

*ON
SEMI-OPEN AND REGULAR-OPEN SETS
TO
NEARLY pT_i -CONTINUOUS MAPPINGS*

A thesis

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<i>Symbol</i>	<i>Meaning</i>
S	Semi
R	Regular
P	Pairwise
nbd	Neighborhood
S.t.	Neighborhood system
\in	Belong to
\notin	Not belong to
\exists	There exists
\forall	For all
$\text{cl}(A)$	Closure of subset A
$\text{int}(A)$	Interior of subset A
$\text{cl}_{\tau_i}(A)$	Closure of subset A with respect to a topological τ_i $i=1$ or 2
$\text{int}_{\tau_i}(A)$	Interior of subset A with respect to a topological τ_i , $i = 1$ or 2
$\text{der}_{\tau_i}(A)$	Derive set of subset A with respect to τ_i
S^*	A definition using semi-open and open sets
$S.O.(X)$	The set of all semi-open sets in X
$S.O_{\tau_i}(X)$	The set of all semi-open sets in X with respect to τ_i
$R.O.(X)$	The set all regular-open set in X
$R.O_{\tau_i}(X)$	The set of all regular-open sets in X with respect to τ_i
ω	Weakly
N	Nearly

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Abstract

The aim of this work is to generalize of semi and regular pairwise continuous mappings and similar generalizations of semi and regular pairwise T_i -spaces and generalizations of semi and regular pairwise $T(i,k)$ -spaces introduce and their relation ship with the concepts of T_i -continuity is studied for $i = 1,2,3$, the main results are:

1- The composite of any two semi-continuous mappings is not semi continuous but this can be hold if either of them is irresolute mapping since the irresolute mapping take the inverse image of a semi-open sets in Y to a semi-open set in X , and this will give a definition of a semi continuous .

2- The composite of any two p -continuous mappings is p -continuous.

3- A mapping $f:(X , \tau_1 , \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ is $SNPT_i$ (resp. $RNPT_i$) continuous if it is $SNPT_i$ (resp. $RNPT_i$) continuous at each point of X where $i = 1,2,3$.

4- Let $f:(X , \tau_1 , \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ be $SNPT_i$ -continuous mapping and (Y, ρ_1, ρ_2) be $SNPT_i$ -space then f is s - p -continuous where $I = 1,2,3$.

5- Let $f:(X , \tau_1 , \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ be $SNPT_1$ (resp. $RNPT_1$) continuous mappings and (Y, ρ_1, ρ_2) be SPR_0 (resp. RPR_0) space then f is semi(resp. regular) P -continuous.

6- Let $f:(X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ be $SNPT_2$ -continuous mapping and (Y, ρ_1, ρ_2) be SPT_i -space then f is s - p -continuous where $i = 1, 2, 3$.

7- Let $f:(X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ be $SNPT_i$ -continuous mappings and (Y, ρ_1, ρ_2) be $SPT(i, k)$ -space then f is SPT_k -continuous where $i, k \in \{1, 2, 3\}$.

8- Let $f:(X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ be $SNPT_3$ -continuous mapping and (X, τ_1, τ_2) be s - p -connected space then (Y, ρ_1, ρ_2) need not be s - p -connected.

Introduction

This research established a relation among bitopological spaces. Bitopological space, initiated by Kelly [8] defined as: A set equipped with two topologies is called a bitopological space, and it is denoted by (X, τ_1, τ_2) where (X, τ_1) and (X, τ_2) are two topological spaces defined on X .

In 1963, N. Levine [11] has first introduced the concept of semi open sets and semi continuous functions, Quite recently, by using semi-open set S.N. Maheshwari and R. Prasad [12] have defined and characterized a new separation axioms, called semi- T_0 , semi- T_1 , semi- T_2 and semi- T_3 .

Let (X, τ) be a topological space and A be any subset of X , let A° , \bar{A} and A^c denoted the interior, closure and complement of A respectively. A is called a semi-open (S. open) in X if, and only if, there exists an open set O , such that $O \subset A \subset \text{cl}(O)$ [11]. A subset N_x of X is called a semi-neighborhood (S. nbd.) of a point $x \in X$ if, and only if there exists a S. open set B in X such that $x \in B \subset N_x$ [2]. A is S. open if, and only if, A is S. nbd of each $x \in A$ [2]. A point $x \in A$ is called a semi-interior (S. interior) point of A if, and only if, there exists a S.nbd. N_x such that $N_x \subset A$ [2]. the set of all S- interior points of A is called semi-interior of A (S. int(A)). The S. int(A) is largest S. open set which is contained in A [2]. A is S. open if, and only if, $A = \text{S. int.}(A)$ [2]. A is called semi-closed (S. closed) if, and only if, A^c is S. open [2]. The smallest S-closed set containing A is called the semi-closure of A (S. cl(A)) [2]. A is semi-closed if, and only if, $A = \text{S. cl}(A)$ [2]. A sub set A of X is called regular open (R. open) if $A = \text{int}(\text{cl}(A))$ and A is called regular closed (R. closed) if its complement is (R. open).

The set of all S. open sets in X is denoted by $S. O.(X)$ and the set of all S. closed sets in X it is denoted by $S. C.(X)$, the set of all r. open sets is denoted by $R. O.(X)$ and the set of all r. closed set it is denoted by $R. C.(X)$.

A bitopological (X, τ_1, τ_2) is called PST_0 -space if $\forall x \neq y \in X$, there is a τ_1 - semi open set or τ_2 -semi open set containing one of them but not the other, and it is called PRT_0 -space if $\forall x \neq y \in X$, there is a τ_1 -regular open set or τ_2 -regular open set containing one of them but not the other.

A bitopological space (X, τ_1, τ_2) is called PST_1 if, $\forall x \neq y \in X$ there are τ_1 - semi open set G_1 and τ_2 - semi open set G_2 such that $x \in G_1, y \notin G_1$ and $x \notin G_2, y \in G_2$, and it is called PRT_1 if, $\forall x \neq y \in X$ there are τ_1 -regular open set G_1 and τ_2 - regular open set G_2 such that $x \in G_1, y \notin G_1$ and $x \notin G_2, y \in G_2$.

A bitopological space (X, τ_1, τ_2) is called PST_2 -space if $\forall x \neq y \in X$, there is τ_1 - semi open set G_1 and τ_2 - semi open set G_2 such that $x \in G_1, y \in G_2$ and $G_1 \cap G_2 = \emptyset$,

A bitopological space (X, τ_1, τ_2) is called PRT_2 -space if $\forall x \neq y \in X$, there is τ_1 - regular open set G_1 and τ_2 - regular open set G_2 such that $x \in G_1, y \in G_2$ and $G_1 \cap G_2 = \emptyset$.

A bitopological space (X, τ_1, τ_2) is called PS^*T_2 -space if $\forall x \neq y \in X$, there is τ_1 - open set G_1 and τ_2 - semi open set G_2 such that $x \in G_1, y \in G_2$ and $G_1 \cap G_2 = \emptyset$.

A bitopological space (X, τ_1, τ_2) is called PS regular if $\forall \tau_1$ - closed set F_1 and $\forall x \notin F_1$ there are τ_1 - semi open set G_1 and τ_2 - semi open set G_2 such that $x \in G_1, F_1 \subset G_2$ and $G_1 \cap G_2 = \emptyset$.

A bitopological space (X, τ_1, τ_2) is called PR regular if $\forall \tau_1$ - closed set F_1 and $\forall x \notin F_1$ there are τ_1 - regular open set G_1 and τ_2 - regular open set G_2 such that $x \in G_1, F_1 \subset G_2$ and $G_1 \cap G_2 = \emptyset$.

A bitopological space (X, τ_1, τ_2) is PS^* - regular space if for each if τ_1 -closed set F_1 and $x \notin F_1$ there are τ_1 - regular open set G_1 and τ_2 - semi open set G_2 such that $x \in G_1, F_1 \subset G_2$ and $G_1 \cap G_2 = \emptyset$ and $\forall \tau_2$ - closed set F_2 and $x \in F_2$ there are τ_1 -regular open set U_1 and τ_2 -semi open set U_2 such that $x \in U_2$ and $F_2 \subset U_1$ and $U_1 \cap U_2 = \emptyset$.

A bitopological space (X, τ_1, τ_2) is called PST_3 - space if it is PS-regular and PST_1 - space.

A bitopological space (X, τ_1, τ_2) is called PRT_3 - space if it is PR-regular and PRT_1 - space.

A bitopological space (X, τ_1, τ_2) is PS^*T_3 - space if it is PS^* - regular space and it is PS^*T_1 - space .

Let (X, τ_1, τ_2) and (Y, ρ_1, ρ_2) be a bitopological spaces , a function $f: X \rightarrow Y$ is said to be S -continuous at $x_0 \in X$ iff, for every S -open set V in Y containing $f(x_0)$, there exists S -open set U in X containing x_0 such that $f(U) \subset V$, and it is called S -continuous on X iff , it is S -continuous at each point $x \in X$. or it is S -continuous iff, the inverse image of any open set in Y is S -open set in X . And f is called R - continuous iff, the inverse image of any open set in Y is R -open in X .

Let (X, τ_1, τ_2) and (Y, ρ_1, ρ_2) be a bitopological spaces , a function $f: X \rightarrow Y$ is said to be nearly continuous of a point $x \in X$ if for every neighborhood U of $f(x)$ in Y , $f^{-1}(U)$ is a neighborhood of x in X , and it is called semi-nearly continuous of $x \in X$ if for every neighborhood U of $f(x)$ in Y , $f^{-1}(U)$ is S -neighborhood of x in X and it is called regular-nearly continuous of $x \in X$ if for every neighborhood U of $f(x)$ in Y , $f^{-1}(U)$ is R -neighborhood of x in X .

A new definition that is called nearly PT_i -continuous mappings which is state that $f: (X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ is said to be nearly PT_i -continuous mappings of a point $x \in X$ if for every PT_i -open cover \mathcal{v} of Y there exists a τ_k -open neighborhood $V \subset X$ of x such that $f(V) \subset \text{st}(f(x), \mathcal{v})$. and we introduced a new definition that is semi nearly PT_i -continuous mappings and a new definition that is regular nearly PT_i -continuous mappings and a new theorem on the . Where $i \in \{1,2,3\}$.

In chapter one , we will give elementary properties and characteristics of semi open sets , semi neighborhood , regular open set semi and regular- separation axioms with some new results and examples .

In chapter two , we will study a pairwise $T(i,k)$ -space $i,k \in \{1,2,3\}$ and we introduced a new definition on these spaces by using semi and regular concepts and study the relation between them.

In chapter three, we will use concepts of chapter one and chapter two to introduce new concept that is nearly pairwise T_i -continuous mappings and we introduce new definitions of this subject by using semi and regular concepts and also studied the relation between them.

There are many papers in a bitopological spaces and we display some of these papers.

In 1975, S. N. Maheshwari and R. Prasad have been published there paper under "Some new separation axioms" in this paper introduced some new separation axioms and studied some of their basic properties. The implications of these new separation axioms among them selves and with the well known axioms T_0 , T_1 and T_2 are obtained.

In 1986, J. Ewert, Slupsk, Poland, with paper "Weak forms of continuity, Quasi-continuity and cliquishness of maps with respect to two topologies" and studied a maps of topological spaces in to a bitopological spaces some weaker forms of continuity-quasi-continuity and cliquishness are defined, various properties of these classes of maps are given.

In 1987, M. N. Mukherjee and S. Ganguly introduce paper "Generalization of almost continuous multi functions to bitopological spaces". He studied in this paper almost continuous multi functions have been introduce and characterized in a bitopological spaces, this also generalized the idea of almost continuity as studied in (Bose and Sinha 1982) finally such multi functions has been investigated in relation to the extended concept of lower and upper semi-continuous multi functions introduce for bitopological spaces.

Luay Abd Al-Hani Al-Swidi and Ali Younis Shaker have paper under "Semi compactness in bitopological spaces" in this paper they are studied a semi-compactness in a bitopological spaces.

In 2001, Yiezi Kadhan Mahdey have a paper under "Separation axioms in bitopological spaces" the main purpose of this paper is to study the separation axioms in bitopological spaces in more veracity specially the translated these axioms by using continuous functions and some conditions in such spaces.

In 2002, Abdul Kafi Abdul Fattah has thesis under "On bitopological spaces" in this thesis studied a separation axioms in a bitopological spaces and the relation ship between these axioms also he studied semi-compactness and semi-connectedness in a bitopological spaces.

In 2003, Iktifa Diaa Jallel , have paper under " δ - open set in bitopological spaces " this paper established a relation between bitopological spaces on the one hand and topological space of type α -open set , on the other , a new definition δ open set in bitopology is formulated based up on δ -open set in bitopology . A new theorem for the relation between topological space and δ open set is also offered. The research gives the specification for the new definition of δ -open set in bitopology.

Chapter One

Some Properties of Semi Open and Regular Open Sets in Topological Spaces

§ (1-1) Semi-open sets

In this section we introduce the notion semi-open sets, let (X, τ) be a topological space and A be any subset of X , let $\text{int}(A)$, $\text{cl}(A)$ and A^c denoted the interior, closure and complement of A respectively. A is called a semi-open (S. open) in X if, and only if, there exists an open set O , such that $O \subset A \subset \text{cl}(O)$ [11]. A subset N_x of X is called a semi-neighborhood (S. nbd.) of a point $x \in X$ if, and only if there exists a S. open set B in X such that $x \in B \subset N_x$. A is S. open if, and only if, A is S. nbd of each $x \in A$. A point $x \in A$ is called a semi-interior (S. interior) point of A if, and only if, there exists a S.nbd. N_x such that $N_x \subset A$. the set of all S- interior points of A is called semi-interior of A ($\text{S. int}(A)$). The $\text{S. int.}(A)$ is largest S. open set, which is contained in A . A is S. open if, and only if, $A = \text{S. int.}(A)$. A is called semi-closed (S. closed) if, and only if, A^c is S. open. The smallest S-closed set containing A is called the semi-closure of A ($\text{S. cl}(A)$). A is semi-closed if, and only if, $A = \text{S. cl}(A)$ [2].

The following proposition will give an equivalent definition of a semi-open sets

Proposition 1.1.1 [2]

A subset A of a topological space (X, τ) is S. open if, and only if, $A \subset \text{cl}(\text{int}(A))$.

Proof: see [11]

Remark 1.1.2 [11]

Let A be a subset of a topological space (X, τ) then

$$(i) \tau \subset \text{S.O}(X) \quad [11]$$

$$(ii) \tau^c \subset \text{S.C}(X)$$

$$(iii) \text{int}(A) \subset \text{S. int}(A) \quad [2]$$

$$(iv) \text{S.cl}(A) \subset \text{cl}(A) \quad [2]$$

The converse of each inclusion of the preceding remark need not be true, as we will see in the following examples.

Example 1.1.3

Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, X, \{c\}, \{a\}, \{a, c\}, \{a, c, d\}\}$ then $\tau^c = \{\emptyset, X, \{a, b, d\}, \{b, c, d\}, \{b, d\}, \{b\}\}$ and hence

$$S.O(X) = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}, \{a,c,d\}, \{a,b\}, \{a,d\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}\}.$$

So

$$S.C(X) = \{\emptyset, X, \{b,c,d\}, \{a,b,d\}, \{b,d\}, \{b\}, \{c,d\}, \{a,d\}, \{b,c\}, \{a,b\}, \{d\}, \{c\}, \{a\}\}$$

Now $\{a,d\} \in S.O(X)$ but $\{a,d\} \notin \tau$

$\{d\} \in S.C(X)$ but $\{d\} \notin \tau^c$

$$S.int(\{b,c\}) = \{b,c\} \not\subset \{c\} = int(\{b,c\})$$

$$cl(\{a\}) = \{a,b,d\} \not\subset \{a\} = S.cl(\{a\})$$

Definition 1.1.4 [17]

A sub set A of a topological space (X, τ) is said to be regular open if, and only if, $A = int(cl(A))$, and it is denoted by $R.O$. In addition, $R.O(X)$ denotes the set of all regular-open sets.

Definition 1.1.5 [17]

A sub set A of a topological space (X, τ) is said to be regular-closed if, and only if, A^c is regular-open set, and the set of all regular-closed sets is denoted by $R.C(X)$.

Remark 1.1.6

Let A be a sub set of a topological space (X, τ) then

(i) $R.O(X) \subset \tau$.

(ii) $R.C(X) \subset \tau^c$.

Remark 1.1.7 [11]

In any topological space we have

(i) Any union of S . open sets is S . open

(ii) Any intersection of S . closed sets is S . closed

Also, the intersection of two S . open sets need not be S . open

Remark 1.1.8

Let A be any open sets in a topological space X then for any subset B of X

(i) $A \cap cl(B) \subset cl(A \cap B)$ [2]

(ii) $A \cap S.cl(B) \subset S.cl(A \cap B)$ [11]

(iii) $A \cap D \in S.O(X)$ for each $D \in S.O(X)$ [17]

The following result above in [11] without proof.

§ (1-2) Semi-Neighborhood system

In this section, we define the S. nbd. System and fundamental system of S. nbds.

Definition 1.2.1 [2]

Let (X, τ) be a topological space and $\{U_x\}$ be the set of all S.nbds. of an arbitrary point $x \in X$, then $\{U_x\}$ is called a S.nbd. system of x in X

Proposition 1.2.2 [2]

In a topological space (X, τ) the S.nbd. system $\{U_x\}$ of a point $x \in X$ has the following properties

- (i) Every subset of X , which contains a set, belongs to $\{U_x\}$ itself belongs to $\{U_x\}$.
- (ii) For each $V \in \{U_x\}$ there exists $W \in \{U_x\}$ such that if $y \in W$ then $V \in \{U_y\}$.

Proof (i) follow directly from the definition

Proof (ii) is true if we take W to be S.open set which contains x in V

Definition 1.2.3 [2]

In a topological space (X, τ) a fundamental system of S.nbds. of a point $x \in X$ is any set $\{U_x\}$ of S.nbds. of x such that for each S. nbd. V of X there exists a S.nbd $U \in \{U_x\}$ such that $U \subset V$.

-

§ (1-3) separation axioms

In this section, we recall the definitions of separation axioms with respect to semi open and regular open sets.

Definition 1.3.1 [1]

A topological space (X, τ) is called semi T_0 -space if $\forall x \neq y \in X$, there is semi-open set G in X such that $x \in G$ and $y \notin G$, and its denoted by ST_0 -space .

Definition 1.3.2 [1]

A topological space (X, τ) is called regular T_0 -space if $\forall x \neq y \in X$, there is regular-open set G in X such that $x \in G$ and $y \notin G$, and its denoted by RT_0 -space .

Definition 1.3.3 [1]

A topological space (X, τ) is called semi T_1 -space if $\forall x \neq y \in X$, there are two semi-open sets G, H such that $x \in G$ and $y \in H$ and $x \notin H$ and $y \notin G$, and its denoted by ST_1 -space .

Definition 1.3.4 [21]

A topological space (X, τ) is called regular T_1 -space if $\forall x \neq y \in X$, there are two regular-open sets G, H such that $x \in G$ and $y \in H$ and $x \notin H$ and $y \notin G$, and its denoted by RT_1 -space .

Theorem 1.3.5 [1]

- (i) A topological space (X, τ) is ST_1 -space if, and only if, $\forall x \in X, \{x\}$ is τ -semi closed .
- (ii) A topological space (X, τ) is RT_1 -space if, and only if, $\forall x \in X, \{x\}$ is τ -regular closed .

Corollary 1.3.6 [1]

- (i) A topological space X is ST_1 -space if, and only if, every finite subset is semi-closed.
- (ii) A topological space X is RT_1 -space if, and only if, every finite subset is regular-closed.

Definition 1.3.7 [18]

A topological space (X, τ) is called semi T_2 - space if $\forall x \neq y \in X$ there are two semi-open sets G, H such that $x \in G$ and $y \in H$ and $G \cap H = \emptyset$, and it is denoted by ST_2 -space .

Definition 1.3.8 [18]

A topological space (X, τ) is called regular T_2 - space if $\forall x \neq y \in X$ there are two regular-open sets G, H such that $x \in G$ and $y \in H$ and $G \cap H = \emptyset$, and it is denoted by RT_2 -space .

Definition 1.3.9 [1]

A topological space (X, τ) is called semi-regular space if \forall semi-closed set F in X and $x \notin F$, there are two semi-open sets G, H in X such that $x \in G$ and $F \subset H$ and $G \cap H = \emptyset$, and it is denoted by S-regular space .

Definition 1.3.10 [1]

A topological space (X, τ) is called regular-regular space if \forall regular-closed set F in X and $x \notin F$, there are two regular-open sets G, H in X such that $x \in G$ and $F \subset H$ and $G \cap H = \emptyset$, and it is denoted by R-regular space .

Theorem 1.3.11 [1]

(i) a topological space (X, τ) is S-regular if, and only if, $\forall x \in X$ and for each open set N containing x in X , \exists semi-open set M containing $x \in X$ such that $cl(M) \subset N$.

(ii) a topological space (X, τ) is R-regular if, and only if, $\forall x \in X$ and for each open set N containing x in X , \exists regular-open set M containing $x \in X$ such that $cl(M) \subset N$.

Definition 1.3.12 [1]

A topological space (X, τ) is called semi T_3 -space if it is S-regular and ST_1 -space, and it's denoted by ST_3 -space.

Definition 1.3.13 [1]

A topological space (X, τ) is called regular T_3 -space if it is R-regular and RT_1 -space, and its denoted by RT_3 -space.

Definition 1.3.14 [21]

A topological space (X, τ) is called semi-normal space if \forall two disjoint closed sets L, M in X there are two S-open sets G, H in X such that $L \subset G$ and $M \subset H$, and it is denoted by S-normal space .

Definition 1.3.15 [1]

A topological space (X, τ) is called regular-normal space if \forall two disjoint closed sets L, M in X there are two R-open sets G, H in X such that $L \subset G$ and $M \subset H$, and it is denoted by R-normal space .

Definition 1.3.16 [1]

A topological space (X, τ) is called semi T_4 -space if it is S-normal and ST_1 -space, and it is denoted by ST_4 -space.

Definition 1.3.17 [1]

A topological space (X, τ) is called regular T_4 -space if it is R-normal and RT_1 -space, and it is denoted by RT_4 -space.

Theorem 1.3.18 [1]

- (i) Every subspace of ST_i -space is ST_i -space, $i=0,1,2,3,4$.
 - (ii) Every subspace of RT_i -space is RT_i -space, $i=0,1,2,3,4$.
 - (iii) Every subspace of S-regular space is S-regular space.
 - (iv) Every subspace of R-regular space is R-regular space.
- A proof is easy.

Theorem 1.3.19

Let (X, τ) be a topological space, a subset A of X is semi-open, if and only if A is semi-neighborhood of each of its points .

§ (1-4) Separation Axioms in Bitopological Space:

In this section we will list new separation axioms in bitopological space and we will study the relation ship between them.

Definition 1.4.1 [23]

A bitopological space (X, τ_1, τ_2) is called pair wise T_0 -space if $\forall x \neq y \in X$ there is τ_1 -open set or τ_2 -open set G containing one of them but not the other , and it is denoted by PTo-space .

Example 1.4.2

Let R be the set of all real numbers and τ_1 be the usual topology on R and τ_2 be any topology on R then (R, τ_1, τ_2) is PTo-space.

Theorem 1.4.3

A bitopological space (X, τ_1, τ_2) is PTo – space if, and only if, $\forall x \neq y \in X$
 $cl_1\{x\} \neq cl_1\{y\}$
 $cl_2\{x\} \neq cl_2\{y\}$ where cl_1 and cl_2 be the closure of x and y with respect to τ_1 and τ_2 respectively.

Proof :

Suppose that (X, τ_1, τ_2) is PTo- space then $\forall x \neq y \in X \exists \tau_1$ -open set or τ_2 -open set G such hat $x \in G$ and $y \notin G$.

1) If G is τ_1 - open set let $x \in G \Rightarrow \{x\} \subset G$ and $y \notin G$.
 Then $y \in G^c \Rightarrow \{y\} \subset G^c$,But $\{x\} \subset cl_1\{x\}$,Also $\{y\} \subset cl_1\{y\}$

To proof $\{x\} \not\subset cl_1\{y\}$

Suppose $\{x\} \subset cl_1\{y\}$, Then $\{x\} \subset \{y\} \cup der_{\tau_1}\{y\}$

Since $x \notin \{y\}$, so $\{x\} \subset der_{\tau_1}\{y\}$

Hence $\{x\} \subset der_{\tau_1}\{y\}$.

Therefore $\forall \tau_1$ -open G_x containing x such that $G_x \cap \{y\} \neq \emptyset$

So $y \in G_x$ hence X is not PTo-space, which is contradiction with the hypothesis, then $\{x\} \not\subset cl_1\{y\}$

2) Similarly if G is τ_2 -open set, we must have $cl_2\{x\} \neq cl_2\{y\}$

Conversely

Let $x \neq y \in X$ such that
 $cl_1\{x\} \neq cl_1\{y\}$ to show that $x \notin cl_1\{y\}$

Let $x \in cl_1\{y\} \Rightarrow cl_1\{x\} \subseteq cl_1\{y\}$

This contradiction (!)

Then $x \notin cl_1\{y\}$

Let $G = X - cl_1\{y\}$ then G is τ_1 -open set and $y \notin G$ and $x \in G$

Hence (X, τ_1, τ_2) is PTo –space

Similarly if $cl_2\{x\} \neq cl_2\{y\}$.

Theorem 1.4.4

A Bitopological space (X, τ_1, τ_2) is PTo–space if (X, τ_1) or (X, τ_2) is To-space

Proof:

Let (X, τ_1) be a To – space then

$\forall x \neq y \in X$ there is τ_1 - open set G ,
such that $x \in G$ and $y \notin G$

Hence (X, τ_1, τ_2) is PTo - space

Similarly if (X, τ_2) is To-space.

Remark 1.4.5

The converse of theorem (1.4.4) need not be true .

Let $X = \{a, b, c\}$

$\tau_1 = \{\emptyset, X, \{a\}, \{b, c\}\}$

$\tau_2 = \{\emptyset, X, \{c\}, \{a, b\}\}$

Then (X, τ_1) and (X, τ_2) are not To- space but (X, τ_1, τ_2) is PTo- space.

Definition 1.4.6 [23]

A bitopological space (X, τ_1, τ_2) is called RPT₀-space if $\forall x \neq y \in X$, there is τ_1 -regular open set or τ_2 -regular open set containing one of them but not the other.

Definition 1.4.7

A bitopological space (X, τ_1, τ_2) is called pair wise S^*PT_0 -space if $\forall x \neq y \in X$, there is τ_1 open set or τ_2 -semi open set G containing one of them but not the other.

Definition 1.4.8 [22]

A bitopological space (x, τ_1, τ_2) is called pair wise ST_0 -space if $\forall x \neq y \in X$, there is a τ_1 - semi open set or τ_2 -semi open set containing one of them but not the other, and its denoted by SPT_0 -space .

Example 1.4.9

Let $X = \{a, b, c\}$

$\tau_1 = \{\emptyset, X\}$, $\tau_2 = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ then

(X, τ_1, τ_2) is SPT_0 -pace .

Example 1.4.10 [1]

Let $X = \{a, b, c\}$

$\tau_1 = \{\emptyset, X, \{b\}, \{b, c\}\}$, $\tau_2 = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$ then

(X, τ_1, τ_2) is RPT_0 -space .

Example 1.4.11

1-Every RPT_0 -space is PTo -space

2-Every PTo –space is SPT_0 -space

The proof of these two facts comes from the following these two facts.

1- Every regular open set is open set [11]

2- Every open set is semi open set [11]

The converse of Remark above need not be true for the following example

Example 1.4.12

Let $X = \{a, b, c\}$

$\tau_1 = \{\emptyset, X\}$, $\tau_2 = \{\emptyset, X, \{c\}\}$

Then (X, τ_1, τ_2) is PTo -space but not RPT_0 -space.

Proposition: 1.4.13

A bitopological space (X, τ_1, τ_2) is SPT_o-space iff $x \neq y \in X$

$$S.cl_1\{x\} \neq S.cl_1\{y\}$$

$$S.cl_2\{x\} \neq S.cl_2\{y\}$$

Proof:

The proof is the same the proof of proposition (1.4.13) if we replaced $R.cl_1$ and $R.cl_2$ by $S.cl_1$ and $S.cl_2$ respectively .

Proof:

Let (X, τ_1, τ_2) be RPT_o-space and let $x \neq y \in X$

Suppose U is τ_1 -regular open set containing x but not y , then

$$y \in R.cl_1\{y\} \subset X - U \quad \text{and so } x \notin R.cl_1\{y\}$$

$$\text{Hence } R.cl_1\{x\} \neq R.cl_1\{y\}.$$

Conversely:

Let $x \neq y \in X$, then either $R.cl_1\{x\} \neq R.cl_1\{y\}$ or

$R.cl_2\{x\} \neq R.cl_2\{y\}$ in these two cases let.

Let p be a point in X such that

$p \in R.cl_1\{y\}$ and $p \notin R.cl_1\{x\}$, we assert that $y \notin R.cl_1\{x\}$ if $y \in$

$R.cl_1\{x\}$ then $R.cl_1\{y\} \subset R.cl_1\{x\}$ but $p \in R.cl_1\{y\}$ and $p \notin R.cl_1\{x\}$

which is contradiction hence $y \notin R.cl_1\{x\}$, thus $U = X - R.cl_1\{x\}$ is a τ_1 -

regular open set containing y but not x , therefore (X, τ) is RPT_o-space.

The case $R.cl_2\{x\} \neq R.cl_2\{y\}$ can be dealt with similarly .

Proposition 1.4.14 [1]

A bitopological space (X, τ_1, τ_2) is RPT_o-space if, and only if

$$\forall x \neq y \in X$$

$$R.cl_1\{x\} \neq R.cl_1\{y\}$$

$$R.cl_2\{x\} \neq R.cl_2\{y\}$$

Theorem 1.4.15

1) A bitopological space (X, τ_1, τ_2) is SPT_o-space if (X, τ_1) or (X, τ_2) is ST_o-space

2) A bitopological space (X, τ_1, τ_2) is RPT_o-space if (X, τ_1) or (X, τ_2) is RT_o-space

Proof:

(1) If (X, τ_1) is RT_0 -space, then $\forall x \neq y \in X$, there is a τ_1 -regular open set containing x but not y .

Similarity if (X, τ_2) is RT_0 -space

And from these we get that (X, τ_1, τ_2) is RPT_0 -space (by def. of RPT_0 -space).

(2) The same proof of (1) if we replaced regular open set by semi open set.

Remark 1.4.16

The converse of part (1) of theorem (1.4.15), need to be true for example

Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\tau_2 = \{\emptyset, X, \{c\}, \{a, b\}\}$

Then clearly the space (X, τ_1, τ_2) is RPT_0 -space but neither (X, τ_1) nor (X, τ_2) is RT_0 -space.

Definition 1.4.17 [9]

A bitopological space (X, τ_1, τ_2) is called pair wise T_1 -space if $\forall x \neq y \in X$ there exist τ_1 -open set H_1 and τ_2 -open set H_2 such that $x \in H_1, y \notin H_1$ and $x \notin H_2, y \in H_2$.

Theorem 1.4.18

A bitopological space (X, τ_1, τ_2) is pair wise T_1 -space then each single point $x \in X, \{x\}$ is τ_1 -closed set or τ_2 -closed set

Proof: Suppose that (X, τ_1, τ_2) is pair wise T_1 -space .

To prove that $\{x\}$ is τ_1 -closed set

i.e To prove that $X - \{x\}$ is τ_1 -open set

Let $y \in X - \{x\}$ then $x \neq y$

Since (X, τ_1, τ_2) is pair wise T_1 -space .

There are τ_1 -open set G_1 and τ_2 -open set G_2 such that $y \in G_1, x \notin G_1$ and $y \notin G_2, x \in G_2$

Since $x \notin G_1$, then $G_1 \subset X - \{x\}$

Hence, every $y \in G_1 \subset X - \{x\}$ and G_1 is T_1 -open set.

Then $X - \{x\}$ is T_1 -open set Therefore $\{x\}$ is T_1 -closed set.

Similarly $\{x\}$ is T_2 - closed set

Theorem 1.4.19

If each single point in a bitopological space (X, τ_1, τ_2) is τ_1 -closed or τ_2 -closed then a bitopological space (X, τ_1, τ_2) is T_1 -with respect to τ_1 or τ_2 or pair wise T_1 -space.

Proof:

Let $x \neq y \in X$ by hypothesis $\{x\}$ is τ_1 -closed or τ_2 -closed set and $\{y\}$ is τ_1 -closed

Let $G_1 = X - \{x\}$ and $G_2 = X - \{y\}$ then $y \in G_1$ and $x \in G_1, x \in G_2$ and $y \notin G_2$ with probable case:

- 1- G_1 and G_2 is τ_1 -open set.
- 2- G_1 is τ_1 -open and G_2 is τ_2 -open set.
- 3- G_1 is τ_2 -open set and G_2 is τ_1 -open set.
- 4- G_1 and G_2 is τ_2 -open set .

Therefore a bitopological space is pair wise T_1 -space or T_1 -space with respect to τ_1 or τ_2 .

Theorem 1.4.20

Every PT_1 -space is PT_0 -space

Proof:

Let (X, τ_1, τ_2) be PT_1 - space then $\forall x \neq y \in X$, There are τ_1 -open set H_1 and τ_2 -open set H_2 such that $x \in H_1, y \notin H_1$ and $x \notin H_2$ and $y \in H_2$.
Now from $x \in H_1, y \notin H_1 \Rightarrow (X, \tau_1, \tau_2)$ is PT_0 -space.

Remark 1.4.21

The converse the theorem (1.4.20) need not true for example

Let $X = \{a, b, c\}$

$\tau_1 = \{\emptyset, X\}, \tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$

Then (X, τ_1, τ_2) is PT_0 -space but is not PT_1 -space

Theorem 1.4.22

A bitopological space (X, τ_1, τ_2) is PT_1 -space, if (X, τ_1) and (X, τ_2) are T_1 -space .

Proof:

Let $x \neq y \in X$ Since (X, τ_1) is T_1 -space there is two τ_1 -open set G_1, H_1 such that $x \in G_1, y \notin G_1$ and $x \notin H_1, y \in H_1$, and since (X, τ_2) is T_1 -

space , there is two τ_2 -open set G_2 , H_2 such that $x \in G_2, y \notin G_2$ and $x \notin H_2, y \in H_2$

Hence, H_1 is τ_1 -open set

G_2 is τ_2 -open set, such that

$x \notin H_1, y \in H_1$ and $x \in G_2, y \notin G_2$

Therefore, (X, τ_1, τ_2) is PT_1 -space

The converse of the theorem (1.4.22) need not true for the following example:

Example 1.4.23

Let $X = \{a, b, c\}$

$\tau_1 = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$.

$\tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.

Then (X, τ_1, τ_2) is PT_1 -space, but (X, τ_1) and (X, τ_2) are not T_1 -space .

Proof:

Suppose that (X, τ_1, τ_2) is pair wise T_1 - space

1) let $x \in X$.to prove $\{x\}$ is τ_1 -closed set

i.e to prove $X - \{x\}$ is τ_1 - open set

Let $y \in X - \{x\}$ then $y \neq x$

Since (X, τ_1, τ_2) is PT_1 -space

There are τ_1 -open set G_1 and τ_2 -open set G_2 such that

$y \in G_1, x \notin G_2$ then $G_1 \subset X - \{x\}$

Hence every $y \in G_1 \subset X - \{x\}$ and G_1 is τ_1 - open set

Then $X - \{x\}$ is τ_1 -open set $\Rightarrow \{x\}$ is τ_1 - closed set.

2) Similarly we can prove that $\{x\}$ is τ_1 - closed set.

Definition 1.4.24[1]

A bitopological space (X, τ_1, τ_2) is called pair wise regular T_1 -space if $\forall x \neq y \in X$ there τ_1 - regular open set G_1 and τ_2 -regular open set G_2 such that $x \in G_1, y \notin G_1$ and $x \notin G_2, y \in G_2$, and denoted by PRT_1 -space.

Definition 1.4.25[23]

A bitopological space (X, τ_1, τ_2) is called pair wise semi T_1 if, $\forall x \neq y \in X$ there are τ_1 - semi open set G_1 and τ_2 - semi open set G_2 such that $x \in G_1, y \notin G_1$ and $x \notin G_2, y \in G_2$ and denoted by PST_1 -space .

Definition 1.4.26[23]

A bitopological space (X, τ_1, τ_2) is called pair wise semi T_1 if, $\forall x \neq y \in X$ there are τ_1 - open set G_1 and τ_2 - semi open set G_2 such that $x \in G_1, y \notin G_1$ and $x \notin G_2, y \in G_2$ and denoted by PS^*T_1 -space .

Theorem 1.4.27

Let (X, τ_1, τ_2) be a bitopological space then

- 1) Every PRT_1 - space PT_1 -space.
- 2) Every PT_1 – space is PS^*T_1 -space.
- 3) Every PS^*T_1 -space is PST_1 -space.

The proof follows from the fact that:

Regular open set \Rightarrow open set \Rightarrow semi open set.

Example 1.4.28

The converse of above Theorem need not be true for example

- 1) Let $X = \{a, b, c\}$ and

$$\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$$

$$\tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, c\}\}$$

(X, τ_1, τ_2) is PT_1 – space but is not PRT_1 -space.

- 2) Let $X = \{a, b, c\}$

$$\tau_1 = \{\emptyset, X, \{a\}\}$$

$$\tau_2 = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$$

(X, τ_1, τ_2) is PST_1 -space but not PT_1 - space

Remark 1.4.29

Let (X, τ_1, τ_2) be a bitopological space then

- 1) Every PRT_1 -space is RT_0 - space
- 2) Every PST_1 -space is PST_0 - space.
- 3) Every PS^*T_1 -space is PS^*T_0 - space.

Proof :

(1) Let (X, τ_1, τ_2) be a PRT_1 -space then $\forall x \neq y \in X$ there are τ_1 - regular open set G_1 and τ_2 - regular open set G_2 such that $x \in G_1, y \notin G_1$ and $y \notin G_2, y \in G_1, \text{ from } x \in G_1, y \notin G_1$ Clearly (X, τ_1, τ_2) is PRT_0 -space.

(2) Let (X, τ_1, τ_2) is PST_1 -space then $\forall x \neq y \in X$ there are τ_1 - semi open set G_1 and τ_2 - semi open set G_2 such that $x \in G_1, y \notin G_1$ and $x \notin G_2, y \in G_2$ from $x \in G_1, y \notin G_1$.

Clearly (X, τ_1, τ_2) is PST_0 - space.

(3) Let (X, τ_1, τ_2) is PS^*T_1 - space , then $\forall x \neq y \in X$, there are τ_1 - open set G_1 and τ_2 - semi open set G_2 such that $x \in G_1, y \notin G_1$ and $x \notin G_2, y \in G_2$ from $x \in G_1, y \notin G_1$

Clearly (X, τ_1, τ_2) is PS^*T_0 - space.

Theorem 1.4.30

1) A bitopological space (X, τ_1, τ_2) is PRT_1 - space (PST_1 -space) if (X, τ_1) and (X, τ_2) are RT_1 -space (ST_1 -space)

2) A bitopological space (X, τ_1, τ_2) is PRT_1 -space (PST_1 -spce) then each single point $x \in X, \{x\}$ is τ_1 - regular closed set or τ_2 - regular closed set (τ_1 - semi closed set or τ_2 - semi closed set).

Proof:

The proof is Similar proof of theorem (1.4.19) and (1.4.22) except change τ_1 -open by τ_1 - semi open and τ_2 -open by τ_2 - semi open.

Definition 1.4.31 [9]

A bitopological space (X, τ_1, τ_2) is called PT_2 -space if $\forall x \neq y \in X$, there is τ_1 - open set G_1 and τ_2 - open set G_2 such that $x \in G_1, y \in G_2$ and $G_1 \cap G_2 = \emptyset$, and it is denoted by PT_2 - space .

Example 1.4.32

Let $X = \{a, b, c\}$

$\tau_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$, $\tau_2 = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$.

Then (X, τ_1, τ_2) is PT_2 -space

Theorem 1.4.33

Let (X, τ_1, τ_2) be a bitopological space then Every PT_2 -space is PT_1 -space.

Proof:

Let (X, τ_1, τ_2) is PT_2 - space if $\forall x \neq y \in X$ there are τ_1 -open set G and τ_2 -open set G_2 , $\exists x \in G_1, y \in G_2$ and $G_1 \cap G_2 = \emptyset$
Hence $x \notin G_2, y \in G_1 \Rightarrow (X, \tau_1, \tau_2)$ is PT_1 - space .

Example 1.4.34

The converse of above theorem need not true for example
Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$.
 $\tau_2 = \{\emptyset, X, \{b\}, \{b, c\}\}$.
 (X, τ_1, τ_2) is PT_1 -space but it is not PT_2 -space.

Theorem 1.4.35

If (X, τ_1, τ_2) is PT_2 -space then $(X, \tau_1), (X, \tau_2)$ are T_0 -space.

Proof:

Let $x \neq y \in X$, since (X, τ_1, τ_2) is PT_2 -space, there are τ_1 - open set G_1 and τ_2 - open set such that
 $x \in G_1, y \in G_2$ and $G_1 \cap G_2 = \emptyset$
Hence $y \notin G_1$ and $G_1 \cap G_2 = \emptyset$
Hence (X, τ_1) is T_0 -space.
And $x \notin G_2$ then (X, τ_2) is T_0 -space.

Remark 1.4.36

The converse of Theorem (1.4.35) need not true for example
Let $X = \{a, b, c\}$
 $\tau_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$.
 $\tau_2 = \{\emptyset, X, \{b\}, \{a, b\}\}$.
Then (X, τ_1) is T_0 -space and (X, τ_2) is T_0 -space but
 (X, τ_1, τ_2) is not PT_2 -space.

Definition 1.4.37 [1]

A bitopological space (X, τ_1, τ_2) is called PRT_2 -space if $\forall x \neq y \in X$, there is τ_1 - regular open set G_1 and τ_2 - regular open set G_2 such that $x \in G_1$, $y \in G_2$ and $G_1 \cap G_2 = \emptyset$, and it is denoted by PRT_2 - space .

Definition 1.4.38 [23]

A bitopological space (X, τ_1, τ_2) is called PST_2 -space if $\forall x \neq y \in X$, there is τ_1 - semi open set G_1 and τ_2 - semi open set G_2 such that $x \in G_1$, $y \in G_2$ and $G_1 \cap G_2 = \emptyset$, and it is denoted by PST_2 - space .

Definition 1.4.39

A bitopological space (X, τ_1, τ_2) is called PS^*T_2 -space if $\forall x \neq y \in X$, there is τ_1 - open set G_1 and τ_2 - semi open set G_2 such that $x \in G_1$, $y \in G_2$ and $G_1 \cap G_2 = \emptyset$, and it is denoted by PS^*T_2 - space .

Example 1.4.40

Let $X = \{a, b, c\}$

$\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.

$\tau_2 = \{\emptyset, X, \{b\}\}$.

(X, τ_1, τ_2) is PRT_1 -space.

Theorem 1.4.41

- 1) Every PRT_2 -space is PT_2 -space.
- 2) Every PT_2 -space is PS^*T_2 -space.
- 3) Every PS^*T_2 -space is PST_2 -space.

The proof of above Theorem follows from the fact that Every Regular - open is open set and every open is semi -open set.

Example 1.4.42 :

Let $X = \{a, b, c\}$

$\tau_1 = \{\emptyset, X, \{a, b\}, \{a\}\}$.

$\tau_2 = \{\emptyset, X, \{a, b\}, \{b\}\}$.

(X, τ_1, τ_2) is PST_1 -space but is not PT_2 - space.

Remark 1.4.43

- 1) Every PRT_2 -space is PRT_1 -space.
- 2) Every PST_2 -space is $PRST_1$ -space.
- 3) Every PS^*T_2 -space is PS^*T_1 -space.

Proof (1):

Let (X, τ_1, τ_2) is PRT_2 -space, then let $x \neq y \in X$, there is τ_1 - regular open set G_1 and τ_2 - regular open set G_2 such that $x \in G_1$, $y \in G_2$ and $G_1 \cap G_2 = \emptyset$, so $x \in G_1$, $y \notin G_1$ and $y \in G_2$, $x \notin G_2$ then (X, τ_1, τ_2) is PRT_1 -space.

Proof of (2) and (3) in remark (1.4.43) is the same proof (1) in the Remark.

Definition: 1.4.44 [1]

A bitopological space (X, τ_1, τ_2) is called P regular if $\forall \tau_1$ - closed set F_1 and $\forall x \notin F_1$ there are τ_1 - open set G_1 and τ_2 open set G_2 such that $x \in G_1$, $F_1 \subset G_2$ and $G_1 \cap G_2 = \emptyset$ and $\forall \tau_2$ - open set U_2 such that $x \in U_2$ there are τ_1 -open set $U_1 \cap U_2 = \emptyset$.

Definition 1.4.45 [23]

A bitopological space is built (X, τ_1, τ_2) is called PT_3 -space if (X, τ_1, τ_2) P-regular space and PT_1 -space.

Example 1.4.46

Let X be any space and let τ_1 be the discrete Topology on X the (X, τ_1, τ_2) is PT_3 - space.

Theorem 1.4.47

Let (X, τ_1, τ_2) be a bitopological space then every PT_3 -space is PT_2 -space

Proof:

Let (X, τ_1, τ_2) be PT_2 -space.

Then (X, τ_1, τ_2) is PT_1 -space

Hence by theorem (1.4.19)

$\{x\}$ is τ_1 -closed set or τ_2 - closed set.

By $x \neq y, y \notin \{x\}$

1) If $\{x\}$ is τ_1 closed set and $y \notin \{x\}$, since (X, τ_1, τ_2) is P regular space there are

τ_1 -open set G_1 and τ_2 - open set G_2 such that

$\{x\} \subset G_2, y \in G_1$ and $G_1 \cap G_2 = \emptyset$

$\{x\} \subset G_2, x \in G_2 \Rightarrow (X, \tau_1, \tau_2)$ is PT_2 - space

2) Similarly if $\{x\}$ is τ_2 - closed set.

Example 1.4.48

The converses of above theorem need not to true for example

Let $X = \{a, b, c\}$

$\tau_1 = \{\emptyset, X, \{b\}, \{a, b\}\}$.

$\tau_2 = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$.

(X, τ_1, τ_2) is PT_2 - space but is not PT_3 - space.

Theorem 1.4.49

A bitopological space (X, τ_1, τ_2) is called P-regular space if, and only if, for each τ_1 - open set G_1 and $x \in G_1$ there is τ_1 - open set U_1 such that $x \in U_1 \subset \text{cl}_2\{U_1\} \subseteq G_1$ and for each τ_2 - open set G_2 and $x \in G_2$, there is τ_2 - open set U_2 such that $x \in U_2 \subset \text{cl}_1\{U_2\} \subseteq G_2$.

Proof :

Suppose that (X, τ_1, τ_2) P- regular space

1) Let G_1 is τ_1 - open set and $x \in G_1$

Then $X - G_1$ is τ_1 -closed set and $x \notin X - G_1$

Since (X, τ_1, τ_2) is P regular space

There are τ_1 - open set U_1 and τ_2 - open set U_2

Such that $X - G_1 \subset U_2$ and $x \in U_1, U_1 \cap U_2 = \emptyset$

Since $U_1 \cap U_2 = \emptyset$ then $U_1 \subset X - U_2$

Thus $\text{cl}_2(U_1) \subset \text{cl}_2(X - U_2) = X - U_2$

But since $X - G_1 \subset U_2$

Hence $X - U_2 \subset G_1$, there for $U_1 \subset \text{cl}_2(U_1) \subset G_1$

2) Similarly for G_2 is τ_2 - open set and $x \in G_2$

Conversely:

To prove that (X, τ_1, τ_2) is P regular space

1) let F_1 is τ_1 - closed set and $x \notin F_1 \Rightarrow x \in X - F_1$ and $X - F_1$ is τ_1 -open set

there is a τ_1 open set U_1 such that

$x \in U_1 \subset \text{cl}_2\{U_1\} \subset X - F_1$

$\text{cl}_2\{U_1\} \subset X - F_1$

$\Rightarrow F_1 \subset X - \text{cl}_2\{U_1\} = U_2$

Since $\text{cl}_2\{U_1\}$ is τ_2 -closed set

Then U_2 is τ_2 - open set and $F_1 \subset U_2$
 Since $U_2 = X - \text{cl}_2 \{U_1\}$ then $U_1 \cap U_2 = \varnothing$
 Hence $x \in U_1, F_1 \subset U_2$ and $U_1 \cap U_2 = \varnothing$
 2) Similarly if F_2 is τ_2 - closed set and $x \notin F_2$
 From (1) and (2)
 We get that (X, τ_1, τ_2) is P- regular space.

Definition 1.4.50 [23]

A bitopological space (X, τ_1, τ_2) is PR- regular space if for each τ_1 - closed set F_1 and $x \notin F_1$, there are τ_1 - regular open set G_1 and τ_2 - regular open set G_2 such that $x \in G_1, F_1 \subset G_2$ and $G_1 \cap G_2 = \varnothing$ and $\forall \tau_2$ - closed set F_2 and $x \in F_2$ there are τ_1 -regular open set U_1 and τ_2 - regular open set U_2 such that $x \in U_2$ and $F_2 \subset U_1$ and $U_1 \cap U_2 = \varnothing$ and denoted by PRR.

Definition 1.4.51

A bitopological space (X, τ_1, τ_2) is pair wise S^* - regular space if for each if τ_1 - closed set F_1 and $x \notin F_1$ there are τ_1 - regular open set G_1 and τ_2 - semi open set G_2 such that $x \in G_1, F_1 \subset G_2$ and $G_1 \cap G_2 = \varnothing$ and $\forall \tau_2$ - closed set F_2 and $x \in F_2$ there are τ_1 -regular open set U_1 and τ_2 - semi open set U_2 such that $x \in U_2$ and $F_2 \subset U_1$ and $U_1 \cap U_2 = \varnothing$, and denoted by PS^*R .

Definition 1.4.52 [1]

A bitopological space (X, τ_1, τ_2) is PS- regular space if for each τ_1 - closed set F_1 and $x \notin F_1$ there are τ_1 - regular open set G_1 and τ_2 - regular open set G_2 such that $x \in G_1, F_1 \subset G_2$ and $G_1 \cap G_2 = \varnothing$ and $\forall \tau_2$ - closed set F_2 and $x \in F_2$ there are τ_1 - S- open set U_1 and τ_2 -S- open set U_2 such that $x \in U_2$ and $F_2 \subset U_1$ and $U_1 \cap U_2 = \varnothing$, and denoted by PSR-space.

Definition 1.4.53 [1]

A bitopological space (X, τ_1, τ_2) is called PRT_3 - space if it is PR- regular and PRT_1 - space .

Definition 1.4.54

A bitopological space (X, τ_1, τ_2) is PS^*T_3 - space if it is PS^* - regular space and it is PS^*T_1 - space .

Remark 1.4.55

- 1) Every PRT_3 -space is PRT_2 -space.
- 2) Every PST_3 -space is PST_2 -space.
- 3) Every PS^*T_3 - space is PS^*T_2 -space.

Proof:

(1) Let (X, τ_1, τ_2) is PRT_3 -space $\Rightarrow (X, \tau_1, \tau_2)$ is R-regular space and PT_1 - space .

Since every R- regular space is regular space (1.4.1)

And since, every PRT_1 -space is PT_1 -space (1.4.2)

From (1.4.1) and (1.4.2) we get (X, τ_1, τ_2) is PR - space .

And PT_1 - space so (X, τ_1, τ_2) is PT_3 - space .

(2) Let (X, τ_1, τ_2) is PRT_3 -space $\Rightarrow (X, \tau_1, \tau_2)$ is R-regular space and PT_1 -space .

Since every P regular space is PS^* -regular space (1.4.3)

And every PT_1 -space is PS^*T_1 -space (1.4.4)

From (1.4.3) and (1.4.4) we get (X, τ_1, τ_2) is PS^*T_1 - space .

(3) Let (X, τ_1, τ_2) is PS^*T_3 -space $\Rightarrow (X, \tau_1, \tau_2)$ is PS^* -regular space and it is PS^*T_1 -space .

Since every PS^* - regular space is PS -regular space ... (1.4.5)

And every PT_1 -space is PS^*T_1 -space ... (1.4.6)

From (1.4.5) and (1.4.6) (X, τ_1, τ_2) is PST_1 - space .

Remark 1.4.56

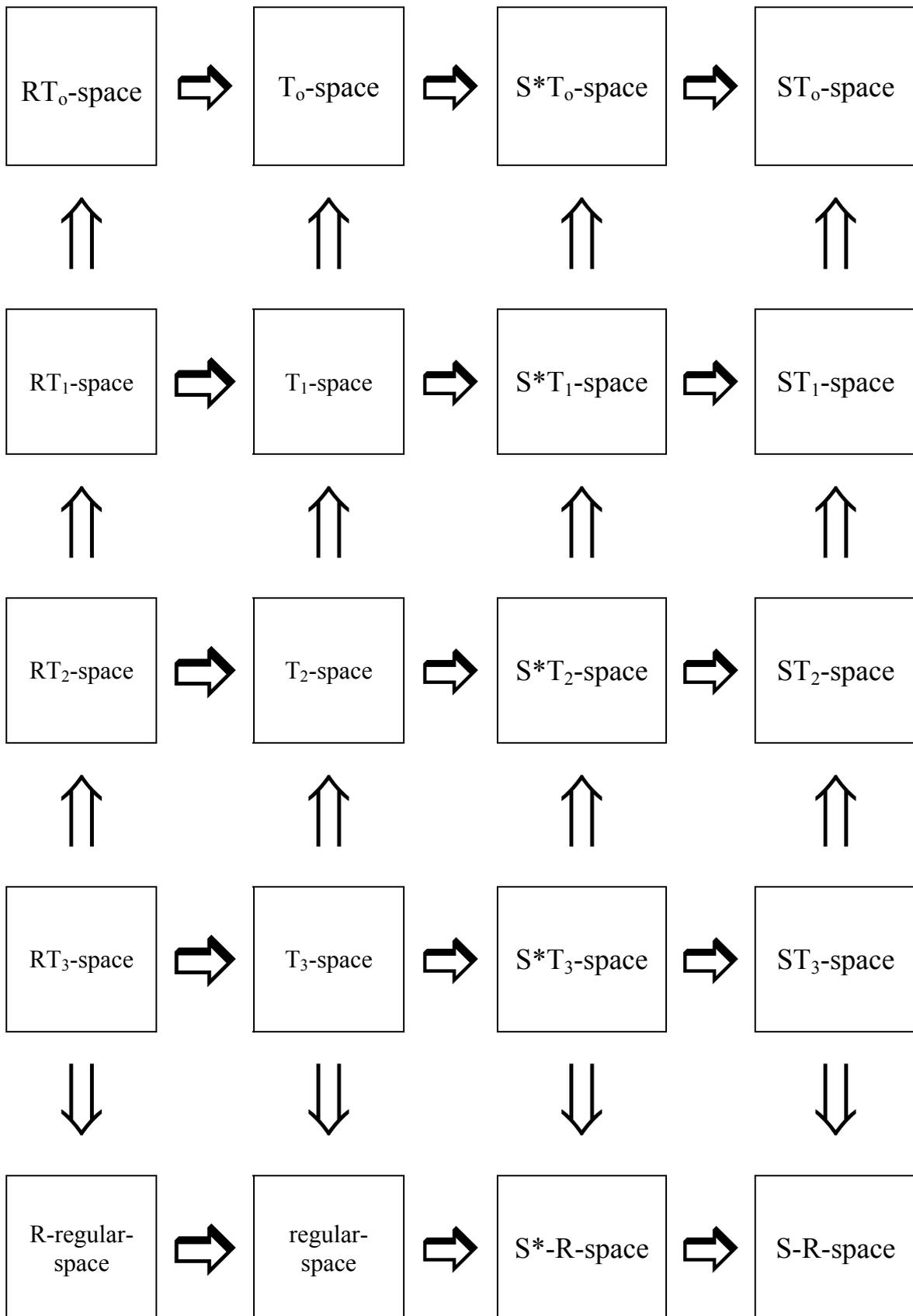
Every PRT_3 -space is PT_3 -space.

Every PT_3 -space is PS^*T_3 -space.

Every PS^*T_3 -space is PST_3 -space.

The proof of this remark is easy by using the fact that every regular open set is open and every open set is semi open.

The following diagram show the relation between the spaces which studies before :



Chapter Two

Some New Bitopological Separation Axioms

§ (2-1) Basic definitions

In this section we will give a new definitions of a cover in bitopological space and we will show the relation between this covers.

Definition 2.1.1 [4]

A cover u of a bitopological space (X, τ_1, τ_2) is called pairwise open if $u \subset \tau_1 \cup \tau_2$ and if furthermore u contains a non – empty member of τ_1 and a non – empty member of τ_2 .

Then a pair wise open cover u is called pT_1 - open.

Definition 2.1.2 [4]

A pairwise open cover u of a bitopological space (X, τ_1, τ_2) is called pT_2 -open if for each $U \in u$, $\text{int}_{\tau_i}(X / U) \neq \emptyset$, for $i=1$ or $i=2$.

Definition 2.1.3 [6]

A pairwise cover ω of a bitopological space (X, τ_1, τ_2) is called pT_3 -open if for each $W \in \omega$ when ever $W \in \tau_i$, there exist τ_j - open sets V_1 and V_2 such that $V_1, V_2 \neq \emptyset$, $\text{cl}_{\tau_i}(V_1) \subset V_2 \subset X/W$, for $i \neq j$ and $i, j = 1, 2$

Definition 2.1.4

A cover u of a bitopological space (X, τ_1, τ_2) is called semi pairwise open if $u \subset S.O_{\tau_1}(X) \cup S.O_{\tau_2}(X)$ and if further more u contains a non – empty member of $S.O_{\tau_1}(X)$ and a non- empty member of $S.O_{\tau_2}(X)$. A semi pair wise open cover is called semi pair wise T_1 – open, an it is denoted by SPT_1 -open.

Definition 2.1.5

A semi pairwise open cover u of a bitopological space (X, τ_1, τ_2) is said to be semi PT_2 - open if for each $U \in u$, $S\text{-int}_{\tau_i}(X/U) \neq \emptyset$, for $i=1$ or 2 , and it is denoted by SPT_2 -open.

Definition 2.1.6

A semi pair wise open cover ω of a bitopological space (X, τ_1, τ_2) is called semi PT_3 -open if for each $W \in \omega$ when ever $W \in \tau_i$, there exist τ_j - semi open sets V_1, V_2 such that $V_1, V_2 \neq \emptyset$, $V_1 \subset Scl_{\tau_i}(V_1) \subset V_2 \subset (X/W)$ for $i \neq j$ and $i = 1, 2$, and it is denoted by SPT_3 -open .

Definition 2.1.7 [13]

A bitopological space (X, τ_1, τ_2) is called mPT_1 if for every pair of distinct points x and y in X following holds

$$cl_{\tau_1}\{x\} \cap cl_{\tau_2}\{y\} = \emptyset \text{ or}$$
$$cl_{\tau_2}\{x\} \cap cl_{\tau_1}\{y\} = \emptyset$$

Definition 2.1.8

A bitopological space (X, τ_1, τ_2) is $SmPT_1$ if for every pair of distinct point x and $y \in X$ the following holds

$$Scl_{\tau_1}\{x\} \cap Scl_{\tau_2}\{y\} = \emptyset \text{ or}$$
$$Scl_{\tau_2}\{x\} \cap Scl_{\tau_1}\{y\} = \emptyset$$

Definition 2.1.9

A bitopological space (X, τ_1, τ_2) is called $RmPT_1$ if $\forall x \neq y \in X$ the following holds $R.cl_{\tau_1}\{x\} \cap R.cl_{\tau_2}\{y\} = \emptyset$ or $R.cl_{\tau_2}\{x\} \cap R.cl_{\tau_1}\{y\} = \emptyset$

Definition 2.1.10 [15]

A bitopological space (X, τ_1, τ_2) is $MNPT_1$ if for every pair distinct points x and y in X there exist a τ_1 - open set or a τ_2 -open set containing x but not y .

Definition 2.1.11

A bitopological space (X, τ_1, τ_2) is $SMNPT_1$ if for every pair distinct points x and y in X there exist a τ_1 - semi open set or a τ_2 - semi open set containing x but not y .

Definition 2.1.12[20]

A bitopological space (X, τ_1, τ_2) is called weakly pair wise T_1 if for each pair of distinct points there is a τ_1 - open set containing one of the

points but not the other and a τ_2 - open set containing the second but not the first , and it is denoted by wPT_1 .

Definition 2.1.13

A bitopological space X is $SwPT_1$ if for each pair of distinct points x and y there is a τ_1 - semi open set containing x but not y and a τ_2 - semi open containing y but not x .

Definition 2.1.14

A bitopological space (X, τ_1, τ_2) is $RwPT_1$ if for each pair wise of distinct points x and y there is a τ_1 - regular open set containing x but not y and a τ_2 - regular open set containing y but not x .

Definition 2.1.15 [7]

A bitopological space (X, τ_1, τ_2) is said to be nearly PT_i - space (briefly nT_i - space) $i \in \{1,2,3\}$ if for each point $x \in X$ and a τ_k -open neighborhood V of x , $k \in \{1,2\}$, there exist a PT_i - open cover u of X such that $st(x, u) \subset V$.

Definition 2.1.16

A bitopological space (X, τ_1, τ_2) is said to be a semi nearly PT_i - space(briefly SnT_i - space) $i \in \{1,2,3\}$ if for each point $x \in X$ and a τ_k - semi open neighborhood V of x , $k \in \{1,2\}$ there exist a PT_i -semi open cover u of X such that $Sst(x, u) \subset V$.

Every $RwPT_1$ - space is RPT_0 , but the converse need not be true, we can see that from this example

Example 2.1.17

Let $X = \{a, b, c\}$
 $\tau_1 = \{ \emptyset, X, \{a\}, \{b, c\} \}$.
 $\tau_2 = \{ \emptyset, X, \{a\}, \{b\}, \{a, b\} \}$.

And let $R.O.(X)_{\tau_1} = \{ \{a\}, \{b, c\} \}$, $R.O.(X)_{\tau_2} = \{ \{a\}, \{b\}, \{a, b\} \}$
 Then clearly the space (X, τ_1, τ_2) is RPT_0 -which it is not $RWPT_1$.

For consider the pair of point's b, c of X , we see that any τ_1 - regular open set containing b also contains c and τ_2 - regular open set containing c also contains b .

Theorem 2.1.18 [7]

Let (X, τ_1, τ_2) be a bitopological space then every SmPT_1 -space is SwPT_1 -space.

Proof:

Let X be SmPT_1 , then by definition (2.1.8) we get that , $\forall x, y \in X$ such that $x \neq y$, we have

$$\text{scl}_{\tau_1}\{x\} \cap \text{scl}_{\tau_2}\{y\} = \emptyset \text{ or } \text{scl}_{\tau_2}\{x\} \cap \text{scl}_{\tau_1}\{y\} = \emptyset$$

if $\text{scl}_{\tau_1}\{x\} \cap \text{scl}_{\tau_2}\{y\} = \emptyset \Rightarrow \exists$ a point $x \in \text{scl}_{\tau_1}\{x\}$ and $x \notin \text{scl}_{\tau_2}\{y\}$ and \exists a point $y \in \text{scl}_{\tau_2}\{y\}$ and $y \notin \text{scl}_{\tau_1}\{x\}$ but $\text{scl}_{\tau_1}\{x\}$ is semi-closed with respect to τ_1 and $\text{scl}_{\tau_2}\{y\}$ is semi-closed with respect to τ_2 .

Now $X/\text{scl}_{\tau_1}\{x\}$ is semi-open and $X/\text{scl}_{\tau_2}\{y\}$ is semi-open which is mean that \exists a τ_1 -semi open set containing y but not x and a τ_2 -semi open set containing x but not y which is mean that X is SwPT_1 -space

Theorem 2.1.19 [7]

Let (X, τ_1, τ_2) be a bitopological space then every SwPT_1 -space is SMNPT_1 -space.

Proof:

Let X is SwPT_1 then by definition (2.1.13) we get that , $\forall x, y \in X$ such that $x \neq y$

\exists a τ_1 -semi open containing x but not y or

\exists a τ_2 - semi open set containing y but not x

And directly from a definition (2.1.11) we get that X is SMNPT_1 .

Theorem 2.1.20 [7]

Let (X, τ_1, τ_2) be a bitopological space then every SPT_1 -space is SmPT_1 -space.

Proof:

Let X be SPT_1 -space then by definition (1.4.25) we get that , $\forall x, y \in X$ such that $x \neq y \in X$, there are τ_1 -semi open set containing x but not y and a τ_1 - semi open set containing y but not x .

Now

$\text{scl}_{\tau_1}\{x\}$ is semi open in τ_1 and

$\text{scl}_{\tau_2}\{y\}$ is semi open in τ_2

If $\text{scl}_{\tau_1}\{x\} \cap \text{scl}_{\tau_2}\{y\} \neq \emptyset \Rightarrow \exists$ a point say $x \in \text{scl}_{\tau_1}\{x\}$

And $x \in \text{scl}_{\tau_2}\{y\}$ which is mean that X is not a SPT_1 -space

Hence $\text{scl}_{\tau_1}\{x\} \cap \text{scl}_{\tau_2}\{y\} = \emptyset$

Similarly $\text{scl}_{\tau_2}\{x\} \cap \text{scl}_{\tau_1}\{y\} = \emptyset$ which is mean that X is SmPT_1 -space.

Theorem 2.1.21[7]

Let (X, τ_1, τ_2) be a bitopological space then every SPT_1 -space is SnPT_1 -space.

Proof :

Let X is SPT_1 – space then by definition (1.4.25) we get that $\forall x, y \in X$ such that $x \neq y$ there are a τ_1 -semi open set containing x but not y and a τ_2 semi open set containing y but not x .

So by theorem (1.3.19) each semi-open is semi-neighborhood of each of its points

Then there exists a SpT_1 – open cover u of x such that $\text{Sst}(x, u) \subset V$ where V is a τ_1 - semi open neighborhood or τ_2 - semi open neighborhood which is mean that X is SnpT_1 - space .

Remark 2.1.22 [7]

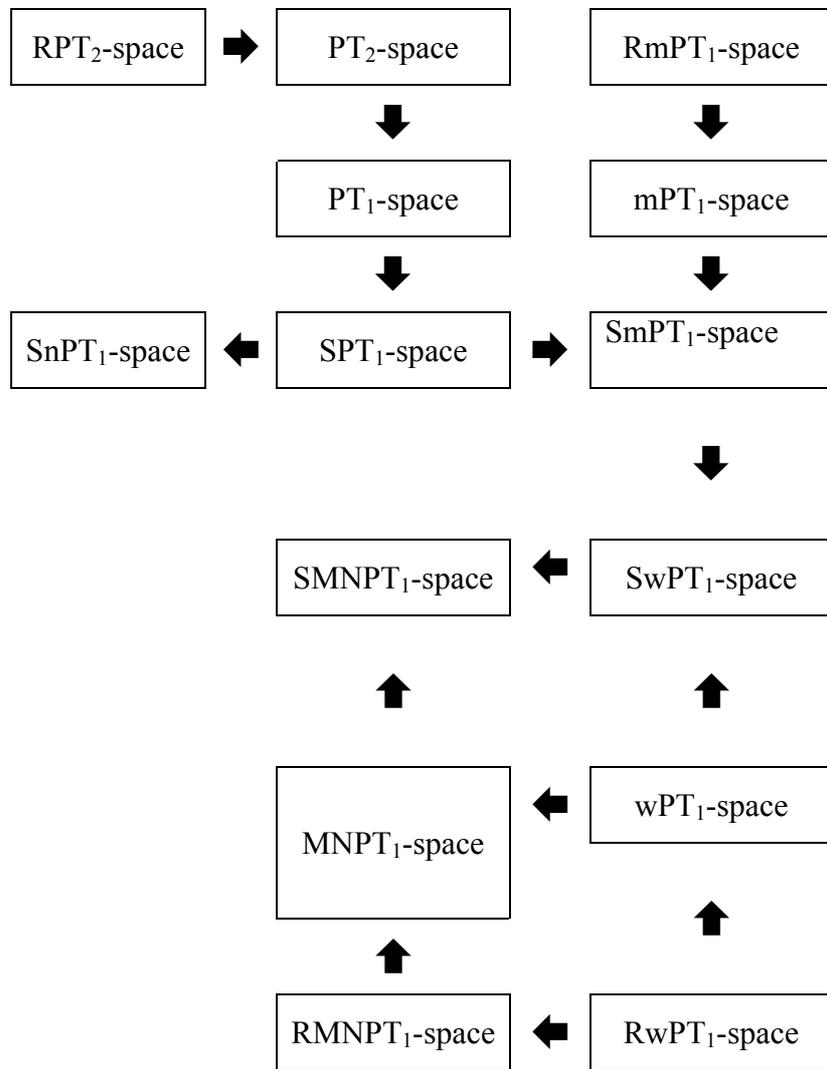
By using above method of the remark above we can prove that every SpT_i -space is SnpT_i -space, $i \in \{1, 2, 3\}$ but the converse is not true which can be show in follow examples.

Example 2.1.22:

Let $X = \{a, b, c\}$
 $\tau_1 = \{\emptyset, X, \{a\}\}$.
 $\tau_2 = \{\emptyset, X, \{b, c\}\}$, and $\text{S.O.}(X)_{\tau_1} = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$,
 $\text{S.O.}(X)_{\tau_2} = \{\emptyset, X, \{c\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$
 Then X is SnpT_1 but not MNPT_1 , so it is not satisfy any of axioms SWPT_1 , SmpT_1 or SpT_1 , Also X is SnpT_2 but not SWPT_2 .

Example 2.1.23 :

Let $X = \{a, b, c\}$
 $\tau_1 = \{\emptyset, X, \{b, a\}, \{a\}, \{a, c\}\}$.
 $\tau_2 = \{\emptyset, X, \{b, c\}\}$, and $\text{S.O.}(X)_{\tau_1} = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$,
 $\text{S.O.}(X)_{\tau_2} = \{\emptyset, X, \{a\}, \{c\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$
 Then X is SMNPT_1 , but not SnpT_1 .
 The following diagram of implications holds and none of those implications revisable.



Moreover, the axiom $SnPT_1$ is independent from any of $SMNPT_1$, $SwPT_1$ and $SmPT_1$.

§ (2-2) pair wise $T(i, k)$ – space

In this section we will give a definition of $PT(i,k)$ - spaces and a relation between them from a hand and between them and a nearly PT_i - spaces from another hand.

Definition 2.2.1 [7]

A bitopological space (X, τ_1, τ_2) is said to be a pair wise $T(i, k)$ space ; $i, k \in \{1, 2, 3\}$, if for every $x \in X$ and every pT_k -open cover \mathcal{u} of X there exists a pT_i - open cover \mathcal{v} of X and $U \in \mathcal{u}$ such st $(x, \mathcal{v}) \subset U$, and it is denoted by $PT(i,k)$ -space .

Lemma 2.2.2

Let (X, τ_1, τ_2) be a bitopological space then every pT_3 -open cover is a pT_2 – open cover.

Proof

Let \mathcal{u} be a pT_3 - open cover, then for each $U \in \mathcal{u}$ whenever $U \in \tau_i$, there exist τ_j - sets V_1, V_2 such that $V_1, V_2 \neq \emptyset, V_1 \subset \text{cl}_{\tau_i}(V_1) \subset V_2 \subset (X/U)$ for $i \neq j$ and $i, j = 1, 2$

Since $V_1 \subset \text{cl}_{\tau_i}(V_1) \subset V_2 \subset X / U$

$\text{int}_{\tau_i}(V_1) \subset ((\text{int}(\text{cl}(V_1)))_{\tau_i}) \subset \text{int}_{\tau_i}(V_2) \subset \text{int}_{\tau_i}(X/U)$

Which is mean $\text{int}_{\tau_i}(X/ U) \neq \emptyset$

Therefore \mathcal{u} is pT_2 – open cover.

Remark 2.2.3

The converse of lemma (2.2.2) need not be true and the following example show that

Example 2.2.4:

Let $X = \{a, b, c\}$

$\tau_1 = \{ \emptyset, X, \{a, b\} \}$

$\tau_2 = \{ \emptyset, X, \{c\} \}$

let $\mathcal{u} = \{ \{a, b\}, \{c\} \}$ is pair wise open cover since $\mathcal{u} \subset \tau_1 \cup \tau_2$

Now let $U = \{a, b\} \in \mathcal{u} \Rightarrow X/U = \{c\}$

$\text{int}_{\tau_2}(\{c\}) = \{c\} \neq \emptyset$

Let $U = \{c\} \in \mathcal{u} \Rightarrow X/U = \{a, b\}$

$$\text{int}_{\tau_1}(\{a,b\}) = \{a,b\} \neq \emptyset$$

Therefore for each $U \in \mathcal{u}$, $\text{int}_{\tau_i}(X/U) \neq \emptyset$, $i=1,2$.

Which is mean \mathcal{u} is pT_2 -open cover but it is not pT_3 -open cover since if

we let $W = \{a,b\}$, $W \in \tau_1$

Let $V_1 = \{c\}$, $V_2 = X$, $V_1, V_2 \neq \emptyset$

$V_1 \subset \text{cl}_1(V_1) \subset V_2 \subset X/W$

Since $\{c\} \subset \{c\} \subset X \not\subset \{c\}$

Lemma 2.2.5

Let (X, τ_1, τ_2) be a bitopological space then every pT_2 – open cover is a pT_1 – open cover

Proof

Let \mathcal{u} be a pT_2 – open cover. Then for each $U \in \mathcal{u}$

$\text{int}_{\tau_i}(X/U) \neq \emptyset$ for $i = 1$ or $i = 2$

let $W \in \tau_i$, $\text{int}_{\tau_i}(X/U)$ for $i = 1$ or $i = 2$

$\Rightarrow W \in \tau_i$ and $W \not\subset \text{int}_{\tau_i}(X/U)$

$\Rightarrow W \in \tau_i$ and $W \in \text{int}_{\tau_i}(U)$

Since U is open $\Rightarrow \text{int}(U) = U$

Since $W \in U$ but $U \in \mathcal{u} \Rightarrow W \in \mathcal{u}$

Therefore \mathcal{u} contains a nonempty member of τ_i , $i = 1$ or 2

Therefore, \mathcal{u} is a pT_1 - open cover

Theorem 2.2.6

Let (X, τ_1, τ_2) be a bitopological space then every $PT(3,3)$ – space is $PT(2,3)$ – space

Proof

Let X be a $PT(3,3)$ – space then $\forall x \in X$ and for every pT_3 – open cover \mathcal{u} of X there exists a pT_3 - open cover \mathcal{v} of X and $U \in \mathcal{u}$ such that $\text{st}(x, \mathcal{v}) \subset U$.

Now

Since \mathcal{v} is a pT_3 – open cover then by lemma (2.2.2) \mathcal{v} is a pT_2 – open cover

Therefore $\forall pT_3$ – open cover \mathcal{u} of X there exists as pT_2 – open cover \mathcal{v} of X and a $U \in \mathcal{u}$ such that $\text{st}(x, \mathcal{v}) \subset U$.

Therefore X is a $PT(2, 3)$ – space.

Theorem 2.2.7

Let (X, τ_1, τ_2) be a bitopological space then every $PT(2, 3)$ – space is a $PT(1, 3)$ – space.

Proof

Let X be a $PT(2, 3)$ - space $\Rightarrow \forall x \in X$ and for every pT_3 – open cover \mathcal{u} of X there exists a pT_2 open cover \mathcal{v} of X and a $U \in \mathcal{u}$ such that $st(x, \mathcal{v}) \subset U$.

By lemma (2.2.4) \mathcal{v} is a pT_1 – open cover. Therefore $\forall pT_3$ – open cover \mathcal{u} of X there exists a $U \in \mathcal{u}$ such that $st(x, \mathcal{v}) \subset U$. Therefore X is a $PT(1, 3)$ – space.

Theorem 2.2.8

Let (X, τ_1, τ_2) be a bitopological space then every $PT(3, 2)$ – space is a $PT(2, 2)$ - space.

Proof

Let X be a $PT(3, 2)$ - space $\Rightarrow x \in X$ and for every pT_2 - open cover \mathcal{u} of X there exists a pT_3 - open cover \mathcal{v} and a $U \in \mathcal{u}$ such that $st(x, \mathcal{v}) \subset U$.

By lemma (2.2.2) \mathcal{v} is a pT_2 – open cover
Therefore X is a $PT(2, 2)$ - space.

Theorem 2.2.9

Let (X, τ_1, τ_2) be a bitopological space then every $PT(2, 2)$ - space is a $PT(1,2)$ - space.

Proof

Let X be a $PT(2, 2)$ - space then $\forall x \in X$ and for every pT_2 – open cover \mathcal{u} of X there exists a pT_2 – open cover \mathcal{v} of X and a $U \in \mathcal{u}$ such that $st(x, \mathcal{v}) \subset U$.

By lemma (2.2.4) \mathcal{v} is a pT_1 – open cover
Therefore X is a $PT(1, 2)$ – space.

Remark 2.2.10

- Let (X, τ_1, τ_2) be a bitopological space then
- 1- Every $PT(3, 1)$ – space is a $PT(2, 1)$ - space.
 - 2- Every $PT(2, 1)$ – space is a $PT(1, 1)$ - space.

Theorem 2.2.11

Let (X, τ_1, τ_2) be a bitopological space then every $PT(3, 1)$ - space is $PT(3, 2)$ - space.

Proof

Let X be a $PT(3, 1)$ - space

And let \mathcal{u} be a pT_2 - open cover of X

By lemma, (2.2.4) \mathcal{u} is a pT_1 - open cover of X

Since X is a $PT(3, 1)$ - space then $\forall x \in X$ and $\forall pT_1$ - open cover \mathcal{u} of X there exists a pT_3 - open cover \mathcal{v} of X and a $U \in \mathcal{u}$ such that $st(x, \mathcal{v}) \subset U$.

But \mathcal{u} is a pT_2 - open cover of X

Therefore X is a $PT(3, 2)$ - space

Similarly we can prove that every $PT(3, 2)$ - space is a $PT(3, 3)$ -space.

Theorem 2.2.12

Let (X, τ_1, τ_2) be a bitopological space then every $PT(2, 1)$ - space is $PT(2, 2)$ - space.

Proof

Let X be a $PT(2, 1)$ - space

And let \mathcal{u} be a pT_2 - open cover of X

By lemma, (2.2.4) \mathcal{u} is a pT_1 - open cover of X

Since X is a $PT(2, 1)$ - space then $\forall x \in X$ and $\forall pT_1$ - open cover \mathcal{u} of X there exists a pT_2 - open cover \mathcal{v} of X and a $U \in \mathcal{u}$ such that $st(x, \mathcal{v}) \subset U$.

But \mathcal{u} is a pT_2 - open cover of X

Therefore X is a $PT(2, 2)$ - space

Similarly we can prove that every $PT(2, 2)$ - space is a $PT(2, 3)$ -space.

Theorem 2.2.13

Let (X, τ_1, τ_2) be a bitopological space then every $PT(1, 1)$ - space is $PT(1, 2)$ - space.

Proof

Let X be a $PT(1, 1)$ - space

And let \mathcal{u} be a pT_2 – open cover of X

By lemma, (2.2.4) \mathcal{u} is a pT_1 – open cover of X

Since X is a $PT(1, 1)$ – space $\Rightarrow \forall x \in X$ and $\forall pT_1$ – open cover \mathcal{u} of X there exists a pT_1 – open cover \mathcal{v} of X and a $U \in \mathcal{u}$ such that $st(x, \mathcal{v}) \subset U$.

But \mathcal{u} is a pT_2 – open cover of X

Therefore X is a $PT(1, 2)$ - space

Similarly we can prove that every $PT(1, 2)$ - space is a $PT(1, 3)$ -space.

Remark 2.2.14

From the definitions and remarks above we can concluding that :

- 1- Every $PT(i, k)$ – space is a $PT(j, k)$ - space, provided $i > j$ and k constant, $i, j, k \in \{1, 2, 3\}$
- 2- Every $PT(i, k)$ – space is a $PT(i, j)$ - space, provided $j > k$ and i constant, $i, j, k \in \{1, 2, 3\}$

Theorem 2.2.15

- 1-Every nPT_3 – space is $PT(3, 1)$ - space
- 2- Every nPT_2 – space is a $PT(2, 1)$ - space.
- 3- Every nPT_1 – space is a $PT(1, 1)$ - space

Proof

(1) Let X be a nPT_3 – space $\Rightarrow \forall x \in X$ and τ_k - open neighborhood \mathcal{u} of x , $k \in \{1, 2\}$, there exists a pT_3 – open cover \mathcal{v} of X such that $st(x, \mathcal{v}) \subset \mathcal{u}$.

Since \mathcal{u} is an open neighborhood of $x \Rightarrow \mathcal{u}$ is open cover of X .

$\Rightarrow \forall x \in X$ and a pT_1 – open cover \mathcal{u} of X , there exist a pT_3 - open cover \mathcal{v} of X such that $st(x, \mathcal{v}) \subset \mathcal{u}$ which is mean that X is $PT(3,1)$ -space.

(2) Same proof of (1).

(3) Same proof of (1) & (2).

Remark 2.2.16

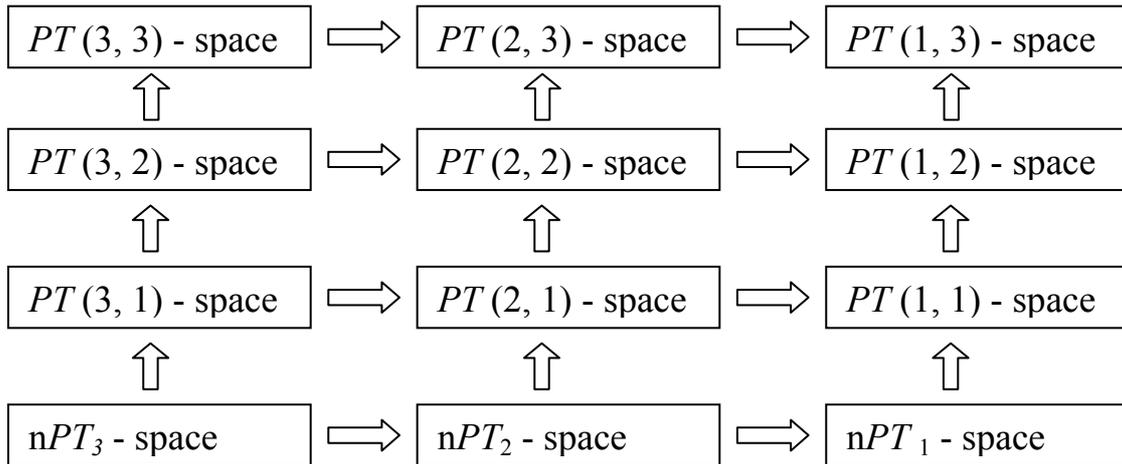
- 1- Every nPT_i – space is a $PT(i, k)$ - space.
 $i, k \in \{1, 2, 3\}$
- 2- Every nPT_i – space is nPT_j – space, $j < i$, $i, j \in \{1, 2, 3\}$

Remark 2.2.17

The following example shows that any of $PT(i, k)$ axioms do not imply $nPT_j, j \in \{1, 2, 3\}$

Example 2.2.18

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \varphi, \{a\}, \{a, c\}\}$ and $\tau_2 = \{X, \varphi, \{b, c\}\}$
Then X is $PT(3, 1)$ but not nPT_1 . And the following diagram hold



$\mathcal{S}(2-3)$ semi PT (i, k) – space, $i, k \in \{1, 2, 3\}$

In this section we will give a definition of SPT(i,k)- spaces and a relation between them from a hand and between them and a nearly SPT_i - spaces from another hand.

Definition 2.3.1

A bitopological space (X, τ_1, τ_2) is said to be semi pair wise T(i, k)-space $i, k \in \{1, 2, 3\}$ if for every $x \in X$ and every SpT_k – open cover \mathcal{u} of X there exist a SpT_i – open cover \mathcal{v} of X and a \mathcal{u} such that $Sst(x, \mathcal{v}) \subset \mathcal{u}$, and it is denoted by SPT(i,k).

Lemma 2.3.2

Let (X, τ_1, τ_2) be a bitopological space then let (X, τ_1, τ_2) be a bitopological space then every SpT_3 – open cover is a SpT_2 – open cover.

Proof

Let \mathcal{u} be spT_3 – open cover, then for each $U \in \mathcal{u}$

Whenever U is semi open with respect to τ_i , there exist a τ_i -semi open sets V_1, V_2 such that $V_1, V_2 \neq \emptyset$.

Since $V_1 \subset Scl_{\tau_i}(V_1) \subset V_2 \subset X/U$ for

$Sint_{\tau_i}(V_1) \subset ((Sint(Scl))_{\tau_i}(V_1)) \subset Sint_{\tau_i}(V_2) \subset Sint_{\tau_i}(X/U)$

This is mean $Sint_{\tau_i}(X/U) \neq \emptyset$

Therefore, \mathcal{u} is a SpT_2 – open cover.

Lemma 2.3.3

Let (X, τ_1, τ_2) be a bitopological space then every SpT_2 – open cover is a SpT_1 – open cover

Proof

Let \mathcal{u} be a SpT_2 – open cover, then for each $U \in \mathcal{u}$, $Sint_{\tau_i}(X/U) \neq \emptyset$ for $i = 1$ or $i = 2$

Let $W \in \tau_i / Sint_{\tau_i}(X/U)$ for $i = 1$ or $i = 2$

$\Rightarrow W \in \tau_i$ and $W \notin Sint_{\tau_i}(X/U)$

$\Rightarrow W \in \tau_i$ and $W \in \text{Sint}_{\tau_i}(U)$

Therefore U is semi open $\Rightarrow \text{Sint}_{\tau_i}(U) = U$

Therefore $W \in \tau_i$ and $W \in U$

Therefore \mathcal{u} contains a non empty member τ_i and $i = 1$ or 2

Therefore, \mathcal{u} is a SpT_1 – open cover.

Lemma 2.3.4

Let (X, τ_1, τ_2) be a bitopological space then every pT_i – open cover is a SpT_i – open cover $i \in \{1, 2, 3\}$

Proof

It is easy by using lemma (2.3.2) and lemma (2.3.3) and the fact that every open set is semi open set.

Theorem 2.3.5

Let (X, τ_1, τ_2) be a bitopological space then every $SPT(3, 3)$ – space is $SPT(2, 3)$ – space.

Proof

Let X be a $SPT(3, 3)$ - space $\Rightarrow \forall x \in X$ and for every SpT_3 – open cover \mathcal{u} of X there exists a SpT_3 – open cover \mathcal{v} of X and $U \in \mathcal{u}$ such that $\text{Sst}(x, \mathcal{v}) \subset U$

Now:

\mathcal{v} is a SpT_3 – open cover then by lemma (2.3.2)

\mathcal{v} is a SpT_2 – open cover \mathcal{u} of X there exist a SpT_2 – open cover \mathcal{v} of X and $U \in \mathcal{u}$ that $\text{Sst}(x, \mathcal{v}) \subset U$.

Theorem 2.3.6

Let (X, τ_1, τ_2) be a bitopological space then every $SPT(2, 3)$ – space is $SPT(1-3)$ – space

Proof

Let x be $SPT(2, 3)$ – space $\Rightarrow \forall x \in X$ and for every SpT_3 – open cover \mathcal{u} of X there exists a SpT_2 -open cover \mathcal{v} of X and a $U \in \mathcal{u}$ such that $\text{Sst}(x, \mathcal{v}) \subset U$.

By lemma (2.3.3) \mathcal{v} is a SpT_1 -open cover

Therefore $\forall SpT_3$ – open cover \mathcal{u} of X there exist a SpT_1 – open cover \mathcal{v} of X and a $U \in \mathcal{u}$ such that $Sst(x, \mathcal{v}) \subset U$.
Therefore X is a $SPT(1, 3)$ – space.

Theorem 2.3.7

Let (X, τ_1, τ_2) be a bitopological space then every $SPT(3, 2)$ – space is a $SPT(2, 2)$ – space

Proof

Let X be a $SPT(3, 2)$ – space $\Rightarrow \forall x \in X$ and for every SpT_2 – open cover \mathcal{u} of X there exists a SpT_3 – open cover \mathcal{V} of X and a $U \in \mathcal{u}$ such that $Sst(x, \mathcal{V}) \subset U$
By lemma (2.3.2) \mathcal{V} is a SpT_2 – open cover
Therefore X is a $SPT(2, 2)$ – space.

Theorem 2.3.8

Let (X, τ_1, τ_2) be a bitopological space then every $SPT(2, 2)$ – space is a $SPT(1, 2)$ – space

Proof

Let X be a $SPT(2, 2)$ – space $\Rightarrow \forall x \in X$ and for every SpT_2 – open cover \mathcal{u} of X there exists a SpT_2 – open cover \mathcal{V} of X and a $U \in \mathcal{u}$ such that $Sst(x, \mathcal{V}) \subset U$.
By lemma (2.3.3) \mathcal{V} is a SpT_1 – open cover
Therefore X is a $SPT(1, 2)$ – space.

Remark 2.3.9

- Let (X, τ_1, τ_2) be a bitopological space then
- 1- Every $SPT(3, 1)$ – space is a $SPT(2, 1)$ – space
 - 2- Every $SPT(2, 1)$ – space is a $SPT(1, 1)$ – space.

Theorem 2.3.10

Let (X, τ_1, τ_2) be a bitopological space then every $SPT(3, 1)$ - space is a $SPT(3, 2)$ – space.

Proof

Let X be a $SPT(3, 1)$ – space and let \mathcal{u} be a SpT_2 – open cover of X by lemma, (2.3.3) \mathcal{u} is a SpT_1 - open cover of X

Since X is a $SPT(3, 1)$ - space $\Rightarrow \forall x \in X$ and $\forall SpT_1$ – open cover \mathcal{u} of X there exists a SpT_3 – open cover V of X and a $U \in \mathcal{u}$ such that $Sst(x, V) \subset U$ but \mathcal{u} is SpT_2 – open cover.

Therefore X is a $SPT(3, 2)$ - space.

Similarly we can prove that every $SPT(3, 2)$ - space is a $SPT(3, 3)$ - space.

Remark 2.3.11

Let (X, τ_1, τ_2) be a bitopological space then every $SPT(2, 1)$ - space is a $SPT(2, 2)$ - space

Proof

Let X be a $SPT(2, 1)$ - space , and let \mathcal{u} is a SpT_2 – open cover of X and by lemma (2.3.3) \mathcal{u} is a SpT_1 -open cover of X .

Since X is a $SPT(2, 1)$ - space $\Rightarrow \forall x \in X$ and $\forall SpT_1$ – open cover \mathcal{u} of X there exists a SpT_2 – open cover V of X and a $U \in \mathcal{u}$ such that $Sst(x, V) \subset U$.

but \mathcal{u} is a SpT_2 – open cover of X .

Therefore X is a $SPT(2, 2)$ - space

Similarly we can prove that every $SPT(2, 2)$ is $SPT(2, 3)$ - space.

Theorem 2.3.12

Let (X, τ_1, τ_2) be a bitopological space then every $SPT(1, 1)$ – space is $SPT(1, 2)$ – space

Proof

Let X be a $SPT(1, 1)$ – space and let \mathcal{u} be a SpT_2 – open cover \mathcal{u} of X by lemma (2.3.3), \mathcal{u} is a SPT_1 – open cover of X

Since X is $SPT(1, 1)$ – space $\Rightarrow \forall x \in X$ and for every SpT_1 – open cover V of X and a $U \in \mathcal{u}$ such that $Sst(x, V) \subset U$.

But \mathcal{u} is a SpT_2 – open cover of X

Therefore X is a $SPT(1, 2)$ – space

Similarity we can prove that every $SPT(1, 2)$ – space is $SPT(1, 3)$ – space.

Remark 2.3.13

Let (X, τ_1, τ_2) be a bitopological space and from above definitions and remarks above we can concluding that

- 1- Every $SPT(i, k)$ - space is a $SPT(j, k)$ - space, provide $i > j$ and k constant, $i, j, k \in \{1, 2, 3\}$.
- 2- Every $SPT(i, k)$ – space is $SPT(i, j)$ – space, provide $k < j$ and i constant $i, j, k \in \{1, 2, 3\}$.

Remark 2.3.14

Let (X, τ_1, τ_2) be a bitopological space then

- 1- Every $SnPT_3$ - space is $SPT(3, 1)$ - space.
- 2- Every $SnPT_2$ - space is a $SPT(2, 1)$ – space.
- 3- Every $SnPT_1$ - space is a $SPT(1, 1)$ – space

Proof

- 1- Let X be a $SnPT_3$ - space then $\forall x \in X$ and τ_k – semi open neighborhood \mathcal{U} of x , $k \in \{1, 2\}$, there exists a SpT_3 – open cover V of X such that $Sst(x, V) \subset \mathcal{U}$

Since \mathcal{U} is a semi-open neighborhood of x then \mathcal{U} is semi open cover of x then $\forall x \in X$ and a SpT_1 – open cover \mathcal{U} of X there exists SpT_3 – open cover V of X such that $Sst(x, V) \subset \mathcal{U}$

Which is mean that X is a $SPT(3, 1)$ – space.

- 2- Same proof of (1)
- 3- Same proof of (1) and (2).

Remark

Let (X, τ_1, τ_2) be a bitopological space then

- 1- Every $SnPT_i$ - space is a $SPT(i, k)$ - space , $i, k \in \{1, 2, 3\}$
- 2- Every $SnPT_i$ - space is a SPT_i -space $j < i$, $i, j \in \{1, 2, 3\}$

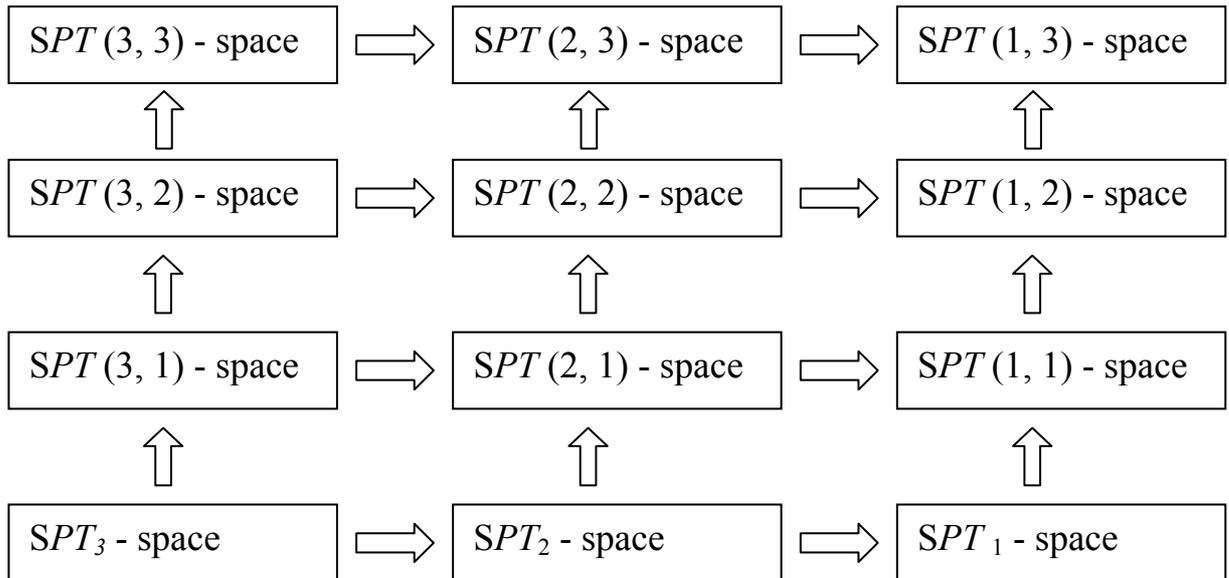
Remark 2.3.15

The following example show that any of $SPT(i, j)$ axioms do not imply to $SnPT_j, i, j \in \{1, 2, 3\}$

Let $X = \{a, b, c\}$ $\tau_1 = \{X, \varphi, \{a\}, \{a, c\}\}$

$\tau_2 = \{X, \varphi, \{b, c\}\}$ then X is $spT(3, 1)$ but not $SnPT_1$

And the following diagram hold.



Chapter Three
Nearly pTi - Continuous
Mappings

§ 3.1. Basic definitions

The aim of this section is to study a nearly PT_i – continuous mapping and give a relation between it and a connected space in bitopological spaces.

Definition 3.1.1 [18]

A function $f: X \rightarrow Y$ is called nearly continuous of a point $x \in X$ if for every neighborhood U of a $f(x)$ in Y , $f^{-1}(U)$ is a neighborhood of x in X .

Definition 3.1.2 [16]

Let (X, τ_1, τ_2) and (Y, ρ_1, ρ_2) be bitopological spaces then a mapping $f: X \rightarrow Y$ is said to be p-continuous if the induced maps $f: (X, \tau_1) \rightarrow (Y, \rho_1)$ and $f: (X, \tau_2) \rightarrow (Y, \rho_2)$ are continuous.

Theorem 3.1.3 [3]

The composite of any two p-continuous mapping is p-continuous.

Proof:

Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ is p-continuous and

$g: (Y, \rho_1, \rho_2) \rightarrow (Z, \mu_1, \mu_2)$ is p-continuous

To prove that $g \circ f: (X, \tau_1, \tau_2) \rightarrow (Z, \mu_1, \mu_2)$ is p-continuous

Since f is p – continuous then f induced continuous maps

$f: (X, \tau_1) \rightarrow (Y, \rho_1)$ and $f: (X, \tau_2) \rightarrow (Y, \rho_2)$ and since

g is p-continuous so g induced continuous maps

$g: (Y, \rho_1) \rightarrow (Z, \mu_1)$ and $g: (Y, \rho_2) \rightarrow (Z, \mu_2)$

Now since the composite of continuous functions is continuous function then

$g \circ f: (X, \tau_1) \rightarrow (Z, \mu_1)$ is continuous and,

$g \circ f: (X, \tau_2) \rightarrow (Z, \mu_2)$ is continuous.

Therefore $g \circ f: (X, \tau_1, \tau_2) \rightarrow (Z, \mu_1, \mu_2)$ is p-continuous .

Definition 3.1.4 [6]

A function f from a bitopological space X into a bitopological space Y is called T_i -pair wise continuous if for every pT_i -open cover ν of Y there exists a τ_k -open cover ω of X such that for every $W \in \omega$ there is a $V \in \nu$ such that $f(W) \subset V$, where $k \in \{1,2\}$ and $i \in \{1,2,3\}$, and its denoted by pT_i -continuous mappings .

Definition 3.1.5 [7]

A mapping $f:(X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ is said to be nearly pT_i -continuous at a point $x \in X$ if for every pT_i -open cover \mathcal{U} of Y there exists a τ_k -open neighborhood $V \subset X$ of x such that $f(V) \subset \text{st}(f(x), \mathcal{U})$.

Note:

A mapping f is nearly pT_i -continuous at each point of X , it is evident that every T_i -pair wise continuous mapping is nearly pT_i -continuous, but the converse, is not necessarily true, as the following example.

Example 3.1.6 [7]

Let $X = \{1, 2, 3, 4\}$, $\tau_1 = \{\emptyset, X, \{1, 4\}, \{1\}, \{2, 3\}, \{4\}, \{1, 2, 3\}, \{2, 3, 4\}\}$,

$\tau_2 = \{\emptyset, X, \{1\}, \{4\}, \{1, 4\}, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 4\}\}$ and $Y = \{a, b, c\}$

$\mathcal{U}_1 = \{\emptyset, Y, \{a, c\}\}$, $\mathcal{U}_2 = \{\emptyset, Y, \{b, c\}\}$. define

$f : X \rightarrow Y$ as follows $f(1) = b$, $f(2) = c$, $f(3) = a$

Then f is nearly pT_i -continuous but not T_i -pair wise continuous.

Proposition 3.1.7 [7]

Let $f:(X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ be nearly pT_i -continuous mapping and (Y, ρ_1, ρ_2) be npT_i -space then f is p -continuous.

Proof:

Let $v \in \mathcal{U}_k$ be a neighborhood of $f(x) \in Y$, since Y is npT_i , there exists a pT_i -open cover \mathcal{U} of Y such that $\text{st}(f(x), \mathcal{U}) \subset v$, since f is nearly pT_i -continuous there exists a τ_k -open neighborhood of x such that $f(U) \subset \text{st}(f(x), \mathcal{U}) \subset v$, thus f is p -continuous.

Remark 3.1.8

A p -continuous mapping $f: X \rightarrow Y$ where Y is an npT_i -space need not be nearly pT_i -continuous, which follows from a bellow example.

Example 3.1.9

Let $X = \{1, 2, 3\}$, $\tau_1 = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}\}$, $\tau_2 = \{\emptyset, X, \{3\}, \{2, 3\}\}$ and $Y = \{a, b, c\}$, $\mathcal{U}_1 = \{\emptyset, Y, \{a\}\}$, $\mathcal{U}_2 = \{\emptyset, Y, \{b, c\}\}$ define the mapping

$f : X \rightarrow Y$ by $f(1) = a$, $f(2) = c$, $f(3) = b$ then Y is npT_i -space, f is p -continuous but not nearly pT_i -continuous.

Corollary 3.1.10

- (a) Let $f: X \rightarrow Y$ be a nearly pT_1 -continuous mappings and Y be a PR_0^* space. Then f is p -continuous.
- (b) Let $f: X \rightarrow Y$ be a nearly pT_i -continuous mapping and Y be a pT_i -space then f is ρ -continuous.

Proof

- (a)- Since every PR_0 space is npT_1 -space, then by proposition (3.1.6) we get that f is p -continuous.
- (b)- Since Y is pT_i -space so Y is npT_i -space and by proposition (3.1.6) we get that f is p -continuous.

Corollary 3.1.11

- (a) Let $f: X \rightarrow Y$ be a pT_i - continuous mapping and Y be an pT_i -space, then f is ρ -continuous and f is nearly pT_i -continuous mapping and f is ρ -continuous.
- (b) Let $f: X \rightarrow Y$ be a pT_i - continuous mapping and Y be a pT_i -space the f is ρ -continuous and f is nearly pT_i -continuous.

Proof

- (a)- Since f is pT_i - continuous so f is nearly pT_i -continuous .
Since Y is pT_i -space so Y is npT_i – space and, by proposition (3.1.7) hence f is ρ -continuous.

Proposition 3.1.12 [7]

Let (X, τ_1, τ_2) be a bitopological space and (Y, ρ_1, ρ_2) be a $pT(i,k)$, for $i \in \{1,2,3\}$ and let $f: X \rightarrow Y$ be a nearly pT_i -continuous mapping .
Then, f is a pT_k - continuous.

Proof:

Let (X, τ_1, τ_2) be a bitopological space and let $f: X \rightarrow Y$ be a nearly pT_i -continuous.

Let \mathbf{u} , be a nearly pT_k – open cover of Y and let $x \in X$

Now, since Y is $PT(i,k)$ -space, then there exists pT_i -open cover \mathbf{v} of Y and $U \in \mathbf{u}$ such that $st(f(x), \mathbf{v}) \in U$

Since f is nearly pT_i -continuous, there is a τ_j -open neighborhood $W \in X$ of x such that

$f(W) \subset st(f(x), \mathbf{v}) \subset U$ for $j \in \{1,2\}$, thus f is pT_k -continuous .

Definition 3.1.13 [17]

A bitopological space (X, τ_1, τ_2) is called pair wise connected, if , and only if , X can not be expressed as the union of two non-empty disjoint sets A and B such that

$(A \cap \text{cl}_{\tau_1} (B)) \cup (B \cap \text{cl}_{\tau_2} (A)) = \emptyset$, and it is denoted by p - connected space.

Theorem 3.1.14

Let (X, τ_1, τ_2) and (Y, ρ_1, ρ_2) be a bitopological spaces and (X, τ_1) (X, τ_2) are connected spaces .If (X, τ_1, τ_2) is p -connected space and $f: X \rightarrow Y$ be a nearly pT_3 -continuous mappings then (Y, ρ_1, ρ_2) is p -connected space.

Remark 3.1.15

Let $f: X \rightarrow Y$ be a nearly pT_3 -continuous mapping and let X be p -connected space then Y need not be p -connected.

Example 3.1.16

Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$, $\tau_2 = \{\emptyset, X, \{b\}\}$ and

$Y = \{1, 2, 3\}$, $\rho_1 = \{\emptyset, Y, \{2\}\}$, $\rho_2 = \{\emptyset, Y, \{3\}, \{1, 3\}\}$

Define $f: X \rightarrow Y$ by

$f(b) = 1$ and $f(a) = f(c) = 3$ then f is nearly pT_3 -continuous , X is p -connected

(X, τ_1) , (X, τ_2) are connected , but Y is not p -connected

§ 3.2. Semi-nearly pTi-continuous mappings

The aim of this section is to study a semi nearly PT_i – continuous mapping and give a relation between it and a semi connected space in bitopological spaces.

Definition 3.2.1 [16]

A mapping $f : X \rightarrow Y$ is semi-continuous if, and only if, f the inverse image of every open set is semi-open , and it is denoted by S-continuous .

Theorem 3.2.2 [16]

A mapping $f: X \rightarrow Y$ is semi-continuous if, and only if, f the inverse image of every closed set is semi-closed .

Proof:

\Rightarrow Suppose $f : X \rightarrow Y$ is S-continuous and G is closed set in Y .

To prove that $f^{-1}(G)$ is semi-closed in X

Since G is closed set in Y

So $Y-G$ is open set in Y . Now since

f is semi-continuous, so we get that

$f^{-1}(Y-G) = f^{-1}(Y) - f^{-1}(G) = X - f^{-1}(G)$ is semi-open , therefore $f^{-1}(G)$ is semi-closed .

\Leftarrow Suppose the inverse image of every closed set in Y is semi closed set in X , G is closed set in Y and $f^{-1}(G)$ is semi-closed in X .

Now let H be a semi open set in Y so $Y-H$ is semi closed in Y hence $f^{-1}(Y-H)$ is semi closed in X , but $f^{-1}(Y-H) = f^{-1}(Y) - f^{-1}(H) = X - f^{-1}(H)$ is semi closed therefore $f^{-1}(H)$ is semi open hence f is S-continuous map .

Definition 3.2.3 [16]

A mapping $f: X \rightarrow Y$ is called irresolute if f the increase image of every semi-open set is semi-open, equivalently, if f the increase image of every semi-closed set is semi-closed.

Remark 3.2.4 [16]

Every continuous mapping is S-continuous but the converse need not be true.

Remark:

The concept of continuous mappings and irresolute mappings are independent, and every irresolute is S-continuous mappings but not conversely.

Remark 3.2.5[3]

The composite of any two S-continuous mappings is not S-continuous but this can hold if either of them is irresolute mapping.

Proof

Let $f : X \rightarrow Y$ is S-continuous and $g: Y \rightarrow Z$ is S-continuous
Let G be open set in $Z \Rightarrow g^{-1}(G)$ is S-open in Y
 $f^{-1}(g^{-1}(G))$ is semi-open (where f is irresolute), but
 $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ which mean $g \circ f$ is S-continuous.

Definition 3.2.6 [7]

A function $f: X \rightarrow Y$ is called semi-nearly continuous and a point $x \in X$ if for every neighborhood U of $f(x)$ in Y , $f^{-1}(U)$ is a semi-neighborhood of x in X .

Definition 3.2.7 [16]

A mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ is said to be semi-p-continuous if f , the induced maps
 $f: (X, \tau_1) \rightarrow (Y, \rho_1)$ and $f: (X, \tau_2) \rightarrow (Y, \rho_2)$ are S-continuous, and it is denoted by S-p-continuous.

Remark 3.2.8[5]

The composite of any two S-p-continuous mapping is not composite of any two S-continuous
Since the composite of any two S-continuous mapping is not S-continuous as remark (3.2.5).

Definition 3.2.9

A function f from a bitopological space X into a bitopological space Y is called semi T_i -pair wise continuous if for every SpT_i -open cover v of Y there exists a semi τ_k - open cover ω of X such that for every $W \in \omega$ there is a $v \in V$ such that $f(W) \subset v$, where $k \in \{1,2\}$ and $i \in \{1,2,3\}$, and it is denoted by SPT_i - continuous mapping.

Definition 3.2.10

A mapping $f:(X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ is said to be semi - nearly pT_i -continuous at point $x \in X$, if for every SPT_i -open cover \mathcal{U} of Y there exists τ_k -s-open neighborhood $V \in X$ of x such that
 $f(V) \subset st(f(x), \mathcal{U})$, and it is denoted by $SNPT_i$ -continuous mapping.

Note:

A mapping f is SNPT_i -continuous if it is SNPT_i -continuous each point of X .

Proposition 3.2.11

Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ by SNPT_i -continuous mapping and (Y, ρ_1, ρ_2) be SNPT_i -space . then f is S-p-continuous.

Proof:

Let $v \in \mathcal{U}_k$ be a semi-neighborhood of $f(x) \in Y$ since Y is SNPT_i -space

\Rightarrow There exists a SPT_i -open cover \mathcal{U} of Y such that $\text{st}(f(x), \mathcal{U}) \subset V$, since f is SNPT_i -continuous , there exists a semi τ_k -open neighborhood of x such that $f(v) \subset \text{st}(f(x), \mathcal{U}) \subset V$, then f is S-p-continuous.

Corollary 3.2.12

(a)- Let $f: X \rightarrow Y$ be SNPT_1 -continuous mappings and Y be a SPR_0 -space then f is S-P-continuous.

(b)- Let $f: X \rightarrow Y$ be SNPT_2 -continuous mappings and Y be a SPT_i -space then f is S-P-continuous.

Proof

(a) -Since every SPR_0 -space is SNPT_1 -space then by proposition (3.2.11) $\Rightarrow f$ is S-P-continuous.

(b)- Since Y is SPT_i -space $\Rightarrow Y$ is SNPT_i -space and by proposition (3.211) $\Rightarrow f$ is SP-continuous.

Corollary 3.2.13

(a)- Let $f: X \rightarrow Y$ be a SPT_i -continuous mapping and Y be a SNPT_i -space, the f is S-P-continuous.

(b)- Let $f : X \rightarrow Y$ be a SPT_i -continuous and Y be a SPT_i -space then f is S-P-continuous.

Proposition 3.2.14

Let Y be a $\text{SPT}(i,k)$ - space , for $i \in \{1,2,3\}$ and let $f : X \rightarrow Y$ be a SNPT_i -continuous mapping then f is SPT_k -continuous.

Proof :

Let X be a bitopological space and let $f: X \rightarrow Y$ be $SNPT_i$ -continuous let U be SPT_k -open cover of Y and let $x \in X$, since Y is $SPT(i,k)$ -space . Then exists SPT_i -open cover \mathcal{U} of Y and $U \in \mathcal{U}$ such that $st(f(x), \mathcal{U}) \subset U$ since f is $SNPT_i$ -continuous , there is τ_i -semi-open neighborhood $W \in X$ of x such that $f(W) \subset st(f(x), \mathcal{U}) \subset U$ for $i \in \{1,2\}$, thus f is SPT_k -continuous.

Definition 3.2.15

A space X is said to be semi-connected between A and B iff, there exists no semi-open set subset F of X such that $A \subset F$ and $F \cap B = \emptyset$ or there is no semi-closed subset F of X such that $A \subset F$ and $F \cap B = \emptyset$, it is also a symmetric relation and it is denoted by S-connected.

Theorem 3.2.16

If space X is S-continuous between A and B then $A \neq \emptyset, B \neq \emptyset$.

Proof

If $A = \emptyset$, then \emptyset being a semi-closed and semi-open subset of X , X cannot be S-connected between A and B , this prove the theorem.

Theorem 3.2.17

If X is S-connected between A and B and if $A \subset A_1$ and $B \subset B_0$ then X is S-connected between A_1 and B_1 .

Proof

Suppose X is not S-connected between A_1 and B_1 then there is a semi-open subsets for X such that $A_1 \subset F$ and $F \cap B_1 = \emptyset$, consequently , X is not S-connected between A and B .

Definition 3.2.18

A bitopological space (X, τ_1, τ_2) is called semi-pair wise connected space iff X can not be expressed as the union of two non-empty disjoint set A and B such that A is τ_1 -semi open and B is τ_2 -semi open and it is denoted by S-P-connected space.

Theorem 3.2.19

Let $f: X \rightarrow Y$ be a $SNPT_3$ -continuous mapping and let X be S-P-connected space, then Y need not be S-P-connected.

Example 3.2.20

Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{c\}, \{c, b\}\}$, $\tau_2 = \{\emptyset, X, \{b\}\}$ and
 $Y = \{1, 2, 3\}$, $\mathcal{U}_1 = \{\emptyset, Y, \{2\}\}$, $\mathcal{U}_2 = \{\emptyset, Y, \{2\}, \{1, 2\}\}$

Defined $f : X \rightarrow Y$ by $f(b) = 1$ and $f(a) = f(c) = 2$

Then f is $SNPT_3$ -continuous, X is s-p-connected (X, τ_1) , (X, τ_2) are connected, but Y is not S-P-connected

§ 3.3. Regular-nearly pT_i -continuous mappings

The aim of this section is to study a regular nearly PT_i – continuous mapping and give a relation between it and a regular connected space in bitopological spaces.

Definition 3.3.1 [16]

A mapping $f : X \rightarrow Y$ is regular-continuous if, and only if, f the inverse image of every open set is regular-open , and it is denoted by R-continuous .

Theorem 3.3.2 [16]

A mapping $f: X \rightarrow Y$ is regular-continuous if, and only if, the inverse image of every closed set is regular-closed .

Proof:

\Rightarrow Suppose $f : X \rightarrow Y$ is R-continuous and G is closed set in Y .

To prove that $f^{-1}(G)$ is regular-closed in X

Since G is closed set in Y , so $Y-G$ is open set in Y . Now since

f is regular-continuous, so we get that

$f^{-1}(Y-G) = f^{-1}(Y) - f^{-1}(G) = X - f^{-1}(G)$ is regular-open therefore $f^{-1}(G)$ is regular-closed .

\Leftarrow Suppose the inverse image of every closed set in Y is regular closed set in X , G is closed set in Y and $f^{-1}(G)$ is regular-closed in X .

Now let H be a regular open set in Y so $Y-H$ is regular closed in Y hence $f^{-1}(Y-H)$ is regular closed in X , but $f^{-1}(Y-H) = f^{-1}(Y) - f^{-1}(H) = X - f^{-1}(H)$ is regular closed therefore $f^{-1}(H)$ is regular open, hence f is R-continuous map .

Definition 3.3.3 [16]

A mapping $f: X \rightarrow Y$ is called irresolute if the inverse image of every regular-open set is regular-open, equivalently, if f the increase image of every regular-closed set is regular-closed.

Remark 3.3.4 [16]

Every continuous mapping is R-continuous but the converse need not be true.

Note:

The concept of continuous mappings and irresolute mappings are independent and every irresolute is R-continuous mappings but not conversely.

Remark 3.3.5

The composite of any two R-continuous mappings is not R-continuous but this can be hold if either of them is irresolute mapping.

Proof

Let $f : X \rightarrow Y$ is R-continuous and $g : Y \rightarrow Z$ is R-continuous
Let G is open set in $Z \Rightarrow g^{-1}(G)$ is R-open in Y
 $f^{-1}(g^{-1}(G))$ is regular-open (where f is irresolute), but
 $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ which mean $g \circ f$ is R-continuous .

Definition 3.3.6 [7]

A mapping $f : X \rightarrow Y$ is called regular-nearly continuous and a point $x \in X$ if for every neighborhood U of $f(x)$ in Y , $f^{-1}(U)$ is a regular-neighborhood of x in X .

Definition 3.3.7 [16]

A mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ will said to be regular-p-continuous if f , the induced maps.
 $f : (X, \tau_1) \rightarrow (Y, \rho_1)$ and $f : (X, \tau_2) \rightarrow (Y, \rho_2)$ are R-continuous , and it is denoted by R-p-continuous.

Remark 3.3.8

The composite of any two R-p-continuous mapping is not composite of any two R-continuous
Since the composite of any two R-continuous mapping is not R-continuous as remark (3.2.5) .

Definition 3.3.9

A mapping f from a bitopological space X into a bitopological space Y is called regular T_i -pair wise continuous if for every regular pT_i -open cover ν of Y there exists a regular τ_k - open cover ω of X such that for every $W \in \omega$ there is a $v \in \nu$ such that $f(W) \subset v$, where $k \in \{1,2\}$ and $i \in \{1,2,3\}$, and it is denoted by SPT_i - continuous mapping.

Definition 3.3.10

A mapping $f:(X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ is said to be regular – nearly pT_i -continuous at point $x \in X$, if for every SPT_i -open cover \mathcal{U} of Y there exists τ_k - R -open neighborhood $V \in X$ of x such that $f(V) \subset st(f(x), \mathcal{U})$, and it is denoted by $SNPT_i$ -continuous mapping.

Note:

A mapping f is $SNPT_i$ -continuous if it is $SNPT_i$ -continuous each point of X .

Proposition 3.3.11

Let $f:(X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ be $SNPT_i$ -continuous mapping and (Y, ρ_1, ρ_2) be $SNPT_i$ -space. then f is R - p -continuous.

Proof:

Let $v \in \mathcal{U}_k$ be a regular-neighborhood of $f(x) \in Y$ since Y is $SNPT_i$ -space

\Rightarrow There exists a SPT_i -open cover \mathcal{U} of Y such that $st(f(x), \mathcal{U}) \subset V$, since f is $SNPT_i$ -continuous, there exists a regular τ_k -open neighborhood of x such that $f(V) \subset st(f(x), \mathcal{U}) \subset V$, then f is R - p -continuous.

Corollary 3.3.12

(a)- Let $f: X \rightarrow Y$ be $SNPT_1$ -continuous mappings and Y be a SPR_0 -space then f is R - P -continuous.

(b)- Let $f: X \rightarrow Y$ be $SNPT_2$ -continuous mappings and Y be a SPT_i -space then f is R - P -continuous.

Proof

(a) -Since every SPR_0 -space is $SNPT_1$ -space then by proposition (3.2.11) $\Rightarrow f$ is R - P -continuous.

(b)- Since Y is SPT_i -space $\Rightarrow Y$ is $SNPT_i$ -space and by proposition (3.2.11) $\Rightarrow f$ is SP -continuous.

Corollary 3.3.13

(a)- Let $f: X \rightarrow Y$ be a SPT_i -continuous mapping and Y be a $SNPT_i$ -space, the f is R - P -continuous.

(b)- Let $f: X \rightarrow Y$ be a SPT_i -continuous and Y be a SPT_i -space then f is R - P -continuous.

Proposition 3.3.14

Let Y be a $SPT(i,k)$ -bitopological space , for $i \in \{1,2,3\}$ and let $f : X \rightarrow Y$ be a $SNPT_i$ -continuous mapping then f is SPT_k -continuous.

Proof :

Let X be a bitopological space and let $f: X \rightarrow Y$ be $SNPT_i$ -continuous let U be SPT_k -open cover of Y and let $x \in X$, since Y is $SPT(i,k)$ -space . Then exists SPT_i -open cover \mathcal{U} of Y and $U \in \mathcal{U}$ such that $st(f(x), \mathcal{U}) \subset U$ since f is $SNPT_i$ -continuous , there is τ_i -regular-open neighborhood $W \in X$ of x such that $f(W) \subset st(f(x), \mathcal{U}) \subset U$ for $i \in \{1,2\}$, thus f is SPT_k -continuous.

Definition 3.3.15

A space X is said to be regular-connected between A and B iff, there exists no regular-open set subset F of X such that $A \subset F$ and $F \cap B = \emptyset$ or there is no-regular-closed subset F of X such that $A \subset F$ and $F \cap B = \emptyset$, it is also a symmetric relation and it is denoted by R -connected.

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If space X is R -continuous between A and B then $A \neq \emptyset, B \neq \emptyset$.

Proof

If any $A = \emptyset$, then \emptyset being a regular-closed or regular-open subset of X , X cannot be R -connected between A and B , this complete prove.

Theorem 3.3.17

If X is R -connected between A and B and if $A \subset A_1$ and $B \subset B_1$ then X is R -connected between A_1 and B_1 .

Proof

Suppose X is not R -connected between A_1 and B_1 then there is a regular-open subset for X such that $A_1 \subset F$ and $F \cap B_1 = \emptyset$, consequently , X is not R -connected between A and B .

Definition 3.3.18

A bitopological space (X, τ_1, τ_2) is called regular-pair wise connected space iff X can not be expressed as the union of two non-empty disjoint set A and B such that A is τ_1 -regular open and B is τ_2 -regular open and it is denoted by R-P-connected space.

Remark 3.3.19

Let $f : X \rightarrow Y$ be a $SNPT_3$ -continuous mapping and let X be R-P-continuous space, then Y need not be R-P-connected.

Example 3.3.20

Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{c\}, \{c, b\}\}$, $\tau_2 = \{\emptyset, X, \{b\}\}$ and $Y = \{1, 2, 3\}$, $\mathcal{U}_1 = \{\emptyset, Y, \{2\}\}$, $\mathcal{U}_2 = \{\emptyset, Y, \{2\}, \{1, 2\}\}$ Defined $f : X \rightarrow Y$ by $f(b) = 1$ and $f(a) = f(c) = 2$

Then f is $SNPT_3$ -continuous, X is R-p-connected (X, τ_1) , (X, τ_2) are connected, but Y is not R-P-connected

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