جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بابل
كلية التربية – قسم الرياضيات

حول الحلول التماثلية للمعادلات التفاضلية من الرتبة الثانية

بحث تخرج مقدم
الى كلية التربية في جامعة بابل وهو جزء من متطلبات نيل
شهادة الماجستير في علوم الرياضيات

من قبل
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الاهداء

الى من رضا الله في رضاهما.... والدتي المتفانية المضحية ووالدي الشهيد

احمد

CERTIFIFCAT

We certify that we read this research entitled "On Similaity Solutions Of Second Order Differential Equations" and, as an examining committee we examined contents and what is connected with, and that in our opinion it meets the standards of a research for the degree of Master of Science in mathematics.

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On Similarity Solutions Of Second Order Differential Equations

A Research Submitted to Department of Mathematics of
The College of education of university of babylon in parial fufillment of the requirement for the degree of master of science in mathematics

BY

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Supervised by

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November 2003

Abstract

Since the inception of integral and differential calculus, the differential equations have played remarkable role in dealing with almost every physical solution. The physical problems pertaining to science and engineering can most of the time be expressed by ordinary and partial differential equation.

The differential equation describing the realistic situation normally are of non-linear type and the corresponding solution can not be had so easily even by computer oriented methods. In this regard there has been some theories based upon coordinate transformation which enable the workers in this field to get exact solution or reduce the degree of difficulty by reducing the order of the equation. In the present dissertation, the author has discussed the Lie continuos group of transformation method to solve the ordinary and partial differential equation of second order.

In the first chapter some basic definitions, regarding differential equations have been mentioned with some basic concept regarding similarity transformation method while in the second chapter, the methodology with necessary details of similarity transformation method for solving second order differential equation based upon the invariance under continuous Lie group of transformation method is described. In the third chapter two applications have been discussed in details with new solutions have been found, while the fourth chapter deals with conclusion.

CANDIDATE'S DECLARATION

I hereby that the work which is presented in the dissertation entitled (On Similarity Solutions of second Order Differential Equations) in partial fulfillment of requirement for the award the degree of Master in Mathematics, submitted in the Department of Mathematics college of Education, Babylon University is an authentic record of my own work carried out under the supervision of Dr. M. K. Jasim, Department of Mathematics, Al-Mustansiryah University

The matter embodied in this dissertation has not been submitted by me for the award any other degree.

Dated: Oct, 15, 2003

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This is to cerfify that the above statement made by the candidate is correct to the best of my knowledge.

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SUPERVISOR CERTIFICATION

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In view of the available recommendations , I forward this research for debate by, examining committee.

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Chairman of Departmental Committee of Graduate Studies in Mathematics.
Date:

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Contents

Chapter Four

Introduction

1.1 Historical Background of the Differential Equation

Mathematical theories in the sciences, if they are to be realistic, must be built on the basis of differential Equations, relations between the derivatives or differentials of varying quantities. It is then the business of the theories to deduce the functional equations between the variables which lie behind the differential equations, i.e. to express the "general laws" whose variations correspond to the given data.

In the physical problem, for example much of the data is expressed interns of variations in "energy", of one form of another, and leads to certain fundamental differential equations from which the laws of conservation of energy are to be derived. Differential equations they occupy a fundamental position in most highly developed mathematical science.

The simplest case can be written

$$\frac{dy}{dx} = f(x) \qquad \textbf{(1.1)}$$

Where f(x) is some given function.

We have seen that this case is easily dealt with by means of the concept of an integral. Thus of the integral to describe the solution of a differential equation.

A more general form of the case is

$$\frac{dy}{dx} = f(x, y) \qquad \textbf{(1.2)}$$

Where the given function f(x,y) involves both x, and y.

The meaning of a differential equation of the form in equation (2) is most clearly seen in diagrammatic terms. At each point (x,y) of the plane OXY, the equation provides a definite value of the derivative $\frac{dy}{dx}$

This is the gradient of the direction in which the variables are allowed to vary from (x,y), i. e. the gradient of the tangent to the curve through (x,y) expressing the relation between x and y implied by the differential equation.

The differential equation from four perspectives, physical, analytical, numerical, and graphical can be introduced. The over riding goal of this is developing the ability to use mathematical idea to provide the answers to the difficult open-ended questions that are part of professional practice in science and engineering [Poul Dauis, 1999].

1.2 Basic concepts Definitions

A differential equation is an equation relating an unknown function or function to one or more of its derivatives.

In many branches of science and engineering we come across equations, which contain different derivatives of the dependent and independent variable with respect to the independent variable. These equations are called differential equations for examples.[sharma, J. N. and Gupta, R. K, 2000].

Examples:

(i) $\frac{d^2y}{dx^2} = ky$

(ii) $(\frac{dy}{dx})^2 = 4(\frac{dy}{dx}) - 2y^2 + 4x$

and

(iii) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz$

1.2.1 Ordinary Differential Equations:

Equations which involve only one independent variable called ordinary differential equations, [krasnov, M. L., 1987] see examples (i) and (ii) above.

1.2.2 Partial Differential Equations

Equations which involve partial differential coefficients with respect to more than one independent variable are called partial differential equations [JACK, G. et al., 1997] see example (iii) above.

1.2.3 order and degree of a differential equations

The order of differential equation is the highest derivative which involved in the differential equation. Example (i) of second order while (ii) of first order.

The degree of differential equation is the highest exponent to the highest derivative which involved in the equation, when the equation has been made rational and integral as for as the derivative are concerned – equation (i) and (iii) are first degree and equation (ii) of second degree.

1.2.4 Solution of the Differential Equation

Any relation between dependent and independent variable which when substituted in the differential equation, reduce it to an identity is called (a solution of differential equation).

Remark:

A solution of differential equation dose not contains the derivative of the dependent variable with respect to the independent variables.

1.2.5 Linear and non-linear Differential Equation

The n^{th} order differential equation is linear if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \quad \textbf{(1.3)}$$

with not all a_n are identically zero.

An equation that is not linear is called non-linear, if the equation is linear, the only operations that may be performed on the dependent variable are:

(i) Differentiation of the dependent variable.

(ii) Multiplication of the dependent variable and its derivatives by functions of the independent variable only.

(iii) Setting equal sums of the terms satisfying conditions (ii) and a function only of the independent variable.

For examples:

$y'' - (2 \tan x)\, y' + 5y = 0$ (which is linear)

and

$y'' - (\sin x)\, y' = y \cos x + (y')''$ (which is non-linear)

1.3 Similarity method

For first-order partial differential equations we take the (restricted) point of view that a sufficiently complete integration theory is given by the theory of characteristics. This connects the solution of partial differential equations with the integration of systems of ordinary differential equations. It may, however, be useful to look at some first order equations directly from the point of view of transformations and invariance. [Bluman, G. W. et al, 1974].

For higher order equations or systems the aim is a reduction in the number of variables. A typical result is the statement that a solution *u(x,y,t)* of a particular partial differential equations in three independent variables must be represents as :

$$u(x,y,t)=\frac{1}{t}F\left(\frac{x}{t},\frac{y}{t}\right) \tag{1.4}$$

This procedure can possibly be repeated more than one time. The special case when a partial differential equation contain only two independent variables is particularly important since the problem is reduced to an ordinary differential equation. In many physical problems of interest the resulting equation which needs to be studied (together with a suitable number of quadratures) is of first order. In this favorable case the structure of all possible solution in the possible solution in the phase plane provides complete information on the structure of a class of solutions to the original partial differential equation. It also may provide the basis for a method or numerical integration.

Another method, which can be used to obtain the same results in special cases, arises not directly from transformation theory but rather from dimensional analysis. The basic idea is that all physical problems must be expressible in dimensionless variable. This idea is applied to the variables entering a problem for a partial differential equation. For example if (x,y), some independent variable, which are space coordinates with the physical dimensions of “length”, enter the problem, then it can be concluded that only the combination *(x | y)* (or equivalent) can enter the problem. Evidently, this represents a reduction in the number of variables.

The role of Lie theory in constructing solutions to partial differential equations differs essentially from its role for solving ordinary differential equations. Invariance under a one-parameter Lie group of transformations reduces by one the number of variable appearing in a partial differential equation rather than the order as is the case for an ordinary differential equation. This method leads to particular (similarity) solutions and not to the “general” solution of a given partial differential equation. Thus boundary conditions play an important role in the applications of Lie theory to partial differential equations. For a second order partial differential equation of first degree in one of the second derivatives note the difference between the general case and the case when one of the independent variables is missing, i.e.,

$u_{xx} = H(u_{xt}, u_{tt}, u_x, u_t, u, x, t)$ general **(1.5)**

and $u_{xx} = H(u_{xt}, u_{tt}, u_x, u_t, u, x)$, t missing **(1.6)**

in second case, there will exit particular solutions $u = f(x)$ where $f(x)$ satisfies the second order ordinary differential equation.

$f''(x) = H(0, 0, f'(x), 0, f(x)x)$ **(1.7)**

As another example consider the wave equation

$u_{xx} = u_{tt}$ **(1.8)**

clearly the one-parameter (α) group of transformations

$x^* = \alpha x$ and $t^* = \alpha t$ **(1.9)**

Leaves invariant equation (15) choosing Canonical Coordinates

$r = \frac{x}{t}$, so that $u = f(r)$ is a particular solution of equation (1.8)

$$u = f\left(\frac{x}{t}\right)$$

$$u_x = \frac{1}{t} f^{/}\left(\frac{x}{t}\right)$$

$$u_{xx} = \frac{1}{t^2} f^{//}\left(\frac{x}{t}\right)$$

$$u_t = \frac{-x}{t^2} f^{/}\left(\frac{x}{t}\right)$$

$$u_{tt} = \frac{x^2}{t^4} f^{//}\left(\frac{x}{t}\right) + \frac{2x}{t^3} f^{/}\left(\frac{x}{t}\right)$$

From equation (1.8), we get

$$f^{//}\left(\frac{x}{t}\right)-\frac{x^2}{t^2}f^{//}\left(\frac{x}{t}\right)+\frac{2x}{t}f^{/}\left(\frac{x}{t}\right)=0$$

or

$$(1-r^2)f^{//}\left(\frac{x}{t}\right)-2rf^{/}\left(\frac{x}{t}\right)=0 \qquad \textbf{(1.10)}$$

In order to generate the particular solution form u = F(r) the important point is the invariance of equation (15) under the one-parameter Lie group of transformation

$r^* = r$

$s^* = s + \varepsilon$

In general, when reducing the number of variables of a partial differential equation from symmetry considerations, the invariants of the group become the new variables. The generated similarity solution satisfies an auxiliary first order PDE (called the invariant surface condition) whose coefficients depends on the infinitesimal of the group. Solving the corresponding characteristic equations of the first order PDE we find the functional form of the similarity solution. For a PDE, with independent variables (x, t) and dependent variable n typically one of the invariants will be of the form $\xi(x, t)$ and the order invariant can be expressed as an arbitary function of ξ, $F(\xi)$.

The functional form for the similarity solution will be

$$u = f(x, t, \xi, F(\xi)) \qquad \textbf{(1.11)}$$

ξ is called the similarity variable and $F(\xi)$ becomes the new dependent variable. The dependence of f on $(x, t, \xi, F(\xi))$ is known explicitly and by substituting from equation (1.8) into the giving PDE we obtain an ordinary differential Equation For $F(\xi)$.

Thus for a given problem we first seek the largest set of infinitesimal leaving invariant the governing PDE. The infinitesimal satisfy a set of "determining equations". Next we analyze the symmetries of the boundary conditions and seek the subgroup (actually the sub-algebra of the infinitesimal) leaving invariant the boundary curves and the boundary condition prescribed on them. For linear PDE it is unnecessary to leave invariant all of the boundary conditions.

The use of group invariance to reduce the number of variables appearing in a PDE does not depend on the governing PDE being linear.

Similarity Transformation Method For Differential Equation of second order

2.1 Similarity method for differential Equations

"Sophus Lie" tried to construct a general integration Theory for Differential Equations [Cohen A., 1911]. He studied the invariance of differential equations under transformations and he introduced the continuos group of transformations based on the infinitesimal properties of the group [Compbell L. S. 1986]. The goal of method is to provide the solutions.

So, we shall provide here the necessary details of the similarity method for solving second order ordinary and partial differential equations, based on invariance under continuous (Lie) group of transformations.

2.1.1 Second Order PDE

Consider the second order partial differential equation (PDE)

$$F(u_{xx}, u_{xt}, u_{tt}, u_x, u_t, u, x, t) = 0 \quad \textbf{(2.1)}$$

Where u is the dependent variable, and x,t are independent variables.

Let us take a one-parameter (say, ε) Lie group of transformations

$$u^* = u^* (x, t, u ; \varepsilon)$$
$$x^* = x^* (x, t, u ; \varepsilon) \quad \textbf{(2.2)}$$
$$t^* = t^* (x, t, u ; \varepsilon)$$

Expanding (2.2) about the identity element $\varepsilon = 0$, we generate the following infinitesimals U, R, T $(0(\varepsilon))$ terms

$$u^* = u + \varepsilon U(x, t, u) + O(\varepsilon^2)$$
$$x^* = x + \varepsilon R(x, t, u) + O(\varepsilon^2) \qquad \textbf{(2.3)}$$
$$t^* = t + \varepsilon T(x, t, u) + O(\varepsilon^2)$$

Also the derivatives of u are transformed according to

$$u^*_{x^*} = u_x + \varepsilon(U_x) + O(\varepsilon^2)$$
$$u^*_{t^*} = u_t + \varepsilon(U_t) + O(\varepsilon^2) \qquad \textbf{(2.4)}$$
$$u^*_{x^*x^*} = u_{xx} + \varepsilon(U_{xx}) + O(\varepsilon^2)$$

and so on, where (U_x), (U_t), (U_{xx}) etc. are the infinitesimals for transformations of the derivatives u_x, u_t, u_{xx} etc., respectively. These are called the first and second extensions depending on the order of the derivative term, the derivation of the extensions can be carried out in the following way:

Let us introduce the total derivative operators

$$\frac{D}{Dx_j} = \frac{\partial}{\partial x_j} + u^{\mu},_j \frac{\partial}{\partial u^{\mu}},$$

$$\frac{D}{Dx^*_j} = \frac{\partial}{\partial x^*_j} + u^{*\mu},_j \frac{\partial}{\partial u^{*\mu}}, u^{\mu},_j = \frac{\partial u^{\mu}}{\partial x_j}$$

$$\frac{Du^i}{Dx_j} = u^i,_j, \frac{Du^{*i\,j}}{Dx^*_j} = u^{*i},_j$$

$$\text{and } \frac{Dx^{\bar{o}}_{\alpha}}{Dx^{o}_{\beta}} = \frac{Dx_{\alpha}}{Dx_{\beta}} = \delta_{\alpha\beta} \qquad \textbf{(2.5)}$$

Where $\delta_{\alpha\beta}$ is the Kronecher delta symbol, which is defined as

$$\delta_{\alpha\beta} = \begin{cases} 0 & \text{for } \alpha \neq \beta \\ 1 & \text{for } \alpha = \beta \end{cases}$$

First extension:

$$u^{*i}_j = \frac{Du^{*i}}{Dx^*_j} = \frac{Du^{*i}}{Dx_{\nu}} \frac{Dx_{\nu}}{Dx^{\circ}_j}$$

$$= \frac{D}{\ldots}[u^i + \varepsilon U^i + 0(\varepsilon^2)) \frac{D}{\ldots}(x^* \ldots \varepsilon R \ldots + 0(\varepsilon^2)]$$

(2.6)

Let us write the infinitesimal of u^{*i} , j by U_j^i , then by using (2.5), we find that the first extension is

$$U_j^i = \frac{\partial U^i}{\partial x_x^\mu} + \frac{\partial U^i}{\partial u_\mu} u^\mu, j - \frac{\partial R_\nu}{\partial x_j} u^i, j - \frac{\partial R_\nu}{\partial u^\mu} u^\mu, j\, u^i, \nu \qquad \textbf{(2.7)}$$

Where I=1,2,…,m,   j=1,2,…,n

Which indicates m dependent and n independent variables.

In the case of (2.2), (2.7) takes the shape

For j = 1; $[U_x] = U_x + (U_u - R_x)u_x - T_x\, u_t - R_u u^2{}_x - T_u u_x u_t$

For j = 2; $[U_t] = U_t + (U_u - T_t)u_t - R_t\, u_x - T_u u^2{}_t - R_u u_x u_t$ **(2.8)**

Second extension:

The second extension can be obtained directly by replacing u^i, j by u^i, jk and u^i by u^i_j in (2.6), thus provides.

$$u^{*i},_{jk} = u^i,_{jk} + \varepsilon\left[\frac{DU_j^i}{Dx_j} - \frac{DR_\nu}{Dx_j}\, u^i, j\nu\right] + O\,(\varepsilon^2). \qquad \textbf{(2.9)}$$

Accordingly, by making the appropriate substitutions in (2.9), the second extension is

$$U_{jk}^i = \frac{\partial^2 U^i}{\partial x_j \partial x_k} + \frac{\partial^2 U^i}{\partial x_j \partial u^\mu} u^\mu, k + \frac{\partial^2 U^i}{\partial x_k \partial u^\mu} u^\mu, j - \frac{\partial^2 R_\nu}{\partial x_j \partial x_k} u^i, \nu$$

$$+ \frac{\partial U^i}{\cdots} u^\mu, jk - \frac{\partial R_\nu}{\cdots} u^i,_k \nu - \frac{\partial R_\nu}{\cdots} u^i, j\nu + \frac{\partial^2 U^i}{\cdots} u^\lambda, j\, u^\mu, k$$

where i = 1,2,………, m; j,k = 1,2,………, n. **(2.10)**

For i = j = k = 1, (2.10) reads as

$$\left[U_{xx}\right] = U_{xx} + (2U_{xu} - R_{xx})u_x - T_{xx}u_t + (U_{uu} - 2R_{xu})u_x^2 - 2T_{xu}u_x u_t - R_{uu}u_x^3 - T_{uu}u_x^2 u_t + (U_u - 2R_x)u_{xx} - 2T_x u_{xt} - 3R_u u_{xx} u_x - T_u u_{xx} u_t - 2T_u u_{xt} u_x. \quad \textbf{(2.11)}$$

For i = j = 1, k = 2, we have

$$\left[U_{xt}\right] = U_{xt} + (U_{xu} - T_{xt})\, u_t + (U_{tu} - R_{xt})\, u_x - T_{xu}u_t^2 + (U_{uu} - 2R_{xu} - T_{ut})\, u_x u_t - R_{tu}u_x^2 - T_{uu}u_x u_x^2 - R_{uu}u_t u_x^2 - T_x U_{tt} + (U_u - R_x - T_t)\, u_{xt} - R_t u_{xx} - 2R_u u_x u_{xt} - 2T_u u_t u_{xt} - T_u u_x u_{tt} - R_u u_t u_{xx}. \quad \textbf{(2.12)}$$

and for i = 1, j = k = 2, (2.10) provides us

$$\left[U_{tt}\right] = U_{tt} + (2U_{tu} - T_{tt})\, u_t - R_{tt}u_x + (U_{uu} - 2T_{tu})\, u_t^2 - 2R_{tu}u_x u_t - T_{uu}\, u_t^3 - R_{uu}u_t^2 u_x + (U_u - 2Tt)u_{tt} - 2R_t u_{xt} - 3T_u u_{tt} u_t - R_u u_{tt} u_x - 2R_u u_{xt} u_t. \quad \textbf{(2.13)}$$

Now, it is necessary for the equation (2.1) to be invariant under the transformation (2.2) that we must have

$$F\,(u^*_{x^* x^*}, u^*_{x^* t^*}, u^*_{t^* t^*}, u^*_{x^*}, u^*, x^*, t^*) = 0 \quad \textbf{(2.14)}$$

On expanding about $\varepsilon = 0$, we get

$$F(u^*_{x^*x^*}, u^*_{x^*t^*}, \ldots, t^*) = F(u_{xx}, u_{xt}, \ldots, x, t)$$

$$+\varepsilon\{(U_{xx})\frac{\partial F}{\partial U_{xx}} + (U_{xt})\frac{\partial F}{\partial u_{xt}} + (U_{tt})\frac{\partial F}{\partial u_{tt}} + (U_x)\frac{\partial F}{\partial u_x} + (U_t)\frac{\partial F}{\partial u_t}$$

$$+(U)\frac{\partial F}{\partial u} + (R)\frac{\partial F}{\partial x} + (T)\frac{\partial F}{\partial t}\} + O(\varepsilon^2) \qquad \textbf{(2.15)}$$

So, for the invariance of (2.1), O(ε) term in (2.15) should be zero, i. e.

$$(U_{xx})\frac{\partial F}{\partial u_{xx}} + (U_{xt})\frac{\partial F}{\partial u_{xt}} + (U_{tt})\frac{\partial F}{\partial u_{tt}} + (U_x)\frac{\partial F}{\partial u_x} + (U_t)\frac{\partial F}{\partial u_t}$$

$$+(U)\frac{\partial F}{\partial u} + (R)\frac{\partial F}{\partial x} + (T)\frac{\partial F}{\partial t} = 0 \qquad \textbf{(2.16)}$$

Hence, the equation (2.16) is the necessary and sufficient condition for (2.1) to be an invariant.

In order to find out the infinitesimal (U, R, T) leaving invariant (2.1), we first of all substitute the PDE (2.1) into (2.16). the resulting equation is treated as a form in the derivatives of u whose coefficients depend on (u, x, t) and unknown (U, R, T). Then we collect the coefficients of like derivative terms in u and set all of them equal to zero. The resulting equations are called the determining equations. On solving the determining equations we obtain the infinitesimal (U, R, T)

For of the solution: after finding out the infinitesimal, we have to determine the form of the solution. This can be done as follows: by the infinitesimal transformation (2.3), we have

$u^*(x^*, t^*) = u + \varepsilon U(x, t, u) + 0(\varepsilon^2),$

$u^*(x + \varepsilon R(x, t, u) + 0(\varepsilon^2), t + \varepsilon T(x, t, u) + 0(\varepsilon^2))$

$= u + \varepsilon U(x, t, u) + 0(\varepsilon^2).$ **(2.17)**

On expanding about ε = 0, and equating the 0(ε) terms on either side, we get

$R(x, t, u)\, u_x + T(x, t, u)\, u_t = U(x, t, u)$ **(2.18)**

The equation (2.18) is called the invariant surface condition and can be solved by Lagrange's condition, i. e.

$$\frac{dx}{R(x,t,u)} = \frac{dt}{T(x,t,u)} = \frac{du}{U} \tag{2.19}$$

the solution of (2.19) involves two constants, one of which becomes independent variable (say, $\sigma(x, t, u)$) and called the similarity variable while the other plays the role of a dependent variable (say $f(\sigma)$). So, the similarity form of the solution is u

$$u(x,t) = f(\sigma) \tag{2.20}$$

After substituting (2.20) in (2.1), the resulting equation is an ordinary differential equation involving only the derivatives with respect to similarity variable σ.

2.1.2 Second Order ODE

First of all, let us introduce the U-symbol of the infinitesimal transformation. Consider the one-parameter (τ) group of transformation

$$x_1 = x + \xi(x,y)\,\tau$$

$$y_1 = y + \eta(x, y)\,\tau \tag{2.21}$$

where ξ and η are the infinitesimal and the corresponding infinitesimal transformation

$$x^* = x_1 + \xi(x_1, y_1)\,\delta\tau,$$

$$y^* = y_1 + \eta(x_1, y_1)\,\delta\tau, \tag{2.22}$$

Now consider a function $f(x^*, y^*)$ such that

$$\delta f = f(x^*, y^*) - f(x_1, y_1) = \frac{\partial f}{\partial x_1}(x_1, y_1)\,\delta x_1 + \frac{\partial f}{\partial y_1}(x_1, y_1)\,\delta y_1$$

$$= [\xi(x_1, y_1)\frac{\partial f}{\partial x_1}(x_1, y_1) + \eta(x_1, y_1)\frac{\partial f}{\partial y_1}(x_1, y_1)]\delta\tau$$

in practical as $\tau \rightarrow 0$; we approach the initial point (x, y), so

$$\lim_{\tau\to 0}\frac{\partial f}{\partial \tau}=\frac{df}{d\tau}=\xi(x,y)\frac{\partial f}{\partial x}+\eta(x,y)\frac{\partial f}{\partial y} \qquad \textbf{(2.23)}$$

We define (1.10.23) by the operator U as

$$Uf=\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}, \qquad \textbf{(2.24)}$$

where

$$U\bullet=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y},$$

On expanding about $\tau = 0$ and using (1.10.24) we have

$f(x_1, y_1) = f(x, y) + \tau\, Uf(x, y) + 0(\tau^2)$.

Evidently, a necessary and sufficient condition for invariance is

Uf = 0 for all (x, y)

i.e.,

$$\xi(x,y)\frac{\partial f}{\partial x}+\eta(x,y)\frac{\partial f}{\partial y}=0 \qquad \textbf{(2.25)}$$

Now, the corresponding transformation in the derivative is

$$y_1'=\frac{dy_1}{dx_1}=\frac{dy+\tau d\eta}{dx+\tau d\varepsilon}=\frac{\frac{dy}{dx}+\tau\frac{d\eta}{dx}}{1+\tau\frac{d\varepsilon}{dx}} \qquad \textbf{(2.26)}$$

$$=(y'+\tau\frac{d\eta}{dx})(1-\tau\frac{d\xi}{dx})+\ldots=y'+\tau[\frac{d\eta}{dx}-y'\frac{d\xi}{dx}]$$

$$=y'+\tau\eta'(x,y,y')$$

(2.27)

Also $\frac{d\xi}{dx} = \xi_x + \xi_y y'$

and $\frac{d\eta}{dx} = \eta_x + \eta_y y',$ **(2.28)**

So $\eta' = \frac{d\eta}{dx} - y' \frac{d\xi}{dx} = \eta_x + y'[\eta_y - \xi_x] - \xi_y y'^2$ **(2.29)**

(2.27) with (2.28) is the infinitesimal transformation in (x,y,y ′). We can thus write the infinitesimal operator U′ which is the extension of U to the (x,y,y ′) spaces as

$$U'F = \xi(x,y)\frac{\partial F}{\partial x} + \eta(x,y)\frac{\partial F}{\partial y} + \eta'(x,y,y')\frac{\partial F}{\partial y'} \quad \textbf{(2.30)}$$

in the same way

$$U''F = \xi(x,y)\frac{\partial F}{\partial x} + \eta(x,y)\frac{\partial F}{\partial y} + \eta'(x,y,y')\frac{\partial F}{\partial y'} + \eta''(x,y,y',y'')\frac{\partial F}{\partial y''}$$

;for F (x, y, y′, y′′) **(2.31)**

Where

$$\eta'' = \frac{d\eta'}{dx}(x,y,y') - y''\frac{d\xi(x,y)}{dx} \quad \textbf{(2.32)}$$

The derivative of (1.10.32) follows from (1.10.27), i.e.

$$y_1'' = \frac{dy_1'}{dx_1} = \frac{dy' + \tau d\eta'}{dx + \tau d\xi} = y'' + \tau\left[\frac{d\eta'}{dx} - y''\frac{d\xi}{dx}\right] = y'' + \tau\eta''(x,y,y',y'')$$

(2.33)

(2.32) with the help of (2.29) takes the form

$$\eta'' = [\eta_{xx} + \eta_{xy}y' + (\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - \xi_{xy})y'^2 + (\eta_y - \xi_x)y'' - \xi_{xy}y'^2 - \xi_{Yy}y'^3$$
$$-2\xi_y y'y''] - y''[\xi_x + \xi_Y y']. \quad \textbf{(2.34)}$$

Now a second order differential equation

$$H, (x, y, y', y'') = y'' - \omega(x, y, y') = 0, \qquad \textbf{(2.35)}$$

Admits all the transformations of a one parameter group

$$UF = \xi(x, y)\frac{\partial F}{\partial x} + \eta(x, y)\frac{\partial F}{\partial y},$$

When

$$U^{''}H = \xi\frac{\partial H}{\partial x} + \eta\frac{\partial H}{\partial y} + \eta^{'}\frac{\partial H}{\partial y^{'}} + \eta^{''}\frac{\partial H}{\partial y^{''}} = 0$$

Which, with reference to (2.29) and (2.34) reads as

$$-\xi\frac{\partial \omega}{\partial x} - \eta\frac{\partial \omega}{\partial y} - [\eta_x + (\eta_y - \xi_x)y^{'} - \xi_y y^{'2}]\frac{\partial \omega}{\partial y^{'}} + \eta_{xx} + (2\eta_{xy} - \xi_{xx})y^{'}$$
$$+ (\eta_{yy} - 2\xi_{xy})y^{'2} - \xi_{yy}y^{'3} + [\eta_y - 2\xi_x - 3\xi_y y^{'}]\,\omega(x, y, y^{'}) = 0 \qquad \textbf{(2.37)}$$

In order to find out (ω, η) the coefficients of different powers of y′ are set to zero, further in the resulting expressions the functions of x which are coefficients of certain power of y are also set to zero.

The equation (2.35) can be reduced to a first order equation by finding out two invariant of the group (u, v) which in turn are found from solving characteristic differential equation.

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)} = \frac{dy^{'}}{\eta^{'}(x, y, y')} \qquad \textbf{(2.37)}$$

As

U(x,y) = a

And

V(x,y,y′) = b

Then, the second order differential equation can be expressed as:

$$\frac{dv}{du} = \phi(u, v) = \frac{v_x + v_y y' + v_{y'} y''}{u_x + u_y y'} \tag{2.39}$$

This way reduces the second order ODE to the first order ODE.

Applications

3.1 Introduction

In physical problems, we always seek a solution of the differential equation, which satisfies some specified conditions known as the boundary conditions. The differential equation together with these boundary conditions constitutes a boundary value problem. [Jordan, D. W et al., 1987].

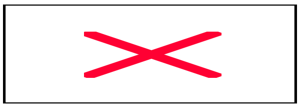
In problems involving ODE, we may first final the general solution and then determine the arbitrary constant from the initial values. But the same process is not applicable to problems involving partial differential equations for the general solution of a partial differential equation contains arbitrary functions, which are difficult to adjust so as to satisfy the given boundary conditions.

Thus the method of similarity transformation which is measured in chapter two is found capable of solving such different equations of ODE and PDE linear and non linear, so the following two application have been discussed in details.

3.2 Solving of Differential Equation of a problem in calculus of variation

3.2.1 Basic Equation

Our aim is to solve the following equation from the field of variatioal problems, so the



(3.1)

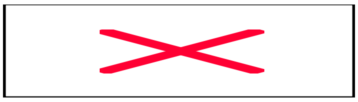
Which is represent a variational problem and can be solved by use of similarity transformation method.

3.2.2 Solution

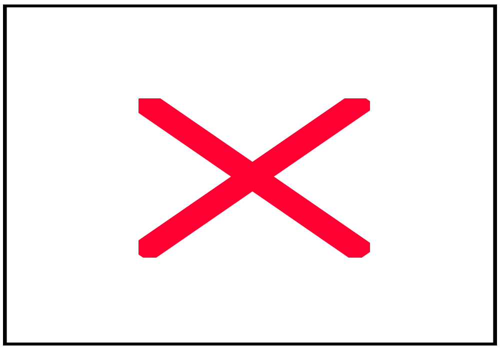
Writing the equation (3.1) in the standard form :

$$H\,(x, y, y', y'') = y'' - w(x, y, y') = 0 \qquad \textbf{(3.2)}$$

Where

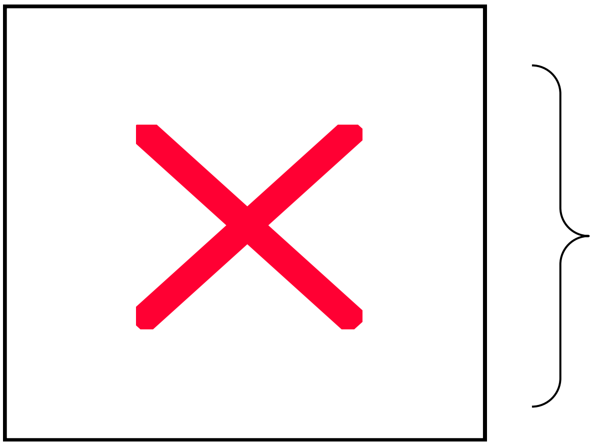


From (3.1)



(3.3)

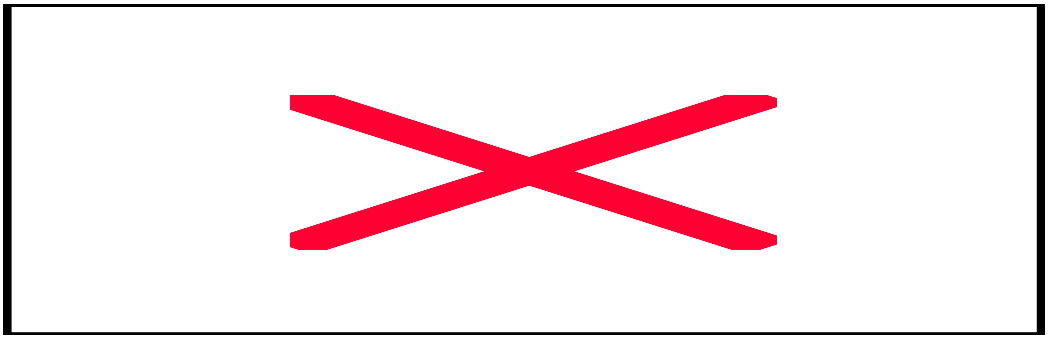
comparing with (3.2), we get



(3.4)

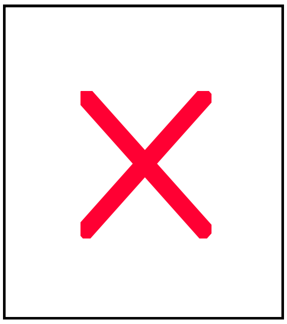
The general criterion for group invariance is

Substitution (3.4) in (3.5), we get



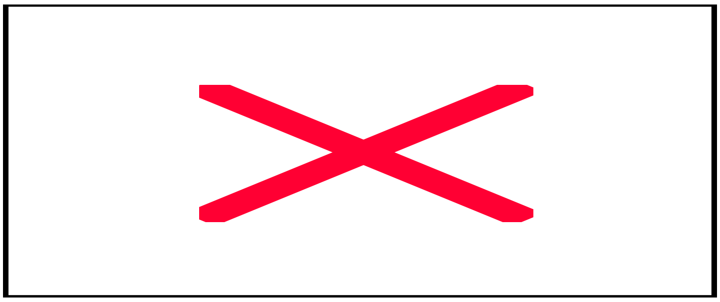
setting the coefficients of different powers of y′ in (3.6) to be zero.

Coefficient of y'^3 **:**



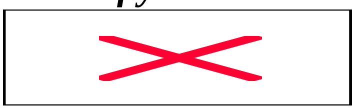
(3.7)

Coefficient of y'^2 **:**



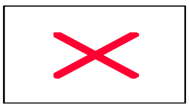
(3.8)

coefficient of y′



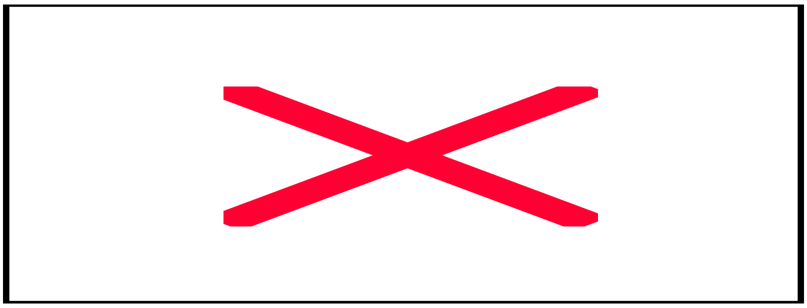
(3.9)

coefficient of 1/y′2



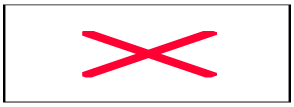
(3.10)

coefficient of 1/y′



(3.11)

coefficient of 1/y



(3.12)

coefficient of y°

$\eta_{xx} = 0$ **(3.13)**

Now, in order to show the equation (3.1) invariant; above system of equations (3.7) – (3.13) should be solved. So from equation (3.7), by integration it, we get

$ln\xi_y = 1/3 \ ln\ y + ln\ A(x)$

$\xi_y = A(x)\ y^{1/3}$

$\xi_{(X,Y)} = ¾\ A(x)\ y^{4/3} + B(x)$ **(3.14)**

From equation (3.8), we get

$\eta_{yy} = 2\xi_{xy} \Rightarrow c'' = 2A'y'^{3}$

Equation (3.9), which can be solved by equating the coefficient of 1/y in (3.9) to zero to get

$\eta_x = 0 \ \Rightarrow \ \eta = c(y)$ **(3.15)**

Equation (3.10) gives

$\eta_x = 0 \ \Rightarrow \ \eta = c(y)$ **(3.16)**

By equating the coefficient of 1/y in (3.11) to the zero, we get

$\xi - 4\eta_y + 6\xi_x = 0$

$¾\ A(x)\ y^{4/3} + B - 4c' + 9/2\ A'y^{4/3} + 6B' = 0$ **(3.17)**

From (3.5), we get

$\eta_{yy} = 2\xi_{xy}$

$\Rightarrow c'' = 2A'y^{1/3}$ **(3.18)**

equation (3.9), which gives

$2\eta_{xy} = \xi_{xx}$

$\Rightarrow 3/4A''y^{4/3} + B'' = 0$ **(3.19)**

And (3.12), gives

$\xi_y = 0 \Rightarrow A(x) = 0,$ **(3.20)**

Using (3.14), with (3.5), we get

$\xi(x,y) = B(x)$ **(3.21)**

Equation (3.17) gives

$B - 4c' + 6B' = 0$ [since A(x) = 0] **(3.22)**

From (3.19), we have

$c'' = 0$ [since A(x) = 0]

$\Rightarrow c(y) = C_3y + C_4$ **(3.23)**

Where C_3 and C_4 are constants of integration using (3.22) and (3.23) in (3.14) and (3.16), we get

$\xi(x,y) = C_1x + C_2$

$\eta(x,y) = C_3y + C_4$ **(3.24)**

Taking

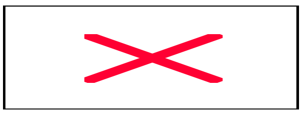
$C_1 = 1,$ $C_2 = 0,$ $C_3 = 5/6,$ $C_4 = 0$ we get

$\xi(x,y) = x$

$\eta(x,y) = 5/6\ y$ **(3.25)**

Thus the differential equation (3.1) is invariant under the group of transformation given by (3.25).

Now, the characteristic differential equation (Lagrange equation) are:

 **(3.26)**

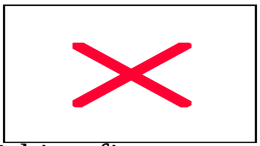
Where

$\eta = \eta_x + y'(\eta_y - \xi_x) - \xi_y y'^2$

$= 0 + y'(5/6 - 1) - 0$

$= -1/6\, y'$

Therefore, from (3.26), we get



(3.27)

Taking first two member of equation (3.27), we get

$\ln x + \ln k_1 = 6/5 \ln y$

$\Rightarrow k_1 = y(x)\, x^{-5/6}$

Therefore,

$$u(x,y) = x^{-5/6}\, y \qquad \textbf{(3.28)}$$

from first and third member of equation (3.27), we get

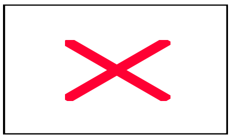
$-1/6 \ln x + \ln K_2 = \ln y'$

$\Rightarrow k_2 = y' x^{1/6}$

Thus,

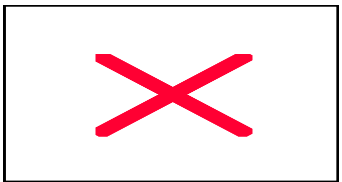
$$v(x,y,y') = x^{1/6}\, y' \qquad \textbf{(3.29)}$$

Now,



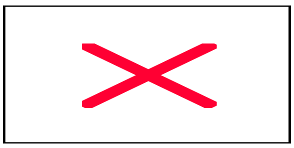
(3.30)

Also,



(3.31)

By using chain Rule, we get



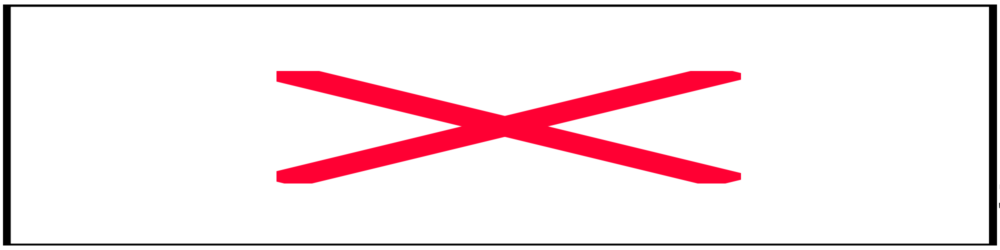
(3.32)

Which is 1st order ODE differential equation in u & v and can be solved by integration.

3.3 similarity solutions for non-linear partial differential equations

3.3.1 Basic equations

Due to the non-linearity of the following equation, which is basically represent a model of perfect fluid distribution of plane symmetric case [Gupta, y. k et al., 1996 and Jasin, M. K. 2000].



33)

so, the differential equations describe the realistic situation normally are non-linear type and corresponding solution can not be had easily. In this regard, the well-known method of Lie point group of transformations shall be used which is already discussed in earlier chapter.

3.3.2 Solutions

Equation (3.) can be written in form of

(3.34)

let

$H = H(u'', \ddot{u}, \grave{u}', \grave{u}, u', u, r, t)$

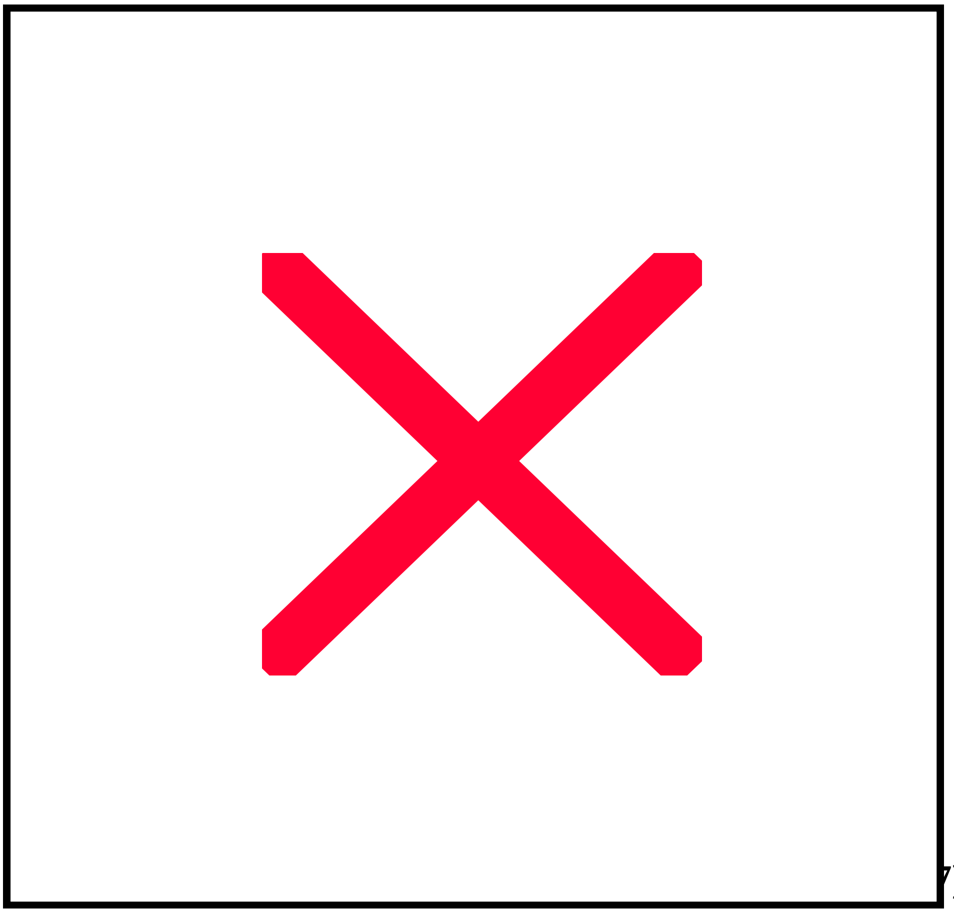
Which is invariant under the transformation

$u = \alpha u \qquad , t = \alpha t \qquad , r = \alpha r$ **(3.35)**

now to solve (3.34) by using similarity transformation method, we have:

(3.36)

Let u = θ (r,t), now (3.36) becomes

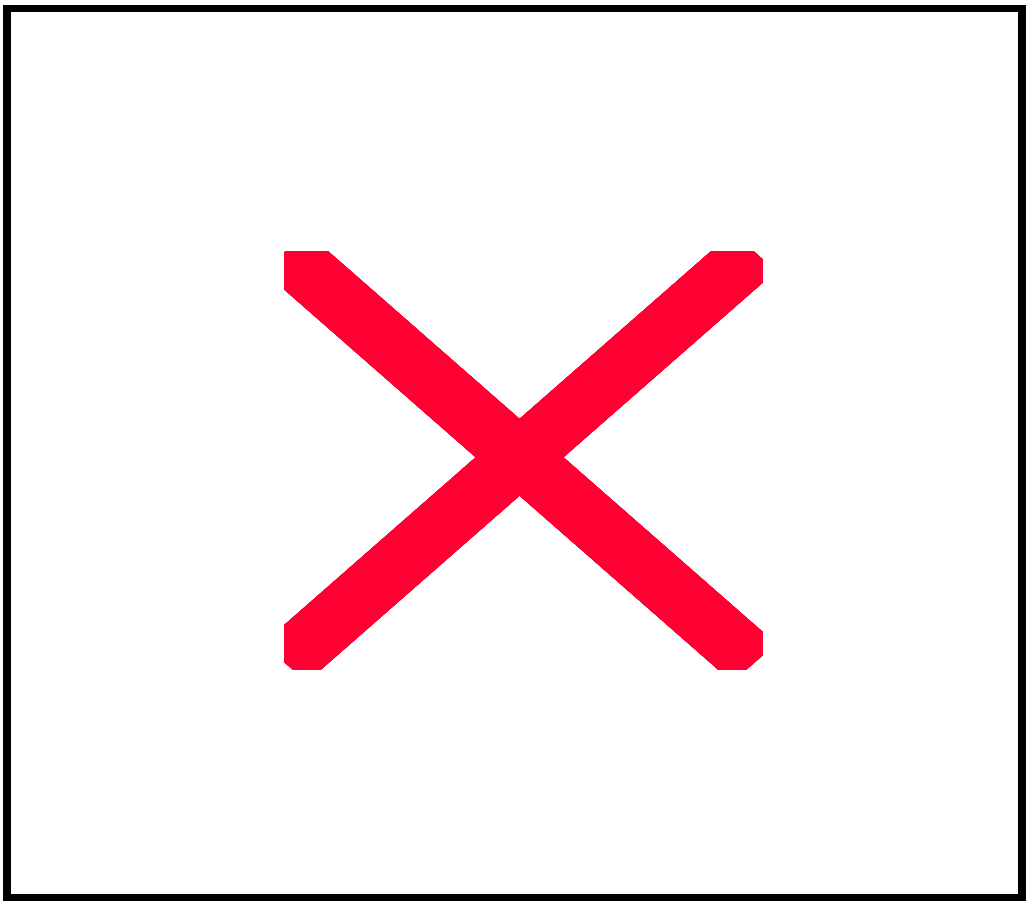


7)

Substituting (3.37) into the recurrance formula

(3.38)

using (3.37), with some simplification, we have:



(3.39)

Now by equation the coefficients of different power of θ to zero, we get the following set of determining equations

$$T_r = U_r = U_t = R_u = T_u = U_{uu} = T_{tt} = R_{rt} = 0$$

$$T + t(U_u - R_r - T_t) = 0$$

$$-T - t(2U_u - 3R_r) + t^2R_{tt} - 2tR_t = 0$$

$$T + t(4U_u = 3R_r - 2T_t) = 0$$

$$4(U_u - R_r) + 2R_t - tR_{tt} = 0$$

(3.40)

Here, for our convenience we have replaced

$u' = u_r$ and

$\acute{u}$ by u_t etc.;

The set (3.40) can be uniquely solved to give

$R = \alpha r + \beta t^3 + \gamma$

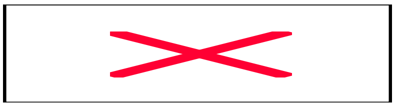
$T = \alpha t$

$U = \alpha u + \delta$ **(3.41)**

Where

α, β, γ and δ are constant

Now we are going to find all the similarity solutions. By using Lagrange method so, we have:

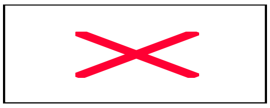
 **(3.42)**

the following two subcases occur

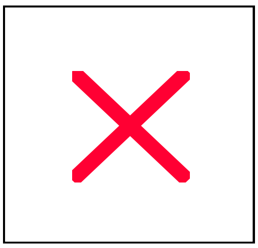
case I : $\alpha = 0$, ***case II*** : $\alpha \neq 0$

<u>case I</u>

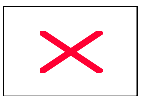
$\alpha = 0$, we have the form

 **(3.43)**

which immediately yield

 **(3.44)**

When insert (3.44) into (3.43) the requirement is that

\-  $[14(at^3 + b)^2]^{-1}\ [8a^3 t^9 - 42ab^2t^3 - 7b^3 - 7b + 14ct^2]$

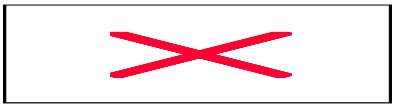
(3.45)

where c is an arbitrary constant,

case II

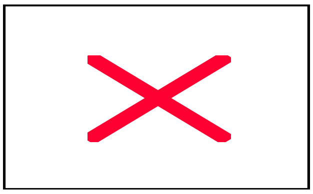
$\alpha \neq 0$

the following equation



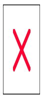
(3.46)

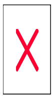
which immediately suggests the form of u as :



(3.47)

on inserting (3.47) into equation (3.46), we get

$y''(y^2 - \alpha x) + 2x\, y'^3 - 2yy'^2 +$  $y' = 0$ **(3.48)**

where $y' =$  and $x = \alpha + \eta$

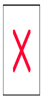
the equation (3.48) can be solved by using any technique as follows:

subcase I

when

$y^2 - \alpha x = 0$

equation (3.48) can be written as

$2x\, y'^3 - 2yy'^2 +$  $y' = 0$ **(3.49)**

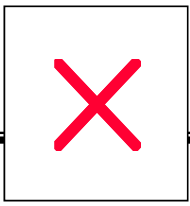
which is easily solvable

subcase II

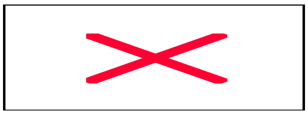
when

$y^2 - \alpha x \neq 0$

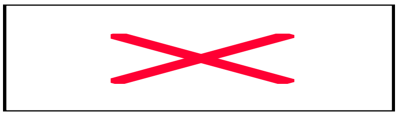
so the equation (3.48), can be solved by using STM, to get



So (3.49) will takes the shape



on integration, we get



which is hyperbolic or elliptic integral

Conclusions

It is clear from the previous work that a systematic study of transformation is useful part of a general integration theory. These transformations must have group properties.

So, such physical problems have been studied in details with new solution have been found.

Hence One should be very careful a bout this method the main advantage of this method is that most of the physical problems , which are non- linear can be solved by this method.

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