وزارة التعليم العالي والبحث العلمي
جامعة بابل
كلية التربية
قسم الرياضيات

أنماط جديدة من التليفات

بحث مقدم
الى مجلس كلية التربية– جامعة بابل وهو جزء من متطلبات نيل
درجة الماجستير في علوم الرياضيات

قدمه الطالب
ظاهر والي فريح الركابي

باشراف
الاستاذ الدكتور هادي جابر مصطفى الحسني

1424هـ 2003م

بسم الله الرحمن الرحيم

﴿اقْرَأْ بِاسْمِ رَبِّكَ الَّذِي خَلَقَ ۞ خَلَقَ
الْإِنسَانَ مِنْ عَلَقٍ ۞ اقْرَأْ وَرَبُّكَ الْأَكْرَمُ ۞
الَّذِي عَلَّمَ بِالْقَلَمِ ۞ عَلَّمَ الْإِنسَانَ مَا لَمْ
يَعْلَمْ﴾

صدق الله العظيم

سورة العلق (1-5)

بسم الله الرحمن الرحيم

﴿وَنُرِيدُ أَنْ نَمُنَّ عَلَى الَّذِينَ اسْتُضْعِفُوا فِي الْأَرْضِ وَنَجْعَلَهُمْ أَئِمَّةً وَنَجْعَلَهُمْ الْوَارِثِينَ﴾

صدق الله العظيم

سورة العلق (1-5)

الإهداء

الى مبدأ الكون وخالقه .... لوجه ربي الكريم
الــى النــور الــذي ارســل رحمــه .... الحبيــب المصطفى
الــى الصــراط المســتقيم وبــاب علمــه .... علــي المرتضى
الــى مــن امرنــي ربــي بالاقتــداء بهــم هــداة الحــق والحجة .... أِمَّتي الاطهار
إلــى مميــت الفتنــة ومحــي الشــريعة .... القــائم المهدي (عج)
الــى مــن انــار طريقــي بفكــره واتبعــت نهجــه .... شهيد الجمعة
إلى من قرن الله طاعتهما بطاعته .... والدَّي
إلــى خيــر مــؤازرٍ وخيــر عضــد.... اخــوتي . أخواتي

ظاهر

المستخلص

في هذا العمل درسنا التليفات وقدمنا مفهوماً جديداً الا وهو التليفات –M.
معظم المبرهنات التي تخص التليفات تكون ايضاً صادقة اذا بدلنا التليفات بالتليفات -M بقية المبرهنات تحقق اذا اضفنا بعض الشروط. ومن بين النتائج التي حصلنا عليها هي :

1- جدار تليفين -M يكون ايضاً تليف -M .
2- جدار تليفين تقريبيين -M يكون ايضاً تليف تقريبي -M.
3- السحب الخلفي -M (M-pull back) الى تليف -M يكون تليف -M
4- السحب الخلفي -M (M-pull back) الى تليف تقريبي -M يكون تليف تقريبي -M.

Abstract

In this work we study fibrations and we Introduce anew concept, namely M-fibrations (Mixed fibrations).

Most of the theorems which are valid for fibrations will be also valid for M-fibrations the others will be valid if we add extra condition. Among the results we obtain are:

1- Aproduct of two Mixed fibration is also a Mixed fibration.

2- Aproduct of two Mixed Approximate fibration is also a Mixed Approximate fibration .

3- The M- pull back of Mixed fibration is also a Mixed fibration.

4- The M- pull back of Mixed approximate fibration is also a Mixed approximate fibrtion.

Contents

Acknowledgments

Praise to God, the lord of all thinks, and let his blessing and peace be upon the prophet of God Mohammad, the master of all massengers, and his family.

If there is any gratitude, it goes to the God, for his mercy and help.

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My sincere thank goes to my family whose patience and support helped me to carry out this work;

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Iam also grateful to my friends because they helped me to yet the references and thanks are also presented to AL-bearot bureau for computer serrices for his care and patience *in printing this research, especially (Muhammad AL - zeeton).*

<u>CERTIFICAT</u>

We certify we have read this research entitled “New types of fibrations” and, as an examining commining committee examined in its contents and that is connected with , and that, in our opinion, it is adequate as a for degree of Master of Science in Mathematics.

Chairman
Signature:
Name:
Date :   /   / 2003

Member
Signature:
Name:
Date :   /   / 2003

Member
Signature:
Name:
Date :   /   / 2003

Member (supervisor)
Signature:
Name: Hadi Jaber Mustaffa
Date :   /   / 2003

Approved by the Dean of the College of Education

Signature:
Name:
Date: /   / 2003

Supervisor Certification

I certify that this research was prepared under my supervision at the Department of Mathematics/ College of Education / University of Babylon as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

Signature:

Name: Dr. Hadi J.Mustafa

(professor)

Date : / / 2003

In view of the available recommendations, I forward this research for debate by examining committee.

Signature:

Name: Dr. Luay AL- Swidi

(Assist. professor)

Data : / / 2003

Introduction :-

Algeraic topology is an important branch of topology, fibrations is one of the important topics of algeraic topology ; the first paper in fibration appears in (1955) ; from that time till now several types of fibration have appeared . In our work , we introduce and study the new concept of M – fibration , this is a M –map , Let Y be any space , $f_1: X_1 \rightarrow Y$, $f_2: X_2 \rightarrow Y$, $\alpha : X_2 \rightarrow X_1$ be map such that $f_1 o\alpha = f_2$, let $\underline{X} = \{X_1, X_2\}$, $\underline{f} = \{f_1, f_2\}$ the ($\underline{X}$, $\underline{f}$, Y , α) has the Mixed homotopy lifting property (M-HLP) w.r.t a space Z iff given a map $k : Z \rightarrow X_2$ and a homotopy $h_t : Z \rightarrow Y$ such that $f_2 ok = h_o$ then there exists a homotopy $g_t: Z \rightarrow X_1$ such that (1) $f_1 og_t = h_t$ (2) $\alpha ok = g_o$ .M- fiber space is called M-fibration for class $\Re$ of space if $\underline{f}$ has (M-HLP) for each $Z \in \Re$. And extra of objection see[6 , 3]

This work consists of two chapters .In chapter one we study fiber space and Hurewicz fibrations . This chapter consists of three sections . In section one, we give the basic definitions and facts concerning Homotopy theory . In section two we study fiber space. We also give several examples and state several theorems . In section three, we study Hurewicz fibration in details . In chapter two we introduce a new concept namely M-fiber space and M-Hurewicz fibration . This chapter also consists of three sections . In section one, we study M-fiber space . In section two , we study M-Hurewicz fibration (Mixed of Hurewicz fibration).

In the last section, we study approximate fibration and we introduce new concepts namely M-approximate fibration (Mixed Approximate fibration).

The word Map in this work means continuous function

Chapter one

In this chapter we give the basic definitions and facts concerning Homotopy theory , fiber space, and Hurewicz fibration .

Section 1.1: __Homotopy theory__

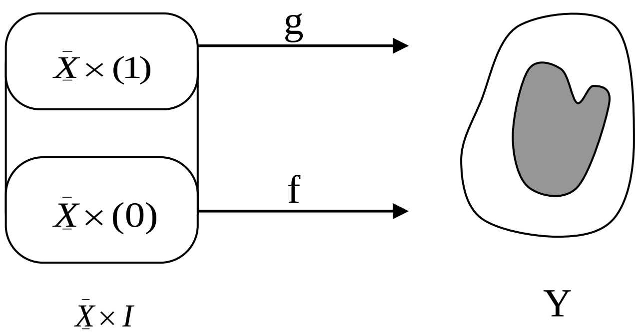
In this section we give basic ideas concerning Homotopy theory, also we study Homotopy Lifting property.

__Definiton 1.1.1:__ [2]

$Let\ X, Y$ be two topological spaces, and , $let\ \ f, g : X \to Y$ be two maps . We say that f is homotopic to g (by symbols $f \cong g$)

iff there exists a continuous function $H = X \times I \to Y$ such that

H (x,0) = f (x), H(x,1) = g (x), $\forall\ x \in$ X



H is called a homotopy.

There is an equivalent definition of homotopic maps as follows:

__Definition 1.1.2 :[2]__

$let\ X, Y$ be two topological spaces, and $let\ \ fig : X \to Y$ be two maps . We say that, *f* is homotopic to g (in symbols $f \cong g$), iff there

$\rightarrow$

exists a family of maps $\{h_t : X \quad Y, t \in I\}$ such that $h_o=f$, $h_1=g$. $\{h_t\}$ is called a homotopy.

The following theorem , gives us plainly examples of homotopic maps

Theorem 1.1.3:

Any two maps f,g : $X \rightarrow lR^n$ are homotopic

Proof:- Define H: $X \times I \rightarrow lR^n$ as following

$H(x,t) = (1-t) f(x) + tg(x)$ forall $x \in X$, $t \in I$

Then $H(x,0) = f(x)$, $H(x,1)= g(x)$

For example f: $X \rightarrow lR$ be defined as $f(x) = 5x$, and $g:X \rightarrow lR$ be defined as $g(x) = x^2$, Then f is homotopic to g.

Detinition 1.1.4:**[2]**

Let X be any space . We say that X is contractible iff $I_x: X \rightarrow X$ is homotopic to a constant map.

Theorem 1.1.5:[2]

Let f: $X \rightarrow S^1$ be map from a space X to the circle S^1 which is not surjective then f is homotopic to a constant map (That is f is called null homotopic).

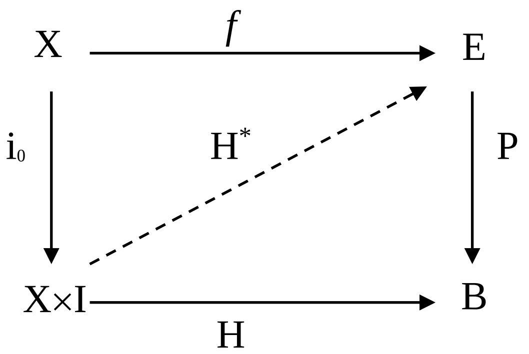
Example 1.1.6:

$let\ I_{lR^n} : lR^n \rightarrow lR^n$ be the identity map , then I_{R^n} is homotopic to a constant map, thus lR^n is contractible , but the identity of circle is not homotopic to constant map (s^1 is not contractible).

We recall the basic definitions of homotopy lifting property.

***Definition 1.1.7:*[7]**

Let P:E $\rightarrow$ B be a map, X be any space. We say that P has the homotopy lifting property (H.L.P by short) w.r.t X, iff given map f: X $\rightarrow$ E and homotopy H: X $\times$ I $\rightarrow$ B such that Pof= Hoi



Then there exist a homotopy $H^* : X \times I \rightarrow E$ such that

(1) H^*o $i_0 = f$ (2) PoH^* = H

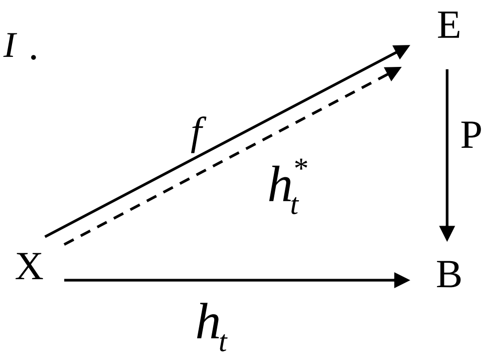
i.e H^*o i_0 $(x) = f(x) \Rightarrow H^*(x,0) = f(x)$

Po$H^*(x,t)$ = H(x,t) for all $x \in X$ and $t \in I$.

We can restate the definiton of homotopy lifting property as follows .

***Defintition 1.1.8:*[5]**

Let P:E$\rightarrow$B be a map .We say that P has H.L.P. w.r.t X iff given a map f: X $\rightarrow$ E and homotopy h_t: X $\rightarrow$ B such that Pof = h_o .Then there exist a homotopy h_t^*: X$\rightarrow$E such that (1) $h_o^* = f$ (2) Poh_t^* = h_t for all $x \in X$ and $t \in I$.



Section 1.2: <u>Fiber Space</u>

In this section we recall the basic definition and fact concerning fiber space and lifting

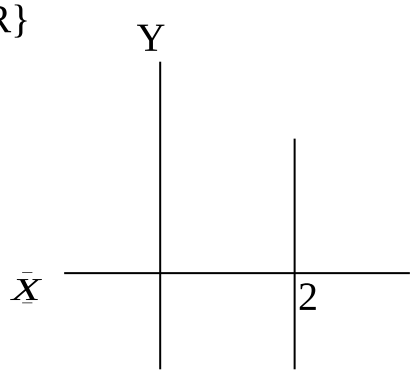
<u>Definition 1.2.1:</u> **<u>[2]</u>**

i- A fiber structure is a triple (E,P,B) consisting of two space E, B and a continuous surjection $P: E \to B$. The space E is called the total (or fibered) space P is termed the projection , and B is the base space let $b_o \in B$ Then $F = P^{-1}(b_o)$ and F is called fiber over b_o .
We refer to (E, P, B) as a fiber structure over B.

<u>Example 1.2.2:</u>

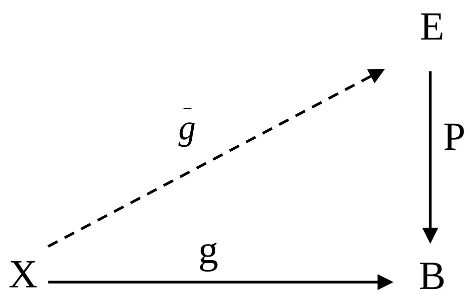
Let $P: \mathbb{R}^2 \to \mathbb{R}$ define by $P(x,y) = x \quad \forall (x, y) \in \mathbb{R}^2$

$$P^{-1}(2) = \{ (2,y) / y \in \mathbb{R} \}$$



<u>Definition 1.2 3:</u> **<u>[4]</u>**

Let (E,P,B) be a fiber structure , X be an arbitrary space, and $g: X \to B$ be continuous ,



A continuous $\bar{g}$: X→E such that Po $\bar{g}$ = g is called lifting (Covering [4]) of g into E .

The following example show that a map g: X → B need not be liftable into E

<u>Example 1.2.4:</u>**<u>[2]</u>**

Let P: 1R → S^1 be a map (fiber structure with projection $P(x) = e^{ix}$ and let X = S^1 it is easy to see that the identity map $I_{s^1} : S^1 \to S^1, I_{s^1}$ can not be lifted to map $f : S^1 \to 1R$

for suppose I_{s^1} can be lifted to map $f : s^1 \to 1R = 1R^1$ then by fact [2] [Each continuous $f : s^n \to 1R^n$ (that is flatting) sends at least one a pair of antipodal points to the same point].

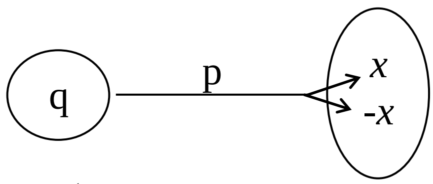
There exist a pair of antipodal points (x, -x) such that

$f(x) = f(-x)$ = q, but $pof = I_{s^1}$

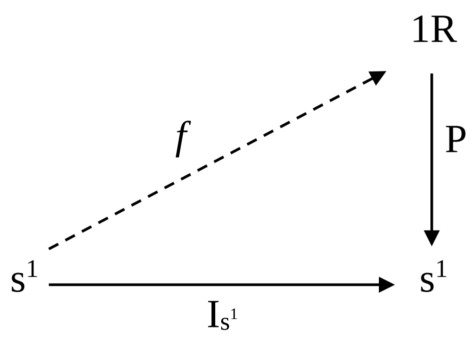
⇒ po$f$ (x) = $I_{s^1}(x)$ ⇒ p(f (x)) = x ⇒ p (q) = x

po$f$ (-x) = $I_{s^1}(-x)$ ⇒ p(f (-x)) = -x ⇒ p (q) = -x

This is contradiction



Then I_s^1: s^1→s^1 can not lifted to map f:s^1→1R



***Definition 1.2.5:*[2]**

A fiber structure (E,P,B) is called a fiber space or (fibration) for class $\Re$ of spaces if P has the homotopy lifting property (H.L.P) for each $X \in \Re$.

Theorem 1.2.6:

Let $E = B \times Y$ where Y is any space , define $P_1: B \times Y \to B$ as followes P_1 (b,y) = b for all $(b,y) \in B \times Y$ (where P_1 is the projection over the first factor) then P_1 has the H.L.P w.r.t every space X.

Proof:- Let $P_2: B \times Y \to Y$ is the projection over the second factor

define as P_2 (b,y) = y for all $(b,y) \in B \times Y$

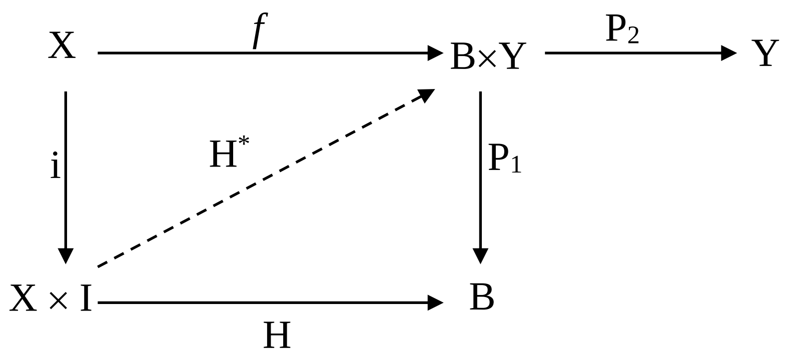
let $f: X \to B \times Y$ be any map and let $H: X \times I \to B$

be any homotopy such that $P_1 o f(x) = Hoi(x)$

Then we can define $H^*: X \times I \to B \times Y$ as follows

$H^*(x,t) = (H(x,t), P_2 o f(x))$ then

(1) $P_1 o H^* = H$ (2) $H^*(x,o) = f(x)$

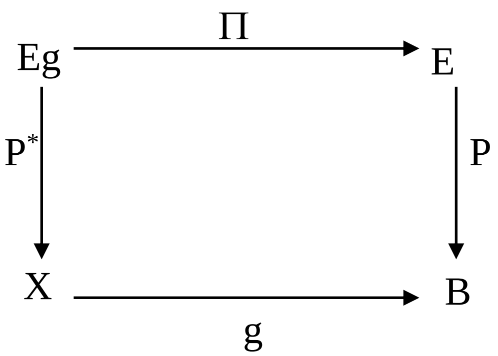


***Definition 1.2.7:*[2]**

Let (E,P,B) be a fiber structure , let X be any space, and let $g: X \to B$ be any continuous map into base B, let $Eg \subset E \times X$ be the subspace , Eg = { (e,x) $\in$ E $\times$ X / p(e) = g (x) } of the cartesian produet.

Let P^*: Eg $\to$ X be the projection P^* (e,x) = x. Then P^* is called the pull back of P by g.

Letting Π : Eg $\to$ E be the projection Π (e,x) = e , it is immediate from the definitions that the diagram is commutative such that Po Π = gop*



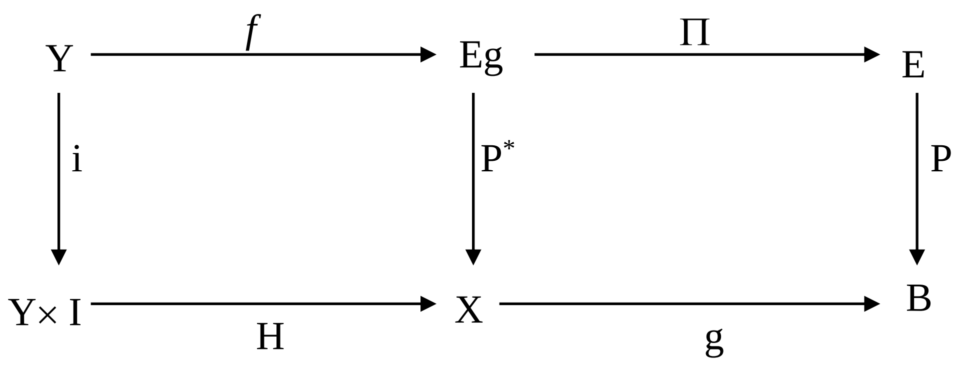
***Theorem 1.2.8:*[2]**

The pull back of a fibration is also fibration

proof: Let Y $\in \Re$, we will show that P^* has the H.L.P w.r.t Y.

let f: Y → Eg be a map and H :Y $\times$ I →X be a homotopy such that

P^*of (y) = Hoi(y) and i = Y $\to$ Y $\times$ I   be inclusion map



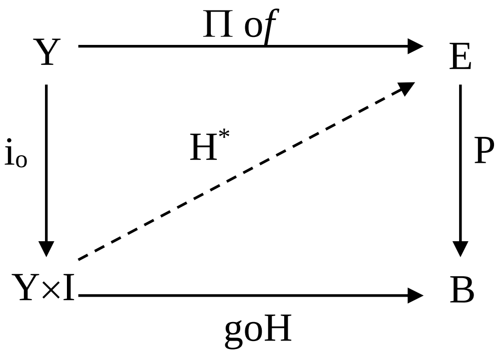
Consider $\Pi o f : Y \to E$ and $goH: Y \times I \to B$ are homotopic

Now $Po\,\Pi\, of = goHoi$

$Po\,\Pi\, of = goH(y,o)$

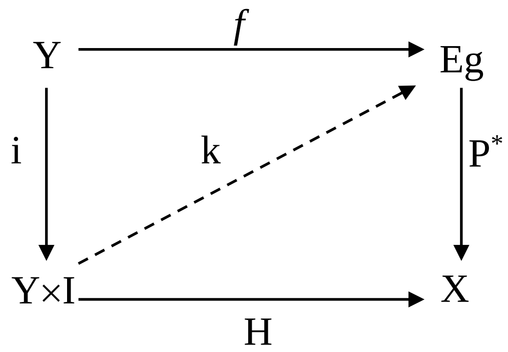
$Po\,\Pi\, of = goP^{*}of$ because $[P^{*}of = H\,(y,o)\,]$

But P is fibration. Then there exist a homotopy $H^{*} : Y \times I \to E$ such that $PoH^{*} = goH$, $H^{*}(y,o) = \Pi\, of$.



Define $k : Y \times I \to Eg$ by

$k\,(y,t) = (\,H^{*}(y,t), H\,(y,t)\,)$



Then (1) $P^{*}ok = H$, (2) $koi = f$

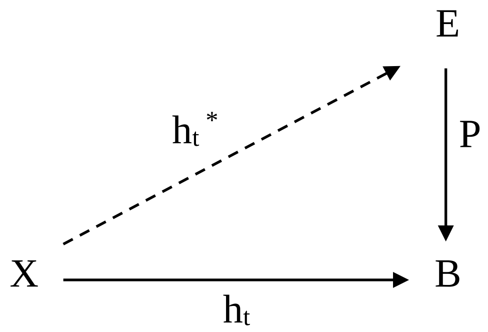
$P^{*}ok\,(y,t) = P^{*}(k\,(y,t)\,) = P^{*}(H^{*}(y,t), H\,(y,t)\,) = H\,(y,t)$

$koi = k\,(y,o) = (H^{*}(y,o), H\,(y,o)\,) = f\,(y)$

There fore (Eg,p^{*}, X) is a fibration for the class $\Re$ of space Y o

Definition 1.2.9:**[2]**

Let $h_t : X \rightarrow B$ be a homotopy , suppose that h_t^* can be lifting of h_t , we say that h_t^* is stationary with h_t , iff $h_t(x_o)$ constant. as a function of t , the function $h_t^*(x_o)$ is also constant as function of t .



Example 1.2.10:**[2]**

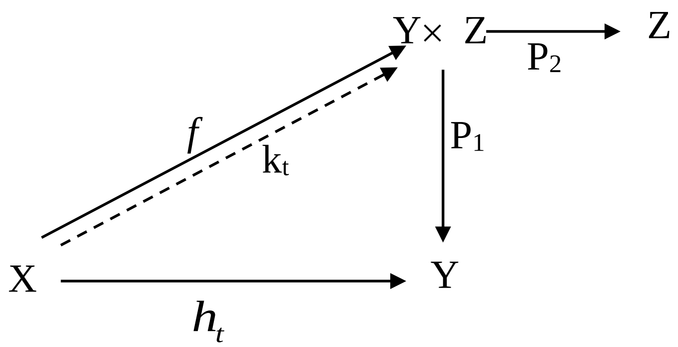
Let P_1: $Y \times Z \rightarrow Y$ be the first projection defined as $P_1(y,z) = y$, Then $(Y \times Z, P_1, Y)$ is fibration . In fact ,

let P_2: $Y \times Z \rightarrow Z$ be the projection defined as $P_2(y,z) = z$; then for any continuous f: $X \rightarrow Y \times Z$ and homotopy $h_t : X \rightarrow Y$ such that $P_1 o f = h_o$ the map k_t: $X \rightarrow Y \times Z$ define, as

$k_t(x) = (h_t(x), P_2 o f(x))$ such that.

(1) $k_o = f$ (2) $p_1 o k_t = h_t$

is H.L.P and is stationary with h_t.

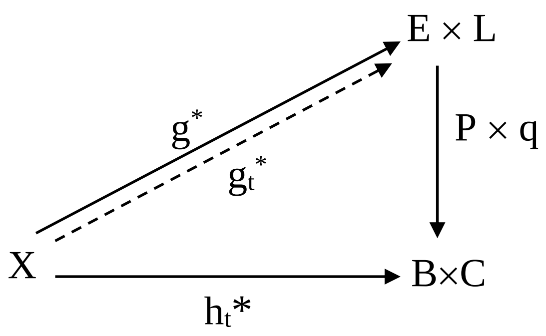


Theorem 1.2.11:

Let (E,P,B) and (L,q,C) be fibrations then $(E\times L, P \times q, B \times C)$ is also fibrtion.

Proof:

Let $g : X \to E$ and $g` : X \to L$ be any maps .Define $g^* : X \to E \times L$ be a map by $g^* (x) = ( g (x), g` (x))$, and $h_t : X \to B$ and $h_t` = X \to C$ be any maps . Define $h_t^* : X \to B \times C$ by $\hat{h}_t (x) = ( h (x), h` (x))$ such that and $(P\times q) o g^* = h_o^*$



Since P,q are fibrations . then there exists

$g_t: X \to E$ such that $Pog_t = h_t$, $g_o = g$ and $g`_t: X \to L$ such that $qog`_t = h_t$, $g`_o = g`$

Now for $\hat{h}_t$ there exist $g^*_t :X \to E \times L$ define as

$g^*_t (x) = (g_t (x), g`_t (x))$ such that

(1) $(P \times q) o g^*_t = h_t^*$, (2) $g^*_o = g^*$

then $P\times q : E\times L \to B\times C$ has HLP w.r.t X.

there for $P\times q$ is afibration o

Section 1.3: **Hurewicz fibration**

In this section we recall the basic definitions and facts concerning Hurewicz fibration .

Definition 1.3.1:**[7]**

i- Let (E, P, B) be a fiber structure, we say that P is a Hurewicz fibration iff P has the HLP w.r.t all spaces .

ii-Let (E,P,B) be a Hurewicz fibration we , say that P is a regular Hurewicz fibration .

The following theorem is an immediate consequenc of theorem 1.2.8

Theorem 1.3.2:

The pull back of a Hurewicz fibration is also a Hurewicz fibration.

The following theorem is an immediate consequenc of theorem 1.2.11

Theorem 1.3.3:

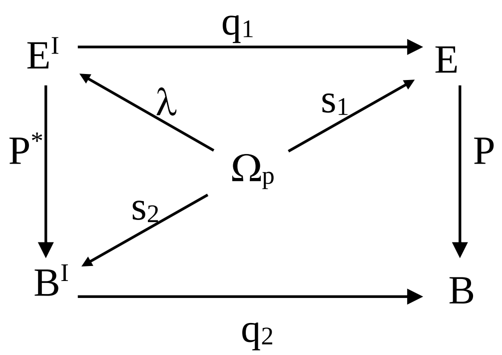
Let (E,P,B) and (L,q,C) are two Hurewicz fibrations then $(E \times L , P \times q, B \times C)$ is also Hurewicz fibration.

Definition 1.3.4:**[2]**

Let (E , P , B) be a fiber structure , and let B^I: ($\alpha : I \to B$) $\Omega_p \subset E \times B^I$ be subspace , $\Omega_p = \{(e, \alpha) \in E{\times}B^I / P(e) = \alpha(o) \}$ of the Cartesian product where

A lifting function for (E,P,B) is a continuous map $\lambda: \Omega_p \to E^I$ such that $\lambda(e,\alpha)(o) = e$ and $po\lambda(e,\alpha)(t) = \alpha(t)$ for each $(e,\alpha) \in \Omega_p$

and t ∈ I , we say that λ is regular if λ (e,α) is a constant path whenever α is a constant path.



Thus a lifting function associated to each e ∈ E and path α in B starting at P(e) , path λ(e,α) in E starting at e , that is ; a lift of α .[7]

Since the c-topology is used in E^I, the countinuity of λ is equivalence to that of associated $\lambda : \Omega_p \times I \to E$, by simpling

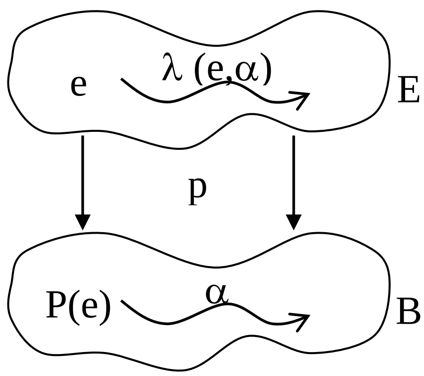
$\lambda (e,\alpha) \in E^I$ and $\lambda (e,\alpha): I\to E$

$\lambda (e,\alpha) (o) = e$, po $\lambda (e,\alpha) (t) = \alpha (t)$

$P^* (\alpha) : I \to B$

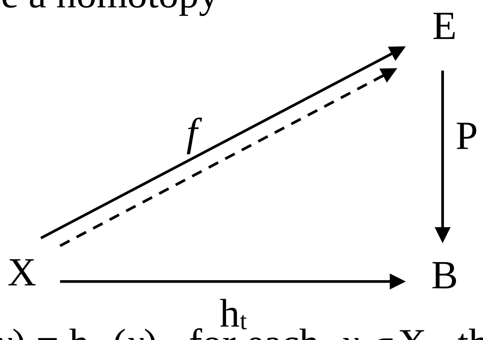
$P^* (\alpha) (t) = p (\alpha (t))$

$\lambda : \Omega_p \times I \to E$ such that $\lambda (e,\alpha) (t) = (\lambda (e, \alpha)) (t)$



Remark 1.3.5:

Let (P,E,B) be a fiber structure , f: X $\to$ E be given a map and let h_t: X $\to$ B be a homotopy



Such that P$\circ f(x) = h_o(x)$ for each $x \in$X , the map α_x : I $\to$ B defined by $\alpha_x(t) = h_t(x)$, defines a path in B

Theorem 1.3.6:

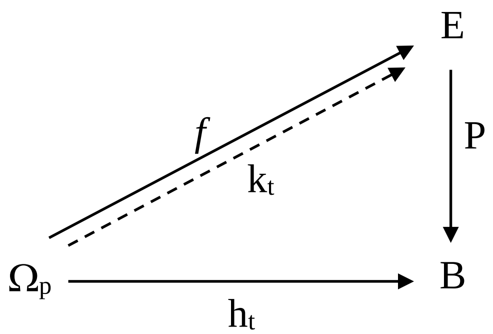
A map P: E $\to$ B is a fibration if and only if there exist a lifting function for p.

Proof: See [7]

***Theorem 1.3.7:*[2] (curtis- Hurewicz)**

The fiber structure (E,P,B) is a (regular) Hurewicz fibration if and only if a (regular) lifting function exists .

Proof :- If P is a(regular) Hurewicz fibration. Let X = Ω_p and f: $\Omega_P \to$E and $h_t : \Omega_p \to$ B defined by f(e, α) = e and $h_t(e,\alpha) = \alpha(t)$ then $h_o(e, \alpha) = \alpha(o)$= P(e) = P$\circ f(e,\alpha)$



there exist a map $k_t : \Omega_p \to E$ be a homotopy lifting such that

$k_o (e, \alpha) = f(e,\alpha) = e$ and $Pok_t = h_t$

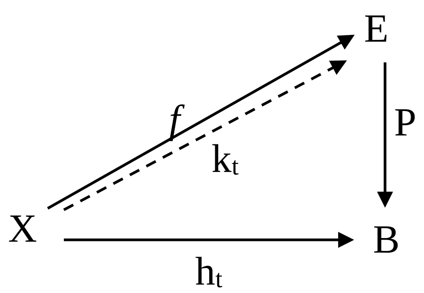
k_t defines a lifting function λ for p by $\lambda (e,\alpha) (t) = k_t (e,\alpha)$

λ is a lifting function which is (regular) whenever k_t is stationary with h_t.

<u>Conversely</u>: If P has a lifting function. Let f: X → E be given and

h_t: X → B be a homotopy such that $pof = h_o$, for each $x \in X$

Iet $\alpha_x : I \to B$ be defined by $\alpha_x(t) = h_t (x)$



Define a map $k_t : X \to E$ as followes :

$k_t (x) = \lambda (f (x) , \alpha_x) (t)$

then (1) $k_o (x) = f(x)$ and (2) $Pok_t = h_t$

therefore P has a (regular) Hurewicz fibration. o

Chapter two

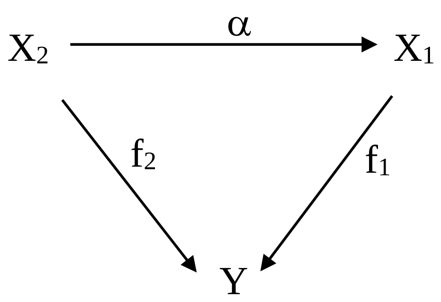
In this chapter we introduce a new concept namely mixed fibration (M-fibration) , Mixed Hurewicz fibration (M- Hurewicz fibration) and Mixed Lifting function.

Section 2.1: M-fibration

In this section we introduce and study a new concept which is namely Mixed fibration (M-fibration) . We start with the following definition .

Definition 2.1.1:

(1) Let X_1 , X_2, Y be three topological space , let $\underline{X}= \{X_1, X_2\}$, $\underline{f} =\{f_1, f_2\}$ where $f_1: X_1 \to Y$, $f_2: X_2 \to Y$ are two maps , and $\alpha: X_2 \to X_1$ such that $f_1 o\alpha = f_2$ then $(\underline{X}, \underline{f}, Y, \alpha)$ is a M-fiber space (Mixed- fiber space).

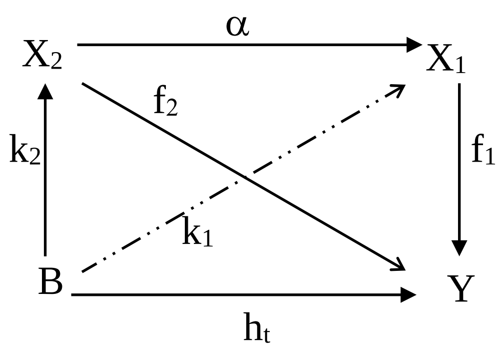


If $X_1 = X_2 =X$, α = identity , $f_1 = f_2 = f$ then (X, f, Y) is the usual fiber space .

(2) Let $(\underline{X}, \underline{f}, Y, \alpha)$ be a M-fiber space , Let $y_o \in Y$ then $F = \{\underline{f}^{-1} (y_o) \}$ is the M- fiber over y_o

Definition 2.1.2:

Let ($\underline{X}$, f, Y,α) be a M- fiber structure . Y, B be any spaces and h_t: B → Y be map . A continuous k_1:B → X_1 and k_2: B → X_2 such that $f_1 o k_1 = h_t$ and $f_2 o k_2 = h_t$, where $\underline{K} = \{k_1, k_2\}$ is called a M-lifting of h_t



Remark 2.1.3:

Let {$\underline{X}$, f, Y, α } be a M-fiber structure over Y.

If f_1,f_2 are homeomorphism then $f = \{f_1, f_2\}$ is called a M-homeomorphism.

Remark 2.1.4:

Let $\underline{X} = \{X_1, X_2\}$, $f = \{f_1, f_2\}$, $f_1: X_1 \to Y$, $f_2: X_2 \to Y$

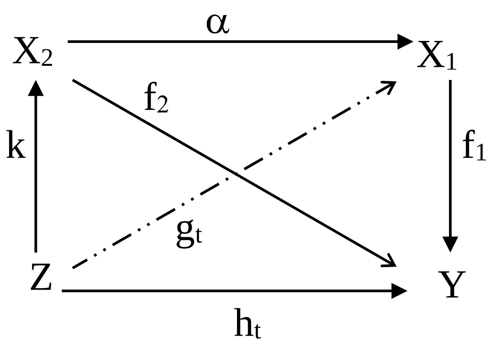
$f` = \{f_1`, f_2`\}$ , $f_1`: X_1 \to Y$, $f_2`: X_2 \to Y$

we say that f is M-homotopic to $f`$ if

(i) $f_1 \cong f_1`$ (ii) $f_2 \cong f_2`$

Definition 2.1.5:-

Let Y be any space , $f_1: X_1 \to Y$, $f_2: X_2 \to Y$, $\alpha : X_2 \to X_1$ be map such that $f_1 o\alpha = f_2$, let $\underline{X} = \{X_1, X_2\}$, $f = \{f_1, f_2\}$ the ($\underline{X}$,f,Y,α) has the Mixed homotopy lifting property (M-HLP) w.r.t a space Z iff given a map k : Z → X_2 and a homotopy h_t : Z → Y such that $f_2 o k = h_o$, then there exists a homotopy $g_t: Z \to X_1$ such that (1) $f_1 o g_t = h_t$ (2) $\alpha o k = g_o$



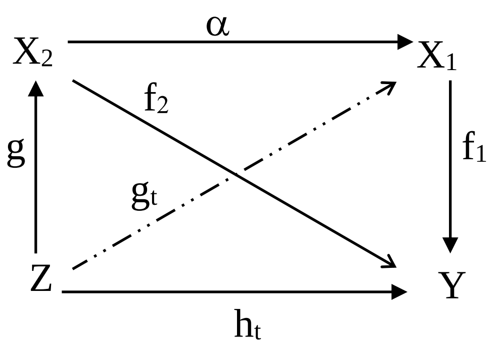
M- fiber space is called M-fibration for class $\Re$ of space if f has (M-HLP) for each $Z \in \Re$

Theorem 2.1.6 :

Every fibration is a Mixed fibration

Proof: Let $\{\underline{X}, f, Y, \alpha\}$ is fiber space such that

$\underline{X} = X_1 = X_2$, $\alpha = I$ (identity) , $f = f_1 = f_2$



let $g: Z \to X_2$ and homotopy $h_t : Z \to Y$ such that

$f_2 o g = h_o$, then there exist $g_t : Z \to X_1$ such that $g_o = \alpha o g$ and $f_1 o g_t = h_t$ for all $z \in Z$ and $t \in I$

Then f has M-HLP w.r.t space Z

Therefore f has M-fibration. o

Remark 2.1.7:

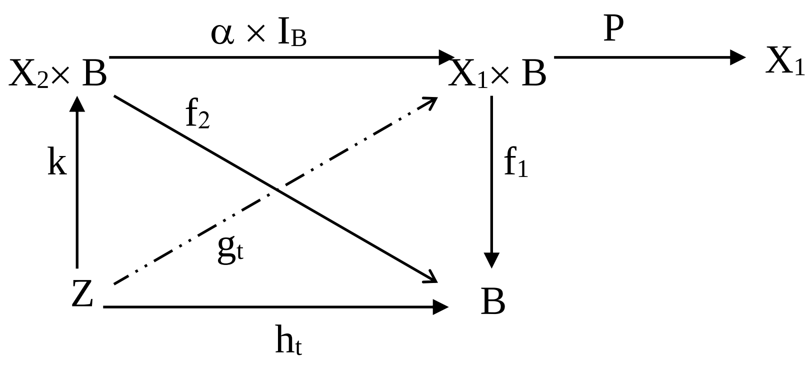
Mixed fibration may not be a fibration.

Example 2.1.8:

Every M- covering projection is aM-fibration in order (the natural M-Projection f:$\underline{X}$×B → B, define as f ($\underline{x}$,b) = b, is aM-fibration) which is not fibration.

for let P: $X_1 \times B \to X_1$ be the projection define as $P(x_1, b) = x_1$

forall $(x_1,b) \in X_1 \times B$, let k: $Z \to X_2 \times B$ be any map, and $h_t: Z \to B$ be any homotopy such that $f_2 o k = h_o$.



Then we may define g_t: $Z \to X_1 \times B$ as follows

$g_t(z) = \{ Po\alpha ok (z) , h_t (z) \}$, then g_t satisfy as condition

(1) $f_1 o g_t = h_t$ (2) $g_o = Po\alpha ok$.

There fore f: $\underline{X} \times B \to B$ is M-fibration, but not fibration.

Definition 2.1.9:

Let {$\underline{X}$, f, Y, α} be M- fiber structure let $\underline{X}$ be any space , and let g: Y\`→ Y be any continuous map into base Y .

Let X_1\` = { ($x_1$, y\`) ∈ X_1 × Y\` : $f_1 (x_1)$ = g (y\`) } and

X_2\` = { ($x_2$, y\`) ∈ X_2 × Y\` : $f_2 (x_2)$ = g (y\`) } then

$\underline{X}` = \{X_1`, X_2\}$ is called a M-pullback of f by g and

$f` = \{f_1`, f_2`\} : \underline{X}` \to Y`$ is called induced M-function of f by g .

Define $\alpha` : X_2` \to X_1`$ by $\alpha`(x_2 , y`) = (\alpha (x_2), y`)$

To show $\alpha`$ is continuous

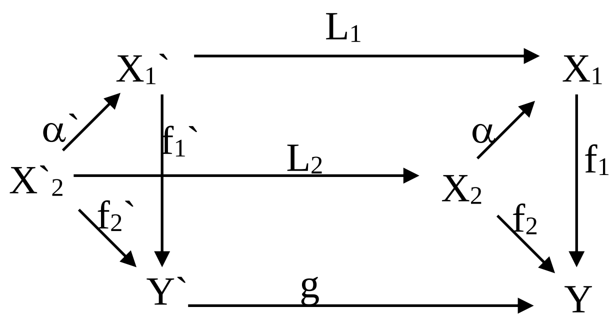
Since $\alpha` = \alpha \times I_{y`}$, α is continuous and $I_{y`}$ is continuous then $\alpha`$ is continuous .

To show is commutative

$f_1` o\ \alpha`(x_2 , y`) = f_1`(\alpha (x_2), y`) = y`$

$f_2` (x_2 , y`) = y`$

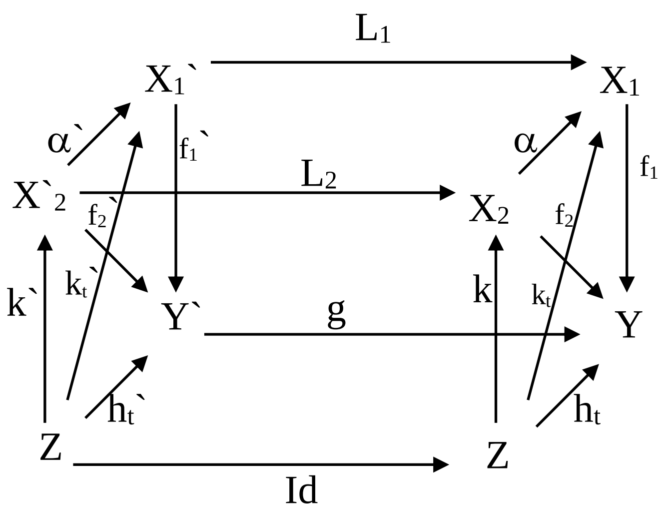
therefore $f_1` o\ \alpha` = f_2`$



Theorem 2.1.10:

The M – pull back of M- fibration is also M- fibration

Proof: Let $k`:Z \to X_2`$ and $k: Z \to X_2$ Define a homotopy $h_t : Z \to Y$ such that $h_o = f_2 o k$



since f is M-fibration

then there exist $k_t : Z \to X_1$ such that $f_1 o k_t = h_t$ and $k_o = \alpha o k$,

Define $h_t` : Z \to Y`$ as $g o h_t` = f_1 o k_t$ and $h_o` = f_2` o k`$.

then there exist $k_t` : Z \to X_1`$, where

$k_t` (z) = (k_t (z), h_t` (z))$, Hence $f_1` o k_t` = h_t`$ and $k_o` = \alpha` o k`$

There for $f` : \underline{X}` \to Y`$ is M- fibration. o

Proposition 2.1.11:

Let f: $\underline{X} \to Y$ and $f`$: $\underline{X}` \to Y$ be two M- fibration then $f \times f`$: $\underline{X} \times \underline{X}` \to Y \times Y`$ is also M-fibration.

Proof:- See proposition 2.2.3.

Section 2.2: M- Hurewicz fibration

In this section we introduce and study a new concept which is namely Mixed Hurewicz fibration (M-Hurewicz fibration) we start with the following definition.

Definition 2.2.1:

Let $\{\underline{X}, f, Y, \alpha\}$ be aM-fiber structure over Y, we say that f is M- Hurewicz fibration iff f has the M- HLP w.r.t all spaces.

The following theorem is an immediate consequence of theorem 1-2-10.

Theorem 2.2.2:

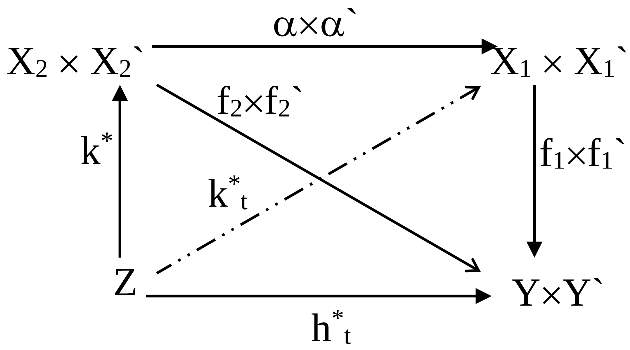
The M-pullback of M-Hurewicz fibration is also M-Hurewicz fibration

Proposition 2.2.3 :

Let f: $\underline{X} \to Y$ and $f`$ : $\underline{X}` \to Y`$ be two M- Hurewicz fibration then $f \times f`$: $\underline{X} \times \underline{X}` \to Y \times Y`$ is also M-Hurewicz fibration.

Proof: Let Z be any arbitrary space

Let $K^* : Z \to X_2 \times X_2`$ be a map, where $K^* (z) = ( k (z), k` (z))$ such that $k : Z \to X_2$ and $k` : Z \to X_2`$ and $h^*_t : Z \to Y \times Y`$ define as $h_t^* (z) = \{h_t (z), h_t` (z) \}$ and $(f_2 \times f_2`)\, o k^* = h_o$



such that $h_t : Z \to Y$ and $h_t` : Z \to Y`$ since $f, f`$ are M-Hurewicz fibration, then there exists a homotopy $k_t : Z \to X_1$ such that $f_1 o k_t = h_t$, $k_o = \alpha o k$ and a homotopy $k_t`: Z \to X_1`$ such that $f_1` o k_t` = h_t`$, $k_o` = \alpha` o k`$

Now, for h^*_t there exist $K^*_t : Z \to X_1 \times X_1`$ define as $K^*_t (z)= \{k_1(z) , k_t`(z)\}$ such that $(f \times f`)\, o K^*_t (z) = h^*_t (z)$ and $K_o^* = (\alpha \times \alpha^1)\, o\, K^*$, since Z be arbitrary .

Therefore $f \times f` : \underline{X} \times \underline{X}` \to Y \times Y`$ is M- Hurewicz . o

Definition 2.2.4:

Let $(\underline{X}, f, B, \alpha)$ be M-fiber structure and $B^I: \{W: I \to B\}$ $\Omega_f \subseteq \underline{X} \times B^I$ be the subspace , $\Omega_f = \{(x,w) \in \underline{X} \times B^I / f(x) = w(o) \}$.

AM –lifting function for$(\underline{X}, f, B, \alpha)$is continuous map $\underline{\lambda}: \Omega_f \to \underline{X}^I$ such that

$\underline{\lambda} (x,w) (o) = x$ and $f o \underline{\lambda} (x,w) (t) = w(t)$

for each $(x,w) \in \Omega_f$ and $t \in I$

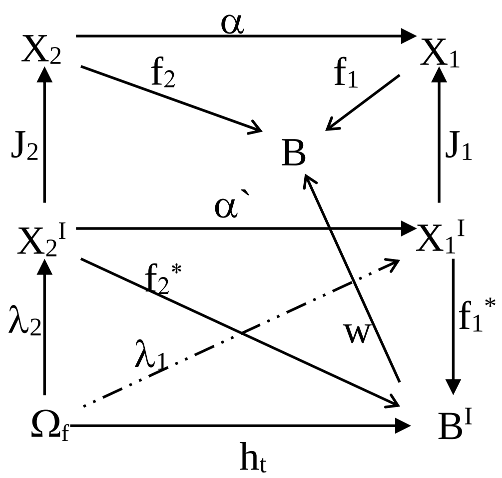
thus $\underline{\lambda} = \{\lambda_1 , \lambda_2 \}$ and $\Omega_f = \{\Omega_{f_1}, \Omega_{f_2}\}$, where

$\lambda_1 : \Omega_{f_1} \rightarrow X_1^I$ and $\lambda_2 : \Omega_{f_2} \rightarrow X_2^I$ defined as

$\lambda_1 (x_1, w) (o) = x_1$, $f_1 o\, \lambda_1 (x_1, w) (t) = w (t)$ and

$\lambda_2 (x_2, w) (o) = x_2$, $f_2 o\, \lambda_2 (x_2, w) (t) = w (t)$

thus a M-lifting function therefore associates



with each $x \in X$, and each bath w in B starting at $f(x)$ a path $\lambda(x_1,w)$ in X_1 and $\lambda_2 (x_2, w)$ in X_2 , starting at x_2 and x_1, and is M-cover of w since the c- topology used in X^I, the continuity of λ is equivalent to that of associated $\underline{\lambda}_i : \Omega_f \times I \rightarrow \underline{X}$

<u>Theorem 2.2.5:</u>

The M- fiber structure $\{\underline{X}, \underline{f}, B, \alpha\}$ is M-Hurewicz fibration if and only if f has M-lifting function.

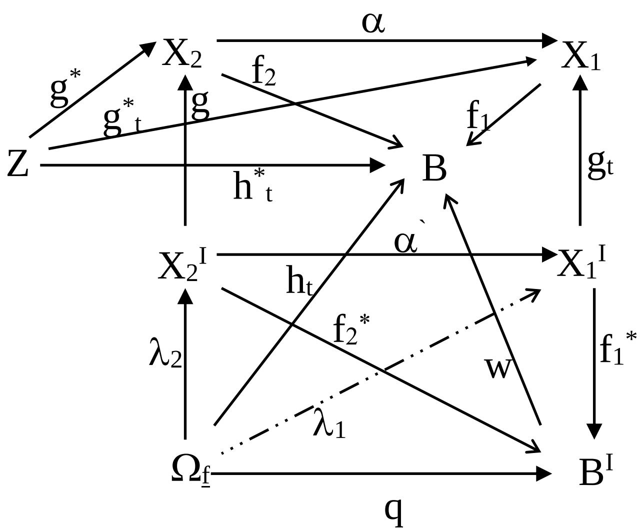
<u>Proof</u>:- if f is M-Hurewicz fibration , let $Z = \Omega_f$

let $g: \Omega_{f_2} \rightarrow X_2$ and $h_t : \Omega_f \rightarrow B$ be defined by $g (x_2,w)(o) = x_2$ and $h_t (x,w) = w (t)$, then $h_o (x,w) = w (o) = f_2 (x_2) = f_2 o g (x_2,w)$ since f is M- Hurewicz fibration . Then there exist $g_t : \Omega_{f_1} \rightarrow X_1$ such that $g_o (x_1,w) = \alpha o g (x_2,w) (o)$, $f_1 o g_t = h_t$. defined as a lifting function

$\underline{\lambda} (x, w) (t) = g_t (x,w)$

i.e: $\lambda_1(x_1,w)$ (t) = $g_t(x,w) \Rightarrow \lambda_1(x_1,w)$ (o) = $g_o(x_1,w) = x_1$

$f_1 o \lambda_1(x_1,w)$ (t) = $f_1 o g_t (x_1,w) \Rightarrow f_1 o \lambda_1(x_1,w) = h_t (x_1,w) = w(t)$



Conversely : If *f* has M-lifting function

Let $g^*:Z \to X_2$ and $h^*_t:Z \to B$ such that $f_2 o g^* = h^*_o$, consider $w_z:I \to B$ such that $w_z (t) = h^*_t (z)$ and define $g^*_t : Z \to X_1$ as $g^*_t(z) = \lambda_1 (\alpha o g^*(z), w_z) (t)$

Thus : $g^*_o = \alpha o g^*$ and $f_1 o g^*_t = h^*_t$

Hence g^*_t is M-HLP of h^*_t w.r.t Z

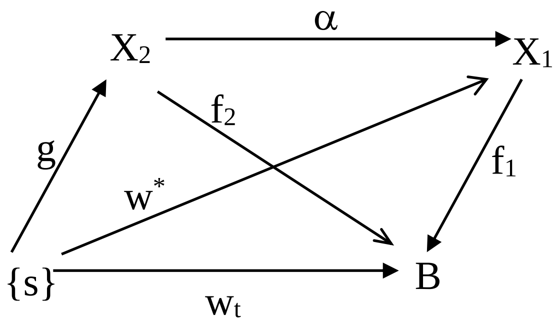
Since Z is arbitrary

Therefore *f* is M-Hurewicz fibration. o

<u>Corollary : 2.2.6</u>:

Let (X, *f*, B,α) is M-Hurewicz fibration, then any w bath in B such that $w_o = f_2 (x_2) = f_1 o \alpha (x_2)$, can be M- lifted to a path in X_1.

Proof:-



A path w: $I \to B$ can be regarded as a homotopy w_t: $\{s\} \to B$ where $\{s\}$ is a point space, and a point $x_2 \in X_2$ such that $f_2(x_2) = w_o$, corresponds to a map g: $\{s\} \to X_2$ such that $(f_2 o g)\{s\} = w_o(\{s\})$, and since f is a M-Hurewicz fibration there exists a path w_t^* in X_1 such that $w^*_o = \alpha(x_2)$ and $f_1 o w_t^* = w_t$

Where w_t^*: $\{s\} \to X_1$ is a M-lifted of w_t: $\{s\} \to B$. o

Section 2.3: Approximate and Mixed Approximate fibration .

In this section we recall the basic defintions and facts concerning Approximate fibration and we introduce a new concept namly Mixed approximate fibration (M-Approximate fibration).

***Definition* 2.3.1:**

Let f,g: $E \to B$ be two mappings and ξ be an open cover of B we say that f,g are ξ- closed iff give $e \in E$, then there exist $w \in \xi$ such that $f(e), g(e) \in w$

***Definition* 2.3.2**

Let P: $E \to B$ be mapping from E into B , we say that P has Approximate homotopy lifting property (A-HLP) w.r.t space X iff given f: $X \to E$ and a homotopy h_t: $X \to B$ such that $Pof = h_o$ and open cover ξ of B then there exists a homotopy g_t :$X \to E$ such that $g_o = f$ and Pog_t, h_t are ξ – closed.

Definition 2.3.3:

Let $P:E \to B$ be a mapping . We say that P is an approximate fibration iff, P has the (A-HLP) w.r.t the class $\Re$ of spaces also we say that P is an approximate Hurewicz fibration iff P has the (A- HLP) w.r.t the class of all space.

***Remark : 2.3.4:*[1]**

1- Every Hurewicz fibration is an approximate Hurewicz fibration .

2- Every fibration is an approximate fibration .

3- Approximate fibration may not be a fibration. for example see [1].

The proof of the following theorem is similar to the proof of theorem 1.2.11

***Theorem 2.3.5*:**

Let (E,P,B) and (L,q,C) be Approximate fibration then $(E\times L, P\times q, B\times C)$ is also Approximate fibrtion . This theorem also is true for approximat Hurewicz fibration.

***Proposition 2.3.6*:[1]**

The Pull back of an approximate fibration is also an approximate fibration.

Definition 2.3.7:

Let $\underline{X}$, Y be a topological space $f_1: X_1 \to Y$, $f_2 : X_2 \to Y$ $\alpha:X_2\to X_1$ be a map such that $f_1 o\alpha=f_2$, where $\underline{X}= \{X_1,X_2\}$, $\underline{f} =\{f_1,f_2\}$, then we say that $\underline{f}$ has the Mixed approximate homotopy lifting property (M- AHLP) iff given a map $k : Z \to X_2$ and homotpy $h_t : Z \to Y$ such that $f_2 ok = h_o$ and all open cover ξ of Y , then there exist a homotopy $g_t:Z \to X_1$ such that $g_o= \alpha o k$ and $f_1 o g_t$, h_t are ξ – close in h_t.

then g_t is called the approximate lifte of h_t

Map with (M-AHLP) w.r.t all class $\Re$ of spaces called Mixed approximate fibration.

Also for all spaces is called Mixed approximat Hurewicz fibration.

For the above definition we have the following result.

Remark. 2.3.8:

1- Every Mixed fibration is an Mixed approximate fibration .

2- Every approximate fibration is an Mixed approximate fibration.

3- A Mixed approximate fibration may not be an approximate fibration.

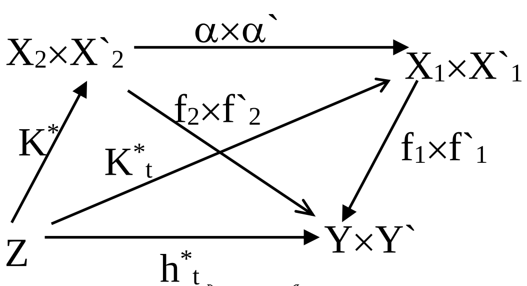
Theorem 2.3.9:

Let f: $\underline{X} \to Y$ and f ` : $\underline{X}$` $\to$ Y` be two M- Approximate fibration then $f \times f$ `: $\underline{X} \times \underline{X}$` $\to Y \times$ Y` is also M- Approximate fibration .

Proof:- Let Y be topological space

Let $K^* : Z \to X_2 \times X`_2$ be a map, where $K^*(z) = \{ k (z), k`(z) \}$ such that $k : Z \to X_2$ and $k` : Z \to X`_2$.

let $h^*_t : Z \to Y \times Y`$ defined as $h^*_t (z) = \{h_t (z), h`_1 (z)\}$and $(f_2 \times f`_2) o k^* = h_o$



such that $h_t : Z \to Y$ and $h`_t : Z \to Y`$ and open cover ξ of Y and open cover $\xi`$ of Y`. since f , $f`$ are M- approximate fibration then there exists a homotopy $k_t$: $Z \to X_1$ such that $k_o = \alpha o k$ and $f_1 o k_t$ , $h_t$ are $\xi$–closed in $h_t$ and homotopy $k_t`: Z \to X`_1$ such that $k`_o = \alpha` o k`$ and $f_1`ok`_t$, $h`_t$ are $\xi`$ - closed in $h`_t$.

Now for h^*_t and open cover $\xi\times\xi`$ of $Y\times Y`$ there exists $K^*_t : Z\to X_1\times X`_1$ defined as $K^*_t (z) = \{k_t(z) , k`_t (z)\}$ such that $K^*_o=(\alpha o \alpha`)oK^*$ and $(f_1\times f_1`)oK^*_t (z)$, $h^*_t (z)$ are $\xi\times\xi`$ closed in $h^*_t$ therefore $f \times f` : \underline{X} \times \underline{X}` \to Y \times Y`$ is M-approximate fibration . o

The proof of the following proposition is similar to the theorem 2.1.10.

<u>Proposition 2.3.10:</u>

The M-pull back of M-approximate fibration is also Mixed approximate fibration.

Future work

Here we refer to some future work .

Let Y be any space and let $f_1: X_1 \to Y$, $f_2: X_2 \to Y$, $f_3 : X_3 \to Y$, $\alpha_{21}: X_2 \to X_1$, $\alpha_{32}: X_3 \to X_2$, $\alpha_{31}: X_3 \to X_1$, such that $\alpha_{21} o \alpha_{32} = \alpha_{31}$, $f_1 o \alpha_{21} = f_2$, $f_2 o \alpha_{32} = f_3$, $f_3 o \alpha_{31} = f_1$, let $\underline{X} = \{X_1, X_2, X_3\}$, $\alpha_{ij} = \{\alpha_{21}, \alpha_{32}, \alpha_{31}\}$, $\underline{f} = \{f_1, f_2, f_3\}$, then $\{\underline{X}, \underline{f}, Y, \alpha_{ij}\}$ has 3-homotopy lifting property (3-HLP) w.r.t a space Z iff , given a map $k_i: Z \to X_I$ and a homotopy $h_t: Z \to Y$ such that $f_i o k_i = h_o$, then there exists a homotopy $L_t^j = Z \to X_j$ where i > j , such that $f_j o L_t^j = h_t$, $\alpha_{ij} o k_i = L_o^j$.

The3-fiber space is called 3-fibration , similarity we find 4-fibration.

So what we had discussed in our work is 2-fibration.

“Reference”

1- Abod , H-M, “Approximat fibration”, M.S.C.thesis, Baghdad university, 1990.

2- Dugundji , J., “Topology”, Allyn and Bacon. Boston, 1966.

3- Hu, S.T, “Elements of general Topology”, Hilden – day ,1966.

4- Munkres, J.,“A First Course in Topology”,Prentice Hall, 2001.

5- Mustafa, H.J., “Some theorems on fibration and Cofibration”, Ph. Dr. thesis, California University , LosAngeles , 1972.

6- Nassar, M.A., “Some results in the theory of fibration and cofibration” Ph. Dr. thesis, Baghdad Unversity, Ibn AL-Haitham, 2003.

7- Spanier, E. H., “Algebraic Topology”, Mc Graw- Hill, 1966.