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ON JULIA SETS

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Abstract

Complex dynamics is the study of iteration of maps which map the complex plane into itself . In general , their dynamics are quite complicated and hard to explain , but for some classes of maps , many interesting results can be proved . For example , one often studies the Julia sets of polynomial maps .

The Julia set of the quadratic map of the form ($z^2 + c$) was studied extensively . In this thesis , we study the Julia set of the quadratic map of the form $Q_\lambda(z)$ = ($\lambda z - \lambda z^{٢}$) .We found :

If $\lambda = 2$, then $J(Q_\lambda)$ is the unit circle .

If $\lambda = 4$, then $J(Q_\lambda)$ is the line segment [٠,١] .

If ١<$|\lambda|$< $1+\sqrt{2}$, then $J(Q_\lambda)$ is simple closed curve such that Julia set which contains no smooth arcs .

If $\lambda = 1 \mp \sqrt{5}$, then $J(Q_\lambda)$ has infinitely many components .

If $Q_\lambda^n(0.5) \to \infty$, then $J(Q_\lambda)$ is totally disconnected .

Otherwise $J(Q_\lambda)$ is connected .

List of Symbols

X	metric space
N	The set of natural numbers
Z	The set of integer numbers
R	The set of real numbers
C	The set of complex numbers
f, g, h	maps
$\overline{A}$	closure of the set A
U	neighborhood
$J(f)$	Julia set of f
C_∞	Riemann sphere
$r(z)$	rational map
$F(f)$	Fatou set of f
Γ	simple closed curve
∂A	boundary of the set A
$K(f)$	filled Julia set of f
$A(z)$	basin of attraction of z
$D_r^o(z_o)$	open disc of radius r about z_0
$D_r(z_o)$	closed disc of radius r about z_0
S^1	unit circle in C
q_c	quadratic polynomial map of the form (z^2+c)
Q_λ	quadratic polynomial map of the form ($\lambda z-\lambda z^{٢}$)

Introduction

In complex dynamics , the iteration theory originated in ١٩١٠ [٢١] . Among the most important concepts in complex dynamics are Julia sets .

Julia sets were studied by the French mathematician Gaston Julia (١٨٩٣ – ١٩٧٨) , who developed much of theory when he was recovering from his wounds in an army hospital during world war I . He published a long paper in French language in [١٦] . Another pioneer figure in the study of complex dynamics was Pierr Fatou (١٨٧٨ – ١٩٢٩) who published a paper in French language in [١٣] . Julia and Fatou looked at the iteration of the simplest quadratic map of the form $(z^2 + c)$. The iteration theory of rational maps also appeared in the work of Julia and Fatou in ١٩٢٠ . The study of dynamics of entire maps essentially started in ١٩٢٦ in the work of Fatou [١٣].If we want to see the work on complex dynamics of rational maps , polynomials and entire maps , we refer to [٢] , [٤] ,[٦] ,[٧],[٨] . In ١٩٨٠ Adrien Douady and J.H.Hubbard studied dynamics of polynomials of degree two of the form $(z^2 + c)$ using computers [١١] .

From a dynamical systems point of view , all of the interesting behavior of a complex analytic map occurs on its Julia set , it is this set that contains the interesting topology [٢١] .

The idea behind the Julia sets is to study whether the absolute value of a point in the complex plane converges towards infinity or not , when it is iterated under a map . All the points that do not go toward infinity , when iterated , are in the Julia set . Each such map f partitions the extended complex plane C_∞ into two regions , one where iteration of the map chaotic and one where it is not . The non – chaotic region , called Fatou set , is the set of all points z such that , under iteration of f , the point z and all its neighbors do approximately the same thing . The remainder of complex plane is called the Julia set and consists of these points which do not behave like all closely neighboring points . Roughly speaking , the Fatou set is the set where iterative behavior is relatively tame in the sense that points close to each other behave similarly while the Julia set is the set where chaotic phenomena take place . For example , consider the map $f(z) = z^2$. Under iteration by f , point's z_0 in the interior of the unit circle are attracted to the origin , while point's z_0 in the exterior are attracted to the point at infinity . Thus both the interior and exterior of the unit circle lie in the Fatou set of f . However , given two points on the unit circle whose angular difference is ε , their iterates both lie on the unit circle but the angular distance between them increases to 2ε under f . Thus any two such points are driven apart under f . Moreover , any neighborhood of a point on the unit circle contains points that converge to ٠ and points that escape to infinity , as well as points that remain on the unit circle . Thus the Julia set is the entire unit circle .

In fact , some of the results in this work for polynomials hold for more general classes of analytic maps such as rational maps or entire maps , [٥] , [١٠] , [١٢] , [٢٢] , but there are some propositions that are not satisfied in general , for example proposition (٢.١.٢) only holds for polynomials .

In general , distinct maps have distinct Julia sets , however , there exist distinct polynomial maps , rational maps and entire maps that have the same Julia sets [١٨] , [٢٠] , [٣١] .

The Julia set of a polynomial typically has a complicated , self – similar structure . Therefore the Julia sets are fractals [١٢] , [٢١] , [٢٥] . However , there exist rational maps whose Julia sets fail to be quasi-self-similar [١٥] .

In this work , we recall three definitions of Julia sets and we study the relation between them . There are other definitions of Julia sets , some of them are equivalent to definition (١.٣.٧) if f is a polynomial [٣] .

The goal of our work is to study Julia sets of quadratic maps of the form $Q\lambda(z) = (\lambda z - \lambda z^{٢})$, where λ is complex constant . Among our special results are the following :

If $\lambda = 2$, the Julia set is the unit circle .

If $\lambda = 4$, the Julia set is the line segment [٠,١] .

If ١ < $|\lambda| < 1+\sqrt{2}$, the Julia set is simple closed curve such that Julia set which contains no smooth arcs .

If $\lambda = 1 \mp \sqrt{5}$,then the Julia set consists of infinitely many components. We also study the topological properties of Julia sets such as total disconnectivity and connectedness .

This work is divided into three chapters . Many of the results of chapter one and chapter two are known , hence we state some of them without proofs , however we proof most theorems and propositions that are stated in the literature without proofs .

In chapter one we recall the different definitions of Julia sets , and we study three of them .We give some fundamental definitions and theorems.

In chapter two we recall the general properties of Julia sets of polynomials as given by one of the previous definitions . We also show that their definitions are equivalent for polynomials of degree ≥ ٢ . In the second section of this chapter , we give the properties of the Julia set of maps of the form $q_c(z) = z^2 + c$. In the third section we study the chaoticity of a map for the Julia set.

Chapter three is divided into two sections , in the first we give the geometric properties of the Julia sets for maps of the form ($\lambda z - \lambda z^{٢}$) , while in the second section we study the topological properties of the Julia set for maps of the form ($\lambda z - \lambda z^{٢}$) .

In the end of this thesis we set an appendix showing the practical part of the subject .

CHAPTER ١

Different definitions of Julia sets

Our goal in this chapter is to study the basic definitions and properties of Julia sets . This chapter consists of three sections . In section one , we recall some fundamental definitions . In section two , we give the notions of normality and Fatou set with their relations to Julia sets . In section three , we give the concept of filled Julia set .

Section One (١.١)

Elementary definitions and theorems

Our goal in this section is the presentation of definitions and theorems that we will use later in this research .

We use the symbol X in this section to denote a metric space with metric d . Special cases are $X = R$ or C and d is the usual metric .

We begin this section with the following definition . For general reference see [١٩] .

Let X be a metric space . All points and sets mentioned below are understood to be elements and subsets of X . Let us recall the following :

(a) The complement of A (denoted by $X - A$) is the set of all points $x \in X$ such that $x \notin A$.

(b) A point x is an interior point of A if there is a neighborhood U of x such that $U \subset A$.

(c) A is open if every point of A is an interior point of A .

(d) A point x is a limit point of the set A if every neighborhood of x contains a point $y \neq x$ such that $y \in A$.

(e) If $x \in A$ and x is not a limit point , then x is called an isolated point of A.

(f) A is closed if every limit point of A is a point of A .

(g) A is perfect if A is closed and if every point of A is a limit point of A .

(h) A is dense in X if $\overline{A} = X$, where $\overline{A}$ = the closure of A = the intersection of all closed sets containing A .

Let $f : X \to X$ be a map . The map is a homeomorphism if f is one – to - one , onto and f , $f^{-١}$ are continuous .

If $f : X \to X$ is smooth , then f is called a C^r - diffeomorphism if f is a C^r- homeomorphism such that $f^{-١}$ is also C^r .

Let $U \subset C$ be an open set . $f : U \to C$ is an analytic in U if it is analytic at each $z_0 \in U$.

Let $f : X \to X$ be a map .If the map is iterated , then the n-th iteration step is denoted by f^n. We put $f^o(x) = x$, $f^{n+1}(x) = f \circ f^n(x)$ for $n \in N$, $x \in X$.

Definition (١.١.١) [٢٦]

Let $f : X \to X$ be a map , then the orbit of a point $x \in X$ is the sequence $\{ f^n(x) \}$, $n \in Z$.

Example(١.١.٢)[٢٤]

Let $f : R \to R$ be a map such that $f(x) = x^2$, then the orbit of $x =$١ is ١,١,١,١,… , and if x = ٢ , the orbit is ٢,٤,١٦,… .

Definition (١.١. ٣) [٢٤]

Let $f : X \to X$ be a map . A point $x \in X$ is called a fixed point if $f(x) = x$. It is a periodic with period n if $f^n(x) = x$, but $f^m(x) \neq x$ for $m < n$.

Example (١.١.٤) [١٠] (pp. ١٨)

Let $g : R \to R$ be a map such that $g(x) = x^2 - 1$, this map has fixed points at (١+$\sqrt{5}$)/٢ , while the points ٠ and -١ lie on a periodic orbit of period ٢ .

A fixed point is periodic of period one .

Definition (١.١.٥) [١٠] (pp.٢٤)

Let $f: X \to X$ be a smooth map , where $X = R$ or C . Let x be a periodic point of period n for f . The point x is hyperbolic if $\left|(f^n)'(x)\right| \neq ١$, x is attracting periodic point if $\left|(f^n)'(x)\right| < ١$ and x is repelling periodic point if $\left|(f^n)'(x)\right| > ١$.

Otherwise $\left|(f^n)'(x)\right| = ١$, x is non- hyperbolic (neutral periodic point) .

Example (١.١.٦) [١٠] (pp.٢٨)

(i) Let $f : R \to R$ be a map such that $f(x) = x^2$, if $x = ٠$ then x is an attracting fixed point. (ii) Let $g : R \to R$ be a map such that $g(x) = x^2 - 1$, if $x = \frac{1+\sqrt{5}}{2}$ then x is a repelling fixed point . (iii) Let $h : R \to R$ be a map such that $h(x) = x^2 + x$, if $x = ٠$ then x is neutral fixed point .

A point x is a critical point of f if $f'(x) = 0$.

Definition (١.١.٧) [١٤] (pp.١٦)

Let $f: X \to X$ be a map . Suppose x is a fixed point of f . Then the basin of attraction of x consists of all y such that $f^n(y)$ converges to x as $n \to \infty$.

Example (١.١.٨) [١٤] (pp.١٦)

Let $f : R \to R$ be a map such that $f(x) = x^2$, $x =$ ٠ is a fixed point . The basin of attraction of ٠ consists of all x such that $|x|$ < ١ is (-١,١).

Definition (١.١.٩) [١٠] (pp. ١٩)

Let $f : X \to X$ be a map . Let x be a periodic point of period n. A point y is forward asymptotic to x if $\lim_{i\to\infty} f^{in}(y) = x$. The stable set of x , denoted by $W^s(x)$, consists of all points forward asymptotic to x.
If x is non – periodic, we may still define forward asymptotic points as those points which $\left|f^i(y) - f^i(x)\right|$ converges to ٠ as $i \to \infty$.If f is invertible , the backward asymptotic points y are those points for which $\left|f^i(y) - f^i(x)\right|$ converges to ٠ as $i \to -\infty$. The set of points, which are backward asymptotic to x is called the unstable set of x and is denoted by $W^u(x)$.

Example (١.١.١٠) [١٠] (pp.١٩)

Let $f : R \to R$ be a map such that $f(x) = x^3$ .Then $W^s(0)$ is the open interval (-١,١) . $W^u(1)$ is the positive real axis , where as $W^u(-1)$ is the negative real axis .

Recall that a subset $G \subset X$ is connected if it can not be represented as the union of two disjoint relatively open sets none of which is empty . Otherwise G is disconnected .

Definition (١.١.١١) [١] (pp.١٣٩)

A region is simply connected if its complement with respect to the extended plane is connected .

Example (١.١.١٢) [٩]

D = { $z \in C$: |z| < ١ } is simply connected .

Definition (١.١.١٣ [٩] (pp.١٠٥)

An arc Γ is defined as Γ = { $z \in C$: $z = x(t) + iy(t)$, $x = x(t)$, $y = y(t)$, $a \le t \le b$ } , where $x(t)$, $y(t)$ are continuous maps of the real parameter t .

Then Γ is simple arc if $z(t_1) \ne z(t_2)$, when $t_1 \ne t_2$. If Γ is simple except for the fact $z(b) = z(a)$ then Γ is simple closed curve .

An arc Γ is smooth if $z'(t)$ exists and continuous in the interval $a \le t \le b$.

Example (١.١.١٤) [٩] (pp.١٠٥)

The Polygonal line $z(t) = \begin{cases} t + it & 0 \le t \le 1 \\ t + i & 1 \le t \le 2 \end{cases}$

consisting of a line segment from ٠ to ١$+i$ followed by one from ١$+i$ to ٢$+i$ is simple arc .The unit circle $z(t) = \cos t + i \sin t$ is simple closed curve , where $0 \le t \le 2\pi$.

Now , we give the fundamental theorem of algebra .

Theorem (١.١.١٥) [١٠] (pp ٢٦٥)

Let $f: C \to C$ be a polynomial map of degree n.

If $n >٠$, then f may be written in the form $f(z) = a(z-\alpha_1). \ldots .(z-\alpha_n)$, where α_i are not necessarily distinct .

For a proof see [١٠].

Definition (١.١.١٦) [١٤](pp ٢٣٥)

Let $f: X \to X$ be a map . f is contraction that has a scaling factor s <١ if $d(f(x), f(y)) \leq sd(x,y)$ for all $x, y \in X$ such f is continuous map on X .

Theorem(١.١.١٧) [١٤] (pp.٢٣٥)

Suppose $f: X \to X$ is a contraction map where X is complete metric space with scaling factor s <١ . Then f has exactly one fixed point x, whose basin of attraction is X .

For a proof see [١٤] .

Definition (١.١.١٨) [٢٤]

Suppose $f : C \to C$ is an analytic map . The Julia set is the closure of all repelling periodic points of f . That is

$J(f)$ = closure { all repelling periodic points of f } .

Section Two (١.٢)

Normality and Fatou Set

Our goal in this section is to give another definition of Julia sets by using the concept of normality .We will talk about families of complex analytic maps .

The following proposition gives the Maximum principle theorem .

Proposition (١.٢.١) [١٠](pp. ٢٦٣)

Let $U \subset C$ be an open set . Suppose $f : U \to C$ is an analytic map in U and continuous on ∂U . If $\overline{U}$ is bounded then $|f(z)|$ assumes its maximum value on the boundary of $\overline{U}$, where $\overline{U} = U \bigcup \partial U$.

For a proof see [١٠] .

From the inverse map theorem , we have the usual result about the existence of a local analytic inverse .

Proposition (١.٢.٢) [١٠] (pp. ٢٦٣)

Suppose $f : C \to C$ is analytic and $f'(z_0) \neq 0$. Then there is an $\varepsilon >$ ٠ and a neighborhood U of z_0 such that f maps U onto $D = \{ z \in C : |z - f(z_0)| < \varepsilon \}$ in a one-to-one formula . Moreover, the inverse map $f^{-١} : D \to U$ is analytic .

For a proof see [١٠] .

Theorem (١.٢.٣) [١٠] (Schwarz lemma)(pp. ٢٦٤)

Suppose f is analytic in the open disk $|z| <$ ١ and satisfies

١- $|f(z)| \leq$ ١ .

٢- $f(0) = 0$.

Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality holds if and only if $f(z) = e^{i\theta} z$.

For a proof see [١٠] .

Remark (١.٢.٤)[٩](pp. ٢٠)

The complex plane together with the point at infinity , denoted by ∞, is called the extended complex plane , it is topologically equivalent to the Riemann sphere.

We put $C_\infty = C \cup \{\infty\}$.

Remark (١.٢.٥)[٢٧]

The metric space of the complex plane is the usual metric , while the metric space of the Riemann sphere is the chordal metric .

Definition (١.٢.٦)[٢٢]

Let $G \subset C_\infty$ be an open and connected set . A map f is conformal on G if f is one –to-one and analytic on G .

Definition (١.٢.٧)[٢٢]

A rational map $r: C_\infty \to C_\infty$ on the Riemann sphere is the quotient of two polynomials $r(z) = \dfrac{f(z)}{g(z)} = \dfrac{a_n z^n + \ldots + a_0}{b_m z^m + \ldots + b_0}$ and we assume that $f(z)$ and $g(z)$ are relatively prime , that is they have no common zeros . The degree of r is max { deg (f) ,deg (g) }.

Definition (١.٢.٨) [٢٢]

Let $r : C_\infty \to C_\infty$ be a rational map such that $r(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha\delta - \beta\gamma \neq 0$ where $\alpha, \beta, \gamma, \delta \in C$, this map is called a mobius map if it is onto and conformal map on C_∞ .

In studying the dynamics of maps , the standard equivalence relation is used to say that two maps have “ same dynamics ” is conjugacy .

Definition (١.٢.٩) [٢٣]

Let X , Y are two metric spaces . If $f: X \to X$ and $g : Y \to Y$ are two maps . Then f is conjugate to g if there exists a homeomorphism $h : X \to Y$ such that $g \circ h = h \circ f$.

Example(١.٢.١٠)[١٤](pp. ١٠٥)

If $g : R \to R$ such that $g(x) = x^{٢} - \dfrac{3}{4}$, for $-\dfrac{3}{2} \leq x \leq \dfrac{3}{2}$

and $f : R \to R$ such that $f(x) = ٣x(١ - x)$, for $٠ \leq x \leq ١$

to show $h \circ g = f \circ h$ where $h(x) = \frac{-1}{3}x + \frac{1}{2}$ is a homemorphism from $\left[\frac{-3}{2}, \frac{3}{2}\right]$ onto [٠,١] , so

$$(h \circ g)(x) = h\left(x^2 - \frac{3}{4}\right) = \frac{-1}{3}\left(x^2 - \frac{3}{4}\right) + \frac{1}{2}$$

$$= \frac{-1}{3}x^2 + \frac{3}{4} \quad ,$$

and $(f \circ h)(x) = f\left(\frac{-1}{3}x + \frac{1}{2}\right)$

$$= 3\left(\frac{-1}{3}x + \frac{1}{2}\right)\left(1 + \frac{1}{3}x - \frac{1}{2}\right)$$

$$= -\frac{1}{3}x^2 + \frac{3}{4}$$

Therefore $h \circ g = f \circ h$.

Theorem (١.٢.١١) [١٤](pp. ١٠٨)

Let $f : R \to R$ and $g : R \to R$ be two maps such that $f(x) = a x^2 + bx + c$ and $g(x) = r x^2 + sx + t$, where $a \neq 0$ and $r \neq 0$, and $c = \frac{b^2 - s^2 + 2s - 2b + 4rt}{4a}$ . . . . (١.٢.١) .

Then f and g are linearly conjugate to one another , with associated homeorphism given by

$h(x) = \frac{a}{r}x + \frac{b-s}{2r}$ . . . . (١.٢.٢)

Proof :

Let $h(x) = dx + e$. We will prove that there are constants $d \neq 0$ and e such that $h \circ f = g \circ h$. Now $h\big(f(x)\big) = g\big(h(x)\big)$, that is

or equivalently , if

$$da\,x^2 + dbx + (dc + e) = r\,d^2 x^2 + (2rde + sd)x + \left(r\,e^2 + se + t\right)$$

Collecting coefficients of like powers of x, and using the hypothesis that $r \neq 0$, we find that

x^2 terms : $da = r\,d^2$, so that if $d \neq 0$, then $d = \frac{a}{r}$

x terms : $db = 2rde + sd$, so that if $d \neq 0$, then $e = \frac{b-s}{2r}$

constant terms : $dc + e = r\,e^2 + se + t$,

substituting for d and e in the last equation , we obtain

$$\frac{a}{r}c + \frac{b-s}{2r} = r\left(\frac{b-s}{2r}\right)^2 + s\left(\frac{b-s}{2r}\right) + t$$. Which yields

$$4ac + 2b - 2s = b^2 - 2bs + s^2 + 2bs - 2\,s^2 + 4rt$$

solving for c and using the fact $a \neq 0$, we conclude that

$$c = \frac{b^2 - s^2 + 2s - 2b + 4rt}{4a}$$ which is (١.٢.١) .

We conclude that for this value of c , f is linearly conjugate to g , with h as given in (١.٢.٢) . ■

We can generalize the previous theorem to maps from C to C .

Proposition (١.٢.١٢)

Let $q(z)=a_1z^2+a_2z+a_3$ be any quadratic polynomial map and $q_c(z)=z^2+c$, where $a_1,a_2,a_3,c\in C$. Then q is linearly conjugate to q_c .

Proof:

If $S = goqog^{-1}$, where $g(z)=a_1z$, thus $S(z)=z^2+a_2z+a_1a_3$ .Then $q_c=hoSoh^{-1}$, where $h(z)=\frac{a_2}{2}+z$, so that $h^{-1}(z)=z-\frac{a_2}{2}$.

$$q_c(z)=hoSoh^{-1}(z)=hoS\left(z-\frac{a_2}{2}\right)=h\left[\left(z-\frac{a_2}{2}\right)^2+a_2\left(z-\frac{a_2}{2}\right)+a_1a_3\right] .$$

Thus
$$q_c(z)=\left(z-\frac{a2}{2}\right)^2+a_2\left(z-\frac{a_2}{2}\right)+a_1a_3+\frac{a_2}{2}$$
$$=z^2-a_2z+\frac{a_2^2}{4}+a_2z-\frac{a_2^2}{2}+a_1a_3+\frac{a_2}{2}$$
$$=z^2+\frac{a_2}{2}-\frac{a_2^2}{4}+a_1a_3 .$$

Therefore $q_c(z)=z^2+c$ for some c , where $c=\frac{4a_1a_3+2a_2-a_2^2}{4}$. By the properties of linearly conjugate , q is linearly conjugate to q_c . ■

Definition (١.٢.١٣) [٣٢]

Let $f : C_\infty \to C_\infty$ be analytic map . A point z_0 in C is called a super attracting fixed point if $|f'(z_0)|=0$.

Next , we looked at the behavior of the point at infinity for the map q_c, where $c \in C$.

Proposition (١.٢.١٤)[٢٤]

Let $r : C_\infty \to C_\infty$ be a mobius map such that $r(z) = \frac{1}{z}$, we shall exchange ٠ with ∞ in order to obtain the behavior of the point ∞ for the map $q_c(z) = z^2 + c$.

Proof:

We will prove that ∞ is super attracting fixed point for q_c .

Consider the map $F_c = r o q_c o r$

$F_c(z) = r o q_c o r(z)$

$$= r o q_c\left(\frac{1}{z}\right)$$

$$= r\left(\left(\frac{1}{z}\right)^2 + c\right) = \frac{1}{\frac{1}{z^2} + c} = \frac{1}{\frac{1 + c z^2}{z^2}} = \frac{z^2}{1 + c z^2} \ .$$

Hence $F_c(0) = 0$. Moreover, $|F'_c(0)| = 0 <$ ١ , which implies that ٠ is an attracting fixed point for F_c and $q'_c(\infty) = \infty$.

Since the derivative of q_c at ∞ is given $\frac{1}{q'_c\left(\frac{1}{z}\right)} = \frac{z}{2}$ .Evaluating at $z = 0$ gives $q'_c(\infty) = 0$. Therefore ∞ is super attracting fixed point for q_c . ■

Remark (١.٢.١٥) [١٠](pp.٢٦٥)

Let $f : C_\infty \to C_\infty$ be a map and suppose $f(\infty) = z_0$, we assume that $z_0 \neq 0$ .Then f is analytic at ∞ if $f o h^{-1}$ is analytic at ٠ , where $h(z) = \frac{1}{z}$.

Note that let $\{f_n\}$ be a family of complex analytic maps defined on an open set U .

Definition (١.٢.١٦) [٢٧]

A sequence $\{u_n\}$ converges uniformly to a limit L , if for any positive ε , no matter how small , there exists an $m \in N$ such that for all $n > m$, $|u_n - L| < \varepsilon$.

Definition (١.٢.١٧) [١٠](pp.٢٧٢)

The family $\{f_n\}$ is said to be normal on U if every sequence of the f_n`s has a subsequence which either

١.converges uniformly on compact subsets of U ,or

٢. converges uniformly to ∞ on U.

Example (١.٢.١٨) [١٠] (pp.٢٧٣)

Let $f : U \to C$ be a map such that $f(z) = az$ with $|a| <$ ١ .

$f(z) = az$

$f^{٢}(z) = a^2 z$

.

.

.

$f^n(z) = a^n z \to 0$ as $n \to \infty$ with $|a| <$ ١ . Hence $\{f_n\}$ converges uniformly to the constant map ٠ on compact subsets of U .

Therefore $\{f_n\}$ forms a normal family of maps on U .

Example (١.٢.١٩)[١٠](pp. ٢٧٣)

Let $f : U \to C$ be a map such that $f(z) = az$ with $|a| >$ ١ , then the above family is normal on any domain which does not include ٠ , but fails to be normal if the domain includes ٠ .Indeed , in any neighborhood of ٠ , there is a point z for which $|f^n(z)|$ is arbitrarily large for some n . In particular , we note that any such neighborhood U satisfies $\bigcup_{n=1}^{\infty} f^n(U) = C$.

In examples(١.٢.١٨) and (١.٢.١٩) , the Julia sets of maps of the form $f(z) = az$, or more generally $f(z) = az + b$ is either empty or consists of a single point (repelling or neutral fixed point).

Definition (١.٢.٢٠) [١٠] (pp. ٢٧٣)

The family $\{f_n\}$ is not normal at z_0 if the family fails to be a normal family in every neighborhood of z_0 .

The following proposition gives one of the most important properties of sequences of analytic maps .

Proposition (١.٢.٢١) [١٠] (pp.٢٧٣)

Suppose $\{f_n\}$ is a sequence of analytic maps which converges uniformly on a domain U to a map f .

Then f is analytic in U and , moreover , $\lim_{n\to\infty} (f_n)^{(k)}(z) = f^{(k)}(z)$.

For a proof see [١٠] .

Proposition (١.٢.٢٢) [١٠] (pp.٢٦٣) .

Let $U \subset C$ be open and suppose $f : U \to C$ is analytic map . Then $f(U)$ is open in C , provided f is non – constant .

For a proof see [١٠] .

Now , we will give the definition of the Fatou set and Julia set.

Definition (١.٢.٢٣) [٢٢]

Let $f : C \to C$ be a map . The Fatou set (stable set) $F(f)$ is the set of points $z \in C$ such that the family of iterates $\{f^n\}$ is normal family in some neighborhood of z.

Definition (١.٢.٢٤) [١٢] (pp. ١٩٩) .

The Julia set $J(f)$ is the complement of the Fatou set , that is $J(f)=\{ z \in C$: the family $\{f^n\}_{n\geq 0}$ is not normal at z } .

That is $J(f) \equiv C \setminus F(f)$.

Also the previous definition can satisfy on the space C_∞ .

Remark (١.٢.٢٥) [٢٢]

If f is a polynomial map , then $\infty \in F(f)$ and there is a neighborhood U of ∞ with $U \subset F(f)$, so $F(f) \neq \phi$. Thus the Julia set of polynomial is never the entire plane .

Lemma (١.٢.٢٦) [١٢] (pp.١٩٩)

Let $f : C \to C$ be a map . Then $F(f)$ is an open set .

Proof :

To show that for all $x \in F(f)$ there exists an open set G such that $x \in G$ and $G \subseteq F(f)$.

Let $x \in F(f)$, thus by definition (١.٢.١٧), $\{f^n\}$ is normal at x . Hence there exists an open set G such that $x \in G$ and $\{f^n\}$ is normal on G .

Therefore $G \subseteq F(f)$, hence $F(f)$ is an open set . ■

Theorem (١.٢.٢٧) [٣٢]

If f is a polynomial map and *h* is a conformal equivalence such that $g = hofo\,h^{-1}$ then $F(g) = h(F(f))$ and $J(g) = h(J(f))$.

Proof :

Let $z_0 \in F(f)$ and suppose that $\{f^{n_i}\}_{i\in\Lambda}$ be a sequence that contains a subsequence which converges uniformly to f on a neighborhood of z_0 . Since h conformal equivalence , hence $h\{f^{n_i}\}_{i\in\Lambda}$ is a sequence that contains a subsequence which converges uniformly to $h(f)$ on a neighborhood of $h(z_0)$. But $h(f)$ in domain g ,thus $h(z_0) \in F(g)$ and $h(F(f)) \subset F(g)$ . . . (١.٢.٣) .Now h^{-1} is a conformal equivalence and $f = h^{-1}ogoh$ .Applying the relation (١.٢.٣) , thus $h^{-1}(F(g)) \subset F(f)$. Therefore $F(g) \subset h(F(f))$ . . . (١.٢.٤) . From (١.٢.٣) and (١.٢.٤) we get $h(F(f)) = F(g)$. By definition (١.٢.٢٤) , $h(J(f)) = J(g)$. ■

The next proposition gives a useful criterion for a family of analytic maps to fail to be normal at a given point .

Proposition (١.٢.٢٨) [١٠](pp.٢٧٣)

Let f be an analytic and suppose that z_0 is repelling periodic points for f . Then the family of iterates of f is not normal at z_0 .

Proof :

First , let z_0 be a repelling fixed point . Assume that $\{f^n\}$ is normal on a neighborhood U of z_0 , since $f^n(z_0)=f(z_0)=z_0$. It follows $f^n(z_0)$ does not converge to ∞ on U . Thus some sequence of the sequence $\{f^n\}$ has subsequence $\{f^{n_i}\}$ which converges uniformly to some map g on U . By definition (١.٢.١٧) , hence $\left|\left(f^{n_i}\right)'(z_0)\right| \to |g'(z_0)|$. Since z_0 is repellor , thus $\left|\left(f^{n_i}\right)'(z_0)\right| \to \infty$. This contradiction with definition (١.٢.١٧) . Therefore $\{f^n\}$ is not normal at z_0 .

Now , let z_0 be a repelling periodic point of period k, that is $\{(f^k)^n\}=\{f^k o f^k o ... o f^k\}=\{f^{kn}\}$. Assume that $\{f^{kn}\}$ is normal on a neighborhood U of z_0 , then $f^{kn}(z_0)=f^k(z_0)=z_0$ for all *n* , it follows that $f^{kn}(z_0)$ does not converge to ∞ on U . Thus some sequence of the sequence $\{f^{kn}\}$ has a subsequence $\{f^{kn_i}\}$ which converges uniformly to some map h on U . By definition (١.٢.١٧) , hence $\left|\left(f^{kn_i}\right)'(z_0)\right| \to |h'(z_0)|$. Since z_0 is repellor point , thus $\left|\left(f^{kn_i}\right)'(z_0)\right| \to \infty$, this is contradiction with definition (١.٢.١٧).

Therefore $\{f^{kn}\}$ is not normal at z_0. ■

Corollary (١.٢.٢٩) [١٠](pp.٢٧٤)

Let f be an analytic map . The family of iterates $\{f^n\}$ fails to be a normal family at any point in $J(f)$.

Proof :

Let $z_0 \in J(f)$, then z_0 is either repellor point or z_0 in limit of repelling periodic points .

If z_0 is a repellor point . By proposition (١.٢.٢٨) $\{f^n\}$ is not normal at z_0 .

If z_0 in limit of repelling periodic points , any neighborhood U of any point contains a repellor point as z_1 . By proposition (١.٢.٢٨) $\{f^n\}$ is not normal at z_1 . Therefore $\{f^n\}$ is not normal at z_0 . ■

Definition (١.٢.٣٠) [٥]

Let f be an analytic map . A complex number z_0 is called an omitted value of the analytic map f , if $f(z) \neq z_0$ for all $z \in C$.

Example (١.٢.٣١)

Let $f: C \to C$ be an analytic map such that $f(z) = e^z$.

٠ is an omitted value of analytic map f .

Definition (١.٢.٣٢) [٥]

Let f be an analytic map . z_0 is called an exceptional point if $\bigcup_{n=1}^{\infty} f^{-n}(z_0)$ is finite , where $f^{-n}(z_0) = \{z \in C : f^n(z) = z_0\}$.

Example (١.٢.٣٣) [١٠] (pp.٢٧٤)

Let $f : C \to C$ be an analytic map such that $f(z) = z^2$, if U is an open set which meets S^1 but does not meet ٠, then $\bigcup_{n=1}^{\infty} f^n(U) = C - \{0\}$.

٠ is an exceptional point .

One of the most important consequences of the failure to be a normal family at a given point is that the family of maps must assume in fact every value in any neighborhood of the point . This result is a variant of a theorem known as Montel’s theorem .

Theorem (١.٢.٣٤) [٣٢] (Montel`s Criterion for Normality)

Let $\{f_n\}$ be a family of analytic maps on a domain G . If there are three values that are omitted be every $f \in \{f_n\}$, then $\{f_n\}$ is normal family . For a proof see [٣٢] .

Definition (١.٢.٣٥) [٣٢]

Let $A \subset X$ and let $f : X \to X$ be a map . A is said to be completely invariant under f if $f(A) = A = f^{-1}(A)$.

Example (١.٢.٣٦) [١٠] (pp. ٢٧٤)

Let $f : R \to R$ be a map such that $f(x) = x(1-x)$, $0 \le x \le 1$. If $A = [٠,١]$, then $f([0,1]) = [0,1]$ and $f^{-1}([0,1]) = [0,1]$.
Therefore $[0,1]$ is completely invariant under f .

Lemma (١.٢.٣٧) [٢٧]

If $f : C_\infty \to C_\infty$ be a map .Let z be any point in $J(f)$, then in every neighborhood of z and for all $n \in N$, the maps $\{f^n\}$ omit at most two points of C_∞ . Moreover these exceptional points , if they exist , are independent of z and don’t belong to $J(f)$.

Proof :

If the lemma is not true then there exist arbitrarily small neighborhoods of z in which each f^n in the sequence $\{f^n\}_{n=1}^{\infty}$ omits at least three values .

If there are arbitrarily small neighborhoods of z which map to neighborhoods of $f^n(z)$ with points omitted then , taking the inverse map , there are arbitrarily small neighborhoods of $f^n(z)$ which have as preimages finite neighborhoods of z with no points omitted . By definition (١.٢.١٧) , $\{f^n\}$ is normal family , which is a contradiction to definition (١.٢.٢٤) since z is in the Julia set .

Consider the exceptional points . As a set , it must be invariant under f , since otherwise they would map to points in the plane which are in the image of the neighborhood of z in contradiction with the statement of the lemma .

If there is only one exceptional point a , then it can have no preimages other than itself . We can move a to ∞ by a mobius map . If there are two exceptional points a and b , then there are the following two possible cases .

١. a has no preimages other than itself and b has no preimages other than itself , each is a fixed point .

٢. a and b from period of ٢ , each maps onto the other and is the preimages of the other .

We move a and b to ٠ and ∞ by a mobius map .

Case ١ means that a polynomial map is of the form $M z^n$, where M is constant , since it leaves ٠ and ∞ invariant .

Case ٢ means that a map is of the form $M z^{-n}$, since this maps the origin to infinity and infinity back to the origin forming the necessary period of z . The exceptional points a , b are fixed points of order ١ or ٢ and , since any attracting fixed points are not in the Julia set , the exceptional points must depend only upon $f(z)$ as required . ■

Lemma (١.٢.٣٨) [١٢](pp. ٢٠١)

Let f be a polynomial map , let $z_0 \in J(f)$ and let U be any neighborhood of z_0 . Then $\bigcup_{n=1}^{\infty} f^n(U)$ is the whole of C , except possibly for a single point . Any such exceptional point is not in $J(f)$, and is independent of z_0 and U.

For a proof see [١٢] .

Theorem (١.٢.٣٩) [١٠] (pp. ٢٧٤)

Let f be a polynomial map . Suppose there is a point $z_0 \in J(f)$ and neighborhood U of z_0 such that $\bigcup_{n=1}^{\infty} f^n(U) = C - \{a\}$. Then $f(z) = a + \lambda(z-a)^n$ for some $\lambda \in C$ and some integer n .

Proof :

Suppose $f(b) = a$. Then b is an exceptional point for f, for there is no z in U which maps to b and then to a . But by lemma (١.٢.٣٨) , $b = a$.

Hence a is fixed point for f and , moreover , a is only preimage . Thus for some n we may write

$$g(z)=\frac{f(z)-a}{(z-a)^n} ,$$

where $g(z)$ is a polynomial map and $g(z)\neq 0$ for any z.

Otherwise , we would have an additional preimage of a .
By fundamental theorem of algebra , thus $g(z)$ reduces to a constant . ■

Proposition (١.٢.٤٠) [١٠] (pp. ٢٧٥)

Suppose f is a polynomial map of degree $n\geq 2$ which has an exceptional point at a . Then f is conjugate to $z\to z^n$.

Proof :

Let $g(z)=z^n$, since a is an exceptional point of f . By theorem (١.٢.٣٩) , thus $f(z)=a+\lambda(z-a)^n$ for some $\lambda\neq 0$. If $\lambda=\mu^{n-1}$ when $h(z)=\mu(z-a)$, where μ is a constant , thus $goh(z)=g(\mu(z-a))=\mu^n(z-a)^n$ and $hof(z)=h(a+\lambda(z-a)^n)=h(a+\mu^{n-1}(z-a)^n)=\mu(\mu^{n-1}(z-a)^n)=\mu^n(z-a)^n=goh(z)$. That is $hof=goh$. ■

Section Three (١.٣)

Filled Julia set

The goal of this section is to give another definition of Julia sets .

Definition (١.٣.١) [٣٣]

Let $f : C \to C$ be a map . The orbit of z under a map f is bounded if there exists $k \in R^+$ such that $|f^n(z)| < k$ for all n . Otherwise , the orbit is unbounded .

Example (١.٣.٢) [٣٣]

Let $f : C \to C$ be a map such that $f(z) = z^2$. For any z such that $|z|<$١ , if $z = |z|(\cos\theta + i\sin\theta)$.

Then $z^n = |z|^n(\cos n\theta + i\sin n\theta) \to 0$ as $n \to \infty$. Thus , any z where $|z| <$١ , the orbit of z is bounded . And if $|z| = 1$, then $|f^n(z)| = 1$ and the orbit is also bounded . However , if $|z|>$١ , the orbit is unbounded .

Definition (١.٣.٣) [٣٠]

Let $f : C_\infty \to C_\infty$ be a polynomial of degree $n \geq 2$. Let $K(f)$ denote the set of points in C whose orbits do not converge to the point at infinity . That is $K(f) = \{ z \in C : \{|f^n(z)|\}_{n=0}^{\infty}$ is bounded $\}$. This set is called filled Julia set .

Definition (١.٣.٤) [٢٧]

Let $f : C_\infty \to C_\infty$ be a map . The escape set $A(\infty)$ of f is all those points that escape to infinity , that is $A(\infty) = \{ z : f^n(z) \to \infty$ as $n \to \infty \}$.

Remark (١.٣.٥)

We can say that $A(\infty)$ is the basin of attraction of ∞ .

Example (١.٣.٦) [٣٣]

Let $f : C_\infty \to C_\infty$ be a map such that $f(z) = z^2$. The set $\{z : |z| \le 1\}$ is the filled Julia set but the set { $z : |z| >$ ١} is $A(\infty)$.

Now we can state another definition for Julia set .

Definition (١.٣.٧) [٢٧]

The Julia set is the boundary of the filled Julia set , that is $J(f) = \partial K(f)$.

Remark (١.٣.٨) [١٧]

The complement of the basin of attraction of ∞ is the filled Julia set of f . That is $C_\infty \setminus A(\infty) = K(f)$.

Theorem (١.٣.٩) [٢٧]

Let f be a polynomial of degree $d \ge 2$, then $K(f)$ is non- empty set .

Proof :

f is a polynomial of degree at least ٢ , thus one can see easily that it fixes at least one point , z_0 say . This is because fixed points are simply solutions to $f(z) = z$, which has d roots and at least one fixed point (taking into account the possibility of repeated roots) .

From definition (١.٣.٤) , one sees that a fixed point can not be in $A(\infty)$ and therefore $z_0 \in K(f)$. Thus $K(f)$ is non-empty . ■

Theorem (١.٣.١٠) [٢٧]

Let f be a polynomial of degree $d \geq 2$, then $K(f)$ is compact (closed and bounded) set .

Proof :

Define V_r to be the set of all points further from the origin than the circle of radius r .

$$V_r = \{ z \in C : |z| > r \} .$$

For sufficiently large r we have $f(V_r) \subset V_r \subset A(\infty)$.

This means that if r is large enough then all points in V_r escape to infinity . Once this is the case , V_r must be contained in $A(\infty)$ from definition (١.٣.٤) . If a point escapes to infinity then its image under the map must also escape to infinity from definition (١.٣.٤), thus the image of V_r must also be contained in $A(\infty)$.

Moreover , we know that all points in V_r are images large than themselves and so $f(V_r) \subset V_r$. $A(\infty)$ is a region containing infinity , V_r is a smaller region containing infinity and each image of V_r is a smaller region again , each region must contain each of the smaller once .

Call the first V_r which satisfy these conditions simply V. For all z_0 in $A(\infty)$ we know that $f^n(z_0) \to \infty$.

Hence there exists a $k \in N$ such that $f^k(z_0) \in V$ so

$$A(\infty) = \bigcup_{n=1}^{\infty} \left(f^n\right)^{-1}(V) \quad . . . (١.٣.١) .$$

In other words , we can choose k to be large enough that the k-th iterate of any escaping point in V .

This proof has effectively split the sphere into areas around the origin . Nearest the centre with the smallest values , is $K(f)$. Outside of $K(f)$ is $A(\infty)$, but this splits into further envelopes . By selecting a suitably large r we can make V_r a region containing infinity wholly contained in $A(\infty)$. This envelope around infinity contains the image of the whole of $A(\infty)$ under f^k . Simply connected region around infinity on the sphere becomes correspondence around the region in the plane .

Each V_r is an open set because there is a strict inequality in the definition , the set does not contain its boundary .

$A(\infty)$ is the union of the preimage of V and so it is the union of the preimages of an open set .

Each preimage of an open set must be an open set since f is a polynomial and therefore is an open map .

The union of open sets is also an open set and we have that $A(\infty)$ is an open set . By remark (١.٣.٨) , the $K(f)$ is closed ,by definition (١.٣.٣) $K(f)$ is bounded . $K(f)$ is compact . ■

Theorem (١.٣.١١) [٢٧]

Let f be a polynomial of degree $d \geq 2$ then the sets $A(\infty)$ and $K(f)$ are completely invariant under f .

Proof :

From (١.٣.١) , we have . $A(\infty) = \bigcup_{n=1}^{\infty}\left(f^n\right)^{-1}(V)$

So that $f^{-1}(A(\infty)) = f^{-1}\left(\bigcup_{n=1}^{\infty} \left(f^n\right)^{-1}(V)\right)$

$$= \bigcup_{n=1}^{\infty}\left(f^{(n+1)}\right)^{-1}(V) = A(\infty) \quad ,$$

Similarly $f\ (A(\infty)) = f\left(\bigcup_{n=1}^{\infty}(f^{n})^{-1}(V)\right) = \bigcup_{n=1}^{\infty}\left(f^{(n-1)}\right)^{-1}(V) = A(\infty)$,

Thus $A(\infty)$ is completely invariant . By remark (١.٣.٨) ,

$K(f)$ is completely invariant . ■

Theorem (١.٣.١٢) [٢٧]

Let f be a polynomial of degree $d \geq 2$, then $K(f)$ is perfect . Moreover every connected component of the interior of $K(f)$ is simply connected.

Proof :

We will prove that $K(f)$ is perfect by contradiction . Assume a point $z_0 \in K(f)$ is an isolated then we can draw a simple closed curve Γ in $A(\infty)$, such that the region W of the plane sphere bounded by Γ and containing the origin , has only the point z_0 in common with $K(f)$. This follows by definition of z_0 being isolated . Choose V as theorem above (١.٣.١٠) , then by (١.٣.١) we can find a $k \in N$ such that $f^k(\Gamma) \subset V$.

This means that we can take a high enough iterate of f to map all points in Γ into V , since Γ is contained in $A(\infty)$. Since $f^k(z_0)$ is not contained in V as $z_0 \in J(f)$, $f^k(W)$ should contain $C \setminus V$.

In other words , since our closed curve maps to a subset of $A(\infty)$, $V \subset A(\infty)$, and W contains a point in $K(f)$, the complement of $A(\infty)$, the whole of W , the region inside of our closed curve , should contain the preimage of $K(f)$. The complete invariance of $K(f)$ means that $K(f) = z_0$.

$f^{-1}(z_0) = \{z_0\}$ and so our polynomial should be of the form $f(z) = z_0 + a(z - z_0)^d$.

This means that z_0 would belong to the interior of $K(f)$ and we have a contradiction , thus there are no isolated points and $K(f)$ is perfect .

If there is a connected component of the interior of $K(f)$ which is not simply connected , then $\overline{A(\infty)} = A(\infty) \cup J(f)$ would not be connected . This contradicts of $A(\infty)$ being an open connected set . ■

CHAPTER ٢

Properties of Julia sets of Polynomials

The goal of this chapter is to drive the basic properties of Julia sets of polynomial maps .

In section one , we study the relation between the three definitions of Julia sets given in this chapter .In section two , we give the properties of Julia sets of maps of the form $(z^2 + c)$, where $c \in C$ .In section three , we study chaoticity on Julia sets .

Section one (٢.١)

General Properties of Julia sets

The goal of this section is to study some properties of polynomials as given by definition of Julia sets , and we also show the equivalence of the three definitions of Julia sets given before in case that the map is a polynomial .

Proposition (٢.١.١) [١٢] (pp. ٢٠٠)

Let $f : C \to C$ be a polynomial of degree $d \geq 2$, and $J(f)=\{z \in C$: the family $\{f^n\}_{n\geq 0}$ is not normal at z }. Then $J(f)$ is non empty set .

Proof :

Suppose $J(f)=\phi$. Then , for all r > ٠ , the family $\{f^n\}$ is normal on the open disc $B_r^0(0)$ with centre at the origin and radius r (since the closed disc $B_r(0)$ is compact , it may be covered by a finite number of open sets on which $\{f^n\}$ is normal) .

Since f is a polynomial , we can extend the range of f , taking r large enough to ensure that $B_r^0(0)$ contains a point z for which $|f^n(z)| \to \infty$ and also contains a fixed point p of f with $f^n(p)=p$ for all n .

Thus it is impossible for any subsequence of $\{f^n\}$ to converge uniformly either to a bounded map or to infinity on any compact subset of $B_r^0(0)$ which contains both z and p , contradicting the normality of $\{f^n\}$. Therefore $J(f)\neq\phi$. ■

Proposition (٢.١.٢) [١٢](pp. ٢٠٠)

Let $f : C \to C$ be a polynomial of degree $d \geq 2$, and $J(f)=\{z \in C$: the family $\{f^n\}_{n\geq 0}$ is not normal at z } .Then $J(f)$ is compact (closed and bounded).

Proof :

By lemma (١.٢.٢٦) , $F(f)$ is open set , and by definition (١.٢.٢٤) , $J(f)$ is closed set . Since f is a polynomial of degree at least two , we may find r such that if $|z| \geq r$, then $|f(z)| \geq 2|z| \geq 2r$, thus

$\left| f^2(z) \right| \geq 2\left| f(z) \right| \geq 2.2|z| = 2^2|z| \geq 2^2 r$ .Thus for n-*th* iterates we get $\left| f^n(z) \right| \geq 2^n r$ if $|z| \geq r$, implying that $\left| f^n(z) \right| > 2^n r$ if $|z| > r$. Thus $f^n(z) \to \infty$ uniformly on the open set $V = \{ z : |z| > r \}$. By lemma (١.٢.٢٦) , $\{f^n\}$ is normal family on V ,so that $V \subset C \setminus J(f)$. Therefore $J(f)$ is bounded , and so $J(f)$ is compact set . ■

Proposition (٢.١.٣) [٢٢]

Let $f : C \to C$ be a polynomial of degree $d \geq 2$, and $J(f) = \{ z \in C$: the family $\{f^n\}_{n\geq 1}$ is not normal at z } .Then $J(f^m) = J(f)$ for every positive integer m .

Proof :

We show , equivalently , that $F(f) = F(f^m)$ for all $m \geq 1$. If every subsequence of $\{f^n\}$ has a subsequence uniformly convergent on a given set , the same is true of $\{f^{mn}\}_{n\geq 1}$. Thus $F(f) \subseteq F(f^m)$ . . . (٢.١.١).
If $\{f^{mn}\}_{n\geq 1}$ is normal , then $\{f^{mn+k}\}_{n\geq 1}$ is normal for $0 \leq k < m$ on $F(f^m)$. But any subsequence of $\{f^n\}_{n\geq 1}$ contains a subsequence of $\{f^{mn+k}\}_{n\geq 1}$ for some $0 \leq k < m$. Hence $\{f^n\}_{n\geq 1}$ is normal , where $\{f^n\} = \bigcup_{k=0}^{m-1} \{f^{mn+k}\}$, thus $F(f^m) \subseteq F(f)$ . . . (٢.١.٢) , from (٢.١.١) and (٢.١.٢) we get $F(f^m) = F(f)$. $J(f^m) = J(f)$. ■

Remark (٢.١.٤)[٢٢]

A set A is completely invariant under f. If f is onto , then $f\left(f^{-1}(A)\right) = A$ thus $A = f\left(f^{-1}(A)\right) = f(A)$.

Proposition (٢.١.٥) [٢٢]

Let $f : C \to C$ be a polynomial of degree $d \geq 2$, and $J(f) = \{ z \in C$: the family $\{f^n\}_{n\geq 0}$ is not normal at z } .Then $J(f)$ is completely invariant . That is $J(f) = f(J(f)) = f^{-1}(J(f))$.

Proof :

We work with the complement $F(f)$. Suppose f is onto , using remark (٢.١.٤) , we can to show $F(f) = f^{-1}(F(f))$ as follows . Let $z_0 \in f^{-1}(F(f))$, $w_0 = f(z_0) \in F(f)$. Suppose that $\{g_n \text{ o } f\}$ is a sequence in $\{f^n\}$. Since $\{g_n\}$ contains a subsequence which converges uniformly on a neighborhood U of w_0 , $\{g_n \text{ o } f\}$ contains a subsequence which converges uniformly on $f^{-1}(U)$,a neighborhood of z_0,that is $z_0 \in F(f)$. Hence $f^{-1}(F(f)) \subset F(f)$ . . .(٢.١.٣).

Let $z_0 \in F(f)$ and $w_0 = f(z_0)$. Suppose $\{g_n\}$ in $\{f^n\}$ with $\{g_n \text{ o } f\}$ contains a subsequence which converges uniformly on a neighborhood U of z_0. Since f is an open map because f is a polynomial , $\{g_n\}$ contains a subsequence which converges uniformly on $f(U)$. It follows that $w_0 = f(z_0) \in F(f)$, and $f(F(f)) \subset F(f)$, since f is onto , thus $F(f) \subset f^{-1}(F(f))$ . . . (٢.١.٤) , from (٢.١.٣) and (٢.١.٤) we get $F(f) = f^{-1}(F(f))$, thus $F(f)$ is completely invariant , since the complement of completely invariant is also completely invariant. Hence $J(f)$ is completely invariant . ■

Proposition (٢.١.٦) [١٢](pp. ٢٠٢)

Let $f : C \to C$ be a polynomial of degree $d \geq 2$,and $J(f)=\{ z \in C$: the family $\{f^n\}_{n\geq 0}$ is not normal at z } .Then $J(f)$ has an empty interior.

Proof :

Suppose $J(f)$ contains an open set U . Then $f^n(U) \subset J(f)$ for all n. By proposition (٢.١.٥) , $J(f)$ is completely invariant , $\bigcup_{n=1}^{\infty} f^n(U) \subset J(f)$ .By lemma (١.٢.٣٨) , $J(f)$ is all of C except possibly for one point , but by proposition (٢.١.٢) , $J(f)$ is bounded set , and this a contradiction . Therefore $J(f)$ has empty interior. ■

Proposition (٢.١.٧) [١٢](pp. ٢٠١)

Let $f : C \to C$ be a polynomial of degree $d \geq 2$, if $z_0 \in J(f)$, and $J(f)$ = closure { all repelling periodic points of f } .Then $J(f) = \overline{\bigcup_{n=1}^{\infty} f^{-n}(z_0)}$.

Proof :

If $z_0 \in J(f)$ then $f^n(z_0) \subset J(f)$. By proposition (٢.١.٥) , so that $\bigcup_{n=1}^{\infty} f^{-n}(z_0)$ and , thus , its closure is contained in the closed set $J(f)$.

On the other hand , for any neighborhood U of some $w \in J(f)$ there is some n such that $z_0 \in f^n(U)$. Thus $\bigcup_{n=1}^{\infty} f^{-n}(z_0)$ meets U , so every $w \in J(f)$ can be approximated arbitrary close by points of $\bigcup_{n=1}^{\infty} f^{-n}(z_0)$ .Therefore $J(f) = \overline{\bigcup_{n=1}^{\infty} f^{-n}(z_0)}$. ■

Proposition (٢.١.٨) [٥]

Let $f : C \to C$ be a polynomial of degree $d \geq 2$, and $J(f)=\{ z \in C$: the family $\{f^n\}_{n\geq 1}$ is not normal at z $\}$. Then $J(f)$ does not contain isolated points .

Proof :

Suppose that $z_0 \in J(f)$, and let U be a neighborhood of z_0 . We can find $z_1, z_2, z_3 \in J(f) \setminus \bigcup_{n=1}^{\infty} f^n(z_0)$, because $\{f^n \setminus U\}$ is not normal from proposition (١.٢.٢٨) , $z_j \in \bigcup_{n=1}^{\infty} f^n(U)$ for some $j \in \{1,2,3\}$.

Hence $\bigcup_{n=1}^{\infty} f^{-n}(z_j) \bigcap U \setminus \{z_0\} \neq \phi$. In particular, $J(f) \bigcap U \setminus \{z_0\} \neq \phi$. Hence z_0 is not an isolated point . ■

Proposition (٢.١.٩) [٥]

Let $f : C \to C$ be a polynomial of degree $d \geq 2$, and $J(f)=\{ z \in C$: the family $\{f^n\}_{n\geq 0}$ is not normal at z $\}$. Then $J(f)$ is a perfect set .

Proof :

By proposition (٢.١.١) , $J(f)$ is a non-empty set . By proposition (٢.١.٢) , $J(f)$ is compact , thus $J(f)$ is a closed set . By proposition (٢.١.٨) , $J(f)$ does not contain isolated points . Then $J(f)$ is perfect set . ■

Proposition (٢.١.١٠) [٢٢]

Let $f: C \to C$ be a polynomial of degree $d \geq 2$. Let U be an open set such that $U \cap J(f) \neq \phi$,and $J(f)$ = closure { all repelling periodic points of f } . Then there is a positive integer k such that $f^k(U \cap J(f)) = J(f)$.

Proof :

By proposition (٢.١.٧) , the repelling periodic points are dense in $J(f)$, there is a repelling periodic point $z_0 \in U \cap J(f)$ of period n , fixed by $g = f^n$. Choose a small neighborhood $V \subset U$ of z_0 with the property $V \subset g(V)$. Then $V \subset g(V) \subset g^2(V) \subset \ . \ . \ .$. But then the union of the open sets $g^m(V)$ contains the entire Julia set . By proposition (٢.١.٣) , $J(g) = J(f)$. By proposition (٢.١.٢) , $J(f)$ is compact , thus $J(f) \subset f^k(V)$ for some finite positive integer k , thus $f^k(V) \cap J(f) = J(f)$ for all $V \subset U$. Therefore $f^k(U \cap J(f)) = J(f)$ for some finite positive integer k . ■

Definition (٢.١.١١) [٢٨]

Let $f : C \to C$ be a diffeomorphism map ,let p and q be neutral periodic points under f . A point in the set $(W^u(p) \cap W^s(q)) \setminus \{p,q\}$ is called a heteroclinic point . If $p = q$ then such a point is called a homoclinic point .

Recall that a homoclinic point z to a repelling fixed point z_0 is one for which there exists $n > ٠$ for which $f^n(z) = z_0$ and for which there is a sequence of inverse image $f^{-i}(z)$ converging to z_0 .

Theorem (٢.١.١٣) [٣٢]

Let $f : C \to C$ be a polynomial of degree $d \geq 2$,the boundary of $A(\infty)$ coincides with the Julia set of f .

Proof :

Suppose $z_0 \in \partial A(\infty)$ and V is a neighborhood of z_0 . Then for $z \in A(\infty) \cap V$, $z \in A(\infty)$ and $z \in V$, thus $f^n(z) \to \infty$. But the iterates of z_0 , $\{f^n(z_0)\}$ remain bounded , thus $\{f^n\}$ is not normal in V so z_0 is in the Julia set . Therefore $\partial A(\infty) \subseteq J(f)$ . . . (٢.١.٥) .

To show that $J(f) \subseteq \partial A(\infty)$ .If $z \in J(f)$,then $f^k(z) \in J(f)$ for all k so it cannot converge to an attracting fixed point , and $z \notin A(\infty)$. However , if U is any neighborhood of z , the set $f^k(U)$ contains points of $A(\infty)$ for some k by lemma (١.٢.٣٨) , so there are points arbitrarily close to z that iterate to ∞ . Thus $z \in \overline{A(\infty)}$ and so $z \in \partial A(\infty)$ .Hence $J(f) \subseteq \partial A(\infty)$ . . . (٢.١.٦) .
From (٢.١.٥) and (٢.١.٦) we get $J(f) = \partial A(\infty)$. ■

The following theorem shows that the three definitions of Julia sets are equivalent for polynomials of degree $d \geq 2$.

Theorem (٢.١.١٤)

Let $f : C \to C$ be a polynomial of degree $d \geq 2$. Then the following statements are equivalent .

١. $J(f)$ is the closure of repelling periodic points .

٢. $J(f)$ is the complement of the Fatou set

٣. $J(f)$ is the boundary of the filled Julia set .

Proof :

١ $\Leftrightarrow 2$ [١٠] (pp. ٢٨٥)

Let $z_0 \in J(f) \Rightarrow \{f^n\}$ is not normal at z_0.

Hence , it suffices to show that there is a repelling periodic point in any neighborhood of a point where $\{f^n\}$ fails to be normal .Toward that end , suppose $\{f^n\}$ is not normal at z and let U be a neighborhood of z .

We will produce a repelling periodic point in U . By proposition (٢.١.١) , $J(f) \neq \phi$, hence z_0 is a repelling periodic point of period n, $f^n(z_0) = z_0$. By proposition (٢.١.٣) , z_0 is a fixed point for f . By proposition (١.٢.٢) , there is a neighborhood U_0 of z_0 such that $f : U \to C$ is a diffeomorphism . Hence f^{-1} is well – defined on U_0 and maps U_0 inside itself . Let $U_i = f^{-i}(U_0)$ and note that $U_{i+1} \subset U_i$ and $\bigcap U_i = \{z_0\}$.

Since $\{f^n\}$ is not normal at z , there is a point $z_1 \in U$ and an integer n such that $f^n(z_1) = z_0$. Similarly , since $\{f^n\}$ is not normal at z_0 , there is a point $z_2 \in U_0$ and an integer m such that $f^m(z_2) = z_1$, this uses the obvious fact that z_1 is not an exceptional point . Hence $f^{m+n}(z_2) = z_0$. For later use , and by definition (٢.١.١١) , z_2 is homoclinic point .

Now , if z_2 is a critical point for f^{m+n} ,$\left(f^{m+n}\right)'(z_2) = 0$,let $f^{m+n}(z_2) = z_0$, and z_0 is a fixed point , thus $\left(f^{m+n+1}\right)(z_2) = f(z_0) = z_0$.

Hence $z_2 \in$ Basin of (z_0), therefore z_0 is an attractor point , this is contradiction with $z_0 \in J(f)$.

If $\left(f^{m+n}\right)'(z_2) \neq 0$, there is a neighborhood V of z_2 which is contained in U_0 and f^{m+n} is a diffeomorphism map from V onto a neighborhood of z_0.

By adjusting V , may be assume that $f^m(V) \subset U$ and that f^{m+n} is a diffeomorphism from V onto U_j for some integer j .But $U_j = f^{-j}(U_0)$, hence f^{m+n+j} is a diffeomorphism from V onto U_0 . By proposition (١.٢.٢), this map has an inverse which contracts U_0 onto V , by theorem (١.١.١٧) , there is a fixed point for f^{m+n+j} in V , and by theorem (١.٢.٣) this point must be repelling . Since $f^m(V) \subset U$, the orbit of this repelling periodic point enters U .

$2 \Leftrightarrow 3$

From theorem (٢.١.١٣) , $J(f) = \partial A(\infty)$. By remark (١.٣.٨) , $J(f) = \partial K(f)$. ■

Section Two (٢.٢)

Properties of the Julia set

Of maps of the form ($z^2 + c$)

The goal of this section is to derive the properties of the Julia set of the polynomial map of the form $q_c(z) = z^2 + c$, $c \in C$.

Example (٢.٢.١) [٢٧]

Consider the map $q_0(z) = z^2$ .When $|z|$ < ١ , $z^n \to 0$ whereas if $|z|$> ١ we have $z^n \to \infty$. The Julia set of q_0 by definition (١.٣.٧), is those points which are neither attracted to infinity nor to zero , thus it is the unit circle , $|z| = 1$. It is both forward and backward invariant under q_0 . This means that it

is both its own image and preimage under q_0, and thus is its own image and preimage under any number of iterations of q_0 . Every point on the unit circle has its entire backward and forward lying on the unit circle .
The two fixed points of q_0 are $z=0$ and $z=1$, the only solution of $q_0(z)=z^2=z$.

If we consider the points of the unit circle of the form exp($\frac{2\pi ir}{2^m}$) for some positive integers r and m , then $q_0^m=1$ and all higher iterates after this are fixed on ١ since it is a fixed point . If we consider points on the unit circle which are not of this form , then the sequence of iterates does not converge to any point . Both points of the form exp($\frac{2\pi ir}{2^m}$) and not of this form are dense on the unit circle
Any arc of the unit circle contains infinitely many points which eventually map to ١ and then stay there and also infinitely many points which map around and around the unit circle and never reach a fixed point , for there is no fixed point other than ١ to be reached . Points can never map off of the unit circle since if $|z|=1$ then $|q_0(z)|=|z^2|=|z|^2=1$.

Next consider the periodic points of q_0 . That is those points that are fixed points of a finite iteration of q_0 say q_0^n then $q_0^n(z)=z^{2n}$, from which we have that fixed points of $q_0^n$of the unit circle . We can also see that for any number of iterations , there are fixed points for any lower number of iteration.

Consider $s=\exp\left(\frac{2\pi i}{2^n-1}\right)$, then s is fixed by q_0^n but not by any lower iterate and so for any positive integer n there are periodic points on the unit circle with period exactly n.

Example (٢.٢.٢)[٣٢]

Let $q_2(z)=z^2-2$ to show the Julia set for q_2 is the line segment [-٢,٢].

Claim ١ : The set [-٢,٢] is completely invariant under q_2 .

Consider $q_2(x)=x^2-2$. $q'_2(x)=2x$ and $q''_2(x)=2$ so $q_2(x)$ has a minimum value of -٢ at $x=0$.

q_2 is also an even map which increases on the interval [٠,٢] , so q_2 has a maximum value of ٢ at $x=\mp 2$.

Thus $q_2([-2,2])\subset[-2,2]$ and $[-2,2]$ is not a subset of $A_2(\infty)$.

Claim ٢ : $\Omega=C_\infty\setminus[-2,2]$ is $A_2(\infty)$.

Let $h(\eta)=\eta+\frac{1}{\eta}$ which maps {$|\eta|$>١} onto $C_\infty\setminus[-2,2]$.

If $q_2oh(\eta)=q_2\left(\eta+\frac{1}{\eta}\right)=\left(\eta+\frac{1}{\eta}\right)^2-2=\eta^2+\frac{1}{\eta^2}=h(\eta^2)$,

Thus $q_2oh=h(\eta^2)$ or $h^{-1}o\,q_2oh=\eta^2$.

Therefore q_2 is conjugate to η^2.

Since the iterates of any η under η^2 tend to ∞ for {$|\eta|$>١} , the iterates of $z\in C_\infty\setminus[-2,2]$ under q_2 also tend to ∞ .

Claim ٣ : [-٢,٢] is the Julia set for q_2 .

By theorem (٢.١.١٣), and since $[-2,2]=\partial A_2(\infty)$, thus [-٢,٢] is the Julia set for q_2 .

Theorem (٢.٢.٣) [٢٦]

Let $z \in C, |z| > |c|, |z| > ٢$. Then $z \in A_c(\infty)$ for $q_c(z) = z^2 + c$.

Proof :

There exists a small number $\varepsilon >$ ٠ with $z = 2 + \varepsilon$ .The triangle inequality for complex numbers implies $|z^2| = |z^2 + c - c| \le |z^2 + c| + |c|$.

Solving this inequality for $|z + c|$, we derive

$$|z^2 + c| \ge |z^2| - |c| = |z|^2 - |c| \ge |z|^2 - |z| = |z|(|z| - 1)$$

$$= |z|(|2 + \varepsilon - 1|)$$

$$= |z||\varepsilon + 1|$$

$$= (\varepsilon + 1)|z| \ .$$

If we iterate once , the absolute will increase by at least a factor $1+\varepsilon$.

The k–th iterate of z will therefore be at least $(1+\varepsilon)^k$ times as large as z in magnitude .Therefore the orbit must escape to infinity if one point in the orbit q_c is greater than max $\{|c|,2\}$. ■

Remark (٢.٢.٤)

The previous theorem can be stated as follows :

If $|z| >$ max $\{|c|,2\}$, then $z \in A_c(\infty)$.

Proposition (٢.٢.٥) [١٤](pp. ٢١٣)

If $z \in J(q_c)$, then $|z| \le$ max $\{|c|,2\}$, so that $J(q_c)$ is a bounded subset of the complex plane .

Proof :

If z is a periodic point , then iterates of z are bounded , so by theorem (٢.٢.٣) , $|z| \leq$ max $\{|c|,2\}$. By proposition (٢.١.٢) , $J(q_c)$ is the closed set containing all repelling periodic points, any $z \in J(q_c)$ also has property $|z| \leq$ max $\{|c|,2\}$ .Consequently $J(q_c)$ is bounded in the complex plane . ■

Proposition (٢.٢.٦) [٢٩]

$A_c(\infty) = C \setminus K(q_c)$.

Proof :

If $z \in C$. Suppose $z \in A_c(\infty)$, by definition (١.٣.٤) ,thus $|q_c^n(z)| \to \infty$ as $n \to \infty$. Hence $\{q_c^n(z)\}$ is unbounded . Therefore $z \notin K(q_c)$, $z \in C \setminus K(q_c)$. Thus $A_c(\infty) \subseteq C \setminus K(q_c)$.

Conversely, suppose that $z \in C \setminus K(q_c)$. Then $\{q_c^n(z)\}$ is unbounded. Therefore , for some $m >$ ٠ we have $|q_c^m(z)| >$ max $\{|c|,2\}$. By theorem (٢.٢.٣) , $|q_c^{m+n}(z)| \to \infty$ as $n \to \infty$, hence $z \in A_c(\infty)$ and $C \setminus K(q_c) \subseteq A_c(\infty)$. Thus $A_c(\infty) = C \setminus K(q_c)$. ■

Proposition (٢.٢.٧) [٢٩]

$A_c(\infty)$ is an open set .

Proof :

Fix any $z_0 \in A_c(\infty)$. Then $|q_c^n(z_0)| \to \infty$ as $n \to \infty$, hence there exists $m >$ ٠ such that $|q_c^m(z)| >$ max $\{|c|,2\}$ + ١ , since q_c is continuous , the map q_c^m is continuous , therefore , we find $\delta > 0$ and U neighborhood of z_0 such that $|z - z_0| < \delta$, thus $|q_c^m(z) - q_c^m(z_0)| <$ ١ . Then by the Triangle Inequality $|q_c^m(z)| \geq |q_c^m(z_0)| - |q_c^m(z_0) - q_c^m(z)| >$ max $\{|c|,2\}$. For all $z \in U$ and by proposition (٢.٢.٣) , $|q_c^{m+n}(z)| \to \infty$, so $z \in A_c(\infty)$, thus $U \subset A_c(\infty)$, since z_0 was arbitrary . Therefore $A_c(\infty)$ is an open set . ■

As an immediate corollary :

Corollary (٢.٢.٨) [٢٩]

$K(q_c)$ is a closed set .

Proposition (٢.٢.٩) [٢٩]

$J(q_c) \subseteq K(q_c)$.

Proof :

Let $z \in J(q_c)$. By definition (١.٣.٧) , $J(q_c)$ is boundary of $K(q_c)$. By proposition (١.٣.١٠) , $K(q_c)$ is closed , thus $z \in K(q_c)$. Therefore $J(q_c) \subseteq K(q_c)$. ■

Proposition (٢.٢.١٠) [٢٩]

$J(q_c)$ is the boundary of $A_c(\infty)$.

Proof :

By theorem (٢.١.١٣), so $J(q_c)$ is the boundary of $A_c(\infty)$. ■

Now , we introduce the next theorem for calculating the Julia sets .

Theorem (٢.٢.١١) [٢٦]

Let $J(q_c)$ be the Julia set for the map $q_c(z) = z^2 + c$, where $c \in C$. Then $J(q_c)$ is connected if and only if the orbit of ٠ is bounded .

The proof can be found in [٢٦] .

Example (٢.٢.١٢) [٣٤]

Let $q_c(z) = z^2 + i$ to show that $J(q_c)$ is connected .To compute the orbit of the critical point at $z = 0$, thus

$$q_c(0) = i$$

$$q_c(i) = -1 + i$$

$$q_c(-1+i) = -i$$

$$q_c(-i) = -1 + i$$

$$q_c(-1+i) = -i$$

$$q_c(-i) = -1 + i$$

$$q_c(-1+i) = -i \ .$$

Thus , the orbit of ٠ is the sequence $\{0, i, -1+i, -i, -1+i, -i, -1+i, -i, \ldots\}$, which is a bounded sequence . Therefore , by theorem (٢.٢.١١) , $J(q_c)$ is connected .

Section Three (٢.٣)

Chaoticity on Julia sets

The goal of this section is to study the chaotic dynamics of the Julia sets .Roughly speaking , a map f is considered to be chaotic if its orbits behave in a very complicated and unpredictable way .

Definition (٢.٣.١) [٢٣]

Suppose that $f:X\to X$ is a map . Then f has sensitive dependence on initial conditions if there exists $\delta>0$ such that for any $x\in X$ and for any neighborhood U of x , there exists $y\in U$ and $n\geq 1$ such that $d\left(f^{n}(x),f^{n}(y)\right)>\delta$.

Example (٢.٣.٢) [١٠](pp. ٥٠)

Let $f:S^1\to S^1$ be a map given by $f(\theta)=2\theta$. If $f(\theta)=2\theta$, then $f^2(\theta)=2(2\theta)=2^2\theta$, thus the n-*th* iterate is $f^n(\theta)=2^n\theta$. Therefore the angular distance between two points is doubled upon iteration of f . Hence f is sensitive to initial conditions .

Definition (٢.٣.٣) [٢٣]

Let $f:X\to X$ be a map , f is transitive if for any two non-empty open sets U , V in X , there exists an integer $n\geq 1$ such that $f^n(U)\cap V\neq\phi$.

Example (٢.٣.٤) [١٠](pp. ٥٠)

Let $f:S^1\to S^1$ be a map given by $f(\theta)=2\theta$. Let U be any small open arc in S^1 , there is $k\in N$ such that $f^k(U)$ covers all of S^1, in particular, $f^k(U)$ intersects any other open arc V in S^1 . This implies $f^k(U)\cap V\neq\phi$, thus f is transitive .

Example (٢.٣.٥) [١٠](pp. ١٨)

Let $f:S^1\to S^1$ be a map defined by $f(\theta)=2\theta$. If $f(\theta)=2\theta$, then $f^2(\theta)=2(2\theta)=2^2\theta$, thus the n-*th* iterate is $f^n(\theta)=2^n\theta$. So that θ is a periodic point of period n if and only if $2^n\theta=\theta+2\pi k$, for some integer k, that is , if and only if $\theta=\dfrac{2\pi k}{2^n-1}$, where $0\le k\le 2^n-1$ is an integer . Hence the periodic points of period n for f are the $(2^n-1)-th$ roots of unity . It follows that the set of periodic points is dense in S^1.

Now , we give the Devaney`s definition of chaos .

Definition (٢.٣.٦) [١٠](pp. ٥٠)

Let $f:X\to X$ be a map . f chaotic on X if

(a) the periodic points for f dense in X .

(b) f is transitive .

(c) f has sensitive dependence on initial conditions .

Example (٢.٣.٧) [١٠] (pp. ٥٠)

Let $f:S^1\to S^1$ be the map given by $f(\theta)=2\theta$.

By example (٢.٣.٢) , f has sensitive dependence on initial conditions.

By example (٢.٣.٤) , f is transitive . The density of periodic points was established in example (٢.٣.٥) . Thus f is chaotic .

Now , we introduce remark conditions the relation between chaotic dynamics and Julia sets .

Remark (٢.٣.٨) [٢٤]

We give an initial point z_0 , and consider its orbit under the action of f , that is , the set $\{z_0, f(z_0), f^2(z_0),...\}$. For some z_0 , the orbit of points near z_0 is very different from that of z_0 . The behavior of iterations in this region is to escape into far situation , while its complementary region is where the iterations are rather tame . The former region (a subset of the complex plane) is called the Julia set , where as the latter is the Fatou set . The situation is something like this : Figure (١) .

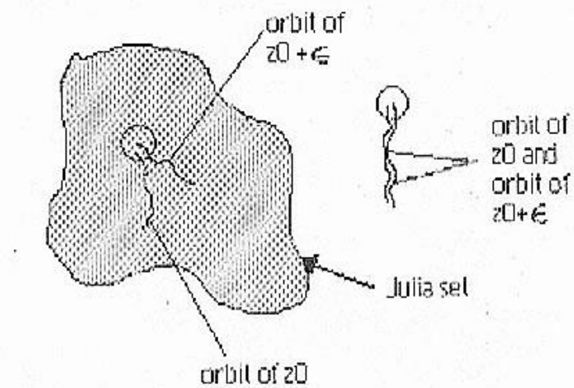

Fig.١

A definition of Julia sets motivated by chaotic dynamics would then be : Julia set of the map f is the chaotic set of f , that is the set of points $\{z: f$ exhibits sensitivity to initial conditions at $z\}$. Hence if $z \in J(f)$, then the iterations of f take points in the neighborhood of z arbitrarily far from the orbit of z . On the other hand , if f is sensitive to initial conditions at a point z , then the family can not be normal in a neighborhood of that point .

Remark (٢.٣.٩) [١٠] (pp. ٢٨٦)

Let z_1 and z_2 be a repelling periodic points for f. By definition (٢.١.١١) , there are heteroclinic orbits connecting them , that is , there is a point z which eventually maps onto z_2 and for which a sequence of backwards iterations of f^n converge to z_1 .

Theorem (٢.٣.١٠) [١٠] (pp. ٢٨٨)

$f : J(f) \to J(f)$ is chaotic .In sence of Devaney.

Proof :

By definition (١.١.١٨) and by proposition (٢.١.٧) , the periodic points are dense in $J(f)$, it suffices to show that f is transitive and also depends sensitively on initial conditions .

Suppose that $z_1, z_2 \in J(f)$ then either z_1 and z_2 are repelling or z_1, z_2 in limit of repelling periodic points . Suppose z_1 and z_2 are repelling and U is a neighborhood of z_1, z_2 .By remark (٢.٣.٩) , there is a heteroclinic orbit connecting z_1 and z_2 . It follows immediately that f is transitive .

Suppose z_1, z_2 in limit of repelling periodic points , any neighborhood U of any point contains z_3 and z_4 are repelling point . By remark (٢.٣.٩) , there is a heteroclinic orbit connecting z_3, z_4 ,thus there is a heteroclinic orbit connecting z_1, z_2 . It follows immediately that f is transitive .

To show that f is a sensitive depends on initial conditions , suppose

$z_1, z_2 \in J(f)$ by above either z_1, z_2 are repelling or not , since this heteroclinic orbit lies in $J(f)$, it follows that f has sensitive depends on initial conditions . Therefore f is chaotic on $J(f)$. ■

Chapter ٣

Properties of Julia sets of maps

Of the form ($\lambda z - \lambda z^2$)

Our goal of this chapter is to study the properties of Julia set for the quadratic polynomial maps of the form $Q_\lambda(z) = \lambda z - \lambda z^2$, where λ is a non zero complex constant . In section one of this chapter we study the geometric properties of the Julia set of the map Q_λ . In section two , we will give the topological properties of Julia set of the map Q_λ .

Section One (٣.١)

Geometric properties of the Julia set

For the map $Q_\lambda(z) = \lambda z - \lambda z^2$

The goal of this section is to study the geometric properties of the Julia set for the quadratic polynomial of the form $Q_\lambda(z) = \lambda z - \lambda z^2$.

Remark (٣.١.١.)

We find the fixed points for Q_λ .

$Q_\lambda(z) = \lambda z - \lambda z^2 = z$, that is $Q_\lambda(z) - z = 0$, hence $\lambda z - \lambda z^2 - z = 0$, thus $\lambda z - (\lambda z^2 + z) = 0$, therefore $z((\lambda - 1) - \lambda z) = 0$, so that the fixed points for Q_λ are $z = 0$ or $z = \frac{\lambda - 1}{\lambda}$.

$|Q'_\lambda(z)| = |\lambda - 2\lambda z|$, if $z = 0$ then $|Q'_\lambda(0)| = |\lambda|$. If $|\lambda|$ < ١ , then $z = 0$ is attracting fixed point . If $|\lambda|$ > ١ , then $z = 0$ is repelling fixed point .

If $z = \frac{\lambda - 1}{\lambda}$ then $\left|Q'_\lambda\left(\frac{\lambda - 1}{\lambda}\right)\right| = |2 - \lambda|$. If ٣ < $|\lambda|$ or $|\lambda|$ < ١, then $z = \frac{|\lambda| - 1}{|\lambda|}$ is repelling fixed point . If ١ < $|\lambda|$ < ٣ , then $z = \frac{|\lambda| - 1}{|\lambda|}$ is attracting fixed point . If $Q'_\lambda(z) = \lambda - 2\lambda z$, thus $Q'_\lambda(z) = \lambda - 2\lambda z = 0$, thus $z = 0.5$. Hence the critical point for Q_λ is ٠.٥ .

Example (٣.١.٢)

Let $q_c : C \to C$ and $Q_\lambda : C \to C$ such that $q_c(z) = z^2 + c$ and $Q_\lambda(z) = \lambda z - \lambda z^2$ to show q_c conjugate to Q_λ . We use theorem (١.٢.١١) with the following substitutions: $a = 1, b = 0, c = c, r = -\lambda, s = \lambda, t = 0$. Using (١.٢.١) we find $c = \frac{-\lambda^2}{4} + \frac{\lambda}{2}$, and using (١.٢.٢) we get $h(z) = \frac{-1}{\lambda} z + \frac{1}{2}$.

$$Q_\lambda \circ h(z) = Q_\lambda\left(\frac{1}{2} - \frac{z}{\lambda}\right) = \lambda\left(\frac{1}{2} - \frac{z}{\lambda}\right) - \lambda\left(\frac{1}{2} - \frac{z}{\lambda}\right)^2$$

$$= \frac{1}{4}\lambda - \frac{z^2}{\lambda} \ .$$

$$h \circ q_c(z) = h(z^2+c) = \frac{1}{2} - \frac{z^2+c}{\lambda}$$

$$= \frac{1}{2} - \frac{z^2 - \frac{\lambda^2}{4} + \frac{\lambda}{2}}{\lambda}$$

$$= \frac{1}{2} - \frac{z^2}{\lambda} + \frac{\frac{\lambda^2}{4}}{\lambda} - \frac{\frac{\lambda}{2}}{\lambda}$$

$$= \frac{1}{4}\lambda - \frac{z^2}{\lambda} = Q_\lambda \circ h(z) \ .$$

Therefore Q_λ is conjugate to q_c and note that $\lambda = 1 \mp \sqrt{1-4c}$.

Example (٣.١.٣)

$J(Q_2)$ is the unit circle of $Q_2(z) = 2z - 2z^2$.

The discussion of this example splits into three claims .

Let $D(a,b) = \{z \in C : |z-a| < b\}$, where $a \in C$ and $0 < b \in R$.

Claim ١ :

Let $z_0 \in D(0,1)$, then $z_0 \in F(Q_2)$.

Let $z_0 \in D(0,1)$, that is $|z_0| <$ ١ . Suppose that $U = D\left(z_0, \frac{1-|z_0|}{2}\right)$. One can see that $U \subseteq D(0,1)$ for all $z \in \overline{U}$ and by using $|z - z_0| \geq |z| - |z_0|$, thus $|z - z_0| < \frac{1-|z_0|}{2}$, hence $|z| - |z_0| \leq |z - z_0| < \frac{1-|z_0|}{2}$,

therefore $|z| - |z_0| < \frac{1-|z_0|}{2}$, thus $|z| < \frac{1}{2} - \frac{|z_0|}{2} + |z_0|$,

hence $|z| < \frac{1}{2} + \frac{|z_0|}{2}$, that is for all $z \in \overline{U}$, $|z| < \frac{1+|z_0|}{2} <$ ١ . Hence $\overline{U} \subset D(0,1)$.

For all $z \in \overline{U}$, $Q_2(z) = 2z - 2z^2$, if

$\left|Q_2(z)\right| = |2z - 2z^2| \leq |2z| + |2z^2| <$ ٢ $|z^2| + 2|z^2| = 4|z^2|$, thus

$\left|Q_2^2(z)\right| = |4z - 12z^2 + 16z^3 - 8z^4|$

$\leq |4z| + |12z^2| + |16z^3| + |8z^4|$

$< 16|z^4| + 16|z^4| + 16|z^4| + 16|z^4|$

$= 4^3|z^4|$,

hence for n -th iterate $\left|Q_2^n(z)\right| \to 0$ as $n \to \infty$.Therefore $\{Q_2^n\}$ is normal in $D(0,1)$, hence $D(0,1) \subseteq F(Q_2)$.

Claim ٢ :

If $|z_0| >$ ١ , then $z_0 \in A_2(\infty)$.

Let $|z_0| >$ ١ . $\left|Q_2(z_0)\right| = |2z_0 - 2z_0^2| \leq 2|z_0| + 2|z_0|^2 < 2|z_0^2| + 2|z_0^2| = 4|z_0^2|$,

Then $\left|Q_2^2(z_{0z})\right| = |4z_0 - 12z_0^2 + 16z_0^3 - 8z_0^4|$

$\leq |4z_0| + |12z_0^2| + |16z_0^3| + |8z_0^4|$

$< 16|z_0^4| + 16|z_0^4| + 16|z_0^4| + 16|z_0^4|$

$= 4^3|z_0^4|$.

Hence , for n-th as $n \to \infty$. Therefore $z_0 \in A_2(\infty)$.

Claim ٣:

If $|z_0| = 1$, then $z_0 \notin F(Q_2)$ and $z_0 \notin A_2(\infty)$.

Let $|z_0| = 1$. Assume $z_0 \in F(Q_2)$ so there exists neighborhood U_{z_0} , which has a subsequence of $\{Q_2^n\}$, and a map f with $Q_2^{n_k} \to f$ uniformly on U_{z_0} . Now for all ε > ٠ there is $D(z_0, \varepsilon) \subset U_{z_0}$, by claim ١ , there is $z_1 \in D(z_0, \varepsilon)$ with $|z_1|$ < ١ . It follows that $Q_2^{n_k} \to 0$ as $n \to \infty$, that is $f(z_1) = 0$.

Since $|z_0| = 1$, $|f(z_1)| = 1$, which is contradicts that f is analytic map (and therefore is continuous). Therefore $z_0 \notin F(Q_2)$. Similarly , we can proof that $z_0 \notin A_2(\infty)$. Therefore $z_0 \in J(Q_2)$ for $|z_0| = 1$.

Hence $J(Q_2)$ is unit circle . ■

Example (٣.١.٤)

$J(Q_4)$ is the line segment [٠,١] for $Q_4(z) = 4z - 4z^2$.

the discussion of this example splits into three claims .

Claim ١:

The set [٠,١] is completely invariant .

Consider $Q_4(x) = 4x - 4x^2$, thus $Q'_4(x) = 4 - 8x$, hence $Q''_4(x) = -8$, therefore $Q_4(x)$ has maximum value ١ at x =٠.٥ since $Q_4(0.5) = 1$.

$Q_4(x)$ is increasing on the interval [٠.٥,١] .

$Q_4(x)$ has minimum value of ٠ at x = ٠ or ١ , since $Q_4(0) = 0$ and $Q_4(1) = 0$.

Thus $Q_4([0,1]) \subset [0,1]$ .Therefore $[0,1]$ is not a subset of $A_4(\infty)$.

Claim ٢:

$W = C_\infty \setminus [0,1]$ is $A_4(\infty)$.

Let $z_0 \in W$ with $|z_0|$>١ .If $|Q_4(z_0)| = |4z_0 - 4z_0^2| \leq 4|z_0| + 4|z_0^2| < 4|z_0^2| + 4|z_0^2| = 8|z_0^2|$,

thus $|Q_4^2(z_0)| = |16z_0 - 80z_0^2 + 128z_0^3 - 64z_0^4|$

$$\leq |16z_0| + |80z_0^2| + |128z_0^3| + |64z_0^4|$$

$$< 128|z_0^4| + 128|z_0^4| + 128|z_0^4| + 128|z_0^4|$$

$$= 8^3|z_0^4|.$$

Hence , for n-th as $n \to \infty$. Therefore $z_0 \in A_4(\infty)$.

Claim ٣ :

[٠,١] is the Julia set for $Q_4(z) = 4z - 4z^2$.

By theorem (٢.١.١٣) , and since $[0,1] = \partial A_4(\infty)$. Hence [٠,١] is the Julia set for Q_4 . ■

Proposition (٣.١.٥)

Suppose that ١<$|\lambda|<1+\sqrt{2}$. Then $J(Q_\lambda)$ is a simple closed curve .

Proof :

$Q_\lambda(z) = |\lambda|z - |\lambda|z^2$, then $|Q'_\lambda(z)| = |\lambda - 2\lambda z| <$ ١ , thus $|\lambda||1-2z| <$ ١ , that is $|1-2z| < \frac{1}{|\lambda|}$, since $|a-b| > |a| - |b|$ thus $1 - 2|z| < \frac{1}{|\lambda|}$, hence $|z| > \frac{1}{2} - \frac{1}{2|\lambda|}$, or $|z| < \frac{1}{2} + \frac{1}{2|\lambda|}$, thus $|z - 0.5| < \frac{1}{2|\lambda|}$, where $\frac{1}{2|\lambda|}$ is the radius and ٠.٥ is the center .

We note if ١<$|\lambda|<1+\sqrt{2}$ =٢.٤١٤٢١٣٥ , then $\frac{1}{2|\lambda|} <$ ٠.٢٠٧١٠٦٧ ,

$0.5 + \frac{1}{2|\lambda|}$ < ٠.٧٠٧١٠٦٧ and $0.5 - \frac{1}{2|\lambda|}$ < ٠.٢٩٢٨٩٣٣ .

The attractor point is $\frac{|\lambda| - 1}{|\lambda|}$ < ٠.٥٨٥٧٨٦٤ , while the critical point of Q_λ and the centre of circle is ٠.٥ .

$|Q_\lambda(0.5)|$ < ٠.٦٠٣٥٥٣٣

$|Q_\lambda(0.5857864)|$ < ٠.٥٨٥٧٨٦٣

$|Q_\lambda(0.7071067)|$ < ٠.٥

$|Q_\lambda(0.2928933)|$ < ٠.٥

$|Q_\lambda(0.5+0.2071067i)| <$ ٠.٧٠٧١٠٦٤

$|Q_\lambda(0.5-0.2071067i)| <$ ٠.٧٠٧١٠٦٤

$|Q_\lambda(0)| = 0$

$|Q_\lambda(2)| <$ ٤.٨٢٨٤٢٧

$|Q_\lambda(0.1)| <$ ٠.٢١٧٢٧٩٢

$|Q_\lambda(0.8)| <$ ٠.٣٨٦٢٧٤١ .

Let Γ_0 be the circle of radius ٠.٢٠٧١٠٦٧ about ٠.٥ . Γ_0 contains both the attracting fixed point (٠.٥٨٥٧٨٦٤) and the critical point ٠.٥ of Q_λ in its interior . Moreover , $|Q'_\lambda(z)|$ > ١ for z in the exterior of Γ_0, where ٠ is repelling fixed point of Q_λ. For each $\theta \in S^1$, we will define a continuous curve $\gamma_\theta : [1,\infty) \to C$ having the property that $z(\theta) = \lim_{t\to\infty} \gamma_\theta(t)$ is a continuous parameterization of $J(Q_\lambda)$.

To define $z(\theta)$, we first note that the preimage Γ_1 of Γ_0 under Q_λ is $Q_\lambda(z) = \lambda z - \lambda z^2 = w$, thus $\lambda^2 - \lambda z + w = 0$, hence $z = \frac{1}{2} \mp \sqrt{\frac{1}{4} - \frac{w}{\lambda}}$.

The preimage with respect to ٠.٧٠٧١٠٦٧ and ٠.٢٩٢٨٩٣٣ are $z = 0.5 \mp 0.2071069$, that is with respect to ٠.٧٠٧١٠٦٧ is $z =$ ٠.٧٠٧١٠٦٩ and $z =$ ٠.٢٩٢٨٩٣١ , also with respect to ٠.٢٩٢٨٩٣٣ is $z =$ ٠.٧٠٧١٠٦٩ and $z =$ ٠.٢٩٢٨٩٣١ , while the preimage with respect to the attracting fixed point (٠.٥٨٥٧٨٦٤) is $z = 0.5 \mp 0.0857869$, that is $z =$ ٠.٥٨٥٧٨٦٩ and $z =$ ٠.٤١٤٢١٣١ , while the preimage with respect to the critical point (٠.٥) is $z =$ ٠.٥٠٠٣١٦٢ and $z =$ ٠.٤٩٩٦٨٣٨ , while the preimage for the points with respect to (٠.٥+٠.٢٠٧١٠٦٧i) and (٠.٥-٠.٢٠٧١٠٦٧i) are $z = 0.5 \mp 0.2071062$, that is

$z=$ ٠.٥+٠.٢٠٧١٠٦٢i and $z=$ ٠.٥-٠.٢٠٧١٠٦٢i and $z=$ ٠.٥+٠.٢٠٧١٠٦٢i and $z=$ ٠.٥-٠.٢٠٧١٠٦٢i ,

each value of the preimages under Q_λ have two values, as follows

$|Q_\lambda(0.7071069)|<$ ٠.٤٩٩٩٩٩٧

$|Q_\lambda(0.2928931)|<$ ٠.٤٩٩٩٩٩٧

$|Q_\lambda(0.5-0.2071062i)|<$ ٠.٧٠٧١٠٥٩

$|Q_\lambda(0.5+0.2071062i)|<$ ٠.٧٠٧١٠٥٩ ,

and the value of the preimages under Q_λ for the critical point , as follows

$|Q_\lambda(0.5003162)|<$ ٠.٦٠٣٥٥٣

$|Q_\lambda(0.4996838)|<$ ٠.٦٠٣٥٥٣١ .

While the value of the preimages under Q_λ for the attracting fixed point , as follows

$|Q_\lambda(0.5857869)|<$ ٠.٥٨٥٧٨٦٢

$|Q_\lambda(0.4142131)|<$ ٠.٥٨٥٧٨٦٢ .

Then preimage Γ_1 of Γ_0 under Q_λ is a simple closed curve which contains Γ_0 in its interior and which is mapped in a two – to – one formula onto Γ_0 . The fact that Γ_1 is a simple closed curve follows from the fact that both the critical point (٠.٥) and its image lie inside Γ_0 . Hence the curves Γ_0 and Γ_1 bound an annular region A_1 (A_1 may be regarded as a fundamental domain for the attracting fixed point for Q_λ)

Let W be the standard annulus defined by

$W=\{\, r e^{i\theta} : 1\le r\le 2, \theta$ arbitrary $\}$.

Choose diffeomorphism $\varphi : W \to A_1$ which maps the inner and outer boundaries of W to the corresponding boundaries of A_1 . See figure (٢) . This allows us to define the initial segment of $\gamma_\theta : [1,2] \to C$ by $\gamma_\theta(r) = \varphi(r e^{i\theta})$. That is , γ_θ is the image of a ray in W under φ .

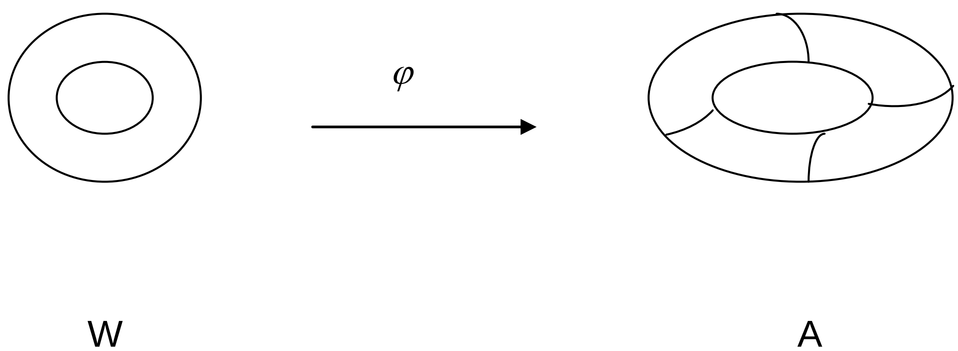

Fig.٢

For $r \geq 2$, may extend γ_θ as follows , since preimage Γ_1 of Γ_0 under Q_λ and the critical point in interior Γ_0 , thus Q_λ has no critical points in the exterior of Γ_1 . The preimages Γ_2 of Γ_1 under Q_λ are $z = 0.5 \mp 0.2071072$, that is $z = 0.7071072$ and $z = 0.2928928$ with respect to ٠.٧٠٧١٠٦٩ , also have the same preimages with respect to ٠.٢٩٢٨٩٣١, while the preimages of critical points (٠.٥٠٠٣١٦٢) and (٠.٤٩٩٦٨٣٨) are $z = 0.5 \mp 0.0004472$ that is $z = 0.5004472$ and $z = 0.4995528$ with respect to ٠.٥٠٠٣١٦٢ and also for ٠.٤٩٩٦٨٣٨, while the preimages for the attracting fixed points are $z = 0.5 \mp 0.0857875$ that is $z = 0.5857875$ and $z = 0.4142125$

for (٠.٥٨٥٧٨٦٩) and also for (٠.٤١٤٢١٣١) , while the preimages of points (٠.٥+٠.٢٠٧١٠٦٢i) and (٠.٥ -٠.٢٠٧١٠٦٢i) are

$z = 0.5 \mp 0.2071057i$ that is $z = 0.5 + 0.2071057i$ and $z = 0.5 - 0.2071057i$ for (٠.٥+٠.٢٠٧١٠٦٢i) and also for (٠.٥-٠.٢٠٧١٠٦٢i),

each value of the preimages under Q_λ have four values, as follows

$|Q_\lambda(0.7071072)|$ < ٠.٤٩٩٩٩٩٥

$|Q_\lambda(0.2928928)|$ < ٠.٤٩٩٩٩٩٥

$|Q_\lambda(0.5 - 0.2071057i)|$ < ٠.٧٠٧١٠٥٥

$|Q_\lambda(0.5 + 0.2071057i)|$ < ٠.٧٠٧١٠٥٥ ,

and the value of the preimages under Q_λ for the critical point , as follows

$|Q_\lambda(0.5004472)|$ < ٠.٦٠٣٥٥٢٨

$|Q_\lambda(0.4995528)|$ < ٠.٦٠٣٥٥٢٨ .

While the value of the preimages under Q_λ for the attracting fixed point , as follows

$|Q_\lambda(0.5857875)|$ < ٠.٥٨٥٧٨٥٩

$|Q_\lambda(0.4142125)|$ < ٠.٥٨٥٧٨٥٩ .

Hence there is a simple closed curve Γ_2 which is mapped in a two – to – one formula onto Γ_1.

Moreover , Q_λ maps the annular region A_2 between Γ_1 and Γ_2 onto A_1 , again in a two –to-one formula . Thus , the preimage of any γ_θ in A_1 is a pair of non –intersection curves in A_2, thus every point $z \in A_2$, imply $f(z) \in A_1$. There is a unique such curve which meets the inner boundary Γ_1 . Hence , for each θ , there is a unique curve in A_2 which contains the point $\gamma_\theta(2)$, that is $\gamma_\theta(1)$ is boundary of Γ_0 and $\gamma_\theta(2)$ is boundary of Γ_1 and $\gamma_\theta(3)$ is boundary of Γ_2 . We may thus sew together these two curves in the obvious way at this point , producing a single curve defined on the interval [١,٣] .

Continuing in this formula , we may extend each γ_θ over the entire interval $[٠,\infty)$.

Now recall that $|Q'_\lambda(z)| > k > ١$ for positive integer k provided z lies in the exterior of Γ_1 . Hence the length of each extension of γ_θ decreases geometrically .

It follows that $\gamma_\theta(t)$ converges uniformly in θ and that $\lim_{t\to\infty}\gamma_\theta(t) = z(\theta)$, since $\lim_{t\to\infty}\gamma_\theta(t)$ is continuous , thus $z(\theta)$ is continuous and is a unique point in C for each θ .

We claim that $z(\theta)$ parameterizes a simple closed curve in C . To show that the image curve is simple , we must prove that if $z(\theta_1) = z(\theta_2)$, then $z(\theta) = z(\theta_1)$ for all θ with $\theta_1 \le \theta \le \theta_2$, see fig. (٣). $z(\theta)$ is a point by substituting $\theta = \theta_1$. However , if this was not the case , the portions of the curves Γ_1 , $\gamma_{\theta_1}(t)$ and $\gamma_{\theta_2}(t)$ would bound a simply connected region containing each $z(\theta)$ in its interior . This implies that there is a neighborhood of $z(\theta)$ whose images under Q_λ^n remains bounded , thus $z(\theta)$ is attracting but not repelling .

Hence $z(\theta) \notin J(Q_\lambda)$ .But this is impossible . Therefore $J(Q_\lambda)$ is simple closed curve . ■

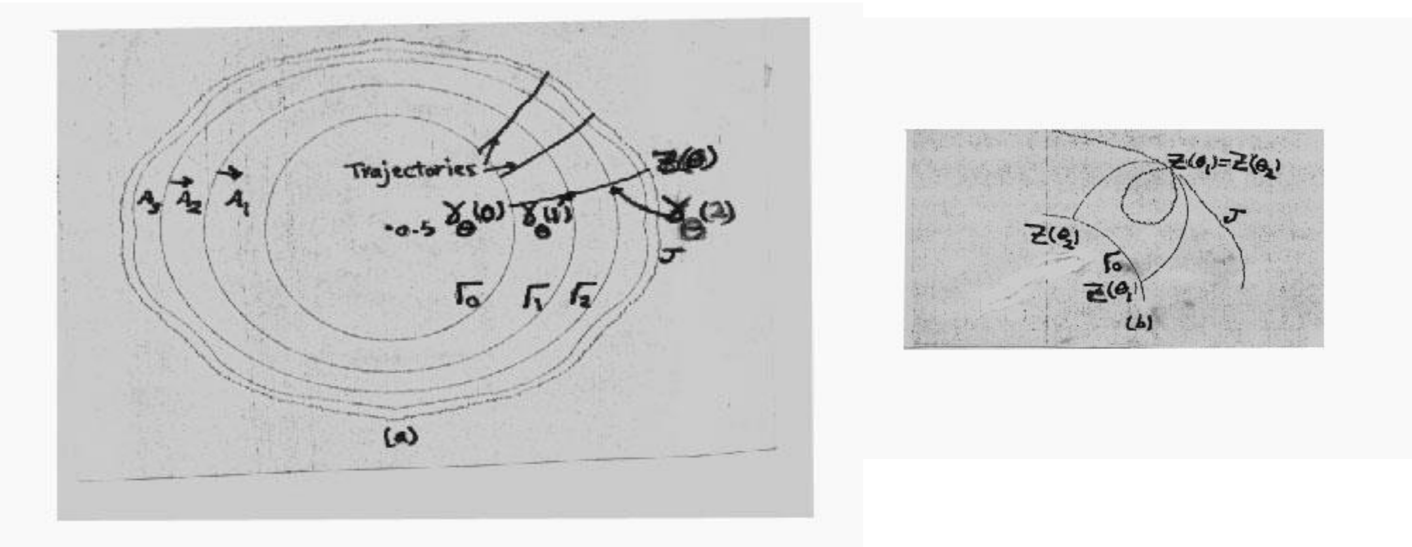

Fig.٣ (a) & (b) the proof of the proposition (for ١<$|\lambda|$ < ١+$\sqrt{2}$)

Proposition (٣.١.٦)

Suppose λ is a complex number and ١<$|\lambda|<1+\sqrt{2}$ .Then $J(Q_\lambda)$ is a simple closed curve such that Julia set which contains no smooth arcs .

Proof :

Suppose that λ is complex , that is $\lambda=\lambda_1+\lambda_2 i$ and satisfies ١<$|\lambda|<1+\sqrt{2}$.

If Q_λ has repelling fixed point at $z_0=0$. Then $|Q'_\lambda(0)|=|\lambda|-2|\lambda|(0)=|\lambda|$, if $\lambda_1\neq 0$ then λ is not pure imaginary , by properties of complex analysis , thus z_0 does not lie in a smooth arc in $z(\theta)$. For if this were the case , then the image of $z(\theta)$ would also be a smooth arc in $J(Q_\lambda)$ passing through z_0 . Since $Q'_\lambda(z_0)$ is complex , the tangents to these two curves $z(\theta_1)$ and $z(\theta_2)$ would not be parallel .

Therefore $z(\theta)$ would not be simple at z_0, that is $z(\theta_1) \neq z(\theta_2)$. Since by proposition (٢.١.٧) , the preimage of z_0 are dense in $J(Q_\lambda)$ .It follows that $J(Q_\lambda)$ contains no smooth arcs . ■

Example (٣.١.١٧)

$J(Q_\lambda)$ is infinitely many different simple closed curves for $\lambda = 1 \mp \sqrt{5}$.

First , let $\lambda = 1+\sqrt{5}$. We now turn to the case of an attracting periodic rather than fixed point .

$Q^2_\lambda(z) = z$, thus $Q^2_\lambda(z) - z = 0$, hence $\lambda^2 z^2 - z(\lambda^2+\lambda) + (\lambda+1) = 0$, therefore $z = \frac{\lambda+1}{2\lambda} \mp \frac{1}{2\lambda}\sqrt{\lambda^2 - 2\lambda - 3}$, thus $z = 0.5$ and $z = 0.809017$, which $Q_\lambda(0.5) = 0.809017$ and $Q_\lambda(0.809017) = 0.5$. Also $Q'_\lambda(z) = \lambda - 2\lambda z$, thus $|Q'_\lambda(0.5)| = 0 <$ ١ is an attracting fixed point .

Therefore ٠.٥ and ٠.٨٠٩٠١٧ lie on an attracting periodic of period ٢ .

The dynamics of Q_λ on the real line relatively straight forward , there are two repelling fixed points at ٠ and ٠.٦٩٠٩٨٢٩ , since Q_λ as two repelling fixed point $z = 0$ or $z = \frac{\lambda - 1}{\lambda} = 0.6909829$, that is $|Q'_\lambda(0)| >$ ١ and $|Q'_\lambda(0.699829)| >$ ١ .

The fixed point at ٠.٦٩٠٩٨٢٩ is the dividing point between the basin of attraction of ٠.٥ and ٠.٨٠٩٠١٧ . By proposition (٣.١.٥) , one may show that there are two simple closed curves γ_0 and γ_1 in $J(Q_\lambda)$ which surround ٠.٥ and ٠.٨٠٩٠١٧ respectively .

The curves γ_0 and γ_1 meet at fixed point ٠.٦٩٠٩٨٢٩ .

There is much more $J(Q_\lambda)$ however . The basin of attraction of ٠.٥ is not completely invariant because one preimage of the interior of γ_0 is γ_1 but there is another surrounding the other preimage of ٠.٥ is ٠.١٩٠٩٨٣ , since $Q_\lambda(z) = 0.5$, thus $3.2360679z^2 - 3.2360679z + 0.5 = 0$, hence $z = 0.809017$ and $z = 0.190983$. Therefore $Q_\lambda(0.190983) = 0.5$. Hence there is a third simple closed curve in $J(Q_\lambda)$ surrounding ٠.١٩٠٩٨٣ as well . Now both ٠.١٩٠٩٨٣ and ٠.٨٠٩٠١٧ must have a pair of distinct preimages , each is surrounded by a simple closed curve in $J(Q_\lambda)$. Continuting in this formula , we get that the Julia set of Q_λ must contain infinitely many different simple closed curves .

In the same way if $\lambda = 1 - \sqrt{5}$ then $z = 0.4999998$ and $z = -0.309017$.

$Q_\lambda(0.4999998) = -0.309017$ and $Q_\lambda(-.0309017) = 0.4999998$, also $|Q'_\lambda(0.4999998)| <$ ١ , thus -٠.٣٠٩٠١٧ and ٠.٤٩٩٩٩٩٨ lie on an attracting periodic of period ٢ , also has two repelling at $z = 0$ and $z = 1.809017$. Hence ٠ is the dividing point between the basin of attraction of -٠.٣٠٩٠١٧ and ٠.٤٩٩٩٩٩٨ . There are two simple closed curves Γ_0 and Γ_1 in $J(Q_\lambda)$ which surrounds ٠.٤٩٩٩٩٩٨ and -٠.٣٠٩٠١٧ respectively .

So that if $-1.2360679z^2 + 1.2360679z + 0.999998 = 0$, then $z = -0.309017$ and $z = 1.3091017$, also $Q_\lambda(1.309017) = 0.4999998$. Hence there is third simple a closed curve in $J(Q_\lambda)$ surrounding ١.٣٠٩٠١٧ . See fig. (٤) .

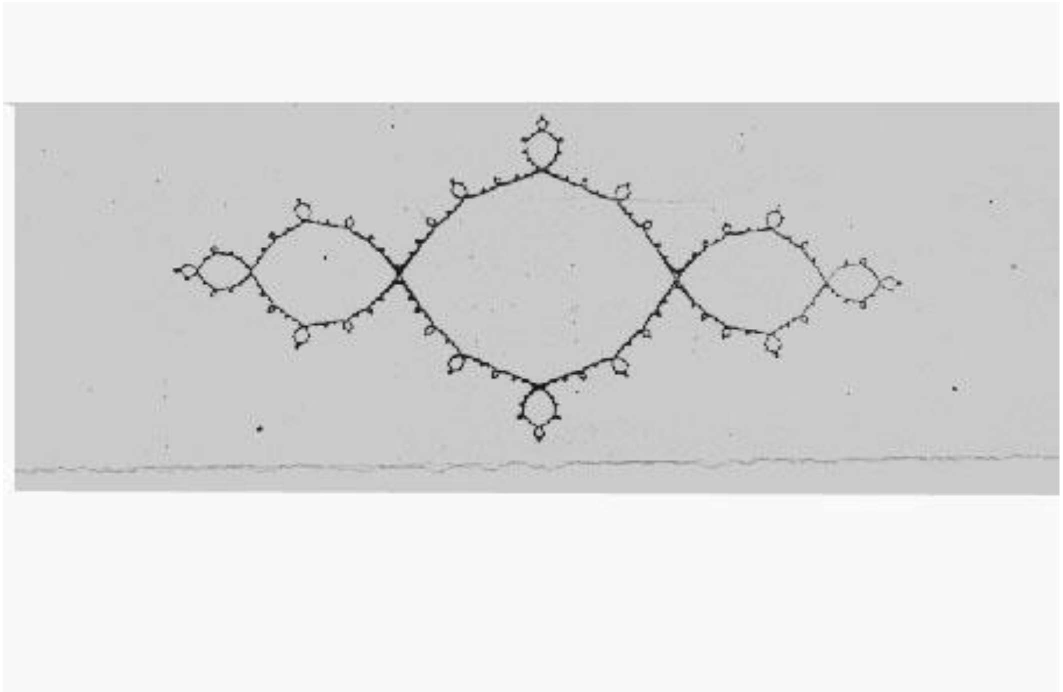

Fig. ٤ Julia set for $\lambda = 1 \mp \sqrt{5}$.

Section Two (٣.٢)

Topological properties of Julia sets of maps Of the form $(\lambda z - \lambda z^2)$

The goal of this section is to derive the topological properties of Julia sets of the map of the form $(\lambda z - \lambda z^2)$.

Theorem (٣.٢.١) [٢٢] (Bottcher theorem)

Let $f : C_\infty \to C_\infty$ be a rational map . If z_0 is super attracting fixed point , $f(z) = z_0 + a(z - z_0)^n + \ldots$, $n \geq 2$ and $a \neq 0$, then f is conjugate to $\varphi \circ f \circ \varphi^{-1} : w \to w^n$ in some neighborhood of z_0 .

The proof can be found in [٢٢] .

Definition (٣.٢.٢) [١٧]

Let $f : C_\infty \to C_\infty$ be a rational map has a super attracting fixed point at ٠ . Let Ω be the basin of attraction of this fixed point . Define $G : \Omega - \{0\} \to R$ by $G(z) = \log|\varphi(z)|$. If $G(z) <$ ٠ . Then the map is called the Green's maps of f .

Remark (٣.٢.٣) [٣٢]

Let $f : C_\infty \to C_\infty$ be a rational map . Define a critical value to be the image of a critical point . And , let a branch of the inverse map $f^{-1}(w)$ be the bijection between a neighborhood of w and a neighborhood of z where $f(z) = w$, w not a critical value of f .

Theorem (٣.٢.٤)

The Julia set $J(Q_\lambda)$, where $Q_\lambda(z) = \lambda z - \lambda z^2$, is connected if and only if there is no finite critical point of Q_λ in the basin of attraction $A_\lambda(\infty)$.

Proof :

By proposition (١.٢.١٤) ,

$F_\lambda(z) = r \circ Q_\lambda \circ r(z) = r \circ Q_\lambda\left(\frac{1}{z}\right) = r\left(\frac{\lambda}{z} - \frac{\lambda}{z^2}\right) = \frac{z^2}{\lambda z - \lambda}$. Thus $F_\lambda(0) = 0$.

Therefore ∞ is super attracting fixed point of Q_λ and $Q_\lambda(\infty) = \infty$, also $Q'_\lambda(\infty) = 0$. Now in a neighborhood of ∞ and by theorem (٣.٢.١) there exists a conformal map φ such that $\varphi(Q_\lambda(z)) = \varphi(z)^2$. . . (٣.٢.١), where $\varphi(z) = z + O\ (1)$

that is the following diagram commutes :

$$\begin{array}{ccc} U_\infty & \xrightarrow{Q_\lambda(z)} & U_\infty \\ \varphi \downarrow & & \downarrow \\ U_\infty & \xrightarrow{z^2} & U_\infty \end{array}$$

Such that $\varphi\, Q_\lambda\, \varphi^{-1} = g$ and $g(z) = z^2$. Next since $\log|\varphi(z)|$ has a logarithmic pole at ∞, $\log|\varphi(z)| = \log|z + O(1)| \cong \log|z| = \log\sqrt{x^2 + y^2}$,

$\frac{\partial^2 \log|\varphi(z)|}{\partial x^2} + \frac{\partial^2 \log|\varphi(z)|}{\partial y^2} = 0$.Thus $\log|\varphi(z)|$ is positive and harmonic .Since $J(Q_\lambda) = \partial A_\lambda(\infty)$ and if $|z| = 1$, then $\log|\varphi(z)| \to 0$ as $z \to \partial A_\lambda(\infty)$.

By definition (٣.٢.٢) , thus $\log|\varphi(z)| = G(z)$ for $A_\lambda(\infty)$.

Thus taking the logarithm of the modulus of (٣.٢.١) for $G(z)$, $\log|\varphi(Q_\lambda(z))| = 2\log|\varphi(z)|$, we have

$$G(Q_\lambda(z)) = 2G(z) \qquad \text{. . . (٣.٢.٢) .}$$

Now a component of the Fatou set map onto another component of the Fatou set since otherwise a boundary point (in element of the julia set) map to a point in the interior of a component of the Fatou set . This is a contradiction because by proposition (٢.١.٥) , $J(Q_\lambda)$ is completely invariant .

Next if a bounded component of $A_\lambda(\infty)$ exists , some iterates of Q_λ maps onto the component of $A_\lambda(\infty)$ which contains ∞ . This means that for some z in the bounded component and integer n , $Q_\lambda^n(z) = \infty$.

This is contradiction because the iterates of a polynomial are polynomials do not have poles . Thus $A_\lambda(\infty)$ is connected .

Define a level curve of $G(z)$ as $\Lambda_a = \{z : G(z) = a\}$, where $a \in R$,since for $z \in \Lambda_a$, $G(Q_\lambda(z)) = 2G(z) = 2a$, then $Q_\lambda(z)$ takes the level curve Λ_a to the level curve Λ_{2a} . So $Q_\lambda(z) \in \Lambda_{2a}$. Define the exterior of level curve Λ_a to be the set $E_a = \{ z : G(z) > a \} = \{ z : |\varphi(z)| > e^a \}$.

Then $Q_\lambda(z)$ maps E_a two –to-one to E_{2a} which is a subset of E_a . To extend $\varphi(z)$, first consider a neighborhood $U = E_r$ of ∞ on which the theorem (٣.٢.١) holds .

Then on $E_{\frac{r}{2}}$ we can define $\varphi(z) = \sqrt{\varphi(Q_\lambda(z))}$ (since $z \in E_{\frac{r}{2}}$, $Q_\lambda(z) \in E_r$) so the right hand side of the equation is defined . Continue in this way defining $\varphi(z)$ on $E_{\frac{r}{2^n}} = \{ z : |\varphi(z)| > \exp(\frac{r}{2^n})\}$ as long as there are no critical point in the extended region . At a critical point a single –valued analytic map can not be defined .

So as $n \to \infty$, φ is defined on $\bigcup_{n=1}^{\infty} E_{\frac{r}{2^n}} = \{ z : |\varphi(z)| > \exp(\frac{r}{2^n})\}$, that is

$\bigcup_{n=1}^{\infty} E_{\frac{r}{2^n}} = \{z : |\varphi(z)| > 1\} = \{z : G(z) > ٠\} = A_\lambda(\infty)$.

$(\Leftarrow)$ Recall that $A_\lambda(\infty)$ is connected . Now φ is homeomorphism which maps $A_\lambda(\infty)$ conformally to the exterior of the unit disk . Since simple connectivity is preserved by homeomorphism and exterior of the unit disk on the Riemann sphere is simply connected , $A_\lambda(\infty)$ must be simply connected . Thus it follows that $\partial A_\lambda(\infty) = J(Q_\lambda)$ is connected .

$(\Rightarrow)$ Assume that there exist $z_0 \in \partial A_\lambda(\infty)$, z_0 is a finite critical of $Q_\lambda(z)$. Let $G(z_0) = r_0$ and consider Λ_{r_0} . Differentiating (٣.٢.٢) at z_0 yields

$\left(\frac{\partial}{\partial z}G(Q_\lambda(z_0))\right).Q'_\lambda(z_0) = 2\frac{\partial}{\partial z}G(z_0)$, since z_0 is a critical point of Q_λ, $Q'_\lambda(z_0) = 0$, thus

$\left(\frac{\partial}{\partial z}G(Q_\lambda(z_0))\right).Q'_\lambda(z_0) = 2\frac{\partial}{\partial z}G(z_0) = 0 = 2\frac{\partial}{\partial z}G(z_0)$, thus $\frac{\partial}{\partial z}G(z_0) = 0$.

So z_0 is a critical point of $G(z)$. Thus the level curve Λ_{r_0} consists of at least two simple closed curves that meet at the critical point z_0 . Within each of these simple curves there exist points in the Julia set. If not , $G(z)$ is harmonic and positive on a non-empty region V within one of the simple curves and the maximum principle applied to $G(z)$ and -$G(z_0)$ gives $G(z) \le r_0$ and -$G(z_0) \le -r_0$ for all $z \in V$. So $G(z) \equiv r_0$ on V . Let f be the analytic map with real part equal to $G(z)$. Then by the uniqueness theorem , $G(z) \equiv r_0$ on $A_\lambda(\infty)$. This contradicts that $A_\lambda(\infty) \to \infty$. Thus $J(Q_\lambda)$ is disconnected . Therefore there is no finite critical point of Q_λ in $A_\lambda(\infty)$. ■

Proposition (٣.٢.٥)

If Q_λ has a critical point in $A_\lambda(\infty)$, then $J(Q_\lambda)$ has uncountably many components.

Proof :

Let z_0 be a critical point for Q_λ.

Let w be an element of the backward orbit of z_0 , that is $Q_\lambda^n(w) = z_0$ for some n . Then by theorem (٣.٢.٤),that G defined on this theorem , $G(Q_\lambda^n(w)) = 2^n G(w)$, or $G(w) = 2^{-n} G(Q_\lambda^n(w))$. Thus $G(w) = 2^n G(z_0)$. Differentiate both sides to get $\frac{\partial}{\partial z}G(w) = 2^n \frac{\partial}{\partial z}G(z_0) = 0$, so that w is a

critical point of $G(z)$ .Thus any level curve consists of at least two simple closed curves that meet at the critical point w . Since the choice of w was arbitrary , the level curves split in each of the w ,so follow the splitting by assigning ٠ to the left branch and ١ to the right branch .
Since there are uncountably many sequences of ٠`s and ١`s there are uncountably many components of $J(Q_\lambda)$. ■

Definition (٣.٢.٦) [١٠] (pp.٣٧)

A set is totally disconnected If it contains no intervals .

Theorem (٣.٢.٧)

Let $Q_\lambda(z) = \lambda z - \lambda z^2$. If $Q_\lambda^n(0.5) \to \infty$, then $J(Q_\lambda)$ is totally disconnected .

Proof :

Since $\infty \in F(Q_\lambda)$ and $F(Q_\lambda)$ is open , there exists a neighborhood D_∞ of ∞ such that $\overline{D}_\infty \subset F(Q_\lambda)$. And since ∞ is an attracting fixed point of Q_λ , $Q_\lambda(\overline{D}_\infty) \subset D_\infty$.

Let $D = C_\infty \setminus \overline{D}_\infty$. Then D is an open set and $J(Q_\lambda) \subset D$.

Now, since $Q_\lambda^n(0.5) \to \infty$ by assumption , choose k large enough so that Q_λ^k maps ٠.٥ to D_∞ . Thus for $n \geq k$, there is no critical value of Q_λ^n in $\overline{D}$, and all the branches of the inverse map Q_λ^{-n} are defined and map $\overline{D}$ in D .

(Else there exists $z \in \overline{D}$ such that $w = Q_\lambda^{-n}(z) \in (C_\infty \setminus D) = D_\infty$. Now $Q_\lambda^n(w) = z \in \overline{D}$, but $Q_\lambda^n : \overline{D}_\infty \to D_\infty$ implies that $z \in D_\infty$. This contradicts the choice of $z \in \overline{D}$) .

Let $z_0 \in J(Q_\lambda)$, then $Q_\lambda^n(z_0) \in J(Q_\lambda)$ since the Julia set is completely invariant under Q_λ from proposition (٢.١.٥) . Define f_n to be the branch of the inverse map Q_λ^{-n} which maps $Q_\lambda^n(z_0)$ to z_0 . That is , $f_n\left(Q_\lambda^n(z_0)\right) = z_0$. Since f_n maps $\overline{D}$ into D , $\{f_n\}$ are uniformly bounded on $\overline{D}$.

Note that by modifying the integer k above , $\{f_n\}$ is uniformly bounded on a neighborhood of $\overline{D}$. Thus $\{f_n\}$ is normal on $\overline{D}$.

Now for all $z \in D \bigcap A_\lambda(\infty)$, $f_n(z)$ accumulates on $J(Q_\lambda)$ since $f_n(z) \to \partial A_\lambda(\infty)$ (except for $z = \infty$) . Then let f be the limit of some subsequence $\{f^{n_m}\}$ of $\{f_n\}$.

Now f maps $D \bigcap A_\lambda(\infty)$ into $J(Q_\lambda)$ since $f^n(z) \to w \in J(Q_\lambda)$ and $f^{n_m} \to f(z)$ implies $f(z) = w \in J(Q_\lambda)$ by the uniqueness of limits .

Then by open map theorem (١.٢.٢٢) f is constant since $J(Q_\lambda) = \partial A_\lambda(\infty)$ from theorem (٢.١.١٣) and the boundary of an open set has empty interior . (If z in the interior of the boundary of an open set U then there exist a neighborhood of z contained entirely in ∂U .

But for any $\varepsilon >$٠ , there exists $z_1 \in D(z,\varepsilon) \bigcap U$, by the definition of the boundary set . This contradicts that U is open) .

Now , diam $\{f_n(D)\} \to 0$. (Suppose not . Then there exists $\varepsilon >$٠ and $\{f_{n_m}\}$ such that diam $\{f_{n_m}(D)\} \geq \varepsilon$.

$\{f_{n_m}\}$ is normal so there exists $\{f_{n_{m_j}}\}$, a subsequence , and f a limit map , such that $f_{n_{m_j}} \to f$ uniformly . By the argument in the previous paragraph , $f \equiv w_0$, a constant . Thus for a fixed branch , $f_{n_{m_j}}$, with $j \geq j_0$,

$\left| f_{n_{m_j}}(z) - w_0 \right| < \dfrac{\varepsilon}{3}$ for all $z \in \overline{D}$, Then diam $\{f_{n_{m_j}}(D)\} < \frac{2\varepsilon}{3}$, contradiction) . Then since f_n is continuous , $f_n(\overline{D}) \subseteq \overline{f_n(D)}$, and $f_n(\overline{D})$ has diameter tending to zero .

Next , by invariance of Fatou set , $\partial D \subset F(Q_\lambda)$ implies that $f_n(\partial D) \subset F(Q_\lambda)$ and it is disjoint from $J(Q_\lambda)$. Now recall that for $z \in J(Q_\lambda)$, f_n was chosen so that $f_n(Q_\lambda^n(z_0)) = z_0$ and $f_n(Q_\lambda^n(z_0)) \subset f_n(D)$ since $Q_\lambda^n(z_0) \in J(Q_\lambda) \subset D$. Also , $f_n(D) \subset f_n(\overline{D})$, so $z_0 \in f_n(\overline{D})$ for all n . Now diam $\{f_n(\overline{D})\} \to 0$ implies that $\{z_0\}$ must be a connected component of .
To see this recall that $\overline{D}$ consists of elements of $J(Q_\lambda)$ and the boundary which is in $F(Q_\lambda)$.
For any $\varepsilon > ٠$,choose k such that diam $\{f_k(\overline{D})\} < \varepsilon$. Within this disc $f_k(\overline{D})$, the boundary is mapped to a curve that winds around elements of the Julia set in the interior . Thus only points in the Julia set are within 2ε of each other will be elements of a connected component of $J(Q_\lambda)$. But since ε can be chosen arbitrarily small , eventually all points of the Julia set will be separated by the Fatou set , $\partial f_k(D)$ for large enough k .
By definition (٣.٢.٦) , $J(Q_\lambda)$ is totally disconnected . ■

Corollary (٣.٢.٨)

Let z be the critical point of $Q_\lambda(z) = \lambda z - \lambda z^2$. If $Q_\lambda^n(z) \to \infty$, then $J(Q_\lambda)$ is totally disconnected . Otherwise, $\{Q_\lambda^n(z)\}$ is bounded , and $J(Q_\lambda)$ is connected .

Proof :

Since $z = 0.5$ is the critical point of $Q_\lambda(z)$.

If $Q_\lambda^n(0.5)$ is bounded then $0.5 \in A_\lambda(\infty)$ and $J(Q_\lambda)$ is connected from theorem (٣.٢.٤) .

Next , if $Q_\lambda^n(0.5) \to \infty$ then by theorem (٣.٢.٧) $J(Q_\lambda)$ is totally disconnected .

■

Appendix : Computer algorithm

The following is a computer algorithm .

The algorithm first introduce , the real and imaginary parts of the parameter c.The lower left coordinates gives the algorithm the lower coordinate of the box in which to calculate the Julia set (thus also giving it the box in which draw the set) .

The side length tells the algorithm how tall and wide to make the box. Thus , if lower left coordinates are given as -١.٥ , -١.٥ and side length is given as ٣ , the Julia set will be calculated for points with real parts from -١.٥ to ١.٥ and imaginary parts from -١.٥ to ١.٥ .

The algorithm does its real work inside the for – loop . The algorithm iterates each x , y combination to see weather it escapes infinity or not .

If it does escape , it moves onto the next point . If it does not , its draws a dot at that point's coordinates on the screen .

The algorithm can not actually check weather a point really goes to infinity or not .

What it actually does is to see weather a point moves outside of a disc of radius ٤ within some number of iterations (specified by the user in the beginning) . If the user wants to change the size of this disc , say to ٦ , he simply has to change the line .

If r > ٤ then exit do

To

If r > ٦ then exit do

The disk of size ٤ does , however , give accurate results .

Similarly , we can introduce the algorithm with respect to parameter λ .

Now , we give the algorithm for c and we give the algorithm for λ .

Algorithm :Plot Julia set ١;

Input :Real part ,Imaginary part ,Low X-coor,Low Y-coor ,Side length, Maxlter,Disk Radius ;

Output :Graphical Repesentation of Julia set;

```
Begin
    High X –coor := Low X-coor + Side length;
    High Y –coor := Low Y-coor + Side length;
    Step size := Side length /٤٠٠
    Set Window (Low X-coor, Low Y-coor)-(High X –coor, High Y –coor);
    For x0 := Low X-coor To High X –coor  Step Step size Do
    For y0 := Low Y-coor To High Y –coor  Step Step size Do
    Begin
        x := x0; y := y0; index := 0;
        While  (Index < Maxlter) Do
        Begin
            Temp:= x*x - y*y + Real part;
            y := 2*x*y + Imaginary part;
            x := temp;
            Radius:= x*x + y*y;
            If Radius > Disk Radius Then
```

Exit while Loop;

Index := index +١;

End;

If index = Maxlter Then

Plot (x_0, y_0);

End;

End;

Algorithm :Plot Julia set ٢;

Input :Low X-coor,Low Y-coor ,Side length, Maxlter, Disk Radius ;

Output :Graphical Repesentation of Julia set;

Begin

L١=١-SQR(٥);

L٢=٠;

High X –coor := Low X-coor + Side length;

High Y –coor := Low Y-coor + Side length;

Step size := Side length /٤٠٠

Set Window (Low X-coor, Low Y-coor)-(High X –coor, High Y –coor);

For x_0 := Low X-coor To High X –coor Step Step size Do

For y_0 := Low Y-coor To High Y –coor Step Step size Do

Begin

$x := x_0; y := y_0; index := 0;$

While (Index < Maxlter) Do

Begin

Temp:= $L1*(x*(1-x)+y*y)+L2*(2*x*y-y)$;

$y := L1*(y-(2*x*y))+L2*(x-x*x+y*y)$;

```
                    x := temp;
                    Radius:= SQR ((x * x) + (y * y)) ;
                    If Radius > Disk Radius Then
                       Exit while Loop;
                    Index := index +١;
            End;
            If  index =  Maxlter Then
                  Plot  ( x0, y0 );
      End;
End;
```

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Some of Julia sets

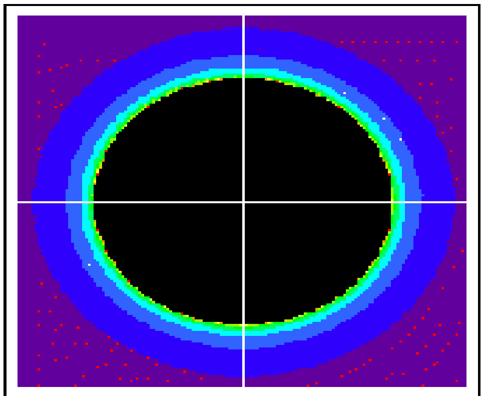

Fig.١ : The Julia set for the map $q_c(z) = z^2 + c$, over the interval [-١.٥ , ١.٥]on the real axis and [-١.٥ , ١.٥] on the imaginary axis for c = ٠ , computed by ١٠٠ iterations .

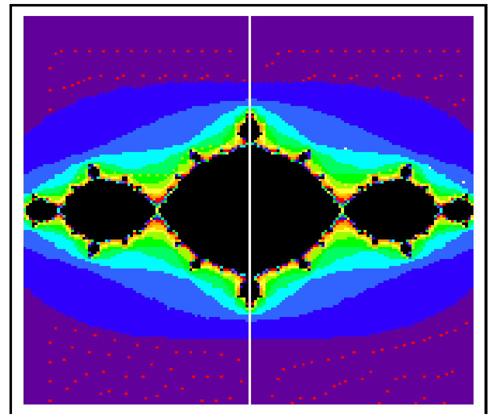

Fig.٢ : The Julia set for the map $q_c(z) = z^2 + c$, over the interval [-١.٥ , ١.٥]on the real axis and [-١.٥ , ١.٥] on the imaginary axis for c = -١ , computed by ١٠٠ iterations .

Fig.٣ : The Julia set for the map $q_c(z) = z^2 + c$, over the interval [-١.٥ , ١.٥]on the real axis and [-١.٥ , ١.٥] on the imaginary axis for c = ٠.٣٢+٠.٠٤٣i , computed by ١٠٠ iterations .

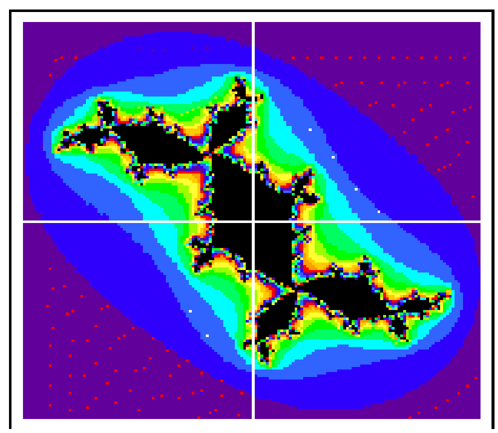

Fig.٤ : The Julia set for the map $q_c(z) = z^2 + c$, over the interval [-١.٥ , ١.٥]on the real axis and [-١.٥ , ١.٥] on the imaginary axis for c = -٠.١+٠.٨i , computed by ١٠٠ iterations .

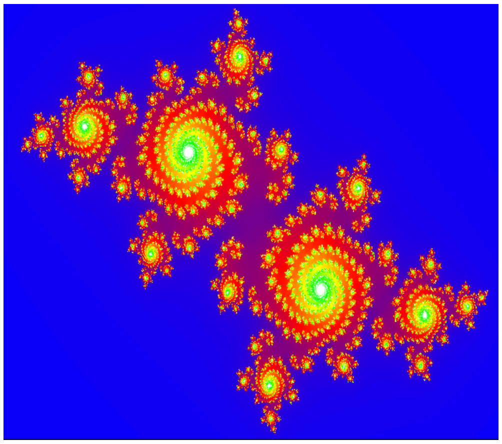

Fig.٥ : The Julia set for the map $q_c(z) = z^2 + c$, over the interval [-١.٥ , ١.٥]on the real axis and [-١.٥ , ١.٥] on the imaginary axis for c = ٠.٦٦i , computed by ١٠٠ iterations .

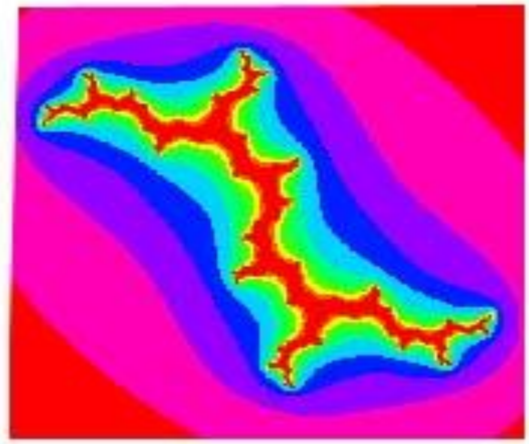

Fig.٦ : The Julia set for the map $q_c(z) = z^2 + c$, over the interval [-١.٥ , ١.٥]on the real axis and [-١.٥ , ١.٥] on the imaginary axis for c = -I , computed by ١٠٠ iterations .

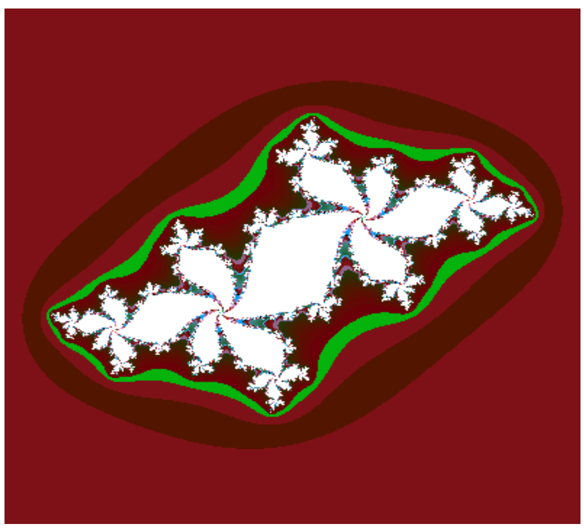

Fig.٧ : The Julia set for the map $q_c(z) = z^2 + c$, over the interval [-١.٥ , ١.٥]on the real axis and [-١.٥ , ١.٥] on the imaginary axis for c = ٠.٢٥-٠.٥٢i , computed by ١٠٠ iterations .