REPUBLIC OF IRAQ
MINISTRY OF HIGHER EDUCATION
AND SCINTIFIC RESEARCH
UNIVERSITY OF BABYLON
COLLEGE OF EDUCATION

δ-*OPEN SET IN BITOPOLOGICAL SPACE*

A RESEARCH

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS OF THE COLLEGE OF EDUCATION OF THE UNIVERSITY OF BABYLON IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

BY

IKTIFA DIAA JALEEL

SUPERVISED By

Dr. LUAY ABD AL-HAIN AL-SWIDI

November 2003

بسم الله الرحمن الرحيم

صنع الله الذي أتقن كل شيء

صدق الله العلي العظيم

سورة النمل/ الآية (٨٨)

CERTIFICATION

We certify that this research was prepared under our supervision at University of Babylon , College of Education in partial fulfillment of the requirements for the degree of Master of Science in Mathematics .

Signature :
Name : **Assist Prof.Dr.Luay Abd Al-Hain Al-Swidi**
Date : **/ / 2003**

Signature :
Name :
Date : **/ / 2003**

In view of the available recommendations .
I forward this research for debate by the Examining Committee .

Signature :
Name : **Assist Prof.Dr.Luay Abd Al-Hain Al-Swidi**
Head of Mathematics Department , College of Education , Babylon University
Date : **/ / 2003**

CERTIFICATION

We certify that we have read this research *entitled "* δ *- Open Set In Bitopological Space " and , as an examining committee examined the student in its content and in what is connected with , and that in our opinion it meets the standards of a research for the degree of Master of Science in Mathematics .*

Signature :
Name : Assist Prof.Dr.Luay Abd Al-Hain Al-Swidi
Date : / / 2003

Supervisor

Signature :
Name :
Date : / / 2003

Supervisor

Signature :
Name :
Date : / / 2003

Member

Signature :
Name :
Date : / / 2003

Member

Signature :
Name :
Date : / / 2003

Chairman

Approved by the Deanery of the College of Education .

Signature :
Name :
Dean , College of Education
University of Babylon
Date : / / 2003

Contents

DEDICATION

To Father , Mother , and All

My Family ,

With Love and Gratitude

ACKNOWLEDGMENTS

The researcher would like to acknowledge her indebtedness and express her thanks to Assist . Prof. Dr. Luay Abd Al-Hain Al-Swidi , Head of the Department of Mathematics , for his advice and remarkable supervision of this research . Many thanks are due to him for his continuous support and generosity in supplying me with the most important references .

The researcher expresses her gratitude to all the professors and lecturers at the Department of mathe-matics , college of education , university of Babylon .

Words of thanks are also extended to my uncle Dr. Hussain Alwan Hussain for proofreading the lingu-istic aspects of this work .

Many thanks are due to father , mother , brother and sisters for their moral support and encouragement .

ABSTRACT

This research establishes a relation between bitopolgical spaces on the one hand and topological space of the type α - open set , on the other . A new definition for δ - open set in bitopology is formulated based up on δ - open set in bitopology . A new theorem for the relation between topological space and δ - open set in bitopology is also offered . The research gives the basic specifications for the new definition of δ - open set in bitopology .

List of Symbols

Symbol	*Definition*
A^{o}	**Interior of A**
A	**Closure of A**
δ -o(X)	**The set of all open sets in (X,T,Ω)**
δ - c(X)	**The set of all closed sets in (X,T,Ω)**
T-int (A)	**The set of all interior point of A in (X,T)**
Ω- cl (A)	**The set of all closure subsets A of (X,Ω)**
δ -int (A)	**The set of all interior point of A in (X,T,Ω)**
δ - cl (A)	**The closure set of X in (X,T,Ω)**
δ- lm (A)	**The set of all limit points of A in (X,T,Ω)**
δ- nbd (A)	**The set of all neighborhoods of X in (X,T,Ω)**
f : X→y	**Single – valued function**

INTRODUCTION

This research establishes a relation between bitopological spaces , initiated by Kelly (1963) [1] , defined as :

A set equipped with two topologies is called a bitopological space , denoted by (X,T,Ω) where (X,T) , (X, Ω) are two topological spaces defined on X ,

and α - open set , defined by Noiril (1974) [2] :

Let (X,T) be a topological space , and A$\subset$ X A is said to be α - open set iff A$\subset$ A^{0-0} .

A new definition for δ - open set in bitopological space is introduced on the basis of α - open set in topological space :

Let (X,T,Ω) be a bitopological space , and A be a subset of X . A is said to be δ - open set iff A$\subset$ T-int (Ω -cl (T-int (A))) .

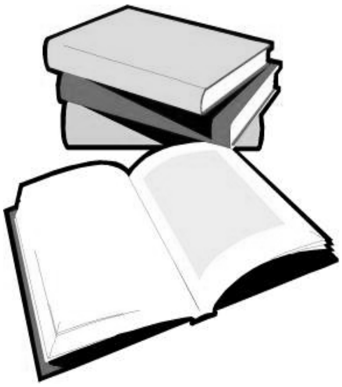
From the relation above , the following generalization is formulated between topological spaces and δ - open set in bitopology :

Let (X,T,Ω) be a bitopological space . Then every T- open set in topological space is δ - open set in bitopological space , but the converse is not true .

The research consists of three chapters . Chapter One offers the basic definitions for neighborhood , interior points , limit points , open set , closed set and closure set in bitopological space .

Chapter Two gives the definitions and theorems of the Separation Axiom in bitopological space , together with explanatory examples .

Chapter Three investigates δ - continuous in bitopological space , formulates the new definitions and theorems mentioned , together with their proofs .



CHPTER ONE

Basic Definitions

This chapter presents some important definitions , theorems , remarks and examples of δ-open set in bitopological space .

1.1 Basic Definition :

1.1.1 Definition :

Let (X,T,Ω) be a bitopological space , and A be a subset of X. A is said to be δ-open (δ-open set in bitopological space) iff A $\subset$ T-int (Ω- cl (T– int (A))) .

1.1.2 Example :

Let x = { a,b,c } , T = { X , Ø , {a} , {a,b} } ,
Ω = { X , Ø }

(X,T) , (X, Ω) are two topologies on X , then (X,T,Ω) is a bitopotogical space . The family of all δ-open set of X is :

δ.o(X) = { X , Ø , {a} , {a,b} , {a,c} } .

If we take Y = {a} , then T-int ({a}) = {a}

Ω-cl (T-int {a}) = X

and T-int (Ω-cl (T-int {a})) =T-int(X) = X ; therefore {a}$\subset$ X .

Hence {a} is δ-open set in (X,T,Ω) , and in general in any bitpological space X , Ø are clearly δ-open sets , so are the other cases {b} , {c} , {a,c} , {a,b} , {b,c} .

1.1.3 Definition :

Let (X,T,Ω) be a bitopological space . A subset Y of X is called δ-closed set of X if the complement of Y is δ-open set of X .

1.1.4 Example :

Let X = {a,b,c} , T = { X , Ø , {b} , {c} , {b,c} } ,

Ω = { X , Ø , {a} , {b} , {a,b} } .

(X,T) , (X, Ω) are two topologies on X . Then (X,T,Ω) is a bitopological space . The set of all δ-open sets on X is :

δ.o(X) = { X , Ø , {b} , {c} {b,c} } .

Y = {a,b}⊂ X is δ-closed since

X-Y = {c}∈ δ.o(X) is δ-open set of X .

Y = {a,c}⊂ X is δ-closed set of X .

Since X-Y = {b}∈δ.o(X) , which is δ-open set of X .

However , Y = {b,c}⊂ X is not δ-closed set of X .

Since X-Y = {a}∉δ.o(X) which is not δ- open of X .

Hence the set of all δ-closed set of X is

δ- c(X) = { X, Ø, { a,c } , { a,b } , { a } }.

1.1.5 Definition :

Let (X,T,Ω) be a bitopological space , and A ⊂ X .
A point x∈X is said to be δ-interior point of A iff there exist an δ-open set G such that x∈G⊂ A , we denote to the set of all interior points of A by δ- int (A) .

1.1.6 Definition :

Let (X,T,Ω) be a bitopological space , and A ⊂ X . The intersection of all δ-closed subsets of A is called δ-closure of A , and denoted by δ-cl (A) ;

i.e δ-cl (A) = ∩{ F : F is δ-closed , A ⊂ F } .

1.1.7 Example :

Let X = {a,b,c} , T = { X , Ø , {a} , {b} , {a,b} } ,

Ω = { X , Ø , {c} , {b} , {b,c} } .

(X,T) , (X, Ω) are two topologies on X, then (X,T,Ω) is a bitopological space such that :

δ- o(X) = { X, Ø, {a} , {b} , {a,b} }.

If we take A = {b,c} , then there exists δ-open set {b} such that b∈{b} ⊂ A , hence b∈δ-int (A) .

Also , there exists only δ-open set X containing c , such that c∈X ⊄ A , hence c ∉ δ-int (A) , hence δ-int (A) = {b} .

And there exists δ-closed sets {b,c} , such that {b,c}⊂ X , and {b,c}⊂ {b,c} .

Hence δ- cl (A) = X ∩ {b,c} = {b,c} .

1.1.8 Definition :

Let (X,T,Ω) be a bitopological space , a point x is δ-limit point of a subset A of X iff each δ-open set G containing a point of A is distinct from x ; that is (G / {x})∩ A ≠ Ø , we denote to the set of δ-limit point of subset A of X by δ-lm(A) .

1.1.9 Example :

Let X = {a,b,c} , T = { x , Ø , {a} , {b} , {a,b} } ,

Ω = { x , Ø , {c} } .

(X,T) , (X, Ω) are two topologies on X , then (X,T,Ω) is a bitopological space .

δ- o(X) = { X, Ø, {a} , {b} , {a,b} }.

If we take A = {a,b} , then c is the only δ-limit point of A .

1.1.10 Definition :

Let (X,T,Ω) be a bitopological space , and let x∈X . A subset A of X is said to be δ-nbd of a point x iff there exists δ-open set U , such that x∈U ⊂ A .

1.1.11 Example :

Let X = {a,b,c,d} , T = { Ø, X , {a} , {d} , {a,d}} ,
Ω = { Ø , X }

(X,T) , (X, Ω) are two topologies space on X, then (X,T,Ω) is a bitopological space .

δ-o(X) = { X , Ø , {a} , {d} , {a,d} , {a,b} , {a,c} , {a,b,c} , {a,b,d} , {b,d} , {c,d} , {a,c,d} , {b,c,d} } .

If we take A = {a,b} , then only a∈{a} ⊂ {a,b} , and a,b∈{a,b} ⊂ {a,b} . Hence δ-nbd (A) = { {a} , {a,b} } .

1.2 Some Important Theorems :

1.2.1 Remark :

(a) T is sub collection of δ-o(X)

(b) A necessary condition for a non empty set to be in δ-o(X) is that its δ- interior is not empty .

(c) A ∈ δ-o(X) iff A ⊂ T- int (Ω-cl (T- int (A))) .

1.2.2 Remark :

The intersection of any two δ-open sets is not necessary δ-open set . The following example shows that there are two δ-open , but the intersection is not δ-open set .

1.2.3 Example :

Let X = { a,b,c,d } , T = { X , Ø , {a} , {d} , {a,d} , {b,c} , {a,b,c} , {b,c,d} } ,
Ω = { X , Ø , {a} , {d} , {a,d} } .

(X,T) , (X, Ω) are two topologies on X , then (X,T,Ω) is a bitpological space .

The set of δ-open set in X is

δ-o(X) = { X , Ø , {a} , {d} , {a,d} , {b,c} , {a,b,c} , {b,c,d} , {a,b} , {a,c} , { b,d} , { a,b,d} , {a,c,d} } .

Hence {b,c} , and {a,c} are two δ-open set , but {b,c}∩{a,c}= {c} is not δ-open set .

1.2.5 Theorem :

Let (X,T,Ω) be a bitopological space , and A ⊂ X , then δ-int (A) = ∪ { G : G is δ-open , G ⊂ A } .

Proof :

Let x ∈ δ-int (A) iff A is δ-nbd of x ,

x ∈ δ-int (A) iff there exists δ-open set G such that x ∈ G ⊂ A ,

x ∈ δ-int (A) iff x ∈∪ { G : G is δ-open } .

Hence δ-int (A) = ∪ { G : G is δ-open , G ⊂ A } .

1.2.6 Theorem :

Let (X,T,Ω) be a bitopological space and A be a subset of X , then :

(a) δ-int (A) is an δ-open set ,
(b) δ-int (A) is the largest δ-open set contained in A ;
(c) A is δ-open iff δ-int (A) = A .

1.2.7 Theorem :

Let (X,T,Ω) be a bitopological space , and A ⊂ X , then δ-int (A) equals the set of all those points of A which are not δ-limit point of X- A .

1.2.8 Theorem :

Let (X,T,Ω) be a bitopological space and A , B be any subsets of X , then :

(a) δ-int (X) = X , δ-int ($\varnothing$) = $\varnothing$,

(b) δ-int (A) $\subset$ A ,

(c) If A $\subset$ B , then δ-int (A) $\subset$ δ-int (B) ,

(d) δ-int (A ∩ B) = δ-int (A) ∩ δ-int (B) ,

(e) δ-int (A) ∪ δ-int (B) $\subset$ δ-int (A ∪ B) ,

(f) δ-int (δ-int (A)) = δ-int (A) .

Proof :

(a) **Since X and $\varnothing$ are δ-open sets , so by theorem (1.2.6)(c) , δ-int (X) = X , δ-int ($\varnothing$) = $\varnothing$.**

(b) **If x$\in\delta$-int (A) , then A is δ-nbd of x , so x$\in$A , hence δ-int (A) $\subset$ A .**

(c) **Let x$\in\delta$-int (A) , then A is δ-nbd of x . Since A $\subset$ B , so B is also a δ-nbd of x ; this implies x$\in\delta$-int (B) , hence δ-int (A) $\subset$ δ-int (B) .**

(d) **Since A∩B$\subset$ A , and A∩B$\subset$ B , we have by theorem (1.2.8 (c)) δ-int (A∩B) $\subset$ δ-int (A) , and**
δ-int (A∩B) $\subset$ δ-int (B) . Hence ,
δ-int (A∩B)$\subset$ δ-int (A) ∩ δ-int (B) ……………………(1)

Let x$\in\delta$-int (A)∩δ-int (B), then x$\in\delta$-int (A) and x$\in\delta$-int (B) ; hence x is δ-interior point of each of the set A and B .
Since A and B are δ-nbd of x , so the intersection A∩B is also δ-nbd of x .
Hence x$\in\delta$-int (A∩B) , thus x$\in\delta$-int (A) ∩ δ-int (B) , x$\in\delta$-int (A∩B) ,

hence δ-int (A)∩ δ-int (B) ⊂ δ-int (A∩B) ……….....…………(2)
From (1) and (2) , we get :
δ-int (A∩B) = δ-int (A) ∩ δ-int (B) .

***(e)* So by (c)we have A⊂ A∪B,then δ-int (A)⊂ δ-int (A∪B) ;**
B⊂ A∪B,then δ-int (B)⊂ δ-int (A∪B)
Hence δ-int (A)∪ δ-int (B) ⊂ δ-int (A∪B) .
But the converse is not true as in the following example .

1.2.9 Example :

Let X = { a,b,c } , T = { Ø , X , {a} , {a,c} } .
And Ω = { Ø , X , {a} } .
(X,T) , (X, Ω) are two topologies on X . Then (X,T,Ω) is a bitpological space , such that :
δ-o(X) = { Ø , X , {a} , {a,c} , {a,b}}
If we take A = {a} , B = {c} then A∪B = {a}∪{c} = {a,c} and δ-int (A∪B) = {a}∪{a,c} = {a,c} ;
and δ-int (A) = {a} , δ-int (B) = Ø .
Hence δ-int (A∪B) ⊄ δ-int (A)∪δ-int (B) ,
i.e {a,c}⊄{a} .

1.2.10 Theorem :

Let A be a subset of a bitopological space , then :
(a) δ-cl (A) is the smallest δ-closed set containing A ,
(b) A is δ-closed iff δ-cl (A) = A .

1.2.11 Theorem :

Let (X,T,Ω) be a bitopological space and let A , B be any subset of X , then :

(a) δ-cl (∅) = ∅ ,

(b) A ⊂ δ-cl (A) ,

(c) If A ⊂ B , then δ-cl (A) ⊂ δ-cl (B) ,

(d) δ-cl (A ∪ B) = δ-cl (A) ∪ δ-cl (B) ,

(e) δ-cl (A ∩ B) ⊂ δ-cl (A) ∩ δ-cl (B) ,

(f) δ-cl (δ-cl (A)) = δ-cl (A) .

Proof :

(a) **Since ∅ is δ-closed , we have δ-cl (∅) = ∅ .**

(b) **By theorem (1.2.10)(a) , hence A ⊂ δ-cl (A) .**

(c) **By (b) , B ⊂ δ-cl (B) . Since A ⊂ B , then A ⊂ δ-cl (B) . But δ-cl (B) is δ-closed set , thus δ-cl (B) is δ-closed set containing A . Since δ-cl (A) is the smallest δ-closed set containing A , we have δ-cl (A)⊂ δ-cl (B) .**

(d) **Since A⊂ A∪B , and B⊂ A∪B ,**

we have δ-cl (A)⊂δ-cl (A∪B),and δ-cl (B)⊂ δ-cl (A∪B) by(c).

Hence δ-cl (A)∪ δ-cl (B) ⊂ δ-cl (A∪B) ……………….(1)

Since δ-cl (A) and δ-cl (B) are δ-closed sets ,

and δ-cl (A) ∪ δ-cl (B) is also δ-closed

and by (b) A⊂ δ-cl (A) , B ⊂ δ-cl (B) .

This implies that A∪B ⊂ δ-cl (A)∪δ-cl (B) . Thus δ-cl (A) ∪ δ-cl (B) is δ-closed set containing A∪B since δ-cl (A∪B) is the smallest δ-closed set containing A∪B .

Therefore δ-cl (A∪B) ⊂ δ-cl (A)∪δ-cl (B) …………………..(2)

From (1) and (2) , we have :

δ-cl (A∪B) = δ-cl (A)∪δ-cl (B) .

(e) **Since A∩B ⊂ A , then δ-cl (A∩B) ⊂ δ-cl (A) by (c) , and A∩B ⊂ B , then δ-cl (A∩B)⊂ δ-cl (B) by (c) .**
Hence δ-cl (A∩B) ⊂ δ-cl (A) ∩ δ-cl (B) , but the converse is not true as in the following example .

1.2.12 Example :

Let X = { a,b,c } , T = { Ø , X , {a} , {a,b} } and Ω = { Ø , X } .

(X,T) , (X, Ω) are two topologies on X . Then (X,T,Ω) is a bitpological space , such that :

δ-o(X) = { X , Ø , {a} , {a,b} , {a,c} } .

If we take A = {a,b} , B = {b,c} , then A∩B = {b} .

Also δ-cl (A∩B) = {b} .

Also δ-cl (A) = X , δ-cl (B) = { b,c} .

Hence δ-cl (A) ∩ δ-cl (B) = {b,c}

∴δ-cl (A) ∩ δ-cl (B) ⊄ δ-cl (A∩B).

1.2.13 Theorem :

Let (X,T,Ω) be a bitopological space , then every T-open in topological space is δ-open set in bitopological space , but the converse is not true .

Proof :

Since A∈T , then A is open set in (X,T) .

Thus T-int (A) = A , and since A⊂ Ω-cl (A) ,

then A⊂ Ω-cl (T-int (A)) , T-int (A)⊂T-int (Ω-cl (T-int (A))) ,

since T-int (A) = A .

Hence A⊂T-int (Ω-cl (T-int (A))) , thus A∈δ-o(x) , but the converse is not true as in the following example .

1.2.14 Example :

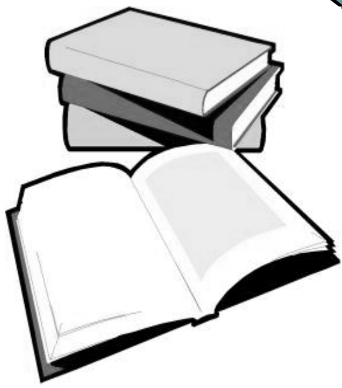
Let X = { a,b,c } , T = { Ø , X , {a} , {b} , {a,b} } ,

and Ω = { Ø , X } .

(X,T) , (X, Ω) are two topologies on X . Then (X,T,Ω) is a bitpological space , such that :

δ-o(X) = { Ø , X , {a} , {b} , {a,b} , {b,c} , {a,c} } , hence its topology space .

Thus δ-o(X) ≠ T-open because δ-o(X) $\not\subset$ T-open .



CHPTER TWO

The Separation Axioms in Bitopological Space

In this chapter we recall the separation axioms in bitopological space and we study the relation between then and give some examples .

2.1.1 Definition :

Let (X,T,Ω) be a bitopological space , then (X,T,Ω) is called δ-T_0 space iff for each pair of points x,y of X , such that x ≠ y , there exists δ-open set G containing x but not containing y or δ-open set H containing y but not containing x .

2.1.2 Example :

Let (X,T, Ω) be a bitopological space , where :

X = { a,b,c,d } ,
T = { X , Ø , {a} , {a,d} , {b,c} , {a,b,c} } ,
Ω = { X , Ø , {a} } .

(X,T) , (X, Ω) are two topologies on X , then :

δ-o(X) = { X , Ø , {a} , {a,d} , {b,c} , {a,b} , {a,b,d } , {a,c} , {a,c,d} , {a,b,c} } and (X,T,Ω) is δ-T_0 space .

If we take a and b , a ≠ b , there exists δ-open {a} contains a but not containing b , and similarly the other cases a ≠ c , a ≠ d , b ≠ c , b ≠ d , c ≠ d , but (X,T) and (X,Ω) are not T_0 space .

2.1.3 Proposition :

Let (X,T,Ω) be a bitopological space , if (X,T) is T_0 space , then (X,T,Ω) is δ-T_0 space .

Proof :

Let x , y ∈ X , such that x ≠ y .

Since (X,T) is T_0-space , then there exists T-open set A in X such that $x \in A$, $y \notin A$.
Since every T-open set is δ-open set by theorem (1.2.13), then A is δ-open set such that $x \in A$, $y \notin A$.
Hence (X,T,Ω) is δ-T_0 space .
The converse of this proposition is not true by the previous example (2.1.2) .

2.1.4 Theorem :

A bitopological space (X,T,Ω) is T_0 space iff for each distinct point x , y of X , δ-cl {x} ≠ δ-cl {y} .

Proof :

Let x , y $\in$X , such that x ≠ y and let δ-cl {x} ≠ δ-cl {y}. Then there exists at least one point Z in X , such that Z $\in$ δ-cl {x} , Z $\notin$ δ-cl {y} .
Suppose x $\in$ δ-cl {y} , since {x} $\subset$ δ-cl {y} .
Then δ-cl {x} $\subset$ δ-cl (δ-cl {y}) .
But δ-cl (cl {y}) = δ-cl {y} , then Z $\in$ δ-cl {y} and this is a contradiction .
So x $\notin$ δ-cl {y} , then x $\in$ X-(δ-cl {y}) and since δ-cl {y} is δ-closed set , so X-(δ-cl {y}) is δ-open set .
Hence X-(δ-cl {y}) is δ-open set containing x but not y . Therefore (X,T,Ω) is δ-T_0 space .
Conversely , since (X,T,Ω) is δ-T_0 space , then for each two distinct point x , y$\in$X , there exist δ-open set G such that x$\in$G , y$\notin$G . X-G is δ-closed set which does not contain x , but contains y . By definition (1.1.6) , δ-cl{y} is the intersection of all δ-closed set containing {y}.Thus δ-cl{y}$\subset$ X-G , then x$\notin$X-G . This implies that x$\notin$$\delta$-cl {y} , so we have x$\in$$\delta$-cl {x} , x$\notin$$\delta$-cl {y} .
Therefore δ-cl {x} ≠ δ-cl {y} .

2.1.5 Theorem :

Every subspace of δ-T_0 space is δ-T_0 space .

Proof :

Let Y be a subspace of δ-T_0 space X , to prove Y is δ-T_0 space , let $y_1 \neq y_2 \in Y$, since $Y \subset X$.
Then $y_1 \neq y_2 \in X$ and X is δ-T_0 space . There exists δ-open set G in X , such that $y_1 \in G$ and $y_2 \notin G$, so $G \cap Y$ is δ-open set in Y and $y_1 \in G \cap Y$, $y_2 \notin G \cap Y$.
Hence Y is δ-T_0 space .

2.1.6 Definition :

A bitopological space (X,T,Ω) is called δ-T_1 space iff for each pair of distinct point x,y of X there exists two δ-open sets G,H such that $x \in G$, but $y \notin G$; and $y \in H$, but $x \notin H$.

2.1.7 Example :

Let (X,T,Ω) be a bitopological space , such that
X = { a,b,c,d } ,
T = { X, ∅ , {a} , {a,d} , {d} , {b,c} , {a,b,c} , {b,c,d} } ,
Ω = { X , ∅ , {a} , {d} , {a,d} } .
(X,T) , (X, Ω) are two topologies on X such that :
δ-o(X) = { X , ∅ , {a} , {d} , {a,d} , {b,c} , {a,b} , {a,b,c} , {a,c} , {b,d} , {a,b,d} , {a,c,d} , {b,c,d} } .
Let a,b $\in$ X , such that a ≠ b . {a} and {b,c} are δ-open sets such that a∈{a} , but b∉{a} ; and b∈{b,c} but a∉{b,c} .
Also there exists δ-open sets {a} , {b,c} such that a∈{a} , but c∉{a} ; and c∈{b,c} , but a∉{b,c} .
Also there exist δ-open sets {a} , {d} such that a∈{a} , but d∉{a} ; and d∈{d} , but d∉{a} .

Also there exists δ-open sets {b,d} , {a,c} such that b$\in${b,d} but c$\notin${b,d} ; and c$\in${a,c} , but b$\notin${a,c} . Also there exists δ-open sets {a,b} , {d} such that b$\in${a,b} , but d$\notin${a,b} ; and d$\in${d} , but b$\notin${d} . And there exists δ-open sets {b,c} , {d} , such that c$\in${b,c} , but d$\notin${b,c} ; and d$\in${d} but c$\notin${d} .
Hence (X,T,Ω) is δ-T_1 space .

2.1.8 Proposition :

If (X,T) is T_1 space , then (X,T,Ω) is δ-T_1 space .

Proof :

Let a,b $\in$ X , a ≠ b . Since (X,T) is T_1 space , then there exists T-open sets A , B in X such that a$\in$A , but b$\notin$A ; and b$\in$B , but a$\notin$B . Since every T-open set is δ-open set by theorem (1.2.13) , then A , B are δ-open sets , such that a$\in$ A , but b$\notin$ A ; and b$\in$ B , but a$\notin$B .
Hence (X,T,Ω) is δ-T_1 space .

2.1.9 Remark :

The converse of the above proposition is not true ; that is if (X,T,Ω) is δ-T_1 space , then it is not true that (X,T) is T_1 space by the previous example (2.1.7) .

2.1.10 Theorem:

Every subspace of δ-T_1 space is δ-T_1 space .

Proof :

Let (X,T,Ω) be a δ-T_1 space and let (Y,T_Y,$Ω_Y$) be a sub space of (X,T,Ω) .
Let $y_1 \neq y_2 \in Y$ and since Y$\subset$ X , then $y_1 \neq y_2 \in X$.
Since X is δ-T_1 space , there exists two δ-open sets G , H in X , such that $y_1\in G$, but $y_2\notin G$; and $y_1\in H$, but $y_2\notin H$. Then

G_1 = G ∩ Y , H_1 = H ∩ Y are δ-open sets in Y and we have $y_1 \in G_1$, but $y_2 \notin G_1$; and $y_1 \notin H_1$, but $y_2 \in H_1$. Hence (Y,T_y,Ω_y) is δ-T_1 space .

2.1.11 Theorem:

A bitopological space (X,T,Ω) is a δ-T_1 space iff every single subset {x} of X is δ-closed .

Proof :

Suppose X is δ-T_1 space , and x be any point of X .
Let y∈X- {x} , then x ≠ y and so there exists δ-open set U containing Y but not x , and δ-open set V containing x but not containing y , y ∈ U ⊂ X- {x} .
Hence X- {x} is δ-open set , then {x} is δ-closed set .
Conversely :
Let x , y∈X , such that x ≠ y . Since {x} is δ-closed set , then X- {x} is δ-open set containing y but not x .
Similarly , X- {y} is δ-open set containing x but not containing y .
Hence (X,T,Ω) is δ-T_1 space .

2.1.12 Theorem:

A bitopological space (X,T,Ω) is δ-T_1 space iff δ-cl {a} = ∅ for each a ∈ X .

Proof :

Let (X,T,Ω) be a δ- T_1 space .
Suppose δ-cl {a} ≠ ∅ , for some a∈X , then there is a point b , such that b∈δ-cl {a} , and b ≠ a . Since X is δ-T_1 space , then there exist δ-open set G such that a∉G , b ∈G . Thus G ∩ {a} =∅ .
Hence b∉δ-cl {a} , which is contradiction.Thus δ-cl {a} = ∅.

Convenersely :

suppose that δ-cl {a} = ∅ , for each a∈X , and let x , y ∈ X , such that x ≠ y .

Then x∉δ-cl {y} , and there exists δ-open set G such that x∈G and G ∩ {y} =∅ , hence G contains x but not containing y . Similarly , there exists δ-open set contains y but not containing x . Thus (X,T,Ω) is δ-T_1 space .

2.1.13 Definition :

A bitopological space (X,T,Ω) is called δ-T_2 space (δ-Hansdorff) iff for each pair of distinct points x ,y of X there exists two δ-open sets G , H Such that x∈G , y ∈ H , G ∩ H = ∅ .

2.1.14 Example :

Let (X,T,Ω) be a bitopological space such that X={a,b,c,d} , T= { X,∅,{a},{b},{a,b},{c,d}} , {a,c,d},{b,c,d} } ,

Ω = { X,∅,{a} }.

(X,T) , (X, Ω) are two topologies on X , such that :

δ-o(X)= {X,∅,{a},{b},{a,b},{c,d},{a,b,c},{a,c},{a,d},{b,c},{b,d}
{a,b,d},{a,c,d},{b,c,d} } .

Then (X,T,Ω) is δ-T_2 space .

Let a,b∈X , a ≠ b. {a} and {b} are δ-open sets such that a∈{a},b∈{b} , and {a} ∩ {b} =∅ , and similarly the other cases a ≠ c , a ≠ d , b ≠ c , b ≠ d , c ≠ d .

2.1.15 Proposition :

Let (X,T,Ω) be a bitopological space . If (X,T) is T_2 space , then (X,T,Ω) is δ-T_2 space .

Proof :

Let $x,y \in X$, $x \neq y$. Since (X,T) is T_2 space , then there exist U , V T-open set in X such that $x \in U$, $y \in V$, $U \cap V = \varnothing$.

Since every T-open set is δ-open (by theorem1.2.13) , then U , V are δ-open sets such that $x \in U$, $y \in V$, $U \cap V = \varnothing$. Therefore (X,T,Ω) is δ-T_2 space .

The converse of this proposition is not true by the previous example (2.1.14) .

2.1.16 Remark :

Every δ-T_2 space is a δ-T_1 space , but the converse is not true as shown by the following example .

2.1.17 Example:

Let (X,T,Ω) be a bitopological space , such that

X = {a,b,c,d} ,

T = { ∅ , X , {c} , {a,b,d} } ,

Ω = { ∅ , X , {c} } .

(X,T) , (X, Ω) are two topologies on X , such that :

δ- o(X) = { X , ∅ , {c} , {a,b,d} , {a,b,c} , {a,c} , {b,c} , {a,c,d} , {b,c,d} ,{c,d} } .

Clearly (X,T,Ω) is δ-T_1 space , but not δ-T_2 space since there exists a ≠ b , such that for all δ- open sets , {a,c} , {b,c} and {a,c} ∩ {b,c} ≠ ∅ .

2.1.18 Theorem:

Every subspace of δ-Hansdorff space is δ- Hansdorff.

Proof :

Let (X,T,Ω) be a δ-Hansdorff , and let y ≠ ∅ subset of X . Let x ≠ y ∈Y , then x ≠ y ∈ X and , since (X,T,Ω) is δ-Hansdorff , there exists two δ-open sets G , H such that x ∈ G , y ∈ H , G ∩ H = ∅ .

So G ∩ Y , H ∩ Y , is δ-open set in Y ,and x∈G ∩ Y, y∈H ∩ Y ; and (G ∩ Y) ∩ (H ∩ Y) = (G ∩ H) ∩ Y = ∅ .

Hence (Y,T_Y,Ω_Y) is δ-T_2 space .

2.1.19 Definition:

A bitopological space (X,T,Ω) is called δ-regular space iff for each δ-closed set F is in X , and each x ∉ F ; and there exist δ-open sets U , V such that x ∈ U , F⊂ V , U ∩ V = ∅ .

2.1.20 Example:

Let (X,T,Ω) be a bitopological space , such that :

X = {a,b,c} ,

T = { X , ∅ , {a} , {c,d} , {c} , {b,c} , {a,c,d} , {a,c} , {a,b,c} , {b,c,d} } ,

Ω = { X,∅,{a} } .

(X,T) , (X, Ω) are two topologies on X .

Then δ-o(X) = { X , ∅ , {a} , {c,d} , {c} , {b,c} , {a,c,d} , {a,b,c} , {a,c} ,{a,b} , {a,d} , {a,b,d},{b,c,d} } .

Take F = {b,c,d} , a ∉ F , then there exists {a} , {b,c,d} δ-open sets such that a∈{a} , {b,c,d} ⊂ {b,c,d} , and {a} ∩ {b,c,d} = ∅ .

And similarly the other cases .

Hence (X,T,Ω) is δ-regular space .

2.1.21 Proposition:

Let (X,T,Ω) be a bitopological space . If (X,T) is regular space , then (X,T,Ω) is δ-regular .

Proof :

Let F be T-closed set in X , x∈X such that x ∉ F .
Since (X,T) is regular space , then there exists G , H are T-open set in X such that x ∈ G , F ⊂ H , G ∩ H = ∅ .
Since every T-open set is δ-open set (by theorem 1.2.13) , then G,H are δ-open sets , such that x∈G ,F⊂ H ,G∩H = ∅ .
Hence (X,T,Ω) is δ -regular .

2.1.22 Remark:

The converse of the above proposition is not true that (X,T) is regular , by the previous example (2.1.20) .

2.1.23 Theorem:

Let (X,T,Ω) be a bitopological space , then (X,T,Ω) is δ-regular iff for each δ-open set U and x∈U , there exists δ-open set V such that x∈V , δ -cl (V)⊂ U .

Proof :

Let (X,T,Ω) be δ-regular space . Let x∈ U where U is δ-open . Let H = X – U , then H is δ-closed , x ∉ H .
Then there exists δ-open sets ω and V such that : x∈V , H⊂ ω , V∩ω = ∅ .
Then V⊂ X – ω , δ-cl (V)⊂δ-cl (X – ω) = X – ω …………(1)
H⊂ ω , then X – ω ⊂ X – H = U , then X – ω ⊂ U …………(2)
From (1) , (2) we have , x∈V , δ-cl (V)⊂ U .
Conversoly :

let H be δ-closed set and x∉H . Let U = X – H , then U is δ-open and x∈U . By hypothesis , there exists δ-open set V such that x∈V , δ-cl (V) ⊂ U , H ⊂ (X – (δ-cl (V))) .
Since x∈V , V∩ (X – (δ-cl (V))) = ∅ .
Hence (X,T,Ω) is δ-regular .

2.1.24 Theorem:

Every subspace of δ-regular space is δ-regular .

Proof :

Let (X,T,Ω) is δ-regular space , let (Y,T_y,Ω_y) be a subspace of X . To prove (Y,T_y,Ω_y) is δ-regular ,
let q∈Y and F be δ-closed set in Y , such that q∉F . Then δ-cl_Y (F) = δ-cl_x (F) ∩ Y , and since F is δ-closed in Y so δ-cl_Y (F) = F . Then , F = δ-cl_x (F)∩ Y .
Since q∉F , then q∉δ–cl_x (F) ∩ Y , q∉δ-cl_x (F) , thus δ-cl_x (F) is δ-closed in X ; and since (X,T,Ω) is δ-regular , then there exist two disjoint δ-open sets G , H in X , such that q∈G , δ-cl_x (F) ⊂ H & G∩H = ∅ .
q∈G∩Y and δ-clx (F)∩ Y⊂ H∩Y , F⊂ H∩Y , since G , H are δ-open in X , then G∩Y , H∩Y are δ-open set in Y . Since G∩H = ∅ , then (G∩Y) ∩ (H∩Y) .
(G∩H)∩ Y = ∅ ∩Y = ∅ .
So (Y,T_y,Ω_y) is δ-regular subspace of (X,T,Ω) .

2.1.25 Definition :

A bitopology space (X,T,Ω) is called δ-normal space iff for each pair of δ-closed set G H in X , such that G ∩ H = ∅ , there exists δ-open sets U , V such that G ⊂ U , H ⊂ V and U ∩ V = ∅ .

2.1.26 Example :

Let (X,T,Ω) be a bitopological space , such that

X = { a,b,c,d } ,

T = { ∅ , X , {c} , {a,c} , {a} } ,

Ω = { X , ∅ } , such that :

δ-o(X) = { ∅ , X , {c} , {a} , {a,b} , {a,c} ,{a,b,c} , {a,d} , {b,c} , {a,b,d} , {a,c,d} , {b,c,d} , {c,d} } .

Then (X,T,Ω) is δ-normal space.

2.1.27 Proposition :

Let (X,T,Ω) be a bitopological space . If (X,T) is normal space , then (X,T,Ω) is δ-normal space .

2.1.28 Remark :

The converse of the above proposition is not true that (X,T) is normal , By the previous example (2.1.26) .

2.1.29 Theorem :

Let (X,T,Ω) be a bitopological space . Then (X,T,Ω) is δ-normal space iff for every δ-closed set H in X and δ-open set U in X containing H , there exist δ-open set V , such that H ⊂ V ⊂ δ-cl (V) ⊂ U .

Proof :

Suppose (X,T,Ω) is δ-normal space , let H be δ-closed in X and U is δ-open in X , such that H ⊂ U . Then X– U is δ-closed in X and H ∩ (X– U) = ∅ . So there exist δ-open sets V,K such that X–K⊂ U , H⊂ V , V ∩ K = ∅ , X–K⊂ U , V⊂ X–K . This implies that δ-cl (V)⊂ δ-cl (X - K) = X–K .

Then H $\subset$ V $\subset$ δ-cl (V) $\subset$ U .

Conversely , H is δ-closed in X , and U is δ-open in X , then X– U is δ-closed in X , and H ∩ (X – U) = ∅ .

By hypothesis , there exist δ-open set V , such that H $\subset$ V , δ-cl (V)$\subset$ U , then X – U $\subset$ (X – (δ-cl (V)) .

So we have H$\subset$ V , X – U $\subset$ (X – (δ–cl (V)) , and

V ∩ (X – (δ–cl (V))) = ∅

Therefore , (X,T,Ω) is δ-normal space .

2.1.30 Corollary :

A bitopological space (X,T,Ω) is δ-normal space iff for each δ-closed set H in X and each δ-open set U in X containing H , there exists a subset A of X , such that H$\subset$ δ-int (A) $\subset$ δ-cl (A)$\subset$ U .

Proof :

By using the last theorem and taking A = U , the proof is satisfied .

2.1.31 Definition :

A bitopological space (X,T,Ω) is called a δ-T_3 space iff X is δ-T_1 and δ-regular .

2.1.32 Example :

Let (X,T,Ω) be a bitopological space , such that :

X = {a,b,c,d} ,

T = { X ,∅ , {a} , {a,d} , {d} , {b,c} , {a,b,c} , {b,c,d} } ,

Ω = { X , ∅ , {a} , {d} , {a,d} } .

(X,T) , (X, Ω) are two topologies on X , such that :

δ-o(X) = { X , ∅ , {a} , {d} , {a,d} , {b,c} ,{a,b} , {a,b,c} , {a,c} , {b,d} , {a,b,d} , {a,c,d} , {b,c,d} }

Then (X,T,Ω) is δ-regular and δ-T_1 space .

Thus (X,T,Ω) is δ-T_3 space .

2.1.33 Remark :

Every δ-T_3 space is δ-regular and the converse is not true as we show in the following example .

2.1.34 Example :

Let (X,T,Ω) be a bitopological space , such that :

X = {a,b,c,d} ,

T = { ∅ , X , {a} , {b,c} } ,

Ω = { ∅ , X , {a} , {c} , {a,c} } , such that :

δ-o(x) = { ∅ , X , {a} , {b,c} }

(X,T,Ω) is δ-regular , but not δ-T_1 space ; hence (X,T,Ω) is not δ-T_3 space .

2.1.35 Definition :

A bitopological space (X,T,Ω) is called δ-T_4 space iff X is δ-normal and δ-T_1 space .

2.1.36 Example :

Let (X,T,Ω) be a bitopological space , such that :

X = {a,b,c} ,

T = { X , ∅ , {a} , {b} , {c} , {a,b } , {b,c} ,{a,c} } ,

Ω = { X , ∅ } . (X,T) , (X, Ω) are two topologies on X such that :

δ-o(X) = { ∅ , X , {a} , {b} , {c} , {a,b} , {b,c} , {a,c} } ,

then (X,T,Ω) is δ-T_1 space and δ-normal space .

Then (X,T,Ω) is δ-T_4 space .

2.1.37 Proposition :

Every δ-T_4 space is also δ-T_3 space .

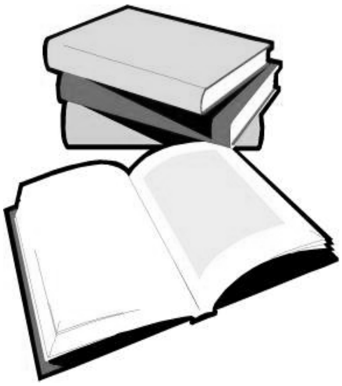
Proof :

Let (X,T,Ω) be a δ-T_4 space , then (X,T,Ω) is δ-normal as well as δ-T_1 space . To prove that the space is δ-T_3 space , it suffices to show that the space is δ-regular .

Let F be a δ-closed subset of X and , let x be a point of X such that $x \notin F$. Since (X,T,Ω) is a δ-T_1 space .

Thus {x} is a δ-closed subset of X , such that $\{x\} \cap F = \varnothing$, then by δ-normarlity , there exist δ-open sets G , H such that $\{x\} \subset G$, $F \subset H$ and $G \cap H = \varnothing$.

Also $\{x\} \subset G$, then $x \in G$, then there exist δ-open sets G , H such that $x \in G$, $F \subset H$ and $G \cap H = \varnothing$. It follows that the space (X,T,Ω) is δ-regular .



CHPTER THREE

Continuous in Bitopological Space

In this chapter we give some important definitions and theorems for continuity .

3.1 δ-Continuous on (δ-open , δ-closed , δ-interior , δ-closure) in bitopological space

3.1.1 Definition :

Let (X,T,Ω) and (Y,T´,Ω´) be a bitopological space , A mapping f : X→Y is said to be δ-continuous at $x_o \in X$ iff every δ-open set V in Y containing $f(x_o)$ there exist δ-open set U in X containing x_o such that $f(U) \subset V$.

3.1.2 Definition :

Let f : X →Y be a mapping , then :

(a) f is said to be δ-open mapping iff f(G) is δ-open in Y for every δ-open set G in X .

(b) f is δ-closed iff f(F) is δ-closed in Y for every δ-closed set F in X .

(c) f is δ-continuous iff f is δ-open and δ-closed .

(d) f is δ-homeomorphism iff (i) f is bijective (1-1 , onto) , (ii) f and f^{-1} are δ-continuous where (X,T, Ω) , (Y,T´,Ω´) are two bitopological spaces .

3.1.3 Example :

Let X = {a,b,c} ,T = { ∅ , X , {a} } ,

Ω = { ∅ , X } .

(X,T) , (X, Ω) are two topologies on X , then (X,T,Ω) is a bitopological space , such that :

δ-o(X) = { ∅ , X , {a} , {a,b} , {a,c} } .

And let Y = {1,2,3} , T´ = { ∅ , X , {1} } ,

Ω´ = { ∅ , X } .

(Y,T´) , (Y, Ω´) are two topologies on Y , then (Y,T´,Ω´) be a bitopological space , such that :

δ-o(Y) = { ∅ , Y , {1} , {1,2} , {1,3} } .

Define f : (X,T,Ω)→(Y,T´,Ω´) by f(a) = 1 , f(b) = 2 , f(c) = 3 . Then f is δ-continuous and δ-open set because f^{-1}(Y) = {a,b,c} = X is δ-open in X , and f^{-1}(∅) = ∅ is δ-open in X . Similarly the other cases f^{-1}({1}) , f^{-1}({1,2}) , f^{-1}({1,3}) are δ-open in X , therefore f is δ-continuous .

And since f({a}) = {1} is δ-open in Y and f({∅}) = ∅ is δ-open in Y , similarly the other cases f({X}) , f({a,b}) , f({a,c}) are δ-open in Y . Therefore f is δ-open mapping .

3.1.4 Example :

Let X = {a,b,c} ,T = { ∅ , X , {a} } ,

Ω = { ∅ , X , {b} , {a,b} } .

(X,T) , (X, Ω) are two topologies on X , then (X,T,Ω) is a bitopological space , such that :

δ-o(X) = { ∅ , X , {a} } .

And let Y = {1,2,3} , T´ = { ∅ , X , {2} , {1} , {1,2} } ,

Ω´ = { ∅ , X , {1} , {1,2} } .

(Y,T´) , (Y, Ω´) are two topologies on Y , then (Y,T´,Ω´) be a bitopological space , such that :

δ-o(Y) = { ∅ , X , {1} , {2} , {1,2} } .

Define f : $(X,T,\Omega) \rightarrow (Y,\acute{T},\acute{\Omega})$ by f(a) = 1 , f(b) = f(c) = 2 . Then f is δ-open but not δ-continuous because $f^{-1}(Y)$ = {a,b,c} = X is δ-open in X , and $f^{-1}(\varnothing) = \varnothing$ is δ-open in X, but $f^{-1}(\{2\})$ = {b,c} is not δ-open in X . Hence f is δ-continuous .

And since f(X) = {1,2} is δ-open in Y , $f(\{\varnothing\}) = \varnothing$ is δ-open in Y . And f(a) = ({1}) , is δ-open in Y therefore f is δ-open .

3.1.5 Theorem :

Let (X,T,Ω) and $(Y,\acute{T},\acute{\Omega})$ be a bitopological space , then a mapping f : $X \rightarrow Y$ is δ-continuous iff for every $x \in X$ the inverse image under f of every δ-open V of f(X) is δ-open set of X .

Proof :

Let f δ-continuous , and V is δ-open in Y to prove $f^{-1}(V)$ is δ-open in X . If $f^{-1}(V) = \varnothing$ so it is δ-open in X . If $f^{-1}(V) \neq \varnothing$, let $x \in f^{-1}(V)$, then $f(x) \in V$. By definition of δ-continuous there exist δ-open G_x in X containing x such that $f(G_x) \subset V$.

$\therefore$ $x \in G_x \subset f^{-1}(V)$.

Hence $f^{-1}(V)$ is δ-open set in X .

Conversely :

Let $f^{-1}(V)$ is δ-open set in X , for each V is δ-open set in Y to prove f is δ-continuous .

Let $x \in X$ and V is δ-open set in Y containing f(x) so $f^{-1}(V)$ is δ-open in X containing x and $f(f^{-1}(V)) \subset V$.

Then f is δ-continuous on X .

3.1.6 Theorem :

Let (X,T,Ω) and (Y,T',Ω') be a bitopological space . A mapping $f : X \to Y$ is δ-continuous iff the inverse image under f of every δ-closed set in Y is δ-closed set in X .

Proof :

(obvious)

3.1.7 Theorem :

A mapping $f : X \to Y$ is δ-continuous iff :
$f(\delta\text{-cl}(A)) \subset \delta\text{-cl}(f(A))$ for every $A \subset X$, where (X,T,Ω) and (Y,T',Ω') are two bitopological spaces .

Proof :

Let f be δ-continuous . Since $\delta\text{-cl}(f(A))$ is δ-closed set in Y .

$\therefore$ $f^{-1}(\delta\text{-cl}(f(A)))$ is δ-closed set in X By (3.1.6) therefore ,

$\delta\text{-cl}(f^{-1}(\delta\text{-cl}(f(A)))) = f^{-1}(\delta\text{-cl}(f(A)))$ ……………………(1)

Now :

$f(A) \subset \delta\text{-cl}(f(A))$, $A \subset f^{-1}(f(A)) \subset f^{-1}(\delta\text{-cl}(f(A)))$.

Then $\delta\text{-cl}(A) \subset \delta\text{-cl}(f^{-1}(\delta\text{-cl}(f(A)))) = f^{-1}(\delta\text{-cl}(f(A)))$ by (1).

Then $f(\delta\text{-cl}(A)) \subset \delta\text{-cl}(f(A))$.

Conversely : let $f(\delta\text{-cl}(A)) \subset \delta\text{-cl}(f(A))$ for every $A \subset X$

Let F be any δ-closed set in Y , so that $\delta\text{-cl}(F) = F$.

Now $f^{-1}(F) \subset X$, by hypothesis

$f(\delta\text{-cl}(f^{-1}(F)) \subset \delta\text{-cl}(f(f^{-1}(F))) \subset \delta\text{-cl}(F) = F$.

Therefore , $\delta\text{-cl}(f^{-1}(F)) \subset f^{-1}(F)$

But $f^{-1}(F) \subset \delta\text{-cl}(f^{-1}(F))$ always .

Hence δ-cl (f^{-1}(F))⊂ f^{-1}(F)and f^{-1}(F) are δ-closed set in X . Hence f is δ-continuous by theorem (3.1.6) .

3.1.8 Theorem :

A mapping f : X → Y is δ-continuous iff : δ-cl (f^{-1}(B)) ⊂ f^{-1}(δ-cl (B)) for every B ⊂ Y, where (X,T,Ω) and (Y,T´,Ω´) are two bitopological spaces .

Proof :

(obvious)

3.1.9 Theorem :

A mapping f : X →Y is δ-continuous iff f^{-1}(δ-int (B)) ⊂ δ-int (f^{-1}(B)) for every B ⊂ Y, where (X,T,Ω) and (Y,T´,Ω´) are two bitopological spaces .

Proof :

(obvious)

3.1.10 Theorem :

Let X , Y and Z be a bitopological space and the mappings f : X→Y and g : Y→ Z be δ-continuous , then the composition map gof : X→ Z is δ-continuous .

Proof :

(obvious)

3.2 δ-Continuous on Separation Axioms in Bitopological Space

3.2.1 Theorem :

Let (Y,T',Ω') be δ-T_0 space , if $f:(X,T,\Omega)\to(Y,T',\Omega')$ is δ-continuous 1-1 function .
Then (X,T,Ω) is δ-T_0 space .

Proof :

Let $x_1 , x_2 \in X$, $x_1 \neq x_2$. Since f is 1-1 function , then $f(x_1) \neq f(x_2)$, $f(x_1) , f(x_2) \in Y$, and Y is δ-T_0 space , then there exists δ-open set G in Y such that $f(x_1)\in G$, $f(x_2) \notin G$. So $x_1 \in f^{-1}(G)$, $x_2 \notin f^{-1}(G)$.
$\therefore f^{-1}(G)$ is δ-open set in X , then (X,T,Ω) is δ-T_0 space .

3.2.2 Theorem :

Let $f:(X,T,\Omega)\to(Y,T',\Omega')$ be an δ-continuous δ-open 1-1 and onto function . If (X,T,Ω) is δ-T_0 space , then (Y,T',Ω') is δ-T_0 space .

Proof :

Suppose that $y_1 , y_2 \in Y$, $y_1 \neq y_2$. Since f is onto , there exists $x_1 , x_2 \in X$, such that $y_1 = f(x_1)$, $y_2 = f(x_2)$ and since f is 1-1 , then $x_1 \neq x_2$, since X is δ-T_0 space .
There exists δ-open set G , such that $x_1\in G$, $x_2\notin G$.
Hence $y_1 = f(x_1) \in f(G)$, $y_2 = f(x_2) \notin f(G)$, since f is δ-open function , then f(G) is δ-open set in Y .
Therefore (Y,T',Ω') is δ-T_0 space .

3.2.3 Theorem :

Let (Y,T',Ω') be δ-T_1 space , if $f:(X,T,\Omega)\to(Y,T',\Omega')$ is δ-continuous 1-1 function , then X is δ-T_1 space .

Proof :

Let $x_1 , x_2 \in X$, $x_1 \neq x_2$. Since f is 1-1 $f(x_1) \neq f(x_2)$ $f(x_1)$, $f(x_2) \in Y$, Y is δ-T_1 space , then there exists U_1 , U_2 δ-

open set in Y such that $f(x_1) \in U_1$, but $f(x_2) \notin U_1$ and $f(x_2) \in U_2$, but $f(x_2) \notin U_2$.
Then $x_1 \in f^{-1}(U_1)$ but $x_2 \notin f^{-1}(U_1)$; and $x_2 \in f^{-1}(U_2)$, but $x_1 \notin f^{-1}(U_2)$; and $f^{-1}(U_1)$, $f^{-1}(U_2)$ are δ-open set in X .
Hence (X,T,Ω) is δ-T_1 space .

<u>3.2.4 Theorem :</u>

Let f : (X,T,Ω)→(Y,T´,Ω´) be δ-continuous 1-1 and onto , δ-open function : If (X,T,Ω) is δ-T_1 space then (Y,T´,Ω´) is δ-T_1 space .

<u>Proof :</u>

Suppose y_1 , $y_2 \in Y$, $y_1 \neq y_2$. Since f is onto , there exists x_1 , $x_2 \in X$ such that $y_1 = f(x_1)$, $y_2 = f(x_2)$. Since f is 1-1 then $x_1 \neq x_2 \in X$, $f(x_1) \neq f(x_2)$, and X is δ-T_1 space , there exists δ-open sets G , H such that $x_1 \in G$ but $x_2 \notin G$ and $x_2 \in H$ but $x_1 \notin H$.
Hence $f(x_1) \in f(G)$, $f(x_2) \in f(H)$, since f is δ-open function , hence f(G) , f(H) are δ-open sets of Y, $y_1 \in f(G)$, but $y_2 \notin f(G)$ and $y_2 \in f(H)$, but $y_1 \notin f(H)$.
Then (Y,T´,Ω´) is δ-T_1 space .

<u>3.2.5 Theorem :</u>

Let (Y,T´,Ω´) be δ-T_2 space , if f : (X,T,Ω)→(Y,T´,Ω´) is δ-continuous 1-1 function , then (X,T,Ω) is δ-T_2 space .

<u>Proof :</u>

Let $x_1 \neq x_2 \in X$, since f is 1-1 , $f(x_1) \neq f(x_2)$.

Let $y_1 = f(x_1)$, $y_2 = f(x_2)$, $y_1 \neq y_2 \in Y$. Since Y is δ-T_2 space , there exist two δ-open sets G , H in Y , such that $y_1 \in G$, $y_2 \in H$, $G \cap H = \emptyset$.
Hence $x_1 \in f^{-1}(G)$, $x_2 \in f^{-1}(H)$ since f is δ-continuous and $f^{-1}(G)$, $f^{-1}(H)$ is δ-open set in X .

Also $f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\varnothing) = \varnothing$.

Thus (X,T,Ω) is δ-T_2 space .

3.2.6 Theorem :

Let $f : (X,T,\Omega) \to (Y,\acute{T},\acute{\Omega})$ is δ-continuous, 1-1 and onto , δ-open function . If (X,T,Ω) is δ-T_2 space , then $(Y,\acute{T},\acute{\Omega})$ is δ-T_2 space .

Proof :

Let $y_1 \neq y_2 \in Y$. Since f is 1-1 and onto , then there exists $x_1 \neq x_2 \in X$ such that $y_1 = f(x_1)$, $y_2 = f(x_2)$. Since X is δ-T_2 space , then there exists δ-open sets G , H such that $x_1 \in G$, $x_2 \in H$, $G \cap H = \varnothing$. Since f is δ-open mapping , then f(G) , f(H) are two δ-open set in Y and

$f(G \cap H) = f(G) \cap f(H) = f(\varnothing) = \varnothing$.

Also $y_1 = f(x_1) \in f(G)$, $y_2 = f(x_2) \in f(H)$

Hence $(Y,\acute{T},\acute{\Omega})$ is δ-T_2 space .

3.2.7 Theorem :

Let (X,T,Ω) be δ-regular space and $f : (X,T,\Omega) \to (Y,\acute{T},\acute{\Omega})$ be δ-homeomorphism . Then $(Y,\acute{T},\acute{\Omega})$ is δ-regular .

Proof :

Let F be δ-closed set in Y , $q \notin F$, $q \in Y$. Since f is 1-1 and onto map , then there exists $p \in X$ such that $f(p) = q$, $p = f^{-1}(q)$. Since f is δ-continuous so $f^{-1}(F)$ is δ-closed in X , $q \notin F$, $p = f^{-1}(q) \notin f^{-1}(F)$. Since (X,T,Ω) is δ-regular , there exists δ-open sets G , H such that $p \in G$, $f^{-1}(F) \subset H$ and $G \cap H = \varnothing$.

So $q = f(p) \in f(G)$, $F \subset f(f^{-1}(F)) \subset f(H)$, since f is δ-open map , hence f(G) , f(H) are δ-open sets in Y and $f(G \cap H) = f(G) \cap f(H) = f(\varnothing) = \varnothing$.

Therefore (Y,T',Ω') is δ-regular space .

3.2.8 Theorem :

δ-normality is a bitopological property .

Proof :

Let (X,T,Ω) be δ-normal space and let (Y,T',Ω') be a δ-homeomorphic image of (X,T,Ω) under a δ-homeomorp-hic f to show that (Y,T',Ω') is also δ- normal space .

Let L,μ be a pair of disjoint δ-closed subsets of Y . Since f is δ-continuous map, then $f^{-1}(L)$ and $f^{-1}(\mu)$ are f isδ-closed subsets of X . Also $f^{-1}(L) \cap f^{-1}(\mu) = f^{-1}(L \cap \mu) = f^{-1}(\varnothing) = \varnothing$. Thus $f^{-1}(L)$, $f^{-1}(\mu)$ are a disjoint pair of δ-closed subsets of X . Since the space (X,T,Ω) is δ-normal , then there exist δ-open sets G and H such that $f^{-1}(L) \subset G$, $f^{-1}(\mu) \subset H$ and $G \cap H = \varnothing$. But $f^{-1}(L) \subset G$, then $f(f^{-1}(L)) \subset f(G)$, $L \subset f(G)$ similarly , $\mu \subset f(H)$. Also since f is an δ-open mapping f(G) and f(H) are δ-open subset of Y , such that :

$$f(G) \cap f(H) = f(G \cap H)$$
$$= f(\varnothing) = \varnothing$$

Thus there exists δ-open subset in Y , $G_1 = f(G)$ and $H_1 = f(H)$ such that $L \subset G_1$, $\mu \subset H_1$, and $G_1 \cap H_1 = \varnothing$. It follows that (Y,T',Ω') is also δ-normal space .

Accordingly , δ-normality is a bitopological property .

REFERENCES

1) *Kelly , J.L . , " General Topology " , Van . Nostrands Princeton , 1955 .*

2) *T. NoIRIL , A.SMASHHOUR , F.H.KHEDR ANDI. A. HSAANBIN – 1974 .*

3) *Sharma , J . N . , " Topology " . Published by Krishna Prakushan Mandir , and Printed at Mano ,1977 .*

الخلاصة

يؤسس هذا البحث العلاقة بين الفضاءات الثنائية – التبولوجيا من ناحية ، والفضاءات التبولوجية من نوع " ألفا المفتوحة " (α-open) ، كما يستنتج تعريفاً جديداً للفضاءات الثنائية – التبولوجيا من نوع " دلتا " المفتوحة (δ-open) بدلالة الفضاءات التبولوجية " ألفا " المفتوحة ، وبالاستناد إلى هذه العلاقة يقدم هذا البحث صياغة للنظرية الأساسية للعلاقة بين الفضاءات التبولوجية والفضاءات الثنائية – التبولوجيا " دلتا " المفتوحة ، كما يفصل البحث الخواص الأساسية للتعريف الجديد للفضاءات الثنائية من النوع الأخير .

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بابل / كلية التربية

" دلتا المفتوحة في الفضاءات الثنائية التبولوجيا"

رسالة مقدمة إلى
كلية التربية في جامعة بابل وهي جزء من متطلبات نيل شهادة الماجستير في علوم الرياضيات

من قبل
اكتفاء ضياء جليل

بأشراف
د. لؤي عبد الهاني السويدي

تشرين الثاني ٢٠٠٣