.t μ_Y by Lemma (١.٢.٨) . Therefore (X, τ_Y ,μ_Y) is m(τ_Y – μ_Y) paracompact w .r .t. μ_Y.

(٢.٢.٦)Corollary.

Let (X, τ ,μ) be a bitopological space and let (Y, τ_Y ,μ_Y) be a τ- closed subspace of (X ,τ, μ).If (X ,τ, μ) is (τ-μ) paracompact w .r .t μ , then (Y, τ_Y ,μ_Y) is (τ_Y-μ_Y) paracompact w .r .t μ_Y .

(٢.٢.٧)Theorem

Let (X, τ ,μ) be a bitopological space and let $\chi = \{ X_i : X_i \in \tau \cap \mu : i \in I \}$ be a partition of X. the space (X,τ,μ) is m(τ-μ)paracompact w .r .t .μ iff (X_i,τ_i,μ_i) is m(τ_i -μ_i) paracompact w .r .t .μ_i . for every i.

Proof

The "only if "part. Since $X_i = X / \bigcup_{j \neq i} X_j$ is τ- closed , then the subspace (X_i , τ_i ,μ_i) is m(τ_i - μ_i) paracompact w .r .t μ_i for every i by Theorem (٢.٢.٥)

The "if" part .

Let $U=\{U_\lambda:\lambda\in\Delta\}$ be a τ- open cover of X with cardinality $\leq$ m .The collection $\Pi=\{U_\lambda\cap X_i:\lambda\in\Delta\}$ is τ_i-open cover of X_i with cardinality $\leq$ m for every i. For ,let $x\in X_i\Rightarrow x\in X\Rightarrow x\in U_\lambda$ for some $\lambda.\Rightarrow x\in U_\lambda\cap X_i$ for some λ .Then

$x\in\bigcup_{\lambda\varepsilon\Delta}(U_\lambda\cap X_i)\Rightarrow\Pi$ cover X_i. Since (X_i, τ_i, μ_i) is $m(\tau_i-\mu_i)$ paracompact w .r .t μ_i . $\forall i$, there exist a μ_i- open refinement $A_i=\{A_{i\lambda}:\lambda\in\Delta\}$ of Π which is locally finite. w.r.t .μ_i.

Let W={ $\bigcup_{i\in I}A_{i\lambda}|\lambda\in\Delta$ }.We claim that W is

(i) μ- open cover of X

(ii) refine U

(iii) locally finite w.r.t. μ.

proof of (i)

Since $A_{i\lambda}$ is μ_i- open and X_i Є μ then $A_{i\lambda}$ is μ-open set by Theorem (١.١.٣), and consequently, W is a collection of μ-open sets. Now since $X=\bigcup_{i\varepsilon I}X_i=\bigcup_{i\varepsilon I}\left(\bigcup_{\lambda\varepsilon\Delta}A_{i\lambda}\right)=\bigcup_{\lambda\varepsilon\Delta}\left(\bigcup_{i\varepsilon I}A_{i\lambda}\right)$. Hence W cover X.

proof of (ii)

Let $\bigcup_{i\varepsilon I}A_{i\lambda}\in W$ .Since A_i refines Π then for every $A_{i\lambda}, A_{i\lambda}\subset U_\lambda\cap X_i$

for some λ . $\Rightarrow\bigcup_{i\varepsilon I}A_{I\lambda}\subset U_\lambda\cap\left(\bigcup_{i\varepsilon I}X_i\right)=U_\lambda\cap X\subset U_\lambda$ .Hence W refines U.

proof of (iii)

Let $x\in X$, if $x\in X_i$, then x has μ_i- open nhd N intersect at most finitely many member of A_i . In other word $N\cap A_{i\lambda}=\phi$ for all but finitely many λ

$.\Rightarrow \bigcup_{i\in I}(N\cap A_{i\lambda})=\phi$ for all but finitely many λ. $\Rightarrow N\cap\left(\bigcup_{i\in I}A_{i\lambda}\right)=\phi$ for all but finitely many λ. Since N is μ_i – open nhd of x and X_i Єμ then N is μ – open nhd of x by Theorem(١.١.٣) and $\bigcup_{i\in I}A_{i\lambda}\in W$ hence W locally finite w.r.t μ . Hence (X,τ,μ) is m(τ - μ) paracompact w .r .t .μ.

(٢.٢.٨) Corollary.

Let (X,τ,μ) be a bitopological space , let $\chi=\{\ X_i : X_i \in \tau\bigcap\mu, i\in I\ \}$ be a partition of X. The space (X, τ ,μ) is (τ -μ) paracompact w .r .t. μ iff the space (X_i , τ_i ,μ_i) is (τ_i -μ_i) paracompact w .r .t. μ_i for every i.

(٢.٢.٩) Theorem

Let (X, τ ,μ) be a m(τ -μ) paracompact w.r.t. μ bitopological space and let (Y,τ_Y ,μ_Y) be a subspace of(X, τ ,μ). If Y is τ-F_σ set then (Y,τ_Y ,μ_Y) is m(τ_Y-μ_Y) semiparacompact w.r.t. μ_Y.

Proof

Suppose Y is τ-F_σ . Then $Y=\bigcup Y_n$ where each Y_n,is τ- closed .

Let $U=\{U_\lambda : \lambda\in\Delta\}$ be a τ_Y-open cover of Y with cardinality ≤ m.Since each U_λ is τ_Y- open subset of Y, we have $U_\lambda = V_\lambda \cap Y$, where V_λ is τ-open subset of X for each λ Є Δ. For each fixed n , $E_n=\{V_\lambda : \lambda\in\Delta\}\bigcup\{X/Y_n\}$ form a τ-open cover of X with cardinality ≤ m.By hypothesis E_n has a μ-open refinement $W=\{W_{\lambda n} : (\lambda,n)\in\Delta\times IN\}$ which is locally finite .w.r.t.μ.For each n, let $B_n=\{W_{\lambda n}\cap Y : W_{\lambda n}\cap Y_n\neq\phi\}$.Let $B=\bigcup B_n$.We claim that B is

(i) collection of μ_Y-open set

(ii) covers Y

(iii) refines U

(iv) σ- locally finite w.r.t. μ_Y

proof of (i) .

Since W_λ is μ-open, then $W_{\lambda n} \cap Y$ is μ_Y –open .

proof of (ii)

If y Є Y ⇒y Є Y_n for some n and then for some set $W_{\lambda n}$ Є W ⇒x Є $W_{\lambda n} \cap Y$; hence B covers Y.

proof of (iii)

Here $W_{\lambda n} \subset X/Y_n$ is impossible so that $W_{\lambda n} \subset V_\lambda$ for some λ ⇒$W_{\lambda n} \cap Y \subset V_\lambda \cap Y = U_\lambda$ for some U_λ Є U intersecting Y_n so that B refines U.

proof of (iv)

Since W is locally finite w.r.t.μ. Then B_n is locally finite .w.r.t. μ_Y by Lemma (١.٢.٨) .Hence B is σ- locally finite w.r.t. μ_Y . There for (X, τ ,μ) is m(τ_Y -μ_Y) semiparacompact w.r.t. μ_Y.

(٢.٢.١٠) Corollary

Let (X, τ ,μ) be a (τ -μ) paracompact w.r.t. μ bitopological space and let (Y,τ_Y ,μ_Y) be a subspace of (X, τ ,μ) . If Y is τ-F_σ then (Y,τ_Y ,μ_Y) is(τ_Y-μ_Y) semiparacompact w.r.t. μ_Y.

(٢.٢.١١) Corollary

Let (X, τ ,μ) be a m (τ -μ) paracompact w.r.t. μ biological space and let (Y,τ_Y,μ_Y) be a subspace of (X, τ ,μ) If Y is τ-F_σ then (Y,τ_Y ,μ_Y) is m(τ_Y –μ_Y)-a-paracompact w.r.t. μ_Y.

Proof

The proof follows immediately from Theorem (٢.٢.٩) and Theorem (٢.١.١١)

(٢.٢.١٢) Corollary .

Let (X, τ ,μ) be a (τ-μ) paracompact w.r.t. μ bitopological space and let(Y,τ_Y ,μ_Y) be a subspace of(X, τ ,μ) .If Y is τ-F_σ , then (Y,τ_Y ,μ_Y) is (τ_Y-μ_Y) –a-paracompact w.r.t. μ_Y

Proof

The proof is immediately follows from Corollary (٢.٢.١٠) and Corollary (٢.١.١٢)

(٢.٢.١٣)Theorem

let (X, τ ,μ) be a bitopological space and let (Y, τ_Y ,μ_Y) be a τ- closed subspace of (X ,τ, μ).If (X ,τ, μ) is m(τ-μ)-a- paracompact w .r .t .μ , then (Y, τ_Y ,μ_Y) is m(τ_Y ,μ_Y)-a- paracompact w .r .t. μ_Y .

Proof

Suppose that (Y,τ_Y,μ_Y) be a τ- closed subspace of m(τ-μ)-a- paracompact w .r .t. μ space (X, τ ,μ). To show (Y,τ_Y,μ_Y) is m(τ_Y -μ_Y)-a- paracompact w .r .t. μ_Y.

Let $U = \{U_\lambda : \lambda \in \Delta\}$ be a τ_Y- open cover of Y with cardinality $\leq$m.

U_λ

Since each is τ_Y-open subset of Y, there is a τ- open subset V_λ of X such that each $U_\lambda = V_\lambda \cap Y$. The collection $\prod = \{V_\lambda : \lambda \in \mathrm{A}\} \cup \{\mathrm{X}/Y\}$ form a τ -open cover of X with cardinality ≤m. For , let $x \in \mathrm{X}$. If $x \notin Y$, then $x \in \mathrm{X}/Y \quad \Rightarrow x \in \bigcup_{\mathrm{A}\varepsilon\ \Delta} V_\lambda \cup (\mathrm{X}/Y)$. If $x \in Y \Rightarrow x \in \bigcup_{\lambda\varepsilon\Delta} U_\lambda$

$\Rightarrow x \in \bigcup_{\lambda\varepsilon\Delta} (V_\lambda \cap Y) = \bigcup_{\lambda\varepsilon\Delta} V_\lambda \cap Y \Rightarrow x \in \bigcup_{\lambda\in\Delta} V_\lambda$

$\Rightarrow x \in \bigcup_{\lambda\varepsilon\Delta} V_\lambda \cup (\mathrm{X}/Y)$ thus $\prod$ is τ – open cover of X. Since $|\Delta| \le$m, then $\prod$ having cardinality ≤ m . By hypothesis $\prod$ has refinement $W = \{W_\gamma : \gamma \in \Gamma\}$ (not necessarily either μ-open or μ-closed) which is locally finite w.r.t μ.

· Now, let $A = \{W_\gamma \cap Y, \gamma \in \Gamma\}$,then A is a collection of subsets of Y (not necessarily either μ_Y-open or μ_Y-closed).We claim that A is

(i) cover Y

(ii) refine U

(iii) locally finite . w.r.t .μ_Y.

Proof of (i)

Let $y \in Y \Rightarrow y \in X$ .Since W covers X, then $y \in \bigcup_{\gamma\in\Gamma} W_\gamma \Rightarrow y \in W_\gamma$ for some $\gamma \Rightarrow y \in W_\gamma \cap Y$ for some $\gamma \Rightarrow x \in \bigcup_{\gamma\in\Gamma} (W_\gamma \cap Y)$; hence A covers Y.

proof of (ii)

Since W refines $\prod$, then $\forall W_\gamma \in W$, there exists $V_\lambda \in \prod$ such that $W_\gamma \subset V_\lambda \Rightarrow W_\gamma \cap Y \subset V_\lambda \cap Y$ .But $U_\lambda = V_\lambda \cap Y$;hence $W_\gamma \cap Y \subset U_\lambda$. Hence for every member of A there exists member of U containing it.Thus A refines U.

proof of (iii)

A is locally finite w .r .t. μ_Y by Lemma (١.٢.٨). Therefore (Y, τ_Y ,μ_Y) is m(τ_Y - μ_Y)-a- paracompact w .r .t. μ_Y.

(٢.٢.١٤)Corollary

Let (X, τ ,μ) be a bitopological space and let (Y, τ_Y ,μ_Y) be a τ- closed subspace of (X ,τ, μ).If (X ,τ, μ) is (τ -μ)-a- paracompact w .r .t. μ , then (Y, τ_Y ,μ_Y) is (τ_Y -μ_Y)-a- paracompact w .r .t .μ_Y .

(٢.٢.١٥)Theorem

Let (X, τ ,μ) be a bitopological space and let $\chi = \{ X_i : X_i \in \tau \cap \mu : i \in I\}$ be a partition of X. The space (X, τ ,μ) is m(τ -μ)-a- paracompact w .r .t. μ ,iff (X_i ,τ_i,μ_i) is m(τ_i -μ_i)-a- paracompact w .r .t. μ_i for every i.

Proof

The "only if "part. Since $Xi = X / \bigcup_{j \neq i} X_j$ is τ- closed , then the subspace (X_i,τ_i,μ_i) is m(τ_i-μ_i)-a-paracompact w.r.t.μ for every i by Theorem (٢.٢.١٣)
The "if" part .

Let $U = \{U_\lambda : \lambda \in \Delta\}$ be a τ-open cover of X with cardinality ≤ m .The collection $\prod = \{U_\lambda \cap X_i : \lambda \in \Delta\}$ is τ_i- open cover of Xi with cardinality ≤ m for every i . For ,let $x \in X_i \Rightarrow x \in X \Rightarrow x \in U_\lambda$ for some $\lambda . \Rightarrow x \in U_\lambda \cap X_i$ for some λ .Then $x \in \bigcup_{\lambda \varepsilon \Delta} (U_\lambda \cap X_i) \Rightarrow \prod$ cover Xi. Since $|\Delta| \le$m, then $\prod$ having cardinality ≤ m. Since (X_i, τ_i ,μ_i) is m(τ_i -μ_i)-a- paracompact w .r .t. μ_i $\forall i$, there exist a refinement $Ai = \{A_{i\lambda} : \lambda \in \Delta \}$ of $\prod$(not necessarily either μ_i-open or μ_i-closed) which is locally finite. w.r.t μ_i.

Let $W=\{\bigcup_{i\in I} A_{i\lambda} | \lambda \in \Delta\}$.We claim that W is

(i) cover of X(not necessarily either μ-open or μ-closed)

(ii) refine U

(iii) locally finite w.r.t .μ.

proof of (i)

Since $A_{i\lambda}$ is (not necessarily either μ_i-open or μ_i-closed) then it is not necessarily either μ-open or μ-closed sets.

Now since $X=\bigcup_{i\varepsilon I} X_i = \bigcup_{i\varepsilon I}\left(\bigcup_{\lambda\varepsilon \Delta} A_{i\lambda}\right) = \bigcup_{\lambda\varepsilon \Delta}\left(\bigcup_{i\varepsilon I} A_{i\lambda}\right)$.

Hence W cover X.

proof of (ii)

Let $\bigcup_{i\varepsilon I} A_{i\lambda} \in W$ .Since A_i refines $\prod$ then for every $A_{i\lambda}, A_{i\lambda} \subset U_\lambda \cap X_i$ for some λ .

$\Rightarrow \bigcup_{i\varepsilon I} A_{I\lambda} \subset U_\lambda \cap \left(\bigcup_{i\varepsilon I} X_i\right) = U_\lambda \cap X \subset U_\lambda$ .Hence W refines U.

proof of (iii).

Let $x \in X$, if $x \in X_i$, then x has μ_i- open nhd N intersect at most finitely many member of A_i . In other word $N \cap A_{i\lambda} = \phi$ for all but finitely many λ .$\Rightarrow \bigcup_{i\in I}(N \cap A_{i\lambda}) = \phi$ for all but finitely many λ.$\Rightarrow N \cap \left(\bigcup_{i\in I} A_{i\lambda}\right) = \phi$ for all but finitely many λ. Since N is μ_i – open nhd of x and $X_i \in \mu$ then N is μ – open nhd of x by Theorem(١.١.٣) and $\bigcup_{i\in I} A_{i\lambda} \in W$ hence W locally finite w.r.t μ . Hence (X,τ,μ) is m(τ-μ) –a- paracompact w.r.t μ.

<u>(٢.٢.١٦) Corollary.</u>

Let (X, τ ,μ) be a bitopological space, let $\chi = \{\ X_i : X_i \in \tau \cap \mu \in I\ \}$ be a partition of X. The space (X, τ ,μ) is (τ -μ)-a- paracompact w .r .t .μ iff the space (X_i , τ_i ,μ_i) is (τ_i -μ_i) –a-paracompact w .r .t μi for every i.

(٢.٢.١٧) Theorem.

If each τ-open set in an m(τ-μ) paracompact w .r .t μ bitopological space (X,τ,μ) is m(τ-μ) paracompact w .r .t. μ , then every subspace (Y, τ_Y, μ_Y) is m(τ_Y-μ_Y) paracompact w .r .t .μ_Y .

Proof

Let $U = \{U_\lambda : \lambda \in \Delta\}$ is a τ_Y-open cover of Y with cardinality $\leq m$. Since each U_λ is τ_Y-open inY,we have $U_\lambda = V_\lambda \cap Y$ where V_λ τ-open subset of X,for every $\lambda \in \Delta$.Then $G = \underset{\lambda\in\Delta}{U} V_\lambda$ is a τ-open set . Let $V = \{V_\lambda, \lambda \in \Delta\}$ be a τ-open cover of G with cardinality $\leq m$.By hypothesis G is m(τ-μ) paracompact w .r .t. μ .Thus V has a μ-open refinement $A = \{A_\gamma, \gamma \in \Gamma\}$ which is locally finite w .r .t μ .

Set

$B = \{B_\gamma, \gamma \in \Gamma\}$, where $B_\gamma = A_\gamma \cap Y$.

We claim that B is

(i) μ_Y-open cover of Y,

(ii) refine U

(iii) locally finite w .r .t μ_Y.

proof of (i).

Since Aγ,is μ-open in X ,then Bγ is μ_Y-open inY . i.e B is a collection of μ_Y-open sets. Clearly that B cover Y.

proof of (ii).

Let $B_\gamma \in B. \Rightarrow \exists A_\gamma \in A \ni B_\gamma = A_\lambda \cap Y$.

Since A refines $V \Rightarrow \forall A_\lambda \in A, \exists V_\lambda \in V \ni A_\gamma \subset V_\lambda \Rightarrow A_\gamma \cap Y \subset V_\lambda \cap Y$. i.e $B_\gamma \subset U_\lambda$

proof of (iii).

Let $y \in Y \Rightarrow y \in X \Rightarrow \exists$ μ-nhd N of $y \ni N \cap A_\gamma = \phi$ for all but finitely many γ $\Rightarrow (N \cap Y) \cap (A_\gamma \cap Y) = \phi$ for all but finitely many γ.Since $M = N \cap Y$ is μ_Y-nhd of y,and $B_\gamma = A_\gamma \cap Y$, then $M \cap B_\gamma = \phi$ for all but finitely many γ.i.e B is locally finite w.r.t.μ_Y. Therefore (Y,τ_Y,μ_Y) is m(τ_Y-μ_Y) paracompact w.r.t.μ_Y.

<u>(٢.٢.١٨) Corollary</u>

If each τ-open set in (τ - μ) paracompact w .r .t μ. bitopological space is (τ - μ) paracompact w .r .t μ. Then every subspace (Y, τ_Y, μ_Y) is (τ_Y-μ_Y) paracompact w .r .t. μ_Y.

<u>(٢.٢.١٩) Theorem.</u>

If *f* is (μ- τ`) closed , (μ-μ`) continuous mapping of a bitopological space (X,τ,μ) onto an m(τ`-μ`) paracompact w.r.t.μ` bitopological space (Y,τ`,μ`) such that $Z = f^{-1}(y), \forall y \in Y$ is m(τ-μ) compact , then (X,τ,μ) is m(τ-μ) paracompact w. r. t. μ .

proof:

Let $U=\{U_\lambda : \lambda \in \Delta\}$ be a τ-open cover of X with cardinality $\leq m$. Then U cover of Z .Since Z is m(τ-μ) compact , there exists a finite subset γ of Δ such that $Z \subset \bigcup_{\lambda\in\gamma} U_\lambda$,where U_λ is μ–open set for every $\lambda \in \gamma$.

Let Γ be the family of all finite sub set γ of Δ , then $|\Gamma| \leq m$.

Set

$$V_\gamma = Y / f\left[X / \bigcup_{\lambda\in\gamma} U_\lambda\right].$$

Since $\bigcup_{\lambda\in\gamma} U_\lambda$ is μ–open set , the set $X / \bigcup_{\lambda\in\gamma} U_\lambda$ is μ – closed , and since f is (μ-τ`) closed , then $f\left[X / \bigcup_{\lambda\in\gamma} U_\lambda\right]$ is τ`– closed in (Y,τ`,μ`) , hence V_γ is τ`–open ,and $y \in V_\gamma$ and $f^{-1}[V_\gamma] \subset \bigcup_{\lambda\in\gamma} U_\lambda$. Therefore $V=\{V_\gamma : \gamma \in \Gamma\}$is τ`– open cover of Y with cardinality $\leq m$. Since (Y,τ`,μ`) is m(τ`- μ`) paracompact w. r. t. μ` , then V has a μ` - open refinement $W=\{W_\delta : \delta \in \Omega\}$ which is locally finite w. r. t μ` .

Set $\Pi = \{f^{-1}[W_\delta] \cap U_\lambda : (\delta,\lambda) \in \Omega \times \gamma_\delta\}$.

We claim that Π is

(i) μ–open cover of X,

(ii) refines U

(iii) locally finite w. r. t μ.

Proof of (i)

Since w_δ is μ` - open $\forall \delta \in \Omega$ and $f$ is (μ-μ`) continuos ,the set $f^{-1}[W_\delta]$ is μ –open $\forall \delta \in \Omega$,and since U_λ is μ–open $\forall \lambda \in \gamma_\delta$, then $f^{-1}[W_\delta] \cap U_\lambda$ is μ–open , for every $(\delta,\lambda) \in \Omega \times \gamma_\delta$.

Now, if $x \in X \Rightarrow \exists U_\lambda \ni x \in U_\lambda$ and $\exists y \in Y \ni y = f(x) \Rightarrow f(x) \in W_\delta$

for some δ $\Rightarrow x \in f^{-1}[W_\delta] \Rightarrow x \in f^{-1}[W_\delta] \cap U_\lambda$ for some(δ,γ_δ).i.e Π covers X.

Proof of (ii)

Let $f^{-1}[W_\delta] \cap U_\lambda \in \Pi$.since $f^{-1}[W_\delta] \cap U_\lambda \subset U_\lambda \Rightarrow \Pi$ refines U.

Proof of (iii)

Let $x \in X \Rightarrow \exists y \in Y \ni y = f(x)$. Since W is locally finite w. r. t μ` ,then there is μ`- nhd N of y such that N intersect at most finitely many W_δ . In other word $N \cap W_\delta = \phi$ for all but finitely many δ $\Rightarrow f^{-1}[N \cap W_\delta] = \phi$ for all but finitely many δ $\Rightarrow f^{-1}[N] \cap f^{-1}[W_\delta] \cap U_\lambda = \phi$ for all but finitely many (δ,λ) since f is (μ- μ`) continuous ,then $f^{-1}[N]$ is μ-nhd of x, and $f^{-1}[W_\delta] \cap U_\lambda \in \Pi$,hence Π is locally finite w. r. t .μ.Therefore(X,τ,μ) is m(τ-μ) paracompact w. r. t μ.

(٢.٢.٢٠) Corollary .

If *f* is (μ- τ`) closed , (μ-μ`) continuous mapping of a bitopological space (X,τ,μ) onto an (τ`-μ`) paracompact w.r.t.μ` bitopological space (Y,τ`,μ`) such that $Z = f^{-1}(y), \forall y \in Y$ is (τ-μ) compact , then (X,τ,μ) is (τ-μ) paracompact w. r. t. μ .

(٢.٢.٢١) Theorem.

If *f* is (μ- τ`) closed , (μ-μ`) continuous mapping of a bitopological space (X,τ,μ) onto an m(τ`-μ`)semiparacompact w.r.t.μ` bitopological space (Y,τ`,μ`) such that $Z = f^{-1}(y), \forall y \in Y$ is m(τ-μ) compact , then (X,τ,μ) is m(τ-μ)semiparacompact w. r. t. μ .

Proof

Let $U=\{U_\lambda:\lambda\in\Delta\}$ be a τ-open cover of X with cardinality $\leq m$ .Then U cover of Z. Since Z is m (τ-μ)compact ,there exists a finite subset γ of Δ such that $Z\subset\bigcup_{\lambda\in\gamma}U_\lambda$,where U_λ is μ-open set for every $\lambda\in\gamma$.

Let Γ be the family of all finite subset γ of Δ, then $|\Gamma|\leq m$.

Set

$$V_\gamma=Y/f\left[X/\bigcup_{\lambda\in\gamma}U_\lambda\right].$$

Since $\bigcup_{\lambda\in\gamma}U_\lambda$ is μ-open set, the set $X/\bigcup_{\lambda\in\gamma}U_\lambda$ is μ-closed ,and since f is (μ- τ`)closed, then $f\left[X/\bigcup_{\lambda\in\gamma}U_\lambda\right]$is τ`-closed in(Y,τ`,μ`) ,hence V_γ is τ` -open and $y\in V_\gamma$ and $f^{-1}[V_\gamma]\subset\bigcup_{\lambda\in\gamma}U_\lambda$ .Therefore $V=\{V_\gamma:\gamma\in\Gamma\}$is τ` -open cover of Y with cardinality $\leq m$.Since (Y,τ`,μ`) is m (τ`-μ`) semiparacompact w. r. t μ`,then V has μ`-open refinement $W=\underset{n}{U}W_n$ where every W_n is locally finite w. r. t μ`.

Set

$$W_n=\{W_{n\delta}:\delta\in\Omega\}.\text{Thus}\,W=\underset{n}{U}\{W_{n\delta}:\delta\in\Omega\}.$$

Set $C=\underset{n}{U}C_n$,where $C_n=\left\{f^{-1}[W_{n\delta}]\cap U_\lambda:(\delta,\lambda)\in\Omega\times\gamma_\delta\right\}$. We claime that C_n is

(i) collection of μ-open set

(ii) locally finite w. r. t. μ.

Proof of (i)

Since $W_{n\delta}$ is μ` -open $\forall \delta \in \Omega$ and $f$ is (μ-μ`) continuous, the set $f^{-1}[W_{n\delta}]$ is μ-open $\forall \delta \in \Omega$,and since U_λ is μ-open $\forall \lambda \in \gamma_\delta$, then $f^{-1}[W_\delta] \cap U_\lambda$ is μ-open $\forall (\delta, \lambda) \in \Omega \times \gamma_\delta$.

Proof of (ii)

Let $x \in X \Rightarrow \exists y \in Y \ni y = f(x)$.Since W_n is locally finite w. r. t. μ $\Rightarrow \exists \mu' - nhd$ N of y such that $N \cap W_{n\delta} = \phi$ for all but finitely many δ$\Rightarrow f^{-1}[N] \cap \left(f^{-1}[W_{n\delta}] \cap U_\lambda \right) = \phi$ for all but finitely many (δ,λ) since f is (μ-μ`) continuous ,then $f^{-1}[N]$is μ-nhd of x .Hence C_n is locally finite w.r.t μ.

Its remains to show that C is :

(i*) cover X

(ii*) refine U

proof of (i*)

Let $x \in X \Rightarrow \exists U_\lambda \ni x \in U_\lambda$ and $\exists y \in Y \ni y = f(x) \Rightarrow \exists W_{n\delta} \ni y \in W_{n\delta}$ for some n,δ$\Rightarrow x \in \overset{-1}{f}[W_{n\delta}]$ for some n,δ $\Rightarrow x \in f^{-1}[W_{n\delta}] \cap U_\lambda$ for some(δ,λ).

Proof of (ii*) .

Since $f^{-1}[W_{n\delta}] \cap U_\lambda \subset U_\lambda, \forall_{n,\delta} \Rightarrow \bigcup_{n=1}^{\infty} \left(f^{-1}[W_{n\delta}] \cap U_\lambda \right) \subset U_\lambda$

i.e Π refine U_λ. Therefore (X,τ,μ) is m(τ-μ) semiparacompact w. r. t μ.

(٢.٢.٢٢) Corollary

If f is (μ- τ`) closed , (μ-μ`) continuous mapping of a bitopological space (X,τ,μ) onto an (τ`-μ`)semiparacompact w.r.t.μ` bitopological space (Y,τ`,μ`) such that $Z = f^{-1}(y), \forall y \in Y$ is (τ-μ) compact , then (X,τ,μ) is (τ-μ)semiparacompact w. r. t. μ .

§(٢-٣) Pairwise Paracompact spaces.

In this section we study the concept of pairwise paracompact spaces.

(٢.٣.١) Definition [١٥]

The bitopological space (X, τ, μ) is called

(i) RR-pairwise (m-) paracompact if the space is (m-)(τ- τ) paracompact w.r.t. μ, and (m-) (μ- μ) paracompact w.r.t τ.

(ii) FHP -pairwise (m-) paracompact if the space is (m-)(τ-μ) paracompact w.r.t μ and (m-)(μ - τ) paracompact w.r.t τ.

(iii) α-pairwise (m-) paracompact if the space is (m-) (τ-(τ v μ)) paracompact w.r.t μ and (m-)(μ -(τ v μ)) paracompact w.r.t τ .

(iv) β- pairwise (m-) paracompact if the space is (m-) (τ-(τ v μ)) paracompact w.r.t τ and (m-)(μ -(τ v μ)) paracompact w.r.t μ.

(٢.٣.٢) Definition

The bitopological space (X, τ, μ) is called

(i) RR-pairwise (m-)semiparacompact if the space is (m-)(τ-τ)semiparacompact w.r.t.μ, and (m-)(μ- μ)semiparacompact w.r.t τ.

(ii) FHP -pairwise (m-)semiparacompact if the space is (m-)(τ-μ)semiparacompact w.r.t.μ and (m-)(μ- τ)semiparacompact w.r.t τ.

(iii) α-pairwise (m-)semiparacompact if the space is (m-)(τ-(τvμ))semiparacompact w.r.t μ and (m-)(μ-(τv μ)) semiparacompact w.r.t τ .

(iv) β- pairwise (m-)semiparacompact if the space is (m-) (τ-(τ v μ)) semiparacompact w.r.t τ and (m-)(μ -(τ v μ))semiparacompact w.r.t μ.

(٢.٣.٣) Definition

The bitoplogical space (X, τ, μ) is called

(i) RR-pairwise (m-)-a-paracompact if the space is (m-)(τ- τ)-a-paracompact w.r.t. μ, and (m-) (μ- μ)-a-paracompact w.r.t τ.

(ii) FHp -pairwise (m-)-a- paracompact if the space is (m-) (τ-μ)-a- paracompact w.r.t μ and (m-)(μ - τ)-a- paracompact w.r.t τ.

(ii) α-pairwise (m-)-a- paracompact if the space is (m-) (τ-(τ v μ))-a- paracompact w.r.t μ and (m-)(μ -(τ v μ))-a- paracompact w.r.t.τ.

(iv) β- pairwise (m-)-a- paracompact if the space is (m-) (τ-(τ v μ))-a- paracompact w.r.t τ and (m-)(μ -(τ v μ))-a- paracompact w.r.t μ.

(٢.٣.٤) Proposition

(i) Every RR-pairwise (semi, *a*-) paracompact is RR-pairwise *m (resp.semi, a-)* paracompact .

(ii) Every FHp – pairwise(*semi, a-)* paracompact is FHP- pairwise m(resp.semi, *a-)* paracompact.

(iii) Every α - pairwise(*semi, a-)* paracompact is α - pairwise m(resp.semi, *a-)* paracompact .

(iv) Every β- pairwise(*semi, a-)* parcompact is β - pairwise m(resp.semi ,*a-)* paracompact.

(٢.٣.٥) Theorem

Every RR- pairwise m- paracompact is α- pairwise m- paracompact.

Proof

Let(X, τ , μ) be RR- pairwise m- paracompact.To show (X, τ , μ) is m (τ –(τvμ)) paracompact w.r.t. μ. Let U={ U_λ: λ Є Δ} be a τ-open cover of X with cardinality ≤ m. By hypothesis (X, τ , μ) is m(τ - τ) paracompact w.r.t. μ. Hence U has a τ-open refinement V={V_γ:γϵ ┌}which is locally finite w.r.t. μ. Since every τ-open set is (τvμ)-open because τ ⊂ τ vμ. Thus V is(τvμ)-open refinement of U which is locally finite w.r.t μ.Hence(X, τ , μ) is m(τ- τvμ)- paracompact w.r.t. μ.To show(X, τ , μ) is m(μ-τvμ) paracompact w.r.t. τ. Let C= {C_α :α ϵ Ω} be a μ-open cover of X with cardinality ≤m By hypothesis (X,τ,μ) is m(μ - μ) paracompact w.r.t. τ. Hence C has a μ-open refinement D= {D_γ:γЄ Γ } which is locally finite w.r.t. τ since every μ-open set is $\tau \vee \mu$ – open because $\mu \subset \tau \vee \mu$. Thus D is a $\tau \vee \mu$ – open refinement of C which is locally finite w.r.t. τ Hence (X, τ , μ) is m (μ- μ) paracompact w.r.t. τ There fore(X, τ , μ) is α- pairwise m – paracompact

(٢.٣.٦) Corollary

Every RR-pairwise paracompactis α- pairwise paracompact.

(٢.٣.٧) Theorem.

Every FHP- pairwise m-paracompactis α- pairwise m-paracompact.

Proof

Suppose that(X, τ , μ) be FHP- pairwise m- paracompact. First we need to show that (X,τ,μ) is an m (τ-$\tau v\mu$) paracompact w.r.t μ. Let $U=\{ U_\lambda: \lambda \in \Delta\}$ be a τ-open cover of X with cardinality $\leq$ m . Since (X, τ , μ)be an m(τ – μ) paracompact w.r.t μ then there is μ-open refinement $V=\{V_\gamma: \gamma \in \Gamma\}$of U which is locally finite w.r.t. μ . Hence (X,τ,μ) is m (τ - $\tau v\mu$) paracompact w.r.t μ. No to show that (X, τ , μ) is m (μ –($\tau v\mu$) paracompact w.r.t. τ. Let $C= \{C_\alpha :\alpha \in \Omega\}$ be a μ-open cover of X with cardinality $\leq$m. Since (X,τ,μ) is m(μ-τ) paracompact w.r.t. τ, then there is a τ-open refinement $D=\{D_\gamma:\gamma \in \Gamma\}$ of C which is locally finite .w.r.t. τ. Since $\tau\subset \tau v \mu$, then D is($\tau v \mu$) –open refinement which is locally finite w. r.t. τ. Hence (X, τ , μ) is m (X, τ , μ) is α-pairwise m – paracompact .

(٢.٣.٨) Corollary

Every FHP-pairwise paracompact is α-pairwise paracompact.

(٢.٣.٩) Theorem.

Every RR-pairwise m-semiparacompact is α- pairwise m–semiparacompact.

Proof

Suppose that(X, τ , μ) be RR- pairwise m- semiparacompact. First we need to show (X, τ , μ) an m (τ - τvμ) semiparacompact w.r.t. μ. Let U={ U λ: λ Є Δ}be a τ-open cover of X with cardinality ≤ m. Since (X,τ, μ) be an m (τ- τ) semiparacompact w.r.t. μ then there is a τ-open refinement V of U which is σ- locally finite w.r.t. μ .Since τ⊂ τvμ, then V is (τvμ) -open refinement of U which is σ- locally finite w.r.t. μ. Hence (X, τ , μ) be an m (τ – τvμ) semiparacompact w.r.t. μ.

Now to show that (X, τ , μ) is an m (μ - τvμ)semiparacompact w.r.t. τ. Let C= {C_α :α ϵ Ω} be a μ -open cover of X with cardinality ≤ m .
Since (X, τ , μ) be an m (μ - μ) semiparacompact w.r.t. τ. Then there is a μ -open refinement D of C which is σ- locally finite w.r.t. τ.Since μ⊂ τvμ, then D is (τvμ)-open refinement of C which is σ- locally finite w.r.t. τ. Hence (X, τ , μ) is an m (μ - τvμ)semiparacompact w.r.t.τ. Hence (X, τ , μ) is α- pairwise m–semiparacompact.

(٢.٣.١٠) Corollary

Every RR-pairwise semiparacompact is α-pairwise semiparacompact.

(٢.٣.١١) Theorem.

Every FHP- pairwise m-semiparacompact is α–pairwise m– semiparacompact.

<u>Proof</u>

Suppose that(X, τ , μ) be FHP – pairwise m- semiparacompact. First we need to show (X, τ , μ) is m (τ - τvμ) semiparacompact w.r.t. μ. Let U={ U_{λ}: λ ϵ Δ}be a τ-open cover of X with cardinality ≤ m . Since (X, τ , μ) be an m (τ- μ) semiparacompact w.r.t. μ ,then there is a μ -open refinement V of U which is σ- locally finite w.r.t. μ .Since μ ⊂ τv μ, then V is (τvμ) -open refinement of U which is σ- locally finite w.r.t. μ. Hence (X, τ , μ) is an m (τ –(τvμ)semi- paracompact w.r.t. μ.

Now to show that(X, τ , μ) be an m (μ - τvμ) semiparacompact w.r.t. τ. Let C= {C_{α} :α ϵ Ω}be a μ -open cover of X with cardinality ≤ m . Since (X, τ , μ) be an m (μ - τ) semiparacompact w.r.t. τ then there is a τ-open refinement of U which is σ- locally finite w.r.t. τ. Since τ ⊂τvμ, then D is (τvμ)-open refinement of C which is σ- locally finite w.r.t.τ. Hence (X,τ, μ) is an m (μ - τvμ) semi Paracompact w.r.t. τ.Thus (X, τ , μ) is α –pairwise m– semiparacompact.

<u>(٢.٣.١٢) Corollary</u>

Every FHP-pairwise paracompact is α –pairwise semiparacompact.

<u>(٢.٣.١٣) Theorem.</u>

The bitopolgical space is RR- pairwise m-a-paracompact iff it is α- pairwise m –a- paracompact

Proof

Suppose that(X, τ , μ) be RR – pairwise m- a-paracompact. First we need to show that (X, τ , μ) is m (τ - τvμ) -a-paracompact w.r.t. μ.

Let U={ U_λ: λ ϵ Δ} be a τ-open cover of X with cardinality ≤ m .

Since (X, τ , μ) be an m (τ-τ) –a-paracompact w.r.t. μ then there is a refinement V of U not necessarily either τ –open or τ-closed which is locally finite w.r.t. μ And also V is not necessarily either (τvμ) –open or (τvμ) – closed . Hence (X, τ , μ) is an m (τ –(τvμ))-a- paracompact w.r.t. μ.

Now to show (X, τ , μ) an m (μ - τvμ) –a- paracompact w.r.t. τ .

Let C= {C_α :α ϵ Ω} be a μ -open cover of X with cardinality ≤ m.

Since (X, τ , μ) be an m (μ- μ) – a- paracompact w.r.t. τ then there is a refinement D of C not necessarily either μ-open or μ-closed which is locally finite w.r.t. τ. And also D not necessarily either τv μ-open or τv μ-closed . Hence (X, τ , μ) is an m(μ –(τvμ))-a- paracompact w.r.t. τ. Thus(X, τ , μ) is α- pairwise m –a- paracompact.

Conversely, suppose that (X, τ , μ) is α- pairwise m –a- paracompact . First we need to show (X, τ , μ) is m(τ - τ) –a-paracompact w.r.t. μ. Let U={ U_λ: λ ϵ Δ} be a τ-open cover of X with cardinality ≤ m. Since (X, τ , μ) be an m (τ-(τvμ)) –a- paracompact w.r.t. μ. Then there is a refinement V of U not necessarily either (τvμ) -open or (τvμ) closed which is locally finite w.r.t. μ And also V is not necessarily either τ-open or τ – closed. Hence (X, τ , μ) is an m(τ – τ)-a- paracompact w.r.t. μ .

Now to show that (X, τ , μ) is an m (μ - μ) –a- paracompact w.r.t. τ .

Let C= {C_α :α ϵ Ω} be a μ -open cover of X with cardinality ≤ m . Since (X, τ , μ) be an m (μ-(τvμ)) –a- paracompact w.r.t. τ , then there is refinement D is not necessarily either (τvμ)-open or (τvμ)-closed . Hence (X,τ,μ) is an m (μ - μ) –a- paracompact w.r.t. τ . Thus (X, τ , μ) is an RR- pairwise m-a-paracompact.

(٢.٣.١٤) Corollary

The bitopological space is RR-pairwise a-paracompact iff it is α- pairwise a - paracompact .

(٢.٣.١٥) Theorem.

A bitopolgical space is RR- pairwise m-a-paracompact iff it is FHP-pairwise m-a-paracompact.

Proof

Suppose that(X, τ , μ) be RR – pairwise m- a-paracompact. First we need to show that (X, τ , μ) is an m (τ - μ) -a-paracompact w.r.t. μ. Let U={ U_λ: $\lambda \in \Delta$}be a τ-open cover of X with cardinality $\leq$ m . Since (X, τ , μ) be an m (τ- τ) - a- paracompact w.r.t. μ then there is a refinement V of U not necessarily either τ –open or τ-closed which is locally finite w.r.t. μ. And so V is not necessarily either μ –open or μ -closed .Hence (X, τ , μ) is an m (τ-μ) –a- paracompact w.r.t. μ.

Now to show (X, τ , μ) an m (μ - τ) –a- paracompact w.r.t. τ .
Let C= {C_α :$\alpha \in \Omega$} be a μ -open cover of X with cardinality $\leq$ m . Since (X, τ , μ) be an m (μ-μ) –a- paracompact w.r.t. τ , then there is refinement D of C not necessarily either μ-open or μ– closed which is locally finite w.r.t.τ. Hence (X,τ,μ) is an (μ-τ)-a-paracompact w.r.t .τ . Thus (X, τ , μ) is FHP- pairwise m-a-paracompact.

Conversely, suppose that(X, τ , μ) be FHP – pairwise m- a-paracompact. To show (X, τ , μ) is m (τ - τ) -a-paracompact w.r.t. μ. Let U={ U_λ: $\lambda \in \Delta$}be a τ-open cover of X with cardinality $\leq$ m . Since (X, τ , μ)

be an m (τ -μ) –a- paracompact w.r.t. τ , then there is refinement V of U not necessarily either μ-open or μ– closed . which is locally finite .w.r. μ.And so V is not necessarily either τ -open or τ – closed . Hence (X, τ , μ) is an (τ- τ)-a -paracompact w.r.t. μ.

Now to show that (X, τ , μ) be an m (μ - μ) –a- paracompact w.r.t. τ . Let C= {C_α :α є Ω} be a μ -open cover of X with cardinality ≤ m . Since (X, τ , μ) be an m (μ- τ) –a- paracompact w.r.t. τ , then there is refinement D of C not necessarily either τ -open or τ – closed which is locally finite w.r.t.τ . And so D is not necessarily either μ -open or μ – closed. Hence (X, τ , μ) is an m (μ - μ) –a- Paracompact w.r.t. τ .Thus (X, τ , μ) is RR- pairwise m-a-paracompact.

(٢.٣.١٦) Corollary

A bitopolgical space is RR- pairwise a-paracompact iff it is FHP- pairwise a-paracompact.

(٢.٣.١٧) Corollary

A bitopological space is FHP pairwise m- a- paracompact iff it is α- pairwise m –a- paracompact .

Proof

The proof follows from Theorem (٢.٣.١٣) and Theorem (٢.٣.١٥) .

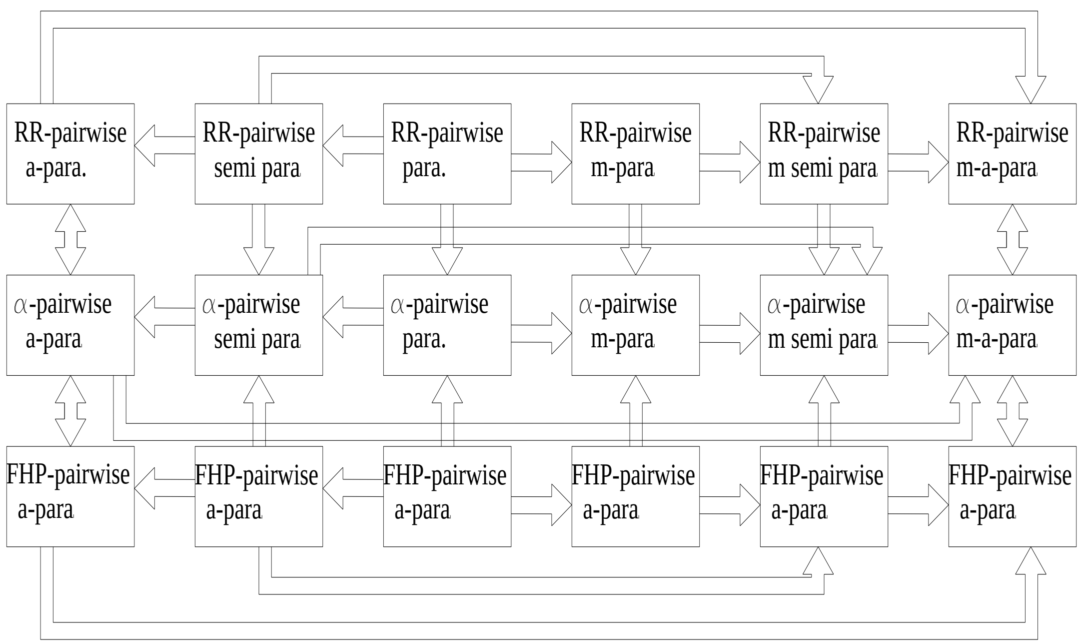
(٢.٣.١٨) Corollary

A bitopological space is FHP-pairwise a- paracompact iff it is α- pairwise a-paracompact .

Proof

The proof follows from Corollary (٢.٣.١٤) and Corollary (٢.٣.١٦) .

The relations among the concepts that were studied a can be showed in the following diagram ;



Note:

Some implications that were given in above diagram can be proved by using the diagram which was given in §(٢-١)

(٢.٣.١٩) Theorem

Let (X, τ , μ) be β - pairwise regular bitopological space. The space (X, τ , μ) is α- pairwise m - paracompact iff it is β- pairwise m- paracompact .

Proof :

The " only if " part. Suppose that(X, τ , μ) be α – pairwise m- paracompact. To show (X, τ , μ) is m(τ - τvμ) paracompact w.r.t. τ. Let U={ U_λ: λ Є Δ}be a τ-open cover of X with cardinality ≤ m . Then U has a (τvμ) -open refinement V which is locally finite w.r.t. μ .Since(X, τ , μ) is β-pairwise regular , then V locally finite w.r.t. τ by Proposition (١.٣.١٤) Hence (X, τ , μ) is an m(τ- τvμ) paracompact w.r.t. τ.

Reversing the role of τ and μ, we obtain that(X, τ , μ) is an m(μ -τvμ) paracompact w.r.t. μ. is β- pairwise m- paracompact .

The "if" part . First we need to show that (X, τ , μ) is an m (τ –τvμ) paracompact w.r.t. μ . Let U τ– open cover of X with cardinality ≤ m . Then U has a (τvμ)-open refinement V which is locally finite w.r.t .τ. Since (X, τ , μ) is β- pairwise regular then V is locally finite w.r.t. μ by proposition (١.٣.١٤). Hence (X, τ , μ) is m(τ -τvμ) paracompact w.r.t. μ.

Reversing the role of τ and μ we obtain that (X, τ , μ) is m(μ -τvμ) paracompact w.r.t. τ. Thus (X, τ , μ) is α- pairwise m-paracompact.

(٢.٣.٢٠) Corollary

Let (X, τ , μ) be β- pairwise regular bitoplogical space . The space (X, τ , μ) is α- pairwise paracompact iff it is β- pairwise paracompact.

(٢.٣.٢١) Theorem.

Let (X, τ,μ) be a β- pairwise regular bitoplogical space . The space (X, τ , μ) is α- pairwise m-a- paracompact iff it is β- pairwise m-a-paracompact .

Proof :

The " only if " part. Suppose that(X, τ , μ) be α – pairwise m- a-paracompact. To show that (X, τ , μ) is m(τ - τvμ) –a- paracompact w.r.t. τ. Let U={ U_λ: λ Є Δ}be a τ-open cover of X with cardinality ≤ m . Then U has a refinement V not necessarily either (τvμ) -open or (τvμ) -closed which is locally finite w.r.t. μ . Since (X, τ , μ) is β-pairwise regular , then V is locally finite w.r.t. τ by Proposition (١.٣.١٤). Hence (X,τ,μ) is an m(τ- τvμ) –a- para compac w.r.t. τ.

Reversing the role of τ and μ, we obtain that(X, τ , μ) is m (μ -τvμ)-a- paracompact w.r.t. μ . So it is β- pairwise m-a- paracompact .

The "if" part . First we need to show that (X, τ , μ) is m (τ –τvμ) –a-paracompact w.r.t. μ . Let U τ– open cover of X with cardinality ≤m . Then U has a refinement V not necessarily either (τvμ)-open or (τvμ)-closed which is locally finite w.r.t .τ. Since(X, τ , μ) is β- pairwise regular then V is locally finite w.r.t. μ by Proposition (١.٣.١٤). Hence (X,τ,μ) is m(τ -τvμ) –a-paracompact w.r.t. μ.

Reversing the role of τ and μ we obtain that (X, τ , μ) is m(μ -τvμ) –a-paracompact w.r.t. τ. Thus (X, τ , μ) is α- pairwise m-a-paracompact.

(٢.٣.٢٢) Corollary

Let (X, τ,μ) be a β- pairwise regular bitoplogical space . The space (X,τ ,μ) is α- pairwise a- paracompact iff it is β- pairwise a-paracompact .

(٢.٣.٢٣) Theorem.

A bitopological space (X, τ,μ) is β- pairwise m-a-paracompact iff it is bi- m-a - paracompact.

Proof

The " only if" part . To show (X, τ) is an m-a- paracompact. Let U be a τ –open cover of X with cardinality ≤ m . Since(X, τ , μ) be an m(τ -τvμ) -a- paracompact w.r.t. τ,then U has a refinement V not necessarily either τvμ-open or τvμ-closed – which is locally finite. w.r.t. τ. And so V is not necessarily either τ-open or τ-closed .Hence (X, τ) is an m-a -paracompact. Now to show (X, μ) is an m-a- paracompact .Let C be μ -open cover of X with cardinality ≤ m . Since (X, τ , μ) is an m (μ -τvμ) – a- Paracompact w.r.t. μ., then C has refinement D is not necessarily either τvμ -open or τvμ -closed which is locally finite w.r.t. μ and so D is not necessarily either μ -open or μ-closed . Hence (X, μ) is an m-a- paracompact. Thus (X, τ , μ) is bi-m-a-Paracompact.

The "if" part .To show (X, τ , μ) is m (τ -τvμ) -a- paracompact w.r.t. τ. Let U be a τ- open cover of X with cardinality ≤m. Since (X, τ) is m-a-paracompact ,then U has a refinement V not necessarily either τ-open or τ-closed which is locally finite w.r.t. τ . And so V is not necessarily either τvμ-open or τvμ-closed. Hence (X, τ , μ) is m (τ- τvμ) - a- paracompact w.r.t. τ.

Now to show (X, τ , μ) is m (μ -τvμ) -a- paracompact w.r.t. μ.

Let C be a μ - open cover of X with cardinality ≤m. Since (X, μ) is m-a-paracompact ,then C has a refinement D not necessarily either μ -open or μ -closed which is locally finite w.r.t. μ. And so D is not necessarily either τvμ-open or τvμ-closed. Hence (X, τ , μ) is m (μ - τvμ) – a- paracompact w.r.t. μ. Thus (X, τ , μ) is β- pairwise m-a-paracompact.

(٢.٣.٢٤) Corollary.

A bitopological space is β- pairwise a-paracompact iff it is bi- a – paracompact.

(٢.٣.٢٥) Theorem.

Every β - pairwise m –paracompact space is bi - m- a- paracompact.

Proof :

Suppose that (X,τ,μ) be a bitopological space . First we need to show that (X, τ) is m-a- paracompact. Let U be a τ-open cover of X with cardinality ≤ m . Since(X,τ,μ) is m(τ -τvμ) paracompact w.r.t. τ, then U has τvμ- open refinement V which is locally finite w.r.t. τ . Hence V is not necessarily either τ –open or τ –closed refinement of U which locally finite w.r.t. τ . Thus (X, τ) is m-a- paracompact.

Reversing the role of τ and μ we obtain that (X, μ) m-a- paracompact. Therefore (X,τ,μ) is bi - m- a- paracompact.

(٢.٣.٢٦) Corollary

Every β- pairwise paracompact space is bi- a paracompact.

(٢.٣.٢٧) Corollary

Every β- pairwise paracompact space is bi-m-a- paracompact.

(٢.٣.٢٨) Theorem.

If (X, τ , μ) be a - pairwise .Hausdorff space such that every τ-closed set and every μ- closed set in (X, τ , μ) having cardinality ≤ m . Then :

(i)Every RR- pairwise m- paracompactspace is β- pairwise m- regular .

(ii)Every FHP- pairwise m- Paracompact space is γ- pairwise m- regular.

Proof:

(i) Let (X, τ , μ)be RR- pairwise m- paracompact and pairwise Hausdorff space . Then (X, τ , μ) is m (τ - τ) paracompact w.r.t. μ and pairwise . Hausedorff. If every τ-closed set having cardinality ≤ m .
Then (X, τ , μ) is m(τ , μ, τ)- regular by Theorem (٢.٢.١).
Also (X, τ , μ) is m (μ - μ) paracompact w.r.t. τ and pairwise . Hausedorff. If every μ -closed set having cardinality ≤ m , then (X, τ , μ) is m(μ , τ, μ)- regular by Theorem (٢.٢.١) . Hence (X,τ,μ) is β-pairwise regular by Definition (١.٣.٨) (ii).

(ii) Let (X, τ , μ) be FHp- pairwise m - paracompact and pairwise Hausedorff space .Then (X, τ, μ) is m(τ- μ) Paracompact w.r.t. μ and pairwise Housedorff. If evry τ-closed set having cardinality ≤ m , then (X, τ , μ) is m(τ, μ, μ)- regular by Theorem (٢.٢.١).Also (X,τ,μ) is m(μ-τ) paracompact w.r.t.τ and pairwise Hausedorff.If every μ -closed set having cardinality ≤ m, then (X, τ , μ) is m(μ,τ,τ) regular by Theorem (٢.٢.١) and so it is γ- pairwise regular by Definition (١.٣.٨) **.**

(٢.٣.٢٩) Corollary

If (X, τ , μ) be a pairwise Hausdorf space. Then:

(i)Every RR- pairwise paracompact space is β-pairwise regular,and consequently α - pairwise regular.

(ii)Every FHP pairwise paracompact space is γ- pairwise regular , and consequently β - pairwise regular .

proof:

(i) Let (X, τ , μ) be RR- pairwise paracompact space, and pairwise Hausdorff .Then (X, τ , μ) is (τ - τ) paracompact w.r.t. μ and pairwise Hausdorff. Hence (X, τ , μ)is (τ ,μ ,τ)- regular by Corollary (٢.٢.٢) . Also (X, τ , μ) is (μ - μ) paracompact w.r.t. τ and pairwise Hausdorff , then (X, τ , μ) is (μ, τ, μ ,)- regular by Corollary (٢.٢.٢) . Hence (X, τ , μ) is β - pairwise regular by Definition (١.٣.٨) (ii) . And consequently α - pairwise regular by Proposition (١.٣.٩) .

(ii) Let (X,τ, μ) be FHP- pairwise paracompact space, and pairwise Hausdorff.Then (X, τ , μ) (τ - μ) paracompact w.r.t. μ and pairwise Hausdorff. Hence (X, τ , μ)is (τ ,μ , μ,) regular by Corollary (٢.٢.٢) . Also (X, τ , μ) is (μ- τ) paracompact w.r.t. τ and pairwise Hausdorff. Hence (X, τ , μ) is (μ , τ ,τ) regular by Corollary (٢.٢.٢) . Hence (X, τ , μ) is γ - pairwise regular by Definition (١.٣.٨) (iii). And consequently (X,τ,μ) is β-pairwise regular by Proposition (١.٣.١٠) .

(٢.٣.٣٠)Theorem.

Let (X, τ , μ) be a pairwise Hausdorff bitopological space . Then the following are equivalent:

(i)(X, τ , μ)is RR- pairwise paracompact.

(ii)(X, τ , μ)is β- pairwise regular and α - pairwise paracompact.

(iii)(X, τ , μ) is β- pairwise regular and β - pairwise para-compact.

(iv) (X, τ , μ)is β- pairwise regular and bi- a- paracompact.

(V) (X, τ , μ)is β- pairwise regular and bi- paracompact.

Proof

(i) ⇒ (iii) follow form Corollary (٢.٣.٦) and Corollary (٢.٣.٢٩) (i) .

(ii) ⇒ (iii) Corollary (٢.٣.٢٠).

(iii)⇒ (iv)Corollary (٢.٣.٢٦).

(iv)⇒ (v) Suppose (iv) .To show (X, τ) and (X, μ) are paracompact. By (iv) (X, τ , μ)is β- pairwise regular.

Then(X, τ) and (X, μ) are regular by Proposition (١.٣.١٢) and both these topological spaces are a-paracompact . Hence they are paracompact by Theorem (١.٣.٣).

(v)⇒ (i) .Suppose (V) .To show (X, τ,μ) is (τ- τ) paracompact w.r.t. μ. Let U be a τ-open cover of X. Since (X, τ) is paracompact , then U has a τ-open refinement V which is locally finite w.r.t. τ. Since(X, τ,μ) is β - pairwise regular , then V is locally finite w r.t. μ by proposition(١.٣.١٤). Hence(X, τ,μ) is (τ- τ) paracompact w.r.t. μ.

Now to show (X, τ,μ) is(μ- μ) paracompact w.r.t. τ. Let C be a μ-open cover of X. Since (X, μ) is Paracompact , then C has a μ - open refinement D which locally finite w.r.t. μ. Since (X, τ, μ) is

β - pairwise regular, then D is locally finite w.r.t.τ by Proposition (١.٣.١٤) .Hence (X, τ,μ) is (μ- μ) paracompact, w.r.t. τ. Therefore the space (X, τ,μ) is RR- pairwise paracompact.

(٢.٣.٣١) Corollary

Let (X, τ,μ) is pairwise Hausedorff. Then

(i) Every FHP- pairwise paracompact is RR- pairwise paracompact.

(ii)Every FHP_ pairwis paracompact is bi-a-paracompact.

proof

(i) Let (X,τ,μ) is FHP- pairwise paracompact and pairwise Hausedorff. Then by corollary (٢.٣.٨) (X, τ,μ) is α - pairwise paracompact. Also it is β - pairwise regular by Corollary (٢.٣.٢٩) (ii).Hence (X, τ, μ) is RR- pairwise paracompact by Theorem (٢.٣.٣٠).

(ii) Follows immediately form (i) and Theorem (٢.٣.٣٠).

(٢.٣.٣٢)Theorem

Every FHP- pairwise paracompact space is bi- paracompact.

Proof :

Let (X , τ , μ) be FHP- pairwise paracompact.Let U be a τ-open cover of X. Since (X , τ , μ) is (τ - μ) paracompact w.r.t. μ, then U has a μ-open refinement V which is locally finite w.r.t. μ .And since (X , τ , μ) is(μ- τ) paracompact w.r.t. τ, then V has a τ-open refinement W which is locally finite w.r.t. τ. But W refines U by Lemma (١.٢.٦) . Thus (X , τ) is paracompact.

Symmetry , let U be a μ -open cover of X . Since (X, τ , μ) is (μ - τ) paracompact w.r.t. τ ,then U has a τ-open refinement V which is locally finite w. r.t. τ .And since(X , τ , μ) is (τ - μ) paracompact w.r.t. μ, then V has a μ –open refinement W which is locally finite w.r.t.μ .But W refine U by Lemma (١.٢.٦) .Thus (X, τ , μ) is paracompact. Therefore (X, τ , μ) is bi- paracompact.

(٢.٣.٣٣) Theorem

Every FHP-pairwise semi paracompact space is bi-semiparacompact.

Proof:

Let (X, τ , μ) be FHP- pairwise semiparacompact. Let U be aτ-open cover of X. Since (X, τ , μ) is (τ - μ) semiparacompact w.r.t. μ, then U has a μ-open refinement V which is σ- locally finite w.r.t. μ. And since (X, τ , μ) is (μ, τ) semiparacompact w.r.t. τ. Then V has a τ-open refinement W which is σ-locally finite w.r.t. τ. (X, τ , μ) is (τ , μ) semiparacompact w.r.t. μ. But W refines U by lemma (١.٢.٦) .Thus (X, τ) is semiparacompact.

Symmetry, let A be μ-open cover of X.Since(X,τ,μ) is (μ-τ) semiparacompact w.r.t. τ, then A has a τ -open refinement B which is σ-locally finite w.r.t. τ. And since (X,τ,μ) is (τ- μ) semiparacompact w.r.t. μ., then B has a μ-open refinement D which is σ-locally finite w.r.t. μ. But D refines A By Lemma (١.٢.٦).Thus is (X, μ) is semiparacompact. Therefore (X, τ , μ) is bi-semiparacompact.

(٢.٣.٣٤) Theorem

If (X, τ , μ) be FHP-pairwise m- paracompact,and pairwise Housdorff, such that every τ- closed set and every μ-closed set in(X, τ , μ) having cardinality ≤ m, then (X, τ , μ) is pairwise m-normal.

Proof :

Let (X, τ , μ) be FHP- pairwise m- paracompact and pairwise Hausdorff space . Then (X, τ , μ) is m (τ - μ) Paracompact w.r.t. μ and pairwise Hausdorff .If every τ- closed set in (X, τ , μ) having cardinality ≤ m , then (X, τ , μ) is m(μ , τ , μ) normal by Theorem (٢.٢.٣).Also (X, τ , μ) is m (μ - τ) paracompact w.r.t. τ and pairwise Hausdorff. lf every μ- closed set in (X,τ,μ) having cardinality ≤ m then (X, τ , μ) is m(μ , τ , τ) normal by Theorem (٢.٢.٣). Hence (X,τ,μ) is pairwise m-normal by Definition (١.٣.١٧).

(٢.٣.٣٥) Theorem

Every FHP-pairwise paracompact pairwise Hausdorff space is pairwise normal.

Proof :

Let (X, τ , μ) be FHP- pairwise paracompact and pairwise Hausdorff space .Then (X,τ,μ)is(τ- μ) paracompact w.r.t.μ and pairwise Hausdorff ,hence(X, τ , μ) is(τ , μ , μ)- normal by Corollary (٢.٢.٤) Also (X, τ , μ) is (μ - τ) paracompact w.r.t. τ and pairwise Hausdorff . Hence (X, τ , μ) is (μ , τ , τ)- normal by Corollary (٢.٢.٤).Thus (X , τ , μ) pairwise normal by Definition (١.٣.١٧).

CHAPTER Three

Paracompact ness in

Tritopological Spaces

§ (٣-١)Some Definitions and results

In this section we define the paracompactness , and semiparacompactness in tritopological spaces.

(٣.١.١)Definition.

A tritopological space (X , τ , μ , ρ) is called(m)(τ-μ) paracompact w.r.t ρ [١٦] if for every τ-open cover $U = \{U_\lambda : \lambda \in \Delta\}$ of X (with cardinality ≤ m) has a μ-open refinement $V = \{V_\gamma : \gamma \in \Gamma\}$ which is locally finite w.r.t ρ.

(٣.١.٢) Proposition.

Every (τ -μ)paracompact.w .r .t ρ. tritopological space (X, τ ,μ ρ) is m (τ -μ)paracompact w .r .t ρ .

Proof

Let (X, τ, μ,ρ) be a (τ -μ) paracompact w .r .t ρ. Let $U = \{U_\lambda : \lambda \in \Delta\}$be a τ -open cover of X with cardinality ≤ m .By hypothesis U has a μ-open refinement $V = \{V_\gamma : \gamma \in \Gamma\}$which is locally finite .w .r .t ρ. Hence (X, τ ,μ,ρ) is m (τ -μ) paracompact w. r. t ρ.

(٣.١.٣). Definition

A tritopological space (X,τ,μ,ρ) is called (m) (τ - μ) semiparacompact w .r .t ρ ,[١٦] if for every τ - open cover $U = \{U_\lambda : \lambda \in \Delta\}$of X (with cardinality ≤ m) has a μ-open refinement $V = \{V_\gamma : \gamma \in \Gamma\}$ which is σ-locally finite. w .r .t ρ .

(٣.١.٤) Proposition.

Every (τ -μ)semiparacompact w.r.t.ρ tritopological space (X,τ,μ,ρ) is m(τ -μ)semiparacompact w .r .t. ρ .

Proof :

The proof is the same as the proof of the Proposition (٣.١.٢) with replacing the word "paracompact " by the word "semiparacompact".

(٣.١.٥) Theorem.

Every m(τ-μ) paracompact w .r .t. ρ tritopological space (X , τ, μ ,ρ) is m (τ-μ) semiparacompact w .r .t .ρ .

Proof

Let (X, τ , μ ,ρ) be m(τ-μ)paracompact w .r .t ρ. Let $U = \{U_\lambda : \lambda \in \Delta\}$be a τ-open cover of X with cardinality ≤ m .By hypothesis there exist a μ-open refinement $V = \{V_\gamma : \gamma \in \Gamma\}$of U which is locally finite .w .r .t ρ. By Remark (١.٢.١٩) , V is σ-locally finite. w . r .t ρ . Hence the space (X,τ ,μ,ρ) is m (τ-μ) semiparacompact w .r. t ρ by Definition (٣.١.٣).

(٣.١.٦) Corollary.

Every (τ-μ) paracompact w .r .t .ρ tritopological space (X , τ ,μ, ρ) is (τ-μ) semiparacompact w.r .t ρ .

(٣.١.٧) Corollary

Every (τ-μ) paracompact w .r .t ρ. tritopological space (X , τ ,μ, ρ) is m(τ-μ) semiparacompact .w .r .t ρ .

Proof

The proof is immediately follow from Proposition (٣.١.٢) and Theorem (٣.١.٥) .

§(٣-٢) Paracompactness and Subspaces.

In this section we give some theorems that related the paracompactness with separation axioms in tritopological spaces-Also we give some theorems concering the subspaces of paracompact tritopological spaces.

(٣.٢.١)Theorem.

If (X, τ , μ ,ρ) be m (τ-μ) paracompact. w .r .t ρ and (X, τ, ρ) pairwise Hausdorff such that every τ- closed set in (X, τ ,μ, ρ) having cardinality $\leq$ m, then (X, τ ,μ, ρ) is m(τ , ρ , μ) - regular .

Proof

Suppose that (X, τ ,μ, ρ) be m(τ ,μ) paracompact.w .r .t ρ,and A be a τ- closed set in (X, τ ,μ, ρ) having cardinality $\leq$ m, and $x \in X / A$.. Since (X, τ , ρ) is pairwise Hausdorff, then for every $y \in A$,we can find τ – open set Vy ,and ρ-open set Uy, such that $x \in U_y , y \in V_y$ and $U_y \cap V_y = \phi$ .The collection

$\Pi = \{V_y : y \in A\} \cup \{X / A\}$ form a τ-open cover of X having cardinality $\leq$ m. For,let $p \in X$,if $p \in A \Rightarrow \exists$τ–open set $V_y \ni p \in V_y \Rightarrow p \in \bigcup_{y \in A} V_y \cup \{X / A\} \Rightarrow \Pi$ cov*ers* X.if $p \notin A \Rightarrow p \in X / A \Rightarrow p \in \bigcup_{y \in A} V_y \cup \{X / A\} \Rightarrow \Pi$ covers X.Hence in any case Π covers X. Since A having cardinality $\leq m$,then the collection $\{V_y : y \in A\}$ having cardinality $\leq m$, thus Π having cardinality $\leq m$. By hypothesis Π has μ – open refinement $W = \{W_\gamma : \gamma \in \Gamma\}$ wich is locally finite.w.r.t.ρ.

Set

$$V = \bigcup_{\gamma\in\Gamma}\{W_\gamma : W_\gamma \cap A \neq \phi\}.$$

Then V is μ– open set containing A . For,let $a \in A \Rightarrow a \notin W_\gamma, \forall \gamma \in \Gamma$ such that $W_\gamma \cap A = \phi \Rightarrow a \in W_\gamma, \exists \gamma \in \Gamma \ni W_\gamma \cap A \neq \phi \Rightarrow a \in \bigcup_{\gamma\in\Gamma}\{W_\gamma : W_\gamma \cap A \neq \phi\}$

Hence $A \subset V$ .Since the μ – open cover W is locally finite . w .r .t ρ , x has a ρ-nhd U^* which meets only finitely many Wγ , say $W\gamma_1$, … , $W\gamma_n$. If some $W\gamma_i$, i=١,٢,…,n meets A it must be subset of some Vy , say Vy_i . In fact if $W_{\gamma i} \cap A \neq \phi, i = 1,2,\ldots,n$

$\Rightarrow W_{\gamma i} \cap (X / A) = \phi, i = 1,2,\ldots,n$

$\Rightarrow W_{\gamma i} \not\subset X / A, i = 1,2,\ldots,n$

$\Rightarrow \exists V_{\gamma i} \ni W_{\gamma i} \subset V_{\gamma i}$ (since W refines Π) , for i=١,٢,…,n .

Set $U = U^* \cap \left(\bigcap_{i=1}^{n} U_{yi} \right)$.

Then $x \in U$ and U is ρ– open set. Finally , we need to show $U \cap V = \phi$.

Assume contrary that, $U \cap V \neq \phi$. then there exist at least one point $p \in U \cap V \Rightarrow p \in U \wedge p \in V$

$\Rightarrow p \in U_{yi}, \forall i, i = 1,2,\ldots,n$

$\Rightarrow \exists V_{yi} \ni p \in V_{yi}$

$\Rightarrow p \in U_{yi} \wedge p \in V_{yi}$

$\Rightarrow p \in U_{yi} \cap V_{yi}$

i.e. $U_{yi} \cap V_{yi} \neq \phi$.

But this is a contradiction. Thus $U \cap V = \phi$.

Therefore the tritopological space (X, τ ,μ, ρ) is m(τ ,ρ,μ)-regular.

(٣.٢.٢) Corollary.

If (X, τ ,μ, ρ) be a (τ-μ) paracompact. w .r .t .ρ .and (X , τ , ρ) is pairwise Hausdorff space then (X, τ ,μ, ρ) is (τ, ρ , μ) –regular.

(٣.٢.٣) Theorem.

If (X, τ ,μ, ρ) be an m(τ , μ) paracompact.w .r .t ρ,and (X, τ , ρ) is pairwise Hausdorff space such that every τ-closed set in (X, τ ,μ, ρ) having cardinality ≤ m , then (X, τ ,μ, ρ) is m (τ ,μ, ρ) – normal .

Proof.

Suppose that (X, τ ,μ, ρ) be an m(τ - μ) paracompact.w .r .t. ρ . Let A and B be disjoint τ-closed sets in (X, τ ,μ, ρ) such that they having cardinality ≤ m. Since(X, τ , ρ) is pairwise Hausdorff space then for each $x \in A, y \in B$ we can find τ-open set U_x and ρ-open set V_x such that $x \in U_x, y \in V_x$ and $U_x \cap V_x = \phi$. Then $\Pi = \{U_x : X \in A\} \cup \{X / A\}$ from a τ-open cover of X having cardinality ≤ m.For, let $p \in X$, if $p \in A \Rightarrow \exists$ τ – open set

$U_x \ni p \in U_x \Rightarrow p \in \bigcup_{x \in A} U_x \bigcup \{X / A\} \Rightarrow \Pi\, covers$ X.

If $p \notin A \Rightarrow p \in X / A \Rightarrow p \in \bigcup_{x \in A} U_x \bigcup \{X / A\} \Rightarrow \Pi$ covers X.Hence in any case Π covers X. since A having cardinality $\leq m$,then the collection $\{U_x : x \in A\}$ having cardinality $\leq m$ thus Π having cardinality $\leq m$. By hypothesis Π has μ – open refinement $W = \{W_\gamma : \gamma \in \Gamma\}$ wich is locally finite . w .r .t ρ . set $U = \bigcup_{\gamma \in \Gamma} \{W_\gamma : W_\gamma \cap A \neq \phi\}$.

Then U is μ– open set containing A .

For,let $a \in A \Rightarrow a \notin W_\gamma, \forall \gamma \in \Gamma$ such that $W_\gamma \cap A = \phi$

$\Rightarrow a \in W_\gamma, \exists \gamma \in \Gamma \ni W_\gamma \cap A \neq \phi \Rightarrow a \in \bigcup_{\gamma\in\Gamma}\{W_\gamma : W_\gamma \cap A \neq \phi\}$

Hence $A \subset U$.

For each y in B, we can find ρ-open nhd H_y of y which meet only finitely many Wγ , say $W\gamma_{i(y)}$ … , $W\gamma_{n(y)}$.(the value of n also dependends on y) . Each $W\gamma_{i(y)}$ which meets A is contained in some U_x , say U_{xi} , for $x_i \in A$.. In fact if $W_{\gamma i}(y) \cap A \neq \phi, i = 1,2,\ldots,n \Rightarrow W_{\gamma i}(y) \cap (X / A) = \phi, i = 1,2,\ldots,n$

$\Rightarrow W_{\gamma i}(y) \not\subset X / A, i = 1,2,\ldots,n$

$\Rightarrow \exists U_{xi}, x_i \in A \ni W_{\gamma i}(y) \subset U_{xi}$ (since W refines Π) , for i=١,٢,…,n .

Set $G_y = H_y \cap \left(\bigcap_{i=1}^{n} V_{xi} \right)$.

The Gy is ρ – open set which containes y but does not meet U . in fact , if $z \in G_y \wedge z \in U \Rightarrow z \in V_{xi}, \forall i \Rightarrow \exists U_{xi} \ni z \in U_{xi} \Rightarrow V_{xi} \cap U_{xi} \neq \phi$. But that contradiction . hence Gy∩U=ϕ . Let $V = \bigcup_{y\in B} G_y$ .then V is ρ – open set containing B . For let

$p \notin V \Rightarrow p \notin G_y, \forall y \in B$

$\Rightarrow p \notin H_y \forall y \in B$

$\Rightarrow p \notin B$

Finally we need to show that $U \cap V = \phi$. Assume contains y that $U \cap V \neq \phi$

$\Rightarrow \exists p \in U \cap V$

$\Rightarrow p \in U \wedge p \in V$

$\Rightarrow p \in U \wedge p \in G_y$

for some y .$\Rightarrow U \cap G_y \neq \phi$ for some y .

But that contradiction since $U \cap G_y = \phi$

for every $y \in B$. Hence $U \cap V = \phi$. Therefore the space (X, τ ,μ, ρ) is m (τ ,μ, ρ)- normal by Definition (١.٣.٢٠) .

(٣.٢.٤) Corollary.

If (X, τ,μ,ρ) be a (τ -μ) paracompact.w .r .t ρ .and(X,τ,ρ)is pairwise Hausdorff space,then (X, τ ,μ, ρ) is (τ,μ, ρ) –normal

proof

The proof is the same as above theorem one just omit the condition "every τ- closed set having cardinality ≤m".

(٣.٢.٥) Theorem.

Let (X, τ ,μ, ρ) be a tritopological space and let (Y, τ_Y , μ_Y,ρ_Y) be a τ- closed subspace of (X ,τ, μ,ρ).If (X ,τ, μ,ρ)be an m(τ-μ) paracompact .w .r .t ρ , then (Y, τ_Y ,μ_Y,ρ_Y) is an m(τ_Y -μ_Y) paracompact.w .r .t ρ_Y .

Proof

Suppose that (Y, τ_Y ,μ_Y,ρ_Y) be a τ- closed subspace of m (τ -μ) paracompact w .r .t ρ space (X, τ ,μ, ρ). To show that (Y, τ_Y ,μ_Y,ρ_Y) is m(τ_Y ,μ_Y) paracompact w .r .t ρ_Y

Let $U = \{U_\lambda : \lambda \in \Delta\}$ be a τ_Y- open cover of Y with cardinality ≤m. since each U_λ $U_\lambda = V_\lambda \cap Y$

is τ_Y-open subset of Y, there is τ- open subset V_λ of X such that

The collection $\Pi = \{V_\lambda : \lambda \in \Delta\} \cup \{X / Y\}$.

form a τ – open over of`` X with cardinality ≤m. For , let $x \in X$ .If $x \notin Y$, then $x \in X / Y \Rightarrow x \in \bigcup_{\lambda \varepsilon \Delta} V_\lambda \cup (X / Y)$. If $x \in Y \Rightarrow x \in \bigcup_{\lambda \varepsilon \Delta} U_\lambda$

$$\Rightarrow x \in \bigcup_{\lambda \varepsilon \Delta} (V_\lambda \cap Y) = \bigcup_{\lambda \varepsilon \Delta} V_\lambda \cap Y \Rightarrow x \in \bigcup_{\lambda \in \Delta} V_\lambda$$

$\Rightarrow x \in \bigcup_{\lambda \varepsilon \Delta} V_\lambda \cup (X/Y)$ thus $\prod$ is τ – open cover of X. Since $|\Delta| \leq$ m, then $\prod$ having cardinality$\leq$m. By hypothesis $\prod$ has μ-open refinement $W = \{W_\gamma : \gamma \in \Gamma\}$ which is locally finite. w.r.t ρ.

Now, let A =$\{\{W_\gamma \cap Y : \gamma \in \Gamma\}$,then A is a collection of μ_Y -open subset of Y.We claim that A is
(i) cover Y,

(ii) refine U,

(iii) locally finite.w.r.t ρ_Y

proof of (i).

Let $y \in Y \Rightarrow y \in X$.Since W covers X, then $y \in \bigcup_{\gamma \varepsilon \Gamma} W_\gamma \Rightarrow y \in W_\gamma$ for some $\gamma \Rightarrow y \in W_\gamma \cap Y$ for some $\gamma \Rightarrow x \in \bigcup_{\gamma \varepsilon \Gamma} (W_\gamma \cap Y)$, hence A cover Y.

proof of (ii)

Since W refines $\prod$,then $\forall W_\gamma \in W$,there exists $V_\lambda \in \prod$such that $W_\gamma \subset V_\lambda \Rightarrow W_\gamma \cap Y \subset V_\lambda \cap Y$

But $U_\lambda = V_\lambda \cap Y$; Hence $W_\gamma \cap Y \subset U_\lambda$.Hence for every member of A there exist member of U containing it. Thus A refines U .

Proof of (iii)

A is locally finite.w.r.t ρ_Y by Lemma (١.٢.٨). Therefore (Y, τ_Y ,μ_Y, ρ_Y) is m(τ_Y – μ_Y) paracompact w .r .t ρ_Y.

(٣.٢.٦)Corollary.

Let (X, τ ,μ, ρ) be a tritopological space and let (Y, τ_Y ,μ_Y,ρ_Y) be a τ- closed subspace of (X ,τ, μ,ρ).If (X ,τ, μ,ρ) is (τ -μ)paracompact. w .r .t .ρ ,then (Y, τ_Y ,μ_Y,ρ_Y) is (τ_Y –μ_Y)paracompact. w .r .t .ρ_Y .

(٣.٢.٧) Theorem.

Let(X,τ,μ,ρ) a tritopological space and , let $\chi = \{ X_i : X_i \in \tau \cap \mu \cap \rho, i \in I \}$ be a partition of X. The space (X, τ ,μ, ρ) is m(τ -μ) paracompact.w .r .t ρ ,iff (X_i ,τ_i ,μ_i,ρ_i) is m(τ_i-μ_i) paracompact w .r .t ρ_i for every $i \in I$

Proof

The "only if "part. Since $Xi = X / \bigcup_{j \neq i} X_j$ is τ- closed , then the subspace (X_i , τ_i ,μ_i, ρ_i) is m(τ_i -μ_i) paracompact w .r .t ρi for every $i \in I$ by Theorem (٣.٢.٥)

The "if" part .Let $U = \{U_\lambda : \lambda \in \Delta\}$ be a τ- open cover of X with cardinality $\leq$m. The collection $\Pi = \{U_\lambda \cap X_i : \lambda \in \Delta\}$ is τi- open cover of Xi with cardinality$\leq$m $\forall i$ .For ,let $x \in X_i \Rightarrow x \in X \Rightarrow x \in U_\lambda$ for some $\lambda. \Rightarrow x \in U_\lambda \cap X_i$ for some λ $\Rightarrow x \in \bigcup_{\lambda \in \Delta} (U_\lambda \cap X_i) \Rightarrow \Pi$ cover Xi.Sine$|\Delta| \leq m$then Π having cardinality$\leq m$. Since (X_i , τ_i ,μ_i,ρ_i) is an m(τ_i -μ_i) paracompact. w .r .t ρ_i ,$\forall_i$, there exist a μ_i- open refinement $A_i = \{ A_{i\lambda} : \lambda \in \Delta \}$ of Π which is locally finite w.r.t ρi. Let

$$W = \left\{ \bigcup_{i \in I} A_{i\lambda} : \lambda \in \Delta \right\}$$.We claim that W is

(i)μ- open cover of X

(ii)refine U

(iii)locally finite .w .r .t ρ.

proof of (i)

If $A_{i\lambda} \in A_i$ there is μ- open set G_λ such that $A_{i\lambda} = G_\lambda \cap X_i$ since $X_i \in \mu$, then $A_{i\lambda}$ is μ- open set in X and consequently W is a collection of μ- open sets .

Now $X = \bigcup_{i \in I} X_i = \bigcup_{i \in I} \left(\bigcup_{\lambda \in \Delta} A_{i\lambda} \right) = \bigcup_{\lambda \in \Delta} \left(\bigcup_{i \in I} A_{i\lambda} \right)$ hence W cover X.

proof of (ii)

Let $\bigcup_{i \varepsilon I} A_{i\lambda} \in W$ .Since Ai refines Π then for every $A_{i\lambda}, A_{i\lambda} \subset U_\lambda \cap X_i$ for some λ $\Rightarrow \bigcup_{i \varepsilon I} A_{I\lambda} \subset U_\lambda \cap \left(\bigcup_{i \varepsilon I} X_i \right) = U_\lambda \cap X \subset U_\lambda$

proof of (iii).

If $x \in X$, then $x \in X_i$, for some i ; hence x has ρ_i- open nhd N intersect at most finitely many member of A_i. In other word $N \cap A_{i\lambda} = \phi$ for all but finitely many λ .Then $\underset{i \in I}{U}(N \cap A_{i\lambda}) = \phi$ for all but finitely many λ

$\Rightarrow N \cap \left(\bigcup_{i \in I} A_{i\lambda} \right) = \phi$ for all but finitely many λ . Since N is ρi – open nhd of x then there exists ρ – open nhd M of x such that $N = M \cap X_i$. But $X_i \in \rho$ thus N is ρ – open nhd of x , and $\bigcup_{i \varepsilon I} A_{i\lambda} \in W$, hence (iii) holds .

Therefore (X, τ ,μ, ρ) is m (τ-μ) paracompact w .r .t ρ .

(٣.٢.٨) Corollary.

Let(X,τ,μ,ρ) be a tritopological space . Let $\chi = \{ X_i : X_i \in \tau \cap \mu \cap \rho, i \in I \}$ be a partition of X. The space (X,τ,μ,ρ) is $(\tau - \mu)$ paracompact.w .r .t. ρ ,iff the space $(X_i, \tau_i, \mu_i, \rho_i)$ is $(\tau_i - \mu_i)$ paracompact w .r .t. ρ_i for every $i \in I$

(٣.٢.٩) Theorem.

If each τ-open set in an m(τ-μ) paracompact w .r .t ρ tritopological space (X,τ,μ,ρ) is m(τ-μ) paracompact w .r .t ρ , then every subspace $(Y, \tau_Y, \mu_Y, \rho_Y)$ is $m(\tau_Y-\mu_Y)$) paracompact w .r .t . ρ_Y .

Proof

Let$U = \{U_\lambda : \lambda \in \Delta\}$ is a τ_Y-open cover of Y with cardinality $\leq m$. Since each U_λ is τ_Y-open in Y,we have$U_\lambda = V_\lambda \cap Y$ whereV_λ τ-open subset of X,for every $\lambda \in \Delta$.Then $G = \bigcup_{\lambda\in\Delta} V_\lambda$ is a τ-open set .Let $V = \{V_\lambda : \lambda \in \Delta\}$ be a τ-open cover of G with cardinality $\leq m$.By hypothesis G is m(τ-μ) paracompact w .r .t ρ .Thus V has a μ-open refinement $A = \{A_\gamma : \gamma \in \Gamma\}$ which is locally finite w .r .t ρ .Set $B = \{B_\gamma : \gamma \in \Gamma\}$, where $B_\gamma = A_\gamma \cap Y$.

We claim that B is

(i) μ_Y-open cover of Y,

(ii) refine U

(iii) locally finite w .r .t ρ_Y.

proof of (i)

Since A_γ,is μ-open in X,then B_γ is μ_Y-open in Y.i.e B is acollection of μ_Y-open sets.And clearly that B cover Y.

proof of (ii)

Let $B_\gamma \in B . \Rightarrow \exists A_\gamma \in A \ni B_\gamma = A_\lambda \cap Y$.

Since A refines $V \Rightarrow \forall A_\lambda \in A, \exists V_\lambda \in V \ni A_\gamma \subset V_\lambda \Rightarrow A_\gamma \cap Y \subset V_\lambda \cap Y$.i.e $B_\gamma \subset V_\lambda$

proof of (iii)

Let $y \in Y \Rightarrow y \in X \Rightarrow \exists$ ρ-nhd N of $y \ni N \cap A_\gamma = \phi$ for all but finitely many γ

$\Rightarrow (N \cap Y) \cap (A_\gamma \cap Y) = \phi$ for all but finitely many γ .Since $M = N \cap Y$ is ρ_Y-nhd of y,and $B_\gamma = A_\gamma \cap Y$ then $M \cap B_\gamma = \phi$ for all but finitely many γ.i.e B is locally finite w .r .t. ρ_Y.Therefore (Y, τ_Y, μ_Y, ρ_Y) is m(τ_Y-μ_Y) paracompact w .r .t ρ_Y.

(٣.٢.١٠) Corollary

If each τ-open set in (τ - μ) paracompact w .r .t. ρ tritopological space (X, τ,μ,ρ) is (τ - μ) paracompact w .r .t ρ. Then every subspace (Y, τ_Y, μ_Y, ρ_Y) is (τ_Y-μ_Y) paracompact w .r .t ρ_Y.

(٣.٢.١١) Theorem.

Let (X, τ,μ,ρ) be m (τ-μ) paracompact w.r.t ρ tritopological space, and let (Y,τ_Y,μ_Y,ρ_Y) be asubspace of (X,τ,μ,ρ).If Y is τ-Fσ set , then (Y,τ_Y,μ_Y,ρ_Y) is m(τ_Y-μ_Y) semiparacompact w.r.t.ρ_Y.

Proof.

Suppose Y is F_σ-set relative τ . Then $Y=\bigcup Y_n$ where each Y_n , is τ- closed .Let $U=\{U_\lambda:\lambda\in\Delta\}$ be a τ_Y-open cover of Y with cardinality ≤ m. Since each U_λ is τ_Y- open subset of Y, we have $U_\lambda = V_\lambda \cap Y$, where V_λ is τ-open subset of X for each λ Є Δ . For each fixed n , $E_n=\{V_\lambda:\lambda\in\Delta\}\bigcup\{X/Y\}$ form a τ-open cover of X with cardinality ≤ m.By hypothesis E_n has a μ-open refinement $W=\{W_{\lambda n}:(\lambda,n)\in\Delta\times IN\}$ which is locally finite .w.r.t.ρ. For each n, let $B_n=\{W_{\lambda n}\cap Y:W_{\lambda n}\cap Y_n\neq\phi\}$.

Let $B=\bigcup B_n$ we claim that B is

(i) collection of μ_Y-open set

(ii) covers Y

(iii) refines U

(iv) σ- locally finite w.r.t. ρ_Y

proof of (i)

Since W_λ is μ-open, then $W_{\lambda n} \cap Y$ is μY–open

proof of (ii)

If y Є Y ⇒y Є Y_n for some n and then for every set $W_{\lambda n}$ Є W ⇒x Є $W_{\lambda n} \cap Y$; hence B covers Y.

proof of (iii)

Here $W_{\lambda n} \subset X/Y_n$ is impossible so that $W_{\lambda n} \subset V_\lambda$ for some V_λ ⇒ $W_{\lambda n} \cap Y \subset V_\lambda \cap Y = U_\lambda$ for some U_λЄU intersecting Y_n so that B refines U.

proof of (iv)

Since W is locally finite w.r.t.ρ. Then B_n is locally finite .w.r.t. ρ_Y by Lemma (١.٢.٨) .Hence B is σ- locally finite .w.r.t.ρ_Y.Therefor $(Y, \tau_Y, \mu_Y, \rho_Y)$ is $m(\tau_Y - \mu_Y)$ semiparacompact w.r.t. ρ_Y.

(٣.٢.١٢) Corollary.

Let (X, τ,μ,ρ) be (τ-μ) paracompact w.r.t ρ tritopological space, and let (Y,τ_Y,μ_Y,ρ_Y) be asubspace of (X,τ,μ,ρ).If Y is τ –Fσ set , then (Y,τ_Y,μ_Y,ρ_Y) is (τ_Y-μ_Y) semiparacompact w.r.t.ρ_Y.

(٣.٢.١٣) Theorem.

If f is (μ- τ\`) closed , (μ-μ\`) continuous and (ρ-ρ\`) continuous mapping of a tritopological space (X,τ,μ,ρ) onto an m(τ\`-μ\`) paracompact w. r. t ρ\` tritopological space (Y,τ\`,μ\`,ρ\`) such that $Z = f^{-1}(y), \forall y \in Y$ is m(τ-μ)compact, then (X,τ,μ,ρ) is m(τ-μ) paracompact w. r. t ρ .

proof:

Let $U = \{U_\lambda : \lambda \in \Delta\}$ be a τ-open cover of X with cardinality $\leq m$. Then U cover of Z .Since Z is m(τ-μ) compact , there exists a finite subset γ of Δ such that $Z \subset \bigcup_{\lambda\in\gamma} U_\lambda$ where U_λ is μ –open set for every $\lambda \in \gamma$. Let Γ be the family of all finite sub set γ of Δ , then $|\Gamma| \leq m$.

Set

$$V_\gamma = Y / f\left[X / \bigcup_{\lambda\in\gamma} U_\lambda\right].$$

Since $\bigcup_{\lambda\in\gamma} U_\lambda$ is μ –open set , the set $X / \bigcup_{\lambda\in\gamma} U_\lambda$ is μ – closed , and since f is (μ-τ\`) closed , then $f\left[X / \bigcup_{\lambda\in\gamma} U_\lambda\right]$ is τ\`– closed in (Y,τ\`,μ\`,ρ\`) , hence $V_\gamma$ is τ\`–open and $y \in V_\gamma$ and $f^{-1}[V_\gamma] \subset \bigcup_{\lambda\in\gamma} U_\lambda$. Therefore $V = \{V_\gamma : \gamma \in \Gamma\}$is τ\`– open cover of Y with cardinality $\leq m$ since (Y,τ\`,μ\`,ρ\`) is m(τ\`- μ\`) paracompact w.r.tρ` , then V has a μ` - open refinement $W=\{W_\delta:\delta\in\Omega\}$ which is locally finite w. r. t ρ` .

Set $\Pi=\left\{f^{-1}[W_\delta]\cap U_\lambda:(\delta,\lambda)\in\Omega\times\gamma_{\chi\delta}\right\}$.

We claim that Π is

(i) μ –open cover of X,

(ii) refines U

(iii) locally finrte w. r. t ρ.

Proof of (i)

Since W_δ is μ` - open $\forall\delta\in\Omega$ and $f$ is (μ-μ`) continuous ,the set $f^{-1}[W_\delta]$ is μ –open $\forall\delta\in\Omega$,and since U_λ is μ–open $\forall\lambda\in\gamma_\delta$,then $f^{-1}[W_\delta]\cap U_\lambda$ is μ–open.

Now, if $x\in X\Rightarrow\exists U_\lambda\ni x\in U_\lambda$ and $\exists y\in Y\ni y=f(x)\Rightarrow f(x)\in W_\delta$ for some δ $\Rightarrow x\in f^{-1}[W_\delta]\Rightarrow x\in \bar{f}^{-1}[W_\delta]\cap U_\lambda$ for some(δ,γ_δ).i.e Π covers X.

Proof of (ii)

Let $f^{-1}[W_\delta]\cap U_\lambda\in\Pi$.Since $f^{-1}[W_\delta]\cap U_\lambda\subset U_\lambda\Rightarrow\Pi$ refines U.

Proof of (iii)

Let $x\in X\Rightarrow\exists y\in Y\ni y=f(x)$. Since W is locally finite w. r. t. ρ` , then there is ρ`- nhd N such that N intersect at most finitely many W_δ. In other word $N\cap W_\delta=\phi$ for all but finitely many δ$\Rightarrow f^{-1}[N\cap W_\delta]=\phi$ forall but finitely many δ$\Rightarrow f^{-1}[N]\cap f^{-1}[W_\delta]\cap U_\lambda=\phi$ for all but finitely many (δ,λ) since f is (ρ- ρ`) continuous ,then $f^{-1}[N]$ is ρ-nhd of x, and $f^{-1}[W_\delta]\cap U_\lambda\in\Pi$,hence Π is locally finite w. r. t. ρ . Therefore (X,τ,μ, ρ) is m(τ-μ) paracompact w. r. t. ρ.

(٣.٢.١٤) Corollary .

If f is (μ- τ`) closed , (μ-μ`) continuous and (ρ-ρ`) continuous mapping of a tritopological space (X,τ,μ,ρ) onto a (τ`-μ`) paracompact w. r. t ρ` tritopological space (Y,τ`,μ`,ρ`) such that $Z = f^{-1}(y), \forall y \in Y$ is (τ-μ)compact, then (X,τ,μ,ρ) is (τ-μ) paracompact w. r. t. ρ .

(٣.٢.١٥) Theorem.

If f is (μ- τ`) closed , (μ-μ`) continuous and (ρ-ρ`) continuous mapping of a tritopological space (X,τ,μ,ρ) onto an m(τ`-μ`) semiparacompact w.r.t ρ` tritopological space (Y,τ`,μ`,ρ`) such that $Z = f^{-1}(y), \forall y \in Y$ is m(τ-μ)compact, then (X,τ,μ,ρ) is m(τ-μ)semiparacompact w. r. t ρ .

Proof

Let $U = \{U_\lambda : \lambda \in \Delta\}$ be aτ-open cover of x with cardinality $\leq m$ .Then U cover of Z.Since Z is m (τ-μ)compact,there exists afinite subset γ of Δ such that $Z \subset \bigcup_{\lambda\in\gamma} U_\lambda$ where U_λ is μ-open set for every $\lambda \in \gamma$.

Let Γ be the family of all finite subset γ of Δ, then $|\Gamma| \leq m$.

Set

$$V_\gamma = Y / f\left[X / \bigcup_{\lambda\in\gamma} U_\lambda \right].$$

Since $\bigcup_{\lambda\in\gamma} U_\lambda$ is μ-open set,the set $X / \bigcup_{\lambda\in\gamma} U_\lambda$ is μ-closed ,and since f is (μ- τ`)closed, then $f\left[ X / \bigcup_{\lambda\in\gamma} U_\lambda \right]$ is τ`-closed in(Y,τ`,μ`,ρ`) ,hence $V_\gamma$ is τ` -open and $y \in V_\gamma$ and $f^{-1}[V_\gamma] \subset \bigcup_{\lambda \in \gamma} U_\lambda$ .therefore $V = \{V_\gamma : \gamma \in \Gamma\}$ is τ` -open cover of Y with cardinality $\leq m$. Since (Y,τ`,μ,`ρ`) is m(τ`-μ`) semiparacompact w. r. t ρ`,then V has μ`-open refinement $W = \bigcup_n W_n$ where every W_n is locally finite w. r. t ρ`.

Set $W_n = \{W_{n\delta} : \delta \in \Omega\}$. Thus $W = \bigcup_n \{W_{n\delta} : \delta \in \Omega\}$.

Set $C = \bigcup_n C_n$,where $C_n = \left\{ f^{-1}[W_{n\delta}] \cap U_\lambda : (\delta, \lambda) \in \Omega \times \gamma_\delta \right\}$. We claim that C_n is

(i) collection of μ-open set

(ii) locally finite w. r. t ρ.

Proof of (i)

Since $W_{n\delta}$ is μ` -open $\forall \delta \in \Omega$ and $f$ is (μ-μ`) continuous, the set $f^{-1}[W_{n\delta}]$ is μ-open $\forall \delta \in \Omega$,and since U_λ is μ-open $\forall \lambda \in \gamma_\delta$,then $f^{-1}[W_\delta] \cap U_\lambda$ is μ-open $\forall (\delta, \lambda) \in \Omega \times \gamma_\delta$.

Proof of (ii)

Let $x \in X \Rightarrow \exists y \in Y \ni y = f(x)$.since W_n is locally finite w.r.t.ρ` $\Rightarrow \exists \rho` - nhd$ N of x such that $N \cap W_{n\delta} = \phi$ for all but finitely many δ$\Rightarrow f^{-1}[N] \cap \left(f^{-1}[W_{n\delta}] \cap U_\lambda \right) = \phi$ for all but finitely many (δ,λ) since f is (ρ-ρ`) continuous,then $f^{-1}[N]$is ρ-nhd of x.Hence C_n is locally finite w.r.t ρ.

Its remain to show that C is:

(i*) cover X

(ii*) refine U

proof of (i*)

Let $x \in X \Rightarrow \exists U_{\lambda} \ni x \in U_{\lambda}$ and $\exists y \in Y \ni y = f(x) \Rightarrow \exists W_{n\delta} \ni y \in W_{n\delta}$ for some n,δ$\Rightarrow x \in f^{-1}[W_{n\delta}]$ for some n,δ$\Rightarrow x \in f^{-1}[W_{n\delta}] \cap U_{\lambda}$ for some(δ,λ).

Proof of (ii*) . Since $f^{-1}[W_{n\delta}] \cap U_{\lambda} \subset U_{\lambda}, \forall_{n,\delta} \Rightarrow U\left(f^{-1}[W_{n\delta}] \cap U_{\lambda}\right) \subset U_{\lambda}$ i.e Π refine U$_{\lambda}$. Therefore (X,τ,μ,ρ) is m(τ-μ) semiparacompact w. r. t ρ.

(٣.٢.١٦) Corollary

If f is (μ- τ\`) closed , (μ-μ\`) continuous and (ρ-ρ\`) continuous mapping of a tritopological space (X,τ,μ,ρ) onto a (τ\`-μ\`) semiparacompact w. r. t ρ\` tritopological space (Y,τ\`,μ\`,ρ\`) such that $Z = f^{-1}(y), \forall y \in Y$ is (τ-μ)compact, then (X,τ,μ,ρ) is (τ-μ)semiparacompact w. r. t. ρ .

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