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السلوك الفوضوي للدالة ذات البعدين

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Two Dimensional Chaotic Mapping

A Research

Submitted to the College of Education /Babylon University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematic

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Abstract

This research deals with the subject of chaos behavior for two Dimensional chaotic mapping .

The aim of this research is to observe the behaviour of this kind of function in the space of chaos .An example has been taken for this subjuct that is a horseshoe map .

It has been found that there are many stable points that remain in the same area even when we make another and another map . These points behave chaos in side the same area therefor we used the program when we started to change the position of these points ,the result was in all cases that they do not go outside their determinate space .

Introduction

Dynamical system are considered as the most important subjects in our daily life . It became now a days something necessary to study chaos behavior ,thus the research takes one aspect of discrete dynamical systems . It takes two dimensional chaos function .

The research consists of three chapters . The first chapter include the definition of dynamical system and explain when this system become discrete and continuous . There the fixed point has been studied and defined . The study has been shown when this point be attracting and repelling . Moreover, this chapter deals with periodic points as well as the tent family (T_μ) .

The second chapter includes chaos . It presents a simple introduction to chaos. A definition of both transitivity and strong chaos has been given in this chapter . Many definition and examples have been suggested concerning chaos . Then it deals with conjugate with some example and proofs regarding it . It also includes an examination of dimensional chaos with some definitions of its concepts . Linear dynamical system has been defined. Eigen value has been explained and eigen vector has been defined as well as characteristic equation. Moreover , non linear function have been defined. This chapter ends up with some important examples of nonlinear function such as quadratic family logistic map and Henon mapping. Finally, it explains bakers function .

The third chapter includes horseshoe map which represents the most impotent examples of dynamical system. The mathematical function of this map has been found and its behavior has been revealed . It has been found that some points remain inside the unit square what so ever we repeat the function they take a chaotic behavior.

ا لاصة (Abstract)

لقد تناولنا ثنا هذا مناقشة موضوع السلوك الفوضوي (Chaos) للدوال ثنائية الابعاد وهدف هذا البحث هو ملاحظة سلوك هذا النوع مـن الدوال الفضاء الفوضوي واخذنا مثال على ذلك هو شبيه دالـة حـذوة الفرس (Quasi Horseshoe Map) ووجدنا نقاط تبقى داخـل ا يز ا دود مهما كررنا الدالة ، وتسلك هذه النقاط سـلوك فوضـوي داخل هذا ا يز وعليه استخدمنا برنامج من برامج ا اسوب وبدأنا بتغـي النقاط فكانت كل ا الات لا رج خارج ا ال ا دد ا .

شكر وتقدير

أود أن أتقدم بشكري وتقديري الجزيل إلى أستاذي ومعلمي الأول الدكتور عادل محمود احمد الذي أنار لي طريق العلم وساعدني في تقديم هذا البحث الذي ظهر بهذه الصورة وكذلك وقوفه بجانبي في كل الظروف الصعبة التي واجهتني في كتابة بحثي هذا .

وكذلك أتقدم بالشكر الجزيل إلى الدكتور لؤي السويدي رئيس قسم الرياضيات في جامعة بابل لما أولاه لي ولزملائي من اهتمام وتسهيل كافة الصعاب التي كنا نواجهها.

وكذلك اتقدم بالشكر الجزيل إلى صديقي الوفي الأستاذ حسين فرج العسافي الذي كان يرافقني طول مسيرتي العلمية والى كافة الأصدقاء الذين ساندوني في مشواري هذا.

وأيضاً اقدم شكري وأمتناني إلى كادر مكتب المورد لخدمات الحاسبات في الرمادي وبالأخص الأستاذ يعرب قحطان حميد الذي كان لي نعم العون في طباعة هذا البحث وأظهاره بالصورة التي ترون.

الباحث

بسم الله الرحمن الرحيم

يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنْكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ)

(المجادلة: من الآية١١)

الأهداء

إلى القلب الذي ينبض في أعماقي

أمي

إلى اقرب الناس لقلبي

أخواني

واختي

C H A P T E R O N E

1.1 DYNAMICAL SYSTEM

In this chapter we explain mathematical definition of a dynamical system and it contains some basic definitions about types of fixed points and at the end of this chapter we illustrate some kinds of periodic points .

Definition 1.1

A topological group is a set G with two structures :

(1) G is group with respect to .

(2) G is a topological space.

Such that the two structures are compactable

Definition 1.2

A dynamical system on a metric space "manifold" X is the triple (X, G , π) where G is a topological group, π is a map from the product space $X \times G$ into X, satisfying the following axioms:

I – $\pi (\chi , 0) = \chi$ for every $\chi \in X$ "identity axiom"

II – $\pi (\pi (\chi, t),S) = \pi (\chi,t + S)$ for every $\chi \in X$ and $t, S \in G$ "group axiom"

III- π is continuous "continuous axiom". [4]

Definition 1.3

A dynamical system (X,G, π) is discrete if the group G is cyclic.

<u>i.e</u> all it's elements are multiples of on element g_o ЄG [13]

If G is R the set of real numbers then it is called <u>a continuous dynamical system</u>. If G is Z the set of integers then it is called <u>a discrete dynamical system</u>. We can restate this definition as follows :-

For every tЄG the mapping π induces a continuous map $f^{(t)} : X \rightarrow X$ such that $f^{(t)} (\chi) = \pi (\chi , t)$. the map $f^{(t)}$ is called transition "or action" on X satisfying the axioms of dynamical system. [6]

I – $f^{(0)} (\chi) = \chi$

II – $f^{S} (f^{(t)} (\chi)) = f^{(S-t)} (\chi)$

III – $f^{(t)}$ is continuous .

Detention 1.4

Let f be a function f: D → D such that $D \subseteq R$. Let χ_o ЄD we say that $f(\chi_o)$ is the first iterate of χ_o for f .

$f(f(\chi_o))$ is the second iterate of χ_o for f, more generally if nЄN and $\chi_o = f^{(n)}(\chi_o)$ then χ_{n+1} is the (n+1)th iterate of χ_o for f. [1]

Remark

We call the sequence $\{f^{(n)}(\chi_o)\}_{n=0}^{\infty}$ of iterates of χ_o "the orbit of χ_o" sometimes we will write χ_n for $f^{(n)}(\chi_o)$, in that case, $\{(\chi_n)\}_{n=0}^{\infty}$ is the orbit of χ_o . [10]

1.2 FIXED POINTS

Definition 1.5

Let $f : X \rightarrow X$ is a map on X, and P be in the domain of f, we say that P is called a fixed point if $f(p) = p$.

Definition 1.6

Let P be a fixed point of f, then :

I- The point P is "an attracting fixed point" if $(\exists \epsilon)$, for all χ $(\chi \in N_\epsilon (P) \cap D)$, $(\lim f^n(\chi)=P)$

II- The point P is "a repelling fixed point " if $(\exists \epsilon)$, for all χ $(\chi \in N_\epsilon (P) \cap D)$ and $\chi \neq P$, then $d(f(\chi) , p) > d (\chi , P)$. [10]

Proposition (1.1)

Let (X,d) be a metric space, f: X→X be a continuous function at χ , and let P be in the domain of f . If $f^n(p) \to \chi$ as n increases without bound "$\lim_{n\to\infty} f^n(P) = \chi$" then χ is a fixed point of f .

Proof : Since $f^{n+1}(\chi) = f(f^n(\chi))$ and f is continuous at $\chi \in X$.

Then $\lim_{n\to\infty} f^{n+1}(P) = \chi = \lim f(f^n(P)) = f(\lim_{n\to\infty} f^n(P)) = f(\chi)$.

Therefore $f(\chi) = \chi$. It means that χ is a fixed point of f .

Theorem 1.1

Let f: J→ R where J=[a,b] and suppose that f is differentiable at the fixed point P

I- If $|f'(P)| < 1$ then P is attracting .

II- If $|f'(P)| > 1$ then P is repelling .

III- If $|f'(P)| = 1$ then P can be attracting , repelling or neither. [10]

Proof : See [10]

We will illustrate these properties in the following examples .

Example (1)

$$Let \quad T_\mu(\chi) = \begin{cases} 2\mu\chi & 0 \le \chi \le \frac{1}{2} \\ 2\mu(1-\chi) & \frac{1}{2} \le \chi \le 1 \end{cases}$$

Case (1)

If $0 < \mu < ½$

I- 0 is the only the fixed point .

II- $|T'_\mu(o)| = |2\mu| < 1$ by theorem (1.1) 0 is attracting fixed point .

Case (2)

If $\mu = ½$

I- $0 \le \chi \le ½$ are fixed point .

II- By definition (1.6) , $0 \le \chi \le ½$ are attracting fixed point.

Case (3)

If $½ < \mu < 1$

I – There are two fixed point. 0 If $0 \le \chi \le ½$ and

$\frac{2\mu}{1+2\mu}$ if $½ < \chi \le 1$.

II- $|T'_\mu(o)| = |2\mu| > 1$, by theorem (1.1) $\frac{2\mu}{1+2\mu}$ is repelling fixed point .

$\left| T'\mu\left(\frac{2\mu}{1+2\mu}\right)\right| = |-2\mu| > 1$, by theorem (1.1) 0 is a repelling fixed point .

Example (2)

Let $f(\chi) = \chi - \chi^2 \quad 0 \le \chi \le 1$

0 is the fixed point of f and $| f'(0) = 1 |$

$| f(\chi) - 0 | = | \chi - \chi^2 - 0 | = | \chi | | 1 - \chi | \le | \chi - 0 |$. By definition (1.6) 0 is attracting fixed point .

Definition (1.7)

Let P be a fixed point of f, then basin of attraction of P is the set:

$$B_P = \{\chi \in D : \lim_{n \to \infty} f^n(\chi) = P\} \quad . \quad [10]$$

Example (3)

Let $f(\chi) = \chi^2$ then basin of attraction B_o of the fixed point 0. because If $| \chi | < 1$, then $f^{(n)}(\chi) = \chi^{(2^n)} \longrightarrow 0$ as increases with out bound, so that χ in B_o by contrast, if $| \chi | \ge 1$ then $|f^{(n)}(\chi)| \ge 1$ so that χ is not in B_o , thus B_o consists of all χ such that $| \chi | < 1$, then is $B_o = (-1, 1)$.

Definition (1.8)

Let (x, f) be a discrete dynamical system, and let P be in the domain of f .

I- P is an eventually fixed point of f if there is a positive integer n such that $f^n(P)$ is a fixed point of f .

II- P is a periodic point of period n if there exists n such that $f^n(P) = P$. [7]

Example (4)

$$Let \quad T(\chi) = \begin{cases} 2\chi & 0 \le \chi \le \frac{1}{2} \\ 2(1-\chi) & \frac{1}{2} < \chi \le 1 \end{cases}$$

then 1/8 is an eventually fixed point because since T (1/8) = ¼ , T (1/4) = ½ , T(½) = 1

T(1) = 0 there for 1/8 is an eventually fixed point .

1.3 PERIODIC POINTS

Definition (1.9)

Let χ be in the domain of f then χ has period n "or is period – n point" if $f^{(n)}(\chi_o) = \chi_o$ and if in addition $\{ \chi_o , f(\chi_o) , f^2(\chi_o) , \ldots, f^{(n-1)}(\chi_o) \}$ are distance. If χ_o has period n, then the orbit of χ_o which is $\{ \chi_o , f(\chi_o) , \ldots, f^{(n-1)}(\chi_o) \}$ is a periodic orbit and called an n- cycle .

Example (5)

Let $f(\chi) = -\chi^3$, f(1) = -1 , f (-1) = 1 then 1 is period 2-point .

Example (6)

The tent function T is given by

$$T(\chi) = \begin{cases} 2\chi & \text{for} \quad 0 \le \chi \le \frac{1}{2} \\ 2-2\chi & \text{for} \quad \frac{1}{2} < \chi \le 1 \end{cases}$$

Then { 2/7, 4/7, 6/7 } is a 3- cycle for T because A routine check yields .

T(2/7) = 4/7, T(4/7) = 2-2(4/7) = 6/4 and T(6/7) = 2-2(6/7) = 2/7 can firming that {2/7,4/7,6/7} is a 3-cycle for T .

Definition (1.10)

Let χ be a period – n point for a function f . Then χ is an attracting period – n point if χ is an attracting fixed point of $f^{(n)}$, also χ is a repelling period – n point If χ is a repelling fixed point of $f^{(n)}$.

Example (7)

Let $f(\chi) = -\chi^{1/3}$ then 1 is an attracting period –2 point of f. because First notice that f(1) =-1 and f(-1) =1 therefore the point 1 has period –2 . Next, observe that $f^{(2)}(\chi) = f(f(\chi)) = -(-\chi^{1/3})^{1/3} = \chi^{1/9}$, so that $(f^2)'(1) = 1/9$, by theorem (1.1) then implies that 1 is an attracting period –2 fixed point of $f^{(2)}$, so that 1 is an attracting period-2 point by definition (1.11) .

Theorem (1.2)

Let {χ,z} be a 2- cycle for f

I- If $| f'(\chi) f'(z) | < 1$ then the cyclic is attracting .

II- If $| f'(\chi) f'(z) | > 1$ then the cyclic is repelling .

Proof see [10]

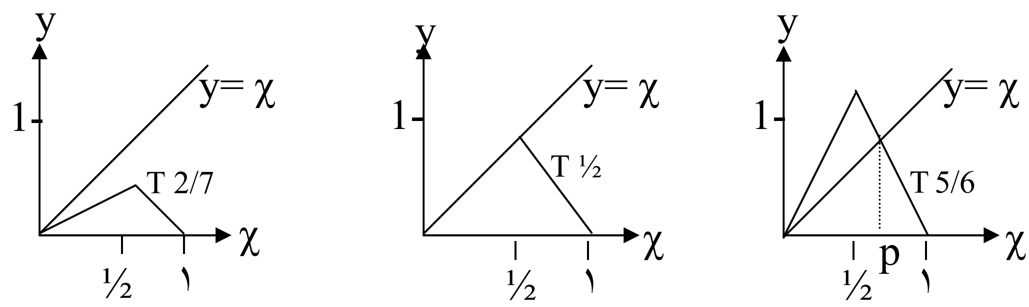
Example (8)

Let $f(\chi) = \chi^2 - 3\chi + 2$ then {0,2} is a repelling 2- cycle. Because since f(0) = 2 and f(2) = 0 . It follows that {0,2} is a 2-cycle, the fact that $f'(\chi) = 2\chi - 3$ implies that $f'(0) = -3$ and $f'(2) = 1$ so that $f'(0) f'(2) = (-3)(1) = -3$ therefore by theorem (1.2) implies that {0,2} is a repelling 2-cycle of f .

1.4 THE TENT FAMILY {Tμ}

The tent family consists of the functions T defined by

$$T_\mu(\chi) = \begin{cases} 2\mu\chi & \text{for} \quad 0 \le \chi \le \frac{1}{2} \\ 2\mu - (1-\chi) & \text{for} \quad \frac{1}{2} \le \chi \le 1 \end{cases}$$



(a) **(b)** **(c)**

Figure (1)

Figure(1) (a.b.c) displays T_μ for $\mu=2/7$, ½ and 5/6 . AS μ increases the height of the graph of T_μ rises because of the factor μ in the formula for T_μ from this observation and the three graphs in Figure (1) .

We deduce that If $0 < \mu < ½$ then $T\mu$ intersects the line $y= \chi$, once (at 0) whereas If $½ < \mu < 1$ then there are two points of intersection . We are led to analyze . Separately the members of $\{T_\mu\}$ for which $0 < \mu < ½$, $\mu = ½$ and $½ < \mu < 1$. Finally we well study T , which is the original tent function T and which has some very interesting features .

<u>Case 1</u>

$0 < \mu < ½$

The graph in figure (1.a) shows that 0 is the only fixed point of T_μ since $0 < \mu < ½$. If follows from the definition of T_μ that if $0 < \chi \leq ½$ then $0 \leq T_\mu(\chi)=2\mu\chi < \chi$ and if $½ < \chi < 1$ then $0 \leq T_\mu(\chi)=2\mu(1-\chi) < 1-\chi < ½ < \chi$.

Consequently for any χ in [0,1] the sequence $\{T_\mu^{(n)}(\chi)\}_{n=0}^{\infty}$ is bounded and decreasing. Then the sequence converges to the fixed point o. " if $[f^{(n)}(\chi)]_{n=0}^{\infty}$ is bounded, monotone sequence then there is a fixed point P such that $f^n(\chi) \longrightarrow P$ as n increases with out bounded"

Therefore 0 is an attracting fixed point whose basin of attraction is [0,1].

Case 2

$\mu = ½$

First we notice that if $0 \le \chi \le ½$ then $T_{½}(\chi) = 2(½)\chi = \chi$ so that χ is a fixed point of $T_{1/2}$ "figure 1.b". Next we calculate that If $½ < \chi \le 1$, then $0 \le T_{1/2}(\chi) = 2(½)(1-\chi) = 1-\chi \le ½$ so that $T_{1/2}(\chi)$ is a fixed point of T .

Consequently every point in [0,1] either is a fixed point of $T_{½}$ or has a fixed point for it's first iterate .

Case 3

$½ < \mu < 1$

In addition to the fixed point 0 there is a second fixed point P that lies in [½ ,1] as you can see in figure (1.c) to evaluate P .We solve the equation $P = T_\mu(P) = 2\mu(1-p)$

Which yields $p = 2\mu/(1+2\mu)$.

As μ increases from ½ toward 1, P increases from ½ toward 2/3 because $|T'_\mu(\chi)| = 2\mu > 1$ on [0,1] except at ½ both 0 and P are repelling fixed points .

Case 4

$\mu = 1$

If $\mu = 1$ then $T_\mu = T$ the Tent function give by

$$T_{(\chi)} = \begin{cases} 2\chi & for \quad 0 \le \chi \le \frac{1}{2} \\ 2(1-\chi) & for \quad \frac{1}{2} < \chi \le 1 \end{cases} \quad . \quad [10]$$

1.5 Eventually periodic and periodic points of T

This is more technical then what has preceded it, and the results are independent from what follows .

Theorem (1.3)

Let χ be in the interval (0,1) then χ is eventually periodic for T if and only if χ is rational .

Proof : see [10]

Lemma (1)

Suppose that P is odd and let χ = K/P be in (0,1) . Then χ is periodic for T if and only if K is event .

Proof : See [10]

Lemma (2)

Suppose that P is even and let χ = K/P be in (0,1) . Then χ is not periodic for T .

Proof :- Since χ is assumed to be in reduced form , with P even it follows that K must be odd , so that χ = (odd integer)/P .

But the T(χ) = 2k/p or T(χ) = 2(p- k) /p , so that in any case , T(χ) = (integer) /(p/2) . Thus the reduced form for T(χ) , like

$T^{(n)}(\chi)$ for any $n > 1$, can not be (odd integer)/p . Thus χ is not periodic □ .

Theorem (1.4)

The rational number χ in (0.1) is periodic for T if and only if χ has the form (even integer)/(odd integer) .

Proof : Because we assume that χ is in reduced form, a moment's reflection tells us that theorem 1.3 , Lemma 1 and Lemma 2 together imply the result.

C H A P T E R T W O

CHAOS

2.1 ONE-DIMENSION , CHAOS

Sensitive dependence on Initial conditions.

Before giving the definition of sensitive dependence on initial conditions, we will write a notation which is writing in the form $f: A \to B$ to indicate that the domain of the function f is A and the range of f is contained in B. Thus $f: J \to J$ signifies that the domain of f is J and the range is contained in J. [10]

Definition (2.1)

Suppose that $f: J \to J$,J be an interval then f has sensitive dependence at x if there is an $\in > 0$ such that for each $\delta > 0$ there is a point(y) in J and a positive integer n such that $|x - y| < \delta$ and $|f^{(n)}(x) - f^{(n)}(y)| > \in$. The above definition says that for any point $x \in J$ there is ((at least)) one point arbitrarily close to x that diverges from x .[5]

2.2 CHAOS

The word “chaos” is familiar in everyday speech. It normally means a lack of order or predictability . Thus ,one says that the weather is chaotic ,or that rising particles of smoke are chaotic . It is the lack of predictability that lies behind the mathematical notion of chaos. Sensitive dependence on initial condition qualify as measures of unpredictability. [10]

Then before giving the definition of chaos, we must know that is meant by "Lyapunov exponent".

Definition (2-2)

Let $f : J \to J$ be continuously differentiable on the bounded interval J . Then $\partial(x) = \lim_{n\to\infty} \frac{1}{n} \ln \left| \left(f^{(n)}\right)'(x) \right|$ is called Lyapunove exponent of f at $x \in J$.[2]

Definition (2-3)

A function f is chaotic if it satisfies at least one of following conditione :

1- f has a positive Lyapuunove exponent at each point in it's domain that is not eventually periodic .

2- f has sensitive dependence on initial condition on it's domain.

2.3 TRANSITIVITY

Definition (2-4)

Suppose that J is an interval and $f : J \to J$ .Then f is transitive if for any pair of non-empty open intervals U and V that lie in side J ,there is a positive integer n such that $f^{(n)}(U)$ and V have a common element.

A subset A of the interval J is <u>dense</u> in J if A intersects every nonempty open subinterval of J. Because every non-empty open interval contains rational numbers, it follows that the collection of rationales in the interval [0,1] is dense in [0,1] . By contrast ,the interval [0,1/2] is not dense in the interval [0,1] because [0,1/2] does not intersect open intervals like (1/2,1).

<u>Example (1)</u>: The set p of periodic points of the tent function T is dense in [0,1] because if we suppose U be in open subinterval in [0,1] with U=(a,b) ,let d=b-a and let n be an odd integer such that n>2/d .

since $\frac{k}{n}-\frac{k-1}{n}=\frac{1}{n}<\frac{d}{2}$

and since U has length d , it follows that two successive numbers in the group 1/n,2/n,…,(n-2)/n lie in U of the two successive numbers , one of them must have .

the form “even integer” /”odd integer” by theorem (1-4) such a number is periodic for T. So is in p, therefore p is dense in [0,1].

<u>Theorem (2-1)</u>

Suppose that J is a closed interval and $f:J\to J$ then f is transitive if and only if there is an x in J whose orbit is dense in J .

<u>Proof</u> : see [10].

Theorem (2-2) **"Hine-Borel theorem"**

Suppose that $B_0, B_1, \{B_i\}_{i=0}^{\infty}$ is a sequence of closed bounded intervals of reals such that $B_{n+1} \subseteq B_n$ for all n. Then there is a point common to all the B_n's.

Proof : See [10].

2.4 Strong chaos

Definition (2-5)

A function f on an interval J is strongly chaotic if:

a. f is chaotic

b. f has a dense set of periodic points.

c. f is transitive .

Theorem (2-3)

a. The tent function is chaotic .

b. The tent function has a dense set of periodic point .

c. The tent function is transitive.

2.5 Conjugate

Definition (2-6)

Let J and K be intervals , the function $f:J\to K$ is a homeomorphism of J on to K provided that h is one-to-one and on to , and provided that both f and f^{-1} are continuous.

Definition (2-7)

Let J and K be intervals and suppose that $f:J\to J$ and $f:K\to K$. Then f and g are conjugate "to one another" if there is a homeomorphisim $h:J\to K$ such that $h\circ f = g\circ h$ in this case we write $f \underset{h}{\approx} g$.

Theorem (2-3)

Suppose that $f \underset{h}{\approx} g$ then :

a. $h\circ f^{(n)} = g^{(n)}\circ h$ for n=1,2,3,…

b. if x^* is period-n point of f, then $h(x^*)$ is a period-n point of g.

c. if f has a dense set of periodic points ,then so does g.

Proof : see [10].

Example(2):

Let $g(x) = x^2-3/4$ for $-3/2\le x\le 3/2$ and $Q(x)=3x(1-x)$ for $0\le x\le 1$ show that $g \underset{h}{\approx} Q$ where $h(x)=-(1/3)x+1/2$

Solution :

We will verify that $h \circ g = Q \circ h$. A routine check verifies that h is a homeomorphism from [-3/2 ,3/2] on to [0,1], and that:

$$(h \circ g)_{(x)} = h(g(x)) = h(x^2 - 3/4) = -1/3(x^2 - 3/4) + 1/2 = -(1/3)\,x^2 + 3/4$$

and

$$(Q \circ h)_{(x)} = Q(h(x)) = Q(-(1/3)\,x + 1/2) = 3(-(1/3)x + 1/2)(1 - (-(1/3)\,x + 1/2))$$
$$= (1/3)\,x^2 + 3/4$$

therefor $h \circ g = Q \circ h$.

Teorem (2-4):

Assume that f $\underset{h}{\approx}$ g and that $f : J \to K$ is transitive , then g is also transitive.

Proof : see [10].

Theorem (2-5)

Let f(x) $=ax^2+bx+c$ and g(x) = $rx^2 +sx +t$ where $a \neq 0$ and $r \neq 0$ and where $c=(b^2-s^2+2s-2b+4rt)/4a$ ………………(1)

Then f and g are linearly conjugate to one another with a ssociated homeomorphism given by h(x) =(a/r)x +(b-s)/2r …………….(2)

Proof: see [10].

Example (3):

Let $g_c(x)=x^2+c$ and $Q_m(x)=m\,x\,(1-x)$ with $0<m<4$ to find c and h so that $g_c \underset{h}{\approx} Q_m$ and h is linear.

We use theorem (2-5) with the following substitutions : $a=1$, $b=0$, $c=c$, $r=-m$, $s=m$, $t=0$ using (1) we find that $c=(b^2-s^{٢}+2s-2b+4rt)/4a \;=-m^2/4 + m/2$ then $g_c \underset{h}{\approx} Q_m$,where by (2) h satisfies :

$$h(x)=(a/r)x+(b-s)/2r=(-1/m)x+1/2$$

This completes the solution.

2.6 TWO – DIMENSIONAL CHAOS

Brief Review of 2×2 matrices

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and let r br a real number, then

$A + B = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$, $rA = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$ and

$AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$.

The matrices on the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is called diagonal matrices

and the matrices on form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the identity matrix

and it is denoted by I.

Now suppose that A is an arbitrary 2x2 matrix . If there is a 2x2 matrix B such that AB=I and BA=I then B is the inverse of A. [11]

If such a matrix B exists for a given matrix A ,then B is unique and is written A^{-1} . Not every matrix has an inverse .

We will show below that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inversely when ad-bc $\neq 0$.

The expression ad-bc is called the **determinant of A** and it is denoted by **det A** or $|A|$ **.**

Thus $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$. [10]

/

Theorem (2-6) :

A_{2x2} matrix A has an inverse if and only if det(A)=0

Proof: see [10].

Proof :-

Let A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. First suppose that det A=ad-bc =0 , then you can check diretly that : $A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Conversely , suppose that det A=0 then

det (AB)=(det A) (det B) =0 (bet B) =0 for all B

If there were a B such that AB=1 then 1=det I =det(AB) =0.

This contradiction show that no such B exists.

Therefor A has no inverse and the proof is complete .

Definition (2-8):

Let V , $w \in R^2$ such that $V = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \; W = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$, then:

$$d(V, W) = \|V - W\| = \sqrt[n]{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$$

If w=0 then $|V| = \sqrt[n]{x_1^n + x_2^n + \ldots + x_n^n}$

Definition (2-9):

Let $\{V_n\}_{n=0}^{\infty}$ be a sequence and W , $V_n \in R^n$. Then $\{V_n\}_{n=0}^{\infty}$ converge to W if $\|V_n - W\| \to 0$ as $n \to \infty$

Definition (2-10)

We say that $L : R^n \to R^n$ is a continuous function at zero if $(\forall \in > 0), (\exists \delta > 0)$ such that If $\|V\| < \delta$ then $\|L(V)\| < \in$. [1]

2.7 Linear dynamical system

Definition (2-11):

The function $L : R^2 \to R^2$ is linear if :

L(bv+cw)=bL(v)+cL(w) , $\forall\ v, w \in R^2$ and $\forall\ b, c \in R$.

A linear function is also called linear map .[10]

Definition (2-12) :

Let $V = \begin{bmatrix} x \\ y \end{bmatrix} \in R^2$ and $L : R^2 \to R^2$ such that :

$L(V) = \begin{bmatrix} ax & by \\ cx & dy \end{bmatrix}$, then the matrix $A_L = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called the "associated matrix" . [10]

Definition (2-13):

Suppose that A is a_{2x2} matrix >Therefor number λ is an eigen value of a provided that there is a non zero V "vector" in R٢ such that $AV = \lambda V$

In this case V is an eigen vector of A "relative to λ"

Example (4):

The vector V=$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is eigen vector of a matrix A=$\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ such that it's eigen value $\lambda = 4$

because AV==$\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 4V$

to find the eigen value for matrix A_{2x2} we write $AV = \lambda V$ which is equivalence $(\lambda I - A)V = 0 \quad \text{where } V \neq 0$,then $|\lambda I - A| = 0$.

This equation is called "characteristic equation" for A . [12]

Example (5):

The eigen value $\lambda = 4$ for $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ and corresponding eigen vector $\lambda = 4$.

because

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} \lambda - 10 & 9 \\ -4 & \lambda + 2 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 10 & 9 \\ -4 & \lambda + 2 \end{vmatrix} = (\lambda - 10)(\lambda + 2) + 36 = 0$$

$\lambda^2 - 8\lambda + 16 = 0$ then $\lambda = 4$ is the eigen value.

To find an eigen vector V for $\lambda = 4$,we will solve $AV = \lambda V$ for V

If $V = \begin{pmatrix} x \\ y \end{pmatrix}$ this means we must find x and y such that :

$$\begin{bmatrix} 10x & -9y \\ 4x & -2y \end{bmatrix} = \begin{bmatrix} 4x \\ 4y \end{bmatrix}$$

Thus 10x-9y=4x , so that 3y=2x ⟶ y=(2/3)x consequently any vector of the form $\begin{pmatrix} x \\ \frac{2}{3}x \end{pmatrix}$ such as $\begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$ is an eigen vector for $\lambda = 4$.

NOT: The general formula for the eigen values of any 2x2 matrix are $\lambda = \frac{1}{2}(a+b) \pm \frac{1}{2}\sqrt{(a-b)^2 + 4bc} \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Theorem (2-7): **"For discrete dynamical system"**

Let p be a fixed point of f . Assume Df(x) exists and continuous in N(p) with λ_i , $i = 1,2,3,4,...,n$ are the eigen values of Df(p) then:

a. If $|\lambda_i| < 1$ for $i = 1,2,3,...,n$ then p is an attracting point.

b. If $|\lambda_i| > 1$ for $i = 1,2,3,...,n$ then p is an repelling point

c. If there exists i such that

$|\lambda_i| < 1$ and $|\lambda_j| > 1$ for each j such that $i \neq j$ and

i= 1,2,3,…,n then p is a saddle point.

<u>Proof:</u> **see.[2]**

<u>Example (6):</u>

Let $f(x) = \begin{pmatrix} y \\ 3\sin x - y \end{pmatrix}$ and o is fixed point of f .

$$Df(0) = \begin{pmatrix} 0 & 1 \\ 3\cos 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & -1 \end{pmatrix}$$

$$\det(Df(0) - \lambda I) = \lambda^2 + \lambda - 3 \Rightarrow \lambda^2 + \lambda - 3 = 0 \Rightarrow \lambda_{1,2} = \frac{-1 \mp \sqrt{13}}{2}$$

i.e. $|\lambda_{1,2}| > 1$

so 0 is repelling fixed point.

<u>Definition (2-14)</u> :

Let $L : R^2 \to R^2$ and $M : R^2 \to R^2$ be two linear functions , then L and M are "**linearly conjugate "** If $\exists p$ such that $p : R^2 \to R^2$ is an inevitable linear function $\ni p \circ L = M \circ p$ or $M = p \circ L \circ p^{-1}$. [10]

Theorem (2-8):

Let $L: R^2 \to R^2$ have the following property :

a. Let A_L have the distinct real eigen values λ

b. M with $|\lambda| < 1$, $|M| < 1$ Then $\forall\, V \in R^2, L^n(V) \to 0$ therefor 0 is “an attracting fixed point” of L and R^2 is the basin of attracting of 0 [10]

Proof:

Let $r \in R^2$ since $L^{(n)}(V) = (A_L)^n_V$. "*If* $L: R^2 \to R^2$ is linear function and A_L is the associated matrix of L ,then $L(V)=A_L(V)$, $\forall\, V \in R^2$"

Now we should show that $(A_L)^n (V) \to 0.$

Let $B = \begin{bmatrix} \lambda & 0 \\ 0 & M \end{bmatrix}$ then $A_L \approx B$ (since A_L has two distinct eigen values)

Then $\exists$ E "invertible matrix"; $A_L \underset{E}{\approx} B$

i.e. $E.A_L=B.E$ then $(A_L)^{(n)}=E^{-1}.B^{(n)}$.(E(V)), to proof will be complete if we show that $(E^{-1}.B^{(n)}$. .(E(V)) . $\longrightarrow$ 0

we Know that $B^n = \begin{bmatrix} \lambda^n & 0 \\ 0 & M^n \end{bmatrix}$ next, Let E V $= \begin{pmatrix} x \\ y \end{pmatrix}$ then:

$$B^n (E(V)) = \begin{bmatrix} \lambda^n & 0 \\ 0 & M^n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda^n x \\ M^n y \end{bmatrix}$$

since $|\lambda| < 1$ and $|M| < 1$ by hypothesis , we find that :

$$\|B^n(E(V))\| = \left\| \begin{pmatrix} \lambda^n x \\ M^n y \end{pmatrix} \right\| = \sqrt{\lambda^{2n} x^2 + M^{2n} y^2} \to 0 \text{ as } n \to \infty \text{ then } A_L^{(n)} \to 0 \text{ as } n \to \infty$$

then 0 is attracting fixed point and the basin of attraction R^2

Example(7):

$L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x/2 \\ y/4 \end{bmatrix}$, show that 0 is an attracting fixed point whose basin in R^2

Solution:

$$L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{. i.e.} \quad A_L = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}$$

then it's eigen values $\lambda = 1/2$ and $M = 1/4$ so $|\lambda| < 1$ and $|M| < 1$, then by theorem (2-8) we have that 0 is attractive fixed point .

2.8 NON-LINER MAPS

Definition (2-15):

Let $V \subseteq R^2$ and let $F : V \to R^2$, the map F can be always be represented in the form $F(V) = \begin{bmatrix} F(v) \\ F(v) \end{bmatrix}$ for all v in V and f ,g are real-valued Coordinate functions such that one of them or both of them is are non –linear . Then F is called a non-linear map.

Definition (2-16):

Let V be a subset of R^2 and consider $F : V \to R^2$ and also assume that the first partials of the coordinates function f and g of F exist at v_0 .Then the differential of F at v_0 is the linear function $DF(v_0)$ which is defined on R^2 .

By: $$[DF(v_0)](v) = \begin{bmatrix} \frac{\partial f}{\partial x}(v_0) & \frac{\partial f}{\partial y}(v_0) \\ \frac{\partial g}{\partial x}(v_0) & \frac{\partial g}{\partial y}(v_0) \end{bmatrix}(v) \quad \text{for all } v \in R^2$$

Since the matrix DF(v_0) is called Jacobian matrix . Therefor we treat the jacobian matrix like an associated matrix of the linear system in order to determine the corresponding dynamical system.

<u>Example (8)</u>:

Let $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ a\sin x + by \end{pmatrix}$, find $DF\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ we have $f\begin{pmatrix} x \\ y \end{pmatrix} = y$ and

$g\begin{pmatrix} x \\ y \end{pmatrix} = a\sin x + by$ so that :

$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 1 , \frac{\partial g}{\partial x} = a\cos x , \frac{\partial g}{\partial y} = b$ therefor $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ the partial are

$\frac{\partial f}{\partial x} = 0 , \frac{\partial f}{\partial y} = 1 , \frac{\partial g}{\partial x} = a\cos x_0$ and $\frac{\partial g}{\partial y} = b$

consequently $DF\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} V = \begin{pmatrix} 0 & 1 \\ a\cos x_0 & b \end{pmatrix} v$ for all v in R^2 or in jacobian matrix from $DF\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a\cos x_0 & b \end{pmatrix} V$

<u>Definition (2-17)</u>:

Let p be a fixed point of F .The p is an attracting fixed point if and only if $\exists$ adisk $o(p)$ which is centered at p such that $F^{(n)}(V) \to p$ for every $v \in o(p)$ then by contrasting ,p is arepeeling if and only if :

$\exists\, o(p)$ such that : $\|F(v) - F(p)\| > \|v - p\| \quad \forall\, v \in o(p)$ and $v \neq p$. [10]

Theorem (2-9):

Let p be a fixed point of f .Assume that DF(p) has eigen values λ and μ such that $|\lambda|<1$ and $|\mu|<1$. Then p is an attracting fixed point.

Proof:

Let $\in>0$ and from the chain rule we have :

$$\left(f^{(n)}\right)'(x)=\left[f'\left(f^{(n-1)}(x)\right)\right]\left[f'\left(f^{(n-2)}(x)\right)\right]\cdots\left[f'\left(f\ \ (x)\right)\right]\left[f'(x)\right]$$

since p is a fixed point so we have $\left(f^{(n)}\right)'(p)=\left(f'(p)\right)^{n}$

If we apply the two dimensional analog to the function F, then we have :

$$\left[DF^{(n)}\right](p)=\left[DF(p)\right]^{(n)}$$

If the partials coordinate of the function f and g are continuous in some neighborhood of v_0 .Then it's possible to have that:-

$$\frac{F(v)-F(v_0)-\left[DF(v_0)\right](v-v_0)}{\|v-v_0\|}\to 0 \quad \text{as} \quad v\to v_0$$

If we replaced F by F^n and v_0 by p , then we find that :

$$\left\|\frac{F^{(n)}(v)-p-\left[DF(p)\right]^{n}(v-p)}{\|v-p\|}\right\|=\left\|\frac{F^{(n)}(v)-F^{(n)}(p)-\left[(DF)^{(n)}(p)\right](v-p)}{\|v-p\|}\right\|<\in$$

If v is in a sufficiently small neighborhood v of p and $v\neq p$ .However the eigen values of $\left[(DF)(p)\right]^{(n)}$ are λ^{n} and μ^{n} , since $|\lambda|<1$ $|\mu|<1$ by hypothesis ,therefor :

$$\left[(DF)(p)\right]^{(n)}\frac{v-p}{\|v-p\|}\to 0$$

Also if n large enough and $v\in V$ with $v\neq p$ then:

$\left\| \frac{F^{(n)}(v)-p}{\|v-p\|} \right\| < 2\in$, it follows that $F^{(n)}(v) \to p, \forall v \in V$, and consequently p is attracting . Using the same argument in the above proof we can show that if $|\lambda|>1$,$|\mu|>1$ then p is repelling fixed point . Also if $|\lambda|>1$,$|\mu|>1$, then p is a saddle point. [10]

2.9 IMPORTANT EXAMPLES

"Quadratic Family-Logistic map"

Quadratic family or Logistic map is an important example for the one dimensional chaos which has the general form :

$Q_m(x)=mx(1-x)$ for $0 \le m \le 1$ and the parameter $m \in R$ which it depend on , so $Q_m(x)$ is a parametric function with $0 < m \le 4$.

To analyze Q_m we should simulate it's main properties as follows:

Suppose $0 \le x \le 1$ and $0 < x \le 4$, Q_m has two fixed points 0 and $1-\frac{1}{m}$ for $0 \le x \le 1$ and $0 < x \le 4$. [10]

a. If $0 < m \le 1$ then:

$Q_m(x)$ has a fixed point which is equal to 0 but $Q'_m(0)=m$, then we have $|Q_m(0)|<1$ and then 0 is an attracting fixed point .

b. $1 < m \le 2$ then Q_m has two fixed points which are equal to 0 and $1-\frac{1}{m}$ such that $Q'_m(0)=0$ then $|Q'_m(0)|>1$ then 0 is a repelling fixed point . Also $Q'_m(1-\frac{1}{m})=-(m-2)$

then $\left|Q_m^{/}(1-1/m)\right|<1$, then 1-1/m is an attracting fixed point .

c. If 2<m<3 then Qm has two fixed points which are equal to 0 and $1-\frac{1}{m}$, and similarly we have that 0 is repelling fixed point , $1-\frac{1}{m}$ is a repelling fixed point.

Henon Mapping

In years ago the preach astronomer mathematician michel Henon we searching for a simple two-dimensional function possessing special properties of more complicated systems. The result was a family of functions which are denoted by $H_{a,b}$. and given by $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}$, where a,b are real numbers .

Hanon map has the following properties : [8] and [9]

.i Let a and b be any fixed real numbers , then

$$\det DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = -b \ , \ \forall x, y \in R^2$$

If $a^2x^2 + b \geq 0$, then the eigen values of $DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ are the real number $-ax \pm \sqrt{a^2x^2 + b}$. B is called the constant Jacobian for $H_{a,b}$ which determines whather the area –expanding or area-contracting for the value of $|b|$ that if:

$|b| < 1$ then the area – contracting .

$|b| > 1$ then the area – expanding.

ii. $H_{a,b}$ is one-to-one.

iii. The map $H_{a,b}$ is composed of the three function H_1,H_2 and H_3 where :

$$H_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}x\\1-ax^2+y\end{bmatrix}$$
$$H_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}bx\\y\end{bmatrix}$$
$$H_3\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}y\\x\end{bmatrix}$$

for 0<b<1 , such that $H_{a,b} = H_3 \circ H_2 \circ H_1$

iv. $H_{a,b}$ is invertible and $H_{a,b}^{-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}\frac{1}{b}y\\-1+\frac{a}{b^2}y^2+x\end{bmatrix}$,

Since H_1,H_2 and H_3 are invertible and then that :

$$H_{a,b}^{-1}=\left(H_3\circ H_2\circ H_1\right)^{-1}=H_1^{-1}\circ H_2^{-1}\circ H_3^{-1}$$

v. Let $a\neq 0$. Then $H_{a,b}$ has a fixed point if $a\geq -\frac{1}{4}(1-b)^2$ in the even that $H_{a,b}$ has two a fixed points p and q .They are given by :

$$p=\begin{cases}\frac{1}{2a}\left(b-1+\sqrt{(1-b)^2+4a}\right)\\ \frac{b}{2a}\left(b-1+\sqrt{(1-b)^2+4a}\right)\end{cases} ,$$

$$q=\begin{cases}\frac{1}{2a}\left(b-1-\sqrt{(1-b)^2+4a}\right)\\ \frac{b}{2a}\left(b-1-\sqrt{(1-b)^2+4a}\right)\end{cases} .$$

2.10 Baker's functions:

We define the function $B_0 : R^2 \rightarrow R^2$ by the two formulas

$$B_0\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{4}x \\ 3y \end{pmatrix} \quad \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le \frac{1}{3}$$

and

$$B_0\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}x \\ \frac{3}{2}\left(y - \frac{1}{3}\right) \end{pmatrix} \quad \text{for } 0 \le x \le 1 \text{ and } \frac{1}{3} \le y \le 1$$

Then B_0 is called a Baker's function because it's linear on two portions of the domain the way the one dimensional Baker's function is. The domain of B_0 consists of the unite square S=[0,1] × [0,1] . The effect of B_0 on S can be seen graphically in Figure (5) where the rectangles in the right graph are the images of the rectangles with similar shading in the left graph .

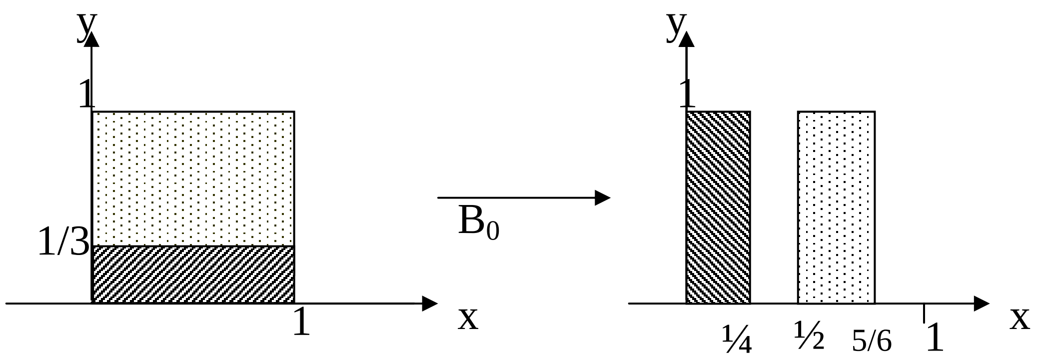


figure (5)

The image $B_0^{(2)}(s)$ in depicted on the right in figure (6)

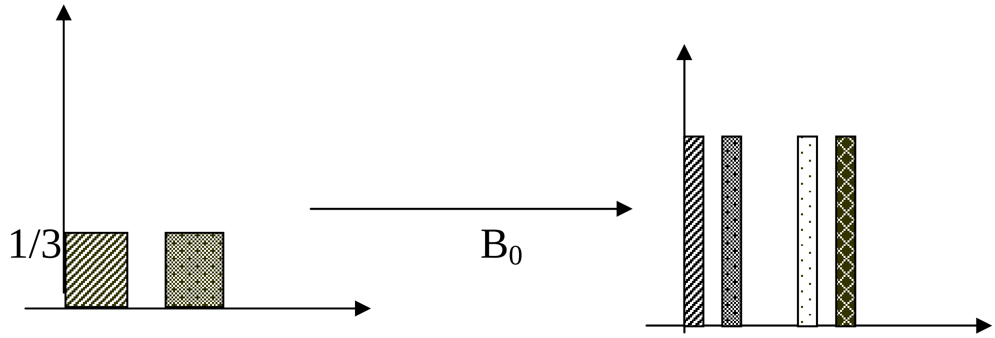


figure (6)

C h a p t e r T h r e e

QUASI HORSESHOE MAP

The horseshoe map is one of the important example on dynamical systems.

The horseshoe map will be denoted by M. It's domain is the set D in R^2 composed of the unit square T=[0,1]x[0,1] and it bounded on the left and right by semicircles A and B where :

$$A = \{(x,y) : x^2 + y^2 \le y \quad \text{and} \quad x \le 0\}$$
$$B = \{(x,y) : (x-1)^2 + y^2 \le y \quad \text{and} \quad x \ge 1\}$$

We assume that D contains A and B. We will partition T as in to C_1 , C_0 and C_2 .

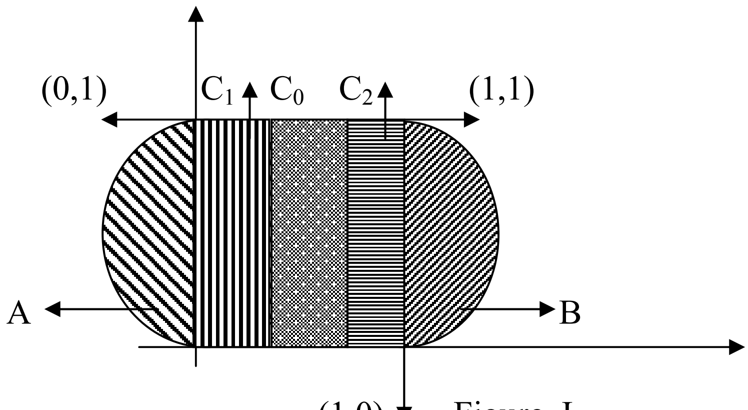


Figure I

The function M Shrinks T vertically by a factor of a<1/3 and expands T horizontally by a factor of b=3 . So that the semicircles are A and B.

So that the function M will shrinks semicircles A and B into the points (0,0) and (3,0) respectly by factor of a<1/3 , i.e. "homothefic" see (Figure II)

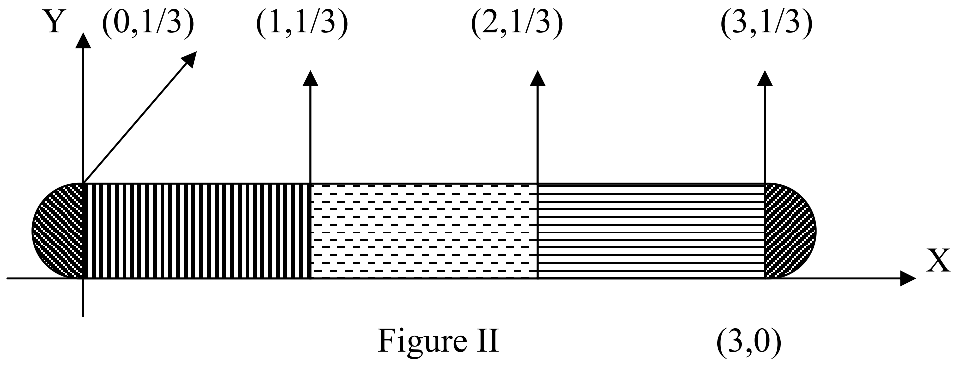


Figure II

The resulting figure is folded by M so that it fits a gain in side D with only the semicircles protruding to the left of T .

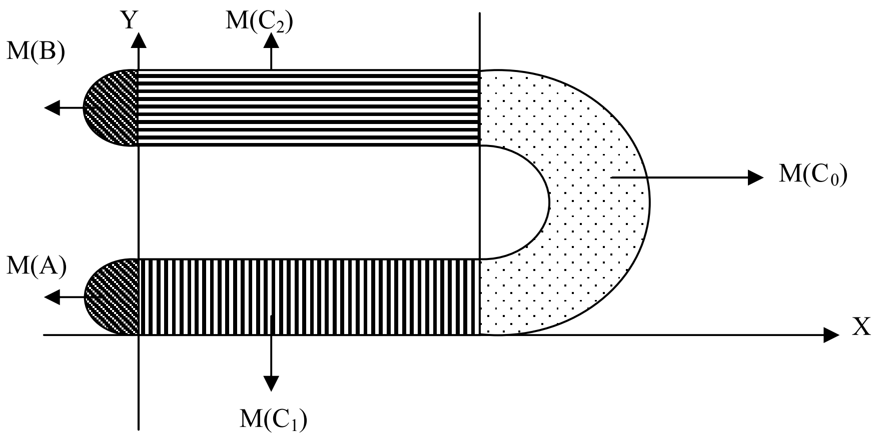


Figure III

and we define M by M:D ⟶ M(D) such that :

$$M(x,y)=\begin{cases} \left(3x,\frac{1}{3}y\right) & 0\le x\le \frac{1}{3} \\ \frac{\cos\frac{\pi}{2}(6x-3)}{4y+2}+1;\frac{\sin\frac{\pi}{2}(6x-3)}{4y+2}+\frac{1}{2} & \frac{1}{3}\le x\le \frac{2}{3} \\ \left(3-3x,1-\frac{1}{3}y\right) \quad , & \frac{2}{3}\le x\le 1 \end{cases}$$

$\forall\, 0\le y\le 1$

Theorem (3-1):

M: D $\longrightarrow$ M(D) is a homeomorphism .

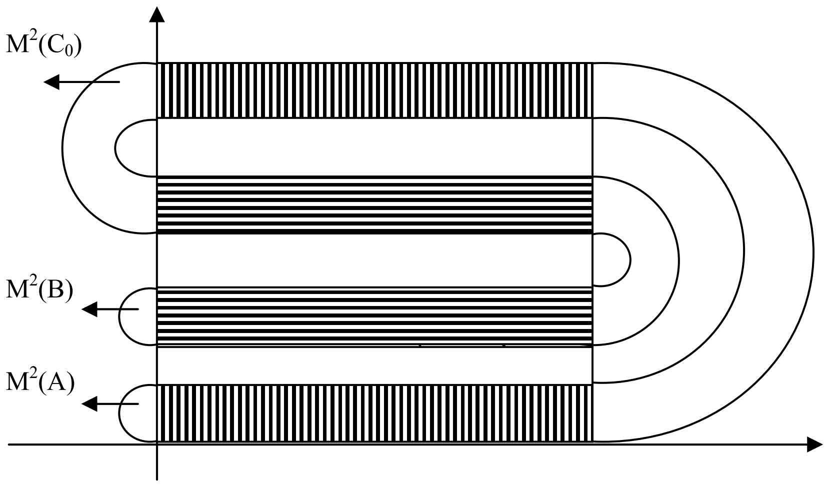
Proof: By definition M is one -to- one and onto. To prove that M is continuous, we compre subrigious in Figure (1) with their corresponding image in Figure III.

The map M expands distances in C_1 and C_2 by factor of 3 and shrinks distances in A and B.

The length of the top boundary of D is $\frac{1}{3}$ and the length of exterior boundary is less than π/2 , it follows that M expands distances in D by no more than a factor of 6. Consequently if V and W one is T and $||V-W||<\in$ then $||M(V)-M(W)||<6\in$ therefore M is continuous the inverse map M is continuous by the same kind of argument therefore M is a homeomorphism.

We notice from the result ,which have shown above that the some point ,which was attracted into the place of T, will explode and convey to the place of B and then exit to the place of A with out return .

When this process is repeated another once shrinks vertically by a factor of a<1/3 ,also examples horizontally by a factor of b=3 ,then we do with fits a gain inside it in the same first side ,we will get on the following shape for the function.



Theorem (3-2):

M has a unique fixed point p in A

Proof: Since M shrinks the domaine vertically by a factor of $a<\frac{1}{3}$, this means that for each n, $M^{(n)}(A)$ is closed and semicircular with diameter a^n.

In addition , $A\supseteq M(A) \supseteq M^2(A) \supseteq \ldots \supseteq M^{(n)}(A)$

By (Heine-Borel) implies that anested sequence of closed bounded set in R^2 has a comment point.

Thuse the iner section B_∞ of the set $M^2(B)$ is a^2 and $\lim_{m\to\infty} a^n \to o$

thus B_∞ contains exactly one element which we denote by P. consequently P=M(P), which means that P is fixed point.

Although p attracts all points in B and E, p is not an attracting fixed point because the iterates of points in the interior of C_1 are drawn away from p . Then we say that p is a saddle point.

(X,d) is a complex metric space ,H(X) is the set of all non empty compact subset of X. First we define the distance d(t,A) between any point t and a given closed bounded sub set A of R^2 d(t,A)=the minimum value of $\|t-a\|$ for a in A .

Next we are preferred to define a distance d from an arbitrary ,closed and bounded subset A in R^2 to an other such set B.

d(A,B)=the maximum value of d(a,B) for a in A.

<u>i.e.</u>

$$d(A,B)=\sup_{x\in A}\ \inf_{y\in B}\ d(x,y)$$

.

$$D(A,B)=\max\left\{\sup\ \inf\ d(x,y),\sup_{y\in A}\ \inf_{x\in B}\ d(x,y)\right\}$$

(H(X),D) is a compact metric space.

$M : H(T) \longrightarrow H(T)$ contraction

$x\in H(T)$ iff $X\subseteq T$

where X non empty and closed

$$T\supseteq X_1=M(X_0)\supseteq X_2=M(X_1)\supseteq\cdots$$

there exist non empty $K=\bigcap_{i=1}^{\infty}X_i$

i.e. $\lim_{n\to\infty} M^n(T) = K \neq \Phi$ and K is closed and bounded

We notice that :

$$K = \lim_{n\to\infty} X_{n+1} = \lim M^{n+1}(X_0)$$

$$= \lim_{n\to\infty} M(M^n(X_0))$$

$$= M \lim_{n\to\infty} (\quad \text{Because M is continuous}$$

$$= M(K)$$

$$K = M(K) \in H(X)$$

then K is a fixed point of M in H(X) invariant subset of T under M.

let $x = (\delta_1, \delta_2, \cdots)_3 \qquad \delta_i \in \{0,2\} \subset K$

$y = (\sigma_1, \sigma_2, \cdots)_3 \qquad \sigma_i \in \{0,1,2\} \subset K$

i.e. $x = \dfrac{\delta_1}{3} + \dfrac{\delta_2}{9} + \dfrac{\delta_3}{27} + \cdots$

$$0 \le x \le \frac{1}{3} \leftrightarrow \delta_1 = 0$$

$$\frac{2}{3} \le x \le 1 \leftrightarrow \delta_1 = 2$$

$$M(x,y) = \begin{cases} \left(3x, \dfrac{1}{3}y\right) & \delta_1 = 0 \\ \left(3 - 3x, 1 - \dfrac{1}{3}y\right) & \delta_1 = 2 \end{cases}.$$

where $\delta_1 = 0 \qquad 0 \le x \le \dfrac{1}{3}$

$$M(x,y)=\left(3x,\frac{1}{3}y\right) \quad , \quad 3\mathrm{x}=0.\delta_2\delta_3\delta_4\cdots$$

$$\frac{1}{3}y=0.\sigma_1\sigma_2\sigma_3\cdots$$

$$y=(\sigma_1,\sigma_2,\sigma_3,\cdots)_3=\frac{\sigma_1}{3}+\frac{\sigma_2}{9}+\frac{\sigma_3}{27}+\cdots$$

$$\frac{1}{3}y=\frac{\sigma_1}{9}+\frac{\sigma_2}{27}+\frac{\sigma_3}{81}+\cdots=.0\sigma_1\sigma_2\sigma_3\cdots$$

i.e. $\mathrm{X}_0=(x_0,y_0)=(.0\delta_2\delta_3\cdots,.0\sigma_1\sigma_2\sigma_3\cdots)$

$$M(X_0)=X_1=(\mathrm{x}_1,y_1)=(.\delta_2\delta_3\cdots,.0\sigma_1\sigma_2\sigma_3\cdots)$$

where $\delta_1=2 \qquad \frac{2}{3}\le x\le 1$

$$M(x,y)=\left(3-3x,1-\frac{1}{3}y\right)$$

$$x=.2\delta_2\delta_3\delta_4\cdots\Rightarrow 3x=2.\delta_2\delta_3\delta_4\cdots$$

$$3-3x=3-2.\delta_2\delta_3\delta_4\cdots=1-.\delta_2\delta_3\delta_4\cdots=0.\overline{\delta_2}\,\overline{\delta_3}\,\overline{\delta_4}\cdots$$

$$1-\frac{1}{3}y=1-0.0\sigma_1\sigma_2\sigma_3\cdots \qquad \sigma_i\in\{0,1,2\}$$

$$M(x,y)=(0.\overline{\delta_2}\,\overline{\delta_3}\,\overline{\delta_4}\cdots,.2\sigma_1\sigma_2\sigma_3\cdots)$$

$$=\cdots\sigma_4\sigma_3\sigma_2\sigma_1\delta_1.\delta_2\delta_3\delta_4\cdots$$

$$M^2(x,y)=\cdots\sigma_4\sigma_3\sigma_2\sigma_1\delta_1\delta_2.\delta_3\delta_4\cdots \qquad [3]$$

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Contents

I certify that this research entitled ***"Two Dimensional Chaotic Mapping"*** *was prepared under my supervision at the University of Babylon, College of Education in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.*

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In view of the available recommendation. I foreword this research for debate by examining committee.

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