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Pre-Open Set in Bitopological Spaces

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كلية التربية

المجموعة المفتوحة تقريبا في الفضاءات ثنائية التبولوجيا

رسالة مقدمة إلى

كلية التربية - قسم الرياضيات - جامعة بابل

كجزء من متطلبات نيل درجة الماجستير في علوم الرياضيات

تَقدّم بها

علاء عرّيس عبد الرؤوف

بإشراف

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شوال 1426 هـ   تشرين الثاني 2005م

بسم الله الرحمن الرحيم

﴿يا أيها الناس قد جاءكم برهان من ربكم

وأنزلنا إليكم نورا مبينا﴾

صدق الله العلي العظيم

النساء (174) آية

الإهداء

إلى أبي . . . صاحب الفضل الأول . . .

إلى أمي . . . منبع الحنان والطيبة . . .

إلى إخوتي . . . أصحاب التشجيع وشد الأزر . . .

إلى أخواتي . . . دعوات وأمنيات لا تنقطع . . .

إلى رفيقة عمري . . . زوجتي . . .

إلى كل من وقف بجانبي وساعدني في الحصول على المصادر العلمية، وأخصّ بالذكر أساتذة قسم الرياضيات في كلية التربية – جامعة بابل ، واساتذة قسم الرياضيات في كلية التربية للبنات – جامعة الكوفة . . .

إلى كل معلم علّمني، وكل مدرس درّسني، وكل أستاذ فهّمني . . .

أهدي جهدي المتواضع هذا . . .

علاء

المستخلص

في هذه الرسالة نقدّم تعريفاً جديداً للمجموعة المفتوحة تقريبا ($pr-$ open set) في الفضاء الثنائي التبولوجيا، والتي تعطي مواصفات أساسية للتعاريف الجديدة بالنسبة للمجموعة المفتوحة تقريبا ($pr-$ open set) في الفضاء الثنائي التبولوجيا.

ولقد حصلنا على النتائج الآتية :-

1- الفضاء الثنائي التبولوجيا (X,τ,ρ) يكون فضاء T_o تقريبا إذا وفقط إذا لكل نقطتين مختلفتين x,y في X يكون $pr-cl(\{y\}) \neq pr-cl(\{x\})$.

2- إذا كانت كل مجموعة أحادية $\{x\}$ جزئية من فضاء ثنائي تبولوجيا (X,τ,ρ) هي مجموعة مغلقة تقريبا ، فإنّ (X,τ,ρ) يكون فضاء T_1 تقريبا.

3- الفضاء الثنائي التبولوجيا (X,τ,ρ) يكون فضاء T_1 تقريبا إذا وفقط إذا: $pr-cl(\{a\}) = \phi$ لكل $X \in a$.

4- الفضاء الثنائي التبولوجيا (X,τ,ρ) يكون فضاء منتظما تقريبا إذا وفقط إذا: لكل مجموعة مفتوحة تقريبا U، $x \in U$ يوجد مجموعة مفتوحة تقريبا V بحيث أن $x \in V$ و $pr-cl(V) \subset U$.

5- الفضاء الثنائي التبولوجيا (X,τ,ρ) يكون سوي تقريبا إذا وفقط إذا لكل مجموعة مغلقة تقريبا H في X ومجموعة مفتوحة تقريبا U في X تحوي H يوجد مجموعة مفتوحة تقريبا V بحيث أن $H \subset V \subset pr-cl(V) \subset U$.

6- ليكن (X,τ,ρ) ، (Y,τ',ρ') فضاءان ثنائيان تبولوجيان، و $f:(X,\tau,\rho) \to (Y,\tau',\rho')$ هي دالة. إذا كانت الصورة العكسية تحت f لكل مجموعة مفتوحة تقريبا في Y هي مجموعة مفتوحة تقريبا في X ، فإنّ الدالة f تكون مستمرة تقريبا .

7- ليكن (X,τ,ρ) ، (Y,τ',ρ') فضاءان ثنائيان تبولوجيان، و $f:(X,\tau,\rho) \to (Y,\tau',\rho')$ هي دالة. إذا كان $f(pr-cl(A)) \subset pr-cl(f(A))$ لكل $A \subset X$ ، فإنّ الدالة f تكون مستمرة تقريبا .

8- إذا كان (X,τ,ρ) ، (Y,τ',ρ') فضاءان ثنائيان تبولوجيان، و $f:(X,\tau,\rho) \to (Y,\tau',\rho')$ هي دالة. إذا كان $f^{-1}(pr-\text{int}(B)) \subset pr-\text{int}(f^{-1}(B))$ لكل $B \subset Y$ ، فإنّ الدالة f تكون مستمرة تقريبا.

9- ليكن (Y,τ',ρ') فضاء T_o تقريبا ، إذا كانت الدالة $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ مستمرة تقريبا ومتباينة، فإنّ الفضاء (X,τ,ρ) يكون فضاء T_2 تقريبا.

10- لتكن الدالة $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ مفتوحة تقريبا وشاملة ، إذا كان فضاء (X,τ,ρ) فضاء T_1 تقريبا فإن (Y,τ',ρ') يكون فضاء T_1 تقريبا.

11- ليكن (Y,τ',ρ') فضاء T_2 تقريبا ، إذا كانت الدالة $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ مستمرة تقريبا ومتباينة، فإنّ (X,τ,ρ) يكون فضاء T_2 تقريبا.

12- ليكن (X,τ,ρ) فضاءً ثنائياً تبولوجياً و (Y,τ') فضاءً تبولوجياً، والدالة $f:(X,\tau,\rho)\to(Y,\tau')$ هي دالة ، إذا كان $pr-cl(f^{-1}(B))\subset f^{-1}(pr-cl(B))$ لكل $B\subset Y$ فإنّ الدالة f تكون مستمرة تقريبا*.

13- لتكن الدالة $f:(X,\tau,\rho)\to(Y,\tau')$ مفتوحة تقريبا* وشاملة ، إذا كان (X,τ,ρ) فضاء T_o تقريبا ، فإنّ (Y,τ') يكون فضاء T_o .

14- ليكن (X,τ) فضاءً تبولوجياً و (Y,τ',ρ') فضاءً ثنائياً تبولوجياً، و $f:(X,\tau)\to(Y,\tau',\rho')$ هي دالة ، إذا كانت الصورة العكسية تحت f لكل مجموعة مغلقة تقريبا في Y هي مجموعة مغلقة في X فإنّ الدالة f تكون مستمرة تقريبا**.

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Alaa

We certify that this thesis was prepared under our supervision at the department of mathematics in the University of Babylon as partial fulfillment of the requirements needed toward the degree of master in mathematics .

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We certify that we have read the thesis entitled ($pr-$open set in bitopological spaces) and as an examining committee , examined the thesis in its content and what is related to it and that in our opinion it is adequate with (   ) attending as a thesis for the degree of master of science in mathematics.

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List of Symbols

Symbol	Meaning
A^c	The complement of A.
$pr.O(X)$	The family of all $pr-$open sets of X with respect to a bitopological space .
τ-open set	The open set with respect to the topology τ.
ρ-open set	The open set with respect to the topology ρ.
$pr.C(X)$	The family of all $pr-$closed sets of X with respect to a bitopological space .
τ-closed set	The closed set of the topology τ.
ρ-closed set	The closed set of the topology ρ.
$pr-N(x)$	The system $pr-$nhd at the point x with respect to a bitopological space .
$\tau-N(x)$	The system nhd at the point x with respect to the topology τ.
$\rho-N(x)$	The system nhd at the point x with respect to the topology ρ.
$pr-\text{int}(A)$	The set of all $pr-$interior points of A with respect to bitopological space .
$\tau-\text{int}(A)$	The set of all interior points of A in the topology τ.
$\rho-\text{int}(A)$	The set of all interior points of A in the topology ρ.
$pr-ext(A)$	The set of all $pr-$exterior points of A with respect to a bitopological space .
$\tau-ext(A)$	The set of all exterior points of A with respect to the topology τ.
$\rho-ext(A)$	The set of all exterior points of A with respect to the topology ρ.

List of Symbols

Symbol	Meaning
$pr-fr(A)$	The set of all $pr-$frontier points of A with respect to a bitopological space .
$\tau-fr(A)$	The set of all frontier points of A with respect to the topology τ .
$\rho-fr(A)$	The set of all frontier points of A with respect to the topology ρ.
$pr-lm(A)$	The set of all $pr-$limit points of A with respect to a bitopological space.
$\tau-lm(A)$	The set of all limit points of A with respect to the topology τ .
$\rho-lm(A)$	The set of all limit points of A with respect to the topology ρ.
$pr-cl(A)$	The $pr-$closure of A with respect to a bitopological space
$\tau-cl(A)$	The $pr-$closure of A with respect to the topology τ .
$\rho-cl(A)$	The closure of A with respect to the topology ρ.
$pr.O(X)_Y$	The relative bitopological space for Y with respect to $pr-$open sets.
τ_Y	The relative topology of τ to Y.
ρ_Y	The relative topology of ρ to Y.
$pr-cl_Y(A)$	The $pr-$closure of A in the subspace Y.
$pr-nhd_Y(A)$	The $pr-$nhd of A in the subspace Y.
$pr-fr_Y(A)$	The set of all $pr-$frontier points of A in the subspace Y.
$pr-lm_Y(A)$	The set of all $pr-$limit points of A in the subspace Y.
$pr-\text{int}_Y(A)$	The set of all $pr-$interior points of A in the subspace Y.

Abstract

In this thesis, we introduce a new definition for $pr-$open set in bitopological space and give the basic specifications for this definition .

We state here some of the main results which, we obtain:

1. A bitopological space (X,τ,ρ) is $pr-T_o$-space iff for each distinct points x,y in X, $pr-cl(\{x\})\neq pr-cl(\{y\})$.
2. If every singleton subset $\{x\}$ of bitopological space (X,τ,ρ) is $pr-$closed , then (X,τ,ρ) is $pr-T_1$-space.
3. A bitopological space (X,τ,ρ) is a $pr-T_1$-space iff $pr-cl\{a\}=\phi$, for each $a\in X$.
4. Let (X,τ,ρ) be a bitopological space, then (X,τ,ρ) is a $pr-$regular iff for each $pr-$open set U and $x\in U$, there exists $pr-$open set V such that $x\in V$, $pr-cl(V)\subset U$.
5. Let (X,τ,ρ) be a bitopological space, then (X,τ,ρ) is a $pr-$ normal iff for each $pr-$closed set H in X and $pr-$open set U in X containing H, there exists $pr-$open set V such that $H\subset V\subset pr-cl(V)\subset U$.
6. Let (X,τ,ρ) and (Y,τ',ρ')be bitopological spaces, and $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be a function. If the inverse image under f of every $pr-$open set in Y is $pr-$open set in X, then f is $pr-$continuous function.
7. Let (X,τ,ρ) and (Y,τ',ρ')be two bitopological spaces, and $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be a function. If $f(pr-cl(A))\subset pr-cl(f(A))$, for every $A\subset X$, then f is $pr-$continuous.
8. Let (X,τ,ρ) and (Y,τ',ρ') be two bitopological spaces, and $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be a function . If $f^{-1}(pr-\text{int}(B))\subset pr-\text{int}(f^{-1}(B))$ for each $B\subset Y$, then f is a $pr-$continuous function .

9. Let (Y,τ',ρ') be a $pr-T_o$-space, and $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ is a $pr-$continuous and 1-1 function , then (X,τ,ρ) is a $pr-T_o$-space.

10. Let $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be onto and $pr-$open function, if $(X,\tau,\rho)$is a $pr-T_1$-space, then $(Y,\tau',\rho')$is a $pr-T_1$-space.

11. Let (Y,τ',ρ')be a $pr-T_2$-space, and $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be a $pr-$continuous and 1-1 function, then (X,τ,ρ) is a $pr-T_2$-space.

12. Let (X,τ,ρ) be a bitopological space, (Y,τ') be a topological space, and $f:(X,\tau,\rho)\to(Y,\tau')$ be a function. If $pr-cl(f^{-1}(B))\subset f^{-1}(\tau'-cl(B))$, for every $B\subset X$, then f is pr^*-continuous function.

13. Let $f:(X,\tau,\rho)\to(Y,\tau')$ be a pr^*-open and onto function and (X,τ,ρ) be a $pr-T_\circ$-space, then $(Y,\tau')$is a $T_\circ$-space.

14. Let (X,τ) be a topological space, (Y,τ',ρ') be a bitopological space, and $f:(X,\tau)\to(Y,\tau',\rho')$ be a function. If the inverse image under f of every $pr-$closed set in Y is a τ-closed set in X, then f is $pr^{**}-$continuous.

Introduction

This thesis establishes a relation between bitopological spaces and topological spaces.

Bitopological space initiated by Kelly [7,8] is defined as : A set equipped with two topologies is called a bitopological space, denoted by (X,τ,ρ) where $(X,\tau),(X,\rho)$ are two topological spaces defined on X.

$\alpha-$open set defined by Njastad (1965) [14] , is as follows: Let (X,τ) be a topological space and $A\subset X$, A is said to be $\alpha-$open set iff $A\subset \text{int}_{\tau}(cl_{\tau}(\text{int}_{\tau}(A)))$. The definition for $pr-$open set in topological space is introduced by Mashhour , Abd El-Monsef , M. E., and El-Deep , S. N. (1981)[11] on the basic of $\alpha-$open set in topological space.

Let (X,τ) be a topological space, A be a subset of X, A is said to be $pre-$open set iff $A\subset \text{int}_{\tau}(cl_{\tau}(A))$ [11] , the family of all $pre-$open sets is denoted by $pr.O(X)$.

In this thesis, the phrase "$pr-$open set" refers to "$pr-$open set in bitopological space (X,τ,ρ)".

A new definition for $pre-$open set in bitopological space is introduced on the basic of $\alpha-$open set in bitopological space and $pre-$open set in bitopological space.

Let (X,τ,ρ) be a bitopological space A be any subset of X. A is said to be $pr-$open set iff $A\subset \text{int}_{\tau}(cl_{\rho}(A))$.

From the relation above, the following generalization is formulated between topological space and $pr-$open set in bitopological spaces.

Let (X,τ,ρ) be a bitopological space, then there is no relationship between ρ-open sets in topological space ρ and $pr-$open set in bitopological space.

Kelly , J. C. (1963) [8] . Introduced the idea of bitopological spaces .

Singal , M. K., and Singal , A.R., (1970) [16] . They introduced some more separation axioms these consider with bitopological spaces .

Reilly , I.L. (1972) [15] . Presented some properties that's deal with separation axioms on bitopological spaces .

Maheshwari , S. N. and Prasad R. (1975) [10] . They introduced some new separation axioms and studied some of their basic properties. The implications of those new separation axioms among themselves and with the well known axioms T_o, T_1, were obtained.

Valeriu , P. (1977) [18] . Introduced some properties of bitopological semi separation space .

Mirevic , M. (1986)[12]. Studied the separation axioms in bitopological space .

Mukherjee , M. N., and Ganguly , S. (1987)[13] . Studied characterized almost continuous multifunctions in a bitopological spaces . Also generalized the idea of almost continuity as studied in (Bose and Sinha (1982)). Finaly such multifunctions had been invesligated in relation to the extended concept of lower and upper semi- continuous multifunctions introduced for bitopological spaces.

Arya , S. P., and Nour , T. M. (1988)[2] . Introduced some separation axioms that's consider with bitopological spaces and some theorems .

Jeli's , M. (1989) [7] . Introduced some T_i-pairwise continuous functions and studied bitopological separation axioms .

Al-Swidi, L. and Shaker A. Y. (1993) [1] . Studied a Semi compactness in bitopological space .

Jelic , M. (1994) [6] . Studied the nearly PT_i-continuous mapping and the relationship among them .

Bosi , G. (1997)[3] . He was present a separation theorem in pairwise normally preordered bitopological spaces, which slightly generalizes both a well known separation theorem by Nachbin in normally preordered topological spaces, and a separation theorem by Kelly in pairwise normal topological spaces. Based on this result, he gave necessary and sufficient conditions for the existence of semi continuous order-preserving functions on such spaces. Further, he discussed the existence of upper semi continuous order-preserving functions on preordered topological spaces.

Dontcher , J. (1998)[4] . Covered most of recent research on preopen sets .

Tallafaha, A. A. , Al-Bsoul, and Fora A. (1999) [17] . Introduced the concept of being countable dense homogeneous bitopological spaces and they defined several kinds of that concept. They gave some results concerning those bitopological spaces satisfying the axioms $p-T_o$, and $p-T_1$ spaces .

Jaleel , I. D. (2003)[5] . Presents the definition of δ-open set in bitopological spaces and some theorems , and results on δ-open set in bitopological spaces

This thesis consists of three chapters:

In chapter one, we introduce new elementary properties and definitions about $pr-$open, $pr-$closed, $pr-$nhd, $pr-$int, $pr-$ext, $pr-$fr, $pr-$lm and $pr-$closure sets in bitopological space.

In chapter two, we present new definitions and theorems for the separation axioms with respect to $pr-$open set in bitopological space, together with examples.

Chapter three contains some new definitions of $pr-$continuous, pr^*- continuous and $pr^{**}-$continuous functions with respect to topological and bitopological spaces. We study some of their basic properties. The implication of these new continuous functions among themselves and with the separation axioms with respect to $pr-$open sets is given .

This chapter includes some basic definitions, theorems, remarks and examples of $pr-$open set in bitopological space.

Basic Definitions:

1.1 Definition:

Let (X,τ,ρ) be a bitopological space, and A be a subset of X, A is called $pr-$**open set** with respect to the two topological spaces τ and ρ if: $A \subset \text{int}_{\tau}[cl_{\rho}(A)]$

The collection of all $pr-$open sets with respect to the two topologies is denoted by $pr.O(X)$.

1.2 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a\}\}$ and $\rho=\{X,\phi,\{a\},\{a,b\}\}$ $(X,\tau),(X,\rho)$be two topological spaces, then (X,τ,ρ) is a bitopological space. The family of all $pr-$open sets of X is:

$pr.O(X)=\{X,\phi,\{a\},\{a,b\},\{a,c\},\}$

If we have $Y=\{a\}$, then $cl_{\rho}(\{a\})=X$

$\text{int}_{\tau}(X)=X$, therefore , $\{a\}\subset X$.

Hence $\{a\}$ is $pr-$open set in (X,τ,ρ) . In general, any bitopological space X,ϕ are clearly $pr-$open sets.

1.3 Remark:

a. Any $\rho-$open set is not necessary to be $pr-$open set. [i.e the topology ρ is not necessary to be subcollection of $pr-$open of X].

b. Any $pr-$open set is not necessary to be of ($\tau-$ open, $\rho-$open) set [i.e the family of $pr-$-open sets is not be necessary to be sub collection from any one of the two topologies τ and ρ.

The following example explains the part (a) of the Remark (1.3).

1.4 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a\}\}$, $\rho=\{X,\phi,\{b\},\{c\},\{b,c\}\}$ $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space. So,

$pr.O(X)=\{X,\phi,\{a\},\{b,c\}\}$

Clearly , ρ is not subcollection of $pr-$open set of X .

The following example explains the part (b) of the Remark (1.3).

1.5 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{c\}\}$, $\rho=\{X,\phi,\{c\},\{b,c\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space. The family of all $pr-$open sets of X is:

$pr.O(X)=\{X,\phi,\{c\},\{a,c\},\{b,c\}\}$

Clearly , $pr-$open set of X is not subcollection from any one of the two topologies τ and ρ.

1.6 Definition:

a. (X,τ,ρ) is called **discrete bitopological space** with respect to $pr-$open if $pr.O(X)$ contains all subsets on X .
b. (X,τ,ρ) is called **indiscrete bitopological space** with respect to $pr-$open if $pr.O(X)=\{X,\phi\}$.

1.7 Remark:

Let X be any finite set, τ and ρ are two topological spaces. If Ω is any topology , neither discrete nor indiscrete, then:

Case	τ	ρ	$pr.O(X)$
1	discrete top.	Ω	contains all subsets on X
2	Ω	discrete top.	Ω
3	indiscrete top.	Ω	contains subsets on X neither discrete nor indiscrete topology
4	Ω	indiscrete top.	contains all subsets on X
5	discrete top.	discrete top.	contains all subsets on X
6	indiscrete top.	indiscrete top.	contains all subsets on X
7	discrete top.	indiscrete top.	contains all subsets on X
8	indiscrete top.	discrete top.	$\{X,\phi\}$

1.8 Remark:

The intersection of any $pr-$open sets is not necessary a $pr-$open set. So the family of all $pr-$open sets of X does not represent a topology on X .

The following example explains the above remark.

1.9 Example:

The example shows that there are two $pr-$open sets, but the intersection is not $pr-$open set.

Let $X=\{a,b,c\}, \tau=\{X,\phi,\{a\},\{a,b\}\}$, $\rho=\{X,\phi,\{c\},\{a,c\}\}$ $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, then:
$pr.O(X)=\{ X,\phi,\{a\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$,

Clearly that , $\{a,b\},\{b,c\}$ are two $pr-$open sets, but $\{a,b\}\bigcap\{b,c\}=\{b\}$is not $pr-$open set.

1.10 Proposition:

The union of any two $pr-$open sets is $pr-$open set with respect to bitopological space.

Proof:

Let A and B be two $pr-$open sets with respect to bitopological space (X,τ,ρ).

(i.e) $A \subset \text{int}_{\tau}[cl_{\rho}(A)]$ and $B \subset \text{int}_{\tau}[cl_{\rho}(B)]$

$A \cup B \subset \text{int}_{\tau}[cl_{\rho}(A)] \cup \text{int}_{\tau}[cl_{\rho}(B)] \subset \text{int}_{\tau}[cl_{\rho}(A) \cup cl_{\rho}(B)]$

Since $cl_{\rho}(A) \cup cl_{\rho}(B) = cl_{\rho}(A \cup B)$

Then $A \cup B \subset \text{int}_{\tau}[cl_{\rho}(A \cup B)]$

So any two $pr-$open sets also $pr-$open set with respect to bitopological space □

1.11 Definition:

Let (X,τ,ρ) be a bitopological space. A subset A of X is called $pr-$**closed set** of X iff the complement of A is $pr-$open set of X, and the collection of all $pr-$closed sets with respect to the two topologies τ and ρ denoted by $pr.C(X)$.

1.12 Remark:

a. Any ρ-closed set is not necessary to be $pr-$closed set [i.e the closed sets of the topology ρ is not necessary to be sub collection of $pr-C(X)$].

b. Any $pr-$closed set is not necessary to be ($\tau-$closed, $\rho-$closed) set [i.e the family $pr-C(X)$ is not necessary to be subcollection from one of the closed sets of any of the two topologies τ and ρ].

The following example explains the part (a) of Remark (1.12).

1.13 Example:

Let $X=\{a,b,c\}, \tau=\{X,\phi,\{a\}\}$, and $\rho=\{X,\phi,\{b\},\{c\},\{b,c\}\}$ $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{a\},\{b,c\}\}$,

Hence , the set of all $pr-$closed sets of X is:

$pr-C(X)=\{X,\phi,\{b,c\},\{a\}\}$.

If we take $A=\{a,c\}$ as a $\rho-$closed set of X, but A is not a subcollection of $pr-C(X)$.

The following example explains the part (b) of Remark (1.12).

1.14 Example:

Let $X=\{a,b,c\}=$, $\tau=\{X,\phi,\{a\},\{a,c\}\}$, $\rho=\{X,\phi,\{a,c\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, then:

$pr.O(X)=\{X,\phi,\{a\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$,

Hence the set of all $pr-$closed sets of X is:

$pr-C(X)=\{X,\phi,\{b,c\},\{a,b\},\{c\},\{b\},\{a\}\}$.

If we take $A=\{a,b\}$ as a $pr-$closed set of X, but A is not to be (τ-closed, or ρ-closed) set.

1.15 Definition:

Let (X,τ,ρ) be a bitopological space, and $x\in X$, a subset N of X is said to be $pr-$**nhd** of a point x with respect to bitopological space (X,τ,ρ) iff there exists a $pr-$open set $U\in pr.O(X)$ such that $x\in U\subseteq N$.

The set of all $pr-$nhds of a point x is denoted by $pr-N(x)$.

1.16 Remark:

If x is any point of X, then $pr-N(x)$ is not necessary to be equal to one of $\tau-N(x)$ [nhd system with respect to the topology τ], or $\rho-N(x)$ [nhd system with respect to the topology ρ].

1.17 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a\}\}$, $\rho=\{X,\phi,\{b\},\{c\},\{b,c\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space.

The family of all $pr-$open sets of X is:

$pr.O(X)=\{X,\phi,\{a\},\{b,c\}\}$,so :

$pr.N(a)=\{X,\{a\},\{a,b\},\{a,c\}\}$

$pr.N(b)=\{X,\{b,c\}\}$

$pr.N(c)=\{X,\{b,c\}\}$, and

$\tau-N(a)=\{X,\{a\},\{a,b\},\{a,c\}\}$

$\tau-N(b)=\{X\}$

$\tau-N(c)=\{X\}$

$\rho-N(a)=\{X\}$

$\rho-N(b)=\{X,\{b\},\{a,b\},\{b,c\}\}$

$\rho-N(c)=\{X,\{c\},\{a,c\},\{b,c\}\}$

1.18 Definition:

Let (X,τ,ρ) be a bitopological space, and $A\subseteq X$. A point $x\in X$ is said to be $pr-$**interior** point of A with respect to two topologies τ and ρ iff there exists a $pr-$open set B such that $x\in B\subseteq A$. The set of all $pr-$interior points of A with respect to two topologies τ and ρ denoted by $pr-\text{int}\,(A)$.

1.19 Remark:

If A be any subset of X, then $pr-\text{int}\,(A)$ is not necessary to be equal to one of $\tau-\text{int}(A)$, $\rho-\text{int}\,(A)$.

The following example explains the above remark.

1.20 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a\}\}$, $\rho=\{X,\phi,\{a\},\{a,b\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space. The family of all $pr-$open sets of X is:

$pr.O(X)=\{X,\phi,\{a\},\{a,b\},\{a,c\}\}$

If we take $A=\{a,c\}$, then $pr-\text{int}(\{a,c\})=\{a,c\}$, and $\tau-\text{int}(\{a,c\})=\{a\}$, $\rho-\text{int}(\{a,c\})=\{a\}$

Clearly $pr-\text{int}(\{a,c\})$ is not equal to any of $\tau-\text{int}(\{a,c\})$, $\rho-\text{int}(\{a,c\})$.

1.21 Remark:

If $x\in pr-\text{int}(A)$, then A is $pr-$nbd of x.

Proof:

Let $x\in pr-\text{int}(A)$, then by Definition (1.18), there exists a $pr-$open set G such that $x\in G\subseteq A$. Hence A is $pr-$nbd of x □

1.22 Definition:

Let (X,τ,ρ) be a bitopological space, and $A\subset X$. A point $x\in X$. x is said to be $pr-$**exterior** point of A with respect to the two topologies τ and ρ iff it is a $pr-$interior point of the complement of A, that is, iff there exists a $pr-$open set B such that $x\in B\subseteq A^c$ or equivalently $x\in B$ and $B\bigcap A=\phi$.

The set of all $pr-$exterior points of A with respect to the two topologies τ and ρ is denoted by $pr-$ext (A).

1.23 Remark:

If A is any subset of X, then $pr-$ext (A) is not necessary equal to one of the τ-ext (A), or ρ-ext (A), as in the following example.

1.24 Example:

Let $X=\{a,b,c,d\}$, $\tau=\{X,\phi,\{b\},\{c\},\{b,c\}\}$, $\rho=\{X,\phi,\{a\},\{a,b\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{a,b,c\},\{a,b,d\},\{a,c,d\}\}$

If we take $A=\{b,d\}$, then $pr-$ext $(\{b,d\})=\{a,c\}$, but τ-ext $(\{b,d\})=\{c\}$, and ρ-ext $(\{b,d\})=\{a\}$.

Clearly , $pr-$ext $(\{b,d\})$ is not equal to any one of τ-ext $(\{b,d\})$, or ρ-ext $(\{b,d\})$.

1.25 Remark:

Let (X,τ,ρ) be a bitopological space, and $A\subseteq X$. Then:

1. $pr-$ext $(A)=pr-$int(A^c).
2. $pr-$ext $(A^c)=pr-$int$(A^{cc})=pr-$int (A).

It follows from the Definition (1.22).

1.26 Definition:

A point x of a bitopological space (X,τ,ρ) is said to be a $pr-$**frontier** point (or $pr-$boundary point) of a subset A of X with respect to the two topologies τ and ρ iff it is neither $pr-$interior nor $pr-$exterior point of A. The set of all $pr-$frontier points of A is denoted by $pr-fr(A)$. [i.e $\forall x\in X\ \&\ \forall U_X\in pr.O(X)\ni U\cap A\neq\phi\ \&\ U\cap A^c\neq\phi$].

1.27 Remark:

If A be any subset of X, then $pr-fr(A)$ is not necessary to be any one of the $\tau-fr(A)$, or $\rho-fr(A)$.

1.28 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{b\},\{a,b\}\}$ and $\rho=\{X,\phi,\{c\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$.

If we take $A=\{b,c\}$, then

$pr-\text{int}(\{b,c\})=\{b,c\}$, $pr-\text{ext}(\{b,c\})=\{a\}$, $pr-fr(\{b,c\})=\phi$

τ-int$(\{b,c\})=\{b\}$, τ-ext $(\{b,c\})=\phi$,

Also $\tau-fr(\{b,c\})=\{a,c\}$

ρ-int $(\{b,c\})=\{c\}$, ρ -ext $(\{b,c\})=\phi$,

therefore $\rho-fr(\{b,c\})=\{a,b\}$

From above, we see $fr-(\{b,c\})$ not equal to any one of $\tau-fr(\{b,c\})$, and $\rho-fr(\{b,c\})$.

1.29 Definition:

Let (X,τ,ρ) be a bitopological space, a point x is called $pr-$**limit point** of subset A of X with respect to two topologies τ and ρ iff for each a $pr-$open set B containing x contains another point different from x in A, that is $(B/\{x\})\bigcap A\neq\phi$. The set of all $pr-$limit points of A be denoted by $pr-lm\,(A)$.

1.30 Remark:

If A is any subset of X, then $pr-lm(A)$ is not necessary equal to any one of the $\tau-lm\,(A)$, or $\rho-lm\,(A)$.

The following example explains the above remark.

1.31 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a\}\}$ and $\rho=\{X,\phi,\{b\},\{c\},\{b,c\}\}$

$(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{a\},\{b,c\}\}$.

If we take $A=\{a,c\}$, then $pr-lm\,(A)=\{b\}$ and

$\tau-lm\;A=\{b,c\}$, $\rho-lm\,(A)=\{a\}$

Clearly that $pr-lm\,(A)$ is not equal to any one of the $\tau-lm\,(A)$, $\rho-lm\,(A)$.

1.32 Definition:

Let (X,τ,ρ) be a bitopological space, and $A\subset X$, the intersection of all $pr-$closed sets containing A is called $pr-$**closure** of A, and is denoted by $pr-cl(A)$; i.e

$pr-cl(A)=\bigcap\{F:F$ is $pr-$closed, $A\subset F\}$.

1.33 Remark:

If A is any subset of X, then $pr-cl(A)$ is not necessary equal to any one of $\tau-cl(A),\ \rho-cl(A)$.

The following example explains the above remark.

1.34 Example:

Let $X=\{a,b,c,d\}$, $\tau=\{X,\phi,\{b\},\{c\}\{b,c\}\}$and $\rho=\{X,\phi,\{a\},\{a,d\}\}$.

$(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space. Hence:

$pr.O(X)=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},(a,b,c\},\{a,b,d\},\{a,c,d\}\}$

$pr.C(X)=\{X,\phi,\{b,c,d\},\{a,c,d\},\{a,b,d\},\{c,d\},\{b,d\},\{b,c\},\{a,d\},\{d\},\{c\},\{b\}\}$

$\tau-C(X)=\{\phi,X,\{a,c,d\},\{a,b,d\},\{a,d\}\}$

$\rho-C(X)=\{\phi,X,\{b,c,d\},\{b,c\}\}$

If we take $A=\{c,d\}$, then

$pr-cl(A)=\{b,c,d\}\bigcap\{a,c,d\}\bigcap\{c,d\}\bigcap X=\{c,d\}$

and $\tau-cl(A)=\{a,c,d\}$, $\rho-cl(A)=\{b,c,d\}$

Clearly that $pr-cl(A)$ is not equal to any one of $\tau-cl(A),\ \rho-cl(A)$.

The following some important theorems that's consider with $pr-$int, $pr-$ext, $pr-$fr, $pr-$cl and $pr-$nhd are discussed .

1.35 Theorem:

Let (X,τ,ρ) be a bitopological space, for each $x\in X$, $pr-N(x)$ be the collection of all $pr-$nhds of x, then:

i.) $\forall x\in X$, $pr-N(x)\neq\phi$ (i.e every point x has at least one $pr-$nhd).

ii.) $\forall N\in pr-N(x)$, then $x\in N$ (i.e every pr-nhd of x contains x).

iii.) If $N\in pr-N(x)\ \&\ N\subset M$, then $M\in pr-N(x)$ (i.e every set containing a $pr-$nhd of x is a $pr-$nhd of x).

iv.) If $N \in pr-N(x)$, then there exists $M \in pr-N(x)$ such that $M \subset N$ and $M \in pr-N(y)$, $\forall y \in M$ (i.e if N is a $pr-$nhd of x, then there exists a $pr-$nhd M of x which is a subset of N such that M is a $pr-$nhd of each of its points).

Proof:

i.) Since X is a $pr-$open set, it is a $pr-$nhd of every $x \in X$. Hence there exists at least one $pr-$nhd (namely x) for every $x \in X$. Hence $pr-N(x) \neq \phi \; \forall \; x \in X$.

ii.) If $N \in pr-N(x)$, then N is $pr-$nhd of x. So by definition of $pr-$nhd, $x \in N$.

iii.) If $N \in pr-N(x)$, then there is a $pr-$open set B such that $x \in B \subseteq N$. Since $N \subset M$, $x \in B \subset M$ and so M is a $pr-$nhd of x. Hence $M \in pr-N(x)$.

iv.) If $N \in pr-N(x)$, then there exists a $pr-$open set M such that $x \in M \subseteq N$. Since M is a $pr-$open set, it is a $pr-$nhd of each of its points. Therefore $M \in pr-N(y)$, $\forall y \in M$ ☐

1.36 Theorem:

Let (X, τ, ρ) be a bitopological space, and $A \subset X$, then: $pr-\text{int}(A) = \bigcup\{G \in pr.O(X) : G \subseteq A\}$.

Proof:

If $x \in pr-\text{int}(A)$ iff A is a $pr-$nhd of x iff there exists a $pr-$open set G such that $x \in G \subset A$ iff $x \in \bigcup\{G \in pr.O(X) : G \subseteq A\}$.

Hence $pr-\text{int}(A) = \bigcup\{G \in pr.O(X) : G \subseteq A\}$ ☐

1.37 Theorem:

Let (X, τ, ρ) be a bitopological space, and A, B be any subsets of X, then:

i.) $pr-\text{int}(X) = X$, $pr-\text{int}(\phi) = \phi$.

ii.) $pr-\text{int}(A) \subset A$.

iii.) If $A \subset B$, then $pr-\text{int}(A) \subseteq pr-\text{int}(B)$

iv.) $pr-\text{int}(A \cap B) \subseteq pr-\text{int}(A) \cap pr-\text{int}(B)$.

v.) $pr-\text{int}(A) \cup pr-\text{int}(B) \subseteq pr-\text{int}(A \cup B)$.

Proof:

i.) Since X is a $pr-$open set with respect to the two topologies τ and ρ, and $X \subseteq X$, so $\forall x \in X \; \exists$ $pr-$open set $X \ni x \in X \subseteq X$. Hence $pr-\text{int}(X) = X$. Also $pr-\text{int}(\phi) = \phi$.

ii.) If $x \in pr-\text{int}(A)$, then x is a $pr-$interior point of A with respect to two topologies τ and ρ , so $\exists G \in pr-O(X) \ni x \in G \subset A$, so $x \in A$.
Hence, $pr-\text{int}(A) \subset A$.

iii.) If $x \in pr-\text{int}(A)$, then x is an $pr-$interior point of A with respect to the two topologies τ and ρ , so $\exists G \in pr.O(X) \ni x \in G \subseteq A$ and $A \subset B$, so $x \in G \subset B$ this implies that $x \in pr-\text{int}(B)$, hence $pr-\text{int}(A) \subseteq pr-\text{int}(B)$

iv.) Since $A \cap B \subseteq A$, and $A \cap B \subseteq B$, [by part (iii) above], we have $pr-\text{int}(A \cap B) \subset pr-\text{int}(A)$,
$pr-\text{int}(A \cap B) \subseteq pr-\text{int}(B)$.
Hence , $pr-\text{int}(A \cap B) \subseteq pr-\text{int}(A) \cap pr-\text{int}(B)$.

v.) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, so by [a part (iii) above], we have:
$pr-\text{int}(A) \subseteq pr-\text{int}(A \cup B)$
$pr-\text{int}(B) \subseteq pr-\text{int}(A \cup B)$
Hence $pr-\text{int}(A) \cup pr-\text{int}(B) \subseteq pr-\text{int}(A \cup B)$ □

The following example explains the converse of the parts (ii) and (iii) of the previous theorem are not true.

1.38 Example:

Let $X=\{a,b,c,d\}, \tau=\{X,\phi,\{a\},\{b\},\{a,b\}\}$ and $\rho=\{X,\phi,\{b\},\{c\},\{b,c\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, such that:

$pr.O(X)=\{X,\phi,\{a\},\{b\},\{a,b\},\{b,c\},\{a,b,c\},\{b,c,d\}\}$.

If we take $A=\{a,c\}$, then $pr-\text{int}(A)=\{a\}$, clearly, $A\not\subset pr-\text{int}(A)$. And if we take $B=\{a,b,c\}$ then $pr-\text{int}(B)=\{a,b,c\}$. Clearly, $A\subset B$ but $pr-\text{int}(B)\not\subset pr-\text{int}(A)$.

1.39 Remark:

The converse of the part (iv) in the previous theorem is not true by Remark (1.8) .

The following example explains the converse of the part (v) in the previous theorem is not true.

1.40 Example:

Let $X=\{a,b,c\}, \tau=\{X,\phi,\{b\},\{a,b\},\{b,c\}\}, \rho=\{X,\phi,\{a\},\{c\},\{b,c\},\{a,c\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$

If we take $A=\{b\}$, and $B=\{a\}$, then:

$(A\bigcup B)=\{a,b\}$ and $pr-\text{int}(A\bigcup B)=pr-\text{int}(\{a,b\})=\{a,b\}$,but $pr-\text{int}(A)\bigcup pr-\text{int}(B)=pr-\text{int}(\{b\})\bigcup pr-\text{int}(\{a\})$

$=\{b\}\bigcup\phi=\{b\}$

Hence , $pr-\text{int}(A\bigcup B)\not\subset pr-\text{int}(A)\bigcup pr-\text{int}(B)$.

1.41 Theorem: This theorem is equivalent to definition (1-22) .

Let (X,τ,ρ) be a bitopological space, and $A\subset X$, then:

$pr-\text{ext }(A)=\bigcup\{G\in pr-O(X):G\subseteq A^c\}$.

Proof:

By Remark (1.25), $pr-\text{ext}(A) = pr-\text{int}(A^c)$. But by Theorem (1.36), $pr-\text{int}(A^c) = \bigcup\{G \in pr.O(X) : G \subseteq A^c\}$

Hence $pr-\text{ext}(A) = \bigcup\{G \in pr.O(X) : G \subseteq A^c\}$ □

1.42 Theorem:

Let (X, τ, ρ) be a bitopological space, and A, B be subsets of X, then:

i.) $pr-\text{ext}(X) = \phi$, $pr-\text{ext}(\phi) = X$.

ii.) $pr-\text{ext}(A) \subset A^c$.

iii.) If $A \subset B$, then $pr-\text{ext}(B) \subset pr-\text{ext}(A)$

iv.) $pr-\text{ext}(A \bigcup B) \subset pr-\text{ext}(A) \bigcap pr-\text{ext}(B)$.

Proof:

i.) By Remark (1.25) and Theorem (1.37):

$pr-\text{ext}(X) = pr-\text{int}(X^c) = pr-\text{int}(\phi) = \phi$.

Also by Remark (1.25) and Theorem (1.37) we have

$pr-\text{ext}(\phi) = pr-\text{int}(\phi^c) = pr-\text{int}(X) = X$.

ii.) By Remark (1.25 and part (ii) of Theorem (1.37), we get

$pr-\text{ext}(A) = pr-\text{int}(A^c) \subset A^c$.

iii.) If $A \subset B$, then $B^c \subset A^c$, by part(iii) of Theorem (1.37) we have $pr-\text{int}(B^c) \subset pr-\text{int}(A^c)$, therefore $pr-\text{ext}(B) \subset pr-\text{ext}(A)$.

iv.) By Remark (1.25) and De-Morgan Law, $pr-\text{ext}(A \bigcup B) = pr-\text{int}(A \bigcup B)^c =$ $pr-\text{int}(A^c \bigcap B^c) \subset (pr-\text{int}(A^c) \bigcap pr-\text{int}(B^c))$[by part(iv) of Theorem (1.37)] $pr-\text{ext}(A) \bigcap pr-\text{ext}(B)$.

Hence $pr-\text{ext}\ (A \bigcup B) \subset pr-\text{ext}(A) \bigcap pr-\text{ext}(B)$ □

The following example explains the converse of the part (ii) in the previous theorem is not true.

1.43 Example:

Let $X=\{a,b,c\}, \tau=\{X,\phi,\{a\},\{c\},\{a,c\}\}, \rho=\{X,\phi,\{a\},\{b\},\{a,b\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so ,

$pr.O(X)=\{X,\phi,\{a\},\{c\},\{a,c\},\{a,b\}\}$

If we take $A=\{a\}$, then $A^c=\{b,c\}$, and,

$pr-\text{ext}(A)=pr-\text{int}(A^c)=pr-\text{int}(\{b,c\})=\{c\}$

Clearly $A^c \not\subset pr-\text{ext}(A)$.

The following example explains the converse of the part (iii) in the previous theorem is not true.

1.44 Example:

Let $X=\{a,b,c\}, \tau=\{X,\phi,\{a\},\{a,b\}\}, \rho=\{X,\phi,\{b\},\{b,c\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so,

$pr.O(X)=\{X,\phi,\{a\},\{b\},\{a,b\},\{b,c\}\}$

If we take $A=\{c\}$, $B=\{b\}$, then we have $pr-\text{ext}(B)\subset pr-\text{ext}(A)$

Since $pr-\text{ext}(B)=pr-\text{ext}(\{b\})=pr-\text{int}(\{b\}^c)=pr-\text{int}(\{a,c\})=\{a\}$,

and $pr-\text{ext}(A)=pr-\text{ext}(\{c\})=pr-\text{int}(\{c\}^c)=pr-\text{int}(\{a,b\})=\{a,b\}$

Hence $pr-\text{ext}(B)\subset pr-\text{ext}(A)$, but $A\not\subset B$.

The following example explains the converse of the part (iv) in the previous theorem is not true .

1.45 Example:

Let $X=\{a,b,c\}, \tau=\{X,\phi,\{b\},\{a,b\}\}, \rho=\{X,\phi,\{a,b\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so,

$pr.O(X)=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}\}$

If we take $A=\{a\}$, $B=\{b\}$, so

$pr-\text{ext}(A) = pr-\text{int}(A^c) = pr-\text{int}(\{b,c\}) = \{b,c\}$
$pr-\text{ext}\ (B) = pr-\text{int}(B^c) = pr-\text{int}(\{a,c\}) = \{a,c\}$, hence ,
$pr-\text{ext}\ (A) \bigcap pr-\text{ext}\ (B) = \{b,c\} \bigcap \{a,c\} = \{c\}$
$pr-\text{ext}\ (A \bigcup B) = pr-\text{ext}\ (\{a,b\}) = pr-\text{int}(\{c\}) = \phi$.
Hence , $pr-\text{ext}\ (A) \bigcap pr-\text{ext}\ (B) \not\subset pr-\text{ext}\ (A \bigcup B)$.

1.46 Theorem:

Let (X,τ,ρ) be a bitopological space, and $A \subset X$, then the point x in X is a $pr-$frontier point of A iff every $pr-$nhd of x intersects both A and A^c.

Proof:

Suppose that $x \in pr-fr(A)$ iff $x \notin pr-\text{int}\ (A)$ and $x \notin pr-ext(A) = pr-\text{int}(A^c)$ iff neither A nor A^c is a $pr-$nhd of x iff no $pr-$nhd of x can be contained in A or A^c iff every $pr-$nhd of x intersects both A and A^c. From Theorem (1.35) part (iii) every set containing a $pr-$nhd of x is a $pr-$nhd of x □

1.47 Corollary:

$pr-fr(A) = pr-fr(A^c)$

Proof:

Suppose that $x \in pr-fr(A)$ iff every $pr-$nhd of x intersects both A and A^c iff every $pr-$nhd of x intersects both $(A^c)^c$ and A^c [since $(A^c)^c = A$] iff $x \in pr-fr(A^c)$ □

1.48 Theorem:

Let A be any subset of a bitopological space (X,τ,ρ) then $pr-\text{int}(A)$, $pr-\text{ext}(A)$ and $pr-fr(A)$ are disjoint and $X = pr-\text{int}(A) \cup pr-\text{ext}(A) \cup pr-fr(A)$.

Proof:

By Definition (1.22), $pr-\text{ext}(A) = pr-\text{int}(A^c)$. Also $pr-\text{int}(A) \subset A$ and $pr-\text{int}(A^c) \subset A^c$.

Since $A \cap A^c = \phi$, it follows that
$pr-\text{int}(A) \cap pr-\text{ext}(A) = pr-\text{int}(A) \cap pr-\text{int}(A^c) = \phi$.

Again by definition of $pr-$frontier, we have:
$x \in pr-fr(A)$ iff $x \notin pr-\text{int}(A)$ and $x \notin pr-\text{ext}(A)$ iff $x \notin [pr-\text{int}(A) \cup pr-\text{ext}(A)]$ iff $x \in [pr-\text{int}(A) \cup pr-\text{ext}(A)]^c$.

Thus , $pr-fr(A) = [pr-\text{int}(A) \cup pr-\text{ext}(A)]^c$

It follows that $pr-fr(A) \cap pr-\text{int}(A) = \phi$, and $pr-fr(A) \cap pr-\text{ext}(A) = \phi$,and
$X = pr-\text{int}(A) \cup pr-\text{ext}(A) \cup pr-fr(A)$ □

1.49 Theorem:

Let (X, τ, ρ) be a bitopological space, and $A \subset X$, then:

i.) $pr-cl(A)$ is the smallest $pr-$closed set containing A.

ii.) A is $pr-$closed iff $pr-cl(A) = A$.

Proof:

i.) This follows directly from the Definition (1.32).

ii.) If A is $pr-$closed, then A itself is the smallest $pr-$closed set containing A , hence $pr-cl(A) = A$.

Conversely, if $pr-cl(A) = A$, by (i), $pr-cl(A)$ is $pr-$closed , so , A is also $pr-$closed □

1.50 Theorem:

Let (X, τ, ρ) be a bitopological space, and A, B be any subsets of X , then:

i.) $pr-cl(\phi) = \phi$, $pr-cl(X) = X$.

ii.) $A \subset pr-cl(A)$.

iii.) If $A \subset B$, then $pr-cl(A) \subset pr-cl(B)$.

iv.) $pr-cl(A \cup B) = pr-cl(A) \cup pr-cl(B)$.

v.) $pr-cl(A \cap B) \subset pr-cl(A) \cap pr-cl(B)$.

vi.) $pr-cl[pr-cl(A)] = pr-cl(A)$.

Proof:

i.) Since ϕ and X are $pr-$closed sets, by Theorem (1.49) we have $pr-cl(\phi)=\phi$ and $pr-cl(X)=X$.

ii.) By Theorem (1.49) part (i), we obtain $A\subset pr-cl(A)$.

iii.) By a part (ii) above, $B\subset pr-cl(B)$, since $A\subset B$, then $A\subset pr-cl(B)$, but $pr-cl(B)$ is $pr-$closed set, thus $pr-cl(B)$ is $pr-$closed set containing A.
Since $pr-cl(A)$ is the smallest $pr-$closed set containing A, hence $pr-cl(A)\subset pr-cl(B)$.

iv.) Since $A\subset A\bigcup B$ and $B\subset A\bigcup B$, by a part (iii) above, we have $pr-cl(A)\subset pr-cl(A\bigcup B)$ and
$pr-cl(B)\subset pr-cl(A\bigcup B)$. hence
$pr-cl(A)\bigcup pr-cl(B)\subset pr-cl(A\bigcup B)$. …… (1-1)

since $pr-cl(A)$ and $pr-cl(B)$ are $pr-$closed sets and $pr-cl(A)\bigcup pr-cl(B)$ is also $pr-$closed, [by a part (ii) above].

$A\subset pr-cl(A)$, $B\subset pr-cl(B)$.

This implies that $A\bigcup B\subset pr-cl(A)\bigcup pr-cl(B)$.

Thus $pr-cl(A)\bigcup pr-cl(B)$ is $pr-$closed set containing $A\bigcup B$, since $pr-cl(A\bigcup B)$ is the smallest $pr-$closed set containing $A\bigcup B$.

Therefore, $pr-cl(A\bigcup B)\subset pr-cl(A)\bigcup pr-cl(B)$ .. (1-2)

From (1-1) and (1-2), we have $pr-cl(A\bigcup B)=pr-cl(A)\bigcup pr-cl(B)$

v.) Since $A\bigcap B\subset A$. then $pr-cl(A\bigcap B)\subset pr-cl(A)$ by a part (iii) above and, then $A\bigcap B\subset B$, $pr-cl(A\bigcap B)\subset pr-cl(B)$ by a part (iii) above.

Hence $pr-cl(A\bigcap B)\subset pr-cl(A)\bigcap pr-cl(B)$.

vi.) Since $pr-cl(A)$ is $pr-$closed set, we have by Theorem(1.49) part (ii), $pr-cl[\,pr-cl(A)]=pr-cl(A)$□

The following example explains the converse of the part (ii) of the previous theorem is not true.

<u>1.51 Example:</u>

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{b\},\{b,c\}\}$, $\rho=\{X,\phi,\{a\},\{c\}\{a,c\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space then:

$pr.O(X)=\{X,\phi,\{b\},\{c\}\{a,c\},\{b,c\}\}$,

and $pr.C(X)=\{\phi,X,\{a,c\},\{a,b\},\{b\},\{a\}\}$.

If we take $A=\{c\}$, then $pr-cl(A)=\{a,c\}$.

Hence, it is clearly that, $pr-cl(A)\not\subset A$.

The following example explains the converse of the part (iii) of the previous theorem is not true.

<u>1.52 Example:</u>

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a\}\}$ and $\rho=\{X,\phi,\{a\},\{a,b\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitoplogical space so:

$pr.O(X)=\{X,\phi,\{a\},\{a,b\},\{a,c\}\}$,

and $pr.C(X)=\{\phi,X,\{b,c\},\{c\},\{b\}\}$.

If we take $A=\{b\}$ and $B=\{b,c\}$, then clearly $A\subset B$, and $pr-cl(A)=\{b\}$, $pr-cl(B)=\{b,c\}$.

Hence $pr-cl(B)\not\subset pr-cl(A)$.

The following example explains the converse of the part (v) in the previous theorem is not true.

<u>1.53 Example:</u>

Let $X=\{a,b,c,d\}$, $\tau=\{\phi,X,\{d\}\},$, $\rho=\{X,\phi,\{a,c\},\{b,d\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, then:

$pr.O(X)=\{X,\phi,\{d\},\{a,b\},\{a,d\},\{b,c\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$

Therefore the family of all $pr-$closed sets is:

$pr.C(X)=\{X,\phi,\{a,b,c\},\{c,d\},\{b,c\},\{a,d\},\{a,b\},\{d\},\{c\},\{b\},\{a\}\}$

If we take $A=\{a,b,d\}$, and $B=(c,d\}$, then: $A\bigcap B=\{d\}$, Also $pr-cl(A\bigcap B)=pr-cl(\{d\})=\{d\}$, but $pr-cl(A)\bigcap pr-cl(B)=X\bigcap\{c,d\}=\{c,d\}$,it is clear that , $pr-cl(A)\bigcap pr-cl(B)\not\subset pr-cl(A\bigcap B)$.

1.54 Definition:

Let (X,τ,ρ) be a bitopological space, and Y be a subset of X. The **relative bitopological space for** Y is denoted by (Y,τ_Y,ρ_Y), Such that:

$\tau_Y=\{G\bigcap Y: G\in\tau\}$ and

$\rho_Y=\{H\bigcap Y: H\in\rho\}$

Then (Y,τ_Y,ρ_Y) is called the subspace of bitopological space (X,τ,ρ), the relative bitopological space for Y with respect to $pr-$open sets is the collection $pr.O(X)_Y$ given by:

$pr.O(X)_Y=\{G\bigcap Y: G\in pr.O(X)\}$

1.55 Remark:

The collection of $pr.O(X)_Y$ does not represent a topology on Y.

1.56 Example:

Let $X=\{a,b,c,d\}$, $\tau=\{X,\phi,\{d\}\}$, $\rho=\{X,\phi,\{a,c\},\{b,d\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{d\},\{a,b\},\{a,d\},\{b,c\},\{c,d\},\{a,b,c\},\{a,b,d\}\{a,c,d\},\{b,c,d\}\}$

And let $Y=\{a,b,d\}$, then $\tau_Y=\{Y,\phi,\{d\}\}$,

and $\rho_Y=\{X,\phi,\{a\},\{b,d\}\}$

$(Y,\tau_Y),(Y,\rho_Y)$ are two topological spaces, then (Y,τ_Y,ρ_Y) is a subspace of bitopological space (X,τ,ρ), such that, $pr.O(X)_Y=\{Y,\phi,\{d\},\{a,b\},\{a,d\},\{b\},\{b,d\}\}$.

Clearly that $pr.O(X)_Y$ does not represent a topology on Y.

1.57 Theorem:

Let (Y, τ_Y, ρ_Y) be a subspace of bitopological space (X, τ, ρ).

Then:

i.) A subset A of Y is $pr-$closed in Y iff there exists $pr-$closed set F such that $A = F \bigcap Y$.

ii.) For every $A \subset Y$, $pr-cl_Y(A) = pr-cl_X(A) \bigcap Y$.

iii.) A subset M of Y is a $pr-$nhd$_Y$ of a point $y \in Y$ iff $M = N \bigcap Y$ for some $pr-$nhd$_x$ N of y.

iv.) A point $y \in Y$ is a limit point of A in a subspace Y iff y is a limit point of A in a bitopological space X. Further; $pr-lm_Y(A) = pr-lm_X(A) \bigcap Y$.

v.) For every $A \subset Y$, $pr-\text{int}_X(A) \subset pr-\text{int}_Y(A)$

vi.) For every $A \subset Y$, $pr-fr_Y(A) \subset pr-fr_X(A)$.

Proof:

i.) A is $pr-$closed in Y iff $Y-A$ is $pr-$open in Y iff $Y-A = G \bigcap Y$ for some $pr-$open subset G of X iff $A = Y-(G \bigcap Y) = (Y-G) \bigcup Y-Y$ [De-Morgan Law] iff $A = Y-G$ iff $A = Y \bigcap (X-G)$ iff $A = Y \bigcap F$ (where $F = X-G$ is $pr-$closed in X since G is $pr-$open in X).

ii.) By Definition (1.32) $pr-cl_Y(A) = \bigcap\{k : k$ is $pr-$closed in Y and $A \subseteq k\} = \bigcap\{F \bigcap Y : F$ is $pr-$closed in X and $A \subset F \bigcap Y\}$[by (i) above] $= \bigcap\{F \bigcap Y : F$ is $pr-$closed in X and $A \subseteq F\} = \{\bigcap F : F$ is $pr-$closed in X and $A \subseteq F\} \bigcap Y = pr-cl_X(A) \bigcap Y$.

iii.) Let M be $pr-$nhd$_Y$ of y, then there exists a $pr-$open set H in Y such that $y \in H \subset M$, then there exists a $pr-$open set G in X such that $y \in H = G \bigcap Y \subset M$.

Let $N = M \bigcup G$. Then N is a nhd$_x$ of y, since G is a $pr-$open set, such that y $\in G \subset N$. Further:

$N \cap Y = (M \cup G) \cap Y = (M \cap Y) \cup (G \cap Y)$, since $M \subset Y$, so $N \cap Y = M \cup (G \cap Y) = M$ because $G \cap Y \subset M$.

Conversely, let $M = N \cap Y$ for some $pr-$nhd$_X$ of y. Then there exists a $pr-$open set G in X such that $y \in G \subset N$ which implies that $y \in G \cap Y \subset N \cap Y = M$.

Since $G \cap Y$ is $pr-$open set in Y, So M is a $pr-$nhd$_Y$ of y.

iv.) y is a $pr-$limit point of A in Y iff $[M/\{y\}] \cap A \neq \phi$, for every $pr-$nhd$_Y$ M of y iff $[N \cap Y/\{y\}] \cap A \neq \phi$, for every $pr-$nhd$_X$ N of y, by (iii) above iff $[N/\{y\}] \cap A \neq \phi$ iff y is $pr-$limit point of A in X.

v.)Let $x \in pr-$int$_X$ (A), then x is a $pr-$interior point of A, then $\exists$ a $pr-$nhd$_X$ G of $x \ni x \in G \subset A$, so $G \cap y$ is a $pr-$nhd$_Y$ of x, so , $x \in G \cap Y \subset A \cap Y = A$.

Hence $pr-$int$_X$ $(A) \subset pr-$int$_Y$ (A).

vi.) Let $y \in pr-fr_Y(A)$, then y is a $pr-$frontier point of A in y, then by Theorem (1.46) every $pr-$nhd$_Y$ of y intersects both A and $Y - A$.

So $N \cap Y$ intersects both A and $Y - A$, for every $pr-$nhd$_X$ N of y, hence every $pr-$nhd$_X$ N of y intersects both A and $X - A$, then y is a $pr-$frontier point of A in X. Therefore, $y \in pr-fr_X(A)$. Hence $pr-fr_Y(A) \subset pr-fr_X(A)$ □

<u>1.58 Theorem:</u>

Let (Y, τ_Y, ρ_Y) be a subspace of a bitopological space (X, τ, ρ).

If a subset A of Y is $pr-$open ($pr-$closed) in X, then A also $pr-$open ($pr-$closed) in Y.

<u>Proof:</u>

Since $A \subset Y$, we have $A = A \cap Y$, so that A is the intersection Y with a set $pr-$open ($pr-$closed) in X ,namely A.

Hence by the Definition (1.54) and Theorem (1.57) part (i), A is $pr-$open ($pr-$closed) in Y □

The Separation Axioms in bitopological space:

In this chapter, we recall the separation axioms in bitopological space and we study the relationship among them with some examples.

2.1 Definition:

Let (X,τ,ρ) be a bitopological space, then (X,τ,ρ) is called $pr-T_o$**-space** iff for each pair of points $x,y\in X$, such that $x\neq y$, there exists $pr-$open set G containing x but not y or $pr-$open set H containing y but not x.

2.2 Remark:

If a bitopological space (X,τ,ρ) is $pr-T_o$-space, then any of two topological spaces (X,τ) and (X,ρ) is not necessary to be T_o-space , and the converse also possible .

2.3 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a\}\}$ and $\rho=\{X,\phi,\{b\}\}$

$(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{a\},\{b\},\{a,b\},\{b,c\}\}$

If we take a and b, $a\neq b$, then there exists $pr-$open set $\{b\}$ contains b but not contains a, and similarly the order cases $b\neq c$, $a\neq c$, therefore (X,τ,ρ) is $pr-T_o$-space, but $(X,\tau),(X,\rho)$ are not T_o-spaces.

2.4 Remark:

If any of two topological spaces $(X,\tau),(X,\rho)$ is T_o-space, then a bitopological space (X,τ,ρ) is not necessary to be $pr-T_o$-space.

2.5 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a\}\}$ $\rho=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$ $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, then:

$pr.O(X)=\{X,\phi,\{a\}\}$

If we take b and c, $b\neq c$, then there is no $pr-$open set such that it contains b but not c or conversely, and it contains c but not b.

Therefore , (X,τ,ρ) is not $pr-T_o$-space, but it is clear that (X,ρ) is T_o-space.

2.6 Theorem:

A bitopological space (X,τ,ρ) is $pr-T_o$-space iff for each distinct points x,y in X, $pr-cl(\{x\})\neq pr-cl(\{y\})$.

Proof:

Let $x,y\in X$, such that $x\neq y$ and $pr-cl(\{x\})\neq pr-cl(\{y\})$. Then there exists at least one point z in X, such that, $z\in pr-cl(\{x\})$ but $z\notin pr-cl(\{y\})$.

Suppose $z\in pr-cl(\{x\})$. To show that $x\notin pr-cl(\{y\})$. If $x\in pr-cl(\{y\})$, then $\{x\}\subset pr-cl(\{y\})$, so $pr-cl(\{x\})\subset pr-cl\,(pr-cl(\{y\}))=pr-cl(\{y\})$, hence $z\in pr-cl(\{x\})$, then $z\in pr-cl(\{y\})$ which is contradiction.

Hence $x\notin pr-cl(\{y\})$, consequently $x\in X-pr-cl(\{y\})$ but $pr-cl(\{y\})$ is $pr-$closed, so $X-pr-cl(\{y\})$ is $pr-$open which contains x but not y.

It follows that (X,τ,ρ) is $pr-T_o$-space.

Conversely, since (X,τ,ρ) is $pr-T_o$-space, then for each two distinct points x, $y\in X$, there exists $pr-$open set G such that $x\in G$, $y\notin G$. $X-G$ is closed set which does not contain x but contains y.

By Definition (1-32), $pr-cl(\{y\})$ is the intersection of all $pr-$closed sets which contain $\{y\}$.

Thus, $pr-cl(\{y\})\subset X-G$, then $x\notin X-G$. This implies that $x\notin pr-cl(\{y\})$, so we have $x\in pr-cl(\{x\})$, $x\notin pr-cl(\{y\})$. Therefore $pr-cl(\{x\})\neq pr-cl(\{y\})$ □

The $pr-T_o$-space is hereditary property .

2.7 Theorem:

Every subspace of $pr-T_o$-space is $pr-T_o$-space .

Proof:

Let (Y,τ_Y,ρ_Y) be a subspace of $pr-T_o$-space (X,τ,ρ).

To prove that the subspace is $pr-T_o$-space, let $y_1\neq y_2\in Y$.

Since $Y\subset X$, then $y_1\neq y_2\in X$ and (X,τ,ρ) is $pr-T_o$-space.

There exists $pr-$open set G in X, such that $y_1\in G$, $y_2\notin G$.

So $G\bigcap Y$ is $pr-$open set in Y and $y_1\in G\bigcap Y$, $y_2\notin G\bigcap Y$.

Hence (Y,τ_Y,ρ_Y) is $pr-T_o$-space □

2.8 Definition:

A bitopological space (X,τ,ρ) is called $pr-T_1$**-space** iff for each pair of distinct points x, y in X, there exist two $pr-$open sets G, H such that G contains x but not y and H contains y but not x.

2.9 Remark:

If a bitopological space (X,τ,ρ) is $pr-T_1$**-space**, then any of two topological spaces (X,τ) and (X,ρ) is not necessary to be T_1-space.

2.10 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a\},\{a,c\}\}$ and $\rho=\{X,\phi,\{a,c\}\}$ $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, such that:

$pr.O(X)=\{X,\phi,\{a\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$

If we take a, $b\in X$, such that $a\neq b$, then there are two $pr-$open sets $\{a\}$, $\{b,c\}$; such that $\{a\}$ contains a but not b and $\{b,c\}$ contains b but not a .

Also , a, $c\in X$; such that $a\neq c$, then there are two $pr-$open sets $\{a\}$, $\{c\}$; such that $\{a\}$ contains a but not c and $\{c\}$ contains c but not a.

Also, b, $c\in X$; such that $b\neq c$, then there are two $pr-$open sets $\{a,b\}$, $\{c\}$; such that $\{a,b\}$ contains b but not c and $\{c\}$ contains c but not b.

Therefore, (X,τ,ρ) is $pr-T_1$-space. But, it is clear that $(X,\tau),(X,\rho)$ are not T_1-spaces.

2.11 Remark:

If any of two topological spaces $(X,\tau),(X,\rho)$ is T_1-space, then a bitopological space (X,τ,ρ) is not necessary to $pr-T_1$-space.

2.12 Example:

Let $X=\{a,b,c\},\tau=\{X,\phi,\{a,c\}\}$, $\rho=\{X,\phi,\{a\},\{b,c\},\{a,c\},\{a,b\},\{b\},\{c\}\}$ $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a topological space, such that:

$pr.O(X)=\{X,\phi,\{a,c\}\}$.

If we take a, c, $a\neq c$, then there are no two $pr-$open sets, such that one of them contains a but not c and the other contains c but not a.

Therefore, (X,τ,ρ) is not $pr-T_1$-space, but it is clear that (X,ρ) is a T_1-space.

2.13 Remark:

Every $pr-T_1$-space is $pr-T_o$-space.

Proof:

The proof is as follows . From Definitions of (2-1) and (2-8) □

2.14 Remark:

The converse of the above remark is not true, as the following example .

2.15 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{c\}\}$ and $\rho=\{X,\phi,\{a\}\}$ $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, hence:

$pr.O(X)=\{X,\phi,\{a\},\{c\},\{a,b\},\{a,c\}\}$.

If we take a, and b, $a\neq b$, then we cannot find two $pr-$open sets, such that one of them contains a but not b and the other contains b but not a.

Therefore, (X,τ,ρ) is not $pr-T_1$-space, but it is clear that (X,τ,ρ) is a $pr-T_o$-space.

The $pr-T_1$-space is hereditary property .

2.16 Theorem:

Every subspace of $pr-T_1$-space is $pr-T_1$-space.

Proof:

Let (Y,τ_Y,ρ_Y) be a subspace of $pr-T_1$-space (X,τ,ρ).

To prove that the subspace Y is $pr-T_1$-space.

Let $y_1\neq y_2$, since $Y\subset X$, then $y_1\neq y_2\in X$ and since (X,τ,ρ) is $pr-T_1$-space.

Then there exist two $pr-$open sets G, H in X, such that $y_1\in G$, but $y_2\notin G$; and $y_2\in H$, but $y_1\notin H$.

Then we obtain two sets $G_1=G\bigcap Y$, $H_1=H\bigcap Y$ are $pr-$open sets in Y , we have $y_1\in G_1$, but $y_2\notin G_1$; $y_2\in H_1$, but $y_1\notin H_1$.

Hence (Y,τ_Y,ρ_Y) is $pr-T_1$-space □

2.17 Theorem:

If every singleton subset $\{x\}$ of bitopological space (X,τ,ρ) is $pr-$closed , then (X,τ,ρ) is $pr-T_1$-space.

Proof:

Let $x, y \in X$ such that $x \neq y$. Since $\{x\}$ is $pr-$closed set, then X-$\{x\}$ is $pr-$open set containing y but not x. similarly, X-$\{y\}$ is $pr-$open set containing x but not y.

Hence , (X,τ,ρ) is $pr-T_1$-space □

2.18 Theorem:

A bitopological space (X,τ,ρ) is a $pr-T_1$-space iff $pr-cl\{a\}=\phi$, for each $a \in X$.

Proof:

Let (X,τ,ρ) be a $pr-T_1$-space. Suppose $pr-cl\{a\} \neq \phi$ for some $a \in X$, then there exists a point b, such that $b \in pr-cl\{a\}$, and $a \neq b$.

Since X is a $pr-T_1$-space, then there exists a $pr-$open set G such that $a \notin G$, $b \in G$, thus $G \cap \{a\} = \phi$.

Hence , $b \notin pr-cl\{a\}$, which is contradiction.

Thus $pr-cl\{a\}=\phi$.

Conversely, suppose that $pr-cl\{a\}=\phi$, for each $a \in X$, and let x, $y \in X$, such that $x \neq y$.

Then $x \notin pr-cl\{y\}$, and there exists $pr-$open set G such that $x \in G$ and $G \cap \{y\} = \phi$.

Hence, G contains x but not y, similarly, there exists $pr-$open set H contains y but not x.

Thus (X,τ,ρ) is $pr-T_1$-space □

2.19 Definition:

A bitopological space (X,τ,ρ) is called a $pr-T_2$**-space (**$pr-$ **Hausdorff)** iff for each pair of distinct points x, y in X, there exist two $pr-$open sets G, H such that $x \in G$, $y \in H$, $G \cap H = \phi$.

2.20 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{b\},\{a,b\}\}$ and $\rho=\{X,\phi,\{a,c\}\}$ $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$. It is clear that (X,τ,ρ) is $pr-T_2$-space.

2.21 Remark:

If a bitopological space (X,τ,ρ) is $pr-T_2$-space, then any one of these topological spaces $(X,\tau),(X,\rho)$ is not necessary to $pr-T_2$-spaces.

2.22 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a\},\{b,c\}\}$and $\rho=\{X,\phi,\{b,c\}\}$ $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$.

Hence , (X,τ,ρ) is $pr-T_2$-space, but it is clear that (X,τ) and (X,ρ) are not T_2-spaces.

2.23 Remark:

If any of two topological spaces, $(X,\tau),(X,\rho)$ are T_2-space, then it is not necessary to be a bitopological space (X,τ,ρ) $pr-T_2$-space.

2.24 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a\},\{a,b\}\}$, $\rho=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{a\},\{a,b\}\}$.

If we take a and c, such that $a\neq c$, then there exists $pr-$open set $\{a,b\}$ containing a but not c , but there is no a $pr-$open set containing c but not a , therefore , (X,τ,ρ) is not $pr-T_2$-space, but it is clear that , (X,ρ) is T_2-space.

2.25 Remark:

Every $pr-T_2$-space is a $pr-T_1$-space.

Proof:

To prove this remark , we use Definitions of (2-8) and (2-19) □

2.26 Remark:

The converse of the remark above is not true , the following example shows that.

2.27 Example:

Let $X=\{a,b,c\}, \tau=\{X,\phi,\{a\},\{b,c\}\}$ and $\rho=\{X,\phi,\{a,c\},\{a,b\},\{a\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, then:
$pr.O(X)=\{X,\phi,\{a\},\{a,b\},\{a,c\}\{b,c\}\}$.

Clearly (X,τ,ρ) is a $pr-T_1$-space, but not $pr-T_2$-space, because there exists $b\neq c$, there is only $pr-$open sets $\{a,b\}$, $\{a,c\}$, such that $b\in\{a,b\}$, $c\in\{a,c\}$, but $\{a,b\}\bigcap\{a,c\}\neq\phi$.

The $pr-T_2$-space is hereditary property .

2.28 Theorem:

Every subspace of $pr-T_2$-space is a $pr-T_2$-space.

Proof:

Let (X,τ,ρ) be a $pr-T_2$-space, and let $Y\neq\phi$ be a subset of X, and $x\neq y\in Y$, then x and $y\in X$, since (X,τ,ρ) is $pr-T_2$-space, there exist two $pr-$open sets G,H such that $x\in G$ and $y\in H$, $G\bigcap H=\phi$.

So $G\bigcap Y$, $H\bigcap Y$ are $pr-$open sets in Y and $x\in G\bigcap Y$, $y\in H\bigcap Y$; and $(G\bigcap Y)\bigcap(H\bigcap Y)=(G\bigcap H)\bigcap Y=\phi$.

Hence (X,τ_Y,ρ_Y) is a $pr-T_2$-space □

2.29 Theorem:

Each singleton subset of $pr-T_2$-space is a $pr-$closed.

Proof:

Let X be a $pr-T_2$-space, and let $x \in X$. To show that $\{x\}$ is a $pr-$closed.

Let y be any arbitrary point in X distinct from x. Since X is $pr-T_2$-space, there exists $pr-$open set N of y; such that $x \notin N$. It follows that y is not a limit point of $\{x\}$ and consequently $pr-lm(\{X\})=\phi$.

Hence, $pr-cl(\{x\})=\{x\}$ and so is $pr-$closed □

2.30 Definition:

A bitopological space (X,τ,ρ) is called $pr-$**regular space** if and only if for each $pr-$closed set F in X, and each $x \notin F$; there exist $pr-$open sets U,V such that $x \in U$, $F \subset V$ and $U \bigcap V=\phi$.

2.31 Remark:

If a bitopological space, (X,τ,ρ) is $pr-$regular space, then any of two topological spaces $(X,\tau),(X,\rho)$ are not to be regular space.

2.32 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a,c\}\}$ and $\rho=\{X,\phi,\{b\}\}$.
$(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:
$pr.O(X)=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\}\{b,c\}\}$, and
$pr.C(X)=\{X,\phi,\{a,b\},\{a,c\}\{b,c\},\{c\},\{b\},\{a\}\}$.

Take $A=\{b,c\}$, $a \notin A$, then there exist $\{a\},\{b,c\}$ $pr-$open sets such that $a \in \{a\},\{b,c\} \subseteq \{b,c\}$, and $\{a\} \bigcap \{b,c\}=\phi$, if we take $A=\{a,c\}$, $b \notin A$, then there exist $\{b\},\{a,c\}$ $pr-$open sets such that $b \in \{b\},\{a,c\} \subseteq \{a,c\}$, and $\{b\} \bigcap \{a,c\}=\phi$.

If we take $A=\{a,b\}$, $c\notin A$, then there exist $\{c\},\{a,b\}$ $pr-$open sets such that $c\in\{c\},\{a,b\}\subseteq\{a,b\}$, and $\{c\}\bigcap\{a,b\}=\phi$. Hence , (X,τ,ρ) is a $pr-$regular space. But it is clear that no one of two topological spaces $(X,\tau),(X,\rho)$ is a regular space.

The following example explains that if any of two topological spaces, $(X,\tau),(X,\rho)$ is a regular space, then a topological space, (X,τ,ρ) is not necessary to be $pr-$regular space.

2.33 Example:

Let $X=\{a,b,c,d\}, \tau=\{X,\phi,\{a,b\},\{c,d\}\}$and $\rho=\{X,\phi,\{a\},\{d\},\{a,b\},\{a,b,d\},\{a,d\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{d\},\{a,b\},\{a,d\},\{c,d\},\{a\},\{a,b,d\},\{a,c,d\}\}$, and

$pr.C(X)=\{X,\phi,\{a,b,c\},\{c,d\},\{b,c\},\{b,c,d\},\{a,b\},\{c\},\{b\}\}$

If we take $A=\{b,c\}$, $a\notin A$, then there are no $pr-$open sets G,H , such that $A\subset G$, $a\in H$, $G\bigcap H=\phi$.

Similary the other cases, if we take $A=\{b,c,d\}$, $a\notin A$, and if we take $A=\{a,b,c\}$, $d\notin A$ so (X,τ,ρ) is not regular apace. But it is clear that (X,τ), is a regular space.

2.34 Example:

Let $X=\{a,b,c,d\}$, $\tau=\{X,\phi,\{b,d\}\}$ and

$\rho=\{X,\phi,\{a\},\{c\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{c,d\},\{a,b,c\},$
$\{a,b,d\},\{a,c,d\},\{b\},\{d\},\{b,d\},\{b,c,d\}\}$

$(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{b,d\}\}$, and $pr.C(X)=\{X,\phi,\{a,c\}\}$.

If we take $A=\{a,c\}$, $b\notin A$, then there exists $pr-$open set G, containing b , but there exists no $pr-$open set H such that $A\subset H$, and $G\bigcap H=\phi$.

So (X,τ,ρ) is not regular space. But, it is clear that (X,ρ), is a regular space.

2.35 Theorem:

Let (X,τ,ρ) be a bitopological space, then (X,τ,ρ) is a $pr-$regular iff for each $pr-$open set U and $x\in U$, there exists $pr-$open set V such that $x\in V$, $pr-cl(V)\subset U$.

Proof:

Let (X,τ,ρ) be a $pr-$regular, $x\in U$ where U is a $pr-$open set.

Let $H=X-U$, then H is $pr-$closed, $x\notin H$.

Then there exist $pr-$open sets W and V such that: $x\in V$, $H\subset W$, $V\bigcap W=\phi$.

Then $V\subset X-W$, $pr-cl(V)\subset pr-cl(X-W)=X-W$…..(2-1)
$H\subset W$, then $X-W\subset X-H=U$, then $X-W\subset U$…….(2-2)

From (2-1),(2-2) we have, $X\in V$, $pr-cl(V)\subset U$.Conversely, let H be $pr-$closed set and $x\notin H$. Let $U=X-H$, then U is a $pr-$open and $x\in U$.

By hypothesis, there exists $pr-$open set V such that $x\in V$, $pr-cl(V)\subset U$, $H\subset(X-(pr-cl(V)))$.Since $x\in V$, $V\bigcap(X-(pr-cl(V)))=\phi$. Hence (X,τ,ρ) is $pr-$regular space □

2.36 Theorem:

Every subspace of a $pr-$regular space is a $pr-$regular space.

Proof:

Let (X,τ,ρ) be a $pr-$regular space, (Y,τ_Y,ρ_Y) be a subspace of Y. To prove (X,τ_Y,ρ_Y) is $pr-$regular space . Let $q\in Y$ and A be $pr-$closed set in Y, such that $q\notin A$, then $pr-cl_Y(A)=pr-cl_X(A)\bigcap Y$, and since A is $pr-$closed set in Y, so $pr-cl_Y(A)=A$

Then $A=pr-cl_X(A)\bigcap Y$. Since $q\notin A$, then $q\notin pr-cl_X(A)\bigcap Y$, $q\notin pr-cl_X(A)$, thus $pr-cl_X(A)$ is $pr-$closed in X, and since (X,τ,ρ) is $pr-$regular, then there exist two disjoint

$pr-$open sets G,H in X, such that $q \in G$, $pr-cl_X(A) \subset H$ and $G \bigcap H = \phi$.

$q \in G \bigcap Y$ and $pr-cl_X(A) \bigcap Y \subset H \bigcap Y$, $A \subset H$, since G,H are $pr-$open in X, then $G \bigcap Y$, $H \bigcap Y$ are $pr-$open set in Y.

Since $G \bigcap Y = \phi$, then $(G \bigcap Y) \bigcap (H \bigcap Y) = (G \bigcap H) \bigcap Y = \phi \bigcap Y = \phi$.

So (Y, τ_Y, ρ_Y) is $pr-$regular subspace of (X, τ, ρ) □

2.37 Definition:

A bitopological space (X, τ, ρ) is called $pr-$**normal space** iff for each pair of $pr-$closed sets G,H in X, such that $G \bigcap H = \phi$, there exist $pr-$open sets U,V such that $G \subset U$, $H \subset V$ and $U \bigcap V = \phi$.

2.38 Remark:

If a bitopological space (X, τ, ρ) is $pr-$normal, then any of two topological spaces $(X, \tau), (X, \rho)$ is not necessary to be normal space.

2.39 Example:

Let $X = \{a,b,c\}$, $\tau = \{X, \phi, \{a\}, \{a,b\}, \{a,c\}\}$and $\rho = \{X, \phi, \{b\}, \{a,b\}, \{b,c\}\}$ $(X, \tau), (X, \rho)$ are two topological spaces, then (X, τ, ρ) is a bitopological space, so:

$pr.O(X) = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}$.

$pr.C(X) = \{X, \phi, \{b,c\}, \{a,c\}, \{c\}, \{b\}, \{a\}\}$.

If we take $\{b,c\},\{a\}$ $pr-$closed sets, such that, $\{b,c\} \bigcap \{a\} = \phi$, then there exist two disjoint $pr-$open sets $\{b,c\},\{a\}$, such that $\{b,c\} \subseteq \{b,c\}$ and $\{a\} \subseteq \{a\}$ and if we take $\{a\},\{b\}$, we can obtain two disjoint $pr-$open sets such that every one containing $pr-$closed set of them, so if we take $\{a,c\},\{b\}$, so if we take $\{b\},\{c\}$, and if we take $\{a\},\{c\}$.

Hence , (X, τ, ρ) is $pr-$normal. But clearly no one of the two topological spaces is normal.

2.40 Theorem:

Let (X,τ,ρ) be a bitopological space, then (X,τ,ρ) is a $pr-$ normal iff for each $pr-$closed set H in X and $pr-$open set U in X containing H, there exists $pr-$open set V such that $H \subset V \subset pr-cl(V) \subset U$.

Proof:

Let (X,τ,ρ) be $pr-$normal. Let H be $pr-$closed in X and U is $pr-$open in X, such that $H \subset U$. then $X-U$ is $pr-$closed in X and $H \bigcap (X-U) = \phi$.

So there exist $pr-$open sets K and V such that,
$X-U \subset K$, $H \subset V$, $V \bigcap K = \phi$, $X-K \subset U$, $V \subset X-K$.

This implies that , $pr-cl(V) \subset pr-cl(X-K) = X-K$.

Then $H \subset V \subset pr-cl(V) \subset U$ $V \subset pr-cl(V) \subset U$.

Conversely, H is a $pr-$closed set in X and U is a $pr-$open in X, then $X-U$ is a $pr-$closed set in X, then $X-U$ is a $pr-$closed in X, and $H \bigcap (X-U) = \phi$.

By hypothesis, there exists $pr-$open set V such that $H \subset V \subset pr-cl(V) \subset U$, then $X-U \subset (X-(pr-cl(V)))$.

So we have $H \subset V$, $X-U \subset (X-(pr-cl(V)))$, and $V \bigcap (X-(pr-cl(V))) = \phi$.

Hence (X,τ,ρ) is $pr-$normal space □

2.41 Corollary:

A bitopological space (X,τ,ρ) is a $pr-$normal iff for each $pr-$closed set H in X and each $pr-$open set U in X containing H, there exists a subset A of X, such that $H \subset pr-\text{int}(A) \subset pr-cl(A) \subset U$.

Proof:

By using the Theorem (2-40) and taking $A = V$, the proof is satisfied □

2.42 Theorem:

Every $pr-$closed subspace of a $pr-$normal space is a $pr-$normal.

Proof:

Let (X,τ,ρ) be a $pr-$normal space and (Y,τ_Y,ρ_Y) be any $pr-$closed subspace of X.

We have to show that (Y,τ_Y,ρ_Y) is also $pr-$normal space.

Let L^*,M^* be disjoint $pr-$closed subsets of Y.

Then there exist $pr-$closed subsets L,M of X, such that $L^*=L\bigcap Y$ and $M^*=M\bigcap Y$. Since Y is $pr-$closed, it follows that L^*,M^* are disjoint $pr-$closed subsets in X. Then by $pr-$normality of X, there exist $pr-$open sets G,H in X, such that $L^*\subset G,\ M^*\subset H$ and $G\bigcap H=\phi$.

Since $L^*\subset Y$ and $M^*\subset Y$, these relations imply:

$L^*\subset G\bigcap Y, M^*\subset M\bigcap Y$ and $(G\bigcap Y)\bigcap H\bigcap Y)=\phi$.

Setting $G\bigcap Y=G^*$ and $H\bigcap Y=H^*$, we see that G^*,H^* are $pr-$open subset of Y such that: $L^*\subset G^*,\ M^*\subset H^*$ and $G^*\bigcap H^*=\phi$.

Accordingly (Y,τ_Y,ρ_Y) is a $pr-$normal space □

2.43 Definition:

A bitopological space (X,τ,ρ) is called a $pr-T_3$-space iff X is a $pr-T_1$ and a $pr-$regular space.

2.44 Example:

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a,b\}\}$ and $\rho=\{X,\phi,\{c\}\}$.

$(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$, and

$pr.C(X)=\{X,\phi,\{b,c\},\{a,c\},\{a,b\},\{c\},\{b\},\{a\}\}$.

Then (X,τ,ρ) is a $pr-$regular and a $pr-T_1$-space.

Thus (X,τ,ρ) is a $pr-T_3$-space.

2.45 Remark:

Every $pr-T_3$- space is a $pr-$regular and the converse is not true as we will show in the following example.

2.46 Example:

Let $X=\{a,b,c\}, \tau=\{X,\phi,\{b,c\}\}$and $\rho=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\}\}$ $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{b,c\}\}$, and $pr.C(X)=\{X,\phi,\{a\}\}$.

Clearly , (X,τ,ρ) is a $pr-$regular, but it is not a $pr-T_1$-space.

Hence (X,τ,ρ) is not a $pr-T_3$-space.

2.47 Theorem:

Every subspace of $pr-T_3$-space is a $pr-T_3$-space.

Proof:

Let X be a $pr-T_3$-space and Y be a subspace of X .

Now X is a $pr-$regular as well as a $pr-T_1$-space and we have shown that both these properties are hereditary.

It follows that Y is a $pr-$regular as well as a $pr-T_1$-space.

Hence Y is a $pr-T_3$-space □

2.48 Definition:

A bitopological space (X,τ,ρ) is called **a** $pr-T_4$ **space** if and only if X is a $pr-$normal and a $pr-T_1$-space.

2.49 Example:

Let $X=\{a,b,c\}, \tau=\{X,\phi,\{a\},\{c\},\{a,c\}\}$ And $\rho=\{X,\phi,\{a,b\}\}$. $(X,\tau),(X,\rho)$ are two topological spaces, then (X,τ,ρ) is a bitopological space, so:

$pr.O(X)=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$, and

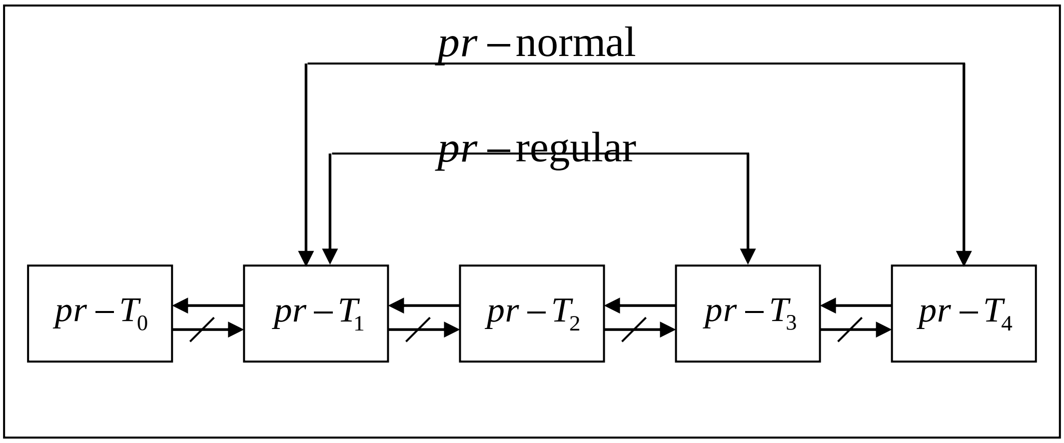
$pr.C(X)=\{X,\phi,\{b,c\},\{a,c\},\{a,b\},\{c\},\{b\},\{a\}\}$.

Then (X,τ,ρ) is a $pr-$normal and a $pr-T_1$-space.
Thus (X,τ,ρ) is a $pr-T_4$-space.

2.50 Remark:

Every $pr-T_4$-space is a $pr-$normal and the converse is not true as we will show in Example (2-46).

The following diagram illustrates the relationship between the properties of spaces discussed in this chapter.



In this chapter, we deal with the continuous open and closed functions in bitopological space. Also , we study the separation axioms that we can translate by a continuous function and the nature of the conditions that we need to satisfy the translation.

In the following, we state the definition of $pr-$continuous function, and some important theorems that deal with ($pr-$open, $pr-$closed, $pr-cl$ sets and $pr-$interior points) in bitopological spaces.

3.1 Definition:

Let (X,τ,ρ) and (Y,τ',ρ') be two bitopological spaces. A function $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ is said to be $pr-$**continuous** at $x\in X$ iff for every $pr-$open set V in Y containing $f(x)$ there exists $pr-$open set U in X containing x such that $f(U)\subset V$. We say that f is $pr-$continuous on X iff f is $pr-$continuous at each $x\in X$.

3.2 Definition:

Let $(X,\tau,\rho),(Y,\tau',\rho')$ be two bitopological spaces and $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be a function, then:

i.) f is said to be $pr-$**open function** iff $f(G)$ is $pr-$open set in Y , for every $pr-$open set G in X .

ii.) f is said to be $pr-$**closed function** iff $f(F)$ is $pr-$closed set F in Y , for every $pr-$closed set F in X .

3.3 Theorem:

Let (X,τ,ρ) and (Y,τ',ρ') be bitopological spaces, and $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be a function. If the inverse image under f of every $pr-$open set in Y is $pr-$open set in X, then f is $pr-$continuous function.

Proof:

Let $f^{-1}(V)$ is $pr-$open set in X, for each V is $pr-$open set in Y to prove f is $pr-$continuous.

Let $x \in X$ and V is $pr-$open set in Y containing $f(x)$, so $f^{-1}(V)$ is a $pr-$open set in X containing x and $f(f^{-1}(V)) \subseteq V$. Then f is $pr-$continuous on X □

3.4 Theorem:

Let (X,τ,ρ) and (Y,τ',ρ') be two bitopological spaces, and $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be a function. If the inverse image under f of every $pr-$closed set in Y is a $pr-$closed set in X, then f is $pr-$continuous function.

Proof:

Let $f^{-1}(A)$ be $pr-$closed in X for every $pr-$closed set A in Y. We want to show that f is a $pr-$continuous function.

Let B be any a $pr-$open set in Y, then $Y-B$ be $pr-$closed in Y, so by hypothesis $f^{-1}(Y-B) = X - f^{-1}(B)$ is $pr-$closed in X, that is, $f^{-1}(B)$ is $pr-$open in X.

Hence , by Theorem (3-3) f is $pr-$continuous □

3.5 Theorem:

Let (X,τ,ρ) and (Y,τ',ρ')be two bitopological spaces, and $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be a function. If $f(pr-cl(A)) \subset pr-cl(f(A))$, for every $A \subset X$, then f is $pr-$continuous.

Proof:

Let $f(pr-cl(A)) \subset pr-cl(f(A))$, for every $A \subset X$.

Let B be any a $pr-$closed set in Y, so that $pr-cl(B) = B$. Now $f^{-1}[B] \subset X$, by hypothesis, $f(pr-cl(f^{-1}(B))) \subset pr-cl(f(f^{-1}(B))) \subset pr-cl(B) = B$

Therefore, $pr-cl(f^{-1}(B)) \subset f^{-1}(B)$. But, $f^{-1}(B) \subset pr-cl(f^{-1}(B))$

always. Hence $pr-cl(f^{-1}(B))=f^{-1}(B)$ and so $f^{-1}(B)$ is $pr-$closed in X. Hence, by Theorem (3.4) f is a $pr-$continuous □

3.6 Theorem:

Let (X,τ,ρ) and (Y,τ',ρ') be two bitopological spaces, and $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be a function. If $pr-cl(f^{-1}(B))\subset f^{-1}(pr-cl(B))$, for every $B\subset Y$, then f is $pr-$continuous.

Proof:

Let A be any $pr-$closed set in Y, so that $pr-cl(A)=A$, by hypothesis, $pr-cl(f^{-1}(A))\subset f^{-1}(pr-cl(A))=f^{-1}(A)$. But $f^{-1}(A)\subset pr-cl(f^{-1}(A))$ always.

Hence $pr-cl(f^{-1}(A))=f^{-1}(A)$, so $f^{-1}(A)$ is $pr-$closed in X .

It follows from Theorem (3.4) that f is a $pr-$continuous function □

3.7 Theorem:

Let (X,τ,ρ) and (Y,τ',ρ') be two bitopological spaces, and $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be a function. If $f^{-1}(pr-\text{int}(B))\subset pr-\text{int}(f^{-1}(B))$, for each $B\subset Y$, then f is a $pr-$continuous function .

Proof:

Let G be any $pr-$open set in Y, so that $pr-\text{int}(G)=G$, by hypothesis, $f^{-1}(pr-\text{int}(G))\subset pr-\text{int}(f^{-1}(G))$, since $f^{-1}(pr-\text{int}(G))=f^{-1}(G)$, then $f^{-1}(G)\subset pr-\text{int}(f^{-1}(G))$.

But $pr-\text{int}(f^{-1}(G))\subset f^{-1}(G)$ always and so $pr-\text{int}(f^{-1}(G))=f^{-1}(G)$.

Therefore $f^{-1}(G)$ is $pr-$open in X and consequently by Theorem (3.3) f is a $pr-$continuous function□

3.8 Theorem:

Let (X,τ,ρ), (Y,τ',ρ') and (Z,τ'',ρ'') be bitopological spaces, the functions $f:(X,\tau,\rho)\to(Y,\tau',\rho')$and $g:(Y,\tau',\rho')\to(Z,\tau'',\rho'')$ be $pr-$continuous, then the composition function $g\circ f:(X,\tau,\rho)\to(Z,\tau'',\rho'')$ is a $pr-$continuous.

Proof:

Let $x\in X$ and $z=g\circ f(x)=g(f(x))$, and W be a $pr-$open subset of Z which contains z, since g is $pr-$continuous, so there exists $pr-$open set V in Y, such that $f(x)\in V$ and $g(V)\subset W$, but f is $pr-$continuous, so there exists any a $pr-$open set U in X, such that $x\in U$ and $f(U)\subset V$. Hence, we get that $g[f(U)]=g\circ f(U)\subset g(V)\subset W$.

Therefore, $g\circ f$ is $pr-$continuous function from X into Z □

In the following, we study some basic theorems on $pr-$continuous function on separation axioms in bitopological spaces.

3.9 Theorem:

Let (Y,τ',ρ') be a $pr-T_o$-space, and $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be a $pr-$continuous and 1-1 function, then (X,τ,ρ) is a $pr-T_o$-space.

Proof:

Let $x_1,x_2\in X, x_1\neq x_2$. Since f is a 1-1 function, then $f(x_1)\neq f(x_2), f(x_1), f(x_2)\in Y$, and Y is $pr-T_o$-space, then there exists $pr-$open G in Y, such that $f(x_1)\in G$, $f(x_2)\notin G$, so there exists $pr-$open set H in X such that $x_1\in H$, $x_2\notin H$ and $f(x_1)\in f(H)\subset G$, $f(x_2)\notin f(H)$.

Hence , (X,τ,ρ) is a $pr-T_o$-space □

3.10 Theorem:

Let $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be a $pr-$open and onto function. If (X,τ,ρ) is a $pr-T_o$-space, then $(Y,\tau',\rho')$is a $pr-T_o$-space.

***Proof*:**

Suppose $y_1, y_2 \in Y, y_1 \neq y_2$, since f is onto, there exist $x_1, x_2 \in X$, such that $y_1 = f(x_1)$, $y_2 = f(x_2)$, if $x_1 = x_2$. Then f is not well defined , which contradiction since f is onto function , then $x_1 \neq x_2$.

Since X is a $pr-T_o$-space, there exists $pr-$open set G, such that $x_1 \in G$, $x_2 \notin G$.

Hence $y_1 = f(x_1) \in f(G)$, $y_2 = f(x_2) \notin f(G)$, since f is $pr-$open function, then $f(G)$is a $pr-$open set in Y.
Therefore, (Y,τ',ρ') is a $pr-T_o$-space□

3.11 Theorem:

Let (Y,τ',ρ')be a $pr-T_1$-space, if $f:(X,\tau,\rho)\to(Y,\tau',\rho')$is a $pr-$continuous and 1-1 function, then X is a $pr-T_1$-space.

Proof:

Let $x_1, x_2 \in X, x_1 \neq x_2$. Since f is 1-1 function, then $f(x_1), f(x_2) \in Y, Y$ is $pr-T_1$-space, then there exist U_1, U_2 $pr-$open sets in Y, such that $f(x_1) \in U_1$, $f(x_2) \notin U_1$, $f(x_2) \in U_2$, $f(x_1) \notin U_2$, since f is $pr-$continuous, so there exist $pr-$open sets H_1, H_2 in X, such that: $x_1 \in H_1$, $x_2 \notin H_1$ and $x_1 \notin H_2$, $x_2 \in H_2$ and $f(H_1) \subset G_1$, $f(H_2) \subset G_2$. Hence $(X,\tau,\rho)$is a $pr-T_1$-space □

3.12 Theorem:

Let $f:(X,\tau,\rho)\to(Y,\tau',\rho')$ be onto and a $pr-$open function, if $(X,\tau,\rho)$is a $pr-T_1$-space, then $(Y,\tau',\rho')$is a $pr-T_1$-space.

Proof:

Suppose $y_1, y_2 \in Y, y_1 \neq y_2$. since f is onto then there exist $x_1, x_2 \in X$, such that $y_1 = f(x_1)$, $y_1 = f(x_1)$ and since f is a function, then $x_1 \neq x_2 \in X$. Since, X is a $pr-T_1$-space, there exist $pr-$open sets G and H in X such that $x_1 \in G$ but $x_2 \notin G$ and $x_2 \in H$ but $x_1 \notin H$.

Hence , $f(x_1) \in f(G)$, $f(x_2) \in f(G)$, since f is $pr-$open function, hence $f(G)$, $f(H)$ are $pr-$open sets of Y, $y_1 \in f(G)$, but $y_2 \notin f(G)$ and $y_2 \in f(H)$, but $y_1 \notin f(H)$.

Hence , $(Y, \tau', \rho')$is a $pr-T_1$-space □

3.13 Theorem:

Let (Y, τ', ρ') be a $pr-T_2$-space, and $f:(X, \tau, \rho) \to (Y, \tau', \rho')$ be a $pr-$continuous and 1-1 function, then (X, τ, ρ) is a $pr-T_2$-space.

Proof:

Let $x_1, x_2 \in X, x_1 \neq x_2$. Since f is a 1-1 function, then $f(x_1) \neq f(x_2)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, $y_1 \neq y_2$. since Y is $pr-T_2$-space, there exist two $pr-$open sets G, H in Y such that $y_1 \in G$, $y_2 \in H$, $G \cap H = \phi$.

Hence, there exist two $pr-$open sets G_1, H_1 in X, such that: $x_1 \in G_1$, and $x_2 \in H_1$ and $f(G_1) \subset G$, $f(H_1) \subset H$. But $f(G_1 \cap H_1) = f(G_1) \cap f(H_1) \subset G \cap H = \phi$, so $f(G_1 \cap H_1) = \phi$, then $G_1 \cap H_1 = \phi$. Therefore (X, τ, ρ) is a $pr-T_2$-space □

3.14 Theorem:

Let $f:(X, \tau, \rho) \to (Y, \tau', \rho')$ be onto and $pr-$open function, if (X, τ, ρ) is a $pr-T_2$-space, then $(Y, \tau', \rho')$is a $pr-T_2$-space.

Proof:

Let $y_1 \neq y_2 \in Y$. since f is onto function, then there exist $x_1 \neq x_2 \in X$, such that $y_1 = f(x_1)$, $y_2 = f(x_2)$, since X is

$pr-T_2$-space, then there exist $pr-$open sets G,H in X such that $x_1 \in G, x_2 \in H$ and $G \cap H = \phi$, and $y_1 = f(x_1) \in f(G)$, $y_2 = f(x_2) \in f(H)$.Since f is a $pr-$open function, then $f(G)$, $f(H)$ are two $pr-$open sets in Y and $(G \cap H) = f(G) \cap f(H) = f(\phi) = \phi$.

Also, $y_1 = f(x_1) \in f(G)$, $y_2 = f(x_2) \in f(H)$. Hence (Y,τ',ρ') is a $pr-T_2$-space □

In the following , we give the definition of pr^*-continuous function, also studying some important theorems that are related with ($pr-$open, $pr-$closed, $pr-$interior, $pr-$closure) in bitopological spaces.

3.15 Definition:

Let (X,τ,ρ) be bitopological space, (Y,τ') be a topological space, a function $f:(X,\tau,\rho) \to (Y,\tau')$ is said to be pr^*-continuous at $x \in X$ iff for every $\tau'-$open set V in Y containing $f(x)$ there exists a $pr-$open set U in X containing x such that $f(U) \subset V$. We say that f is pr^*-continuous on X iff f is pr^*-continuous at each $x \in X$.

3.16 Definition:

Let (X,τ,ρ) be bitopological space, (Y,τ') be topological space, and $f:(X,\tau,\rho) \to (Y,\tau')$be a function, then:

i.) f is said to be a pr^*-**open function** iff $f(G)$ is τ'-open in Y , for every $pr-$open set G in X .

ii.) f is a pr^*-**closed function** iff $f(A)$ is τ'- closed in Y , for every $pr-$ closed set A in X .

3.17 Theorem:

Let (X,τ,ρ) be bitopological space, (Y,τ') be a topological space, and $f:(X,\tau,\rho) \to (Y,\tau')$ be a function. If the inverse image under f of every τ'-open set V of Y is a $pr-$open set of X, then f is pr^*-continuous function .

Proof:

Let $f^{-1}(V)$ be $pr-$open set in X for each V a τ'-open set in Y to prove f is $pr^{*}-$continuous.

Let $x \in X$ and V be a τ'-open set in Y containing $f(x)$, $f^{-1}(V)$ is $pr-$open in X containing x and $f(f^{-1}(V)) \subseteq V$.

Then f is $pr^{*}-$continuous function □

3.18 Theorem:

Let (X,τ,ρ) be a bitopological space, (Y,τ') be a topological space, and $f:(X,\tau,\rho)\to(Y,\tau')$be a function. If the inverse image under f of every τ'-closed set in Y $pr-$ closed set in X, then f is $pr^{*}-$continuous function .

Proof:

Let $f^{-1}(A)$ be $pr-$closed in X for every τ'-closed set A in Y, to show that f is $pr^{*}-$continuous function. Let G be any τ'-open set in Y, then $Y-G$ is τ'-closed in Y so by hypothesis;

$f^{-1}(Y-G)=X-f^{-1}(G)$ is $pr-$closed in X, that is $f^{-1}(G)$ is $pr-$open in X. Hence, by Theorem (3-3) f is $pr^{*}-$continuous function □

3.19 Theorem:

Let (X,τ,ρ) be a bitopological space, (Y,τ') be a topological space, and $f:(X,\tau,\rho)\to(Y,\tau')$ be a function. If: $f(pr-cl(A))\subset\tau'-cl(f(A))$ for every $A\subset X$, then f is a $pr^{*}-$continuous function .

Proof:

Let $f(pr-cl(A))\subset\tau'-cl(f(A))$, for every $A\subset X$.

Let B be any τ'- closed in Y, so that $\tau'-cl(f(B))=B$.

Now, $f^{-1}(B)\subset X$, by hypothesis,

$f(pr-cl(f^{-1}(B)))\subset\tau'-cl(f(f^{-1}(B)))\subset\tau'-cl(B)=B$.

Hence, $pr-cl(f^{-1}(B)) = f^{-1}(B)$, so $f^{-1}(B)$ is $pr-$closed in X. Hence, by Theorem (3-18), f is a $pr^{*}-$continuous function □

3.20 Theorem:

Let (X,τ,ρ) be a bitopological space, (Y,τ') be a topological space, and $f:(X,\tau,\rho)\to(Y,\tau')$ be a function. If $pr-cl(f^{-1}(B)) \subset f^{-1}(\tau'-cl(B))$, for every $B\subset Y$, then f is $pr^{*}-$continuous function.

Proof:

Let the condition holds and A be any τ'- closed in Y, so that $\tau'-cl(A) = A$ by hypothesis , $pr-cl(f^{-1}(A)) \subset \tau'-cl(f(A)) = f^{-1}(A)$. But $f^{-1}(A) \subset pr-cl(f^{-1}(A))$ always for every $A\subset X$.

Hence $pr-cl(f^{-1}(A)) = f^{-1}(A)$, so $f^{-1}(A)$ is $pr-$closed in X. It follows from Theorem (3-18) that f is $pr^{*}-$continuous function □

3.21 Theorem:

Let (X,τ,ρ) be a bitopological space, (Y,τ') be topological space, and $f:(X,\tau,\rho)\to(Y,\tau')$ be a function. If $f^{-1}(\tau'-\text{int}(B)) \subset pr-\text{int}(f^{-1}(B))$, for every $B\subset Y$, then f is $pr^{*}-$continuous function.

Proof:

Let the condition holds and G be any τ'-open set in Y, so that $\tau'-\text{int}(G) = G$, by hypothesis $f^{-1}(\tau'-\text{int}(G)) \subset pr-\text{int}(f^{-1}(G))$, since $f^{-1}(\tau'-\text{int}(G)) = f^{-1}(G)$, then $f^{-1}(G) \subset pr-\text{int}(f^{-1}(G))$.
But $pr-\text{int}(f^{-1}(G)) \subset f^{-1}(G)$ always, so $pr-\text{int}(f^{-1}(G)) = f^{-1}(G)$.

Therefore, $f^{-1}(G)$ is a $pr-$open in X and consequently by Theorem (3-17) f is $pr^{*}-$continuous function □

The following ,we are studying some main theorems those deal with $pr^{*}-$continuous functions on separation axioms in bitopological spaces**.**

3.22 Theorem:

Let (Y,τ')be a $T_{\circ}$-space, if $f:(X,\tau,\rho)\to(Y,\tau')$is a $pr^{*}-$continuous and 1-1 function, then (X,τ,ρ) is a $pr-T_{\circ}$-space.

Proof:

Let $x_1,x_2\in X,x_1\neq x_2$. Since f is 1-1 function, then $f(x_1)\neq f(x_2)$, $f(x_1), f(x_2)\in Y$, and (Y,τ') is a T_0-space, then there exists $\tau'-$open G in Y, such that $f(x_1)\in G$, $f(x_2)\notin G$, so there exists $pr-$open set H in X such that $x_1\in H$, $x_2\notin H$ and $f(x_1)\in f(H)\subset G$, $f(x_2)\notin f(H)$. Hence , (X,τ,ρ) is a $pr-T_{\circ}$-space □

3.23 Theorem:

Let $f:(X,\tau,\rho)\to(Y,\tau')$ be $pr^{*}-$open and onto function (X,τ,ρ) be a $pr-T_{\circ}$-space, then $(Y,\tau')$is a $T_{\circ}$-space.

Proof:

Suppose $y_1,y_2\in Y,y_1\neq y_2$ since f is onto, there exist $x_1,x_2\in X$, such that $y_1=f(x_1)$, $y_2=f(x_2)$ since f is onto function, then $x_1\neq x_2$. Since, X is a $pr-T_{\circ}$-space, then there exists $pr-$open set G, such that $x_1\in G$, $x_2\notin G$. Hence, $y_1=f(x_1)\in f(G)$, $y_2=f(x_2)\notin f(G)$, since f is $pr^{*}-open$ function, then $f(G)$ is a τ'-open set in Y.

Therefore, $(Y,\tau')$is T_0-space □

3.24 Theorem:

Let (Y,τ') be a T_1-space, $f:(X,\tau,\rho)\to(Y,\tau')$ be a $pr^{*}-$continuous and 1-1 function, then$(X,\tau,\rho)$is a $pr-T_1$-space.

Proof:

Let $x_1, x_2 \in X, x_1 \neq x_2$. Since f is a 1-1 function, then $f(x_1), f(x_2) \in Y$, Y is T_1-space, then there exist U_1, U_2 τ'-open sets in Y, such that $f(x_1) \in U_1$, $f(x_2) \notin U_1$, and $f(x_2) \in U_2$, $f(x_1) \notin U_2$. Since f is a pr^*-continuous, so there exist two $pr-$open sets H_1, H_2 in X such that $x_1 \in H_1$, $x_2 \notin H_1$ and $x_1 \notin H_2$, $x_2 \in H_2$ and $f(H_1) \subset G_1$, $f(H_2) \subset G_2$. Hence, $(X, \tau, \rho)$is a $pr-T_1$-space □

3.25 Theorem:

Let $f:(X, \tau, \rho) \to (Y, \tau')$ be onto and pr^*-open function. If (X, τ, ρ) is a $pr-T_1$-space, then (Y, τ') is a T_1-space.

Proof:

Suppose $y_1, y_2 \in Y, y_1 \neq y_2$ since f is onto function , there exist $x_1, x_2 \in X$, such that $y_1 = f(x_1)$, $y_2 = f(x_2)$ and since f is a function, then $x_1 \neq x_2 \in X$. Since X is $pr-T_1$-space, there exist $pr-$open sets G, H in X, such that $x_1 \in G$, but $x_2 \notin G$, and $x_2 \in H$ but $x_1 \notin H$.

Hence, $f(x_1) \in f(G)$, $f(x_2) \in f(H)$, since f is pr^*-open function, hence, $f(G)$, $f(H)$ are τ'-open sets of Y, such that $y_1 \in f(G)$, but $y_2 \notin f(G)$, and $y_2 \in f(H)$, but $y_1 \notin f(H)$.

Then $(Y, \tau')$is a T_1-space □

3.26 Theorem:

Let (Y, τ')be a T_2-space, $f:(X, \tau, \rho) \to (Y, \tau')$ be a pr^*-continuous and 1-1 function, then $(X, \tau, \rho)$is a $pr-T_2$-space.

Proof:

Let $x_1, x_2 \in X, x_1 \neq x_2$. Since f is a 1-1 function, then $f(x_1) \neq f(x_2)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, $y_1 \neq y_2$. Since (Y, τ') is T_2-space, there exist two τ'-open sets G, H in Y, such that $y_1 \in G$, $y_2 \in H$, $G \bigcap H = \phi$.

Hence, there exist two $pr-$open sets G_1, H_1 in X, such that $x_1 \in G_1$ and $x_2 \in H_1$, and $f(G_1) \subset G$, $f(H_1) \subset H$. But $f(G_1 \cap H_1) = f(G_1) \cap f(H_1) \subset G \cap H = \phi$, so $f(G_1 \cap H_1) = \phi$, then $G_1 \cap H_1 = \phi$. Therefore , (X, τ, ρ) is a $pr - T_2$-space □

3.27 Theorem:

Let $f:(X,\tau,\rho) \to (Y,\tau')$ be onto and a $pr^* -$open function. If (X,τ,ρ) is a $pr - T_2$-space, then $(Y,\tau')$is a T_2-space.

Proof:

Let $y_1 \neq y_2$. Since f is an onto function, then there exist $x_1 \neq x_2 \in X$, such that $y_1 = f(x_1)$, $y_2 = f(x_2)$. Since X is $pr - T_2$-space, then there exist $pr-$open sets G, H ,$G \cap H = \phi$. Since f is a $pr^* -$open function, then $f(G)$, $f(H)$ are two τ'-open sets in Y, and $f(G) \cap f(H) = f(G \cap H) = f(\phi) = \phi$.

Also, $y_1 = f(x_1) \in f(G)$, $y_2 = f(x_2) \in f(H)$.

Hence, $(Y,\tau')$is a T_2-space □

In the following, we state a definition for $pr^{**} -$continuous function, also mention some important theorems on ($pr-$open, $pr-$closed, $pr-$interior, and $pr-$closure) in bitopological space.

3.28 Definition:

Let (X,τ) be topological space, (Y,τ',ρ')be bitopological space, a function $f:(X,\tau) \to (Y,\tau',\rho')$ is said to be $pr^{**} -$**continuous** at $x \in X$ iff for every $pr-$open set V in Y containing $f(x)$ there exists τ-open set U in X containing x such that $f(U) \subset V$.

We say f is $pr^{**} -$continuous on X iff f is $pr^{**} -$continuous at each $x \in X$.

3.29 Definition:

Let $f:(X,\tau)\to(Y,\tau',\rho')$ be a function, then:

i.) f is said to be $pr^{**}-$open function iff $f(G)$ is $pr-$open set in Y , for every τ-open set G in X .

ii.) f is $pr^{**}-$closed function set iff $f(A)$ is $pr-$closed set in Y , for every τ-closed set A in X .

3.30 Theorem:

Let (X,τ) be a topological space, (Y,τ',ρ') be a bitopological space, and $f:(X,\tau)\to(Y,\tau',\rho')$be a function. If the inverse image under f of every $pr-$open set V of Y is a τ-open set of X, then f is $pr^{**}-$continuous function .

Proof:

Let $f^{-1}(V)$ be a τ-open set in X, for each a $pr-$open set V in Y. To prove f is $pr^{**}-$continuous.

Let $x\in X$ and V be $pr-$open in Y containing $f(x)$, so $f^{-1}(V)$ is τ-open set in X containing x and $f(f^{-1}(V))\subseteq V$. Then f is $pr^{**}-$continuous on X□

3.31 Theorem:

Let (X,τ) be a topological space, (Y,τ',ρ') be a bitopological space, and $f:(X,\tau)\to(Y,\tau',\rho')$ be a function. If the inverse image under f of every $pr-$closed set in Y is a τ-closed set in X, then f is $pr^{**}-$continuous.

Proof:

Let $f^{-1}(A)$ be a τ-closed in X for every $pr-$ closed set A in Y. To show that f is $pr^{**}-$continuous function. Let G be any $pr-$open set in Y. Then $Y-G$ is $pr-$closed in Y , so by hypothesis , $f^{-1}(Y-G)=X-f^{-1}(G)$ is τ-closed in X, that is $f^{-1}(G)$ is τ-open in X. Hence, by Theorem (3-30) f is $pr^{**}-$continuous function□

3.32 Theorem:

Let (X,τ) be a topological space, (Y,τ',ρ') be a bitopological space, and $f:(X,\tau)\to(Y,\tau',\rho')$be a function. If $f(\tau-cl(A)\subset pr-cl(f(A))$, for every $A\subset X$, then f is $pr^{**}-$continuous function.

Proof:

Let $f(\tau-cl(A))\subset pr-cl(f(A))$, for every $A\subset X$,and B be any $pr-$ closed set in Y, so that $pr-cl(B)=B$.Now, $f^{-1}(B)\subset X$, by hypothesis, $f(\tau-cl(f^{-1}(B)))\subset pr-cl(f(f^{-1}(B))\subset pr-cl(B)=B$. Therefore, $\tau-cl(f^{-1}(B))\subset f^{-1}(B)$.But $f^{-1}(B)\subset\tau-cl(f^{-1}(B))$ always. Then $\tau-cl(f^{-1}(B))=f^{-1}(B)$ and so $f^{-1}(B)$ is τ-closed set in X.Hence, by Theorem (3-31) f is $pr^{**}-$continuous function□

3.33 Theorem:

Let (X,τ) be a topological space, (Y,τ',ρ') be a bitopological space, and $f:(X,\tau)\to(Y,\tau',\rho')$be a function. If $\tau-cl(f^{-1}(B))\subset f^{-1}(pr-cl(B))$, for every $B\subset Y$, then f is $pr^{**}-$continuous function.

Proof:

Suppose $\tau-cl(f^{-1}(B))\subset f^{-1}(pr-cl(B))$, for every $B\subset Y$ that satisfy, let A be any $pr-$ closed set in Y, so that $pr-cl(A)=A$. By hypothesis, $\tau-cl(f^{-1}(A))\subset f^{-1}(pr-cl(A))=f^{-1}(A)$

But $f^{-1}(A)\subset\tau-cl(f^{-1}(A))$ always. Hence $\tau-cl(f^{-1}(A))=f^{-1}(A)$, so $f^{-1}(A)$ is τ-closed in X. It follows from Theorem (3-4), f is $pr^{**}-$continuous function□

3.34 Theorem:

Let (X,τ) be a topological space, (Y,τ',ρ') be a bitopological space, and $f:(X,\tau)\to(Y,\tau',\rho')$be a function. If $f^{-1}(pr-\text{int}(B))\subset\tau-\text{int}(f^{-1}(B))$, for every $B\subset Y$, then f is $pr^{**}-$continuous function.

Proof:

Let $f^{-1}(pr-\text{int}(B)) \subset \tau-\text{int}(f^{-1}(B))$, for every $B \subset Y$, that satisfy G be any $pr-$open set in Y, so that $pr-\text{int}(G)=G$. By hypothesis, $f^{-1}(pr-\text{int}(G)) \subset \tau-\text{int}(f^{-1}(G))$, since $f^{-1}(pr-\text{int}(G)) = f^{-1}(G)$, then $(f^{-1}(G)) \subset \tau-\text{int}(f^{-1}(G))$.
But $\tau-\text{int}(f^{-1}(G)) \subset f^{-1}(G)$ always and so $\tau-\text{int}(f^{-1}(G)) = f^{-1}(G)$.Therefore, $f^{-1}(G)$ is τ-open in X and consequently by Theorem (3-30) , f is a $pr^{**}-$continuous function □

In the following, we are discussion some important theorems that deal with $pr^{**}-$continuous functions on separation axioms in bitopological spaces**.**

3.35 Theorem:

Let (Y,τ',ρ') be $pr-T_\circ$-space. if $f:(X,\tau)\rightarrow(Y,\tau',\rho')$is a $pr^{**}-$continuous, 1-1 function, then (X,τ) is a T_o-space.

Proof:

Let $x_1 \neq x_2$ in X, such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$, since f is 1-1, so $y_1 \neq y_2$, but Y is $pr-T_\circ$-space. Then there exists $pr-$open set V in Y, such that $y_1 \in V$, and $y_2 \notin V$, since f is $pr^{**}-$continuous, then there exists τ-open set U in X, such that $x_1 \in U$, $x_2 \notin U$ and $f(U) \subset V$. Therefore (X,τ) is T_o-space □

3.36 Theorem:

Let $f:(X,\tau)\rightarrow(Y,\tau',\rho')$ be $pr^{**}-$open and onto function, if (X,τ) is a T_o-space, then (Y,τ',ρ') is a $pr-T_\circ$-space.

Proof:

Suppose $y_1, y_2 \in Y, y_1 \neq y_2$ since f is onto, there exist $x_1, x_2 \in X$, such that $y_1 = f(x_1)$, $y_2 = f(x_2)$, so $x_1 \neq x_2$.

Since X is T_o-space, then there exists τ-open set G, such that $x_1 \in G$, $x_2 \notin G$. Hence $y_1 = f(x_1) \in f(G)$, $y_2 = f(x_2) \notin f(G)$, since f is $pr^{**}-$open function, then $f(G)$ is a $pr-$open set in Y.

Therefore $(Y, \tau', \rho')$is a $pr-T_\circ$-space □

3.37 Theorem:

Let (Y, τ', ρ') be $pr-T_1$-space, if $f:(X,\tau) \to (Y, \tau', \rho')$is a $pr^{**}-$continuous and 1-1 function, then (X,τ) is a T_1-space.

Proof:

Let $x_1, x_2 \in X, x_1 \neq x_2$. Since f is a 1-1 function, then $f(x_1), f(x_2) \in Y$, Y is a $pr-T_1$-space, there exist V_1, V_2 $pr-$open sets in Y, such that $f(x_1) \in V_1$, $f(x_2) \notin V_1$ and $f(x_2) \in V_2$, $f(x_1) \notin V_2$. Since f is a $pr^{**}-$continuous function, then there exist two τ -open sets U_1, U_2 in X, such that $x_1 \in U_1$, , $x_2 \notin U_1$, and $x_1 \notin U_2$, $x_2 \in U_2$ and $f(U_1) \subset V_1$, $f(U_2) \subset V_2$. Hence, (X,τ) is T_1-space □

3.38 Theorem:

Let $f:(X,\tau) \to (Y, \tau', \rho')$ be $pr^{**}-$open and onto function, if (X,τ) is a T_1-space, then (Y, τ', ρ') is a $pr-T_1$-space.

Proof:

Suppose $y_1, y_2 \in Y, y_1 \neq y_2$ since f is onto, there exist $x_1, x_2 \in X$, such that $y_1 = f(x_1)$, $y_2 = f(x_2)$ and $x_1 \neq x_2 \in X$. Since X is T_1-space, there exist τ-open sets G, H in X such that $x_1 \in G$, but $x_2 \notin G$ and $x_2 \in H$ but $x_1 \notin H$. Hence, $f(x_1) \in f(G)$, $f(x_2) \in f(H)$, since f is $pr^{**}-$open function, hence, $f(G)$, $f(H)$ are $pr-$ open sets of Y. $y_1 \in f(G)$, but $y_2 \notin f(G)$ and $y_2 \in f(H)$, but $y_1 \notin f(H)$.

Then (Y,τ',ρ') is $pr-T_1$-space □

3.39 Theorem:

Let (Y,τ',ρ') be $pr-T_2$-space, $f:(X,\tau)\to(Y,\tau',\rho')$ be a $pr^{**}-$continuous and 1-1 function, then (X,τ) is a T_2-space.

Proof:

Let $x_1,x_2 \in X, x_1 \neq x_2$. Since f is 1-1 function, then $f(x_1)\neq f(x_2)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, $y_1 \neq y_2$. since Y is $pr-T_2$-space, there exist two $pr-$open sets G,H, in Y, such that $y_1 \in G$, $y_2 \in H$, $G \cap H = \phi$.

Since f is a $pr^{**}-$continuous function, so there exist two τ-open sets G_1,H_1, in X, such that $x_1 \in G_1$, $x_2 \in H_1$, and $y_1 \in f(G_1)\subset G$, $y_2 \in f(H_1)\subset H$, but $G \cap H = \phi$, so $f(G_1 \cap H_1) = f(G_1)\cap f(H_1) = \phi$. Then $G_1 \cap H_1 = \phi$ .Therefore, (X,τ) is a T_2 -space □

3.40 Theorem:

Let $f:(X,\tau)\to(Y,\tau',\rho')$ be $pr^{**}-$open and onto function, if (X,τ) be a T_2-space, then (Y,τ',ρ') is a $pr-T_2$-space.

Proof:

Let $y_1 \neq y_2$ since f is onto, then there exist $x_1 \neq x_2 \in X$, such that $y_1 = f(x_1)$, $y_1 = f(x_2)$ since X is T_2 space, then there exist τ-open sets G,H in X, such that $x_1 \in G$, $x_2 \in H$, $G \cap H = \phi$.

Since, f is a $pr^{**}-$open function, then $f(G)$, $f(H)$ are two $pr-$open sets in Y, and $f(G)\cap f(H) = f(G \cap H) = f(\phi) = \phi$. Also, $y_1 = f(x_1) \in f(G)$, $y_2 = f(x_2) \in f(H)$. Hence, (Y,τ',ρ') is $pr-T_2$-space □

References

[1] Al-Swidi, L. ,and Shaker , A. Y. "Semi compactness in bitopological spaces" , to appear in the magazine of technical institutes establishment (1993).

[2] Arya , S. P., and Nour, T. M. " Separation Axioms for Bitopological spaces" , Indian , J. pure appl. Math. 19(1988) .

[3] Bosi, G. "On the existence of Semicontinuous order-preserving functions on bitopological preodered spaces", Essen (FRG), October (1997), 22-24.

[4] Dontcher, J. "Survey On Preopen sets", ar 14, Math 30 (1998), 1-18.

[5] Jaleel, I.D. "S-openset in bitopological spaces", M.sc University of Baghdad, 2003.

[6] Jelic. M. "Nearly PTi-Continuous Mapping", Mathematics Bechure 46 (1994), 25-28.

[7] Jeli's , M., " T_i-pairwise continuous Functions and Bitopological Separation Axioms" , Mat. Vesnik 41, 3(1989) , 155-159 .

[8] Kelly, J. C. "Bitopological spaces", proc. London Math. Soc 13 (1963), 71-89.

[9] Kelly, J. C. " Bitopological space", publ Math Debrecen, 15, 1968, 87-90.

[10] Maheshwari , S. N., and Prusad , R., "Some New Separation Axioms" , Ann. Soc. Sci. Bruxelles , 89(1975) , 395-402 .

[11] Mashhour , A. S., Abd El-Monsef , M. E., and EL-Deep , S. N. "On Precontinuous and Weak precontinuous mapping" proc. Math. and phys. soc . Egypt . 51(1981) .

[12] Mirevic. M. "On Bitopological Separation Axioms", Math vesnik 38 (1986) , 313 -318 .

[13] Mukherjee. M.N. and anguly. S.G. "Generalization of Almost Continuous multi-Function to Bitopological spaces", Bull. Cal. Soc. (1987).

[14] Njastad , O., "ON some classes of nearly open sets" , pacific, J. Math. 15(1965) , 961-970 .

[15] Reilly , I. L., "On Bitopological Separation Properties" Nanta Math . (1972) , 14-25 .

[16] Singal , M. K. and Singal , A. R., "Some more Separation Axioms in Bitopological spaces" . Ann. Soc. Sci. Bruxelles 84 II(1970) , 203-207 .

[17] Tallafha, A. A. Al-Bsoul, and A. Fora, "Countable Dense Homogeneous Bitopological spaces", Tr. J. of Mathematics, 23 (1999), 233-242.

[18] Valeriu , P., "On Some Properties of Bitopological Semi Separation space" , (1977) , 71-77 .