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On The Riesz Spaces

A thesis
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Abstract

In order to draw a meaningful picture in our minds for the Riesz spaces and prepare some information that we need in our work and motivate our result. We have to know the point of origin for the theory of Riesz Spaces, recall some definitions and results related to the basic concepts of this thesis.

The first result that we obtain in this work is "If a Riesz Spaces is Dedekind complete or order separable then we prove that any ideal of Riesz space has the same property".

And we prove a known remark that says "The linear map is isomorphism if this linear map and its inverse are positive ".

Now we prove by a simple example that the converse of the theorem that say " If $x_n \uparrow$ and $\| x- x_n\| \Rightarrow \theta$

then $x = \sup x_n$ " is impossible.

Also throught out the generalization of the Hahn-Banach theorem by interchanging the range in the classical Hahn-Banach theorem by Dedekind complete Riesz spaces we find and prove some important results that related by the Hahn-Banach theorem.

Finally we get a new result of the classical Cauchy-Schwarz Inequality.

List of symbols

Symbols	Mean
R	The set of real numbers
R^n	The Euclidean space
R_+	$=[٠,\infty)$
$u_\alpha \to u$	Order convergence
$u_\alpha \uparrow u$	Increasing net with supremum
C[a,b]	Space of real valued continuous function on [a,b]
$\mid x \mid$	Absolute value of x
$\|x\|$	Norm of x
$(X,\|.\|)$	Normed space
sup	Least upper bound
inf	Greatest lower bound
٠	Zero element
θ	Zero vector
◈	The end of proof

List of definitions

Chapter One:

Introduction and preliminaries

§ ١.١. The origin point of the theory of Riesz spaces

Our interest in the theory of Riesz spaces stems from its beauty its utility and its rich history. The authenticity of the Riesz spaces due to Riesz F. in ١٩٢٨ at the international mathematical congress in Bolo, Riesz F. in a short address [٩] triggered the investigation of what is today called the Riesz spaces.

In mid-thirties, Freudenthal H. [٦] and Kantorvich L.V. [٧,٨] independently set-up the axiomatic foundation and derived a number of properties dealing with the lattice structure of Riesz spaces . In the forties and early fifties the Japanese school, led by Nakano H., Ogasaware T. and Yosida K., and the Russian school , led by Kantrovich L.V., Judin A.I. and Vulikh B.Z., made fundamental contributions on a Riesz spaces .At the same time a number of books started to appear in this field. Most of the early work on Riesz spaces, as well as most of books, were devoted to the so-called "algebraic part" of the theory and little attention was given to the "analytic part".

The bulk of the existing work on the analytic part is in the area of normed Riesz spaces, and in particular in the theory of Banach lattice. The work of Luxemburg W.A.J. and Zaanen A.C. appeared in a series of articles [١٦] and the book of Schaefer H.H. [١٠]. The general theory of topological Riesz spaces seems some how to have neglected. The book by Fremlin D.H. [٥] devoted to this subject.

The thesis is organized as follows :

In chapter one we give the basic definitions and facts concerning the set theory, linear and topological spaces. This chapter consists of four sections. In § ١.١. the origin point of the theory of Riesz space is given. In § ١.٢. the necessary for subsequent account basic definitions , notations and result concerned with the set theory , linear and topological spaces is given . In § ١.٣. the definition of Riesz spaces and some important properties that's we needed in this work are cited. In § ١.٤. several examples about Riesz space are considered.

Chapter two deals with the lattice structure of the Riesz spaces. A series of properties of order completeness and order transformation of Riesz spaces are obtained. This chapter consists of three sections. In § ٢.١. the order completeness properties and some results are obtained. In § ٢.٢. the order bounded transformation of the Riesz spaces is considered. in § ٢.٣. the normed Riesz spaces and some properties are given .

Chapter three is devoted to study some theorems in the theory of Riesz space and we introduced a new result in this field. In § ٣.١. we generalized the Hahn Banach Theorem for the Riesz spaces. In § ٣.٢. we prove some important properties as results of the Hahn-Banach theorem.

In the last section we proved the main corollaries concerning the classical Cauchy –Schwarz inequality.

§ ١.٢. Technical result from the set theory, relations, linear spaces structures and topological spaces.

In this section series of known notations and facts about the set theory, topological and linear spaces are cited.

Definition ١.٢.١[١١]

Let X,Y are linear spaces over the same field of scalar F

A mapping $T:X \longrightarrow Y$ is said to be ***linear*** if

(i) T is additive: $T(x+y) = T(x)+T(y) \quad \forall\, x,y \in X$

(ii) T is homogenous: $T(\alpha x) = \alpha T(x) \quad \forall\, x \in X, \alpha \in F$

Definition ١.٢.٢[١١]

Let X,Y are linear spaces over the same field of scalar F, a mapping $T:X \to Y$ is said to be ***sublinear*** if

(i) $T(x+y) \leq T(x) + T(y) \quad \forall\, x,y \in X$

(ii) $T(\alpha x) = \alpha T(x) \quad \forall\, x \in X, \forall\, \alpha \in F$

Definition ١.٢.٣[٢]

Let X ,Y and Z be a linear spaces and f is a map from XxY into Z . Associate to each $a \in X$ and each $b \in Y$ the mapping $f_a:Y \longrightarrow Z$, $f_b:X \longrightarrow Z$ be defining $f_a(b)=f(a,b)=f_b(a)$,f is said to be ***bilinear*** , if f_a and f_b are linear .

Example ١.٢.٤

Let X be the linear space of n-tuples $x=(\lambda_1,\ldots\ldots,\lambda_n)$, $y=(\mu_1,\ldots\ldots.,\mu_n)$ over a field of scalar F such that λ_i , $\mu_i \in F$ $i=1, \ldots,n$. Define $\Phi:X\text{x}X \longrightarrow X$ by $\Phi(x,y)=\sum\lambda_k\mu_k$.

It is easy to see that, Φ is bilinear map.

Definition ١.٢.٥[٢]

Let A be a nonempty subset of a linear space X. The intersection of the family of all the linear subspaces of X that contain A is called ***linear hull*** of A.

Definition ١.٢.٦[٢]

A relation $\leq$ on a set X is called partial order on X if satisfy the following conditions:

(i) Reflexive, i.e $x\leq x \quad \forall x \in X$;

(ii) Antisymmetric, i.e. $x\leq y$ and $y\leq x$ implies $x=y$;

(iii) Transitive, i.e. $x\leq y$ and $y\leq z$ implies $x\leq z$.

We shall say that a partial order relation $\leq$ on a set X is linear order or (total order), if and only if, for each pair $x,y \in X$, we have either $x\leq y$ or $y\leq x$, when $\leq$ is linear order relation on X , we shall say that X is linearly ordered by $\leq$.

Definition ١.٢.٧[٢]

An element x of a set X that is partially ordered by ≤ is ***maximal element*** of X if $y \in X$ and $x \leq y$ imply $x=y$

Theorem ١.٢.٨[٢] "Zorn's Lemma"

Let X be a non-empty partially ordered set with property that each non-empty totally ordered subset of X has an upper bound, then X has a maximal element.

Definition ١.٢.٩[١٢]

A ***sequence*** in a set X is a mapping S of natural number N into X. It is customary to denote the sequence S by $<S_n>$, $n \in N$.

Theorem ١.٢.١٠[١١]

Let $x=<x_n>$ and $y=<y_n>$ be sequences of scalars, such that $\lim x_n=x_o$ and $\lim y_n=y_o$, where x_o and y_o are scalars and let α be any scalar, then

$$(i) \lim_{n\to\infty}(x_n + y_n) = x_o + y_o$$

(ii) $x_n < y_n \; \forall \; n \in N \;\; \Rightarrow \;\; x_o \leq y_o$

Definition ١.٢.١١[١٢]

Let A be a non-empty set. We say that a binary relation $\geq$ on A directs the set A if

(i) $a \in A \Rightarrow a \geq a$;

(ii) $a \geq b$ and $b \geq c \Rightarrow a \geq c$ $(a,b,c \in A)$

(iii) For any two members a and b of A. There exist a member $c \in A$, such that $c \geq a$ and $c \geq b$. The pair $(A,\geq)$ is then called ***directed set***.

Definition ١.٢.١٢ [١٢]

A ***net*** in X is a mapping u of a direct set A into a set X and is denoted by $<u_\alpha>$, $\alpha \in A$.

§ ١.٣. The elementary properties of the Riesz Spaces

In this section, we shall define the Riesz Spaces also we offer an elementary facts and properties of Riesz spaces that we need in this work. For more details we refer to [٤, ١٤, ١٥].

Definition ١.٣.١ [٣]

Let X be a linear space. An ordering $\leq$ is called linear

if $\forall$ x, y, z $\in$ X.

(i) $x\leq y$ implies $x+z\leq y+z$;

(ii) $x\leq y$ implies $\alpha x\leq\alpha y$ $\forall$ $\alpha\in R_{+}=[٠,\infty)$.

A real linear space together with the linear ordering is called an ordered linear space, an ordered linear space is called (***Riesz Spaces or linear lattice***) if the supremum and infimum of every finite non empty subset of X exists .

Now let us introduce classical lattice notations. We denote the supremum of the set $\{x,y\}$ by $x \vee y$, which means that $x \vee y = \sup\{x,y\}$. Similarly $x\wedge y$ denotes the infimum of the set $\{x, y\}$ that is $x\wedge y=\inf\{x, y\}$.

The linear lattices are also a well studied object. Their properties are detailed accounted in [٧, ١٣] for instance. We denoted the zero element of a linear space X by θ.

The element x of X with $x \geq \theta$ are called ***positive element***, and those satisfying $x > \theta$ are called strictly positive elements.

The set $X^+=\{x \in X: x \geq \theta\}$ is called the ***positive cone*** of X.

The positive cone of X satisfies the following properties:

(i) $X^+ + X^+ \subseteq X^+$, where $X^+ + X^+ = \{x+y: x, y \in X^+\}$

(ii) $\alpha X^+ \subseteq X^+$ for $٠ \leq \alpha \in R$ where $\alpha X^+=\{\alpha x: x \in X^+\}$.

(iii) $X^+ \cap (-X^+)=\{\theta\}$ where $-X^+ =\{-x: x \in X^+\}$.

Any subset A of real linear space X satisfying properties (i), (ii) and (iii) is called a ***cone*** of X.

If A is a cone of X, note that the relation $x \geq y$ whenever $x-y \in A$ makes X an order linear space whose positive cone is precisely A.

For an element x of a Riesz Space X the Positive part of x is defined by $x^+ = x \vee \theta$, the negative part of x is defined by $x^- = (-x) \vee \theta$ and the absolute value by $|x| = x \vee (-x)$ .For fundamental relations and additional ones see [١,١٣,١٤,١٥].

Theorem ١.٣.٢ [١]

If x, y and z are elements of a Riesz space X, then we have

١. $x+y = x \vee y + x \wedge y$;

٢. $x=x^+-x^-$ and $x^+ \wedge x^- = \theta$;

٣. $x \vee y =(x-y)^+ + y = (y-x)^+ - x$;

٤. $|x| = x^+ + x^-$ (hence $|x| = \theta$, if and only if $x= \theta$) ;

٥. $||x| - |y|| \leq |x+y| \leq |x| + |y|$ (the triangle inequality);

٦. $x+y \vee z = (x+y) \vee (x+z)$ and $x+y \wedge z = (x+y) \wedge (x+z)$

٧. $\alpha (x \vee y) = (\alpha x) \vee (\alpha y)$ and $\alpha (x \wedge y) = (\alpha x) \wedge (\alpha y)$ if $\alpha \geq$ ٠

٨. $|\alpha x| = |\alpha||x|$ for all $\alpha \in R$;

٩. $x-y \wedge z = (x-y) \vee (x-z)$ and $x-y \vee z=(x-y) \wedge (x-z)$

١٠. $|x-y| = x \vee y - x \wedge y$

١١. $|x \vee z - y \vee z| + |x \wedge z - y \wedge z| = |x-y|$ (Birkhoff's identity)

١٢. $|x \vee z - y \vee z| \leq |x-y|$ and $|x \wedge z - y \wedge z| \leq |x-y|$ (Birkhoff's inequality)

١٣. If $x,y,z \in X^+$, then $(x+y) \wedge z \leq (x \wedge z)+(y \wedge z)$.

Definition ١.٣.٣[١]

A subset S of a Riesz space X is said to be a solid set , if it follows from $|x| \leq |y|$ in X and $y \in S$ that $x \in S$.

Let A be a subset of X, the smallest solid set of X contain A is called a ***solid hull*** of A .

Definition ١.٣.٤ [١]

Let A be a linear subspace of a Riesz space X. We say that

(i) A is a ***Riesz subspace*** of X if any two elements x and y in A the supremum of x and y taken in X belongs to A.

(ii) A is an ***ideal*** of X, if A is solid subset of X.

by theorem (١.٣.٢)[٣] a linear subspace A of a Riesz space X is a Riesz subspace , if and only if for every $x \in A$ we have $x^+ \in A$.

Remark ١.٣.٥

Every ideal of a Riesz space is a Riesz subspace but the converse is not true.

Every subset D of a Riesz space X which is contained in a smallest ideal A is called the ideal generated by D; A is the intersection of all ideals containing D.

Definition ١.٣.٦[١]

The ideal A of a Riesz space X is called a ***Band*** if whenever a subset of A has a supremum, this supremum belongs to A.

A net $<u_\alpha>$ of a Riesz space X is called increasing (in symbols $u_\alpha\uparrow$) if $u_\beta \geq u_\alpha$ wherever the indices $\alpha,\beta \in A$ (A is directed set) satisfy $\beta \geq \alpha$.

The symbol $u_\alpha \downarrow$ denotes a decreasing net and its definition is analogous. The symbol $u_\alpha \uparrow u$ means that the net $<u_\alpha>$ is increasing and the supremum of $<u_\alpha>$ exists and equals u the meaning of $u_\alpha \downarrow u$ is defined similarly.

Definition ١.٣.٧[١٥]

A net $<u_\alpha>$, $\alpha \in A$ (A is directed set) of a Riesz spaces X is said to be ***order convergent*** to $u\in X$ (in symbol $u_\alpha \to u$) if there exists a net $<v_\alpha>$ (with the same indexed set) such that $|u_\alpha-u|\leq v_\alpha$ holds $\forall$ α and with $v_\alpha\downarrow\theta$ (we shall write the last two conditions symbolically as $|u_\alpha-u|\leq v_\alpha\downarrow\theta$)

The basic properties of order convergence are summarized in the next theorem.

Theorem ١.٣.٨[١٥]

If we, assume that the nets $<u_\alpha>$, $\alpha \in A$ (A is directed set) and $<v_\beta>$, $\beta \in A$ (A is directed set) of a Riesz space X satisfy $u_\alpha \to u$ and $v_\beta \to v$, then we have

(i) $|u_\alpha| \to |u|$; $u^+_\alpha \to u^+$ and $u^-_\alpha \to u^-$;

(ii) $\lambda u_\alpha + \mu v_\beta \to \lambda u + \mu v \quad \forall\ \lambda, \mu \in R$;

(iii) $u_\alpha \vee v_\beta \to u \vee v$ and $u_\alpha \wedge V_\beta \to u \wedge v$;

(iv) if $u_\alpha \le w$ for all $\alpha \ge \alpha_0$ then $u \le w$.

§ ١.٤. Examples

In this section, we give the following examples by which we explain that the linear space C [a,b] and the set of real numbers R with the given conditions are Riesz spaces .

(i) The linear space C[a,b] of real valued continuous functions defined on the interval [a,b] with order relation :

$f \geq g$ if $f(t) \geq g(t)$ $\forall t \in [a,b]$

This relation is partially ordered and the space of these vectors is ordered linear space, since:

$f(t) \geq g(t)$ $\forall t \in [a,b]$,

so $\forall f,g,h \in c[a,b]$, we get

$f(t) + h(t) - h(t) \geq g(t)$, then

$f(t) + h(t) \geq g(t) + h(t)$ and

$(f+h)(t) \geq (g+h)(t)$ $\forall t \in [a,b]$.

Thus $f+h \geq g+h$.

Also

$f \geq g \Rightarrow f(t) \geq g(t)$ $\forall t \in [a,b]$

$f(t) - h(t) \geq ٠ \Rightarrow$

$(f-h)(t) \geq ٠ \Rightarrow$

$\alpha (f-h)(t) \geq ٠ \Rightarrow$

$\alpha f(t) - \alpha h(t) \geq ٠ \Rightarrow$

$\alpha f(t) \geq \alpha h(t) \Rightarrow$

$\alpha f \geq \alpha g$

Now it is easy to see that $\forall f,g \in c[a,b]$ the

Supremum and infimum of f , g exist, hence C[a,b] is a Riesz space .

(ii) The set of real numbers R is a real linear space with natural ordering. We can easily show that R is a linear lattice whose positive cone is R^+.

Chapter Two

The lattice structure of Riesz spaces

§ ٢.١. Order completeness properties and some results.

In this section, some important definitions and facts about order completeness are given. The main result in this section is the proposition "If *X is a Dedekind complete or order separable, then any ideal in X has the same property*" ٢.١.٥ and for details sees [١, ١٤, ١٥].

Definition ٢.١.١[١]

A Riesz space X is said to be:

(i) Dedeking complete , if every non empty subset that is bounded from above has a supremum

(ii) σ-Dedeking complete, if every countable subset that is bounded from above has a supremum.

Remark ٢.١.٢

The sufficient condition for Dedekind completeness is that if $\theta \leq u_\alpha \uparrow \leq g$ in X, $\alpha \in$ A then the supremum of u_α exists, similarly,

it is sufficient for Dedeking σ-completeness that if $\theta \leq u_n \uparrow \leq g$ (n=١,٢,...) then the supremum of u_n exists .

Definition ٢.١.٣[١٧]

The Riesz space X is said to be ***order separable***, if for every subset S of X whose supremum exists in X, there exists an at most a countable subset of S having the same supremum as S in X.

Definition ٢.١.٤[١٧]

Any Riesz space, which is Dedeking complete and order separable is said to be ***super Dedeking complete***.

Proposition ٢.١.٥

If X is a Dedeking complete or order separable, then any ideal in X has the same property.

Proof: Let X be a Dedeking complete, and A is an ideal in X so, A is bounded above and its supremum exists, hence any subset of A is bounded above and its supremum exists. So A is Dedeking complete.

Let X is an order separable space, and B is an ideal in X so it has a supremum, and it contains at most a countable subset having the same supremum of B. Hence any subset of B possessing a supremum contains an at most a countable subset having the same supremum as a subset it self. Thus B is an order separable ◈

§ ٢.٢. Order bounded transformation on the Riesz spaces

In this section, some informations about the order bounded transformations on Riesz spaces are considered and for detail see [١, ٤, ١٤, ١٥]

Definition ٢.٢.١[١]

A linear mapping T from a Riesz space X into a Riesz space Y is called:

(i) A ***Riesz homomorphism***, if $x \wedge y = \theta$ in X implies $T(x) \wedge T(y) = \theta$ in Y ;

(ii) A ***Riesz σ-homomorphism***, if T is a Riesz homomorphism and $u_n \to \theta$ in X implies $T(u_n) \to \theta$ in Y n = (١,٢,...);

(iii) A ***normal Riesz homomorphism***, if T is a Riesz homomorphism and $u_\alpha \to \theta$ in X implies $T(u_\alpha) \to \theta$ in Y $\alpha \in A$;

(iv) A ***Riesz isomorphism (into)***, if T is bijective Riesz homomorphism.

Riesz homeomorphisms preserve the lattice structure as the following theorem.

Theorem ٢.٢.٢[١٥]

For a linear mapping $T: X \to Y$ between two Riesz spaces X and Y, the following statements are equivalent:

(i) T is a ***Riesz homomorphism***;

(ii) $T(x \wedge y) = T(x) \wedge T(y)$ $\forall\, x, y \in X$;

(iii) $T(x \vee y) = T(x) \vee T(y)$ $\forall\, x, y \in X$;

(iv) $T(x \vee y) = T(x) \vee T(y)$ whenever $x \wedge y = \theta$ holds in X

(v) $T(x^+) = [T(x)]^+$ $\forall\, x \in X$;

(vi) $T(|x|) = |T(x)|$ $\forall\, x \in X$;

Definition ٢.٢.٣[١]

A linear mapping $T: X \to Y$ between two Riesz spaces X and Y is said to be an ***order bounded linear transformation*** if T(A) is an order bounded subset of Y whenever A is an order bounded subset of X , also
A linear mapping $T: X \to Y$ is said to be ***positive*** (denoted symbolically by $T \geq \theta$) if $T(x) \geq \theta$ holds in Y whenever $x \geq \theta$ holds in X .

Remark ٢.٢.٤

Every Riesz homomorphism is positive.

Proposition ٢.٢.٥:

The linear map T from a Riesz space X into a Riesz space Y is a Riesz isomorphism, if and only if T and $T^{-١}$ are positive

Proof: since the linear map T is bijective then $T^{-١}$ is also bijective and therefore it is linear and since T is a Riesz homomorphism then $T^{-١}$ is a Riesz homomorphism and by Remark (٢.٢.٤) we find that T and $T^{-١}$ are positive ◈

Proposition ٢.٢.٦

If X is a Riesz space and F an order bound linear functional on X , then there are positive linear functionals G and H such that F=G –H

Proof: suppose that $x \geq \theta$ and let

$G(x)=\sup\{F(y):\theta \leq y \leq x\}$

For $x_1, x_2 \geq \theta$, we have $G(x_1+x_2)=G(x_1)+G(x_2)$

Since $G(x_1)+G(x_2) = \sup\{F(y_1):\theta \leq y_1 \leq x_1\}+$

$$\sup\{F(y_2) : \theta \leq y_2 \leq x_2\}$$

$$= \sup\{F(y_1+y_2):\theta \leq y_1 \leq x_1 , \theta \leq y_2 \leq x_2\}$$

$$= G(x_1+x_2) \qquad (١)$$

Also $G(\alpha x) = \sup\{F(\alpha y) \quad :\theta \leq x \leq y\}$ since F is linear, so

$$G(\alpha x) = \sup\{\alpha F(y) \quad :\theta \leq x \leq y\}$$

$$= \alpha \sup\{F(y) :\theta \leq x \leq y\}$$

$$= \alpha G(x) \qquad (٢)$$

From (١) and (٢), we note that, G is a positive linear functional on X.

Moreover $G(x) \geq F(x) \ \forall \ x \geq \theta$ and therefore H=G-F is positive on X ◈

§ ٢.٣. Normed Riesz spaces and some properties

In this section, some order and topological properties of normed Riesz space which are analogous to the corresponding properties of usual normed space are cited. Moreover we shall show that the Riesz space with this norm becomes Banach Riesz spaces. For details we refer to the books [١, ٥, ١٧].

Definition ٢.٣.١[١٧]

The Riesz space X is called a ***normed Riesz space*** , if there exists a norm in X with the property that $|x| \leq |y|$ in X implies $\|x\| \leq \|y\|$, the norm is called Riesz norm in X , if X is a norm complete with respect to a Riesz norm in X , then X is called ***Banach lattice***.

Example ٢.٣.٢

The Euclidean Riesz space R^n forms a Banach lattice when ordering is defined as follows:

$x=(x_١,x_٢,\ldots\ldots,x_n)$, $y=(y_١,y_٢,\ldots\ldots,y_n)$

$x,y \in R^n$, $x \leq y$ if and only if $x_i \leq y_i$ for i=١ ,٢,…,n .

We Know that the norm of the Euclidean space R^n is

$$\| x \| = (\sum_{i=1}^{n} | x_i |^2)^{1/2}$$

Since the condition of a norm in above definition is satisfied in R^n, then $(R^n, \| \cdot \|)$ is a normed Riesz space .

Now we want to show that R^n is a norm complete with respect to Riesz norm in R^n (the proof is given in [١١])

Let $<x_١, x_٢, \ldots, x_m, \ldots>$ be a Cauchy sequence in R^n. Since each x_m is an n-tuple of real numbers we shall write $x_m = (x_١^{(m)}, x_٢^{(m)}, \ldots, x_n^{(m)})$ so that $x_k^{(m)}$ is the K-th coordinates of x_m.

Let $\epsilon >$ ٠ be given, since $<x_m>$ is a Cauchy sequence, then there exists a positive integer m_o such that for each

$l, m \geq m_o$

We have $\|x_m - x_l\| < \epsilon$ then $\|x_m - x_l\|^٢ < \epsilon^٢$ so

$\sum |x_i^{(m)} - x_i^{(l)}| < \epsilon^٢$ (١) and so for i=١,٢,…n

We have $\|x_i^{(m)} - x_i^{(l)}\|^٢ < \epsilon^٢$ Then

$\|x_i^{(m)} - x_i^{(l)}\| < \epsilon$

This shows that the sequence $<x_i^{(m)}>_{m=١}^{\infty}$ is a Cauchy sequence of real numbers for each i .

Since R is complete, each of these sequences converges to a point say z_i in R so that

$$\lim_{m \to \infty} x_i^{(m)} = z_i \qquad i = 1, 2, \ldots, n \qquad (2)$$

We now show that the cauchy sequence $<x_m>$ converges to the point $z = (z_١, \ldots, z_n) \in R^n$

To prove this , let $l \to \infty$ in (١) , then by (٢) and by theorem (١.٢.١١) we obtain $\sum |x_i^{(m)} - z_i|^٢ < \epsilon^٢$ Then $\|x_m - z\| < \epsilon$ it follows that the Cauchy sequence $<x_m>$

converges to z in R^n and hence R^n is complete with respect to Riesz norm in R^n . Therefore $(R^n,\|.\|)$ is a Banach lattice.

Definition ٢.٣.٣[١٧]

Let $(X, \|.\|_x)$, $(Y, \|.\|_y)$ be a normed Riesz spaces. An isomorphism T from X into Y is said to be ***isometrically*** if $\|T(x)\|_y = \|x\|_x \ \forall \ x \in X$, and we say that X and Y are isometrically isomorphic .

Definition ٢.٣.٤[٤]

The Riesz norm $\|.\|$ is said to be ***σ-order continuous***, if it follows from $x_n \downarrow \theta$ (n=١,٢,…) that $\|x_n\| \downarrow$ ٠ and $\|.\|$ is called ***order continuous*** if $\|x_r\| \downarrow$ ٠ hold for any $x_r \downarrow \theta$, $r \in A$ (A is directed set).

Definition ٢.٣.٥[٧]

The normed Riesz spaces X and Y are called ***isometrically Riesz isomorphic***, if there exists a Riesz isomorphism between X and Y which is at the same time an isometry (notation $X \approx Y$).

Definition ٢.٣.٦ [١٥]

A subset A of $(X,\|.\|)$ is said to be ***dense*** in X , if for every $x\in X$ there exists a sequence $<x_n>$ in A such that $x_n \to x$, $n \in N$.

In the following theorem, we shall show that, any isomorphism between two Riesz subspaces implies to isomorphism between Riesz spaces, if the Riesz sub space has the norm dense property

Theorem ٢.٣.٧ [١٧]

The two Banach lattices X and Y satisfy $X\approx Y$ if and only if there exist normed dense subspaces A and B such that $A \approx B$

Proof : Let T be an isometric Riesz isomorphism between A and B, given $x \in X$, there exists a sequence $< x_n>$ in A such that $\|x_n-x\| \to ٠$. Since T is an isometry and Y is a norm complete, the element $\lim Tx_n$ exists in Y and depends only on x and not on the choice of the sequence $<x_n>$.

Writing $T'x = \lim Tx_n$, we thus obtain a linear isometry T', such that T' and T coincide on A. It remains to prove that T' and its inverse $(T')^{-١}$ are positive. Let $x\in X$, $x \geq \theta$, $<x_n>\subseteq A$,n=١,٢,... and $x_n\to x$ such that $|x_n| \to |x|=x$ so $T'(x)=\lim T(|x_n|) \geq \theta$ that is T' is positive . Now let $y\in Y$, $y\geq \theta$,$<y_n> \subseteq B$, n=(١,٢,...) and $y_n\to y$ such that

$|y_n| \to |y| = y$ so $(T')^{-١}(y) = \lim T'(|y_n|) \geq ٠$ that is $(T')^{-١}$ is positive ◈

Theorem ٢.٣.٨[١٧]

If $x_n \uparrow$ and $\|x_n - x\| \to ٠$ then $x = \sup x_n$

Proof: For m>n, we have by Birkhoff's inequality.

$$\theta \leq x_n - (x \wedge x_n) = (x_m \wedge x_n) - (x \wedge x_n)$$
$$\leq |x_m - x|$$

Since $\|x_m - x\| \to ٠$ as $m \to \infty$, it follows that

$\|x_n - (x \wedge x_n)\| = ٠$ so $x_n = (x \wedge x_n)$

which shows that x is an upper bound of the sequence $<x_n>$. Let y be another upper bound of the sequence, then

$$\theta \leq (x \vee y) - y = (x \vee y) - (x_n \vee y)$$
$$\leq |x_n - x|$$

Since $\|x_n - x\| \to ٠$ as $n \to \infty$, it follows that $\|x \vee y - y\| = ٠$ and this implies that $x \vee y = y$ so $x \leq y$, this shows that x is the least upper bound of $<x_n>$. ◈

In the following example we shall show that the converse of above theorem is not true

Example ٢.٣.٩

The space of all bounded sequences l_∞ with

$\| x\| = \sup | x_n|$ and the ordering defined by

$x_1 \le x_2 \le \ldots \le x_n \le \ldots$ It is clear that, $\|x_n - x\| \nrightarrow \theta$.

In the following theorem we want to show the relation between the super Dedekind complete, order continuous σ- Dedekind complete and σ- order continuous also we shall consider the case that, every bounded increasing sequence in norm Riesz space X has a norm limit.
We first prove a lemma. For the formulation, we recall the definition.

Definition ٢.٣.١٠ [١٧]

The up wards directed set $(x_r : r \in\{\tau\})$ in X is called an ***e-Cauchy net*** if, given $\in > o$, there exists an index $r_o \in \{\tau \}$ Such that $\|x_r - x_{ro}\| < \in$ for $x_r \ge x_{ro}$

Lemma ٢.٣.١١ [١٧]

(i) Given the up ward directed e- Cauchy net $< x_r >$, $r\in \{ \tau \}$ and the sequence $<\in_n>$, (n=١,٢,٣,...) of real numbers such that , $\in_n \downarrow$ ٠ , there exists an increasing subsequence $< x_{rn}>$, (n=١,٢,...) such that $\| x_r \vee x_{rn} - x_{rn}\| < \in_n \ \forall$ n and r . Furthermore, any upper bound of the sub sequence is an upper bound of the given net.

(ii) If every increasing order bounded e-Cauchy sequence in X has a norm limit , then any order bounded up wards directed e-Cauchy net $< x_r>$,$r \in \{\tau\}$ has supremum x and there exists an increasing subsequence $<x_{r_n}>$, n=١,٢,... such that x =sup x_{r_n} . Finally we have $\| x- x_r \| \to$ ٠ .

Proof : (i) since $x_r \uparrow$, there is an increasing sub sequence $<x_{r_n}>$ n=١,٢,... such that for n= ١,٢,٣,... we have $\|x_r - x_{r_n}\| < \in_n$ for $x_r \geq x_{r_n}$. For any fixed n and for any fixed r there exists an element $x_r \geq (x_r \vee x_{r_n})$ so, $\|(x_r \vee x_{r_n}) - x_{r_n}\| \leq \| x_r - x_{r_n} \| < \in_n$.

To prove the second assertion, we assume that y be an upper bound of the set $<x_{r_n}>$. We have to show that $y \vee x_r = y$ for every x. This follows from:

$$\theta \leq y \vee x_r - y \leq y \vee (x_r \vee x_{r_n}) - (y \vee x_{r_n})$$

$$\leq (x_r \vee x_{r_n}) - x_{r_n} , \quad \text{for } n = ١,٢,٣,\dots .$$

So $\|y \vee x_r - y\| =$ ٠ which implies $(y \vee x_r) = y$.

(ii) Let $<x_r >$, $r \in \{\tau\}$ be an order bounded up wards directed e- Cauchy net and let $<x_{r_n} >$, n =١,٢ ... be an increasing subsequence as in part (i). This sequence is an order bounded increasing e – Cauchy sequence so by assumption it has a norm limit x. According to theorem (٢.٣.٨), x is the supremum and by part (i) we have x= sup x_r. ◈

Now we shall prove the theorem that we reffered to it in page ٢٤.

Theorem ٢.٣.١٢ [١٧]

The following conditions for a normed Riesz space are equivalent

(i) X is super Dedekind complete and ‖. ‖ is order continuous

(ii) X is Dedekind σ- complete and ‖. ‖ is σ- order continuous

(iii) Every increasing order bounded sequence has a norm limit.

Proof: (i) ⇒(ii) is evident from the definition of super Dedekind complete and order continuous.

(ii) $\Rightarrow$ (iii) let $x_n \uparrow \leq x$. Then z =sup x_n exists by assumption that X is Dedekind σ- complete , so z- $x_n \downarrow \theta$, which implies $\|z - x_n\| \to$ ٠

(iii)$\Rightarrow$ (i) Any order bounded up wards directed set $<x_r>$ is e-Cauchy net , since other-wise there exists an increasing subsequence without a norm limit , contrary to hypothesis .from lemma ٢.٣.١١(ii) it follows that x = sup x_r exists and the net contains a countable subset having the same supremum as the net itself . Hence X is super Dedekind complete. Now suppose that $x_r \downarrow \theta$ Since we have to prove that $\|x_r\| \downarrow$ ٠ . We may assume the existence of an element x_o satisfying $x_o \geq x_r \downarrow \theta$.

Setting yr = $x_o - x_r$, we have $\theta \leq y \uparrow x_o$ .By the super Dedekind completeness of X there is a subsequence $<y_{rn}>$ (n= ١,٢,٣,…) such that $y_{rn} \uparrow x_o$. Since this sequence has a norm limit by hypothesis, it follows from theorem (٢.٣.٨) that $\|x_o - y_{rn}\| \to$ ٠ as $n \to \infty$. But then inf$\|x. - y_r\|$ = ٠ , so inf $\|x_r\|$= ٠ ◈

In Banach lattice the condition of the following theorem is equivalent to the condition in theorem ٢.٣.١٢

The next lemma is due to P. Meyer – Nieberg we shall use it in the proof of theorem ٢.٣.١٤

Lemma ٢.٣.١٣ [١٧]

Let <X_n> n=١, ٢, ٣,… be a sequence of positive elements in the normed Riesz space X, satisfying the following conditions :

(i) There exists a number Є > ٠ such that $\|x_n\| \geq$ ١+Є $\forall$ n

(ii) There exists a number Є > o such that

$$\left\| \sum_{k=1}^{n} x_k \right\| \leq \varepsilon \ \forall n$$

(iii) The sequence is order bounded i.e., $\theta \leq x_n \leq x_o$ for some $x_o \in X$ then there exists a subsequence <$x_{k١}$,$x_{k٢}$,…> and a sequence <$y_١$,$y_٢$,…> of disjoint elements such that $\theta \leq y_n \leq x_{k_n}$ and $\|y_n\| \geq$ ١ for n=١,٢,… .

Theorem ٢.٣.١٤[١٧]

For a normed Riesz space X the following conditions are equivalent.

(i) Every order bounded increasing sequence in X is e-Cauchy sequence.

(ii) For every order-bounded sequence of disjoint positive elements the norm tends to zero.

Proof: (i) $\Rightarrow$ (ii) Given an order bounded sequence $< x_n >$ disjoint positive elements.

We set

$$.S_m = \sum_{n=1}^{m} x_n$$

Then $<S_m>$, m=١,٢, … is an order bounded increasing sequence. By assumption, this is a e-Cauchy sequence. It follows that $\|x_n\| \Rightarrow ٠$ as $n \to \infty$

(ii) $\Rightarrow$ (i) let $<y_n>$ n=١,٢,…. be an order bounded increasing sequence of positive elements,

i.e. $\theta \le y_n \uparrow \le x_o$

and assume that this is not e-Cauchy sequence. Then there exists a number $\delta > ٠$ and a subsequence $<z_١ = y_{n١}, z_٢ = y_{n٢}, \ldots>$ with $n_١ < n_٢ < \ldots$. such that

$$\|z_{k+١} - z_k\| > \delta \qquad \text{for} \quad k=١,٢,\ldots$$

Let $x_k = \delta^{-١} (١+\in) (z_{k+١} - z_k)$ for k=١,٢….

Then $\theta \le x_k \le \delta^{-١} (١+\in) x_o \ \forall\, k$.

Furthermore $\|x_k\| \ge ١+\in$ and

$$\|\Sigma x_k\| = \delta^{-١} (١+\in) \|z_{m+١} - x_m\|$$

$$\le \delta^{-١} (١+\in) \| x_o\|$$

$$= C \qquad \forall m$$

The sequence $<x_١, x_٢, \ldots>$ satisfies the conditions of Meyer. Nieberg's lemma, so there exists a disjoint element sequence <$p_١,p_٢,p_٣$……> of positive elements ,majored by $\delta^{-١}(١+\epsilon)x_o$ and satisfying $\|p_n\| \geq ١$ for every n , This contradicts our hypothesis.

Hence <$y_١$, $y_٢$,...> is a e-Cauchy sequence ◈

CHAPTER THREE

Some Theorems in the Theory of Riesz Spaces

This chapter deals with a modified Hahn-Banach theorem. Some properties of this theorem are also given.
In additional, new results concerning Cauchy-Schwarz inequality are included.

§ ٣.١. Hahn-Banach Theorem

Theorem ٣.١.١[٢] [Classical Hahn-Banach theorem]

Let p be a sublinear functional on a linear space X, let M be a linear subspace of X and let f_o be a linear functional on M for which $f_o(x) \leq p(x) \forall x \in M$. Then there exists a linear functional f on X such that

a) $f(x) \leq p(x) \; \forall x \in X$ b) $f(x) = f_o(x) \; \forall x \in M$.

Now to prove the new Hahn Banach theorem, we must prove the following lemma

Lemma ٣.١.٢

Let X be a linear space and M its subspace for $x_o \in X-M$, let $N = [M \cup \{x_o\}]$. Furthermore, suppose that f is a linear mapping on M to the Dedeking complete Riesz space F and ρ is a sublinear mapping on X such that $f(x) \le \rho(x)$ $\forall x \in M$ then there exists a linear mapping h defined on N such that $h(x) = f(x)\ \forall\ x \in M$ and $h(x) \le \rho(x)\ \forall\ x \in N$.

Proof: since $f(x) \le \rho(x)$ for all $x \in M$ and f is linear we have for arbitrary $y_1, y_2, \in M$,

$$f(y_1) + f(y_2) = f(y_1+y_2) \le \rho(y_1+y_2) \qquad \text{or}$$

$$f(y_1) + f(y_2) \le \rho(y_1 + x_o + y_2 - x_o)$$

$$\le \rho(y_1 + x_o) + \rho(y_2 - x_o).$$

So $f(y_2) - \rho(y_2 - x_o) \le \rho(y_1 + x_o) - f(y_1)$ (١)

Let $A = \{f(y_2) - \rho(y_2 - x_o) : y_2 \in M\}$

$B = \{\rho(y_1 + x_o) - f(y_1) : y_1 \in M\}$

Inequality (١) shows that $\alpha \le \beta\ \forall\ \alpha \in A$ and $\beta \in B$.

Thus A is bounded above , B is bounded below and $\sup A \le \inf B$ in the Dedeking complete Riesz space F suppose ν be an element in F with

$\sup A \le \nu \le \inf B$ (2)

It may be observed that if $\alpha = \beta$ then $\nu = \alpha = \beta$

Therefore; for all $y \in M$

$f(y) - \rho(y - x_o) \le \nu \le \rho(y + x_o) - f(y)$ (٣)

From definition of N, it is clear that, every element x of N can be Written as

$x = y + \lambda x_0$ (٤)

Where $x_0 \in X - M$, λ is scalar and x are uniquely determined element in M.

We now define a mapping h on N as follows:

$h(x) = h(y+\lambda x_o) = f(y) + \lambda\nu.$ (٥)

where ν is given in (٢) and x given in (٤).

We shall now verify that the mapping h(x) satisfies the desired conditions i.e.

١. h is linear

٢. $h(x) = f(x) \; \forall \; x \in M$.

٣. $h(x) \le p(x) \; \forall \; x \in N$.

Now

١:h is linear ,for $z_1, z_2 \in N$ such that

$z_1 = y_1 + \lambda_1 x_o$, $z_2 = y_2 + \lambda_2 x_o$ then

$$\begin{aligned} h(z_1+z_2) &= h(y_1+\lambda_1 x_o + y_2 + \lambda_2 x_o) \\ &= h((y_1+y_2)+(\lambda_1+\lambda_2)x_o) \\ &= f(y_1+y_2) + (\lambda_1+\lambda_2)\nu \\ &= f(y_1) + f(y_2) + \lambda_1\nu + \lambda_2\nu \quad \text{as f is a linear} \\ &= [f(y_1) + \lambda_1\nu] + [f(y_2) + \lambda_2\nu] \\ &= h(z_1) + h(z_2) \end{aligned}$$

Similarly:

$$\begin{aligned} h(\alpha z) &= h(\alpha(y + \lambda x_o)) \\ &= h(\alpha y + \alpha\lambda x_o) \\ &= f(\alpha y) + \alpha\lambda\nu \\ &= \alpha f(y) + \alpha\lambda\nu \\ &= \alpha(f(y) + \lambda\nu) \\ &= \alpha h(z) \end{aligned}$$

٢: if $x \in M$, then λ must be zero in (٤) and the equality (٥) gives $h(x) = f(x)$

٣: to verify condition ٣ we consider two cases (see eq. (٤))

Case ١: if λ =٠ we have seen that h(x) = f(x) ,
and since f(x) ≤ p(x) therefore h(x) ≤ p(x).

Case ٢: if λ>٠ we get from eq. (٣)

$\nu \le p(y+x_o) - f(y)$ (٦)

Since N is a subspace of X, $y/\lambda \in N$, then replacing y by y/λ inequality (٦) we have

$$\nu \le \quad p(\frac{y}{\lambda}+x_o) - f(\frac{y}{\lambda}) \quad or \quad \nu \le \quad p(\frac{1}{\lambda}(y+\lambda x_o)) \quad - f(\frac{y}{\lambda}) .$$

Since p is a sublinear map, then

$$p(\frac{1}{\lambda}(y+\lambda x_o)) = \frac{1}{\lambda}p(y+\lambda x_o) \qquad \text{for} \quad \lambda > 0 \quad \text{and}$$

$$f(\frac{y}{\lambda}) = \frac{1}{\lambda}f(y).$$

As f is linear therefore $\lambda\nu \le p(y+\lambda x_o) - f(y)$,
or $f(y) + \lambda\nu \le p(y+\lambda x_o)$.Thus from (٤) and (٥) h(x) ≤ p(x) $\forall x \in N$ ◈

Theorem ٣.١.٣ Let p be a sublinear mapping from the linear space X to the Dedeking complete Riesz space F, M is a linear subspace of X and let f_o be a linear mapping from M to F for which $f_o(x) \le p(x)$ for all $x \in M$ then there exists a linear mapping f from X to F that extends f_o

a) $f(x) \le p(x) \ \forall \ x \in X$

b) $f(x) = f_o(x) \ \forall \ x \in M$

Proof: Let φ be the family of order pairs (G,g) in which G is a linear subspace of X , G contains M and g is a linear mapping on G such that $g(x)=f_o(x)$ for all $x \in M$ and $g(x) \leq p(x)$ for all $x \in G$ we define the relation $\leq$ on φ by $(G_1,g_1) \leq (G_2,g_2)$ if and only if $G_1 \subseteq G_2$ and $g_1(x)= g(x)$ for all $x \in G$, it is easy to show that $\leq$ is a partial order relation on φ . We shall prove that every linearly order subset of φ has an upper bound.

Suppose that A is a non-empty subfamily of φ , that is linearly ordered by $\leq$

Let $G= \bigcup \ G'$, suppose that (G_1,g_1) and (G_2,g_2) are pairs in A and let $x \in G_1 \cap G_2$.

Since A is linear ordered , then either $(G_1,g_1) \leq (G_2,g_2)$ or $(G_2,g_2) \leq (G_1,g_1)$, in either cases , by definition on the relation $\leq$ we have $g_1(x) = g_2(x)$.

For this we shall define a function g on G as follows :

Given $x \in G$ and let $g(x)=g'(x)$ where (G',g') is any pair of A with $x \in G'$ .we shall prove that G is a linear subspace in X.

Let $x,y \in G$. $\alpha,\beta \in R$, then $x \in G_1$ and $y \in G_2$ for some pairs (G_1,g_1) and (G_2,g_2) in A .

Since A is a linearly ordered, we may suppose that, $(G_1,g_1) \leq (G_2,g_2)$, then $G_1 \subset G_2$ and hence $x,y \in G_2$.

Since G_2 is a linear subspace of X, we have

$\alpha x + \beta y \in G_2 \subseteq G$ this shows that G is a linear subspace of X. Also by definition of g, we have

$g(\alpha x+\beta y)=g_{٢}(\alpha x+\beta y)$

$$= g_{٢}(\alpha x)+g_{٢}(\beta y)$$

$$= \alpha g_{٢}(x) +\beta g_{٢}(y)$$

$$= \alpha g(x) +\beta g(y)$$

This shows that g is a linear mapping on G , it is obvious that $M\subset G$ and $g(x)=f_o(x)$ for all $x\in M$ and $g(x) \le p(x)$ for all $x\in G$ consequently $(G,g) \in A$,clearly $(G',g') \le (G,g)$ for all $(G',g')\in A$ so (G,g) is an upper bound for A it follows now by Zorn's Lemma (١.٢.٨) that , φ has at least a maximal element, let (H,f') be a maximal element subset of φ we shall prove that H=X.

Suppose $H\neq X$ and choose $x_{١}\in X - H$ and let $H_{١}$ be a linear hull of the set $H \cup \{x_{١}\}$, since $(H,f') \in \varphi$ we have $f'(x) \le p(x)$ for all $x\in H$ and therefore , by previous lemma (٣.١.٢) , there is a linear map f on $H_{١}$ such that $f_{١}(x)=f'(x)$ for all $x\in H$, it follows that $(H_{١},f_{١})\in\varphi$ and $(H, f') \le (H_{١},f_{١})$ this contradicts the maximalty of (H, f') in φ because $(H, f') \neq (H_{١},f_{١})$ and we must have H=X ◈

§ ٣.٢. A modified Hahn-Banach theorem and some new Results

In this section we obtain a new propositions related to the Hahn-Banach theorem.

Theorem ٣.٢.١

Let X is a linear space, F is a Riesz space M is a proper subspace of X. And let T be linear transformation from M into F, then there exists a linear transformation T' from X into F such that T' is an extension of T

Proof: take any element $x_o \in X-M$ i.e. $x_o \in X$ and $x_o \notin M$.

Let M_o be the subspace of X spanned by the set $M \cup \{x_o\}$ then $M_o = M+[x_o]$ where $[x_o]$ is the subspace of X generated by x_o.

If $z \in M_o$ then z can be represented as $z = x+\alpha x_o$, $x \in M$ and α is some scalar. Also this representation for z is unique. For that suppose $z_{١} = x_{١}+\alpha_{١}x_o$ and $z_{٢}=x_{٢}+\alpha_{٢} x_o$ where $x_{١},x_{٢} \in M$ and $\alpha_{١}, \alpha_{٢}$ are some scalars then, $x_{١} + \alpha_{١}x_o = x_{٢} + \alpha_{٢}x_o$ .This gives $x_{١}-x_{٢} =(\alpha_{١}-\alpha_{٢}) x_o$

Now $x_{١},x_{٢} \in M \Rightarrow x_{١} - x_{٢} \in M$ because M is a subspace of X therefore $(\alpha_{١}-\alpha_{٢})x_o \in M$. Suppose $\alpha_{١} - \alpha_{٢} \neq ٠$.

Then $(\alpha_{١}-\alpha_{٢}) x_o \in M$

i.e. $x_o \in M$. But this is a $\Rightarrow \frac{1}{(\alpha_1 - \alpha_2)}(\alpha_1 - \alpha_2)x_o \in M$

contradiction, because $x_o \notin M$, hence we must have $\alpha_{١}-\alpha_{٢} = ٠$ i.e. $\alpha_{١}=\alpha_{٢}$ then $x_{١}+\alpha_{١} x_o = x_{٢} +\alpha_{٢}x_o$ gives $x_{١}=x_{٢}$ and thus $z=x+\alpha x_o$, $x\in M$, α scalar (١)

For the uniqueness representation of $z \in M$ we now define a mapping T'_o from M_o into Riesz space F by the formula $T'_o(x+\alpha x_o)=T(x)+\alpha y_o$ where y_o is some fixed vector in F obviously T'_o is an extension of T because, if $z \in M_o$ such that z is also in M, then the representation for z in the form (١) is $z = z+\cdot x_o$ and so by define of T'_o , we have $T'_o(z)=T(z)+\cdot y_o = T(z)$ also T'_o is linear transformation from M_o into F , suppose $z_١=x_١+\alpha_١ x_o$, $z_٢=x_٢+\alpha_٢ x_o$ any two members of M_o and α,β are any two scalars then $T'_.$

$$(\alpha z_١+\beta z_٢) = T'_o[\alpha(x_١+\alpha_١x_o)+\beta(x_٢+\alpha_٢x_.)]$$

$$=T'_.[(\alpha x_١+\beta x_٢)+(\alpha\alpha_١+\beta\alpha_٢)x_o]$$

$$=T(\alpha x_١+\beta x_٢)+(\alpha\alpha_١+\beta\alpha_٢)y_o \quad \text{Since } \alpha x_١+\beta x_٢\in M$$

$$=\alpha T(x_١)+\beta T(x_٢)+(\alpha\alpha_١+\beta\alpha_٢)y_o$$

$$=\alpha[T(x_١)+\alpha_١y_o]+\beta[(T(x_٢)+\alpha_٢y_o]$$

$$=\alpha T'_o(x_١+\alpha_١x_o)+\beta T'_o(x_٢+\alpha_٢x_o)$$

$$=\alpha T'_o(z_١)+\beta T'_o(z_٢),$$

Thus T'_o is a linear transformation from M_o into F and T'_o is an extension of T.

Now let h be a linear transformation with domain D (h) as a subspace of X and range R(h) as a subspace of F. Suppose that M is a proper subspace of D (h) and h is an extension of T. Let p be the class of all such linear transformation h then p is non-empty because $T'o \in p$

If $h_1, h_2 \in p$ let us define the relation $h_1 \leq h_2$ to mean that h_2 is an extension of h_1 i.e. $D(h_1) \subset D(h_2)$ and $h_2(x) = h_1(x)$ $\forall\ x \in D(h_1)$.

Obviously this relation defines a partial ordering of p.

Now suppose that Q is a totally ordered subset of p. We shall show that there exists an element $U \in p$ which is an upper bound of Q. Let D(U) be union of the domains of each element h in Q. Then D(U) is a subset of X and M is contained properly in D(U) .

Also D(U) is a subspace of X . For suppose α, β are scalars and $x_1, x_2 \in D(U)$. Then there exists two elements $h_1, h_2 \in Q$ such that $x_1 \in D(h_1)$, $x_2 \in D(h_2)$.

Since Q is totally ordered. We may take $h_1 \leq h_2$, then $D(h_1) \leq D(h_2)$. Therefore, $x_1, x_2 \in D(h_2)$ but $D(h_2)$ is a subspace of X.

Now if $x \in D(U)$, then $x \in D(h)$ for some $h \in Q$.

Define the mapping U from D(U) into F by the formula

$U(x) = h(x)$ when $h \in Q$ and x is in the domain of h.

If $x \in D(h_1) \cap D(h_2)$ where $h_1, h_2 \in Q$ we may take $h_1 \leq h_2$ because Q is totally ordered.

Then $h_1(x) = h_2(x)$ also U is an extension of T because $h \in Q$ is an extension of T. Further U is a linear transformation from D(U) into F , suppose α, β are scalars and $x_1, x_2 \in D(U)$,

Then as shown previously there exists $h_2 \in Q$ such that $\alpha x_1 + \beta x_2 \in D(h_2)$. Now by definition of U

we have

$U(\alpha x_1 + \beta x_2) = h(\alpha x_1+\beta x_2)$

$= \alpha h(x_1) + \beta h(x_2)$

$= \alpha U(x_1) + \beta U(x_2)$,

Hence U is a linear transformation from D(U) into F.
Since D(U) is a subspace of X,M is contained properly in D(U) and U is an extension of T , therefore $U \in P$ also obviously U is an extension of h $\forall h \in Q$ therefore $h \leq U$ $\forall h \in Q$ and so U is an upper bound of Q .
Now all the conditions of Zorn's Lemma are satisfied in p. Hence p contains a maximal element say T' we must have D(T') =X. If D(T') is a proper subspace of X, then we can regard D(T') as M in the first part of the proof and thus we shall get an element $h \in p$ such that $h \neq T'$ and $T' \leq h$. This is contrary to the maximalty of T'. Hence D(T') = X. thus the linear transformation T' has properties required in the theorem ◈

Proposition ٣.٢.٢

Let X be a linear space, then there exists a mapping T' defined from X into a Riesz space F, such that this mapping is linear.

Proof: Let x_o be any element of X and consider the subspace Z of X contains of all elements $x=\alpha x_o$, where α is scalar.
On Z we define a mapping $T: Z \to F$ by
$T(x) = T(\alpha x_o) = \alpha T(x_o)$

To use the previous theorem (٣.٢.١) we must prove, that T is a linear mapping.

Let $x, y \in Z, \alpha, \beta$ are scalars and let $x = \lambda_1 x_o$, $y = \lambda_2 x_o$

$$\alpha x + \beta y = (\alpha\lambda_1 x_o + \beta\lambda_2 x_o)$$

$$= (\alpha\lambda_1 + \beta\lambda_2)\, x_o$$

$$T(\alpha x + \beta y) = T\,((\alpha\lambda_1 + \beta\lambda_2)\, x_o)$$

$$= (\alpha\lambda_1 + \beta\lambda_2)\, T\,(x_o)$$

$$= \alpha\lambda_1 T\,(x_o) + \beta\lambda_2 T\,(x_o)$$

$$= \alpha T\,(\lambda_1 x_o) + \beta T\,(\lambda_2 x_o)$$

$$= \alpha T(x) + \beta T(y)$$

Then by theorem ٣.٢.١ there exists an extension T': $X \rightarrow F$ which is linear ◈

Proposition ٣.٢.٣

Let M be a subspace of a linear space X and let x_o be a vector not in M then there exists a mapping $T: X \rightarrow F$ (F is a Riesz space)such that $T(M) = \{\theta\}$ and $T(x_o) \neq \theta$

Proof: Consider the natural map $\Pi: X \rightarrow X / M$, such that $\Pi(x) = x + M$, it is easy to show that Π is a linear map.

If $m \in M \Rightarrow \Pi(m) = m + M = \theta$ (i.e the zero vector M in X /M) since $x_o \notin M$. We have $\Pi(x_o) = x_o + M \neq M$, then by proposition ٣.٢.٢, since X/M is linear space and $x_o + M \in X/M$ so, there exists a linear map $T' : X/M \rightarrow F$ such that $T'(x_o + M) \neq M \in F$.

Now we define T by $T(x) = T'\Pi(x)$, for every $x \in X$ then we have

$$T(\alpha x+\beta y) = T'\Pi(\alpha x+\beta y)$$
$$= T'((\alpha x+\beta y+M))$$
$$= T'(\alpha x+M+\beta y+M)$$
$$= \alpha T'(x+M)+\beta T'(y+M)$$
$$=\alpha T'\Pi(x) +\beta T'\Pi(y)$$
$$=\alpha T(x)+\beta T(y)$$

Thus T is a linear.

Furthermore, if $m\in M$ then, $T(m) = T'\Pi(m)=T'(M)=\{\theta\}$

Also $T(x_o) = T'\Pi(x_o)=T'(x_o+M) \neq\theta$ ◈

§ ٣.٣. A modified Cauchy-Schwarz inequality

In this section we obtain a new Cauchy-Schwarz inequality by interchanging the Range and Domain of the classical Cauchy-Schwarz inequality.
Now the classical Cauchy-Schwarz inequality is given in the following theorem.

Theorem ٣.٣.١[١١]
If X is a linear space and $A:X \times X \to R$ is a bilinear map with $A(x,x) \geq ٠$ and $A(x,y) = A(y,x)$ for all $x \in X$ then $A(x,y)^{٢} \leq A(x,x)\, A(y,y)$.

Definition ٣.٣.٢[٣]
An associative real lattice ordered algebra X is called ***almost f-algebra*** if $x \wedge y = \theta \Rightarrow xy = \theta \ \forall \ x,y \in X$.

Definition ٣.٣.٣[٣]
If X a Riesz space, a bilinear map A of $X \times X$ into a linear space F is called ***orthosymmetric*** if $x,y \in X$, $x \wedge y = \theta \Rightarrow A(x,y) = \theta$.

Proposition ٣.٣.٤ (Modified Cauchy Schwarz inequality)

Let X be a Riesz space and let F positive partially order cone be a field. If $A:X\times X \rightarrow F$ is an bilinear map with $A(x,x)\geq\theta$ and $A(x,y)=A(y,x)$ then $A(x,y)^{٢} \leq A(x,x)\,A(y,y)$

Proof: Since $A(x+\lambda y,x+\lambda y) \geq \theta$

So $A(x+\lambda y,x+\lambda y) = A_{x+\lambda y}(x+\lambda y)$

$= A_{x+\lambda y}(x)+A_{x+\lambda y}(\lambda y)$

$= A(x,x+\lambda y) + A(\lambda y,x+\lambda y)$

$= A_x(x+\lambda y)+\lambda A_y(x+\lambda y)$

$= A_x(x) + \lambda A_x(y) +\lambda[A_y(x)+\lambda A_y(y)]$

$= A(x,x)+\lambda A(x,y)+\lambda A(y,x) +\lambda^{٢}A(y,y)$

Let $a=A(x,x)$, $b=A(x,y)$, $c=A(y,y)$

Then $a+٢\lambda b+\lambda^{٢}c \geq \theta$

$$\lambda^2+\frac{b}{c}\lambda+\frac{a}{c}\geq \theta \Rightarrow (\lambda+\frac{b}{c})^2+\frac{ac-b^2}{c^2}\geq \theta$$

Since the final result in F^+, therefore

$$\frac{ac-b^2}{c^2}\geq \theta$$

$\Rightarrow ac-b^{٢} \geq c^{٢}\theta$

$\Rightarrow ac-b^{٢} \geq \theta$

$\Rightarrow ac \geq b^{٢}$

$\Rightarrow A(x,y)^{٢} \leq A(x,x)\,A(y,y)$ ◈

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