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تقدم بها

حبيب عبد كاظم

بإشراف

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DEPARTMENT OF MATHEMATICS

ON COCONVEX APPROXIMATION

A thesis submitted to the Department of Mathematics College of Education Babylon University as a partial fulfillment of requirements for the degree of Master of Science in Mathematics

By

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Supervised by
Assist. Professor
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المُستَخْلَص

ناقشنا في تلك الرسالة الدوال التي تنتمي إلى الفضاء $L_p(I),\ 0 < p \le \infty$ والتي يتغير تحدبها مرات منتهية على الفترة $I = [-1,1]$. وكان اهتمامنا في تلك الرسالة بتقدير درجة تقارب تلك الدالة المتغيرة التحدب بواسطة متعددات الحدود التي تحافظ على شكل الدالة الأصلية وبالاعتماد على مقياس النعومة المرجّح لكل من دتيزين وتوتك , $\omega_{k,r}^{\varphi}$.

تضمنت رسالتنا ثلاثة فصول ,في الفصل الأول مقدمة احتوت على نظرة عامة حول نظرية التقريب وكذلك تضمن هذا الفصل بعض التعريفات , المتراجحات وبعض النظريات التي استخدمناها في هذا العمل .

في الفصل الثاني درسنا مقياس دتيزين- توتك المرجّح لنعومة الدالة وبعض الخواص عليه, وتم تقريب الدوال المتغيرة التحدب بواسطة متعددات الحدود الجبرية والمقطعية ,وبعد ذلك قمنا بقياس أفضل درجة للتقريب المتغير التحدب للدالة $f \in L_p(I)$ بالاعتماد على مقياس النعومة $\omega_{k,r}^{\varphi}$.

في الفصل الثالث قمنا بدراسة تقريب الدالة $f \in \Delta^2(Y_s) \cap L_p(I)$ المعرفة على الفترة المنتهية , والذي يكون حافظاً لشكلها , حيث أن هذا الفصل تضمن ثلاثة أقسام ,في القسم الأول تم تقدير درجة تقريب الدالة f بواسطة استخدام متعددات الحدود الجبرية والمقطعية مع تغير التحدب عند النقاط التي تنتمي إلى Y_s .في القسم الثاني استخدمنا فرق القسمة ومقياس النعومة المرجّح في تقريب الدالة $f \in L_{\varphi}^{r}(I) \cap \Delta^2(Y_s)$, وفي القسم الأخير قمنا بتقدير درجة التقريب للدالة المعرفة في القسم الثاني.

Supervisor Certification

We certify that the preparation for this thesis entitled "**On Coconvex Approximation**" was made under our supervision at the University of Babylon, College of Education as a partial fulfillment of the requirements of degree Master of Science in Mathematics.

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Examining committee certification

We certify that we have read this thesis as examining committee examined the student in its contents and that in our opinion it is adequate for the partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Dedication

To

Lover of Allah, Mohammed

And Kinsfolk his home ...

My Mother and My Father..

My Brothers and My Sisters ..

My wife and My Son..

✍***Habeeb***

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I am thankful to God the most merciful as this thesis has been completed .

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✍ *Habeeb A. K.*

Table of Contents

ABSTRACT

This thesis discuss the functions $f \in L_p,\ 0 < p \leq \infty$ which change its convexity finitely many times, on a finite interval . We are interest in the estimating of the degree of approximation of f by polynomials which are coconvex with it and which shape preserving, involving the weighted Ditzian–Totik modulus of smoothness $\omega_{k,r}^{\varphi}$.

The thesis is divided into three chapters. Chapter one is an introduction in which definitions, inequalities and theorems of researchers related to this study are introduced.

In chapter two we study and consider the properties of weighted Ditzian–Totik modulus of smoothness and we deal with approximation of f in L_p ,$0 < p \leq \infty$ on $[-1,1]$, which changes convexity finitely many times by algebraic or piecewise polynomials (splines).Also we are interested in estimating the degrees of best coconvex approximation of $f \in L_p[-1,1]$ involve the weighted Ditzian–Totik modulus of smoothness.

In chapter three we discuss the coconvex approximation of function f which defined a finite interval

,and there are three sections, in the first one , we estimate the degree of approximation of f by using algebraic and piecewise polynomials which change convexity exactly at the points of Y_s, in the second, using the divided difference and weighted D. T. modulus of smoothness to approximate $f \in L_{\varphi}^{r}(I) \cap \Delta^2(Y_s)$ and in the third we estimate the degree of approximation of f by dividing difference .

بسم الله الرحمن الرحيم

﴿يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنْكُمْ
وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ﴾

صَدق الله العَليُ العَظيم

سورة المجادلة. . . الآية 11

Chapter One

Introduction and Preliminaries

1.1 Approximations Theory and Overview

Our interest in approximation theory stems from its beauty, its utility, and its rich history. There are also many connections that can be drawn to questions in both classical and modern analysis.

The main idea of approximation includs in finding for a complicated function f from a large space X a close by, a simple function ψ from a small subset Y of X. There are three elements here. The large space X is usually a normed space, such as C, L_p with $0 < p < \infty$ or one of the other Banach spaces of functions. The distance between ψ and f can be measured by $\|f - \psi\|$ in X. Finally, we have to define the special functions of Y.

The authenticity of the approximation method due to Chebyshev can be thought of as well established . This make it all the more interesting to investigate the source of the impetus that led our genius fellow countryman to his brilliant and deep constructions.

There is nothing mysterious about it . Chebyshev himself said in perfectly clear way, about the purpose of his research in approximation theory and names the person whose results were his starting point.

In 1853 , the great Russian mathematician , P. L. Chebyshev , while working on a problem of linkages , devices which translate the linear motion of a steam engine into the circular motion of a wheel , considered the following problem:

Given a continuous function f defined on a closed interval $[a,b]$ and a positive integer n , can we "represent" f by a polynomial $p(x)=\sum_{k=0}^{n} a_k x^k$, of degree at most n , in such a way that, the maximum error at any point x in $[a,b]$ is controlled ? In particular , is it possible to construct p in such a way that the error $\max_{a\le x\le b}|f(x)-p(x)|$ is minimized?

This problem raises several questions , the first of which Chebyshev himself ignored :

- Why should such a polynomial even exist?
- If it does , can we hope to construct it?
- If it exists , is it also unique?
- What happens if we change the measure of the error to, say , $\int_a^b |f(x)-p(x)|\,dx$?

Chebyshev's problem is perhaps best understood by rephrasing it in modern terms.

What we have here is a problem of linear approximation in a normed linear space . Recall that a norm on a (real) vector space X is a nonnegative function on X satisfying

$$\|x\|=0 \iff x=0$$

$$\|\alpha x\|=|\alpha|\|x\| \quad for\ all\ \alpha\in R$$

$$\|x+y\|\le\|x\|+\|y\| \quad for\ any\ x,y\in X\ .$$

Any norm on X induced a metric or distance function by setting $dist(x,y) = \|x-y\|$. The abstract version of our problems can now be restated:

Given a subset (or even a subspace) Y of X and a point $x \in X$, is there an element $y \in Y$ which is "nearest" to x ; that is , can we find a vector $y \in Y$ such that $\|x-y\| = inf_{z\in Y}\|x-z\|$? If there is such a "best approximation " to x ; from elements of Y, is it unique?

Now, we state some results for the existence and uniqueness of best approximation, answer for the above questions .

Let X be a normed linear space and $X \supseteq Y$, is a finite-dimensional subspace of X, then for each element $x \in X$ there exists not necessary a unique element $y_x \epsilon Y$ that is a best approximation from Y to x ,and map $x \to y_x$,and let Y be a subspace of a normed linear space X , and let $x \in X$ the set Y_x consisting of all best approximation to x out of Y is bounded , convex and closed set, also in case Y be a compact and strictly convex set in a normed linear space, X then for each $x \in X$, there is just one best approximation from Y to x, see [17] .

A new content was given to the theory of best approximations of functions in the first decade of the current century by Weierstrass the head of Berlin school of mathematics.The proposition which he established in 1885, is that, for any $f \in C[a,b]$ and for any $\varepsilon > 0$ there exists an algebraic polynomial of the form $p(x) = c_0 + c_1 x + \cdots + c_n x_n$, $a \le x \le b$, such that: the bound $\|f-p\|_\infty \le \varepsilon$ is satisfied, see[38]

The basic of the approximation theory of functions of real variable is a theorem study by Weierstrass which is of great importance in the development of the whole of mathematical analysis, see [41].

Note that, Fundamental theorem of approximation theory , let $f \in C[a,b], -\infty < a < b < \infty$ Given $\varepsilon > 0$, there exists an algebraic polynomial p for which

$$|f(x) - p(x)| < \varepsilon \qquad \text{for all } x \in [a,b] ,$$

there are more than proofs to Fundamental theorem ,see [37].

Given the values of a function $f(x)$ at two distinct values of x ,say x_0 and x_1 , we could approximate f by a linear function p that satisfies the conditions

$$p(x_0) = f(x_0) \qquad and \qquad p(x_1) = f(x_1) .$$

We can construct the linear interpolating polynomial directly , writing $p(x) = ax + b$ and using the above two conditions to give two linear equations to determine a and b . On solving these equastions , we obtain

$$p(x) = \frac{x_1 f(x) - x_0 f(x_1)}{x_1 - x_0} + x\left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right).$$

This can also be expressed in the Lagrange symmetric form

$$p(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right) f(x_1) ,$$

or in Newton's divided difference form

$$p(x) = f(x_0) + (x - x_0)\left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right) ,$$

to which we will return later . Observe that if we write $x_1 = x_0 + h$ in above equation , the limit of $p(x)$ as $h \to 0$ gives the first two terms of the taylor series for f is differentiable .

It is convenient to denote the set of all polynomials of degree at most n by Π_n . Given the values of a function $f(x)$ at $n+1$ distinct values of x, say $x_0, x_1, \dots, x_n$, can we find a polynomial $p_n \in \Pi_n$,say

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \,,$$

Such that $p_n(x_j) = f(x_j)$, , for $j = 0,1,\dots,n$ this means that we require

$$a_0 + a_1 x_j + a_2 x_j^2 + \cdots + a_n x_j^n = f(x_j), \quad 0 \le j \le n \ .$$

In the other hand ,we obtain

$$a_0 = f(x_0) \qquad \text{and} \qquad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \ ,$$

we will write

$$a_j = [x_0, x_1, \dots, x_j; f], \quad 0 \le j \le n,$$

to emphasize its dependence on f and $x_0, x_1, \dots, x_j$,and refer to a_j as a jth divided difference . Thus we may write p_n in the form

$$p_n(x) = [x_0; f]\pi_0(x) + [x_0, x_1; f]\pi_1(x) + \cdots + [x_0, x_1, \dots, x_n; f]\pi_n(x)\,,$$

which is *Newton's divided difference formula* for the interpolating polynomial . Observe that $[x_0; f] = f(x_0)$,in general we will find it more appropriate to use another notation for divided differences, where we write $[x_0, x_1, \dots, x_n; f]$ instead of $f[x_0, x_1, \dots, x_n]$ is a symmetric function of its arguments $x_0, x_1, \dots, x_n$.

Where,

$$\pi_i(x) = \begin{cases} 1\,, & i = 0, \\ (x - x_0)(x - x_1)\dots(x - x_{i-1}), & 1 \le i \le n. \end{cases}$$

The divided difference $[x_0, x_1, \dots, x_n; f]$ can be expressed as the following symmetric sum of multiples of $f(x_j)$,

$$[x_0, x_1, \dots, x_n; f] = \sum_{r=0}^{n} \frac{f(x_r)}{\prod_{\substack{j \neq r \\ j=0}}^{n} (x_r - x_j)} \quad ,$$

where in the above product of n factors , r remains fixed and j takes all values from 0 to n , excluding r .

In recent years they have been extensively studied by many authors most notably Ditzian , DeVore, Hristov, Jiang, Ivanov, Leviatan and Kopotun [8] , [12] , [13] , [27] , [36]) .

We shall give a simple example about the idea of approximation theory, let us construct the interpolating polynomial $p_4(x)$ for the function 2^x based on the points $-2, -1, 0, 1$ and 2 ; hence estimate $2^{1/2} = \sqrt{2}$ by evaluating $p_4\left(\frac{1}{2}\right)$.

For $x = -2, -1, 0, 1, 2$, we have $2^x = \frac{1}{4}, \frac{1}{2}, 1, 2, 4$, respectively .Then , the following method discussed above ,we find that the forward difference form of the interpolating polynomial is

$$p_4(x) = \frac{1}{4} + \frac{1}{4}(x+2) + \frac{1}{8}(x+2)(x+1) + \frac{1}{24}(x+2)(x+1)x$$

$$+ \frac{1}{96}(x+2)(x+1)x(x-1).$$

On evaluating $p_4(x)$ at $x = \frac{1}{2}$, we obtain

$$\frac{723}{512} \approx 1.4121.$$

Since $\sqrt{2} \approx 1.4142$,this interpolation method has provided an approximation whose error is in the third digit after the decimal point.

There are many important applications of approximation theory, as it was begun itself by application, by the Great Russian scientist P. L. Chebyshev, and he said about the relationship between a mathematical theory and its applications illustrate quite clearly not just the source of creativity, but also scientific and philosophical positions. In this work, we give some of its applications in mathematics, which are:

A) Solving algebraic equations . In the memoir [22], there are about ten theorems (6-10, 15-19) that are derived from the basic propositions on best approximation. These theorems state that, under certain conditions the polynomial of interest has at least one zero in some interval. The length of the interval depends on the one hand, on the value of the polynomial at the center of the interval, on the other hand, on specific assumption on the coefficients or on the zeros of the polynomial. For example that: if the polynomial $P(x) = x^{2n+1} + \cdots + k$,does not contain any even power of x, then it has at least one zero in interval $|x| \leq 2\left(\frac{|k|}{2}\right)^{\frac{1}{2n+1}}$. Let us also formulate theorem which uses a different assumption: if a polynomial $P(x)$of degree n with leading coefficient equals to one, and has only real roots, then for any t there is a real root in the interval $|x - t| \leq 4\left(\frac{P(t)}{4}\right)^{\frac{1}{n}}$. Later in his 1872 memoir, using a result on monotone polynomials, Chebyshev narrowed the interval by replacing with $|x-t| \leq 4\left(\frac{P(t)}{2(n-1)\pi}\right)^{\frac{1}{n}}$.A further improvement of this result is due to A. A. Markove [30].

B) *Interpolation (remainder estimate)*. To minimize the error in the Lagrange interpolation formula, Chebyshev suggests taking the nodes of the interpolation (say in the interval $(-1,1)$) to be the zero of the polynomial $T_n(x) = \cos(n \arccos(x))$, since for a given function $f(x)$ the remainder has the form $R(x) = \frac{f^{(n+1)}(\lambda)}{(n+1)!} P_n(x)$, where $\lambda \in (-1,1)$, $P_n(x) = \prod_{k=1}^{n}(x - x_k)$ and x_k are the nodes.

Therefore with $R(x) \leq \max\left|\frac{f^{(n+1)}(\lambda)}{(n+1)!}\right| \max|P_n(x)|$,

the choice $P_n(x) = T_n(x)$ is the most profitable. Here Chebyshev partly envisions the later result of Range that says that as $n \to \infty$ Chebyshev interpolation converges for any function that is regular in the basic interval, while this is not true for Newton interpolation with equally space nodes.

C) *A rule for finding approximately distance on the surface of the Earth .*

Let us quote it completely :"(1) Take the distance between the two latitudes and the two longitude ,and express them in minutes,(2) Double the difference of the latitudes ,(3) Out of these two numbers the difference of longitudes and the doubled difference latitudes , multiply the smaller by three, multiply the longer one by seven and then add the two products,(4) the result divided by three will give the desired distance in verests" . A further improvement is due to Markov[30].

D) *Constructing geographic maps* . If one needs to draw on a map some piece of the Earth surface with a given boundary , then there is a choice among infinitely many projections that provide the infinitesimal similarity and presentation of the scale in each point and in all directions .There are the so called conformal projections . However as follows from Gauss " Theorma egregium ", among all conformal projections of a ball to a plane , it is impossible to find a projections that would preserve the scale for all points of the surface .In a talk given on 30 January1865 published under the title "Sur la constructions des cartes geographiques "[31], Chebyshev posed the problem of finding of finding a conformal projection for which the logarithm of the scale would vary within tightest possible deviation from some average value. Without a proof , he claimed that if the specified condition is satisfied , then the scale should be constant on the boundary of the map. Chebyshev statement was proved much later by academician D. A. Grave in [28] and also in [11].

Recently ,approximation theory has also many other applications both numerical and analytical. The most prominent of these are to image processing , statistical estimation and the numerical and analytic treatment of differential equations.

1.2 The Spaces L_p, $p<1$.

The degree of constrained and unconstrained approximation of a function f in either the uniform norm or in the $L_p(I)$, $0<p<\infty$ quasi-norm will be studied,where $I=[-1,1]$. The degree of approximation will be measured by the appropriate quasi-norm which we denote by $\|.\|_p=\|.\|_{L_p(I)}$. Since we need the L_p quasi-norm on other intervals we will in all cases of an interval $J\neq I$, indicate that by writing $\|.\|_{L_p(J)}$. Also we refer to ($\|.\|_p$with $p=\infty$), the uniform norm on I and we denote by $\|.\|$ and on the interval J by $\|.\|_J$ Furthermore, $\|.\|_p$ is a norm for $1\le p<\infty$. Characteristic for L_p, $p\ge 1$ are the inequalities of Holder and Minkowski.

The dual space of L_p,$1\le p<\infty$, is the spaces L_q with the conjugate exponent q of p. Thus the spaces L_p with $1\le p<\infty$ are reflexive.

The different structure of the spaces L_p with $0<p<1$ and the numerous questions by others lead us to understand the need for the following few facts about L_p for $p<1$.

We consider the space $L_p(I)$, consisting of all measurable functions f on I, for which

$\|f\|_p^p=\int_I |f(x)|^p\,dx<\infty$. Recall that $\|f\|_p\le 2^{\frac{1}{p}-1}\|f\|_1$, that is $L_1 \subset L_p$.

As we will see shortly, the L_p is not actually a norm space for $p<1$. Nevertheless, it is not hard to see that $L_p(I)$ is a complete metric space.

Theorem 1.1 **[3]**

$$\|f+g\|_p \le \left(\|f\|_p^p + \|g\|_p^p\right)^{\frac{1}{p}} \le 2^{\frac{1}{p}}\left(\|f\|_p + \|g\|_p\right), \text{ for any } f,g \in L_p(I).$$

Thus $\mathrm{d}(f,g) = \|f-g\|_p^p$ defines a translation invariant metric on L_p. It is a complete metric, because convergence (respectively Cauchy) in L_p implies convergence (respectively Cauchy) in measure since $\varepsilon^p meas\{t : |f(t)| > \varepsilon\} \le \int_{-1}^{1} |f(t)|^p dt$, where by "*meas*" , we mean the measure of a set.

What this means to us is that :

(i) A linear map on L_p is continuous if and only if it is bounded (continuous at zero).

(ii) The open mapping and closed graph theorems still apply.

(iii) The Hahn Banach theorem may fail!

Indeed, as we will see shortly, L_p is not locally convex. In fact, it is impossible to define a norm on L_p which gives the same topology as the usual metric. There are several ways to see that $L_p(I)$ is not normable for $0 < p < 1$. Most useful from our point of view is

***Theorem* 1.2** [6]

L_p, $0 < p < 1$ has a trivial dual.

The Hahn Banach theorem allows a much fancier sounding version of this result:

***Corollary* 1.3** [6]

There are no non zero continuous linear map from L_p into any normed space.

In any event, it should now be clear that there can be no norm on L_p which gives the same topology as the usual metric and that the Hahn Banach theorem evidently fails in $L_p(I)$, for $0 < p < 1$. The fact that L_p has a trivial dual with the theorem from [45] states that there exists a nonzero continuous linear function on a linear space X, if and only if there is at least one convex set that is not all of X, imply another rather strange result that would be hard to believe otherwise.

***Corollary* 1.4** [6]

If U is any neighborhood of zero in $L_p(I)$, then

$$L_p(I) = conv(U)$$

In particular

$$L_p(I) = conv\{f : \|f\|_p^p < 1\},$$

where $conv(U)$ is a smallest convex neighborhood of zero contains U.

Now, we will settle the question of which $L_p, p < 1$, embed into L_q for $q \geq 1$. Or which subspaces of $L_p(I)$ on which all of the various $L_p(I)$ quasi-norms for $0 < p < q$ are equivalent. The key in this article is from [11].

For $0 < \varepsilon < 1$and $0 < p < \infty$, consider the following subset of $L_p(I)$ $M(p,\varepsilon) = \{f \in L_p(I) : meas\{x : |f(x)| \geq \varepsilon \|f\|_p\} \geq \varepsilon\}$.

Notice that if $\varepsilon_1 < \varepsilon_2$, then $M(p,\varepsilon_2) \subset M(p,\varepsilon_1)$. Also $\bigcup_{\varepsilon>0} M(p,\varepsilon) = L_p(I)$, since for any nonzero $f \in L_p(I)$we have $meas\{|f| \geq \varepsilon\} \to meas\{f \neq 0\}$ as $\varepsilon \to 0$. In fact, any finite subset of $L_p(I)$is contained in an $M(p,\varepsilon)$for same $\varepsilon > 0$. Finally note that $meas\{|f| \geq \|f\|_p\} \geq 1$ implies $|f| = \|f\|_p$ almost everywhere.

The following theorem puts this observation to good use

***Theorem* 1.5** [4]

For a subset S of $L_p(I)$, the following are equivalent

(i) $S \subset M(p,\varepsilon)$ for some $\varepsilon > 0$.

(ii) For each $0 < p < q$, there exists a constant $c(q) < \infty$ such that

$$\|f\|_q \le \|f\|_p \le c(q)\|f\|_q \text{ for all } f \in S.$$

(iii) For some $0 < p < q$, there exists a constant $c(q) < \infty$ such that

$$\|f\|_q \le \|f\|_p \le c(q)\|f\|_q, \text{ for all } f \in S.$$

In our thesis, we will use the notations c and C to denote such absolute constants which are of no significance to us and may differ on different occurrences , even in the same line. In order to emphasize that c or C depends only in parameters $\nu_1,\nu_2,.......\nu_k$, the notation $c(\nu_1,\nu_2,.......,\nu_k)$(respectively $C(\nu_1,\nu_2,.......,\nu_k)$) is used, if one of those ν_i is $= p$, we mean that constant depends on p, only in the case $p<1$. Also we will have constants $c_1, c_2, ...$ when we have a reason to keep trace of them in the computations that we have to curry in the proofs.

1.3 Moduli of Smoothness

Moduli of smoothness are intended for mathematicians working in approximation theory, numerical analysis and real analysis, it is very useful for measuring the smoothness of a function by differentiability is too crude for many purpose in approximation theory. More subtle measurement is proved by the moduli of smoothness. We will use moduli of smoothness which are connected with difference of higher orders.

The rth symmetric difference of f is given by:

$$\Delta_h^r(f,x)=\Delta_h^r(f,x,[a,b])$$

$$\Delta_h^k(f,x)=\begin{cases}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}f\left(x-\frac{kh}{2}+ih\right) & , \ x\pm\frac{kh}{2}\in I\\ 0, & o.w.\end{cases}$$

Then, *the rth usual modulus of smoothness of* $f\in L_p[a,b]$ is defined by

$$\omega_r(f,t,I)_p=\sup_{0<h\le t}\left\|\Delta_h^r(f,.)\right\|_{L_p(I)},t\ge 0.$$

A new way of measuring smoothness was introduced by Ditzian and Totik [43]. The need for this new concept arises from the failure of the classical moduli of smoothness to solve some basic problems, such as characterizing the behavior of best polynomial approximation in $L_p(I)$.

The Ditzian-Totik modulus of smoothnessof $f\in L_p(I), 0<p\le\infty$, [10] which is defined for such an f as follows

$$\omega_r^{\varphi}(f,t,I)_p=\sup_{0<h\le t}\left\|\Delta_{h\varphi(\cdot)}^r(f,.)\right\|_{L_p(I)}.$$

In the applications the φ usually used

$$\varphi(x)=(1-x^2)^{\frac{1}{2}} \text{ for } I=[-1,1]$$

$$\varphi(x)=(x(1-x))^{\frac{1}{2}} \text{ and } \varphi(x)=\sqrt{x}(1-x) \text{ for } I=[0,1]$$

$$\varphi(x)=\sqrt{x},\ \varphi(x)=(x(1+x))^{\frac{1}{2}} \text{ and } \varphi(x)=x \text{ for } I=(0,\infty),$$

we have

$$\omega_r^{\varphi}(f,t)_p \le \omega_r(f,t)_p, \quad 0<p<\infty.$$

The weighted Ditzian-Totik kth modulus of smoothness of function $f \in L_p(I), 0<p\le\infty,$ is defined by

$$\omega_{k,r}^{\varphi}(f,t)_p = \sup_{0<h\le t}\left\|\varphi^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f,t)\right\|_{L_p(I)}$$

where $\varphi(x)=\sqrt{1-x^2}$.If $r=0,$ then $\omega_{k,0}^{\varphi}(f,t)_p=\omega_k^{\varphi}(f,t)_p.$

We shall prove the relationship between $\omega_r^{\varphi}(f,t)_p$ and $\omega_{k,r}^{\varphi}(f,t)_p$ in chapter two in case $0<p\le\infty.$

We collect below some properties of $\omega_r(f,t)_p$ and $\omega_r^{\varphi}(f,t)_p$, that are also true for $0<p<1$ and for the D.T. modulus of smoothness, we are referred to [7, equation (12.5)], [9], [43] for references

$$\omega_r(f,t)_p \le \omega_r(f,\lambda t)_p \le c\omega_r(f,t)_p,$$

$$\omega_r^{\varphi}(f,t)_p \le \omega_r^{\varphi}(f,\lambda t)_p \le c\omega_r^{\varphi}(f,t)_p,$$

$$\omega_r^{\varphi}(f,t)_p \le c\|f\|_p,$$

$$\omega_r(f+g,t)_p^q \le \omega_r(f,t)_p^q + \omega_r(g,t)_p^q,$$

$$\omega_r^{\varphi}(f+g,t)_p^q \le \omega_r^{\varphi}(f,t)_p^q + \omega_r^{\varphi}(g,t)_p^q.$$

where $0<p\le\infty, \lambda>1, q=\min(1,p)$,and c is a constant depending only on r , q and also on λ if applicable.

Also now let us compare between the classical modulus of smoothness and its extension to ω_r^{φ} by the following properties :The following most basic facts about $\omega_r(f,t)_p$ and $\omega_r^{\varphi}(f,t)_p$ satisfied where $0<p\le\infty$

(a) $\lim_{t\to\infty}\omega_r(f,t)_p = 0$ for all $f \in L_p(I), 0<p<\infty.$

(b) $\omega_r(f,t)_p$ is a nondecreasing function of t.

($\bar{\text{b}}$) $\omega_r^{\varphi}(f,t)_p$ is a nondecreasing function of t.

(c) $\omega_r(f,\lambda t)_p \le c(p)\lambda^r \omega_r(f,t)_p$ for $\lambda \ge 1$.

(d) $\omega_{r+1}(f,t)_p \le 2\omega_r(f,t)_p$.

which is generalized using the following inequality in [35]

$(\bar{\mathrm{d}})\,\omega^{\varphi}_{r+1}(f,t)_p \le c(p)\omega^{\varphi}_r(f,t)_p$.

Another inequality about the classical modulus of smoothness is

(e) $\omega_r(f,\lambda t)_p \le c\lambda^r \omega_r(f,t)_p$, for $\lambda \ge 1$.

$(\bar{\mathrm{e}})$ Another inequality about $\omega^{\varphi}_r(f,t)$ from [3]

$$\omega^{\varphi}_r(f,\lambda t)_p \le c\lambda^r \omega^{\varphi}_r(f,t)_p, \text{ for } \lambda \ge 1.$$

There is another basic property of the classical modulus of smoothness:

(f) $\omega_{k+r}(f,t,J)_p \le ct^r \omega_k(f^{(r)},t,J)_p,\ p \ge 1$.

Which is valid if $f^{(r)} \in L_p(J)$ and $f^{(r-1)}$ is absolutely continuous in every closed interval $J = [a,b]$. The inequality f is not true if $0 < p < 1$.

Where the constants involved depending on the location of the points of change of convexity, also we show that in some cases the constants may be taken an independent of the points of change of convexity, but that in some other cases this dependence is essential.

We will introduce some theorems about properties of weighted Ditzian-Totik modulus of smoothness in the case $0 < p \leq \infty$, in Chapter two.

1.4 Shape Preserving Approximation

The fundamental problem of the theory of approximation of functions interested with the connection between the structural properties of a function and its degree of approximation. It is to relate the smoothness of the function to the rate of decrease of the degree of approximation to zero. We are concerned in our work of examining these questions for algebraic polynomial approximation and piecewise polynomial approximation (spline).

In many applications, it is desirable that the mathematical moduli preserve certain geometric properties of the data such as monotenicity, convexity and in general $k-$ monotenicity. This is the subject that the so called shape preserving approximation deals with.

Recently there has been more attention given to approximation with constrains. The appearance of constrains can make it more difficult to obtain direct estimates. Still the general lines of attack development for the non constrained problem can be very useful. We want to indicate what modifications are necessary to push the approach through. We do this for comonotone approximation.

In coconvex polynomial approximation, we give a function f that changes its convexity finitely many times in the interval I. We are interested in estimating the degree of approximation of f by polynomials which are convex with it, namely polynomials that change their convexity exactly at the points where f does. Question of this nature first appeared in the work of D.J Newman see [1], [33] and [34]. They did not deal with a function f that changes convexity, but rater with f changes monotonicity finitely many times in I and they were able to obtain weaker Jackson type estimates the degree of approximation of that function by polynomials which truly comonotone with it as well as some proper Jackson estimates when the polynomials were comonotone with f except near the points where a change of monotenicity of f occurred. Later Newman [32], and also f I live [15], obtained the proper Jackson estimated involving the modulus of continuity of f, for the approximation by polynomials that were truly comonotoned with f. Then Beatson and Leviatan [1] obtained the desired estimates under the assumption that f. Possesses a continuous derivative in I.

Afterwards, in general this topic was developed by a number of good and convincing results which were a accomplished (for more detail see references). Regarding "co convex polynomial approximation" several results have been obtained in the uniform space, but in $L_p(I)$ with $0 < p < \infty$, it seems that nothing like has been achieved.

Let $s \in N_0 = N \cup \{0\}$ and let $Y_s = \{y_i\}_{i=1}^{s}$ be the set of points such that

$$-1=y_0<y_1<y_2<\cdots<y_s<y_{s+1}=1$$

where, $s=0, Y_0=\phi$. for Y_s we set $\pi(x)=\pi(x,Y_s)=\prod_{i=1}^{s}(x-y_i)$ and $\delta(x)=\operatorname{sgn}(\pi(x))$, where the empty product is equal to 1.

Let us now introduce the definition of k $-$monotone functions and discuss some of their properties.

A function $f:[a,b]\to R$ is said to be k $-$monotone ,$k\geq 1$,on $[a,b]$ if and only if for all choices of $k+1$ distinct points $x_0,x_1,\dots,x_k$in $[a,b]$ the inequality $[x_0,x_1,\dots,x_k;f]>0$, holds where

$$[x_0,x_1,\dots,x_k;f]=\sum_{j=0}^{k}\frac{f(x_j)}{\pi_i(x_j)}$$

denoted the *kth divided difference* of f at $x_0,x_1,\dots,x_k$, and

$$\pi_i(x)=\prod_{i=0}^{k}(x-x_i).$$

Note that 1 $-$monotone and 2 $-$monotone functions are just nondecreasing and convex functions , respectively .We denote the class of all k $-$monotone on $[a,b]$, by Δ^k.If $f\in C^k[a,b]$,then $f\in\Delta^k$ if and only if $f^{(k)}(x)\geq 0$,$x\in[a,b]$.Also we denote by $\Delta^k(a,b)$ the classof all k $-$monotone functions that are not required to be defined at the end points of $[a,b]$ (and thus, do not have to be bounded on $[a,b]$) see[3] . For example

$\frac{(-1)^k}{x} \in \Delta^k(0,1)$ for all $k \in N$.

Now ,for $Y_s = \{y_i\}_{i=1}^{s}$ we denote by $\Delta^0(Y_s)$ the set of all functions $f \in L_p$,such that $(-1)^{s-k} f(x) \geq 0$ for $x \in [y_k, y_{k+1}]$,$k = 0, \dots, s$.If $f \in \Delta^0(Y_s)$ has $0 \leq s < \infty$ sign changes at the points in Y_s and is nonnegative. In particular, if $s = 0$,then $\Delta^0 = \Delta^0(Y_s)$ denotes the set of all nonnegative functions on I,we note that $f \in \Delta^0$ iff $f(x) \geq 0$, for all $x \in I$.And a function g is said to be copositive with f if $f(x)g(x) \geq 0$, for all $x \in I$, see[21].

Let $\Delta^2(Y_s)$ be the set of all functions f that change convexity at the points $y_i \in Y_s$, and are convex near 1. In particular, if $s = 0$ then f is convex on I, and write $f \in \Delta^2$, that is the divided differences $[x_0, x_1, x_{2;} f]$ are nonnegative for all choices of three distinct points x_0, x_1 and x_2, where the divided difference of a function f ,see [2] ,at the points $x_0, x_1, \dots, x_n$ are defined by ,

$$[x_0, x_1, x_2, \dots, x_n; f] = \sum_{j=0}^{n} \frac{f(x_j)}{\prod_{\substack{i=0 \\ i \neq j}} (x_j - x_i)},$$

Moreover, if f is twice differentiable function on I (i.e. $f \in C^2(I)$), then $f \in \Delta^2(Y_s)$ if and only if $f''(x)\pi(x) \geq 0, \forall x \in I$. or

if and only if $f''(x)\delta(x) \geq 0, \forall x \in I$.

We are interested in approximating functions from $\Delta^2(Y_s)$ and Δ^2 by polynomials P_n of degree $\leq n$ and implies with no more than n. For $f \in L_p(I),\ p > 0$; let

$$E_n(f, \Pi_n)_p = E_n(f)_p = \inf_{p_n \in \Pi_n} \left\| f - p_n \right\|_p,$$

denoted the degree of unconstrained approximation, or degree of best polynomial approximation of f. And

$$E_n^{(2)}(f, Y_s)_p = \inf_{p_n \in \Pi_n \cap \Delta^2(Y_s)} \left\| f - p_n \right\|_{L_p(I)}$$

The degree of best coconvex polynomial approximation of f (degree of constrained approximation).

In particular, if $Y_0 = \phi, s = 0$ then,

$$E_n^{(2)}(f, Y_0)_p = E_n^{(2)}(f)_p = \inf_{p_n \in \Pi_n \cap \Delta^2} \left\| f - p_n \right\|_{L_p(I)}$$

the degree of the best convex polynomial approximation of f.

Also, let $\Sigma_{r,n}$ be the space of all piecewise polynomial functions of degree$r$ (order $r+1$) with the knots

$$Z_n = (z_i)_{i=0}^{n} = -1 = z_0 < z_1 < \dots < z_n = 1.$$

In other words, let $s_n \in \sum_{r,n}$ and Π_n denotes the space of algebraic polynomials of degree n define on the Chebyshev partition $\{z_i\}_{i=0}^{n}$, we have the degree of f is

$$\delta_{n,r}(f)_p = E(f, \sum_{r,n})_p = \inf\{\left\| f - s_n \right\|_{L_p(I)}, s_n \in \sum_{r,n}\}$$

$$= \inf_{s_n \in \sum_{r,n}} \left\| f - s_n \right\|_{L_p(I)}$$

Note that, for a given $f \in \Delta^2$, take $s \in \sum_{r,n}$ a best (unconstrained free) knot spline approximation to f, i.e.

$$\left\| f - s \right\|_{L_p(I)} = \delta_{n,r}(f)_p,$$

and then correct s to $s^* \in \Delta^2$, a best approximation to s from Δ^2, hence

$$\left\| s - s^* \right\|_{L_p(I)} = \inf_{g \in \Delta^2} \left\| s - g \right\|_{L_p(I)} \le \left\| s - f \right\|_{L_p(I)}$$

and so

$$c(p)\left\| f - s^* \right\|_{L_p(I)} \le \left\| f - s \right\|_{L_p(I)} + \left\| s - s^* \right\|_{L_p(I)} \le 2\left\| f - s \right\|_{L_p(I)} = 2\delta_{n,r}(f)_p.$$

The degree of best piecewise polynomial approximation is define

$$\text{by}\,\delta^{(2)}_{n,r}(f,Y_s)_p=\inf\{\|f-s_n\|_{L_p(I)},s_n\in\textstyle\sum_{k,n}(Y_s)\cap\Delta^2(Y_s)\}$$

also is called the degree of coconvex approximation of f, by spline of order $\le n$.

In particular if $s=0$ then,

$$\delta^{(2)}_{n,r}(f)_p=\inf_{s_n\in\sum_{r,n}\cap\Delta^2}\|f-s_n\|_{L_p(I)}.$$

The degree of convex approximation, by spline of order $r\le n$ in L_p (quasi) norm. And , let

$$\delta^{(2)}_{n,r}(f,Y_s)_p=\inf_{S_n\in\sum_{r,n}\cap\Delta^2(Y_s)\cap C^{r-2}}\|f-s_n\|_{L_p(I)}$$

We can write the above equations by the following forms.

$$\delta^{(2)}_{n,r}(f,Y_s)_p=E\ (f,\textstyle\sum_{r,n}(Y_s)\cap\Delta^2(Y_s))_p$$

also, $$\delta^{(2)}_{n,r}(f)_p=E\ (f,\textstyle\sum_{r,n}\cap\Delta^2)_p$$

We node that Leviadan and Shadrin in 1997 and Petrov in 1998 are proved the following two theorems.

Theorem 1.4.1 **[19]**

Let $r, n \in N, r \geq 2,$ and $0 < p \leq \infty,$ then there exist constants $c_0 = c_0(r)$ and $c_1 = c_1(r, p)$ such that, for all $f \in \Delta^2 \cap L_p(I)$

$$\delta^{(2)}_{c_0 n,r}(f)_p \leq c_1 \delta_{n,r}(f)_p .$$

Theorem 1.4.2 **[19]**

Let $r, n \in N, r \geq 2,$ and $0 < p \leq \infty,$ then there exist constants $c_0 = c_0(r)$ and $c_1 = c_1(r, p)$ such that, for all $f \in \Delta^2(Y_s) \cap L_p(I)$

$$\delta^{(2)}_{c_0 n,r}(f, Y_s)_p \leq c_1 \delta_{n,r}(f, Y_s)_p .$$

Lemma 1.4.3 **[43]**

For $k \in N$ and $1 \leq p \leq \infty$, there exists constant c which depends only on k such that for any $f \in L_p(I)$ and $n \geq k-1$ there exists a polynomial $P_n \in \Pi_n$ such that

$$\left\| f - P_n \right\|_{L_p(I)} \leq c\omega^{\varphi}_k(f, n^{-1})_p .$$

Lemma 1.4.4

Let $f \in L_p(I), 0 < p \leq \infty$, then

$$E_n(f)_p \le E_n^{(2)}(f,Y_s)_p .$$

Proof:

Since, clearly

$$\inf_{p\in\Pi_n} \left\| f-P_n \right\|_{L_p(I)} \le \inf_{q_n\in\Pi_n\cap\Delta^2(y_s)} \left\| f-q_n \right\|_{L_p(I)}$$

then

$$E_n(f)_p \le E_n^{(2)}(f,Y_s)_p \;♦$$

Theorem 1.4.5

Let $f\in L_p(I),\ 0<p\le\infty$ and $P_n\in\Pi_n$ is the best polynomial approximation from Π_n

$$\left\| f-P_n \right\|_{L_p(I)} \le cE_n^{(2)}(f,Y_s)_p$$

where c independent of n and f.

Proof:

By Lemma 1.4.3 and Lemma 1.4.4 ,we have

$$\left\| f-P_n \right\|_{L_p(I)} \le cE_n(f)_p \le cE_n^{(2)}(f,Y_s)_p ♦$$

1.5 Equivalence of Moduli of Smoothness

For a partition $Z_n = \{z_0, \ldots, z_n; -1 = z_0 < z_1 < \cdots < z_n = 1\}$ of the interval $[-1,1]$, denote the scale of the partition Z_n by $v = v(Z_n) = \max\limits_{0 \le j \le n-1} \frac{|J_{j\pm1}|}{|J_j|}$, where , $J_j = [z_j, z_{j+1}]$ with $z_j = -1, j < 0,$ and $z_j = 1, j > n$ and $|J| = measJ.$ And , suppose that $\delta_{\max} = \delta_{\max}(Z_n) = \max\limits_{0 \le j \le n-1} |J_j|$ and $\delta_{\min} = \delta_{\min}(Z_n) = \min\limits_{0 \le j \le n-1} |J_j|$.

We say that Z_n is $\Delta-$quasi uniform if $\Delta = \frac{\delta_{\max}}{\delta_{\min}}$ is bounded by a constant independent of n , and denote such partition by u_n^{Δ}. Note that the 1-quasi uniform partition $u_n = u_n^1$ is just the uniform partition of I into n subintervals of equal lengths.

If $Z_n = u_n^{\Delta}$, then clearly $\frac{z}{(n\Delta)} \le \delta_{\min} \le \frac{z}{n} \le \delta_{\max} \le \frac{z\Delta}{n}$, and $\varphi(Z_n) \le \Delta.$ Therefore, $\delta_{\min} \sim \delta_{\max} \sim n^{-1}$ with equivalence constants depending only on Δ.

We say that α is equivalent to β and write $\alpha \sim \beta$ if there exist a positive constant c such that $c^{-1}\alpha \le \beta \le c\alpha$, we refer to this constant c as equivalence constant.

Theorem 1.5.1 (Local Estimates) [22]

Let $f \in \sum_{r,n} \cap C^{m}(I)$, and $r \in N$, $0 \le m \le r-1$, $J = [z_{M_1}, z_{M_2}]$, with $M_2 - M_1 \le c_0$, for some constant c_0. Then for any $1 \le k \le r+1$ and $0 < p \le \infty$, we have

$$|J|^{v} \omega_{k-v}(f^{(v)}, |J|)_{p} \sim \omega_{k}(f, |J|)_{p},$$

where $1 \le v \le \min\{k, m+1\}$.

Equivalence constants above depend on r, v, c_0 and p as $p \to 0$.

Theorem 1.5.2 (Quasi-Uniform partition) [22]

Let $\overset{\Delta}{u}_{n}, n \in N$, be a $\Delta-$quasi uniform partition of I , and let

$s \in \sum_{r,n}(\overset{\Delta}{u_n}) \cap C^m(I)$, and $r \in N$, $0 \le m \le r-1$, Then, for any $1 \le k \le r+1$ and $0 < p \le \infty$, we have

$$n^{-v}\omega_{k-v}(s^{(v)}, n^{-1})_p \sim \omega_k(s, n^{-1})_p, 1 \le v \le \min\{k, m+1\} \ .$$

Equivalence constants above depend on r, Δ and p as $p \to 0$.

We say that Z_n is a Chebyshev partition (and $z's$ are Chebyshev Knots) if $Z_n = t_n = (t_i)_{i=0}^{n}$ where $t_i = \cos\frac{(n-i)\pi}{n}$, $0 \le i \le n$.

Theorem 1.5.3 (Chebyshev partition) [23]

Let $s \in \sum_{r,n}(t_n) \cap C^m(I)$, and $r \in N$, $0 \le m \le r-1$ Then, for any $1 \le k \le r+1$, $1 \le v \le \min\{k, m+1\}$ and $0 < p \le \infty$, we have

$$n^{-v}\omega^{\varphi}_{k-v,v}(s^{(v)}, n^{-1})_p \sim \omega^{\varphi}_k(s, n^{-1})_p .$$

Equivalence constants above depend on r and p as $p \to 0$.

Theorem 1.5.4 (Chebyshev knots) [10]

Let t_n be a Chebyshev partition of I if $s \in \sum_{r,n}(t_n), r \in N$ then for any $1 \le k \le r+1$ and $0 < p \le \infty$, we have

$$\omega^{\varphi}_{k-v,v}(s^{(v)}, n^{-1})_p \le c(r,v) n^{v} \omega^{\varphi}_k(s, n^{-1})_p ,$$

for all $1 \le v \le k$.

Theorem 1.5.5 [14]

Let $\overset{\Delta}{u}_n, n \in N$ be a $\Delta-$quasi uniform partition of I , and let $s \in \sum_{r,n}(\overset{\Delta}{u}_n), r \in N$. Then, for any $1 \le k \le r+1$ and $0 < p \le \infty$, we have

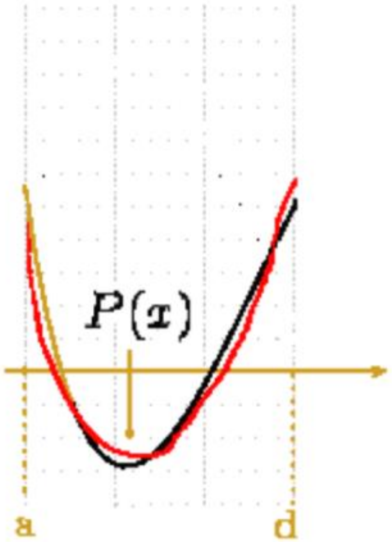
$$\omega_{k-v}(s^{(v)}, n^{-1})_p \le c(r,\Delta,p) n^v \omega_k(s, n^{-1})_p$$

for all $v = 1,2,...,k.$

1.6 Geometric Meaning

We can explain shapes of the convex polynomial approximation and the coconvex polynomial approximation in Figure 1 and Figure 2, respectively and we explained the error between the origin function and polynomial approximation to it ,see [19].

Figure 1



CONVEX
POLYNOMIAL
APPROXIMATION

$P'' \in \Delta^0$

$$E_n^{(2)}(f)_p = \inf\{\|f - P\|_p \mid P''(x) \geq 0\}$$

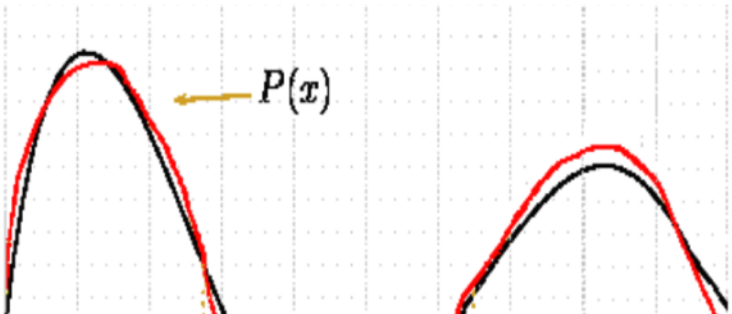


Figure 2

$$E_n^{(2)}(f,Y_s)_p = \inf\left\{\|f-P\|_p \,\middle|\, P''(x).f'(x) \ge 0\right\}$$

COCONVEX POLYNOMIAL APPROXIMATION

$$Y_s = \{u\,,w\}\,, s = 2\,, P'' \in \Delta^0(Y_s)$$

We can notice that the error in the convex (or Coconvex) polynomial approximation is less than the error in monotone (or Commonotone) polynomial approximation , respectively as well as this less than the error in the positive (or Copositive) polynomial approximation respectively.

Chapter One

Introduction and Preliminaries

1.1 Approximations Theory and Overview

Our interest in approximation theory stems from its beauty, its utility, and its rich history. There are also many connections that can be drawn to questions in both classical and modern analysis.

The main idea of approximation includs in finding for a complicated function f from a large space X a close by, a simple function ψ from a small subset Y of X. There are three elements here. The large space X is usually a normed space, such as C, L_p with $0 < p < \infty$ or one of the other Banach spaces of functions. The distance between ψ and f can be measured by $\|f - \psi\|$ in X. Finally, we have to define the special functions of Y.

The authenticity of the approximation method due to Chebyshev can be thought of as well established . This make it all the more interesting to investigate the source of the impetus that led our genius fellow countryman to his brilliant and deep constructions.

There is nothing mysterious about it . Chebyshev himself said in perfectly clear way, about the purpose of his research in approximation theory and names the person whose results were his starting point.

In 1853 , the great Russian mathematician , P. L. Chebyshev , while working on a problem of linkages , devices which translate the linear motion of a steam engine into the circular motion of a wheel , considered the following problem:

Given a continuous function f defined on a closed interval $[a,b]$ and a positive integer n , can we "represent" f by a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$, of degree at most n , in such a way that, the maximum error at any point x in $[a,b]$ is controlled ? In particular , is it possible to construct p in such a way that the error $\max\limits_{a \le x \le b} |f(x) - p(x)|$ is minimized?

This problem raises several questions , the first of which Chebyshev himself ignored :

- Why should such a polynomial even exist?
- If it does , can we hope to construct it?
- If it exists , is it also unique?
- What happens if we change the measure of the error to, say , $\int_a^b |f(x) - p(x)|\, dx$?

Chebyshev's problem is perhaps best understood by rephrasing it in modern terms.

What we have here is a problem of linear approximation in a normed linear space . Recall that a norm on a (real) vector space X is a nonnegative function on X satisfying

$$\|x\| = 0 \;\Leftrightarrow\; x = 0$$

$$\|\alpha x\| = |\alpha|\|x\| \quad for\ all\ \ \alpha \in R$$

$$\|x + y\| \le \|x\| + \|y\| \quad for\ any\ x, y \in X\ .$$

Any norm on X induced a metric or distance function by setting $dist(x,y) = \|x - y\|$. The abstract version of our problems can now be restated:

Given a subset (or even a subspace) Y of X and a point $x \in X$, is there an element $y \in Y$ which is “nearest” to x ; that is , can we find a vector $y \in Y$ such that $\|x - y\| = inf_{z \in Y} \|x - z\|$? If there is such a “best approximation “ to x ; from elements of Y , is it unique?

Now, we state some results for the existence and uniqueness of best approximation, answer for the above questions .

Let X be a normed linear space and $X \supseteq Y$, is a finite-dimensional subspace of X, then for each element $x \in X$ there exists not necessary a unique element $y_x \in Y$ that is a best approximation from Y to x ,and map $x \to y_x$,and let Y be a subspace of a normed linear space X , and let $x \in X$ the set Y_x consisting of all best approximation to x out of Y is bounded , convex and closed set, also in case Y be a compact and strictly convex set in a normed linear space, X then for each $x \in X$, there is just one best approximation from Y to x, see [17] .

A new content was given to the theory of best approximations of functions in the first decade of the current century by Weierstrass the head of Berlin school of mathematics.The proposition which he established in 1885, is that, for any $f \in C[a, b]$ and for any $\varepsilon > 0$ there exists an algebraic polynomial of the form $p(x) = c_0 + c_1 x + \cdots + c_n x_n$, $a \leq x \leq b$, such that: the bound $\|f - p\|_\infty \leq \varepsilon$ is satisfied, see[38]

The basic of the approximation theory of functions of real variable is a theorem study by Weierstrass which is of great importance in the development of the whole of mathematical analysis, see [41].

Note that, Fundamental theorem of approximation theory , let

$f \in C[a,b], -\infty < a < b < \infty$ Given $\varepsilon > 0$, there exists an algebraic polynomial p for which

$$|f(x) - p(x)| < \varepsilon \qquad \text{for all } x \in [a,b] ,$$

there are more than proofs to Fundamental theorem ,see [37].

Given the values of a function $f(x)$ at two distinct values of x ,say x_0 and x_1 , we could approximate f by a linear function p that satisfies the conditions

$$p(x_0) = f(x_0) \qquad and \qquad p(x_1) = f(x_1) .$$

We can construct the linear interpolating polynomial directly , writing $p(x) = ax + b$ and using the above two conditions to give two linear equations to determine a and b . On solving these equastions , we obtain

$$p(x) = \frac{x_1 f(x) - x_0 f(x_1)}{x_1 - x_0} + x\left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right).$$

This can also be expressed in the Lagrange symmetric form

$$p(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right) f(x_1) ,$$

or in Newton's divided difference form

$$p(x) = f(x_0) + (x - x_0)\left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right) ,$$

to which we will return later . Observe that if we write $x_1 = x_0 + h$ in above equation , the limit of $p(x)$ as $h \to 0$ gives the first two terms of the taylor series for f is differentiable .

It is convenient to denote the set of all polynomials of degree at most n by Π_n . Given the values of a function $f(x)$ $at\, n+1$ distinct values of x, say $x_0, x_1, \dots, x_n$, can we find a polynomial $p_n \in \Pi_n$,say

$$p_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \ ,$$

Such that $p_n(x_j) = f(x_j)$, , for $j = 0,1,\ldots,n$ this means that we require

$$a_0 + a_1x_j + a_2x_j^2 + \cdots + a_nx_j^n = f(x_j), \quad 0 \le j \le n \ .$$

In the other hand ,we obtain

$$a_0 = f(x_0) \qquad \text{and} \qquad a_1 = \frac{f(x_1)-f(x_0)}{x_1-x_0} \ ,$$

we will write

$$a_j = [x_0, x_1, \ldots, x_j; f], \quad 0 \le j \le n ,$$

to emphasize its dependence on f and $x_0, x_1, \ldots, x_j$,and refer to a_j as a jth divided difference . Thus we may write p_n in the form

$$p_n(x) = [x_0; f]\pi_0(x) + [x_0, x_1; f]\pi_1(x) + \cdots + [x_0, x_1, \ldots, x_n; f]\pi_n(x) ,$$

which is *Newton's divided difference formula* for the interpolating polynomial . Observe that $[x_0; f] = f(x_0)$,in general we will find it more appropriate to use another notation for divided differences, where we write $[x_0, x_1, \ldots, x_n; f]$ instead of $f[x_0, x_1, \ldots, x_n]$ is a symmetric function of its arguments $x_0, x_1, \ldots, x_n$.

Where,

$$\pi_i(x) = \begin{cases} 1 , & i = 0, \\ (x - x_0)(x - x_1) \ldots (x - x_{i-1}), & 1 \le i \le n. \end{cases}$$

The divided difference $[x_0, x_1, \ldots, x_n; f]$ can be expressed as the following symmetric sum of multiples of $f(x_j)$,

$$[x_0, x_1, \ldots, x_n; f] = \sum_{r=0}^{n} \frac{f(x_r)}{\prod_{\substack{j \ne r \\ j=0}}^{n} (x_r - x_j)} \ ,$$

where in the above product of n factors , r remains fixed and j takes all values from 0 to n , excluding r .

In recent years they have been extensively studied by many authors most notably Ditzian , DeVore, Hristov, Jiang, Ivanov, Leviatan and Kopotun [8] , [12] , [13] , [27] , [36]) .

We shall give a simple example about the idea of approximation theory, let us construct the interpolating polynomial $p_4(x)$ for the function 2^x based on the points $-2,-1,0,1$ and 2 ; hence estimate $2^{1/2}=\sqrt{2}$ by evaluating $p_4\left(\frac{1}{2}\right)$.

For $x=-2\,,-1,0,1,2$, we have $2^x=\frac{1}{4},\frac{1}{2},1,2,4$, respectively .Then , the following method discussed above ,we find that the forward difference form of the interpolating polynomial is

$$p_4(x)=\frac{1}{4}+\frac{1}{4}(x+2)+\frac{1}{8}(x+2)(x+1)+\frac{1}{24}(x+2)(x+1)x$$

$$+\frac{1}{96}(x+2)(x+1)x(x-1).$$

On evaluating $p_4(x)$ at $x=\frac{1}{2}$, we obtain

$$\frac{723}{512}\approx 1.4121.$$

Since $\sqrt{2}\approx 1.4142$,this interpolation method has provided an approximation whose error is in the third digit after the decimal point.

There are many important applications of approximation theory, as it was begun itself by application, by the Great Russian scientist P. L. Chebyshev, and he said about the relationship between a mathematical theory and its applications illustrate quite clearly not just the source of creativity, but also scientific and philosophical positions. In this work, we give some of its applications in mathematics, which are:

$\mathcal{A}$) *Solving algebraic equations* . In the memoir [22], there are about ten theorems (6-10, 15-19) that are derived from the basic propositions on best approximation. These theorems state that, under certain conditions the polynomial of interest has at least one zero in some interval. The length of the interval depends on the one hand, on the value of the polynomial at the center of the interval, on the other hand, on specific assumption on the coefficients or on the zeros of the polynomial. For example that: if the polynomial $P(x) = x^{2n+1} + \cdots + k$,does not contain any even power of x, then it has at least one zero in interval $|x| \leq 2\left(\frac{|k|}{2}\right)^{\frac{1}{2n+1}}$. Let us also formulate theorem which uses a different assumption: if a polynomial $P(x)$of degree n with leading coefficient equals to one, and has only real roots, then for any t there is a real root in the interval $|x - t| \leq 4\left(\frac{P(t)}{4}\right)^{\frac{1}{n}}$. Later in his 1872 memoir, using a result on monotone polynomials, Chebyshev narrowed the interval by replacing with $|x - t| \leq 4\left(\frac{P(t)}{2(n-1)\pi}\right)^{\frac{1}{n}}$.A further improvement of this result is due to A. A. Markove [30].

$\mathcal{B}$) *Interpolation (remainder estimate)*. To minimize the error in the Lagrange interpolation formula, Chebyshev suggests taking the nodes of the interpolation (say in the interval $(-1,1)$) to be the zero of the polynomial $T_n(x) = \cos(n \arccos(x))$, since for a given function $f(x)$ the remainder has the form $R(x) = \frac{f^{(n+1)}(\lambda)}{(n+1)!} P_n(x)$, where $\lambda \in (-1,1)$, $P_n(x) = \prod_{k=1}^{n} (x - x_k)$ and x_k are the nodes.

Therefore with $R(x) \le \max \left| \frac{f^{(n+1)}(\lambda)}{(n+1)!} \right| \max \left| P_n(x) \right|$,

the choice $P_n(x) = T_n(x)$ is the most profitable. Here Chebyshev partly envisions the later result of Range that says that as $n \to \infty$ Chebyshev interpolation converges for any function that is regular in the basic interval, while this is not true for Newton interpolation with equally space nodes.

C) A rule for finding approximately distance on the surface of the Earth .

Let us quote it completely :"(1) Take the distance between the two latitudes and the two longitude ,and express them in minutes,(2) Double the difference of the latitudes ,(3) Out of these two numbers the difference of longitudes and the doubled difference latitudes , multiply the smaller by three, multiply the longer one by seven and then add the two products,(4) the result divided by three will give the desired distance in verests" . A further improvement is due to Markov[30].

D) *Constructing geographic maps* . If one needs to draw on a map some piece of the Earth surface with a given boundary , then there is a choice among infinitely many projections that provide the infinitesimal similarity and presentation of the scale in each point and in all directions .There are the so called conformal projections . However as follows from Gauss " Theorma egregium ", among all conformal projections of a ball to a plane , it is impossible to find a projections that would preserve the scale for all points of the surface .In a talk given on 30 January1865 published under the title “Sur la constructions des cartes geographiques ”[31], Chebyshev posed the problem of finding of finding a conformal projection for which the logarithm of the scale would vary within tightest possible deviation from some average value. Without a proof , he claimed that if the specified condition is satisfied , then the scale should be constant on the boundary of the map. Chebyshev statement was proved much later by academician D. A. Grave in [28] and also in [11].

Recently ,approximation theory has also many other applications both numerical and analytical. The most prominent of these are to image processing , statistical estimation and the numerical and analytic treatment of differential equations.

1.2 The Spaces L_p, $p<1$.

The degree of constrained and unconstrained approximation of a function f in either the uniform norm or in the $L_p(I)$, $0<p<\infty$ quasi-norm will be studied,where $I=[-1,1]$. The degree of approximation will be measured by the appropriate quasi-norm which we denote by $\|.\|_p=\|.\|_{L_p(I)}$. Since we need the L_p quasi-norm on other intervals we will in all cases of an interval $J\neq I$, indicate that by writing $\|.\|_{L_p(J)}$. Also we refer to ($\|.\|_p$with $p=\infty$), the uniform norm on I and we denote by $\|.\|$ and on the interval J by $\|.\|_J$ Furthermore, $\|.\|_p$ is a norm for $1\leq p<\infty$.

Characteristic for $L_p, p \geq 1$ are the inequalities of Holder and Minkowski.

The dual space of $L_p, 1 \leq p < \infty$, is the spaces L_q with the conjugate exponent q of p. Thus the spaces L_p with $1 \leq p < \infty$ are reflexive.

The different structure of the spaces L_p with $0 < p < 1$ and the numerous questions by others lead us to understand the need for the following few facts about L_p for $p < 1$.

We consider the space $L_p(I)$, consisting of all measurable functions f on I, for which

$\| f \|_p^p = \int_I | f(x) |^p dx < \infty$. Recall that $\| f \|_p \leq 2^{\frac{1}{p}-1} \| f \|_1$, that is $L_1 \subset L_p$.

As we will see shortly, the L_p is not actually a norm space for $p < 1$. Nevertheless, it is not hard to see that $L_p(I)$ is a complete metric space.

Theorem 1.1 **[3]**

$$\| f + g \|_p \leq \left(\| f \|_p^p + \| g \|_p^p \right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}} \left(\| f \|_p + \| g \|_p \right), \text{ for any } f, g \in L_p(I).$$

Thus $\mathrm{d}(f, g) = \| f - g \|_p^p$ defines a translation invariant metric on

L_p. It is a complete metric, because convergence (respectively Cauchy) in L_p implies convergence (respectively Cauchy) in measure since $\varepsilon^p meas\{t : |f(t)| > \varepsilon\} \le \int_{-1}^{1} |f(t)|^p dt$, where by "*meas*" , we mean the measure of a set.

What this means to us is that :

(i) A linear map on L_p is continuous if and only if it is bounded (continuous at zero).

(ii) The open mapping and closed graph theorems still apply.

(iii) The Hahn Banach theorem may fail!

Indeed, as we will see shortly, L_p is not locally convex. In fact, it is impossible to define a norm on L_p which gives the same topology as the usual metric. There are several ways to see that $L_p(I)$ is not normable for $0 < p < 1$. Most useful from our point of view is

***Theorem* 1.2** [6]

L_p, $0 < p < 1$ has a trivial dual.

The Hahn Banach theorem allows a much fancier sounding version of this result:

***Corollary* 1.3** [6]

There are no non zero continuous linear map from L_p into any normed space.

In any event, it should now be clear that there can be no norm on L_p which gives the same topology as the usual metric and that the Hahn Banach theorem evidently fails in $L_p(I)$, for $0 < p < 1$. The fact that L_p has a trivial dual with the theorem from [45] states that there exists a nonzero continuous linear function on a linear space X, if and only if there is at least one convex set that is not all of X, imply another rather strange result that would be hard to believe otherwise.

***Corollary* 1.4** [6]

If U is any neighborhood of zero in $L_p(I)$, then

$$L_p(I) = conv(U)$$

In particular

$$L_p(I) = conv\{f : \|f\|_p^p < 1\},$$

where $conv(U)$ is a smallest convex neighborhood of zero contains U.

Now, we will settle the question of which $L_p, p < 1$, embed into L_q for $q \geq 1$. Or which subspaces of $L_p(I)$ on which all of the various $L_p(I)$

quasi-norms for $0 < p < q$ are equivalent. The key in this article is from [11].

For $0 < \varepsilon < 1$ and $0 < p < \infty$, consider the following subset of $L_p(I)$

$$M(p,\varepsilon) = \{ f \in L_p(I) : meas\{x : |f(x)| \geq \varepsilon \|f\|_p\} \geq \varepsilon\}.$$

Notice that if $\varepsilon_1 < \varepsilon_2$, then $M(p,\varepsilon_2) \subset M(p,\varepsilon_1)$. Also $\bigcup_{\varepsilon>0} M(p,\varepsilon) = L_p(I)$, since for any nonzero $f \in L_p(I)$ we have $meas\{|f| \geq \varepsilon\} \to meas\{f \neq 0\}$ as $\varepsilon \to 0$. In fact, any finite subset of $L_p(I)$ is contained in an $M(p,\varepsilon)$ for same $\varepsilon > 0$. Finally note that $meas\{|f| \geq \|f\|_p\} \geq 1$ implies $|f| = \|f\|_p$ almost everywhere.

The following theorem puts this observation to good use

***Theorem* 1.5** [4]

For a subset S of $L_p(I)$, the following are equivalent

(i) $S \subset M(p,\varepsilon)$ for some $\varepsilon > 0$.

(ii) For each $0 < p < q$, there exists a constant $c(q) < \infty$ such that

$$\|f\|_q \leq \|f\|_p \leq c(q)\|f\|_q \text{ for all } f \in S.$$

(iii) For some $0 < p < q$, there exists a constant $c(q) < \infty$ such that

$$\|f\|_q \leq \|f\|_p \leq c(q)\|f\|_q, \text{ for all } f \in S.$$

In our thesis, we will use the notations c and C to denote such absolute constants which are of no significance to us and may differ on different occurrences , even in the same line. In order to emphasize that c or C depends only in parameters $v_1, v_2, \ldots\ldots., v_k$, the notation $c(v_1, v_2, \ldots\ldots., v_k)$ (respectively $C(v_1, v_2, \ldots\ldots., v_k)$) is used, if one of those v_i is $= p$, we mean that constant depends on p, only in the case $p < 1$. Also we will have constants $c_1, c_2, \ldots$ when we have a reason to keep trace of them in the computations that we have to curry in the proofs.

1.3 Moduli of Smoothness

Moduli of smoothness are intended for mathematicians working in approximation theory, numerical analysis and real analysis, it is very useful for measuring the smoothness of a function by differentiability is too crude for many purpose in approximation theory. More subtle measurement is proved by the moduli of smoothness. We will use moduli of smoothness which are connected with difference of higher orders.

The rth symmetric difference of f is given by:

$$\Delta_h^r(f, x) = \Delta_h^r(f, x, [a, b])$$

$$\Delta_h^k(f, x) = \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f\left(x - \frac{kh}{2} + ih\right) & , \ x \pm \frac{kh}{2} \in I \\ 0, & o.w. \end{cases}$$

Then, *the rth usual modulus of smoothness of* $f \in L_p[a, b]$ is defined by

$$\omega_r(f,t,I)_p = \sup_{0<h\le t}\left\|\Delta_h^r(f,.)\right\|_{L_p(I)}, t \ge 0.$$

A new way of measuring smoothness was introduced by Ditzian and Totik [43]. The need for this new concept arises from the failure of the classical moduli of smoothness to solve some basic problems, such as characterizing the behavior of best polynomial approximation in $L_p(I)$.

The Ditzian-Totik modulus of smoothnessof $f \in L_p(I), 0 < p \le \infty$, [10] which is defined for such an f as follows

$$\omega_r^{\varphi}(f,t,I)_p = \sup_{0<h\le t}\left\|\Delta_{h\varphi(\cdot)}^r(f,.)\right\|_{L_p(I)}.$$

In the applications the φ usually used

$\varphi(x) = (1-x^2)^{\frac{1}{2}}$ for $I = [-1,1]$

$\varphi(x) = (x(1-x))^{\frac{1}{2}}$ and $\varphi(x) = \sqrt{x}(1-x)$ for $I = [0,1]$

$\varphi(x) = \sqrt{x}$, $\varphi(x) = (x(1+x))^{\frac{1}{2}}$ and $\varphi(x) = x$ for $I = (0,\infty)$,

we have

$$\omega_r^{\varphi}(f,t)_p \le \omega_r(f,t)_p, \quad 0 < p < \infty.$$

The weighted Ditzian-Totik kth modulus of smoothness of function $f \in L_p(I), 0 < p \le \infty,$ is defined by

$$\omega_{k,r}^{\varphi}(f,t)_p = \sup_{0<h\le t} \left\| \varphi^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f,t) \right\|_{L_p(I)}$$

where $\varphi(x) = \sqrt{1-x^2}$. If $r = 0,$ then $\omega_{k,0}^{\varphi}(f,t)_p = \omega_k^{\varphi}(f,t)_p$.

We shall prove the relationship between $\omega_r^{\varphi}(f,t)_p$ and $\omega_{k,r}^{\varphi}(f,t)_p$ in chapter two in case $0 < p \le \infty.$

We collect below some properties of $\omega_r(f,t)_p$ and $\omega_r^{\varphi}(f,t)_p$, that are also true for $0 < p < 1$ and for the D.T. modulus of smoothness, we are referred to [7, equation (12.5)], [9], [43] for references

$$\omega_r(f,t)_p \le \omega_r(f,\lambda t)_p \le c\omega_r(f,t)_p,$$

$$\omega_r^{\varphi}(f,t)_p \le \omega_r^{\varphi}(f,\lambda t)_p \le c\omega_r^{\varphi}(f,t)_p,$$

$$\omega_r^{\varphi}(f,t)_p \le c\|f\|_p,$$

$$\omega_r(f+g,t)_p^q \le \omega_r(f,t)_p^q + \omega_r(g,t)_p^q,$$

$$\omega_r^{\varphi}(f+g,t)_p^q \le \omega_r^{\varphi}(f,t)_p^q + \omega_r^{\varphi}(g,t)_p^q.$$

where $0 < p \le \infty, \lambda > 1, q = \min(1,p)$,and c is a constant depending only on r , q and also on λ if applicable.

Also now let us compare between the classical modulus of smoothness and its extension to ω_r^{φ} by the following properties :The following most basic facts about $\omega_r(f,t)_p$ and $\omega_r^{\varphi}(f,t)_p$ satisfied where $0 < p \le \infty$

(a) $\lim\limits_{t\to\infty} \omega_r(f,t)_p = 0$ for all $f \in L_p(I), 0 < p < \infty.$

(b) $\omega_r(f,t)_p$ is a nondecreasing function of t.

($\bar{\mathrm{b}}$) $\omega_r^{\varphi}(f,t)_p$ is a nondecreasing function of t.

(c) $\omega_r(f,\lambda t)_p \le c(p)\lambda^r \omega_r(f,t)_p$ for $\lambda \ge 1.$

(d) $\omega_{r+1}(f,t)_p \le 2\omega_r(f,t)_p$.

which is generalized using the following inequality in [35]

($\bar{\mathrm{d}}$)$\omega_{r+1}^{\varphi}(f,t)_p \le c(p)\omega_r^{\varphi}(f,t)_p$.

Another inequality about the classical modulus of smoothness is

(e) $\omega_r(f,\lambda t)_p \le c\lambda^r \omega_r(f,t)_p$, for $\lambda \ge 1$.

($\bar{e}$) Another inequality about $\omega_r^{\varphi}(f,t)$ from [3]

$$\omega_r^{\varphi}(f,\lambda t)_p \le c\lambda^r \omega_r^{\varphi}(f,t)_p, \text{ for } \lambda \ge 1.$$

There is another basic property of the classical modulus of smoothness:

(f) $\omega_{k+r}(f,t,J)_p \le ct^r \omega_k(f^{(r)},t,J)_p,\ p \ge 1.$

Which is valid if $f^{(r)} \in L_p(J)$ and $f^{(r-1)}$ is absolutely continuous in every closed interval $J = [a,b]$. The inequality f is not true if $0 < p < 1$.

Where the constants involved depending on the location of the points of change of convexity, also we show that in some cases the constants may be taken an independent of the points of change of convexity, but that in some other cases this dependence is essential.

We will introduce some theorems about properties of weighted Ditzian-Totik modulus of smoothness in the case $0 < p \le \infty$, in Chapter two.

1.4 Shape Preserving Approximation

The fundamental problem of the theory of approximation of functions interested with the connection between the structural properties of a function and its degree of approximation. It is to relate the smoothness of the function to the rate of decrease of the degree of approximation to zero. We are concerned in our work of examining these questions for algebraic polynomial approximation and piecewise polynomial approximation (spline).

In many applications, it is desirable that the mathematical moduli preserve certain geometric properties of the data such as monotenicity, convexity and in general $k-$ monotenicity. This is the subject that the so called shape preserving approximation deals with.

Recently there has been more attention given to approximation with constrains. The appearance of constrains can make it more difficult to obtain direct estimates. Still the general lines of attack development for the non constrained problem can be very useful. We want to indicate what modifications are necessary to push the approach through. We do this for comonotone approximation.

In coconvex polynomial approximation, we give a function f that changes its convexity finitely many times in the interval I. We are interested in estimating the degree of approximation of f by polynomials which are convex with it, namely polynomials that change their convexity exactly at the points where f does. Question of this nature first appeared in the work of D.J Newman see [1], [33] and [34]. They did not deal with a function f that changes convexity, but rater with f changes monotonicity finitely many times in I and they were able to obtain weaker Jackson type estimates the degree of approximation of that function by polynomials which truly comonotone with it as well as some proper Jackson estimates when the polynomials were comonotone with f except near the points where a change of monotenicity of f occurred. Later Newman [32], and also f I live [15], obtained the proper Jackson estimated involving the modulus of continuity of f, for the approximation by polynomials that were truly comonotoned with f. Then Beatson and Leviatan [1] obtained the desired estimates under the assumption that f. Possesses a continuous derivative in I.

Afterwards, in general this topic was developed by a number of good and convincing results which were a accomplished (for more detail see references). Regarding "co convex polynomial approximation" several results have been obtained in the uniform space, but in $L_p(I)$ with $0 < p < \infty$, it seems that nothing like has been achieved.

Let $s \in N_0 = N \cup \{0\}$ and let $Y_s = \{y_i\}_{i=1}^{s}$ be the set of points such that

$$-1 = y_0 < y_1 < y_2 < \cdots < y_s < y_{s+1} = 1$$

where, $s = 0, Y_0 = \phi.$ for Y_s we set $\pi(x) = \pi(x, Y_s) = \prod_{i=1}^{s}(x - y_i)$ and $\delta(x) = \operatorname{sgn}(\pi(x))$, where the empty product is equal to 1.

Let us now introduce the definition of $k-$monotone functions and discuss some of their properties.

A function $f\colon [a,b] \to R$ is said to be $k-$monotone ,$k \geq 1$,on $[a,b]$ if and only if for all choices of $k+1$ distinct points $x_0, x_1, \dots, x_k$in $[a,b]$ the inequality $[x_0, x_1, \dots, x_k; f] > 0$, holds where

$$[x_0, x_1, \dots, x_k; f] = \sum_{j=0}^{k} \frac{f(x_j)}{\pi_i(x_j)}$$

denoted the *kth divided difference* of f at $x_0, x_1, \dots, x_k$, and

$$\pi_i(x) = \prod_{i=0}^{k} (x - x_i).$$

Note that $1-$monotone and $2-$monotone functions are just nondecreasing and convex functions , respectively .We denote the class of all $k-$monotone on $[a,b]$, by Δ^k.If $f \in C^k[a,b]$,then $f \in \Delta^k$ if and only if $f^{(k)}(x) \geq 0$, $x \in [a,b]$.Also we denote by $\Delta^k(a,b)$ the classof all $k-$monotone functions that are not required to be defined at the end points of $[a,b]$ (and thus, do not have to be bounded on $[a,b]$) see[3] . For example $\frac{(-1)^k}{x} \in \Delta^k(0,1)$ for all $k \in N$.

Now ,for $Y_s = \{y_i\}_{i=1}^{s}$ we denote by $\Delta^0(Y_s)$ the set of all functions $f \in L_p$,such that $(-1)^{s-k} f(x) \geq 0$ for $x \in [y_k, y_{k+1}]$,$k = 0, \dots, s$.If $f \in \Delta^0(Y_s)$ has $0 \leq s < \infty$ sign changes at the points in Y_s and is nonnegative. In particular, if $s = 0$,then $\Delta^0 = \Delta^0(Y_s)$ denotes the set of all nonnegative functions on I,we note that $f \in \Delta^0$ iff $f(x) \geq 0$, for all $x \in I$.And a function g is said to be copositive with f if $f(x)g(x) \geq 0$, for all $x \in I$, see[21].

Let $\Delta^2(Y_s)$ be the set of all functions f that change convexity at the points $y_i \in Y_s$, and are convex near 1. In particular, if $s = 0$ then f is convex on *I*, and write $f \in \Delta^2$, that is the divided differences $[x_0, x_1, x_{2;} f]$ are nonnegative for all choices of three distinct points x_0, x_1 and x_2, where the divided difference of a function f ,see [2] ,at the points $x_0, x_1, \ldots, x_n$ are defined by ,

$$[x_0, x_1, x_2, \ldots, x_n; f] = \sum_{j=0}^{n} \frac{f(x_j)}{\prod\limits_{\substack{i=0 \\ i \neq j}} (x_j - x_i)},$$

Moreover, if f is twice differentiable function on I (i.e. $f \in C^2(I)$), then $f \in \Delta^2(Y_s)$ if and only if $f''(x)\pi(x) \geq 0, \forall x \in I$. or

if and only if $f''(x)\delta(x) \geq 0, \forall x \in I$.

We are interested in approximating functions from $\Delta^2(Y_s)$ and Δ^2 by polynomials P_n of degree $\leq n$ and implies with no more than n. For $f \in L_p(I),\ p > 0$; let

$$E_n(f, \Pi_n)_p = E_n(f)_p = \inf_{p_n \in \Pi_n} \left\| f - p_n \right\|_p,$$

denoted the degree of unconstrained approximation, or degree of best polynomial approximation of f. And

$$E_n^{(2)}(f,Y_s)_p = \inf_{p_n \in \Pi_n \cap \Delta^2(Y_s)} \left\| f - p_n \right\|_{L_p(I)}$$

The degree of best coconvex polynomial approximation of f (degree of constrained approximation).

In particular, if $Y_0 = \phi, s = 0$ then,

$$E_n^{(2)}(f,Y_0)_p = E_n^{(2)}(f)_p = \inf_{p_n \in \Pi_n \cap \Delta^2} \left\| f - p_n \right\|_{L_p(I)}$$

the degree of the best convex polynomial approximation of f.

Also, let $\sum_{r,n}$ be the space of all piecewise polynomial functions of degree$r$ (order $r+1$) with the knots

$$Z_n = (z_i)_{i=0}^{n} = -1 = z_0 < z_1 < \ldots < z_n = 1.$$

In other words, let $s_n \in \sum_{r,n}$ and Π_n denotes the space of algebraic polynomials of degree n define on the Chebyshev partition $\{z_i\}_{i=0}^{n}$, we have the degree of f is

$$\delta_{n,r}(f)_p = E(f,\textstyle\sum_{r,n})_p = \inf\{\left\| f - s_n \right\|_{L_p(I)}, s_n \in \textstyle\sum_{r,n} \}$$

$$= \inf_{s_n \in \sum_{r,n}} \left\| f - s_n \right\|_{L_p(I)}$$

Note that, for a given $f \in \Delta^2$, take $s \in \sum_{r,n}$ a best (unconstrained free) knot spline approximation to f, i.e.

$$\|f-s\|_{L_p(I)} = \delta_{n,r}(f)_p ,$$

and then correct s to $s^* \in \Delta^2$, a best approximation to s from Δ^2, hence

$$\left\|s-s^*\right\|_{L_p(I)} = \inf_{g\in\Delta^2}\|s-g\|_{L_p(I)} \le \|s-f\|_{L_p(I)}$$

and so

$$c(p)\left\|f-s^*\right\|_{L_p(I)} \le \|f-s\|_{L_p(I)} + \left\|s-s^*\right\|_{L_p(I)} \le 2\|f-s\|_{L_p(I)} = 2\delta_{n,r}(f)_p .$$

The degree of best piecewise polynomial approximation is define by

$$\delta^{(2)}_{n,r}(f,Y_s)_p = \inf\{\left\|f-s_n\right\|_{L_p(I)}, s_n \in \textstyle\sum_{k,n}(Y_s)\cap\Delta^2(Y_s)\}$$

also is called the degree of coconvex approximation of f, by spline of order $\le n$.

In particular if $s=0$ then,

$$\delta^{(2)}_{n,r}(f)_p = \inf_{s_n\in\Sigma_{r,n}\cap\Delta^2}\left\|f-s_n\right\|_{L_p(I)} .$$

The degree of convex approximation, by spline of order $r \le n$ in L_p (quasi) norm. And , let

$$\delta^{(2)}_{n,r}(f,Y_s)_p = \inf_{S_n \in \Sigma_{r,n} \cap \Delta^2(Y_s) \cap C^{r-2}} \| f - s_n \|_{L_p(I)}$$

We can write the above equations by the following forms.

$$\delta^{(2)}_{n,r}(f,Y_s)_p = E\ (f, \sum_{r,n}(Y_s) \cap \Delta^2(Y_s))_p$$

also, $$\delta^{(2)}_{n,r}(f)_p = E\ (f, \sum_{r,n} \cap \Delta^2)_p$$

We node that Leviadan and Shadrin in 1997 and Petrov in 1998 are proved the following two theorems.

Theorem 1.4.1 **[19]**

Let $r,n \in N, r \geq 2,$ and $0 < p \leq \infty,$ then there exist constants $c_0 = c_0(r)$ and $c_1 = c_1(r,p)$ such that, for all $f \in \Delta^2 \cap L_p(I)$

$$\delta^{(2)}_{c_0 n,r}(f)_p \leq c_1 \delta_{n,r}(f)_p .$$

Theorem 1.4.2 **[19]**

Let $r,n \in N, r \geq 2,$ and $0 < p \leq \infty,$ then there exist constants $c_0 = c_0(r)$ and $c_1 = c_1(r,p)$ such that, for all $f \in \Delta^2(Y_s) \cap L_p(I)$

$$\delta^{(2)}_{c_0 n,r}(f,Y_s)_p \leq c_1 \delta_{n,r}(f,Y_s)_p .$$

***Lemma 1.4.3* [43]**

For $k \in N$ and $1 \le p \le \infty$, there exists constant c which depends only on *k* such that for any $f \in L_p(I)$ and $n \ge k-1$ there exists a polynomial $P_n \in \Pi_n$ such that

$$\left\| f - P_n \right\|_{L_p(I)} \le \ c\omega_k^{\varphi}(f, n^{-1})_p .$$

Lemma 1.4.4

Let $f \in L_p(I), 0 < p \le \infty$, then

$$E_n(f)_p \le E_n^{(2)}(f, Y_s)_p .$$

Proof:

Since, clearly

$$\inf_{p \in \Pi_n} \left\| f - P_n \right\|_{L_p(I)} \le \inf_{q_n \in \Pi_n \cap \Delta^2(y_s)} \left\| f - q_n \right\|_{L_p(I)}$$

then

$$E_n(f)_p \le E_n^{(2)}(f, Y_s)_p \ \blacklozenge$$

Theorem 1.4.5

Let $f \in L_p(I),\ 0 < p \le \infty$ and $P_n \in \Pi_n$ is the best polynomial approximation from Π_n

$$\left\| f - P_n \right\|_{L_p(I)} \le cE_n^{(2)}(f, Y_s)_p$$

where c independent of n and f.

Proof:

By Lemma 1.4.3 and Lemma 1.4.4 ,we have

$$\left\| f - P_n \right\|_{L_p(I)} \le cE_n(f)_p \le cE_n^{(2)}(f, Y_s)_p \blacklozenge$$

1.5 Equivalence of Moduli of Smoothness

For a partition $Z_n = \{z_0, ..., z_n; -1 = z_0 < z_1 < \cdots < z_n = 1\}$ of the interval $[-1,1]$, denote the scale of the partition Z_n by

$v = v(Z_n) = \max\limits_{0 \le j \le n-1} \dfrac{\left| J_{j\pm1} \right|}{\left| J_j \right|}$, where , $J_j = [z_j, z_{j+1}]$ with $z_j = -1, j < 0,$

and $z_j = 1, j > n$ and $\left| J \right| = measJ$. And , suppose that

$\delta_{\max} = \delta_{\max}(Z_n) = \max\limits_{0\le j\le n-1}\left|J_j\right|$ and $\delta_{\min} = \delta_{\min}(Z_n) = \min\limits_{0\le j\le n-1}\left|J_j\right|$.

We say that Z_n is $\Delta-$quasi uniform if $\Delta = \frac{\delta_{\max}}{\delta_{\min}}$ is bounded by a constant independent of n, and denote such partition by u_n^{Δ}. Note that the 1-quasi uniform partition $u_n = u_n^{1}$ is just the uniform partition of I into n subintervals of equal lengths.

If $Z_n = u_n^{\Delta}$, then clearly $\frac{z}{(n\Delta)} \le \delta_{\min} \le \frac{z}{n} \le \delta_{\max} \le \frac{z\Delta}{n}$, and $\varphi(Z_n) \le \Delta.$ Therefore, $\delta_{\min} \sim \delta_{\max} \sim n^{-1}$ with equivalence constants depending only on $\Delta.$

We say that α is equivalent to β and write $\alpha \sim \beta$ if there exist a positive constant c such that $c^{-1}\alpha \le \beta \le c\alpha,$ we refer to this constant c as equivalence constant.

Theorem 1.5.1 (Local Estimates) [22]

Let $f \in \sum_{r,n} \cap C^{m}(I),$ and $r \in N$, $0 \le m \le r-1$, $J = [z_{M_1}, z_{M_2}],$ with $M_2 - M_1 \le c_0,$ for some constant $c_0.$ Then for any $1 \le k \le r+1$ and $0 < p \le \infty,$ we have

$$|J|^{v}\omega_{k-v}(f^{(v)},|J|)_{p} \sim \omega_{k}(f,|J|)_{p},$$

where $1 \le v \le \min\{k, m+1\}$.

Equivalence constants above depend on r, v, c_{0} and p as $p \to 0.$

Theorem 1.5.2 (Quasi-Uniform partition) [22]

Let $\overset{\Delta}{u}_{n}, n \in N,$ be a $\Delta-$quasi uniform partition of I, and let $s \in \sum_{r,n}(\overset{\Delta}{u}_{n}) \cap C^{m}(I),$ and $r \in N$, $0 \le m \le r-1$, Then, for any $1 \le k \le r+1$ and $0 < p \le \infty$, we have

$$n^{-v}\omega_{k-v}(s^{(v)}, n^{-1})_{p} \sim \omega_{k}(s, n^{-1})_{p}, 1 \le v \le \min\{k, m+1\} .$$

Equivalence constants above depend on r, Δ and p as $p \to 0.$

We say that Z_{n} is a Chebyshev partition (and $z's$ are Chebyshev Knots) if $Z_{n} = t_{n} = (t_{i})_{i=0}^{n}$ where $t_{i} = \cos\frac{(n-i)\pi}{n}$, $0 \le i \le n.$

Theorem 1.5.3 (Chebyshev partition) [23]

Let $s \in \sum_{r,n}(t_n) \cap C^m(I),$ and $r \in N$, $0 \le m \le r-1$ Then, for any $1 \le k \le r+1,\ 1 \le v \le \min\{k, m+1\}$ and $0 < p \le \infty,$ we have

$$n^{-v}\omega^{\varphi}_{k-v,v}(s^{(v)}, n^{-1})_p \sim \omega^{\varphi}_k(s, n^{-1})_p .$$

Equivalence constants above depend on r and p as $p \to 0$.

Theorem 1.5.4 (Chebyshev knots) [10]

Let t_n be a Chebyshev partition of I if $s \in \sum_{r,n}(t_n), r \in N$ then for any $1 \le k \le r+1$ and $0 < p \le \infty$, we have

$$\omega^{\varphi}_{k-v,v}(s^{(v)}, n^{-1})_p \le c(r,v) n^v \omega^{\varphi}_k(s, n^{-1})_p ,$$

for all $1 \le v \le k$.

Theorem 1.5.5 [14]

Let $\overset{\Delta}{u}_n, n \in N$ be a $\Delta-$quasi uniform partition of I , and let $s \in \sum_{r,n}(\overset{\Delta}{u}_n), r \in N.$ Then, for any $1 \le k \le r+1$ and $0 < p \le \infty$, we have

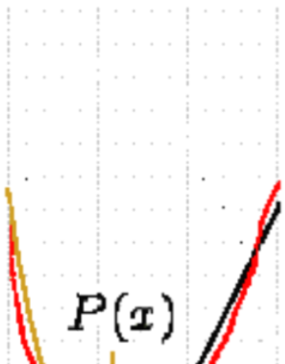
$$\omega_{k-v}(s^{(v)}, n^{-1})_p \le c(r, \Delta, p) n^v \omega_k(s, n^{-1})_p$$

for all $v = 1,2,\ldots,k$.

1.6 Geometric Meaning

We can explain shapes of the convex polynomial approximation and the coconvex polynomial approximation in Figure 1 and Figure 2, respectively and we explained the error between the origin function and polynomial approximation to it ,see [19].

Figure 1



CONVEX
POLYNOMIAL
APPROXIMATION

$P'' \in \Delta^0$

$$E_n^{(2)}(f)_p = \inf\left\{\|f - P\|_p \,\middle|\, P''(x) \geq 0\right\}$$

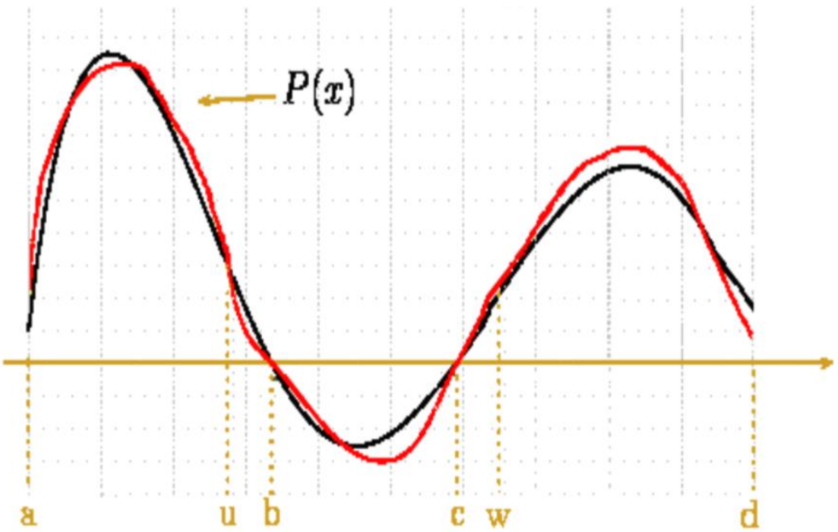


Figure 2

$$E_n^{(2)}(f, Y_s)_p = \inf \left\{ \|f - P\|_p \middle| P''(x).f'(x) \geq 0 \right\}$$

COCONVEX POLYNOMIAL APPROXIMATION

$$Y_s = \{u , w\} , s = 2 , P'' \in \Delta^0(Y_s)$$

We can notice that the error in the convex (or Coconvex) polynomial approximation is less than the error in monotone (or Commonotone) polynomial approximation , respectively as well as this less than the error in the positive (or Copositive) polynomial approximation respectively.

Chapter Two

Coconvex Polynomial Approximation in $L_p(I)$, $0 < p \leq \infty$, by using Weighted D. T. Modulus of Smoothness of f

In (2006) K. Kopotun , D. Leviatan and I. A. Shevchuk introduced some results about the coconvex polynomial approximation in the uniform norm and they dealt with approximation of a continuous function .

In this chapter we shall deal with approximation of f in L_p ,$0 < p \leq \infty$ on $[-1,1]$, which changes convexity finitely many times , by algebraic polynomials and piecewise polynomials and we are interested in estimating the degree of approximation of f. These estimates involve the wighted Ditzian Totik modulus of smoothness.

2.1 Introduction and Main Results

Given $n \in N$, we set $x_{-1} = 1$, $x_n = -1$ and $x_j = x_{j,n} = \cos\left(\frac{j\pi}{n}\right)$,

Throughout this chapter we use the following notations, the Chebyshev partition of interval I, we denote $I_j = I_{j,n} = [x_j, x_{j-1}]$,

$$h_j = \left|I_j\right| = x_{j-1} - x_j \text{ , and } \psi_j = \psi_{j,n} = \frac{h_j}{\left|x - x_j\right| + h_j} \text{ , } j = 0,1,\dots,n \text{ .}$$

Let $\sum_{k,n}$ denotes the collection of all piecewise polynomials of degree $k-1$, on the Chebyshev partition and let $\sum_{k,n}^1 \subseteq \sum_{k,n}$, be the subset of all continuously differentiable piecewise polynomials on the Chebyshev partition. That is, if $S \in \sum_{k,n}$, then

$$S \mid I_j = \mathrm{P}_j \text{ , } j = 1,2,\dots,n \text{ where } P_j \in \Pi_{k-1},$$

$$P_j(x_j) = P_{j+1}(x_j) \text{ , } j = 1,2,\dots,n-1,$$

and if $S \in \sum_{k,n}^1$, then in addition,

$$P'_j(x_j) = P'_{j+1}(x_j) \text{ , } j = 1,2,\dots,n-1.$$

Given $0 \le s < \infty$, let

$$O_i = O_{i,n}(Y_s) = (x_{j+1}, x_{j-2}) \text{ , if } y_i \in [x_j, x_{j-1}) \text{ ,}$$

and,

$O = O(n,Y_s) = \bigcup_{i=1}^{s} O_i$, $O(n,\phi) = \phi$.We write

$$j \in \mathrm{H} = \mathrm{H}(n, \mathrm{Y}_s) \text{ , if } I_j \cap O = \phi.$$

Finally, let $\sum_{k,n}(Y_s)\subseteq\sum_{k,n}$ and $\sum^1_{k,n}(Y_s)\subseteq\sum^1_{k,n}$ be denote the subsets of such piecewise polynomials for which

$P_j \equiv P_{j+1}$, whenever both $j, j+1 \notin H$,

where, Π_{k-1} denotes the space of algebraic polynomials of degree $\leq k-1$, usually , piecewise polynomials are called Splines.

To define the notion of coconvexity more precise , we first denote by Y_s , $s \geq 1$ the set of all collection $Y_s = \{y_i\}_{i=1}^{s}$ such that

$$Y_s = \{y_1, y_2, \ldots, y_s : y_0 = -1 < y_1 < \ldots < y_{s+1} = 1\}.$$

In particular if $s = 0$,then $Y_0 = \phi$.

Let $\Delta^2(\mathrm{Y_s})$, be the set of all coconvex functions on $L_p(I), 0 < p < \infty$.

If f be twice differentiable function, then $f \in \varDelta^2(Y_s)$ if and only if $f''(x)\pi(x) \geq 0, x \in (-1,1),$ where

$$\pi(x) = \prod_{i=1}^{s}(x - y_i)$$

It is possible for a function to belong to more than one class $\varDelta^2(Y_s)$ there is an example about that , for example , $f \equiv 0$ is in $\varDelta^2(Y_s)$ for all set Y_s , $s \geq 1$.

Let $C^m[-1,1]$ denote the set of m –times continuously differentiable functions on $[-1,1]$.

For $1 \le p \le \infty$,then the Sobolev space $W^{v}(L_p)$ is a collection of all functions f defined on I ,such that $f^{(v-1)}$ is absolutely continuous and $f^{(v)} \in L_p(I)$.

Our main results are the following:

Theorem 2.1.1

Let $f \in L_p(I)$, $0 < p \le \infty$, and $k, r \in N_0$ then

i. $\omega^{\varphi}_{k,r}(f,t)_p \le \omega^{\varphi}_{k}(f,t)_p$.

ii. $\omega^{\varphi}_{k+1,r}(f,t)_p \le c\, \omega^{\varphi}_{k,r}(f,t)_p$.

iii. $\omega^{\varphi}_{k,r}(f,t)_p \le c\, \|\varphi^{v} f\|_{L_p(I)}$.

From Theorem 2.1.1 (iii) ,we get that

Corollary 2.1.2

Let $f \in L_p(I)$,$0 < p \le \infty$, $k, r \in N_0$,then $\omega^{\varphi}_{k,r}(f,t)_p$is bounded.

Theorem 2.1.3

Let,$f \in \sum_{r,n} \cap C^{l-1}(I)$,then for any $0 < p \le \infty$ and , $t > 0$,$1 \le l \le r$,we have

$$\omega^{\varphi}_{r-l,l}(f^{(l)},t)_p \le C t^{r-l} \left\|\varphi^{r} f^{(r)}\right\|_{L_p(I)}.$$

Theorem 2.1.4

For every $r, n \in N$ and $f \in \sum_{r,n} \cap C^{l}(I)$,with $0 < p \le \infty$,

$0 < l < r$,and $k \ge 1$, $t > 0$, we have

$\omega^{\varphi}_{k+r-l,l}(f^{(l)}, t)_p \le C t^{r-l}\, \omega^{\varphi}_{k,r}(f^{(r)}, t)_p$.

Theorem 2.1.5

For every $s \in N_0$,with , $1 < p \le \infty$, and, such that ,

if $f \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap C^r(I)$,we have

$$\tilde{\delta}^{2}_{n}(f, Y_s)_p \le c n^{-1} \omega^{\varphi}_{k-r,r}(f^{(r)}, n^{-1})_p, n \ge \eta,$$

where c and η are constants > 0.

Corollary 2.1.6

From theorem 2.1.5 , if $k = r$,then

$$\tilde{\delta}^{2}_{n}(f, Y_s)_p \le c n^{-r} \| \varphi^{r} f^{(r)} \|_{L_p(I)}, n \ge \eta$$

where c and η are constants > 0.

Theorem 2.1.7

Let $f \in L_p(I)$, $1 \le p \le \infty$ and for any $k \ge 1, r \ge 0$ we have

$$\omega^{\varphi}_{k,r}(f, \lambda t)_p \le c(\lambda + 1)^{k} \omega^{\varphi}_{k,r}(f, t)_p, \lambda > 0.$$

Theorem 2.1.8

Let $n, m, v \in N$, ,and $v \le m$ and let $f \in L_p(I) \cap \Delta^2(Y_s)$, $0 < p \le \infty$also, W_p^v be some smoothness class of functions which is contained in the Sobolev space $W^v(L_p)$ Suppose that following

i. For any $g \in W_p^{\ v} \cap \Delta^2(Y_s)$ there exists a polynomial

$q_n \in \Pi_n \cap \Delta^2(Y_s)$,such that,

$$\| g - q_n \|_{L_p(I)} \le c_1 n^{-v} \omega^{\varphi}_{m-v,v}(g^{(v)}, n^{-1})_p .$$

ii. For some $r \ge m-1$,there exists a spline

$$h \in \Sigma_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap W_p^{\ v} ,$$

Such that,

$$\| f - h \|_{L_p(I)} \le c_2 \omega^{\varphi}_{m,v}(f, n^{-1})_p ,$$

then there exists a polynomial $p_n \in \Pi_n \cap \Delta^2(Y_s)$, such that,

$$\| f - p_n \|_{L_p(I)} \le c\omega^{\varphi}_{m,n}(f, n^{-1})_p ,$$

where, $c = c(c_1, c_2, r, m, v, p)$.

Theorem 2.1.9

Let $n, m, v, d \in N$, ,and$v \le m$ and let $f \in L_p(I) \cap \Delta^2(Y_s)$,$0 < p \le \infty$,$s \ge 1$,and let W_p^v and $W^v(L_p)$ be as in Theorem

2.1.7,and suppose that also the following are satisfies

i. for any $g \in W_p^v \cap \Delta^2(Y_s)$,and any $r \geq m-1$,there exists a spline $\tilde{h} \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap C^d[-1,1]$,such that

$$\| g-\tilde{h} \|_{L_p(I)} \leq c_1 n^{-v} \omega_{m-v,v}^{\varphi}(g^{(v)}, n^{-1})_p .$$

ii. For some $r \geq m-1$,there exists a spline $h \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap W_p^v$,such that

$$\| f-h \|_{L_p(I)} \leq c_2 \omega_{m,v}^{\varphi}(f, n^{-1})_p ,$$

then for any $r \geq m-1$, there exists a spline

$s_n \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap C^d[-1,1]$, such that

$$\| f-s_n \|_{L_p(I)} \leq c\omega_{m,v}^{\varphi}(f, n^{-1})_p ,$$

where, $c = c(c_1, c_2, r, m, d, v, p)$.

2.2 The Weighted Ditizian-Totik Modulus of Smoothness

Moduli of smoothness are intended for mathematician working in approximation theory ,numerical analysis and real analysis. Measuring the smoothness of a function by differentiability is too crude for many purposes in approximation theory. The modulus of a function f can be defined when f is given on any metric space X, but we will restrict ourselves to $I = [-1,1]$.

We will use moduli of smoothness which are connected with difference of higher orders. The kth symmetric difference of f is given , for $k \in N_0$, let

$$\Delta_h^k(f,x) = \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f\left(x - \frac{kh}{2} + ih\right) , & x \pm \frac{kh}{2} \in I \\ 0, & o.w. \end{cases}$$

and

$$\Delta_h^k(f,x) = \Delta_h^k(f,x,I).$$

We will have parameters k,r of which will denoted nonnegative integers , with $k + r > 0$, and let $\varphi(x) = \sqrt{1-x^2}$, $x \in I$, and $\varphi(x) \le 1$, $\forall x \in I$.

The weighted Ditizian – Totik modulus of smoothness of a function $f \in L_p(I), 0 < p \le \infty$, is defined by

$$\omega_{k,r}^{\varphi}(f,t)_p = sup_{0<h\le t}\left\|\varphi^r(x)\Delta_{h\varphi(x)}^k(f,x)\right\|_{L_p(I)}.$$

If $r = 0$,then

$$\omega_{k,0}^{\varphi}(f,t)_p = \omega_k^{\varphi}(f,t)_p = sup_{0<h\le t}\left\|\Delta_{h\varphi(x)}^k(f,x)\right\|_{L_p(I)}$$

is the usual Ditzian – Totik modulus of smoothness. Also, we note that, if $k = 0$,then

$$\omega_{0,r}^{\varphi}(f,t)_p = \|\varphi^r f\|_{L_p(I)}.$$

2.3 Auxiliary Lemmas

***Lemma* 2.3.1** [22] ,[40]

Let $f \in L_p(I)$, $1 \le p \le \infty, \alpha, \beta > 0, h \in R$, then

i. $\Delta_h^{\alpha}(\Delta_h^{\beta} f(x)) = \Delta_h^{\alpha+\beta} f(x)$, for almost every x.

ii. $\| \Delta_h^{\alpha+\beta} f(x) \|_p \le c(\alpha) \| \Delta_h^{\beta} f(x) \|_p$,

iii. $\lim_{h\to 0^+} \| \Delta_h^{\alpha} f(x) \|_p = 0.$

Lemma 2.3.2 (classical moduli of smoothness) [23]

Consider the following properties of moduli of smoothness

a. For $f \in L_p(I), 0 < p \le \infty$, we have

$$\omega_{k+1}(f,t,I)_p \le 2^{\max\{1,\frac{1}{p}\}} \omega_k(f,t,I)_p, k \in N.$$

b. $\omega_k(f,\lambda t,I)_p \le c(k,p)(\lambda+1)^{k+\max\{1,\frac{1}{p}\}-1} \omega_k(f,t)_p, \lambda > 0.$

c. If $f^{(v-1)} \in AC(I), v \in N, and, f^{(v)} \in L_p(I), 1 \le p \le \infty$, then

$$\omega_{k+v}(f,t,I)_p \le t^v \omega_k(f^{(v)},t,I)_p$$

d. If $f \in L_p[a,b], 0 < p < \infty, and, r \in N$, then ,

$$\omega_r(f,t,[a,b])_p^p \le \frac{c(r,p)}{t} \int_0^t \int_a^b |\Delta_h^r(f,x)|^p \, dxdh$$

e. Suppose that $f \in L_p[-1,1], 0 < p < \infty, and, k, m \in N$, then,

$$\sum_{j=0}^{n-m-1} \omega_k\left(f, \bigcup_{i=j}^{j=m} J_i\right)_p^p \le \begin{cases} c(k,\Delta.m,p)\omega_k(f,n^{-1})_p^p \\ c(k,m,p)\omega_k^{\varphi}(f,n^{-1})_p^p \end{cases}$$

Lemma 2.3.3 (Whitney's Inequality) [22]

For any $f\in L_p[a,b], 0<p\le\infty$, there exists $q_{k-1}\in \Pi_{k-1}$,such that

$$\| f-q_{k-1}\|_{L_p[a,b]} \le c\omega_k(f,[a,b])_p .$$

Lemma 2.3.4 [43]

If for any $f\in L_p[a,b], p>0,$ there is $q_j\in \Pi_{k+v-1}$ be such that

$$\| f-q_j\|_{L_p(J)} \le c\omega_{k+v}(f,J_j)_p \text{, then}$$

$$\sup_{0<h\le\frac{1}{n}} \|\Delta_{h\varphi(x)}^{k+v}(f,x)\|_p^p = \sup_{0<h\le\frac{1}{n}} \int_{J_j} |\Delta_{h\varphi(x)}^{k+v}(f-q_j,x)|^p\, dx$$

Lemma 2.3.5 [23]

Let $n,r\in N$,$k\in N_0$, $1\le v\le r$ and let n of the Chebyshev partition of I If $f\in \Sigma_{r,n}\cap C^{v-1}(I)$,then for any $0<p\le\infty$ and $t>0$,we have

$$\omega_{k+v}^{\varphi}(f,t)_p \le C(r,k,p)t^v\omega_{k,v}^{\varphi}(f^{(v)},t)_p.$$

In particular ,in the case $k = 0$,

$$\omega_v^{\varphi}(f,t)_p \le C(r,p)t^v\left\|\varphi^v f^{(v)}\right\|_{L_p(I)}.$$

Lemma 2.3.6

Let $f \in L_p(I)$, $J \subseteq (-1,1)$, we have

$$\omega_k^{\varphi}(f,|J|,J)_p \le c(r,p)\omega_{k,r}^{\varphi}(f,|J|)_p \quad .$$

Proof:-

Let $x \pm \frac{k}{2}h \in J$, then $h \le \frac{|J|}{k}$,and we observe that ,If $x \pm \frac{k}{2}h \in J$ and $x \pm \frac{k}{2}h\varphi(x) \in I$ and since $\varphi(x) \le 1$, $\left[x-\frac{r}{2}h, x+\frac{r}{2}h\right] \subseteq J$ we have $J \subseteq I$,then

By using the following inequality from [20],$|J_1| \le |J|$ for $J_1 \subseteq J$,we obtain

$$h \le \frac{|J|}{r}\varphi(x)$$

$$\omega_k^{\varphi}(f,|J|,J)_p \le sup_{0<h\le\frac{|J|}{r}\varphi(.)}\left\|\Delta_{h\varphi}^k(f,x)\right\|_{L_p(J)}$$

$$\le sup_{0<h\le\frac{|J|}{r}}\left\|\Delta_{h\varphi(.)}^k(f,x)\right\|_{L_p(J)}$$

$$\le sup_{0<h\le\frac{|J|}{r}}\frac{1}{\varphi^r(.)}\left\|\varphi^r(.)\,\Delta_{h\varphi(x)}^k(f,x)\right\|_{L_p(J)}$$

$$\le c(r,p)\; sup_{0<h\le\frac{|J|}{r}}\left\|\varphi^r(x)\,\Delta_{h\varphi(x)}^k(f,x)\right\|_p$$

$$\le c(r,p)\omega_{k,r}^{\varphi}(f,|J|)_p \;♦$$

Lemma 2.3.7

For a function $f \in L_p(J), 1 \le p \le \infty$, where $J = [a,b] \subset I$, we have

$$\omega^{\varphi}_{k,r}(f,t,J)_p \approx \widetilde{K}_{k,r,\varphi}(f,t^k)_p ,$$

where $\widetilde{K}_{k,r,\varphi}$ is the weighted Ditzian-Totik $\widetilde{K}$ - functional defined by

$$\widetilde{K}_{k,r,\varphi}(f,t^k)_p = \inf_{\substack{p_n \in \pi_n \\ n=\left[\frac{1}{t}\right]}} \{\left\|\varphi^r(f-p_n)\right\|_{L_p(J)} + t^k \left\|\varphi^k P_n^{(k)}\right\|_{L_p(J)}\}$$

Proof:

We shall prove $\omega^{\varphi}_{k,r}(f,t,J) \le c\widetilde{K}_{k,r,\varphi}(f,t^k)_p$

$$\omega^{\varphi}_{k,r}(f,t,J) = \omega^{\varphi}_{k,r}(f-P_n+P_n,t)_p$$

$$\le \omega^{\varphi}_{k,r}(f-P_n,t)_p + \omega^{\varphi}_{k,r}(P_n,t)_p$$

$$= \mathrm{A} + \mathrm{B}$$

$$A = \omega^{\varphi}_{k,r}(f-P_n,t)^p_p = \sup_{0<h\le t}\left\|\varphi^r(.)\Delta^k_{h\varphi(\cdot)}(f-P_n,t)\right\|^p_{L_p(J)}$$

$$= \sup_{0<h\le t}\int_J \left|\varphi^r(\cdot)\Delta^k_{h\varphi(\cdot)}(f-P_n,.)\right|^p dx$$

$$\le c(k,p)\sup_{0<h\le t}\int_J \left|\varphi^r(f-P_n)\right|^p dx$$

$$\le c(k,p)\left\|\varphi^r(f-P_n)\right\|_{L_p(J)}.$$

For the second part we will use Lemma 2.3.5

$$B=\omega_{k,r}^{\varphi}(P_n,t)_p \le \omega_k^{\varphi}(P_n,t)_p \le ct^k\left\|\varphi^k P_n^{(k)}\right\|_{L_p(J)}$$

Thus,

$$\omega_{k,r}^{\varphi}(f,t)_p \le c \inf_{P_n\in\Sigma_{r,n}\cap C^{k-1}[-1,1]}\{\left\|\varphi^r(f-P_n)\right\|_{L_p(J)} + t^k\left\|\varphi^k P_n^{(k)}\right\|_{L_p(J)}\}$$

$$\le c\tilde{K}_{k,r,\varphi}(f,t^k)_p.$$

Now, we shall prove $\tilde{K}_{k,r,\varphi}(f,t^k)_p \le c\omega_{k,r}^{\varphi}(f,t)_p$

by Lemma 2.3.6 and Lemma 1.4.3,we have

$$\tilde{K}_{k,r,\varphi}(f,t^k)_p = \inf_{\substack{p_n\in\pi_n\\ n=\left[\frac{1}{t}\right]}}\{\left\|\varphi^r(f-p_n)\right\|_{L_p(J)} + t^k\left\|\varphi^k P_n^{(k)}\right\|_{L_p(J)}\}$$

$$\le \left\|f-P_n\right\|_{L_p(J)} + \left\|\varphi^k P_n^{(k)}\right\|_{L_p(J)}$$

$$\le c\omega_{k,r}^{\varphi}(f,t)_p+\left\|\varphi^k P_n^{(k)}\right\|_{L_p(J)}.$$

Let us write $P_n^{(k)}$ as a difference of two polynomials each of them as a best approximation to f and by Lemma 2.3.6 ,i.e. $P_n^{(k)}=P_m^{(k)}-P_w^{(k)}$,then

$$\left\|\varphi^k P_n^{(k)}\right\|_p=\left\|\varphi^k(P_m^{(k)}-P_w^{(k)})\right\|_p\le\left\|P_m^{(k)}-f\right\|_p+\left\|P_w^{(k)}-f\right\|_p$$

$$\le c\omega_{k,r}^{\varphi}(f,t)_p$$

hence, $\tilde{K}_{k,r,\varphi}(f,t^k)_p\le c\omega_{k,r}^{\varphi}(f,t)_p$ ♠

2.4 The proof of the main results

Proof of theorem 2.1.1

Since $\varphi(x)\le 1,\ \forall x\in I,$ then

$$\omega_{k,r}^{\varphi}(f,t)_p^p=\sup_{0<h\le t}\left\|\varphi^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f,.)\right\|_{L_p(I)}^p$$

$$=\sup_{0<h\le t}\int_I\left|\varphi^r(x)\Delta_{h\varphi(x)}^k(f,x)\right|^p dx$$

$$\le\sup_{0<h\le t}\int_I\left|\Delta_{h\varphi(x)}^k(f,x)\right|^p dx$$

$$= \sup_{0<h\le t} \left\| \Delta^{k}_{h\varphi(\cdot)}(f,.) \right\|^{p}_{L_p(I)}$$

$$= \omega^{\varphi}_{k}(f,t)_{p}.$$

This is satisfying (i), we shall now prove (ii).

$$\omega^{\varphi}_{k+1,r}(f,t)^{p}_{p} = \sup_{0<h\le t} \left\| \varphi^{r}(.)\ \Delta^{k+1}_{h\varphi(\cdot)}(f,.) \right\|^{p}_{L_p(I)}$$

$$= \sup_{0<h\le t} \int_I \left| \varphi^{r}(x) \Delta^{k+1}_{h\varphi(x)}(f,x) \right|^{p} dx.$$

By Lemma 2.3.1 we have

$$\omega^{\varphi}_{k+1,r}(f,t)^{p}_{p} \le \sup_{0<h\le t} \int_I \left| \varphi^{r}(x) \Delta^{k}_{h\varphi(x)}(f,x) \right|^{p} dx$$

$$\le c \sup_{0<h\le t} \left\| \varphi^{r}(\cdot) \Delta^{k}_{h\varphi(\cdot)}(f,.) \right\|^{p}_{L_p(I)}$$

$$= c\,\omega^{\varphi}_{k,r}(f,t)_{p}.$$

Now, by (Theorem 2.1.1 (ii)), we have

$$\omega^{\varphi}_{k,r}(f,t)_{p} \le c_1 \omega^{\varphi}_{k-1,r}(f,t)_{p} \le \cdots \le c\,\omega^{\varphi}_{0,r}(f,t)_{p}$$

$$= c \left\| \varphi^{r} f \right\|_{L_p(I)} \blacklozenge$$

Proof of Corollary 2.1.2

By Theorem 2.1.1 (iii), we have, the necessary and sufficient condition for $\omega^{\varphi}_{k,r}(f,t)_p$ is bounded, is that $\omega^{\varphi}_{k,r}(f,t)_p < \infty,$ since c is positive constant and $\left\|\varphi^{r} f\right\|_{L_p(I)} < \infty$ ♦

Proof of Theorem 2.1.3

Using Lemma 1.5.4, ,we have

$$\omega^{\varphi}_{r-l,l}(f^{(l)},t)_p \le ct^{-l}\omega^{\varphi}_{r}(f,t)_p$$

$$\le ct^{-l}t^{r}\left\|\varphi^{r} f^{(r)}\right\|_{L_p(I)}$$

$$= ct^{r-l}\left\|\varphi^{r} f^{(r)}\right\|_{L_p(I)}$$ ♦

Proof of Theorem 2.1.4

Let $J_i \subseteq I,$ for each $0 \le j \le n-1,$ and let $q_j \in \Pi_{k-r-1,}$ such that

$$\left\|f^{(r)} - q_j\right\|_{L_p(J_j)} \le c\omega_{k-r}(f^{(r)},J_j)_p$$

where q_j exists by Lemma 2.3.3 (Whitney's Inequality) and assume the inequality $\varphi(x) \le \left|J_j\right| / t,\ x \in J_j$ by using Lemma 2.3.2 , Lemma 2.3.5 and Lemma 1.5.4, we have

$$\omega^{\varphi}_{k+r-l,l}(f^{(l)},t)^{p}_{p} = \sup_{0<h\le t}\left\|\varphi^{l}(\cdot)\Delta^{k+r-l}_{h\varphi(\cdot)}(f^{(l)},.)\right\|^{p}_{L_p(I)}$$

$$= \sup_{0<h\le t}\sum_{j=0}^{n-1}\left\|\varphi^{l}(\cdot)\Delta^{k+r-l}_{h\varphi(\cdot)}(f^{(l)},.)\right\|^{p}_{L_p(J_j)}$$

$$\le t^{-lp}\sum_{j=0}^{n-1}\left|J_j\right|^{lp}\sup\left\|\varphi^{l}(\cdot)\Delta^{k+r-l}_{h\varphi(\cdot)}(f^{(l)}-q_j,.)\right\|^{p}_{L_p(J_j)}$$

$$\le c(k,p,r)t^{-lp}\sum_{j=0}^{n-1}\left|J_j\right|^{lp}\left\|f^{(l)}-q_j\right\|^{p}_{L_p(J_j)}$$

$$\le c(k,p,r)t^{-lp}\sum_{j=0}^{n-1}\left|J_j\right|^{lp}\omega_{k+r-l}(f^{(l)},J_j)^{p}_{p}$$

$$\le c(r,p)t^{-lp}\sum_{j=0}^{n-1}\omega_{k+r}(f\ ,J_j)^{p}_{p}$$

$$\le c(r,p)t^{-lp}\omega^{\varphi}_{k+r}(f\ ,t)^{p}_{p}$$

$$\le c(r,p)t^{-lp}t^{rp}\omega^{\varphi}_{k,r}(f\ ,t)^{p}_{p}$$

$$\le c(r,p)t^{(r-l)p}\omega^{\varphi}_{k,r}(f\ ,t)^{p}_{p} \blacklozenge$$

Proof of Theorem 2.1.5

From Theorem 1.5.3 (Chebyshev partition) , and theorem 1.4.3 ,then, there exists a polynomial $P_n \in \Pi_n$ such that

$$\left\| f - P_n \right\|_{L_p(I)} \le c\omega_k^{\varphi}(f, n^{-1})_p ,$$

since $f \in \Sigma_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap C^r(I)$,we have

$$\tilde{\delta}_n^2(f, Y_s)_p = \inf_{P_n \in \Pi_n \cap \Delta^2(Y_s)} \left\| f - P_n \right\|_{L_p(I)}$$

$$\le c\omega_k^{\varphi}(f, n^{-1})_p$$

$$\le cn^{-r} \omega_{k-r,r}^{\varphi}(f^{(r)}, n^{-1})_p , n \ge \eta$$

where $c =$ constant > 0, and η constant > 0 ♦

Proof of Corollary 2.1.6

In the case $k = 0$, and since $f \in \Sigma_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap C^r[-1,1]$then by Theorem 2.1.5 ,we have

$$\tilde{\delta}_n^2(f, Y_s)_p \le cn^{-r} \omega_{0,r}^{\varphi}(f^{(r)}, n^{-1})_p$$

$$= cn^{-r} \sup_{0<h\le t} \left\| \varphi^r \Delta_{h\varphi(\cdot)}^0 (f^{(r)}, .) \right\|_{L_p(I)}$$

$$= cn^{-r}\left\|\varphi^{r} f^{(r)}\right\|_{L_p(I)}, n \ge \eta \quad \blacklozenge$$

Proof of Theorem 2.1.7

By using Lemma 2.3.7 , we have

$$\omega^{\varphi}_{k,r}(f,\lambda t,J)_p \le \tilde{K}_{k,r,\varphi}(f,(\lambda t)^k)_p$$

$$\le c\lambda^k \tilde{K}_{k,r,\varphi}(f,t^k)_p$$

$$\le c(\lambda+1)^k \tilde{K}_{k,r,\varphi}(f,t^k)_p$$

$$\le c(\lambda+1)^k \omega^{\varphi}_{k,r}(f,t)_p \quad \blacklozenge$$

Proof of Theorem 2.1.8

By (ii), $h \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap W^v$

$$\|f-h\|_{L_p(I)} \le c_2\omega^{\varphi}_{m,v}(f,n^{-1})_p \le \omega^{\varphi}_{m,v}(h,n^{-1})_p$$

Therefore, using Lemma 1.5.4 (Chebyshev knots) , we have

$$\|f-P_n\|_{L_p(J)} \le c\|f-h\|_{L_p(I)} + c\|h-P_n\|_{L_p(I)}$$

$$\le c\|f-h\|_{L_p(I)} + cn^{-v}\omega^{\varphi}_{m-v,v}(h^{(v)},n^{-1})_p$$

$$\le c\|f-h\|_{L_p(I)} + c\omega^{\varphi}_{m,v}(h\ ,n^{-1})_p$$

$$\le c\|f-h\|_{L_p(I)} + c\omega^{\varphi}_{m,v}(f,n^{-1})_p$$

$$\le c\omega^{\varphi}_{m,v}(f,n^{-1})_p .$$

where, $c = c(c_1, c_2, r, m, v, p)$ ♦

Proof of Theorem 2.1.9

For $f \in L_p[-1,1] \cap \Delta^2(Y_s)$, $0 < p \le \infty$, and by (ii) then $h \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap W^v$, such that

$$\|f-h\|_{L_p(I)} \le c_2\omega^{\varphi}_{m,v}(f,n^{-1})_p ,$$

we use (i) to conclude that there exists a spline (piecewise polynomial) $S_n \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap C^d[-1,1]$ such that, by using Lemma 1.5.4 (Chebyshev knots) , we use same manner in (proof of theorem 2.1.8) ,we have

$$\left\| f - S_n \right\|_{L_p(I)} \le c\left\| f - h \right\|_{L_p(I)} + c\left\| h - S_n \right\|_{L_p(I)}$$

$$\le c\left\| f - h \right\|_{L_p(I)} + cn^{-v}\omega^{\varphi}_{m-v,v}(h^{(v)}, n^{-1})_p$$

$$\le c\left\| f - h \right\|_{L_p(I)} + c\omega^{\varphi}_{m,v}(h\ , n^{-1})_p$$

$$\le c\left\| f - h \right\|_{L_p(I)} + c\omega^{\varphi}_{m,v}(f, n^{-1})_p$$

$$\le c\omega^{\varphi}_{m,v}(f, n^{-1})_p .$$

where, $c = c(c_1, c_2, r, m, d, v, p)$ ♦

Chapter Two

Coconvex Polynomial Approximation in $L_p(I)\,, 0 < p \leq \infty$, by using Weighted D. T. Modulus of Smoothness of f

In (2006) K. Kopotun , D. Leviatan and I. A. Shevchuk introduced some results about the coconvex polynomial approximation in the uniform norm and they dealt with approximation of a continuous function .

In this chapter we shall deal with approximation of f in $L_p\,, 0 < p \leq \infty$ on $[-1,1]$, which changes convexity finitely many times , by algebraic polynomials and piecewise polynomials and we are interested in estimating the degree of approximation of f. These estimates involve the wighted Ditzian Totik modulus of smoothness.

2.1 Introduction and Main Results

Given $n \in N$, we set $x_{-1} = 1$, $x_n = -1$ and $x_j = x_{j,n} = \cos\left(\frac{j\pi}{n}\right)$,

Throughout this chapter we use the following notations, the Chebyshev partition of interval I, we denote $I_j = I_{j,n} = [x_j, x_{j-1}]$,

$$h_j = \left|I_j\right| = x_{j-1} - x_j \text{ , and } \quad \psi_j = \psi_{j,n} = \frac{h_j}{\left|x - x_j\right| + h_j} \text{ , } j = 0,1,\dots,n \text{ .}$$

Let $\sum_{k,n}$ denotes the collection of all piecewise polynomials of degree $k-1$, on the Chebyshev partition and let $\sum_{k,n}^{1} \subseteq \sum_{k,n}$, be the subset of all continuously differentiable piecewise polynomials on the Chebyshev partition. That is, if $S \in \sum_{k,n}$, then

$$S \mid I_j = P_j \text{ , } j = 1,2,\dots,n \text{ where } P_j \in \Pi_{k-1},$$

$$P_j\left(x_j\right) = P_{j+1}\left(x_j\right) \text{ , } j = 1,2,\dots,n-1,$$

and if $S \in \sum_{k,n}^{1}$, then in addition,

$$P_j'\left(x_j\right) = P_{j+1}'\left(x_j\right) \text{ , } j = 1,2,\dots,n-1.$$

Given $0 \le s < \infty$, let

$$O_i = O_{i,n}(Y_s) = (x_{j+1}, x_{j-2}) \text{ , if } y_i \in [x_j, x_{j-1}) \text{ ,}$$

and,

$$O = O(n, Y_s) = \bigcup_{i=1}^{s} O_i \text{ , } O(n,\phi) = \phi.$$We write

$$j \in H = H(n, Y_s) \text{ , if } I_j \cap O = \phi.$$

Finally, let $\sum_{k,n}(Y_s) \subseteq \sum_{k,n}$ and $\sum_{k,n}^{1}(Y_s) \subseteq \sum_{k,n}^{1}$ be denote the subsets of such piecewise polynomials for which

$P_j \equiv P_{j+1}$, whenever both $j, j+1 \notin H$,

where, Π_{k-1} denotes the space of algebraic polynomials of degree $\leq k-1$, usually , piecewise polynomials are called Splines.

To define the notion of coconvexity more precise , we first denote by Y_s , $s \geq 1$ the set of all collection $Y_s = \{y_i\}_{i=1}^{s}$ such that

$$Y_s = \{y_1, y_2, \ldots, y_s : y_0 = -1 < y_1 < \ldots < y_{s+1} = 1\}.$$

In particular if $s = 0$,then $Y_0 = \phi$.

Let $\Delta^2(\mathrm{Y_s})$, be the set of all coconvex functions on $L_p(I), 0 < p < \infty$. If f be twice differentiable function, then $f \in \Delta^2(Y_s)$ if and only if $f''(x)\pi(x) \geq 0, x \in (-1,1),$ where

$$\pi(x) = \prod_{i=1}^{s}(x - y_i)$$

It is possible for a function to belong to more than one class $\Delta^2(Y_s)$ there is an example about that , for example , $f \equiv 0$ is in $\Delta^2(Y_s)$ for all set Y_s , $s \geq 1.$

Let $C^m[-1,1]$ denote the set of $m-$times continuously differentiable functions on $[-1,1]$.

For $1 \leq p \leq \infty$,then the Sobolev space $W^{v}(L_p)$ is a collection of all functions f defined on I ,such that $f^{(v-1)}$ is absolutely continuous and $f^{(v)} \in L_p(I)$.

Our main results are the following:

Theorem 2.1.1

Let $f \in L_p(I)$, $0 < p \leq \infty$, and $k, r \in N_0$ then

i. $\omega_{k,r}^{\varphi}(f,t)_p \leq \omega_k^{\varphi}(f,t)_p$.

ii. $\omega_{k+1,r}^{\varphi}(f,t)_p \leq c\, \omega_{k,r}^{\varphi}(f,t)_p$.

iii. $\omega_{k,r}^{\varphi}(f,t)_p \leq c\, \|\varphi^{v} f\|_{L_p(I)}$.

From Theorem 2.1.1 (iii) ,we get that

Corollary 2.1.2

Let $f \in L_p(I)$,$0 < p \leq \infty$, $k, r \in N_0$,then $\omega_{k,r}^{\varphi}(f,t)_p$is bounded.

Theorem 2.1.3

Let, $f \in \sum_{r,n} \cap\, C^{l-1}(I)$,then for any $0 < p \leq \infty$ and , $t > 0$, $1 \leq l \leq r$,we have

$$\omega_{r-l,l}^{\varphi}(f^{(l)},t)_p \leq C t^{r-l} \left\|\varphi^{r} f^{(r)}\right\|_{L_p(I)}.$$

Theorem 2.1.4

For every r , $n \in N$ and $f \in \sum_{r,n} \cap\, C^{l}(I)$,with $0 < p \leq \infty$,

$0 < l < r$,and $k \geq 1$, $t > 0$, we have

$$\omega^{\varphi}_{k+r-l,l}(f^{(l)}, t)_p \leq Ct^{r-l}\, \omega^{\varphi}_{k,r}(f^{(r)}, t)_p .$$

Theorem 2.1.5

For every$s \in N_0$,with , $1 < p \leq \infty$, and, such that ,

if $f \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap C^r(I)$,we have

$$\tilde{\delta}^{2}_{n}(f, Y_s)_p \leq cn^{-1}\omega^{\varphi}_{k-r,r}(f^{(r)}, n^{-1})_p, n \geq \eta,$$

where c and η are constants > 0.

Corollary 2.1.6

From theorem 2.1.5 , if $k = r$,then

$$\tilde{\delta}^{2}_{n}(f, Y_s)_p \leq cn^{-r} \| \varphi^{r} f^{(r)} \|_{L_p(I)}, n \geq \eta$$

where c and η are constants > 0.

Theorem 2.1.7

Let $f \in L_p(I)$, $1 \leq p \leq \infty$ and for any $k \geq 1, r \geq 0$ we have

$$\omega^{\varphi}_{k,r}(f, \lambda t)_p \leq c(\lambda + 1)^{k}\omega^{\varphi}_{k,r}(f, t)_p, \lambda > 0.$$

Theorem 2.1.8

Let $n, m, v \in N,$,and $v \leq m$ and let $f \in L_p(I) \cap \Delta^2(Y_s)$, $0 <$

$p \leq \infty$also, W_p^v be some smoothness class of functions which is contained in the Sobolev space $W^v(L_p)$ Suppose that following

i. For any $g \in W_p^{\ v} \cap \Delta^2(Y_s)$ there exists a polynomial $q_n \in \Pi_n \cap \Delta^2(Y_s)$,such that,

$$\| g - q_n \|_{L_p(I)} \leq c_1 n^{-v} \omega^{\varphi}_{m-v,v}(g^{(v)}, n^{-1})_p .$$

ii. For some $r \geq m-1$,there exists a spline

$$h \in \textstyle\sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap W_p^{\ v},$$

Such that,

$$\| f - h \|_{L_p(I)} \leq c_2 \omega^{\varphi}_{m,v}(f, n^{-1})_p ,$$

then there exists a polynomial $p_n \in \Pi_n \cap \Delta^2(Y_s)$, such that,

$$\| f - p_n \|_{L_p(I)} \leq c\omega^{\varphi}_{m,n}(f, n^{-1})_p ,$$

where,$c = c(c_1, c_2, r, m, v, p)$.

Theorem 2.1.9

Let $n, m, v, d \in N,$,and $v \leq m$ and let $f \in L_p(I) \cap \Delta^2(Y_s)$,$0 < p \leq \infty$,$s \geq 1$,and let W_p^v and $W^v(L_p)$ be as in Theorem 2.1.7,and suppose that also the following are satisfies

i. for any $g \in W_p^{\ v} \cap \Delta^2(Y_s)$,and any $r \geq m-1$,there exists a spline

$\widetilde{h} \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap C^d[-1,1]$, such that

$$\| g-\widetilde{h} \|_{L_p(I)} \le c_1 n^{-v} \omega^{\varphi}_{m-v,v}(g^{(v)}, n^{-1})_p .$$

ii. For some $r \ge m-1$, there exists a spline

$h \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap W_p^v$, such that

$$\| f-h \|_{L_p(I)} \le c_2 \omega^{\varphi}_{m,v}(f, n^{-1})_p ,$$

then for any $r \ge m-1$, there exists a spline

$s_n \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap C^d[-1,1]$, such that

$$\| f-s_n \|_{L_p(I)} \le c\omega^{\varphi}_{m,v}(f, n^{-1})_p ,$$

where, $c = c(c_1, c_2, r, m, d, v, p)$.

2.2 The Weighted Ditizian-Totik Modulus of Smoothness

Moduli of smoothness are intended for mathematician working in approximation theory ,numerical analysis and real analysis. Measuring the smoothness of a function by differentiability is too crude for many purposes in approximation theory. The modulus of a function f can be defined when f is given on any metric space X, but we will restrict ourselves to $I = [-1,1]$.

We will use moduli of smoothness which are connected with difference of higher orders. The kth symmetric difference of f is given , for $k \in N_0$, let

$$\Delta_h^k(f,x) = \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f\left(x - \frac{kh}{2} + ih\right) , & x \pm \frac{kh}{2} \in I \\ 0, & o.w. \end{cases}$$

and

$$\Delta_h^k(f,x) = \Delta_h^k(f,x,I).$$

We will have parameters k, r of which will denoted nonnegative integers , with $k + r > 0$, and let $\varphi(x) = \sqrt{1-x^2}$, $x \in I$, and $\varphi(x) \le 1$, $\forall x \in I$.

The weighted Ditizian – Totik modulus of smoothness of a function $f \in L_p(I), 0 < p \leq \infty$, is defined by

$$\omega_{k,r}^{\varphi}(f,t)_p = sup_{0<h\leq t}\left\|\varphi^r(x)\Delta_{h\varphi(x)}^{k}(f,x)\right\|_{L_p(I)}.$$

If $r = 0$,then

$$\omega_{k,0}^{\varphi}(f,t)_p = \omega_k^{\varphi}(f,t)_p = sup_{0<h\leq t}\left\|\Delta_{h\varphi(x)}^{k}(f,x)\right\|_{L_p(I)}$$

is the usual Ditzian – Totik modulus of smoothness. Also, we note that, if $k = 0$,then

$$\omega_{0,r}^{\varphi}(f,t)_p = \|\varphi^r f\|_{L_p(I)}.$$

2.3 Auxiliary Lemmas

***Lemma* 2.3.1** [22] ,[40]

Let $f \in L_p(I),\ 1 \leq p \leq \infty, \alpha, \beta > 0, h \in R$, then

i. $\varDelta_h^{\alpha}(\varDelta_h^{\beta} f(x)) = \varDelta_h^{\alpha+\beta} f(x)$, for almost every x.

ii. $\| \varDelta_h^{\alpha+\beta} f(x) \|_p \leq c(\alpha) \| \varDelta_h^{\beta} f(x) \|_p$,

iii. $\lim\limits_{h\to 0^+} \| \varDelta_h^{\alpha} f(x) \|_p = 0.$

Lemma 2.3.2 (classical moduli of smoothness) [23]

Consider the following properties of moduli of smoothness

a. For $f \in L_p(I), 0 < p \le \infty$, we have

$$\omega_{k+1}(f,t,I)_p \le 2^{\max\{1,\frac{1}{p}\}} \omega_k(f,t,I)_p, k \in N.$$

b. $\omega_k(f,\lambda t,I)_p \le c(k,p)(\lambda+1)^{k+\max\{1,\frac{1}{p}\}-1} \omega_k(f,t)_p, \lambda > 0.$

c. If $f^{(v-1)} \in AC(I), v \in N, and, f^{(v)} \in L_p(I), 1 \le p \le \infty$, then

$$\omega_{k+v}(f,t,I)_p \le t^v \omega_k(f^{(v)},t,I)_p$$

d. If $f \in L_p[a,b], 0 < p < \infty, and, r \in N$, then ,

$$\omega_r(f,t,[a,b])_p^p \le \frac{c(r,p)}{t} \int_0^t \int_a^b |\Delta_h^r(f,x)|^p \, dxdh$$

e. Suppose that $f \in L_p[-1,1], 0 < p < \infty, and, k, m \in N$, then,

$$\sum_{j=0}^{n-m-1} \omega_k\left(f, \bigcup_{i=j}^{j=m} J_i\right)_p^p \le \begin{cases} c(k,\Delta.m,p)\omega_k(f,n^{-1})_p^p \\ c(k,m,p)\omega_k^{\varphi}(f,n^{-1})_p^p \end{cases}$$

Lemma 2.3.3 (Whitney's Inequality) [22]

For any $f \in L_p[a,b], 0 < p \le \infty$, there exists $q_{k-1} \in \Pi_{k-1}$,such that

$$\| f - q_{k-1} \|_{L_p[a,b]} \le c\omega_k (f,[a,b])_p .$$

Lemma 2.3.4 [43]

If for any $f \in L_p[a,b], p>0,$ there is $q_j \in \Pi_{k+v-1}$ be such that

$$\| f - q_j \|_{L_p(J)} \le c\omega_{k+v} (f, J_j)_p \text{, then}$$

$$\sup_{0<h\le \frac{1}{n}} \| \Delta^{k+v}_{h\varphi(x)} (f,x) \|^p_p = \sup_{0<h\le \frac{1}{n}} \int_{J_j} | \Delta^{k+v}_{h\varphi(x)} (f - q_j , x) |^p \, dx$$

Lemma 2.3.5 [23]

Let $n, r \in N$, $k \in N_0$, $1 \le v \le r$ and let n of the Chebyshev partition of I If $f \in \sum_{r,n} \cap \, C^{v-1}(I)$,then for any $0 < p \le \infty$ and $t > 0$,we have

$$\omega^{\varphi}_{k+v}(f,t)_p \le C(r,k,p) t^v \omega^{\varphi}_{k,v}(f^{(v)},t)_p .$$

In particular ,in the case $k = 0$,

$$\omega^{\varphi}_{v}(f,t)_p \le C(r,p) t^v \left\| \varphi^v f^{(v)} \right\|_{L_p(I)} .$$

Lemma 2.3.6

Let $f \in L_p(I)$, $J \subseteq (-1,1)$, we have

$$\omega_k^{\varphi}(f,|J|,J)_p \le c(r,p)\omega_{k,r}^{\varphi}(f,|J|)_p \quad .$$

Proof:-

Let $x \pm \frac{k}{2}h \in J$, then $h \le \frac{|J|}{k}$,and we observe that ,If $x \pm \frac{k}{2}h \in J$ and $x \pm \frac{k}{2}h\varphi(x) \in I$ and since $\varphi(x) \le 1$, $\left[x-\frac{r}{2}h, x+\frac{r}{2}h\right] \subseteq J$ we have $J \subseteq I$,then

By using the following inequality from [20],$|J_1| \le |J|$ for $J_1 \subseteq J$,we obtain

$$h \le \frac{|J|}{r}\varphi(x)$$

$$\omega_k^{\varphi}(f,|J|,J)_p \le sup_{0<h\le\frac{|J|}{r}\varphi(.)} \left\|\Delta_{h\varphi}^k(f,x)\right\|_{L_p(J)}$$

$$\le sup_{0<h\le\frac{|J|}{r}}\left\|\Delta_{h\varphi(.)}^k(f,x)\right\|_{L_p(J)}$$

$$\le sup_{0<h\le\frac{|J|}{r}} \frac{1}{\varphi^r(.)}\left\| \varphi^r(.)\, \Delta_{h\varphi(x)}^k(f,x)\right\|_{L_p(J)}$$

$$\le c(r,p)\, sup_{0<h\le\frac{|J|}{r}}\left\| \varphi^r(x)\, \Delta_{h\varphi(x)}^k(f,x)\right\|_p$$

$$\le c(r,p)\omega_{k,r}^{\varphi}(f,|J|)_p \blacklozenge$$

Lemma 2.3.7

For a function $f \in L_p(J), 1 \le p \le \infty$,where $J = [a,b] \subset I$,we have

$$\omega_{k,r}^{\varphi}(f,t,J)_p \approx \tilde{K}_{k,r,\varphi}(f,t^k)_p \ ,$$

where $\tilde{K}_{k,r,\varphi}$ is the weighted Ditzian-Totik $\tilde{K}$- functional defined by

$$\tilde{K}_{k,r,\varphi}(f,t^k)_p = \inf_{\substack{p_n \in \pi_n \\ n=\left[\frac{1}{t}\right]}} \{\left\|\varphi^r(f-p_n)\right\|_{L_p(J)} + t^k\left\|\varphi^k P_n^{(k)}\right\|_{L_p(J)}\}$$

Proof:

We shall prove $\omega_{k,r}^{\varphi}(f,t,J) \le c\tilde{K}_{k,r,\varphi}(f,t^k)_p$

$$\omega_{k,r}^{\varphi}(f,t,J) = \omega_{k,r}^{\varphi}(f-P_n+P_n,t)_p$$

$$\le \omega_{k,r}^{\varphi}(f-P_n,t)_p + \omega_{k,r}^{\varphi}(P_n,t)_p$$

$$= \mathrm{A} + \mathrm{B}$$

$$A = \omega_{k,r}^{\varphi}(f-P_n,t)_p^p = \sup_{0<h\le t}\left\|\varphi^r(.)\Delta_{h\varphi(\cdot)}^k(f-P_n,t)\right\|_{L_p(J)}^p$$

$$= \sup_{0<h\le t}\int_J\left|\varphi^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f-P_n,.)\right|^p dx$$

$$\le c(k,p)\sup_{0<h\le t}\int_J\left|\varphi^r(f-P_n)\right|^p dx$$

$$\le c(k,p)\left\|\varphi^r(f-P_n)\right\|_{L_p(J)}.$$

For the second part we will use Lemma 2.3.5

$$B=\omega_{k,r}^{\varphi}(P_n,t)_p\le\omega_k^{\varphi}(P_n,t)_p\le ct^k\left\|\varphi^kP_n^{(k)}\right\|_{L_p(J)}$$

Thus,

$$\omega_{k,r}^{\varphi}(f,t)_p\le c\inf_{P_n\in\Sigma_{r,n}\cap C^{k-1}[-1,1]}\{\left\|\varphi^r(f-P_n)\right\|_{L_p(J)}+t^k\left\|\varphi^kP_n^{(k)}\right\|_{L_p(J)}\}$$

$$\le c\tilde{K}_{k,r,\varphi}(f,t^k)_p.$$

Now, we shall prove $\tilde{K}_{k,r,\varphi}(f,t^k)_p\le c\omega_{k,r}^{\varphi}(f,t)_p$

by Lemma 2.3.6 and Lemma 1.4.3,we have

$$\tilde{K}_{k,r,\varphi}(f,t^k)_p=\inf_{\substack{p_n\in\pi_n\\ n=\left[\frac{1}{t}\right]}}\{\left\|\varphi^r(f-p_n)\right\|_{L_p(J)}+t^k\left\|\varphi^kP_n^{(k)}\right\|_{L_p(J)}\}$$

$$\le\left\|f-P_n\right\|_{L_p(J)}+\left\|\varphi^kP_n^{(k)}\right\|_{L_p(J)}$$

$$\le c\omega_{k,r}^{\varphi}(f,t)_p+\left\|\varphi^kP_n^{(k)}\right\|_{L_p(J)}.$$

Let us write $P_n^{(k)}$ as a difference of two polynomials each of them as a best approximation to f and by Lemma 2.3.6 ,i.e. $P_n^{(k)}=P_m^{(k)}-P_w^{(k)}$,then

$$\left\|\varphi^{k} P_{n}^{(k)}\right\|_{p}=\left\|\varphi^{k}(P_{m}^{(k)}-P_{w}^{(k)})\right\|_{p} \leq \left\|P_{m}^{(k)}-f\right\|_{p}+\left\|P_{w}^{(k)}-f\right\|_{p}$$

$$\leq c\omega_{k,r}^{\varphi}(f,t)_{p}$$

hence, $\tilde{K}_{k,r,\varphi}(f,t^{k})_{p} \leq c\omega_{k,r}^{\varphi}(f,t)_{p}$ ♦

2.4 The proof of the main results

Proof of theorem 2.1.1

Since $\varphi(x) \leq 1,\ \forall x \in I,$ then

$$\omega_{k,r}^{\varphi}(f,t)_{p}^{p}=\sup_{0<h\leq t}\left\|\varphi^{r}(\cdot)\Delta_{h\varphi(\cdot)}^{k}(f,.)\right\|_{L_{p}(I)}^{p}$$

$$=\sup_{0<h\leq t}\int_{I}\left|\varphi^{r}(x)\Delta_{h\varphi(x)}^{k}(f,x)\right|^{p}dx$$

$$\leq \sup_{0<h\leq t}\int_{I}\left|\Delta_{h\varphi(x)}^{k}(f,x)\right|^{p}dx$$

$$=\sup_{0<h\leq t}\left\|\Delta_{h\varphi(\cdot)}^{k}(f,.)\right\|_{L_{p}(I)}^{p}$$

$$=\omega_{k}^{\varphi}(f,t)_{p}.$$

This is satisfying (i), we shall now prove (ii).

$$\omega^{\varphi}_{k+1,r}(f,t)^{p}_{p} = \sup_{0<h\le t}\left\|\varphi^{r}(.)\ \Delta^{k+1}_{h\varphi(\cdot)}(f,.)\right\|^{p}_{L_p(I)}$$

$$= \sup_{0<h\le t}\int_I \left|\varphi^{r}(x)\Delta^{k+1}_{h\varphi(x)}(f,x)\right|^{p} dx.$$

By Lemma 2.3.1 we have

$$\omega^{\varphi}_{k+1,r}(f,t)^{p}_{p} \le \sup_{0<h\le t}\int_I \left|\varphi^{r}(x)\Delta^{k}_{h\varphi(x)}(f,x)\right|^{p} dx$$

$$\le c \sup_{0<h\le t}\left\|\varphi^{r}(\cdot)\Delta^{k}_{h\varphi(\cdot)}(f,.)\right\|^{p}_{L_p(I)}$$

$$= c\omega^{\varphi}_{k,r}(f,t)_{p}.$$

Now, by (Theorem 2.1.1 (ii)), we have

$$\omega^{\varphi}_{k,r}(f,t)_{p} \le c_1\omega^{\varphi}_{k-1,r}(f,t)_{p} \le \cdots \le c\omega^{\varphi}_{0,r}(f,t)_{p}$$

$$= c\left\|\varphi^{r} f\right\|_{L_p(I)} \blacklozenge$$

Proof of Corollary 2.1.2

By Theorem 2.1.1 (iii), we have, the necessary and sufficient condition for $\omega^{\varphi}_{k,r}(f,t)_{p}$ is bounded, is that $\omega^{\varphi}_{k,r}(f,t)_{p} < \infty$, since c is positive constant and $\left\|\varphi^{r} f\right\|_{L_p(I)} < \infty$ ♦

Proof of Theorem 2.1.3

Using Lemma 1.5.4, ,we have

$$\omega_{r-l,l}^{\varphi}(f^{(l)},t)_p \le ct^{-l}\omega_r^{\varphi}(f,t)_p$$

$$\le ct^{-l}t^{r}\left\|\varphi^{r} f^{(r)}\right\|_{L_p(I)}$$

$$= ct^{r-l}\left\|\varphi^{r} f^{(r)}\right\|_{L_p(I)}$$ ♦

Proof of Theorem 2.1.4

Let $J_i \subseteq I,$ for each $0 \le j \le n-1,$ and let $q_j \in \Pi_{k-r-1,}$ such that

$$\left\|f^{(r)} - q_j\right\|_{L_p(J_j)} \le c\omega_{k-r}(f^{(r)},J_j)_p$$

where q_j exists by Lemma 2.3.3 (Whitney's Inequality) and assume the inequality $\varphi(x) \le |J_j|/t,\ x \in J_j$ by using Lemma 2.3.2 , Lemma 2.3.5 and Lemma 1.5.4, we have

$$\omega_{k+r-l,l}^{\varphi}(f^{(l)},t)_p^p = \sup_{0<h\le t}\left\|\varphi^{l}(\cdot)\Delta_{h\varphi(\cdot)}^{k+r-l}(f^{(l)},.)\right\|_{L_p(I)}^p$$

$$= \sup_{0<h\le t} \sum_{j=0}^{n-1} \left\| \varphi^{l}(\cdot)\Delta_{h\varphi(\cdot)}^{k+r-l}(f^{(l)},.)\right\|_{L_p(J_j)}^{p}$$

$$\le t^{-lp} \sum_{j=0}^{n-1} \left|J_j\right|^{lp} \sup \left\| \varphi^{l}(\cdot)\Delta_{h\varphi(\cdot)}^{k+r-l}(f^{(l)}-q_j,.)\right\|_{L_p(J_j)}^{p}$$

$$\le c(k,p,r)t^{-lp} \sum_{j=0}^{n-1} \left|J_j\right|^{lp} \left\| f^{(l)}-q_j\right\|_{L_p(J_j)}^{p}$$

$$\le c(k,p,r)t^{-lp} \sum_{j=0}^{n-1} \left|J_j\right|^{lp} \omega_{k+r-l}(f^{(l)},J_j)_p^{p}$$

$$\le c(r,p)t^{-lp} \sum_{j=0}^{n-1} \omega_{k+r}(f\ ,J_j)_p^{p}$$

$$\le c(r,p)t^{-lp} \omega_{k+r}^{\varphi}(f\ ,t)_p^{p}$$

$$\le c(r,p)t^{-lp} t^{rp} \omega_{k,r}^{\varphi}(f\ ,t)_p^{p}$$

$$\le c(r,p)t^{(r-l)p} \omega_{k,r}^{\varphi}(f\ ,t)_p^{p} \blacklozenge$$

Proof of Theorem 2.1.5

From Theorem 1.5.3 (Chebyshev partition) , and theorem 1.4.3 ,then, there exists a polynomial $P_n \in \Pi_n$ such that

$$\left\| f-P_n \right\|_{L_p(I)} \le c\omega_k^{\varphi}(f,n^{-1})_p ,$$

since $f \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap C^r(I)$,we have

$$\tilde{\delta}_n^2(f,Y_s)_p = \inf_{P_n \in \Pi_n \cap \Delta^2(Y_s)} \left\| f-P_n \right\|_{L_p(I)}$$

$$\le c\omega_k^{\varphi}(f,n^{-1})_p$$

$$\le cn^{-r}\omega_{k-r,r}^{\varphi}(f^{(r)},n^{-1})_p , n \ge \eta$$

where $c =$ constant > 0, and η constant > 0 ♦

Proof of Corollary 2.1.6

In the case $\mathrm{k} = 0$, and since $f \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap C^r[-1,1]$then by Theorem 2.1.5 ,we have

$$\tilde{\delta}_n^2(f,Y_s)_p \le cn^{-r}\omega_{0,r}^{\varphi}(f^{(r)},n^{-1})_p$$

$$= cn^{-r} \sup_{0<h\le t} \left\| \varphi^r \Delta_{h\varphi(\cdot)}^{0}(f^{(r)},.) \right\|_{L_p(I)}$$

$$= cn^{-r} \left\| \varphi^r f^{(r)} \right\|_{L_p(I)} , n \ge \eta \ ♦$$

Proof of Theorem 2.1.7

By using Lemma 2.3.7 , we have

$$\omega_{k,r}^{\varphi}(f,\lambda t,J)_p \le \tilde{K}_{k,r,\varphi}(f,(\lambda t)^k)_p$$

$$\le c\lambda^k \tilde{K}_{k,r,\varphi}(f,t^k)_p$$

$$\le c(\lambda+1)^k \tilde{K}_{k,r,\varphi}(f,t^k)_p$$

$$\le c(\lambda+1)^k \omega_{k,r}^{\varphi}(f,t)_p \ \blacklozenge$$

Proof of Theorem 2.1.8

By (ii), $h \in \sum_{r,n}(Y_s) \cap \Delta^2(Y_s) \cap W^v$

$$\|f-h\|_{L_p(I)} \le c_2\omega_{m,v}^{\varphi}(f,n^{-1})_p \le \omega_{m,v}^{\varphi}(h,n^{-1})_p$$

Therefore, using Lemma 1.5.4 (Chebyshev knots) , we have

$$\left\|f-P_n\right\|_{L_p(J)} \le c\|f-h\|_{L_p(I)} + c\left\|h-P_n\right\|_{L_p(I)}$$

$$\le c\|f-h\|_{L_p(I)} + cn^{-v}\omega_{m-v,v}^{\varphi}(h^{(v)},n^{-1})_p$$

$$\le c\|f-h\|_{L_p(I)}+c\omega^{\varphi}_{m,v}(h\ ,n^{-1})_p$$

$$\le c\|f-h\|_{L_p(I)}+c\omega^{\varphi}_{m,v}(f,n^{-1})_p$$

$$\le c\omega^{\varphi}_{m,v}(f,n^{-1})_p.$$

where, $c=c(c_1,c_2,r,m,v,p)$ ♦

Proof of Theorem 2.1.9

For $f\in L_p[-1,1]\cap\Delta^2(Y_s)$, $0<p\le\infty$, and by (ii) then $h\in\sum_{r,n}(Y_s)\cap\Delta^2(Y_s)\cap W^v$, such that

$$\|f-h\|_{L_p(I)}\le c_2\omega^{\varphi}_{m,v}(f,n^{-1})_p,$$

we use (i) to conclude that there exists a spline (piecewise polynomial) $S_n\in\sum_{r,n}(Y_s)\cap\Delta^2(Y_s)\cap C^d[-1,1]$ such that, by using Lemma 1.5.4 (Chebyshev knots) , we use same manner in (proof of theorem 2.1.8) ,we have

$$\|f-S_n\|_{L_p(I)}\le c\|f-h\|_{L_p(I)}+c\|h-S_n\|_{L_p(I)}$$

$$\le c\|f-h\|_{L_p(I)}+cn^{-v}\omega^{\varphi}_{m-v,v}(h^{(v)},n^{-1})_p$$

$$\le c\|f-h\|_{L_p(I)}+c\omega^{\varphi}_{m,v}(h\ ,n^{-1})_p$$

$$\le c\|f-h\|_{L_p(I)}+c\omega^{\varphi}_{m,v}(f,n^{-1})_p$$

$$\le c\omega^{\varphi}_{m,v}(f,n^{-1})_p.$$

where, $c=c(c_1,c_2,r,m,d,v,p)$ ♦

Chapter Three

Shape Preserving Coconvex Approximation in $L_p(I),\ 0 < p \leq \infty$

When we approximate a function $f \in L_p(I)$ which changes convexity finitely many times at Y_s ,we wish some times that the approximating polynomials follow these changes in convexity . We discuss in this chapter three sections , in the first one , we estimate the degree of approximation of f by algebraic and piecewise polynomials which change convexity exactly at the points where f does , and the second, using the divided difference and weighted D. T modulus of smoothness in Theorem 3.1.4 ,finally we estimate the degree of approximation of f by divided difference .

3.1 Introduction and Main Results

Our main interest in this chapter is the approximation of $f \in L_p(I)$, $I = [-1,1]$ which changes convexity finitely many times ,by it. We are interested in estimating the degree of approximation of f by polynomials which are coconvex with it ,i.e. if $f \in \Delta^2(Y_s)$, $s \geq 1$. We intend here to obtain the analogous results for coconvex approximation .

Denote by $\mathbf{Y}_s$,$s \in N$,the set of all collection $Y_s = \{y_i\}_{i=1}^{s}$,such that $-1 < y_s < \cdots < y_1 < 1$,for $s = 0$,we write $Y_s = \emptyset$. For later reference set $y_0 = 1$ and $y_{s+1} = -1$.

Finally ,let $\Delta^2(Y_s)$ denote the collection of all functions $f \in L_p[-1,1]$ that change convexity at the set Y_s. Given $n \in N$,$n > 1$,we set $x_j = x_{j,n} = \cos\left(\frac{j\pi}{n}\right)$,and we denote $I_j = I_{j,n} = [x_j, x_{j-1}]$, $j = 1,2,\dots,n$.

Let $\Sigma_{k,n}$,be the collection of all piecewise polynomials of degree $k-1$, on the Chebyshev partition of I .

Also ,given $Y_s \in \mathbf{Y}_s$, and let

$$(3.1.1) \qquad O_i = O_{i,n}(Y_s) = (x_{j+1}, x_{j-2}) \quad \text{if } y_i \in [x_j, x_{j-1})$$

where , $x_{n+1} = -1$, $x_{-1} = 1$,and denote

$$O = O(n, Y_s) = \cup_{i=1}^{s} O_i \ , \quad O(n, \emptyset) = \emptyset \ .$$

Finally ,we write $j \in H$, if $I_j \cap O = \emptyset$.

Let $L_p^r[-1,1]$ be the set of r −times differentiable functions on $[-1,1]$, where

$$L_p^r[-1,1] = \{f \in L_p[-1,1] : f^{(r)} \in L_p[-1,1]\} \ ,$$

also , we define

$$L_\varphi^r(I) = \left\{f \in L_p^r(-1,1) \cap L_p(I) \ \backslash \ \lim_{x \to \pm 1} \varphi^r(x) f^{(r)}(x) = 0\right\}$$

where $r \geq 1$, in particular .

The following notion of the length of an interval $J = [a,b] \subseteq I$ relative to its position in I ,was introduced in [26] and [18].

We always have that $\omega_r^{\varphi}(f,t,J)_p \le \omega_r(f,t,J)_p$, $0 < p \le \infty$ .But the converse is not true in general , however in [3] ,that moduli of smoothness ω_r^{φ} and ω_r for a function f defined on $J \subseteq I$ are equivalent , if $|J| \sim \Delta_n(a)$,where

$$\Delta_n(a) = \frac{1}{n}\sqrt{(1-a^2)} + \frac{1}{n^2} ,$$

$$\omega_r(f,\Delta_n(a),J)_p \ \sim \ \omega_r^{\varphi}(f,n^{-1},J)_p .$$

Let $L_{m-1}(g;z_0,z_1,\dots,z_{m-1})$ denote the polynomial of degree $\le m-1$ which interpolates a function g at the points $z_0,z_1,\dots,z_{m-1}$. We remind the reader that $[z_0,z_1,\dots,z_m;g]$ stands for the $m-th\ divided\ difference$ of a function g at the knots $z_0,z_1,\dots,z_m$ defined by

$$(3.1.2) \qquad [z_0,z_1,\dots,z_m;g] = \frac{g(z_m) - L_{m-1}(g;z_0,z_1,\dots,z_{m-1})(z_m)}{(z_m-z_0)(z_m-z_1)\dots(z_m-z_{m-1})},$$

the following Newton formula for interpolating polynomials is well known:

$$(3.1.3) \qquad L_{m-1}(g;z_0,z_1,\dots,z_{m-1}) = \sum_{i=0}^{m-1}(z_m-z_0)(z_m-z_1)\dots(z_m - z_{i-1})\,[z_0,z_1,\dots,z_{i-1};g]$$

Also , assuming that $z_0,z_1,\dots,z_{m-1}$ from either a non_increasing or a non_decreasing sequence such that

$$\min_{0\le i\le m-1}|z_{i+1}-z_i| \ \sim \ \max_{0\le i\le m-1}|z_{i+1}-z_i|$$

and using Whitney’s inequality ,we have the following estimate:

$$(3.1.4) \quad |[z_0,z_1,\dots,z_m;g]| \le c|z_m-z_0|^{-m}\omega_m(g,|z_m-z_0|,[min\{z_0,z_m\}\ ,max\{z_0,z_m\}])_p$$

Where, c depends on m and the ratio

$$\min_{0\le i\le m-1}|z_{i+1}-z_i| / \max_{0\le i\le m-1}|z_{i+1}-z_i| \ .$$

To know more information about the divided difference ,see in [3] and [35].

We note that for a set $Y_s \in \mathbf{Y}_s$,$s \ge 1$,if

$$n \ge 4\left(\min_{1\le j\le s}\{y_{j-1}-y_j\}\right)^{-1} = \mathcal{M}(Y_s),$$

then , there is at least one knot x_i between y_{j-1} and y_j for all $1 \le j \le s+1$.

Our main results are the following:

Theorem 3.1.1

If $f \in \Delta^2(Y_s) \cap L_p^2(I)$, then for each $k \le 3$

$$E_n^{(2)}(f,Y_s)_p \le \frac{c}{n^2}\,\omega_k^{\varphi}(f'',n^{-1})_p \qquad ,n \ge \eta,$$

where c= $c(s)$ and $\eta = \eta(Y_s)$.

Theorem 3.1.2

Let $r \ge 3$ and assume that $f \in \Delta^2(Y_s) \cap L_p^r(I)$, then

$$E_n^{(2)}(f,Y_s)_p \le \frac{c}{n^r}\,\omega_{k-r}^{\varphi}(f^{(r)},n^{-1})_p \qquad , \quad n \ge \eta$$

for each $k \ge 3$,with constants c= $c(k,r,s)$ and $\eta = \eta(k,r,Y_s)$.

Theorem 3.1.3

Let $k \ge 1$, $r = 5$,$s \ge 1$, and $Y_s \in \mathbf{Y}_s$, be given .

If $f \in L_\varphi^5(I) \cap \Delta^2(Y_s)$,then

$$E_n^{(2)}(f,Y_s)_p \le cn^{-5}\omega_{k,5}^{\varphi}(f^{(5)},n^{-1})_p \quad ,n \ge \eta(k,Y_s)$$

where $\eta(k,Y_s)$ is constant depends on k and Y_s .

Theorem 3.1.4

Let $f \in L_p(I)$, $1 \le p < \infty$,and let $k \ge 1$, $r \ge 0$ be such that $k+r \ge 3$ and $1 \le \mu \le n-k$ be fixed . Then , for all $1 \le j \le \mu$,

(3.1.5) $$\left|[x_\mu, x_{\mu+1}, \dots, x_{\mu+k-1}; f] - [x_j, x_{j+1}, \dots, x_{j+k-1}; f]\right|$$

$$\le cn^{2k+r-2}\left(\frac{1}{\min\{j,n-\mu\}}\right)^{k+r-2} \omega_{k,r}^{\varphi}(f,n^{-1})_p$$

Also ,if $k+r \ge 5$,then for all v and j such that $1 \le j \le v \le \mu$,we also have ,

(3.1.6) $$\varepsilon\left(\left|[x_v, x_{v+1}, \dots, x_{v+k-1}; f] - [x_j, x_{j+1}, \dots, x_{j+k-1}; f]\right|\right)$$

$$\le cn^{2k+r-4}\left(1+\frac{n^2}{(n-\mu)^{k+r-2}}\right)\omega_{k,r}^{\varphi}(f,n^{-1})_p$$

where $\varepsilon = sgn\{[x_\mu, x_{\mu+1}, \dots, x_{\mu+k-1}; f]\}$.

Corollary 3.1.5

Let $f \in L_\varphi^2(I)$

(i) For any index $1 \le \mu \le n-3$,if $sgn\{[x_\mu, x_{\mu+1}, x_{\mu+2}; f'']\} = \varepsilon$

then

(3.1.7) $$-\varepsilon[x_1, x_2, x_3; f''] \le cn^6\omega_{3,2}^{\varphi}(f'',n^{-1})_p$$

Moreover ,if an index $1 \le v \le \mu$ is such that

$sgn\{[x_v, x_{v+1}, f'']\} = \varepsilon$,then

we also have ,

(3.1.8) $$-\varepsilon[x_1, x_2; f''] \le cn^4\left(1 + \frac{n^2}{(n-\mu)^3}\right)\omega^{\varphi}_{3,2}(f'', n^{-1})_p$$

(ii) For any index $1 \le \mu \le n-3$,if

$Sgn\{[x_{n-\mu}, x_{n-\mu-1}, x_{n-\mu-2}; f'']\} = \varepsilon$, then

$$-\varepsilon[x_{n-1}, x_{n-2}, x_{n-3}; f''] \le cn^6\omega^{\varphi}_{3,2}(f'', n^{-1})_p$$

Moreover ,if an index $1 \le v \le \mu$ is such that

$sgn\{[x_{n-v}, x_{n-v+1}, f'']\} = -\varepsilon$,then we have

$$-\varepsilon[x_{n-1}, x_{n-2}; f''] \le cn^4\left(1 + \frac{n^2}{(n-\mu)^3}\right)\omega^{\varphi}_{3,2}(f'', n^{-1})_p \ .$$

Corollary 3.1.6

Let $s = 1$, $f \in L^2_{\varphi}[-1,1] \cap \Delta^2(Y_s)$,and $n \ge 7(\varphi(y_s))^{-3}$,then

$$max\{|[x_1, x_2, x_3; f'']|, [x_{n-1}, x_{n-2}, x_{n-3}; f'']\}$$
$$\le cn^6\omega^{\varphi}_{3,2}(f'', n^{-1})_p + cn^2\omega^{\varphi}_{2,2}(f'', n^{-1})_p ,$$

And,

$$max\{-[x_1, x_2; f''], [x_{n-1}, x_{n-2}; f'']\}$$
$$\le cn^4\omega^{\varphi}_{3,2}(f'', n^{-1})_p + cn^2\omega^{\varphi}_{2,2}(f'', n^{-1})_p \ .$$

Theorem 3.1.7

Let $s \ge 2$ and $Y_s \in \mathbf{Y}_s$ be given .If $f \in L^2_{\varphi}[-1,1] \cap \Delta^2(Y_s)$,$1 \le p < \infty$, then

(3.1.9) $$E^{(2)}_n(f, Y_s)_p \le cn^{-2}\omega^{\varphi}_{3,2}(f'', n^{-1})_p \qquad , n \ge N ,$$

where ,N is constant depends on Y_s .

Theorem 3.1.8

Let $s=1$ and $Y_s \in \mathbf{Y}_s$ be given .If $f \in L_\varphi^2[-1,1] \cap \Delta^2(Y_s)$,$1 \le p < \infty$, then

(3.1.10)
$$E_n^{(2)}(f,Y_s)_p \le cn^{-2}\omega_{3,2}^{\varphi}(f'',n^{-1})_p + cn^{-4}\omega_{2,2}^{\varphi}(f'',n^{-1})_p ,$$

$n \ge \eta$, where η =const. ,depends on Y_s .Hence

(3.1.11) $$E_n^{(2)}(f,Y_s)_p \le cn^{-2}\omega_{2,2}^{\varphi}(f'',n^{-1})_p , n \ge \eta$$

Moreover ,

(3.1.12) $$E_n^{(2)}(f,Y_s)_p \le cn^{-2}\omega_{3,2}^{\varphi}(f'',n^{-1})_p + cn^{-6}\|f''\|_{L_p[-\frac{1}{2},\frac{1}{2}]} ,$$

where , $n \ge \eta$.

3.2 The Degree of Coconvex Approximation by D. T. Modulus of Smoothness

In this section we study the approximation of coconvex function in $L_p[-1,1]$,in terms of Ditizian_Totik modulus of smoothness.

To prove our theorems in this section ,we need the following Lemmas:

Lemma 3.2.1 [5]

For any $f \in L_p(J)$,$0 < p < \infty$,where $J = [a,b] \subseteq I$, we have

$\omega_k(f,|J|,J)_p \le c(r,p)\omega_k^{\varphi}(f,|J|)_p$,

and for $t \ge 0$

$\omega_k(f,t^2)_p \le c(r,p)\omega_k^{\varphi}(f,t)_p$.

Lemma 3.2.2

Let $f \in L_{\varphi}^{r}[-1,1]$, $J = (a,b) \subseteq (-1,1)$, we have

$$\omega_k(f^{(r)},|J|,J)_p \le \frac{1}{m^r(a,b)}\omega_{k,r}^{\varphi}(f^{(r)},|J|)_p \quad ,$$

where $m(a,b) = \sqrt{(1+a)(1-b)}$.

Proof:-

Let $x \pm \frac{k}{2}h \in J$, then $h \le \frac{|J|}{k}$,and we observe that ,If $x \pm \frac{k}{2}h \in J$ and $x \pm \frac{k}{2}h\varphi(x) \in I$, we have $J \subseteq I$,then

By using the following inequality from [20],$|J_1| \le |J|$ for $J_1 \subseteq J$,we obtain

$h \le \frac{|J|}{r}\varphi(x)$,

and,

$$\omega_k(f^{(r)},|J|,J)_p \le sup_{0<h\le\frac{|J|}{r}\varphi(x)} \left\|\Delta_h^k(f^{(r)},x)\right\|_{L_p(J)}$$

$$\leq sup_{0<h\leq\frac{|J|}{r}}\left\|\Delta_{h\varphi(x)}^{k}(f^{(r)},x)\right\|_{L_p(J)}$$

$$\leq sup_{0<h\leq\frac{|J|}{r}}\frac{1}{m^{r}(a,b)}\left\|\varphi^{r}(x)\,\Delta_{h\varphi(x)}^{k}(f^{(r)},x)\right\|_{L_p(J)}$$

$$\leq \frac{1}{m^{r}(a,b)}\,sup_{0<h\leq\frac{|J|}{r}}\left\|\varphi^{r}(x)\,\Delta_{h\varphi(x)}^{k}(f^{(r)},x)\right\|_{L_p(J)}$$

$$\leq \frac{1}{m^{r}(a,b)}\,\omega_{k,r}^{\varphi}(f^{(r)},|J|)_{p}$$ ♦

Lemma 3.2.3

Let $k \geq 1$,and let $f\epsilon L_p^2[a,a+h]$, $h > 0$ be a convex polynomial P of degree $\leq k+1$satisfying$P(a)=f(a)$,$P(a+h)=f(a+h)$, $P'(a) \geq f'(a)$ and $P'(a+h) \leq f'(a+h)$ and such that

$$\|f-P\|_{L_p[a,a+h]} \leq ch^2\omega_{k-2}(f'',h,[a,a+h])_p.$$

Proof:-

Let $J=[a,a+h]$, and $|J|=h$ and since $f\epsilon L_p^2[a,a+h]$, $P_{k+1} \in \Pi_{k+1}$,then by (Lemma 2.3.3 ,Whitney's Inequality) and by (Theorem 1.4.1 , Local Estimates) ,we have

$$\|f-P_{k+1}\|_{L_p[a,a+h]} \leq c\,\omega_k(f,|J|,J)_p$$

$$\leq c|J|^2\omega_{k-2}(f'',|J|,J)_p$$

$$\leq c|J|^2\omega_{k-2}(f'',h,J)_p$$ ♦

Lemma 3.2.4

Let $k \geq 1$, and let $a < \beta \leq a + h$ be fixed , and assume that , $f \epsilon L_p^2[a, a+h]$ is such that $f''(x)(x-\beta) \geq 0$, $a \leq x \leq a+h$ .If a polynomial $p \in \Pi_{k-1}$ satisfies $p(x)(x-\beta) \geq 0$, $a \leq x \leq a+h$ then there exists a polynomial $P_{k+1} \in \Pi_{k+1}$ such that $P''_{k+1} = p$, $P_{k+1}(a) = f(a)$, $P'_{k+1}(a) \leq f'(a)$,also $P'_{k+1}(a+h) \leq f'(a+h)$ and we have

$$\|f - P_{k+1}\|_{L_p[a,a+h]} \leq ch^2 \|f'' - p\|_{L_p[a,a+h]} .$$

Proof:-

Let $P_{k+1} \in \Pi_{k+1}$ and by (Lemma 2.3.3 ,Whitney's Inequality and Lemma 2.3.1) ,and by (Theorem 1.4.1 , Local Estimates) ,we have

$$\begin{aligned}\|f - P_{k+1}\|_{L_p[a,a+h]} &\leq c\omega_k(f,h,J)_p \\ &\leq ch^2 \omega_{k-2}(f'',h,J)_p \\ &= ch^2 sup_{0<h\leq|J|} \left\|\Delta_h^{k-2}(f'',.)\right\|_{L_p[a,a+h]} \\ &= ch^2 sup_{0<h\leq|J|} \left\|\Delta_h^{k-2}(f'' - P''_{k+1},.)\right\|_{L_p[a,a+h]} \\ &\leq c(k)h^2 \|f'' - P''_{k+1}\|_{L_p[a,a+h]} \\ &= c(k)h^2 \|f'' - p\|_{L_p[a,a+h]} \spadesuit\end{aligned}$$

Lemma 3.2.5 [5]

For every $s \in N_0$, $0 < p < \infty$ and $k, n \in N$,there are constants $c(k,s,p)$ and $c(s,p)$,such that , if $f \in \sum_{k,n}(Y_s) \cap \Delta^2(Y_s)$,then there exists $P_n \in \Delta^2(Y_s)$ of degree not exceeding $c(s,p)n$, satisfies

$$\|f-P_n\|_p \le c(k,s,p)\,\omega_k^{\varphi}(f,n^{-1})_p .$$

Lemma 3.2.6 [10]

Let $f \in L_p^r[-1,1]$,$0<p<\infty$, $r \ge 1$,then

$$\omega_k^{\varphi}(f,t)_p \le ct^r \omega_{k-r}^{\varphi}(f^{(r)},t)_p \ ,$$

where c is const. depends on k and r.

Lemma 3.2.7 [3]

For a function , $f \in L_p[-1,1]$, $0<p<\infty$,the following inequalities hold

$$\sum_{i=1}^{n} \omega_r(f,h_i,I_i)_p \le c\omega_k^{\varphi}(f,n^{-1})_p \ .$$

Lemma 3.2.8

Let $f \in L_\varphi^r[-1,1]$, $1<j<n$ and $0 \le l \le r$, then

$$|I_j|^l\,\omega_{k+r-l}(f^{(l)},|I_j|,I_j)_p \le cn^{-r}\omega_{k,r}^{\varphi}(f^{(r)},n^{-1})_p \ .$$

Proof:-

For , $1<j<n$, we have by using Lemma 3.2.1 and Lemma 3.2.2 ,then

$$|I_j| \le c\, m(x_j, x_{j-1}) n^{-1}$$

therefore $1 < j < n$ and $0 \le l \le r$,

$$|I_j|^l\, \omega_{k+r-l}(f^{(l)}, |I_j|, I_j)_p \quad \le C|I_j|^r\, \omega_k^{\varphi}(f^{(r)}, |I_j|, I_j)_p$$

$$\le C \frac{|I_j|^r}{m^r(x_j, x_{j-1})}\, \omega_{k,r}^{\varphi}(f^{(r)}, |I_j|)_p$$

$$\le Cn^{-r} \omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_p$$ ♦

Proof of Theorem 3.1.1

Given $f \in \Delta^2(Y_s) \cap L_p^2[-1,1]$, $r \ge 2$,we take $\eta(Y_s)$ so big that if $n \ge \eta$, then for each $1 \le i \le s$,the set O_i define in (3.1.1) contains only y_i ,also and O_i and O_{i+1} , $1 \le i \le s-1$, are separated by at least one interval of the partition .

Thus ,we have no restriction on η,if $s = 1$. Then we have s intervals $O_i = (a_i, b_i)$, $i = 1,2,\dots,s$; such that either

(3.2.1) $$f''(x)(x - y_i) \ge 0 , \qquad a_i < x < b_i$$

or

(3.2.2) $$f''(x)(x - y_i) \le 0 , \qquad a_i < x < b_i$$

we define polynomials $P_{k-1,i} \in \Pi_{k-1}$,$k = 1,2,3$; which satisfy , respectively

(3.2.3) $$P_{k-1,i}(x)(x-y_i) \geq 0\,, \qquad a_i < x < b_i$$

or

(3.2.4) $$P_{k-1,i}(x)(x-y_i) \leq 0\,, \qquad a_i < x < b_i$$

and are closed to f''.To this end we take $P_{0,i} \equiv 0$ and $P_{1,i}$ to be the linear polynomial Interpolating f'' at y_i and a_i or b_i whichever is farther from y_i , and finally $P_{2,i}$ to be the quadratic polynomial interpolating f'' at a_i , y_i and b_i.

By (Lemma 2.3.3 ,Whitney's Inequality ,Ch. 2),we know that

(3.2.5) $$\|f'' - P_{k-1,i}\|_{L_p(O_i)} \leq c\omega_k(f'', |O_i|, O_i)_p\ ,\ k = 1,2,3;$$

where c constant depends on the ratios between $|O_i|$ and the distances between the points of interpolation .

Thus , c is an absolute constant , for $k = 1,2$.

We obtain , for each $k \geq 2$, the existence of $P_{k-1,i} \in \Pi_{k-1}$, such that (3.2.1) and (3.2.2) hold respectively , and

(3.2.6) $$\|f'' - P_{k-1,i}\|_{L_p(O_i)} \leq c|O_i|\omega_{k-1}(f^{(3)}, |O_i|, O_i)_p\ ,$$

Thus ,in all cases we coclude by Lemma 3.2.4 , (3.2.4) and (3.2.5),that there exists a polynomial $P_{k+1,i} \in \Pi_{k+1}$,which is coconvex with f on O_i ,

$P_{k+1,i}(a_i) = f(a_i) + \alpha_i$,where α_i is an arbitrary constant to be prescribed ,and such that

(3.2.7) $$\|f - P_{k+1,i}\|_{L_p(O_i)} \leq |\alpha_i| + \tfrac{3}{2}|O_i|^2\|f'' - P_{k-1,i}\|_{L_p(O_i)}\ ,$$

by (3.2.6) and (3.2.7),we have an estimate on the second term on the right .

Note that (3.2.7) implies that,

(3.2.8) $$\left|P_{k+1,i}(b_i)-f(b_i)\right| \le |\alpha_i| + \tfrac{3}{2}|O_i|^2\|f''-P_{k-1,i}\|_{L_p(O_i)} ,$$

also if (3.2.1) holds ,then

(3.2.9) $$P'_{k+1,i}(a_i) \le f'(a_i) \quad \text{and} \quad P'_{k+1,i}(b_i) \le f'(b_i) ,$$

and if (3.2.2) holds ,then

(3.2.10) $$P'_{k+1,i}(a_i) \ge f'(a_i) \quad \text{and} \quad P'_{k+1,i}(b_i) \ge f'(b_i) .$$

In all other intervals I_j ,$j \in H$, f is either convex in I_j or concave there.

If $g_j = f + \beta_j$,where β_j is an arbitrary constant to be prescribed , then by Lemma 3.2.3, there exists a polynomial $p_{k+1,j} \in \Pi_{k+1}$, coconvex with f and satisfying

(3.2.11) $$p_{k+1,j}(x_j) = g_j(x_j) \quad \text{and} \quad p_{k+1,j}(x_{j-1}) = g_j(x_{j-1}) ,$$

also , if f is convex ,then we have

(3.2.12) $$p'_{k+1,j}(x_j) \ge f'(x_j) \quad \text{and} \quad p'_{k+1,j}(x_{j-1}) \le f'(x_{j-1}),$$

and if f is cocave , then

(3.2.13) $$p'_{k+1,j}(x_j) \le f'(x_j) \quad \text{and} \quad p'_{k+1,j}(x_{j-1}) \ge f'(x_{j-1}) .$$

By Lemma 3.2.6,

(3.2.14) $$\begin{aligned}\|f-p_{k+1,j}\|_{L_p(I_j)} &\le |\beta_j| + c|I_j|^2\omega_k(f'',|I_i|,I_i)_p \\ &\le |\beta_j| + cn^{-2}\omega_k^{\varphi}(f'',n^{-1})_p ,\end{aligned}$$

Since $|I_j| \le \frac{c}{n}$.

We construct the piecewise polynomial $f \in \sum_{k+2,n}(Y_s) \cap \Delta^2(Y_s)$, sweeping $[-1,1]$ from left and right .Let $a_s = x_{j0}$, where $O_s = (a_s, b_s)$, and let $a_s = 0$.

Then for $j_0 < j \leq n$, we take $\beta_j = 0$ and set

$$S/I_j = p_{k+1,j} \quad ,j_0 < j \leq n$$

and $$S/O_s = P_{k+1,s} .$$

Note that , $S \in L_p[-1, b_s)$ and by (3.2.11) and (3.2.12) or (3.2.10) and (3.2.13) respectively , it is coconvex with f there.

Suppose that we have defined S in $[-1, b_i)$, $1 < i \leq s$, let $b_i = x_{j_1}$ and $a_{i-1} = x_{j_2}$. then we take ,

$\beta_{i-1} = \sum_{m=i}^{s}(P_{k+1,m}(b_m) - f(b_m))$, and for $j_2 < j \leq j_1$,$\alpha_j = \beta_{i-1}$.

Then we set

$$S/I_j = p_{k+1,j} \quad , j_2 < j \leq j_1$$

$$S/O_{i-1} = P_{k+1,i-1}$$

this $S \in L_p[-1, b_{i-1})$, coconvex with f there.

Finally , if $b_1 = x_{j_3}$, then for 1< $j \leq j_3$,we take

$\alpha_j = \sum_{m=1}^{s}(P_{k+1,m}(b_m) - f(b_m))$,and we set

$$S/I_j = p_{k+1,j} \quad ,1 < j \leq j_3 .$$

It is readily seen that we have obtained an $f \in \sum_{k+2,n}(Y_s) \cap \Delta^2(Y_s)$,again we deel with $f \in L_p^2[-1,1]$.

Since $|O_i| \leq \frac{c}{n}$,it follows by (3.1.5) that is

$$\left\|f'' - P_{k-1,i}\right\|_{L_p(O_i)} \leq c\omega_k^{\varphi}(f'', n^{-1})_p \ , \ k = 1,2,3;$$

hence , combining with (3.2.7) , (3.2.8) and(3.2.14),also by Lemma 3.2.7, yields

$$\|f-S\|_{L_p(I)} = \left(\int_{-1}^{1} |f(x)-S(x)|^p dx\right)^{1/p}$$

$$= \left(\sum_i \int_{I_i\cup O_i} |f(x)-S(x)|^p dx\right)^{1/p}$$

$$\leq \left(\sum_i \int_{I_i} |f(x)-S(x)|^p dx + \sum_i \int_{O_i} |f(x)-S(x)|^p dx\right)^{1/p}$$

$$\leq C(p)\left(\sum_i \int_{I_i} |f(x)-S(x)|^p dx\right)^{1/p}$$

$$+c(p) \left(\sum_i \int_{O_i} |f(x)-S(x)|^p dx\right)^{1/p}$$

$$\leq C(p)\left(\sum_i \|f-S\|_{L_p(I_i)} + \sum_i \|f-S\|_{L_p(O_i)}\right)$$

$$\leq c(p)\left(\sum_i \left\|f-P_{k+1,i} \ +P_{k+1,i}-S\right\|_{L_p(I_i)} + \sum_i \left\|f-P_{k+1,i} \ +P_{k+1,i}-S\right\|_{L_p(O_i)}\right)$$

$$\le c(p)\sum_i \Big(\|f-P_{k+1,i}\|_{L_p(I_i)} + \|P_{k+1,i}-S\|_{L_p(I_i)} + \|f-P_{k+1,i}\|_{L_p(O_i)} + \|P_{k+1,i}-S\|_{L_p(O_i)} \Big).$$

Hence, by direct substituting ,and simplification ,we have ,

(3.2.15) $$\|f-S\|_{L_p(I)} \le \frac{c}{n^2}\,\omega_k^{\varphi}(f'',n^{-1})_p \qquad , k=1,2,3$$

this in turn implies

$$\omega_{k+2}^{\varphi}(S,n^{-1})_p = \omega_{k+2}^{\varphi}(S-f+f,n^{-1})_p$$

$$\le \omega_{k+2}^{\varphi}(f,n^{-1})_p + c\|S-f\|_{L_p(I)}$$

$$\le \omega_{k+2}^{\varphi}(f,n^{-1})_p + c\,n^{-1}\,\omega_k^{\varphi}(f'',n^{-1})_p$$

(3.2.16) $$\le c\,n^{-1}\,\omega_k^{\varphi}(f'',n^{-1})_p \ .$$

Therefore ,we apply (3.2.15) and (3.2.16) ,also by Lemma 3.2.5 ,to obtain a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$ such that

$$\|f-P_n\|_{L_p(I)} = \|f-S+S-P_n\|_{L_p(I)}$$

$$\le \|f-S\|_{L_p(I)} + \|S-P_n\|_{L_p(I)}$$

$$\le \frac{c}{n^2}\,\omega_k^{\varphi}(f'',n^{-1})_p + \omega_{k+2}^{\varphi}(S,n^{-1})_p$$

$$\le \frac{c}{n^2}\,\omega_k^{\varphi}(f'',n^{-1})_p + c\,n^{-1}\,\omega_k^{\varphi}(f'',n^{-1})_p$$

$$\le c\,n^{-1}\,\omega_k^{\varphi}(f'',n^{-1})_p$$

where , $c = c(k,s,p)$ and $\eta = \eta(Y_s)$ ♦

Proof of Theorem 3.1.2

Assume that $f \in L_p^r[-1,1]$, $r \geq 3$ and let $k \geq 2$. Then ,it follows by ((3.2.6) in the proof of Theorem 3.1.1)

$$\|f'' - P_{k-1,i}\|_{L_p(O_i)} \leq c\, n^{-1}\omega_{k-1}^{\varphi}(f^{(3)}, n^{-1})_p \,. \tag{3.2.17}$$

Hence ,combining with, (3.2.7) ,(3.2.8) and (3.2.12) are in proof of Theorem 3.1.1 , yields

$$\|f - S\|_{L_p[-1,1]} \leq c\, s\, n^{-3}\omega_{k-1}^{\varphi}(f^{(3)}, n^{-1})_p \,. \tag{3.2.18}$$

This is turn gives

$$\omega_{k+2}^{\varphi}(S, n^{-1})_p = \omega_{k+2}^{\varphi}(f - f + S, n^{-1})_p$$

$$\leq \omega_{k+2}^{\varphi}(f, n^{-1})_p + c\|f - S\|_{L_p(I)}$$

$$\leq \omega_{k+2}^{\varphi}(f, n^{-1})_p + c\, s\, n^{-3}\omega_{k-1}^{\varphi}(f^{(3)}, n^{-1})_p$$

$$\leq c\, n^{-3}\omega_{k-1}^{\varphi}(f^{(3)}, n^{-1})_p \,, \tag{3.2.19}$$

where $c = c(k, s)$.

Therefore , to apply (3.2.18), (3.2.19) , Lemma 3.2.5 and Lemma 3.2.6 ,to obtain polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$, such that

$$\|f - P_n\|_{L_p[-1,1]} \leq cs\, n^{-3}\omega_k^{\varphi}(f^{(3)}, n^{-1})_p$$

Since , $f \in L_p^r[-1,1]$, $r \geq 3$, it follows by Lemma 3.2.5 and Lemma 3.2.6 ,we have

$$\|f - P_n\|_{L_p[-1,1]} = \|f - S + S - P_n\|_{L_p(I)}$$

$$\leq \|f - S\|_{L_p(I)} + \|S - P_n\|_{L_p(I)}$$

$$\leq c\, s\, n^{-3}\omega_k^{\varphi}(f^{(3)}, n^{-1})_p + \omega_k^{\varphi}(S, n^{-1})_p$$

$$\leq c\, s\, n^{-3}\omega_k^{\varphi}(f^{(3)}, n^{-1})_p + \omega_k^{\varphi}(S - f + f, n^{-1})_p$$

$$\leq c\, s\, n^{-3}\omega_k^{\varphi}(f^{(3)}, n^{-1})_p + \omega_k^{\varphi}(f, n^{-1})_p$$

$$\leq c\, s\, n^{-r}\omega_{k-r}^{\varphi}(f^{(r)},n^{-1})_p .$$

That is,

$$E_n^{(2)}(f,Y_s)_p \leq \frac{c}{n^r}\,\omega_{k-r}^{\varphi}(f^{(r)},n^{-1})_p \quad ,n \geq N$$

is valid for all $r \geq 3$ and $k \geq 1$, with $c = c(k,r,s)$ and $\eta = \eta(k,r,Y_s)$ ♦

Proof of Theorem 3.1.3

From Lemma 3.2.5 ,we estimate that

$$\omega_{k+5}^{\varphi}(P_n,n^{-1})_p = \omega_{k+5}^{\varphi}(f - f + P_n,n^{-1})_p \ ,$$

$$\leq \omega_{k+5}^{\varphi}(f,n^{-1})_p + c\|f - P_n\|_{L_p(I)} ,$$

$$\leq cn^{-5}\omega_{k,5}^{\varphi}(f^{(5)},n^{-1})_p + c\|f - P_n\|_{L_p(I)} .$$

We note that from theorem 2.1.4,when $l = 0$,then, we only need to construct a spline $P_n \in \sum_{k+5,n}(Y_s) \cap \Delta^2(Y_s)$, such that

(3.2.19) $$\|f - P_n\|_{L_p(I)} \leq cn^{-5}\omega_{k,5}^{\varphi}(f^{(5)},n^{-1})_p$$

Also by Lemma 3.2.8 with $l = 3$ and $r = 5$,implies

(3.2.20) $$|I_j|^3\, \omega_{k+2}(f^{(3)},|I_j|,I_j)_p \quad \leq \omega_{k,5}^{\varphi}(f^{(5)},n^{-1})_p ,$$

For $1 < j < n$, while with $l = 2$ and$r = 5$, it implies

$$|I_j|^2\, \omega_{k+3}(f'',|I_j|,I_j)_p \quad \leq cn^{-5}\omega_{k,5}^{\varphi}(f^{(5)},n^{-1})_p \quad .$$

Taking these estimates into account ,the same construction as in Theorem 3.1.1 and Theorem 3.1.2, yields a spline $P_n \in \sum_{k+5,n}(Y_s)$ which is coconvex with f on $[-1,1]$ and such that (3.2.19) holds .

For the sake of completeness, to describe this construction .We take $N(Y_s)$ to be so large that ,for $n \ge N$, the sets O_i , $1 \le i \le s$, are all disjoint and do not contain the endpoints of the interval $[-1,1]$.

Now if $I_j \cap O = \emptyset$,then f does not change its convexity on I_j and Lemma 3.2.3 implies that there exists a polynomial $P_j \in \Pi_{k+5}$ with coconvex with f , interpolates it at the endpoints of I_j , and such that

$P_j'(x_j) \ge f'(x_j)$ and $P_j'(x_{j-1}) \le f'(x_{j-1})$ if f is convex on I_j ,or

$P_j'(x_j) \le f'(x_j)$ and $P_j'(x_{j-1}) \ge f'(x_{j-1})$ if f is cocave on I_j ,and satisfies

$$\|f-P_j\|_{L_p(I_j)} \le c|I_j|^2 \omega^{\varphi}_{k+3}(f'',|I_j|,I_j)_p$$

$$\le cn^{-5}\omega^{\varphi}_{k,5}(f^{(5)},n^{-1})_p \quad .$$

Now , it is convenient to denote the endpoints of O_i ,by a_i and b_i i.e. $O_i = (a_i\,,b_i)$, $1 \le i \le s$.

Also , for each $1 \le i \le s$,there exists a polynomial $\widetilde{P}_i \in \Pi_{k+3}$,which is copositive with f''on O_i (i.e. $\widetilde{P}_i(x)f''(x) \ge 0$ for all $x \in O_i$) and such that,

$$\|f''-\widetilde{P}_i\|_{L_p(O_i)} \le c|O_i| \;\; \omega^{\varphi}_{k+2}(f^{(3)},|O_i|,O_i)_p \,.$$

And ,Lemma 3.2.4 implies that there exists a polynomial $\overline{P}_i \in \Pi_{k+5}$ such that $\overline{P}_i'(a_i) \le f'(a_i)$ and $\overline{P}_i'(b_i) \le f'(b_i)$,

(if f is such that $f''(x)(x-y_i) \ge 0$,for $x \in O_i$) ,and satisfying

$$\|f-\overline{P}_i\|_{L_p(O_i)} \le c|O_i|^2 \|f''-\widetilde{P}_i\|_{L_p(O_i)}$$

$$\le c|O_i|^3 \,\omega_{k+2}(f^{(3)},|O_i|,O_i)_p$$

$$\le cn^{-5}\omega^{\varphi}_{k,5}(f^{(5)},n^{-1})_p \quad .$$

Where ,the last inequality follows from (3.2.20),the observation that $|O_i| \sim |I_j|$,where j is such that $y_i \in I_i$, and the fact that O_i is far from ± 1.

Now ,the piecewise polynomial approximation $P_n \in \sum_{k+5,n}(Y_s) \cap \Delta^2(Y_s)$ is constructed from the polynomial pieces P_j and $\overline{P}_i$ in such a way that ,if P_n is constructed for all $x \le x_v$,then on $[x_v, x_{v-1}]$ (or $[x_v, x_{v-3}] = O_\vartheta$ if x_v happens to be the left endpoint of some interval O_ϑ),it is defined to be P_v (or $\bar{P}_\vartheta + \alpha$,where the constant α is chosen in such a way as to make $P_n \in L_p[-1,1]$), it is not difficult to see now P_n is coconvex with f and (3.2.20) holds ♦

3.3 The Divided Difference and the Weighted D.T. Modulus of Smoothness

In this section ,we will use the divided difference and weighted D. T. modulus of smoothness in Theorem 3.1.5 .

To prove our theorems in this section ,we need the following Lemmas:

Lemma 3.3.1

For any $f \in L_p(J)$, $0 < p < \infty$, $J \subseteq I$ and $1 \le r < k$,we have

$$\omega_k(f, |J|, J)_p \le c\omega^{\varphi}_{k,r}(f, |J|)_p ,$$

where c depends on r.

Proof :-

We observe that if $x \pm \frac{r}{2}h \in J$ and $x \pm \frac{r}{2}h\varphi(x) \in I$, let $\left[x-\frac{r}{2}h, x+\frac{r}{2}h\right] \subseteq J$,then by using ,$|J_1| \le |J|$ for $J_1 \subseteq J$,we obtain

(3.3.1) $$h \le \frac{|J|}{r}\varphi(x)$$

and we have $x \pm \frac{r}{2}h \in J$ if and only if

(3.3.2) $$h \le \frac{|J|}{r}$$

also , let $\varphi(x) \le |J|$ .And by using the two inequality (3.3.1)and (3.3.2), we obtain,

$$\omega_k(f,|J|,J)_p \le sup_{0<h\le\frac{|J|}{r}\varphi(x)} \left\|\Delta_h^k(f,x)\right\|_{L_p(J)}$$

$$\le sup_{0<h\le\frac{|J|}{r}} \varphi^{-r}(x) \quad \left\|\varphi^r(x)\,\Delta_{h\varphi(x)}^k(f,x)\right\|_{L_p(J)}$$

$$\le \frac{c}{|J|^r}\, sup_{0<h\le\frac{|J|}{r}} \left\|\varphi^r(x)\,\Delta_{h\varphi(x)}^k(f,x)\right\|_{L_p(J)}$$

$$\le c\omega_{k,r}^{\varphi}(f,\tfrac{|J|}{r})_p$$

$$\le c\omega_{k,r}^{\varphi}(f,|J|)_p$$

where c depends on r ♦

Lemma 3.3.2

Let $k \ge 2$, $r \in N$,$1 \le m \le k-1$ and $n \ge 3m$,and let $j \in N$ be such that $[x_{j+m}, x_j] \subseteq \left[-\frac{1}{2},\frac{1}{2}\right]$,(which is equivalent to $\frac{n}{3} \le j \le \frac{2n}{3-m}$). Then for any $f \in L_p(I)$,$1 \le p < \infty$,

$$\left\|[x_j, x_{j+1},\dots,x_{j+m};f]\right\| \le c\left(n^k\omega_{k,r}^{\varphi}(f,n^{-1})_p + \|f\|_{L_p\left[-\frac{1}{2},\frac{1}{2}\right]}\right),$$

Where c depends only on k and r.

Proof :-

From the Marchaud inequality for a function $f \in L_p[a,b]$ and $1 \le m < k$,it was shown in [42] and [44] that for $1 \le p < \infty$,

$$\omega_m(f,t,[a,b])_p \le ct^m \left\{ \int_t^{b-a} \frac{\omega_k(f,s,[a,b])_p ds}{s^{m+1}} + (b-a)^{-m}\|f\|_{L_p[a,b]} \right\}$$

Now ,taking $[a,b] = \left[-\frac{1}{2},\frac{1}{2}\right]$ in the above estimate , using (3.1.4) and taking $|x_j - x_{j+m}| \sim \frac{1}{n}$,for all j such that $x_j \in \left[-\frac{1}{2},\frac{1}{2}\right] = J$,and By using Lemma 3.3.1,and since $J \subseteq [-1,1]$,we have

$$|[x_j, x_{j+1}, \ldots, x_{j+m}; f]| \le c\left(x_j - x_{j+m}\right)^{-m} \omega_m(f, x_j - x_{j+m}, [x_{j+m}, x_j])_p$$

$$\le cn^m \omega_m\left(f, 1/n, \left[-\frac{1}{2},\frac{1}{2}\right]\right)_p$$

$$\le \int_{1/n}^{1} \frac{\omega_k\left(f, s, \left[-\frac{1}{2},\frac{1}{2}\right]\right)_p ds}{s^{m+1}} + c\|f\|_{L_p(J)}$$

$$\le c \int_{1/n}^{1} \frac{\omega_k\left(f, s, \left[-\frac{1}{2},\frac{1}{2}\right]\right)_p ds}{s^k} s^{k-1-m}\, ds + c\|f\|_{L_p(J)}$$

$$\le cn^k \omega_k(f, |J|, J)_p \int_{1/n}^{1} s^{k-1-m}\, ds + c\|f\|_{L_p(J)}$$

$$\le cn^k \omega_k(f, |J|, J)_p + c\|f\|_{L_p(J)}$$

$$\le c\left(n^k \omega_{k,r}^{\varphi}(f, n^{-1})_p + \|f\|_{L_p(J)}\right) ♦$$

Lemma 3.3.3

Let $k \geq 2$, $n \geq 3k$ and $1 \leq m \leq k-1$,and let $r \in N_0$, $r > k-2m$,then for $f \in L_p[-1,1]$ and for every $1 \leq j \leq n-m-1$, $J = \left[-\frac{1}{2}, \frac{1}{2}\right]$

$$\left\|[x_j, x_{j+1}, \dots, x_{j+m}; f]\right\| \leq c\left(n^k \left(\frac{n}{min\{j, n-j\}}\right)^{r+2m-k} \omega_{k,r}^{\varphi}(f, n^{-1})_p + \|f\|_{L_p(J)}\right).$$

Proof:-

For $1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$, for all $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $\varphi(x_i) \sim \frac{i}{n}$ and $|I_i| \sim \frac{i}{n^2}$.

Now for all $1 \leq j \leq n-k-1$,the inequalities (3.1.4) and Lemma 3.2.2 , imply

$$\left\|[x_j, x_{j+1}, \dots, x_{j+k}; f]\right\| \leq c\left|I_j\right|^{-k} \omega_k\left(f, x_j - x_{j+k}, [x_{j+k}, x_j]\right)_p$$

$$\leq c\left|I_j\right|^{-k} m^{-r}\left(x_{j+k}, x_j\right) \omega_{k,r}^{\varphi}\left(f, \left|[x_{j+k}, x_j]\right|\right)_p$$

$$\leq c\left|I_j\right|^{-k} \varphi^{-r}\left(x_j\right) \omega_{k,r}^{\varphi}(f, n^{-1})_p.$$

Therefore , if $1 \leq j \leq min\left\{n-k-1, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, then

$$\left\|[x_j, x_{j+1}, \dots, x_{j+m}; f]\right\| \leq cn^k \left(\frac{n}{j}\right)^{k+r} \omega_{k,r}^{\varphi}(f, n^{-1})_p. \tag{3.3.3}$$

Let $1 \leq \mu \leq \left\lfloor \frac{n}{2} \right\rfloor$ be an index ,and let $[x_{\mu+m}, x_\mu] \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$and Lemma 3.3.2 implies that

$$\left\|[x_\mu, x_{\mu+1}, \dots, x_{\mu+m}; f]\right\| \leq c\left(n^k \omega_{k,r}^{\varphi}(f, n^{-1})_p + \|f\|_{L_p(J)}\right)$$

Also,

$$-[x_j,x_{j+1},\dots,x_{j+m};f] = \left([x_\mu,x_{\mu+1},\dots,x_{\mu+m};f] - [x_j,x_{j+1},\dots,x_{j+m};f]\right)$$

$$-[x_\mu,x_{\mu+1},\dots,x_{\mu+m};f],$$

$$=\sum_{i=j}^{\mu-1}(x_{j+m+1}-x_i)\,[x_i,x_{i+1},\dots,x_{i+m+1};f]$$

$$-[x_\mu,x_{\mu+1},\dots,x_{\mu+m};f],$$

therefore ,

$$\left|[x_j,x_{j+1},\dots,x_{j+m};f]\right| \le c\sum_{i=j}^{\left\lfloor\frac{n}{2}\right\rfloor}|I_i|\,[x_i,x_{i+1},\dots,x_{i+m+1};f]$$

$$+c\,n^k\,\omega_{k,r}^{\varphi}(f,n^{-1})_p + c\|f\|_{L_p(J)},$$

by using (3.3.3) ,we have

$$\left|[x_j,x_{j+1},\dots,x_{j+m};f]\right|$$

$$\le c\sum_{i=j}^{\left\lfloor\frac{n}{2}\right\rfloor}|I_i|\left(n^k\left(\frac{n}{min\{i,n-i\}}\right)^{r+2(m+1)-k}\omega_{k,r}^{\varphi}(f,n^{-1})_p\right)$$

$$+c\,n^k\,\omega_{k,r}^{\varphi}(f,n^{-1})_p + c\|f\|_{L_p(J)}$$

$$\le c\,n^k\,\omega_{k,r}^{\varphi}(f,n^{-1})_p\sum_{i=j}^{\left\lfloor\frac{n}{2}\right\rfloor}|I_i|\,n^k\left(\frac{n}{i}\right)^{r+2(m+1)-k}$$

$$+c\,n^k\,\omega_{k,r}^{\varphi}(f,n^{-1})_p + c\|f\|_{L_p(J)}$$

$$\leq c\, n^k\, \omega_{k,r}^{\varphi}(f, n^{-1})_p \sum_{i=j}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n^{r+2m-k}}{i^{r+2m-k+1}}$$

$$+c\, n^k\, \omega_{k,r}^{\varphi}(f, n^{-1})_p \ + c\|f\|_{L_p(J)}$$

$$\leq c\left(n^k \left(\frac{n}{j}\right)^{r+2m-k} \omega_{k,r}^{\varphi}(f, n^{-1})_p + \|f\|_{L_p(J)} \right)$$ ♦

Corollary 3.3.4

Let $n \geq 9$, $m = 1$ or $m = 2$ and $f \in L_p[-1,1]$.

Then ,

$$\left|[x_j, x_{j+1}, \dots, x_{j+m}; f]\right| \leq Cn^{2m+2}\, \omega_{k,r}^{\varphi}(f, n^{-1})_p + C\|f\|_{L_p[-\frac{1}{2},\frac{1}{2}]}.$$

The proof is clearly ,by Lemma3.3.3.

Lemma 3.3.5

Let $f \in L_{\varphi}^{r}[-1,1]$,$n \geq k+1$,and let a polynomial P_{k+r} of degree $\leq k+r-1$,be such that $P_{k+r}^{(i)}(x_1) = f^{(i)}(x_1)$ for all $i = 0,1,\dots,r-1$ and $P_{k+r}^{(r)}(x) = L_{k-1}\left(f^{(r)}; x_1, x_2, \dots, x_k\right)(x)$

Then ,

$$\|f - P_{k+r}\|_{L_p(J)} \leq \frac{c}{n^r}\, \omega_{k,r}^{\varphi}(f, n^{-1})_p \,.$$

Proof:-

We denote $L_{k-1}\left(f^{(r)};x_1,x_2,\dots,x_k\right)$ and note that it follows by (Lemma 2.3.3 ,Whitney's inequality),that for any $\alpha\in[x_1,1]$ and the interval $J=[x_k,\alpha]$ and by Theorem 3.1.3,we have

$$\left\|L_{k-1}-f^{(r)}\right\|_{L_p(J)}\le\omega_k\left(f^{(r)},|J|,J\right)_p$$

$$\le\frac{c}{(1-\alpha)^{r/2}}\,\omega_{k,r}^{\varphi}\left(f^{(r)},|J|\right)_p$$

$$\le\frac{c}{(1-\alpha)^{r/2}}\,\omega_{k,r}^{\varphi}\left(f^{(r)},n^{-1}\right)_p.$$

And ,in particular,

(3.3.4) $$\left|L_{k-1}(\alpha)-f^{(r)}(\alpha)\right|\le\frac{c}{(1-\alpha)^{r/2}}\,\omega_{k,r}^{\varphi}\left(f^{(r)},n^{-1}\right)_p$$

Since ,for any $g\in L_p^r[a,b]$ and $x\in[a,b]$,

$$g(x)=\sum_{i=0}^{r-1}\frac{g^{(i)}(a)}{i!}(x-a)^i+\frac{1}{(r-1)!}\int_a^x(x-u)^{r-1}g^{(r)}(u)du$$

by using (3.3.4) , for any $x_1\le x<1$,the following holds:

$$|f(x)-P_{k+r}(x)|=\frac{1}{(r-1)!}\left|\int_{x_1}^{x}(x-u)^{r-1}\left(f^{(r)}(u)-P_{k+r}^{(r)}(u)\right)du\right|$$

$$\le c\int_{x_1}^{x}(x-u)^{r-1}\left|f^{(r)}(u)-L_{k-1}(u)\right|du$$

$$\le c\,\omega_{k,r}^{\varphi}\left(f^{(r)},n^{-1}\right)_p\int_{x_1}^{x}\frac{(x-u)^{r-1}}{(1-u)^{r/2}}\,du$$

$$\leq c\,\omega_{k,r}^{\varphi}\left(f^{(r)},n^{-1}\right)_p \int_{x_1}^{x} (1-u)^{\frac{r}{2}-1}\,du$$

$$\leq c(1-x_1)^{\frac{r}{2}}\,\omega_{k,r}^{\varphi}\left(f^{(r)},n^{-1}\right)_p$$

$$\leq cn^{-r}\omega_{k,r}^{\varphi}\left(f^{(r)},n^{-1}\right)_p \blacklozenge$$

Proof of Theorem 3.1.4

For all $1 \leq i \leq n-1$, $\varphi(x_i) \sim min\{i, n-i\}/n$ and $|I_i| \sim min\{i, n-i\}/n^2$,where as usual ,$\alpha_i \sim \beta_i$ means that $\frac{\alpha_i}{\beta_i}$ is bounded a way from 0 and ∞.

The following inequality is contained in the proof of Lemma 3.3.3;

(3.3.5) $$\left|\left[x_j, x_{j+1}, \dots, x_{j+k}; f\right]\right| \leq cn^k \left(\frac{n}{min\{j,n-j\}}\right)^{k+r} \omega_{k,r}^{\varphi}(f, n^{-1})_p$$

,

for all $1 \leq j \leq n-k-1$

now ,for any $m \geq 0$ and $1 \leq j \leq \delta < n-m$,

(3.3.6) $$[x_\delta, x_{\delta+1}, \dots, x_{\delta+m}; f] - \left[x_j, x_{j+1}, \dots, x_{j+m+1}; f\right]$$

$$= \sum_{i=j}^{\delta-1} (x_{i+m+1} - x_i)[x_i, x_{i+1}, \dots, x_{i+m+1}; f],$$

This , with $m = k-1$, $\delta = \mu$, together with the inequality (3.3.5) for $1 \leq j < \mu \leq n-k$ *,implies*

$$\left|[x_\mu, x_{\mu+1}, \dots, x_{\mu+k-1}; f] - [x_j, x_{j+1}, \dots, x_{j+k-1}; f]\right|$$

$$= \left|\sum_{i=j}^{\mu-1} (x_{i+k} - x_i)[x_i, x_{i+1}, \dots, x_{i+k}; f]\right|$$

$$\leq c \sum_{i=j}^{\mu-1} |I_i| n^k \left(\frac{n}{min\{i, n-i\}}\right)^{k+r} \omega_{k,r}^{\varphi}(f, n^{-1})_p$$

$$\leq cn^{2k+r-2} \omega_{k,r}^{\varphi}(f, n^{-1})_p \sum_{i=j}^{\mu-1} \left(\frac{1}{min\{i, n-i\}}\right)^{k+r-1}$$

$$\leq cn^{2k+r-2} \omega_{k,r}^{\varphi}(f, n^{-1})_p \sum_{i=\{j, n-\mu\}}^{\infty} \frac{1}{i^{k+r-1}}$$

$$\leq cn^{2k+r-2} \left(\frac{1}{min\{j, n-\mu\}}\right)^{k+r-2} \omega_{k,r}^{\varphi}(f, n^{-1})_p,$$

where ,for the last inequality we used $k + r \geq 3$ .Thus ,(3.1.5) is proved .Now ,to prove (3.1.6) , suppose that $k + r \geq 5$. Applying (3.3.6) with $m = k - 2$ and $\delta = v$ and (3.1.5) ,for all $1 \leq j \leq v \leq \mu$, yields

$$\varepsilon\big([x_v, x_{v+1}, \dots, x_{v+k-2}; f] - [x_j, x_{j+1}, \dots, x_{j+k-2}; f]\big)$$

$$= \varepsilon \sum_{i=j}^{v-1} (x_{i+k} - x_i)[x_i, x_{i+1}, \dots, x_{i+k-1}; f]$$

$$= \varepsilon \sum_{i=j}^{v-1} (x_i - x_{i+k})([x_i, x_{i+1}, \dots, x_{i+k-1}; f] - [x_i, x_{i+1}, \dots, x_{i+k-1}; f])$$

$$- \varepsilon [x_i, x_{i+1}, \dots, x_{i+k-1}; f] \sum_{i=j}^{v-1} (x_i - x_{i+k})$$

$$\leq \sum_{i=j}^{v-1}(x_i - x_{i+k})\left|[x_\mu, x_{\mu+1}, \dots, x_{\mu+k-1}; f]\right.$$
$$\left. - [x_i, x_{i+1}, \dots, x_{i+k-1}; f]\right|$$

$$\leq cn^{2k+r-2}\omega_{k,r}^{\varphi}(f, n^{-1})_p \sum_{i=j}^{v-1}|I_i|\left(\frac{1}{min\{i, n-\mu\}}\right)^{k+r-2}$$

$$\leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f, n^{-1})_p \sum_{i=j}^{v-1}\frac{min\{i, n-i\}}{(min\{i, n-\mu\})^{k+r-2}}$$

$$\leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f, n^{-1})_p \sum_{i=j}^{\mu-1}\frac{min\{i, n-i\}}{(min\{i, n-i\})^{k+r-2}} \quad = \Phi\,.$$

Now ,since $k + r \geq 5$,if $\mu \leq \left\lfloor \frac{n}{2} \right\rfloor$,then

$$\Phi \leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f, n^{-1})_p \sum_{i=1}^{\infty}\frac{1}{i^{k+r-3}}$$

$$\leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f, n^{-1})_p.$$

And ,if $\mu > \left\lfloor \frac{n}{2} \right\rfloor$,then

$$\Phi \leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f, n^{-1})_p \left(\sum_{i=1}^{n-\mu}\frac{1}{i^{k+r-3}} + \sum_{i=j}^{\mu-1}\frac{min\{i, n-i\}}{(n-\mu)^{k+r-2}}\right)$$

$$\leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f, n^{-1})_p \left(1 + \frac{1}{(n-\mu)^{k+r-2}}\sum_{i=1}^{n} i\right)$$

$$\leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f,n^{-1})_p\left(1+\frac{n^2}{(n-\mu)^{k+r-2}}\right)\blacklozenge$$

We note that the proof of Corollary 3.1.5 and Corollary 3.1.6 are direct by Theorem 3.1.4 .

3.4 The Degree of Coconvex Approximation and Divided Difference by Using Weighted D. T. Modulus of Smoothness

In this section we estimate the degree of approximation of f by divided difference. And to prove our theorems in this section ,we need the following Lemmas:

Lemma 3.4.1

Let $n \geq 9$, $m = 1$ or $m = 2$, and $f \in L_\varphi^2$ .Then ,

$$max\{|[x_1, x_2, \dots, x_{m+1}; f'']|, |[x_{n-1}, x_{n-2}, \dots, x_{n-m-1}; f'']|\}$$

$$\leq cn^{2m+2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p + c\|f''\|_{L_p[-\frac{1}{2},\frac{1}{2}]}.$$

Proof of this Lemma is an immediate consequence of Corollary 3.3.4.

Lemma 3.4.2

Let $s \geq 3$ and $f \in L_\varphi^2 \cap \Delta^2(Y_s)$,and

$n \geq max\{\mathcal{M}(Y_s), (min\{\varphi(y_i) / 1 \leq i \leq s\})^{-3}\}$,

then,

(3.4.1) $$max\{|[x_1, x_2, x_3; f'']|, |[x_{n-1}, x_{n-2}, x_{n-3}; f'']|\} \leq cn^6\, \omega_{3,2}^{\varphi}(f'', n^{-1})_p$$

And ,

(3.4.2) $$max\{|[x_1, x_2; f'']|, |[x_{n-1}, x_{n-2}; f'']|\} \leq cn^4\, \omega_{3,2}^{\varphi}(f'', n^{-1})_p.$$

Lemma 3.4.3

Let $s = 2$, $f \in L_\varphi^2 \cap \Delta^2(Y_s)$,and

$n \geq max\{\mathcal{M}(Y_s), (min\{\varphi(y_1), \varphi(y_2)\})^{-3}\}$, then

(3.4.3) $$max\{-[x_1, x_2, x_3; f''], -[x_{n-1}, x_{n-2}, x_{n-3}; f'']\} \leq cn^6\, \omega_{3,2}^{\varphi}(f'', n^{-1})_p$$

And,

(3.4.4) $$max\{-[x_1, x_2; f''], -[x_{n-1}, x_{n-2}; f'']\}$$

$$\leq cn^4\ \omega_{3,2}^{\varphi}(f'',n^{-1})_p.$$

Proof of Lemma 3.4.2 and Lemma 3.4.3.

Let $\mathbb{N} = \mathbb{N}(Y_s) = min\{\varphi(y_i) /\ 1 \leq i \leq s\}$. Also ,let $s \geq 2$, $f \in L_\varphi^2 \cap \Delta^2(Y_s)$ be given ,and observe that if an index i is such that $y_s \leq x_i \leq y_1$,then

$$4\,min\{i, n-i\} \geq n\ \sin(i\pi/n) = n\varphi(x_i) \geq n\,min\{\varphi(y_s), \varphi(y_1)\}$$

Now , let the indices μ_1, v_1, v_2 and μ_2 (if $s \geq 3$) be such that

$$f''(x_{\mu_1+1}) = min\{f''(x_i) \mid y_2 \leq x_i \leq y_1\},$$

$x_{v_1+1} \leq y_1 < x_{v_1}$, $x_{v_2+1} \leq y_2 < x_{v_2}$,and

$$f''(x_{\mu_2+1}) = max\{f''(x_i) \mid y_3 \leq x_i \leq y_2\}.$$

Then , using $f''(x)(x-y_1)(x-y_2) \geq 0$ for all $x \geq y_3$,we conclude that the following inequality holds: $1 \leq v_1 \leq \mu_1 < v_2 \leq n-2$, $v_2 \leq \mu_2 \leq n-3$ (if $s \geq 3$),

$$[x_{\mu_1}, x_{\mu_1+1}, x_{\mu_1+2}; f''] \geq 0\ \ ,[x_{v_1}, x_{v_1+1}; f''] \geq 0$$

and

$$[x_{\mu_2}, x_{\mu_2+1}, x_{\mu_2+2}; f''] \leq 0\ \ ,[x_{v_2}, x_{v_2+1}; f''] \leq 0\,.$$

By Corollary 3.1.5 (i) with $\mu = \mu_1$ and $v = v_1$,taking into account that $n - \mu_1 + 1 \geq \mathbb{N}n/4$, it follows that

(3.4.5) $$-[x_1, x_2, x_3; f''] \leq cn^6\ \omega_{3,2}^{\varphi}(f'',n^{-1})_p.$$

And,

(3.4.6) $$-[x_1, x_2; f''] \leq cn^4 \left(1 + \frac{n^2}{(n-\mu_2)^3}\right) \omega_{3,2}^{\varphi}(f'',n^{-1})_p.$$

If $s \geq 3$,then Corollary 3.1.5 (i) with $\mu = \mu_2$ and $v = v_2$,and the observation that $n - \mu_2 \geq \mathbb{N}n/4$,imply

(3.4.7) $$[x_1, x_2, x_3; f''] \leq cn^6\ \omega_{3,2}^{\varphi}(f'', n^{-1})_p$$

And,

$$[x_1, x_2; f''] \leq cn^4 \left(1 + \frac{n^2}{(n - \mu_2)^3}\right) \omega_{3,2}^{\varphi}(f'', n^{-1})_p$$

(3.4.8) $$\leq cn^4\ \omega_{3,2}^{\varphi}(f'', n^{-1})_p).$$

This in turn implies that

$$|[x_1, x_2, x_3; f'']| \leq cn^6\ \omega_{3,2}^{\varphi}(f'', n^{-1})_p\ ,$$

And ,

$$|[x_1, x_2; f'']| \leq cn^4\ \omega_{3,2}^{\varphi}(f'', n^{-1})_p.$$

This complete the proof of Lemma 3.4.2 , and to complete the proof Lemma 3.4.3,it suffices to use Corollary 3.1.5 (ii) with $\mu = n - \mu_1 - 2$ and $v = n - v_2 - 1$,and the estimate $\mu_1 + 1 \geq \mathbb{N}n/4$,and to combine the resulting inequalities with (3.4.5)and (3.4.6) ♦

Proof of theorem 3.1.7 and theorem 3.1.8

Suppose that n is such that

$$n \geq max\left\{4\left(\min_{1\leq j\leq s+1}\{y_{i-1} - y_i\}\right)^{-1}, \left(\min_{1\leq j\leq s}\{\varphi(y_i)\}\right)^{-3}\right\}.$$

We use the same construction as in proof of Theorem 3.1.1,which we described in the proof of Theorem 3.1.3. The only difference now is that , on each interval O_i , $1 \le i \le s$,the polynomial $\tilde{p}_i$ is defined to be the quadratic polynomial interpolating $f''\ at\ a_i, y_i$ and b_i ,whence ,by (Lemma 2.3.3 ,Whitney's inequality),

$$\|f'' - \tilde{p}_i\|_{L_p(O_i)} \le c\omega_3(f'', |O_i|, O_i)_p.$$

Hence using the inequality

$$|I_j|^2 \omega_3(f'', |I_i|, I_i)_p \le cn^{-2}\omega^{\varphi}_{3,2}(f'', n^{-1})_p \ ,1 < j < n$$

which follows from Lemma 3.2.8, we conclude that there exists a spline $P_n \in \sum_{5,n}(Y_s)$ which is coconvex with f on $[x_{n-1}, x_1]$, satisfies the inequality

$$\text{(3.4.9)} \qquad \|f - P_n\|_{L_p[x_{n-1},x_1]} \le cn^{-2}\omega^{\varphi}_{3,2}(f'', n^{-1})_p,$$

and is such that

$$P_n(x^+_{n-1}) = f(x_{n-1}) \quad , \quad (-1)^{s+1}P_n'(x^+_{n-1}) \le (-1)^{s+1}f'(x_{n-1})$$

$$P_n'(x_1^-) \le f'(x_1) .$$

We now extend the construction of P_n to the intervals I_1 and I_n preserving its coconvexity with the original function f , as well as keeping it close to f.

On I_1 and I_n , P_n is defined as follows

$$P_n(x_1^+) = P_n(x_1^-) \quad ,P_n'(x_1^+) = f'(x_1),$$

and $P_n^{(i)}(x^-_{n-1}) \le f^{(i)}(x_{n-1})$,$i = 0,1$

$$\begin{aligned} P_n''(x) = f''(x_1) &+ (x - x_1)\, max\{0, [x_1, x_2; f'']\} \\ &+ (x - x_1)(x - x_2)\ max\{0, [x_{n-1}, x_{n-2}, x_{n-3}; f'']\}, x \in I_1 \end{aligned}$$

$$
\begin{aligned}
P_n''(x) = f''(x_{n-1}) & \\
& + (x - x_{n-1})(-1)^{s+1} max\{0, (-1)^{s+1}[x_{n-1}, x_{n-2}; f'']\} \\
& + (x - x_{n-1})(x \\
& \quad - x_{n-2})\, (-1)^s\, max\{0, (-1)^s[x_{n-1}, x_{n-2}, x_{n-3}; f'']\} \\
& , x \in I_n.
\end{aligned}
$$

We wish to emphasize that in the case $s \geq 3$,we could alternatively define $P_n''(x) = f''(x_1)$, $x \in I_1$ and $P_n''(x) = f''(x_{n-1})$, $x \in I_n$, which is some what simpler than the current construction ,but would force us to consider the case $s \leq 2$ separately. Evidently , $P_n \in L_p[-1,1]$,and is in $\Delta^2(Y_s)$ (since P_n' and $(-1)^s P_n'$ are non-decreasing on $I_1$and I_n, respectively ,we have that

$$(-1)^s P_n'(x_{n-1}^-) \leq (-1)^s P_n'(x_{n-1}^+) \text{ and } \quad P_n'(x_1^-) \leq P_n'(x_1^+) \text{).}$$

Hence ,it remains to estimate $\|f - P_n\|_{L_p(I_1)}$ and $\|f - P_n\|_{L_p(I_n)}$ .We note that (3.4.9) implies that $\alpha = f(x_1) - P_n(x_1^+)$,satisfies

$$|\alpha| \leq cn^{-2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p \, .$$

Therefore , by Lemma 3.3.5 ,we have for every $x \in I_1$, assume that $p_1''(x) = l_1(x)$, $p_n''(x) = l_n(x)$,where p_1 and p_n are tow polynomials of degree ≤ 4 ,then

$$\|f - p_1\|_{L_p(I_1)} \leq cn^{-2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p \, ,$$

$$\|f - p_n\|_{L_p(I_n)} \leq cn^{-2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p \, .$$

We have,

$$
\begin{aligned}
|f(x) - P_n(x)| &\leq \|f - p_1\|_{L_p(I_1)} + |p_1(x) - P_n(x)| \\
&\leq cn^{-2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p
\end{aligned}
$$

$$+\left|f(x_1)-P_n(x_1^+)+\int_{x_1}^{x}(x-u)(l_1(u)-P_n''(u))du\right|$$

$$\leq cn^{-2}\omega_{3,2}^{\varphi}(f'',n^{-1})_p+|\alpha|+\left|\int_{x_1}^{x}(x-u)(l_1(u)-P_n''(u))du\right|$$

$$\leq cn^{-2}\omega_{3,2}^{\varphi}(f'',n^{-1})_p+cn^{-4}\|l_1-P_n''\|_{L_p(I_1)}.$$

Similarly except that $P_n(x_{n-1}^-)=f(x_{n-1})=p_n(x_{n-1})$, for every $x\in I_n$,we have

$$|f(x)-P_n(x)|\leq cn^{-2}\omega_{3,2}^{\varphi}(f'',n^{-1})_p+cn^{-4}\|l_n-P_n''\|_{L_p(I_n)}.$$

Now , for $x\in I_1$

$$0\leq P_n''(x)-l_1(x)$$
$$=(x-x_1)\,(max\{0,[x_1,x_2,x_3;f'']\}-[x_1,x_2,x_3;f''])$$
$$=(x-x_1)\,max\{0,-[x_1,x_2;f'']\}$$
$$+(x-x_1)(x-x_2)\;max\{0,-[x_1,x_2,x_3;f'']\}.$$

Hence ,for $s\geq 2$,we conclude by Lemma 3.4.2 and Lemma 3.4.3,then

$$0\leq P_n''(x)-l_1(x)\leq(x-x_1)cn^4\omega_{3,2}^{\varphi}(f'',n^{-1})_p$$
$$+(x-x_1)(x-x_2)cn^6\omega_{3,2}^{\varphi}(f'',n^{-1})_p$$
$$\leq cn^2\omega_{3,2}^{\varphi}(f'',n^{-1})_p\,,x\in I_1.$$

For $s=1$, we apply Corollary 3.1.6 and similarly conclude that

$$0\leq P_n''(x)-l_1(x)\leq cn^2\omega_{3,2}^{\varphi}(f'',n^{-1})_p+c\omega_{2,2}^{\varphi}(f'',n^{-1})_p\,,x\in I_1,$$

analogously ,for $x\in I_n$,

$$0 \le (-1)^s\big(P_n''(x) - l_n(x)\big)$$

$$= (x - x_{n-1})(-1)^{s+1} max\{0,(-1)^s[x_{n-1},x_{n-2};f'']\}$$

$$+(x - x_{n-1})(x - x_{n-2})\,(-1)^s\, max\{0,(-1)^{s+1}[x_{n-1},x_{n-2},x_{n-3};f'']\}.$$

Hence ,for $s \geq 2$,by Lemma 3.4.2 and Lemma 3.4.3 instead Corollary 3.1.6 ,we get

$$0 \le \big(P_n''(x) - l_n(x)\big) \le cn^2\omega_{3,2}^{\varphi}(f'',n^{-1})_p + c\omega_{2,2}^{\varphi}(f'',n^{-1})_p\,, x \in I_n.$$

Also, in the case $s = 1$,applying Lemma 3.4.1 ,instead Corollary 3.1.6 ,we have ,for $x \in I_1$

$$|P_n''(x) - l_1(x)| \le (x - x_1)|[x_1,x_2;f'']| + (x - x_1)(x - x_2)|[x_1,x_2,x_3;f'']|$$

$$\le n^{-2}|[x_1,x_2;f'']| + n^{-4}|[x_1,x_2,x_3;f'']|$$

$$\le cn^2\omega_{3,2}^{\varphi}(f'',n^{-1})_p + cn^{-2}\|f''\|_{L_p[-\frac{1}{2},\frac{1}{2}]}\ ,$$

and the estimate for $\|l_n - P_n''\|_{L_p(I_n)}$ is derived analogously.

To summarize , in the case $s \geq 2$,we have

$$\|f - P_n\|_{L_p(I)} \le cn^{-2}\omega_{3,2}^{\varphi}(f'',n^{-1})_p\,,$$

and in the case $s = 1$,we have

$$\|f - P_n\|_{L_p(I)} \le cn^{-2}\omega_{3,2}^{\varphi}(f'',n^{-1})_p + cn^{-6}\|f''\|_{L_p[-\frac{1}{2},\frac{1}{2}]}\,,$$

and,

$$\|f - P_n\|_{L_p(I)} \le cn^{-2}\omega_{3,2}^{\varphi}(f'',n^{-1})_p + cn^{-4}\omega_{2,2}^{\varphi}(f'',n^{-1})_p\,.$$

By virtue of Lemma 3.2.5 and the estimate

$$\omega_5^{\varphi}(P_n,n^{-1})_p \le c\|f - P_n\|_{L_p(I)} + c\,\omega_5^{\varphi}(f,n^{-1})_p$$

$\leq c\|f-P_n\|_{L_p(I)}+cn^{-2}\omega^{\varphi}_{3,2}(f'',n^{-1})_p$,

we conclude that there exists a polynomial $p_n \in \Delta^2(Y_s)$ of degree $\leq cn$ such that ,

$$\begin{aligned}\|f-p_n\|_{L_p(I)} &\leq \|f-P_n\|_{L_p(I)}+\|P_n-p_n\|_{L_p(I)}\\ &\leq \|f-P_n\|_{L_p(I)}+cn^{-2}\omega^{\varphi}_{3,2}(f'',n^{-1})_p\,.\\ &\leq cn^{-2}\omega^{\varphi}_{3,2}(f'',n^{-1})_p+cn^{-6}\|f''\|_{L_p\left[-\frac{1}{2},\frac{1}{2}\right]}\end{aligned}$$

♦

Chapter Three

Shape Preserving Coconvex Approximation in $L_p(I),\ 0 < p \le \infty$

When we approximate a function $f \in L_p(I)$ which changes convexity finitely many times at Y_s ,we wish some times that the approximating polynomials follow these changes in convexity . We discuss in this chapter three sections , in the first one , we estimate the degree of approximation of f by algebraic and piecewise polynomials which change convexity exactly at the points where f does , and the second, using the divided difference and weighted D. T modulus of smoothness in Theorem 3.1.4 ,finally we estimate the degree of approximation of f by divided difference .

3.1 Introduction and Main Results

Our main interest in this chapter is the approximation of $f \in L_p(I)$, $I = [-1,1]$ which changes convexity finitely many times ,by it. We are interested in estimating the degree of approximation of f by polynomials which are coconvex with it ,i.e. if $f \in \Delta^2(Y_s)$, $s \ge 1$. We intend here to obtain the analogous results for coconvex approximation .

Denote by $\mathbf{Y}_s$,$s \in N$,the set of all collection $Y_s = \{y_i\}_{i=1}^{s}$,such that $-1 < y_s < \cdots < y_1 < 1$,for $s = 0$,we write $Y_s = \emptyset$. For later reference set $y_0 = 1$ and $y_{s+1} = -1$.

Finally ,let $\Delta^2(Y_s)$ denote the collection of all functions $f \in L_p[-1,1]$ that change convexity at the set Y_s. Given $n \in N$,$n > 1$,we set $x_j = x_{j,n} = \cos\left(\frac{j\pi}{n}\right)$,and we denote $I_j = I_{j,n} = [x_j, x_{j-1}]$, $j = 1,2,\dots,n$.

Let $\sum_{k,n}$,be the collection of all piecewise polynomials of degree $k-1$, on the Chebyshev partition of I .

Also ,given $Y_s \in \mathbf{Y}_s$, and let

$$(3.1.1) \qquad O_i = O_{i,n}(Y_s) = (x_{j+1}, x_{j-2}) \quad \text{if } y_i \in [x_j, x_{j-1})$$

where , $x_{n+1} = -1$, $x_{-1} = 1$,and denote

$$O = O(n, Y_s) = \bigcup_{i=1}^{s} O_i \ , \quad O(n, \emptyset) = \emptyset \ .$$

Finally ,we write $j \in H$, if $I_j \cap O = \emptyset$.

Let $L_p^r[-1,1]$ be the set of $r-$times differentiable functions on $[-1,1]$, where

$$L_p^r[-1,1] = \{f \in L_p[-1,1] : f^{(r)} \in L_p[-1,1]\} \ ,$$

also , we define

$$L_\varphi^r(I) = \left\{f \in L_p^r(-1,1) \cap L_p(I) \ \backslash \ \lim_{x \to \pm 1} \varphi^r(x) f^{(r)}(x) = 0\right\}$$

where $r \geq 1$, in particular .

The following notion of the length of an interval $J = [a,b] \subseteq I$ relative to its position in I ,was introduced in [26] and [18].

We always have that $\omega_r^{\varphi}(f,t,J)_p \leq \omega_r(f,t,J)_p$, $0 < p \leq \infty$ .But the converse is not true in general , however in [3] ,that moduli of smoothness

ω_r^{φ} and ω_r for a function f defined on $J \subseteq I$ are equivalent , if $|J| \sim \Delta_n(a)$,where

$$\Delta_n(a) = \frac{1}{n}\sqrt{(1-a^2)} + \frac{1}{n^2} \, ,$$

$$\omega_r(f, \Delta_n(a), J)_p \ \sim \ \omega_r^{\varphi}(f, n^{-1}, J)_p \, .$$

Let $L_{m-1}(g; z_0, z_1, \dots, z_{m-1})$ denote the polynomial of degree $\leq m-1$ which interpolates a function g at the points $z_0, z_1, \dots, z_{m-1}$. We remind the reader that $[z_0, z_1, \dots, z_m; g]$ stands for the $m-th\ divided\ difference$ of a function g at the knots $z_0, z_1, \dots, z_m$ defined by

$$\text{(3.1.2)} \qquad [z_0, z_1, \dots, z_m; g] = \frac{g(z_m) - L_{m-1}(g; z_0, z_1, \dots, z_{m-1})(z_m)}{(z_m - z_0)(z_m - z_1) \dots (z_m - z_{m-1})},$$

the following Newton formula for interpolating polynomials is well known:

$$\text{(3.1.3)} \qquad L_{m-1}(g; z_0, z_1, \dots, z_{m-1}) = \sum_{i=0}^{m-1} (z_m - z_0)(z_m - z_1) \dots (z_m - z_{i-1}) \, [z_0, z_1, \dots, z_{i-1}; g]$$

Also , assuming that $z_0, z_1, \dots, z_{m-1}$ from either a non_increasing or a non_decreasing sequence such that

$$\min_{0 \leq i \leq m-1} |z_{i+1} - z_i| \ \sim \ \max_{0 \leq i \leq m-1} |z_{i+1} - z_i|$$

and using Whitney's inequality ,we have the following estimate:

$$\text{(3.1.4)} \quad |[z_0, z_1, \dots, z_m; g]| \leq c|z_m - z_0|^{-m} \omega_m(g, |z_m - z_0|, [min\{z_0, z_m\} \ , max\{z_0, z_m\}])_p$$

Where, c depends on m and the ratio

$$\min_{0 \leq i \leq m-1} |z_{i+1} - z_i| / \max_{0 \leq i \leq m-1} |z_{i+1} - z_i| \ \ .$$

To know more information about the divided difference ,see in [3] and [35].

We note that for a set $Y_s \in \mathbf{Y}_s$,$s \geq 1$,if

$$n \geq 4\left(\min_{1\leq j\leq s}\{y_{j-1} - y_j\}\right)^{-1} = \mathcal{M}(Y_s),$$

then , there is at least one knot $x_i$between $y_{j-1}$and y_j for all $1 \leq j \leq s+1$.

Our main results are the following:

Theorem 3.1.1

If $f \in \Delta^2(Y_s) \cap L_p^2(I)$, then for each $k \leq 3$

$$E_n^{(2)}(f, Y_s)_p \leq \frac{C}{n^2}\, \omega_k^{\varphi}(f'', n^{-1})_p \qquad ,n \geq \eta,$$

where c= $c(s)$ and $\eta = \eta(Y_s)$.

Theorem 3.1.2

Let $r \geq 3$ and assume that $f \in \Delta^2(Y_s) \cap L_p^r(I)$, then

$$E_n^{(2)}(f, Y_s)_p \leq \frac{C}{n^r}\, \omega_{k-r}^{\varphi}(f^{(r)}, n^{-1})_p \qquad , \quad n \geq \eta$$

for each $k \geq 3$,with constants c= $c(k, r, s)$ and $\eta = \eta(k, r, Y_s)$.

Theorem 3.1.3

Let $k \geq 1$, $r = 5$,$s \geq 1$, and $Y_s \in \mathbf{Y}_s$, be given .

If $f \in L_\varphi^5(I) \cap \Delta^2(Y_s)$,then

$$E_n^{(2)}(f, Y_s)_p \leq cn^{-5}\omega_{k,5}^{\varphi}(f^{(5)}, n^{-1})_p \quad ,n \geq \eta(k, Y_s)$$

where $\eta(k, Y_s)$ is constant depends on k and Y_s .

Theorem 3.1.4

Let $f \in L_p(I)$, $1 \le p < \infty$,and let k ≥ 1 , $r \ge 0$ be such that $k + r \ge 3$ and $1 \le \mu \le n - k$ be fixed . Then , for all $1 \le j \le \mu$,

$$(3.1.5) \qquad \left|\left[x_\mu, x_{\mu+1}, \dots, x_{\mu+k-1}; f\right] - \left[x_j, x_{j+1}, \dots, x_{j+k-1}; f\right]\right|$$

$$\le cn^{2k+r-2}\left(\frac{1}{\min\{j,n-\mu\}}\right)^{k+r-2} \omega^{\varphi}_{k,r}(f, n^{-1})_p$$

Also ,if $k + r \ge 5$,then for all v and j such that $1 \le j \le v \le \mu$,we also have ,

$$(3.1.6) \qquad \varepsilon\left(\left|[x_v, x_{v+1}, \dots, x_{v+k-1}; f] - \left[x_j, x_{j+1}, \dots, x_{j+k-1}; f\right]\right|\right)$$

$$\le cn^{2k+r-4}\left(1 + \frac{n^2}{(n-\mu)^{k+r-2}}\right) \omega^{\varphi}_{k,r}(f, n^{-1})_p$$

where $\varepsilon = sgn\{\left[x_\mu, x_{\mu+1}, \dots, x_{\mu+k-1}; f\right]\}$.

Corollary 3.1.5

Let $f \in L^2_\varphi(I)$

(i) For any index $1 \le \mu \le n - 3$,if $sgn\{\left[x_\mu, x_{\mu+1}, x_{\mu+2}; f''\right]\} = \varepsilon$ then

$$(3.1.7) \qquad -\varepsilon[x_1, x_2, x_3; f''] \le cn^6 \omega^{\varphi}_{3,2}(f'', n^{-1})_p$$

Moreover ,if an index $1 \le v \le \mu$ is such that

$sgn\{[x_v, x_{v+1}, f'']\} = \varepsilon$,then

we also have ,

$$(3.1.8) \qquad -\varepsilon[x_1, x_2; f''] \le cn^4\left(1 + \frac{n^2}{(n-\mu)^3}\right) \omega^{\varphi}_{3,2}(f'', n^{-1})_p$$

(ii) For any index $1 \leq \mu \leq n-3$,if

$Sgn\{[x_{n-\mu}, x_{n-\mu-1}, x_{n-\mu-2}; f'']\} = \varepsilon$, then

$$-\varepsilon[x_{n-1}, x_{n-2}, x_{n-3}; f''] \leq cn^6 \omega_{3,2}^{\varphi}(f'', n^{-1})_p$$

Moreover ,if an index $1 \leq v \leq \mu$ is such that

$sgn\{[x_{n-v}, x_{n-v+1}, f'']\} = -\varepsilon$,then we have

$$-\varepsilon[x_{n-1}, x_{n-2}; f''] \leq cn^4 \left(1 + \frac{n^2}{(n-\mu)^3}\right) \omega_{3,2}^{\varphi}(f'', n^{-1})_p \ .$$

Corollary 3.1.6

Let $s = 1$, $f \in L_{\varphi}^2[-1,1] \cap \Delta^2(Y_s)$,and $n \geq 7(\varphi(y_s))^{-3}$,then

$$\begin{aligned} &max\{|[x_1, x_2, x_3; f'']|, [x_{n-1}, x_{n-2}, x_{n-3}; f'']\} \\ &\qquad \leq cn^6 \omega_{3,2}^{\varphi}(f'', n^{-1})_p + cn^2 \omega_{2,2}^{\varphi}(f'', n^{-1})_p \ , \end{aligned}$$

And,

$$\begin{aligned} &max\{-[x_1, x_2; f''], [x_{n-1}, x_{n-2}; f'']\} \\ &\qquad \leq cn^4 \omega_{3,2}^{\varphi}(f'', n^{-1})_p + cn^2 \omega_{2,2}^{\varphi}(f'', n^{-1})_p \ . \end{aligned}$$

Theorem 3.1.7

Let $s \geq 2$ and $Y_s \in \mathbf{Y}_s$ be given .If $f \in L_{\varphi}^2[-1,1] \cap \Delta^2(Y_s)$,$1 \leq p < \infty$, then

(3.1.9) $$E_n^{(2)}(f, Y_s)_p \leq cn^{-2} \omega_{3,2}^{\varphi}(f'', n^{-1})_p \qquad , n \geq N \ ,$$

where ,N is constant depends on Y_s .

Theorem 3.1.8

Let $s = 1$ and $Y_s \in \mathbf{Y}_s$ be given .If $f \in L_\varphi^2[-1,1] \cap \Delta^2(Y_s)$,$1 \le p < \infty$, then

(3.1.10) $$E_n^{(2)}(f, Y_s)_p \le cn^{-2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p + cn^{-4}\omega_{2,2}^{\varphi}(f'', n^{-1})_p ,$$

$n \ge \eta$, where η =const. ,depends on Y_s .Hence

(3.1.11) $$E_n^{(2)}(f, Y_s)_p \le cn^{-2}\omega_{2,2}^{\varphi}(f'', n^{-1})_p , n \ge \eta$$

Moreover ,

(3.1.12) $$E_n^{(2)}(f, Y_s)_p \le cn^{-2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p + cn^{-6}\left\|f''\right\|_{L_p\left[-\frac{1}{2},\frac{1}{2}\right]} ,$$

where , $n \ge \eta$.

3.2 The Degree of Coconvex Approximation by

D. T. Modulus of Smoothness

In this section we study the approximation of coconvex function in $L_p[-1,1]$,in terms of Ditizian_Totik modulus of smoothness.

To prove our theorems in this section ,we need the following Lemmas:

Lemma 3.2.1 [5]

For any $f \in L_p(J)$, $0 < p < \infty$,where $J = [a, b] \subseteq I$, we have

$$\omega_k(f,|J|,J)_p \le c(r,p)\omega_k^{\varphi}(f,|J|)_p\,,$$

and for $t \ge 0$

$$\omega_k(f,t^2)_p \le c(r,p)\omega_k^{\varphi}(f,t)_p\,.$$

Lemma 3.2.2

Let $f \in L_{\varphi}^{r}[-1,1]$, $J=(a,b) \subseteq (-1,1)$, we have

$$\omega_k(f^{(r)},|J|,J)_p \le \frac{1}{m^r(a,b)}\omega_{k,r}^{\varphi}(f^{(r)},|J|)_p \quad ,$$

where $m(a,b)=\sqrt{(1+a)(1-b)}$.

Proof:-

Let $x \pm \frac{k}{2}h \in J$, then $h \le \frac{|J|}{k}$,and we observe that ,If $x \pm \frac{k}{2}h \in J$ and $x \pm \frac{k}{2}h\varphi(x) \in I$, we have $J \subseteq I$,then

By using the following inequality from [20],$|J_1| \le |J|$ for $J_1 \subseteq J$,we obtain

$$h \le \frac{|J|}{r}\varphi(x)\,,$$

and,

$$\omega_k(f^{(r)},|J|,J)_p \le sup_{0<h\le\frac{|J|}{r}\varphi(x)}\left\|\Delta_h^k(f^{(r)},x)\right\|_{L_p(J)}$$

$$\le sup_{0<h\le\frac{|J|}{r}}\left\|\Delta_{h\varphi(x)}^k(f^{(r)},x)\right\|_{L_p(J)}$$

$$\le sup_{0<h\le\frac{|J|}{r}}\frac{1}{m^r(a,b)}\left\|\varphi^r(x)\,\Delta_{h\varphi(x)}^k(f^{(r)},x)\right\|_{L_p(J)}$$

$$\le \frac{1}{m^r(a,b)}\,sup_{0<h\le\frac{|J|}{r}}\left\|\varphi^r(x)\,\Delta_{h\varphi(x)}^k(f^{(r)},x)\right\|_{L_p(J)}$$

$$\leq \frac{1}{m^r(a,b)}\omega_{k,r}^{\varphi}(f^{(r)},|J|)_p$$ ♦

Lemma 3.2.3

Let $k \geq 1$,and let $f\epsilon L_p^2[a,a+h]\,,h>0$ be a convex polynomial P of degree $\leq k+1$ satisfying $P(a)=f(a)\,,P(a+h)=f(a+h)\,,$ $P'(a)\geq f'(a)$ and $P'(a+h)\leq f'(a+h)$ and such that

$$\|f-P\|_{L_p[a,a+h]}\leq ch^2\omega_{k-2}(f'',h,[a,a+h])_p.$$

Proof:-

Let $J=[a,a+h]$, and $|J|=h$ and since $f\epsilon L_p^2[a,a+h]$, $P_{k+1}\in\Pi_{k+1}$,then by (Lemma 2.3.3 ,Whitney's Inequality) and by (Theorem 1.4.1 , Local Estimates) ,we have

$$\begin{aligned}\|f-P_{k+1}\|_{L_p[a,a+h]} &\leq c\,\omega_k(f,|J|,J)_p\\ &\leq c|J|^2\omega_{k-2}(f'',|J|,J)_p\\ &\leq c|J|^2\omega_{k-2}(f'',h,J)_p\end{aligned}$$ ♦

Lemma 3.2.4

Let $k\geq 1$, and let $a<\beta\leq a+h$ be fixed , and assume that , $f\epsilon L_p^2[a,a+h]$ is such that $f''(x)(x-\beta)\geq 0$, $a\leq x\leq a+h$ .If a polynomial $p\in\Pi_{\mathrm{k}-1}$ satisfies $p(x)(x-\beta)\geq 0$, $a\leq x\leq a+h$ then there exists a polynomial $P_{k+1}\in\Pi_{\mathrm{k}+1}$ such that $P''_{k+1}=p$, $P_{k+1}(a)=f(a)$,$P'_{k+1}(a)\leq f'(a)$,also $P'_{k+1}(a+h)\leq f'(a+h)$ and we have

$$\|f-P_{k+1}\|_{L_p[a,a+h]}\leq ch^2\|f''-p\|_{L_p[a,a+h]}\ .$$

Proof:-

Let $P_{k+1} \in \Pi_{k+1}$ and by (Lemma 2.3.3 ,Whitney's Inequality and Lemma 2.3.1) ,and by (Theorem 1.4.1 , Local Estimates) ,we have

$$\|f - P_{k+1}\|_{L_p[a,a+h]} \le c\omega_k(f,h,J)_p$$

$$\le ch^2\omega_{k-2}(f'',h,J)_p$$

$$= ch^2 sup_{0<h\le|J|}\left\|\Delta_h^{k-2}(f'',.)\right\|_{L_p[a,a+h]}$$

$$= ch^2 sup_{0<h\le|J|}\left\|\Delta_h^{k-2}(f'' - P''_{k+1},.)\right\|_{L_p[a,a+h]}$$

$$\le c(k)h^2\|f'' - P''_{k+1}\|_{L_p[a,a+h]}$$

$$= c(k)h^2\|f'' - p\|_{L_p[a,a+h]}$$ ♦

Lemma 3.2.5 [5]

For every $s \in N_0$, $0 < p < \infty$ and $k,n \in N$,there are constants $c(k,s,p)$ and $c(s,p)$,such that , if $f \in \sum_{k,n}(Y_s) \cap \Delta^2(Y_s)$,then there exists $P_n \in \Delta^2(Y_s)$ of degree not exceeding $c(s,p)n$, satisfies

$$\|f - P_n\|_p \le c(k,s,p)\,\omega_k^{\varphi}(f,n^{-1})_p \,.$$

Lemma 3.2.6 [10]

Let $f \in L_p^r[-1,1]$,$0 < p < \infty$, $r \ge 1$,then

$$\omega_k^{\varphi}(f,t)_p \le ct^r\omega_{k-r}^{\varphi}(f^{(r)},t)_p \ ,$$

where c is const. depends on k and r.

Lemma 3.2.7 [3]

For a function , $f \in L_p[-1,1]$, $0 < p < \infty$,the following inequalities hold

$$\sum_{i=1}^{n} \omega_r(f, h_i, I_i)_p \leq c\omega_k^{\varphi}(f, n^{-1})_p \ .$$

Lemma 3.2.8

Let $f \in L_{\varphi}^{r}[-1,1]$, $1 < j < n$ and $0 \leq l \leq r$, then

$$|I_j|^l \, \omega_{k+r-l}(f^{(l)}, |I_j|, I_j)_p \quad \leq cn^{-r} \omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_p \quad .$$

Proof:-

For , $1 < j < n$, we have by using Lemma 3.2.1 and Lemma 3.2.2 ,then

$$|I_j| \leq c \, m(x_j, x_{j-1}) n^{-1}$$

therefore $1 < j < n$ and $0 \leq l \leq r$,

$$|I_j|^l \, \omega_{k+r-l}(f^{(l)}, |I_j|, I_j)_p \quad \leq C|I_j|^r \, \omega_k^{\varphi}(f^{(r)}, |I_j|, I_j)_p$$

$$\leq C \frac{|I_j|^r}{m^r(x_j, x_{j-1})} \, \omega_{k,r}^{\varphi}(f^{(r)}, |I_j|)_p$$

$$\leq Cn^{-r} \omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_p \quad \blacklozenge$$

Proof of Theorem 3.1.1

Given $f \in \Delta^2(Y_s) \cap L_p^2[-1,1]$, $r \geq 2$,we take $\eta(Y_s)$ so big that if $n \geq \eta$, then for each $1 \leq i \leq s$,the set O_i define in (3.1.1) contains only y_i ,also and O_i and O_{i+1} , $1 \leq i \leq s-1$, are separated by at least one interval of the partition .

Thus ,we have no restriction on η,if $s = 1$. Then we have s intervals $O_i = (a_i, b_i)$, $i = 1,2,\dots,s$; such that either

(3.2.1) $$f''(x)(x - y_i) \geq 0 , \qquad a_i < x < b_i$$

or

(3.2.2) $$f''(x)(x - y_i) \leq 0 , \qquad a_i < x < b_i$$

we define polynomials $P_{k-1,i} \in \Pi_{k-1}$, $k = 1,2,3$; which satisfy , respectively

(3.2.3) $$P_{k-1,i}(x)(x - y_i) \geq 0 , \qquad a_i < x < b_i$$

or

(3.2.4) $$P_{k-1,i}(x)(x - y_i) \leq 0 , \qquad a_i < x < b_i$$

and are closed to f''.To this end we take $P_{0,i} \equiv 0$ and $P_{1,i}$ to be the linear polynomial Interpolating f'' at y_i and a_i or b_i whichever is farther from y_i , and finally $P_{2,i}$ to be the quadratic polynomial interpolating f'' at a_i , y_i and b_i.

By (Lemma 2.3.3 ,Whitney's Inequality ,Ch. 2),we know that

(3.2.5) $$\left\| f'' - P_{k-1,i} \right\|_{L_p(O_i)} \leq c\omega_k(f'', |O_i|, O_i)_p \ , \ k = 1,2,3;$$

where c constant depends on the ratios between $|O_i|$ and the distances between the points of interpolation .

Thus , c is an absolute constant , for $k = 1,2$.

We obtain , for each $k \geq 2$, the existence of $P_{k-1,i} \in \Pi_{k-1}$, such that (3.2.1) and (3.2.2) hold respectively , and

$$\|f'' - P_{k-1,i}\|_{L_p(O_i)} \leq c|O_i|\omega_{k-1}(f^{(3)}, |O_i|, O_i)_p \ , \tag{3.2.6}$$

Thus ,in all cases we coclude by Lemma 3.2.4 , (3.2.4) and (3.2.5),that there exists a polynomial $P_{k+1,i} \in \Pi_{k+1}$,which is coconvex with f on O_i , $P_{k+1,i}(a_i) = f(a_i) + \alpha_i$,where α_i is an arbitrary constant to be prescribed ,and such that

$$\|f - P_{k+1,i}\|_{L_p(O_i)} \leq |\alpha_i| + \tfrac{3}{2}|O_i|^2\|f'' - P_{k-1,i}\|_{L_p(O_i)} \ , \tag{3.2.7}$$

by (3.2.6) and (3.2.7),we have an estimate on the second term on the right .

Note that (3.2.7) implies that,

$$|P_{k+1,i}(b_i) - f(b_i)| \ \ \leq |\alpha_i| + \tfrac{3}{2}|O_i|^2\|f'' - P_{k-1,i}\|_{L_p(O_i)} \ , \tag{3.2.8}$$

also if (3.2.1) holds ,then

$$P'_{k+1,i}(a_i) \leq f'(a_i) \quad \text{and} \quad P'_{k+1,i}(b_i) \leq f'(b_i) \ , \tag{3.2.9}$$

and if (3.2.2) holds ,then

$$P'_{k+1,i}(a_i) \geq f'(a_i) \quad \text{and} \quad P'_{k+1,i}(b_i) \geq f'(b_i) \ . \tag{3.2.10}$$

In all other intervals I_j , $j \in H$, f is either convex in I_j or concave there.

If $g_j = f + \beta_j$,where β_j is an arbitrary constant to be prescribed , then by Lemma 3.2.3, there exists a polynomial $p_{k+1,j} \in \Pi_{\mathrm{k+1}}$, coconvex with f and satisfying

$$p_{k+1,j}(x_j) = g_j(x_j) \quad \text{and} \quad p_{k+1,j}(x_{j-1}) = g_j(x_{j-1}) \ , \tag{3.2.11}$$

also , if f is convex ,then we have

$$p'_{k+1,j}(x_j) \geq f'(x_j) \quad \text{and} \quad p'_{k+1,j}(x_{j-1}) \leq f'(x_{j-1}), \tag{3.2.12}$$

and if f is cocave , then

(3.2.13) $p'_{k+1,j}(x_j) \leq f'(x_j)$ and $p'_{k+1,j}(x_{j-1}) \geq f'(x_{j-1})$.

By Lemma 3.2.6,

$$(3.2.14) \quad \|f - p_{k+1,j}\|_{L_p(I_j)} \leq |\beta_j| + c|I_j|^2 \omega_k(f'', |I_i|, I_i)_p$$

$$\leq |\beta_j| + cn^{-2}\omega_k^{\varphi}(f'', n^{-1})_p ,$$

Since $|I_j| \leq \frac{C}{n}$.

We construct the piecewise polynomial $f \in \sum_{k+2,n}(Y_s) \cap \Delta^2(Y_s)$, sweeping $[-1,1]$ from left and right .Let $a_s = x_{j0}$, where $O_s = (a_s, b_s)$, and let $a_s = 0$.

Then for $j_0 < j \leq n$, we take $\beta_j = 0$ and set

$$S/I_j = p_{k+1,j} \quad , j_0 < j \leq n$$

and $$S/O_s = P_{k+1,s} .$$

Note that , $S \in L_p[-1, b_s)$ and by (3.2.11) and (3.2.12) or (3.2.10) and (3.2.13) respectively , it is coconvex with f there.

Suppose that we have defined S in $[-1, b_i)$, $1 < i \leq s$, let $b_i = x_{j_1}$ and $a_{i-1} = x_{j_2}$. then we take ,

$\beta_{i-1} = \sum_{m=i}^{s}(P_{k+1,m}(b_m) - f(b_m))$, and for $j_2 < j \leq j_1$,$\alpha_j = \beta_{i-1}$.

Then we set

$$S/I_j = p_{k+1,j} \quad , j_2 < j \leq j_1$$

$$S/O_{i-1} = P_{k+1,i-1}$$

this $S \in L_p[-1, b_{i-1})$, coconvex with f there.

Finally , if $b_1 = x_{j_3}$, then for 1< $j \leq j_3$,we take

$\alpha_j = \sum_{m=1}^{s}(P_{k+1,m}(b_m) - f(b_m))$,and we set

$$S/I_j = p_{k+1,j} \quad , 1 < j \leq j_3 .$$

It is readily seen that we have obtained an $f \in \sum_{k+2,n}(Y_s) \cap \Delta^2(Y_s)$,again we deel with $f \in L_p^2[-1,1]$.

Since $|O_i| \le \frac{c}{n}$,it follows by (3.1.5) that is

$$\left\|f'' - P_{k-1,i}\right\|_{L_p(O_i)} \le c\omega_k^{\varphi}(f'', n^{-1})_p \ , \ k = 1,2,3;$$

hence , combining with (3.2.7) , (3.2.8) and(3.2.14),also by Lemma 3.2.7, yields

$$\|f - S\|_{L_p(I)} = \left(\int_{-1}^{1} |f(x) - S(x)|^p dx\right)^{1/p}$$

$$= \left(\sum_i \int_{I_i \cup O_i} |f(x) - S(x)|^p dx\right)^{1/p}$$

$$\le \left(\sum_i \int_{I_i} |f(x) - S(x)|^p dx + \sum_i \int_{O_i} |f(x) - S(x)|^p dx\right)^{1/p}$$

$$\le C(p)\left(\sum_i \int_{I_i} |f(x) - S(x)|^p dx\right)^{1/p}$$

$$+c(p) \left(\sum_i \int_{O_i} |f(x) - S(x)|^p dx\right)^{1/p}$$

$$\le C(p)\left(\sum_i \|f - S\|_{L_p(I_i)} + \sum_i \|f - S\|_{L_p(O_i)}\right)$$

$$\leq c(p)\left(\sum_i \left\|f-P_{k+1,i}\ +P_{k+1,i}-S\right\|_{L_p(I_i)} + \sum_i \left\|f-P_{k+1,i}\ +P_{k+1,i}-S\right\|_{L_p(O_i)}\right)$$

$$\leq c(p)\sum_i\left(\left\|f-P_{k+1,i}\ \right\|_{L_p(I_i)}+\left\|P_{k+1,i}-S\right\|_{L_p(I_i)} + \left\|f-P_{k+1,i}\ \right\|_{L_p(O_i)}+\left\|P_{k+1,i}-S\right\|_{L_p(O_i)}\right).$$

Hence, by direct substituting ,and simplification ,we have ,

$$\|f-S\|_{L_p(I)} \leq \frac{C}{n^2}\ \omega_k^{\varphi}(f'',n^{-1})_p \qquad , \text{k=1,2,3} \tag{3.2.15}$$

this in turn implies

$$\begin{aligned}\omega_{k+2}^{\varphi}(S,n^{-1})_p &= \omega_{k+2}^{\varphi}(S-f+f,n^{-1})_p\\ &\leq \omega_{k+2}^{\varphi}(f,n^{-1})_p + c\|S-f\|_{L_p(I)}\\ &\leq \omega_{k+2}^{\varphi}(f,n^{-1})_p + c\,n^{-1}\,\omega_k^{\varphi}(f'',n^{-1})_p\\ &\leq c\,n^{-1}\,\omega_k^{\varphi}(f'',n^{-1})_p\ \ .\end{aligned} \tag{3.2.16}$$

Therefore ,we apply (3.2.15) and (3.2.16) ,also by Lemma 3.2.5 ,to obtain a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$ such that

$$\begin{aligned}\|f-P_n\|_{L_p(I)} &= \|f-S+S-P_n\|_{L_p(I)}\\ &\leq \|f-S\|_{L_p(I)} + \|S-P_n\|_{L_p(I)}\\ &\leq \frac{C}{n^2}\ \omega_k^{\varphi}(f'',n^{-1})_p\ +\ \omega_{k+2}^{\varphi}(S,n^{-1})_p\\ &\leq \frac{C}{n^2}\ \omega_k^{\varphi}(f'',n^{-1})_p\ +\ c\,n^{-1}\,\omega_k^{\varphi}(f'',n^{-1})_p\\ &\leq c\,n^{-1}\,\omega_k^{\varphi}(f'',n^{-1})_p\end{aligned}$$

where , $c = c(k,s,p)$ and $\eta = \eta(Y_s)$ ♦

Proof of Theorem 3.1.2

Assume that $f \in L_p^r[-1,1]$, $r \geq 3$ and let $k \geq 2$. Then ,it follows by ((3.2.6) in the proof of Theorem 3.1.1)

$$\left\|f'' - P_{k-1,i}\right\|_{L_p(O_i)} \leq c\, n^{-1} \omega_{k-1}^{\varphi}(f^{(3)}, n^{-1})_p \,. \tag{3.2.17}$$

Hence ,combining with, (3.2.7) ,(3.2.8) and (3.2.12) are in proof of Theorem 3.1.1 , yields

$$\|f - S\|_{L_p[-1,1]} \leq c\, s\, n^{-3} \omega_{k-1}^{\varphi}(f^{(3)}, n^{-1})_p \,. \tag{3.2.18}$$

This is turn gives

$$\begin{aligned} \omega_{k+2}^{\varphi}(S, n^{-1})_p &= \omega_{k+2}^{\varphi}(f - f + S, n^{-1})_p \\ &\leq \omega_{k+2}^{\varphi}(f, n^{-1})_p + c\|f - S\|_{L_p(I)} \\ &\leq \omega_{k+2}^{\varphi}(f, n^{-1})_p + c\, s\, n^{-3} \omega_{k-1}^{\varphi}(f^{(3)}, n^{-1})_p \\ &\leq c\, n^{-3} \omega_{k-1}^{\varphi}(f^{(3)}, n^{-1})_p \,, \qquad (3.2.19) \end{aligned}$$

where $c = c(k, s)$.

Therefore , to apply (3.2.18), (3.2.19) , Lemma 3.2.5 and Lemma 3.2.6 ,to obtain polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$, such that

$$\|f - P_n\|_{L_p[-1,1]} \leq cs\, n^{-3} \omega_k^{\varphi}(f^{(3)}, n^{-1})_p$$

Since , $f \in L_p^r[-1,1]$, $r \geq 3$, it follows by Lemma 3.2.5 and Lemma 3.2.6 ,we have

$$\begin{aligned} \|f - P_n\|_{L_p[-1,1]} &= \|f - S + S - P_n\|_{L_p(I)} \\ &\leq \|f - S\|_{L_p(I)} + \|S - P_n\|_{L_p(I)} \\ &\leq c\, s\, n^{-3} \omega_k^{\varphi}(f^{(3)}, n^{-1})_p + \omega_k^{\varphi}(S, n^{-1})_p \\ &\leq c\, s\, n^{-3} \omega_k^{\varphi}(f^{(3)}, n^{-1})_p + \omega_k^{\varphi}(S - f + f, n^{-1})_p \end{aligned}$$

$$\le c\, s\, n^{-3}\omega_k^{\varphi}(f^{(3)},n^{-1})_p + \omega_k^{\varphi}(f,n^{-1})_p$$

$$\le c\, s\, n^{-r}\omega_{k-r}^{\varphi}(f^{(r)},n^{-1})_p\,.$$

That is,

$$E_n^{(2)}(f,Y_s)_p \le \frac{C}{n^r}\,\omega_{k-r}^{\varphi}(f^{(r)},n^{-1})_p \quad ,n \ge N$$

is valid for all $r \ge 3$ and $k \ge 1$, with $c = c(k,r,s)$ and $\eta = \eta(k,r,Y_s)$ ♠

Proof of Theorem 3.1.3

From Lemma 3.2.5 ,we estimate that

$$\omega_{k+5}^{\varphi}(P_n,n^{-1})_p = \omega_{k+5}^{\varphi}(f-f+P_n,n^{-1})_p \ ,$$

$$\le \omega_{k+5}^{\varphi}(f,n^{-1})_p + c\|f-P_n\|_{L_p(I)}\,,$$

$$\le cn^{-5}\omega_{k,5}^{\varphi}(f^{(5)},n^{-1})_p + c\|f-P_n\|_{L_p(I)}\,.$$

We note that from theorem 2.1.4,when $l = 0$,then, we only need to construct a spline $P_n \in \sum_{k+5,n}(Y_s) \cap \Delta^2(Y_s)$, such that

(3.2.19) $$\|f-P_n\|_{L_p(I)} \le cn^{-5}\omega_{k,5}^{\varphi}(f^{(5)},n^{-1})_p$$

Also by Lemma 3.2.8 with $l = 3$ and $r = 5$,implies

(3.2.20) $$|I_j|^3\,\omega_{k+2}(f^{(3)},|I_j|,I_j)_p \quad \le \omega_{k,5}^{\varphi}(f^{(5)},n^{-1})_p\ ,$$

For $1 < j < n$, while with $l = 2$ and$r = 5$, it implies

$$|I_j|^2\,\omega_{k+3}(f'',|I_j|,I_j)_p \quad \le cn^{-5}\omega_{k,5}^{\varphi}(f^{(5)},n^{-1})_p \quad .$$

Taking these estimates into account ,the same construction as in Theorem 3.1.1 and Theorem 3.1.2, yields a spline $P_n \in \sum_{k+5,n}(Y_s)$ which is coconvex with f on $[-1,1]$ and such that (3.2.19) holds .

For the sake of completeness, to describe this construction .We take $N(Y_s)$ to be so large that ,for $n \geq N$, the sets O_i, $1 \leq i \leq s$, are all disjoint and do not contain the endpoints of the interval $[-1,1]$.

Now if $I_j \cap O = \emptyset$,then f does not change its convexity on I_j and Lemma 3.2.3 implies that there exists a polynomial $P_j \in \Pi_{k+5}$ with coconvex with f , interpolates it at the endpoints of I_j , and such that

$P_j'(x_j) \geq f'(x_j)$ and $P_j'(x_{j-1}) \leq f'(x_{j-1})$ if f is convex on I_j ,or

$P_j'(x_j) \leq f'(x_j)$ and $P_j'(x_{j-1}) \geq f'(x_{j-1})$ if f is cocave on I_j ,and satisfies

$$\|f - P_j\|_{L_p(I_j)} \leq c|I_j|^2 \omega_{k+3}^{\varphi}(f'', |I_j|, I_j)_p$$

$$\leq cn^{-5}\omega_{k,5}^{\varphi}(f^{(5)}, n^{-1})_p \quad .$$

Now , it is convenient to denote the endpoints of O_i ,by a_i and b_i i.e. $O_i = (a_i \,, b_i)$, $1 \leq i \leq s$.

Also , for each $1 \leq i \leq s$,there exists a polynomial $\widetilde{P_i} \in \Pi_{k+3}$,which is copositive with f''on O_i (i.e. $\widetilde{P_i}(x)f''(x) \geq 0$ for all $x \in O_i$) and such that,

$$\|f'' - \widetilde{P_i}\|_{L_p(O_i)} \leq c|O_i| \quad \omega_{k+2}^{\varphi}(f^{(3)}, |O_i|, O_i)_p \,.$$

And ,Lemma 3.2.4 implies that there exists a polynomial $\overline{P_i} \in \Pi_{k+5}$ such that $\overline{P_i}'(a_i) \leq f'(a_i)$ and $\overline{P_i}'(b_i) \leq f'(b_i)$,

(if f is such that $f''(x)(x - y_i) \geq 0$,for $x \in O_i$) ,and satisfying

$$\|f - \overline{P_i}\|_{L_p(O_i)} \leq c|O_i|^2 \|f'' - \widetilde{P_i}\|_{L_p(O_i)}$$

$$\leq c|O_i|^3 \, \omega_{k+2}(f^{(3)}, |O_i|, O_i)_p$$

$$\leq cn^{-5}\omega_{k,5}^{\varphi}(f^{(5)}, n^{-1})_p \quad .$$

Where ,the last inequality follows from (3.2.20),the observation that $|O_i| \sim |I_j|$,where j is such that $y_i \in I_i$, and the fact that O_i is far from ± 1.

Now ,the piecewise polynomial approximation $P_n \in \sum_{k+5,n}(Y_s) \cap \Delta^2(Y_s)$ is constructed from the polynomial pieces P_j and $\overline{P}_\iota$ in such a way that ,if P_n is constructed for all $x \le x_v$,then on $[x_v, x_{v-1}]$ (or $[x_v, x_{v-3}] = O_\vartheta$ if x_v happens to be the left endpoint of some interval O_ϑ),it is defined to be P_v (or $\overline{P}_\vartheta + \alpha$,where the constant α is chosen in such a way as to make $P_n \in L_p[-1,1]$), it is not difficult to see now P_n is coconvex with f and (3.2.20) holds ♦

3.3 The Divided Difference and the Weighted D.T. Modulus of Smoothness

In this section ,we will use the divided difference and weighted D. T. modulus of smoothness in Theorem 3.1.5 .

To prove our theorems in this section ,we need the following Lemmas:

Lemma 3.3.1

For any $f \in L_p(J)$, $0 < p < \infty$, $J \subseteq I$ and $1 \le r < k$,we have

$$\omega_k(f,|J|,J)_p \le c\,\omega_{k,r}^{\varphi}(f,|J|)_p ,$$

where c depends on r.

Proof :-

We observe that if $x \pm \frac{r}{2}h \in J$ and $x \pm \frac{r}{2}h\varphi(x) \in I$,

let $\left[x - \frac{r}{2}h, x + \frac{r}{2}h\right] \subseteq J$,then by using ,$|J_1| \le |J|$ for $J_1 \subseteq J$,we obtain

(3.3.1) $$h \le \frac{|J|}{r}\varphi(x)$$

and we have $x \pm \frac{r}{2} h \in J$ if and only if

(3.3.2) $$h \le \frac{|J|}{r}$$

also , let $\varphi(x) \le |J|$ .And by using the two inequality (3.3.1)and (3.3.2), we obtain,

$$\omega_k(f, |J|, J)_p \le sup_{0<h\le\frac{|J|}{r}\varphi(x)} \left\|\Delta_h^k(f, x)\right\|_{L_p(J)}$$

$$\le$$

$$sup_{0<h\le\frac{|J|}{r}} \varphi^{-r}(x) \quad \left\|\varphi^r(x)\, \Delta_{h\varphi(x)}^k(f, x)\right\|_{L_p(J)}$$

$$\le \frac{c}{|J|^r}\; sup_{0<h\le\frac{|J|}{r}} \left\|\varphi^r(x)\, \Delta_{h\varphi(x)}^k(f, x)\right\|_{L_p(J)}$$

$$\le c\omega_{k,r}^{\varphi}(f, \frac{|J|}{r})_p$$

$$\le c\omega_{k,r}^{\varphi}(f, |J|)_p$$

where c depends on r ♦

Lemma 3.3.2

Let $k \ge 2$, $r \in N$,$1 \le m \le k-1$ and $n \ge 3m$,and let $j \in N$ be such that $[x_{j+m}, x_j] \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]$,(which is equivalent to $\frac{n}{3} \le j \le \frac{2n}{3-m}$). Then for any $f \in L_p(I)$,$1 \le p < \infty$,

$$\left|[x_j, x_{j+1}, \dots, x_{j+m}; f]\right| \le c\left(n^k \omega_{k,r}^{\varphi}(f, n^{-1})_p + \|f\|_{L_p\left[-\frac{1}{2},\frac{1}{2}\right]}\right),$$

Where c depends only on k and r.

Proof :-

From the Marchaud inequality for a function $f \in L_p[a,b]$ and $1 \le m < k$,it was shown in [42] and [44] that for $1 \le p < \infty$,

$$\omega_m(f,t,[a,b])_p \le ct^m\left\{\int_t^{b-a}\frac{\omega_k(f,s,[a,b])_p ds}{s^{m+1}}+(b-a)^{-m}\|f\|_{L_p[a,b]}\right\}$$

Now ,taking $[a,b]=\left[-\frac{1}{2},\frac{1}{2}\right]$ in the above estimate , using (3.1.4) and taking $|x_j-x_{j+m}| \sim \frac{1}{n}$,for all j such that $x_j\in\left[-\frac{1}{2},\frac{1}{2}\right]=J$,and By using Lemma 3.3.1,and since $J\subseteq[-1,1]$,we have

$$\left|\left[x_j,x_{j+1},\dots,x_{j+m};f\right]\right| \le c\left(x_j-x_{j+m}\right)^{-m}\omega_m(f,x_j-x_{j+m},[x_{j+m},x_j])_p$$

$$\le cn^m\omega_m\left(f,1/n,\left[-\frac{1}{2},\frac{1}{2}\right]\right)_p$$

$$\le \int_{1/n}^{1}\frac{\omega_k\left(f,s,\left[-\frac{1}{2},\frac{1}{2}\right]\right)_p ds}{s^{m+1}}+c\|f\|_{L_p(J)}$$

$$\le c\int_{1/n}^{1}\frac{\omega_k\left(f,s,\left[-\frac{1}{2},\frac{1}{2}\right]\right)_p ds}{s^k}s^{k-1-m}\,ds+c\|f\|_{L_p(J)}$$

$$\le cn^k\omega_k(f,|J|,J)_p\int_{1/n}^{1}s^{k-1-m}\,ds+c\|f\|_{L_p(J)}$$

$$\le cn^k\omega_k(f,|J|,J)_p+c\|f\|_{L_p(J)}$$

$$\le c\left(n^k\omega_{k,r}^{\varphi}(f,n^{-1})_p+\|f\|_{L_p(J)}\right)$$ ♦

Lemma 3.3.3

Let $k\ge 2$, $n\ge 3k$ and $1\le m\le k-1$,and let $r\in N_0$, $r>k-2m$,then for $f\in L_p[-1,1]$ and for every $1\le j\le n-m-1$, $J=\left[-\frac{1}{2},\frac{1}{2}\right]$

$$\left|\left[x_j, x_{j+1}, \dots, x_{j+m}; f\right]\right| \le c\left(n^k \left(\frac{n}{min\{j,n-j\}}\right)^{r+2m-k} \omega_{k,r}^{\varphi}(f, n^{-1})_p + \|f\|_{L_p(J)}\right).$$

Proof:-

For $1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor$, for all $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$, $\varphi(x_i) \sim \frac{i}{n}$ and $|I_i| \sim \frac{i}{n^2}$.

Now for all $1 \le j \le n-k-1$,the inequalities (3.1.4) and Lemma 3.2.2 , imply

$$\left|\left[x_j, x_{j+1}, \dots, x_{j+k}; f\right]\right| \le c\left|I_j\right|^{-k} \omega_k\left(f, x_j - x_{j+k}, \left[x_{j+k}, x_j\right]\right)_p$$

$$\le c\left|I_j\right|^{-k} m^{-r}\left(x_{j+k}, x_j\right) \omega_{k,r}^{\varphi}\left(f, \left|\left[x_{j+k}, x_j\right]\right|\right)_p$$

$$\le c\left|I_j\right|^{-k} \varphi^{-r}\left(x_j\right) \omega_{k,r}^{\varphi}(f, n^{-1})_p.$$

Therefore , if $1 \le j \le min\left\{n-k-1, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, then

$$\left|\left[x_j, x_{j+1}, \dots, x_{j+m}; f\right]\right| \le c n^k \left(\frac{n}{j}\right)^{k+r} \omega_{k,r}^{\varphi}(f, n^{-1})_p. \tag{3.3.3}$$

Let $1 \le \mu \le \left\lfloor \frac{n}{2} \right\rfloor$ be an index ,and let $\left[x_{\mu+m}, x_\mu\right] \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$and Lemma 3.3.2 implies that

$$\left|\left[x_\mu, x_{\mu+1}, \dots, x_{\mu+m}; f\right]\right| \le c\left(n^k \omega_{k,r}^{\varphi}(f, n^{-1})_p + \|f\|_{L_p(J)}\right)$$

Also,

$$-\left[x_j, x_{j+1}, \dots, x_{j+m}; f\right] = \left(\left[x_\mu, x_{\mu+1}, \dots, x_{\mu+m}; f\right] - \left[x_j, x_{j+1}, \dots, x_{j+m}; f\right]\right) - \left[x_\mu, x_{\mu+1}, \dots, x_{\mu+m}; f\right],$$

$$= \sum_{i=j}^{\mu-1} \left(x_{j+m+1} - x_i\right) \left[x_i, x_{i+1}, \dots, x_{i+m+1}; f\right]$$

$$-[x_\mu, x_{\mu+1}, \dots, x_{\mu+m}; f],$$

therefore ,

$$\left|[x_j, x_{j+1}, \dots, x_{j+m}; f]\right| \le c \sum_{i=j}^{\left\lfloor \frac{n}{2} \right\rfloor} |I_i| \, [x_i, x_{i+1}, \dots, x_{i+m+1}; f]$$

$$+c\, n^k\, \omega_{k,r}^{\varphi}(f, n^{-1})_p + c\|f\|_{L_p(J)},$$

by using (3.3.3) ,we have

$$\left|[x_j, x_{j+1}, \dots, x_{j+m}; f]\right|$$

$$\le c \sum_{i=j}^{\left\lfloor \frac{n}{2} \right\rfloor} |I_i| \left(n^k \left(\frac{n}{min\{i, n-i\}} \right)^{r+2(m+1)-k} \omega_{k,r}^{\varphi}(f, n^{-1})_p \right)$$

$$+c\, n^k\, \omega_{k,r}^{\varphi}(f, n^{-1})_p + c\|f\|_{L_p(J)}$$

$$\le c\, n^k\, \omega_{k,r}^{\varphi}(f, n^{-1})_p \sum_{i=j}^{\left\lfloor \frac{n}{2} \right\rfloor} |I_i|\, n^k \left(\frac{n}{i} \right)^{r+2(m+1)-k}$$

$$+c\, n^k\, \omega_{k,r}^{\varphi}(f, n^{-1})_p + c\|f\|_{L_p(J)}$$

$$\le c\, n^k\, \omega_{k,r}^{\varphi}(f, n^{-1})_p \sum_{i=j}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n^{r+2m-k}}{i^{r+2m-k+1}}$$

$$+c\, n^k\, \omega_{k,r}^{\varphi}(f, n^{-1})_p + c\|f\|_{L_p(J)}$$

$$\le c \left(n^k \left(\frac{n}{j} \right)^{r+2m-k} \omega_{k,r}^{\varphi}(f, n^{-1})_p + \|f\|_{L_p(J)} \right) ♦$$

Corollary 3.3.4

Let $n \geq 9$, $m = 1$ or $m = 2$ and $f \in L_p[-1,1]$.

Then ,

$$\left|[x_j, x_{j+1}, \dots, x_{j+m}; f]\right| \leq Cn^{2m+2}\, \omega_{k,r}^{\varphi}(f, n^{-1})_p + C\|f\|_{L_p\left[-\frac{1}{2},\frac{1}{2}\right]}.$$

The proof is clearly ,by Lemma3.3.3.

Lemma 3.3.5

Let $f \in L_{\varphi}^{r}[-1,1]$, $n \geq k+1$,and let a polynomial P_{k+r} of degree $\leq k+r-1$,be such that $P_{k+r}^{(i)}(x_1) = f^{(i)}(x_1)$ for all $i = 0,1,\dots,r-1$ and $P_{k+r}^{(r)}(x) = L_{k-1}\left(f^{(r)}; x_1, x_2, \dots, x_k\right)(x)$

Then ,

$$\|f - P_{k+r}\|_{L_p(J)} \leq \frac{C}{n^r}\, \omega_{k,r}^{\varphi}(f, n^{-1})_p\ .$$

Proof:-

We denote $L_{k-1}\left(f^{(r)}; x_1, x_2, \dots, x_k\right)$ and note that it follows by (Lemma 2.3.3 ,Whitney's inequality),that for any $\alpha \in [x_1, 1]$ and the interval $J = [x_k, \alpha]$ and by Theorem 3.1.3,we have

$$\begin{aligned}\left\|L_{k-1} - f^{(r)}\right\|_{L_p(J)} &\leq \omega_k\left(f^{(r)}, |J|, J\right)_p \\ &\leq \frac{c}{(1-\alpha)^{r/2}}\, \omega_{k,r}^{\varphi}\left(f^{(r)}, |J|\right)_p \\ &\leq \frac{c}{(1-\alpha)^{r/2}}\, \omega_{k,r}^{\varphi}\left(f^{(r)}, n^{-1}\right)_p.\end{aligned}$$

And ,in particular,

(3.3.4) $\left|L_{k-1}(\alpha)-f^{(r)}(\alpha)\right| \leq \frac{C}{(1-\alpha)^{r/2}} \omega_{k,r}^{\varphi}\left(f^{(r)}, n^{-1}\right)_p$

Since ,for any $g \in L_p^r[a,b]$ and $x \in [a,b]$,

$$g(x)=\sum_{i=0}^{r-1}\frac{g^{(i)}(a)}{i!}(x-a)^i+\frac{1}{(r-1)!}\int_a^x (x-u)^{r-1}g^{(r)}(u)du$$

by using (3.3.4) , for any $x_1 \leq x < 1$,the following holds:

$$|f(x)-P_{k+r}(x)|=\frac{1}{(r-1)!}\left|\int_{x_1}^{x}(x-u)^{r-1}\left(f^{(r)}(u)-P_{k+r}^{(r)}(u)\right)du\right|$$

$$\leq c\int_{x_1}^{x}(x-u)^{r-1}\left|f^{(r)}(u)-L_{k-1}(u)\right|du$$

$$\leq c\,\omega_{k,r}^{\varphi}\left(f^{(r)},n^{-1}\right)_p\int_{x_1}^{x}\frac{(x-u)^{r-1}}{(1-u)^{r/2}}\,du$$

$$\leq c\,\omega_{k,r}^{\varphi}\left(f^{(r)},n^{-1}\right)_p\int_{x_1}^{x}(1-u)^{\frac{r}{2}-1}\,du$$

$$\leq c(1-x_1)^{\frac{r}{2}}\,\omega_{k,r}^{\varphi}\left(f^{(r)},n^{-1}\right)_p$$

$$\leq cn^{-r}\omega_{k,r}^{\varphi}\left(f^{(r)},n^{-1}\right)_p$$ ♦

Proof of Theorem 3.1.4

For all $1 \leq i \leq n-1$, $\varphi(x_i) \sim min\{i, n-i\}/n$ and $|I_i| \sim min\{i, n-i\}/n^2$,where as usual ,$\alpha_i \sim \beta_i$ means that $\frac{\alpha_i}{\beta_i}$ is bounded a way from 0 and ∞.

The following inequality is contained in the proof of Lemma 3.3.3;

(3.3.5) $$\left|\left[x_j, x_{j+1}, \dots, x_{j+k}; f\right]\right| \le cn^k \left(\frac{n}{min\{j,n-j\}}\right)^{k+r} \omega_{k,r}^{\varphi}(f, n^{-1})_p$$

,

for all $1 \le j \le n-k-1$

now ,for any $m \ge 0$ and $1 \le j \le \delta < n-m$,

(3.3.6) $$[x_\delta, x_{\delta+1}, \dots, x_{\delta+m}; f] - \left[x_j, x_{j+1}, \dots, x_{j+m+1}; f\right]$$

$$= \sum_{i=j}^{\delta-1} (x_{i+m+1} - x_i)[x_i, x_{i+1}, \dots, x_{i+m+1}; f],$$

This , with $m = k-1$, $\delta = \mu$, together with the inequality (3.3.5) for $1 \le j < \mu \le n-k$ $, implies$

$$\left|\left[x_\mu, x_{\mu+1}, \dots, x_{\mu+k-1}; f\right] - \left[x_j, x_{j+1}, \dots, x_{j+k-1}; f\right]\right|$$

$$= \left|\sum_{i=j}^{\mu-1} (x_{i+k} - x_i)[x_i, x_{i+1}, \dots, x_{i+k}; f]\right|$$

$$\le c \sum_{i=j}^{\mu-1} |I_i| n^k \left(\frac{n}{min\{i, n-i\}}\right)^{k+r} \omega_{k,r}^{\varphi}(f, n^{-1})_p$$

$$\le cn^{2k+r-2} \omega_{k,r}^{\varphi}(f, n^{-1})_p \sum_{i=j}^{\mu-1} \left(\frac{1}{min\{i, n-i\}}\right)^{k+r-1}$$

$$\le cn^{2k+r-2} \omega_{k,r}^{\varphi}(f, n^{-1})_p \sum_{i=\{j,n-\mu\}}^{\infty} \frac{1}{i^{k+r-1}}$$

$$\leq cn^{2k+r-2}\left(\frac{1}{min\{j,n-\mu\}}\right)^{k+r-2} \quad \omega_{k,r}^{\varphi}(f,n^{-1})_p,$$

where ,for the last inequality we used $k+r\geq 3$ .Thus ,(3.1.5) is proved .Now ,to prove (3.1.6) , suppose that $k+r\geq 5$. Applying (3.3.6) with $m=k-2$ and $\delta=v$ and (3.1.5) ,for all $1\leq j\leq v\leq \mu$, yields

$$\varepsilon\big([x_v,x_{v+1},\dots,x_{v+k-2};f]-[x_j,x_{j+1},\dots,x_{j+k-2};f]\big)$$

$$=\varepsilon\sum_{i=j}^{v-1}(x_{i+k}-x_i)[x_i,x_{i+1},\dots,x_{i+k-1};f]$$

$$=\varepsilon\sum_{i=j}^{v-1}(x_i-x_{i+k})([x_i,x_{i+1},\dots,x_{i+k-1};f]-[x_i,x_{i+1},\dots,x_{i+k-1};f])$$

$$-\varepsilon[x_i,x_{i+1},\dots,x_{i+k-1};f]\sum_{i=j}^{v-1}(x_i-x_{i+k})$$

$$\leq\sum_{i=j}^{v-1}(x_i-x_{i+k})\,\big|[x_\mu,x_{\mu+1},\dots,x_{\mu+k-1};f]$$

$$-[x_i,x_{i+1},\dots,x_{i+k-1};f]\big|$$

$$\leq cn^{2k+r-2}\omega_{k,r}^{\varphi}(f,n^{-1})_p\sum_{i=j}^{v-1}|I_i|\,\left(\frac{1}{min\{i,n-\mu\}}\right)^{k+r-2}$$

$$\leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f,n^{-1})_p\sum_{i=j}^{v-1}\frac{min\{i,n-i\}}{(min\{i,n-\mu\})^{k+r-2}}$$

$$\leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f,n^{-1})_p\sum_{i=j}^{\mu-1}\frac{min\{i,n-i\}}{(min\{i,n-i\})^{k+r-2}}\quad=\Phi\,.$$

Now ,since $k+r \geq 5$,if $\mu \leq \left\lfloor\frac{n}{2}\right\rfloor$,then

$$\Phi \leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f,n^{-1})_p \sum_{i=1}^{\infty}\frac{1}{i^{k+r-3}}$$

$$\leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f,n^{-1})_p.$$

And ,if $\mu > \left\lfloor\frac{n}{2}\right\rfloor$,then

$$\Phi \leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f,n^{-1})_p \left(\sum_{i=1}^{n-\mu}\frac{1}{i^{k+r-3}} + \sum_{i=j}^{\mu-1}\frac{min\{i,n-i\}}{(n-\mu)^{k+r-2}}\right)$$

$$\leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f,n^{-1})_p \left(1+\frac{1}{(n-\mu)^{k+r-2}}\sum_{i=1}^{n} i\right)$$

$$\leq cn^{2k+r-4}\omega_{k,r}^{\varphi}(f,n^{-1})_p \left(1+\frac{n^2}{(n-\mu)^{k+r-2}}\right)$$ ♠

We note that the proof of Corollary 3.1.5 and Corollary 3.1.6 are direct by Theorem 3.1.4 .

3.4 The Degree of Coconvex Approximation and Divided Difference by Using Weighted D. T. Modulus of Smoothness

In this section we estimate the degree of approximation of f by divided difference. And to prove our theorems in this section ,we need the following Lemmas:

Lemma 3.4.1

Let $n \geq 9$, $m = 1$ or $m = 2$, and $f \in L_{\varphi}^{2}$ .Then ,

$$max\{|[x_1, x_2, \dots, x_{m+1}; f'']|, |[x_{n-1}, x_{n-2}, \dots, x_{n-m-1}; f'']|\}$$

$$\leq cn^{2m+2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p + c\|f''\|_{L_p\left[-\frac{1}{2},\frac{1}{2}\right]}.$$

Proof of this Lemma is an immediate consequence of Corollary 3.3.4.

Lemma 3.4.2

Let $s \geq 3$ and $f \in L_{\varphi}^{2} \cap \Delta^{2}(Y_s)$,and

$$n \geq max\{\mathcal{M}(Y_s), (min\{\varphi(y_i) \,/\, 1 \leq i \leq s\})^{-3}\},$$

then,

(3.4.1) $$max\{|[x_1, x_2, x_3; f'']|, |[x_{n-1}, x_{n-2}, x_{n-3}; f'']|\}$$

$$\leq cn^{6}\, \omega_{3,2}^{\varphi}(f'', n^{-1})_p$$

And ,

(3.4.2) $$max\{|[x_1, x_2; f'']|, |[x_{n-1}, x_{n-2}; f'']|\}$$

$$\leq cn^4\, \omega_{3,2}^{\varphi}(f'', n^{-1})_p.$$

Lemma 3.4.3

Let $s = 2$, $f \in L_{\varphi}^{2} \cap \Delta^2(Y_s)$,and

$n \geq max\{\mathcal{M}(Y_s)\,, (min\{\varphi(y_1), \varphi(y_2)\})^{-3}\ \}$, then

$$max\{-[x_1, x_2, x_3; f''], -[x_{n-1}, x_{n-2}, x_{n-3}; f'']\} \leq cn^6\, \omega_{3,2}^{\varphi}(f'', n^{-1})_p \tag{3.4.3}$$

And,

$$max\{-[x_1, x_2; f''], -[x_{n-1}, x_{n-2}; f'']\} \leq cn^4\, \omega_{3,2}^{\varphi}(f'', n^{-1})_p. \tag{3.4.4}$$

Proof of Lemma 3.4.2 and Lemma 3.4.3.

Let $\mathbb{N} = \mathbb{N}(Y_s) = min\{\varphi(y_i) \,/\, 1 \leq i \leq s\}$. Also ,let $s \geq 2$, $f \in L_{\varphi}^{2} \cap \Delta^2(Y_s)$ be given ,and observe that if an index i is such that $y_s \leq x_i \leq y_1$,then

$$4\, min\{i, n-i\} \geq n\ \sin(i\pi/n) = n\varphi(x_i) \geq n\, min\{\varphi(y_s), \varphi(y_1)\}$$

Now , let the indices μ_1, v_1, v_2 and μ_2 (if $s \geq 3$) be such that

$$f''\big(x_{\mu_1+1}\big) = min\{f''(x_i) \mid y_2 \leq x_i \leq y_1\}\,,$$

$x_{v_1+1} \leq y_1 < x_{v_1}$, $x_{v_2+1} \leq y_2 < x_{v_2}$,and

$$f''\big(x_{\mu_2+1}\big) = max\{f''(x_i) \mid y_3 \leq x_i \leq y_2\}.$$

Then , using $f''(x)(x-y_1)(x-y_2) \geq 0$ for all $x \geq y_3$,we conclude that the following inequality holds: $1 \leq v_1 \leq \mu_1 < v_2 \leq n-2$, $v_2 \leq \mu_2 \leq n-3$ (if $s \geq 3$),

$$\big[x_{\mu_1}, x_{\mu_1+1}, x_{\mu_1+2}; f''\big] \geq 0\ \ , \big[x_{v_1}, x_{v_1+1}; f''\big] \geq 0$$

and

$$\left[x_{\mu_2}, x_{\mu_2+1}, x_{\mu_2+2}; f''\right] \leq 0 \quad , \left[x_{v_2}, x_{v_2+1}; f''\right] \leq 0 \,.$$

By Corollary 3.1.5 (i) with $\mu = \mu_1$ and $v = v_1$,taking into account that $n - \mu_1 + 1 \geq \mathbb{N}n/4$, it follows that

(3.4.5) $$-[x_1, x_2, x_3; f''] \leq cn^6 \, \omega_{3,2}^{\varphi}(f'', n^{-1})_p.$$

And,

(3.4.6) $$-[x_1, x_2; f''] \leq cn^4 \left(1 + \frac{n^2}{(n-\mu_2)^3}\right) \omega_{3,2}^{\varphi}(f'', n^{-1})_p.$$

If $s \geq 3$,then Corollary 3.1.5 (i) with $\mu = \mu_2$ and $v = v_2$,and the observation that $n - \mu_2 \geq \mathbb{N}n/4$,imply

(3.4.7) $$[x_1, x_2, x_3; f''] \leq cn^6 \, \omega_{3,2}^{\varphi}(f'', n^{-1})_p$$

And,

$$[x_1, x_2; f''] \leq cn^4 \left(1 + \frac{n^2}{(n - \mu_2)^3}\right) \omega_{3,2}^{\varphi}(f'', n^{-1})_p$$

(3.4.8) $$\leq cn^4 \, \omega_{3,2}^{\varphi}(f'', n^{-1})_p).$$

This in turn implies that

$$|[x_1, x_2, x_3; f'']| \leq cn^6 \, \omega_{3,2}^{\varphi}(f'', n^{-1})_p \,,$$

And ,

$$|[x_1, x_2; f'']| \leq cn^4 \, \omega_{3,2}^{\varphi}(f'', n^{-1})_p.$$

This complete the proof of Lemma 3.4.2 , and to complete the proof Lemma 3.4.3,it suffices to use Corollary 3.1.5 (ii) with $\mu = n - \mu_1 - 2$ and $v = n - v_2 - 1$,and the estimate $\mu_1 + 1 \geq \mathbb{N}n/4$,and to combine the resulting inequalities with (3.4.5)and (3.4.6) ♦

Proof of theorem 3.1.7 and theorem 3.1.8

Suppose that n is such that

$$n \geq max\left\{4\left(\min_{1\leq j\leq s+1}\{y_{i-1}-y_i\}\right)^{-1},\left(\min_{1\leq j\leq s}\{\varphi(y_i)\}\right)^{-3}\right\}.$$

We use the same construction as in proof of Theorem 3.1.1,which we described in the proof of Theorem 3.1.3. The only difference now is that , on each interval O_i , $1 \leq i \leq s$,the polynomial $\tilde{p}_i$ is defined to be the quadratic polynomial interpolating $f''\ at\ a_i, y_i$ and b_i ,whence ,by (Lemma 2.3.3 ,Whitney's inequality),

$$\|f''-\tilde{p}_i\|_{L_p(O_i)} \leq c\omega_3(f'',|O_i|,O_i)_p.$$

Hence using the inequality

$$\left|I_j\right|^2\omega_3(f'',|I_i|,I_i)_p \leq cn^{-2}\omega_{3,2}^{\varphi}(f'',n^{-1})_p\ \ ,1<j<n$$

which follows from Lemma 3.2.8, we conclude that there exists a spline $P_n \in \sum_{5,n}(Y_s)$ which is coconvex with f on $[x_{n-1},x_1]$, satisfies the inequality

(3.4.9) $$\|f-P_n\|_{L_p[x_{n-1},x_1]} \leq cn^{-2}\omega_{3,2}^{\varphi}(f'',n^{-1})_p,$$

and is such that

$P_n(x_{n-1}^{+}) = f(x_{n-1})$, $(-1)^{s+1}P_n'(x_{n-1}^{+}) \leq (-1)^{s+1}f'(x_{n-1})$

$P_n'(x_1{}^{-}) \leq f'(x_1)$.

We now extend the construction of P_n to the intervals I_1 and I_n preserving its coconvexity with the original function f , as well as keeping it close to f.

On I_1 and I_n , P_n is defined as follows

$P_n(x_1{}^{+}) = P_n(x_1{}^{-})$, $P_n'(x_1{}^{+}) = f'(x_1)$,

and $P_n^{(i)}(x_{n-1}^{-}) \leq f^{(i)}(x_{n-1})$, $i = 0,1$

$$P_n''(x) = f''(x_1) + (x - x_1)\, max\{0, [x_1, x_2; f'']\}$$
$$+ (x - x_1)(x - x_2)\ \ max\{0, [x_{n-1}, x_{n-2}, x_{n-3}; f'']\}\ , x \in I_1$$

$$P_n''(x) = f''(x_{n-1})$$
$$+ (x - x_{n-1})(-1)^{s+1} max\{0, (-1)^{s+1}[x_{n-1}, x_{n-2}; f'']\}$$
$$+(x - x_{n-1})(x$$
$$- x_{n-2})\ (-1)^s\ max\{0, (-1)^s [x_{n-1}, x_{n-2}, x_{n-3}; f'']\}$$
$$, x \in I_n.$$

We wish to emphasize that in the case $s \geq 3$,we could alternatively define $P_n''(x) = f''(x_1)$, $x \in I_1$ and $P_n''(x) = f''(x_{n-1})$, $x \in I_n$, which is some what simpler than the current construction ,but would force us to consider the case $s \leq 2$ separately. Evidently , $P_n \in L_p[-1,1]$,and is in $\Delta^2(Y_s)$ (since P_n' and $(-1)^s P_n'$ are non-decreasing on I_1 and I_n, respectively ,we have that

$$(-1)^s P_n'(x_{n-1}^-) \leq (-1)^s P_n'(x_{n-1}^+) \ \text{ and } \ \ P_n'(x_1{}^-) \leq P_n'(x_1{}^+)\ ).$$

Hence ,it remains to estimate $\|f - P_n\|_{L_p(I_1)}$ and $\|f - P_n\|_{L_p(I_n)}$ .We note that (3.4.9) implies that $\alpha = f(x_1) - P_n(x_1{}^+)$,satisfies

$$|\alpha| \leq cn^{-2} \omega_{3,2}^{\varphi}(f'', n^{-1})_p\ .$$

Therefore , by Lemma 3.3.5 ,we have for every $x \in I_1$, assume that $p_1''(x) = l_1(x)$, $p_n''(x) = l_n(x)$,where p_1 and p_n are tow polynomials of degree ≤ 4 ,then

$$\|f - p_1\|_{L_p(I_1)} \leq cn^{-2} \omega_{3,2}^{\varphi}(f'', n^{-1})_p\ ,$$

$$\|f - p_n\|_{L_p(I_n)} \leq cn^{-2} \omega_{3,2}^{\varphi}(f'', n^{-1})_p\ .$$

We have,

$$|f(x) - P_n(x)| \leq \|f - p_1\|_{L_p(I_1)} + |p_1(x) - P_n(x)|$$

$$\leq cn^{-2} \omega_{3,2}^{\varphi}(f'', n^{-1})_p$$

$$+\left|f(x_1)-P_n(x_1^{+})+\int_{x_1}^{x}(x-u)(l_1(u)-P_n''(u))du\right|$$

$$\leq cn^{-2}\omega_{3,2}^{\varphi}(f'',n^{-1})_p+|\alpha|+\left|\int_{x_1}^{x}(x-u)(l_1(u)-P_n''(u))du\right|$$

$$\leq cn^{-2}\omega_{3,2}^{\varphi}(f'',n^{-1})_p+cn^{-4}\|l_1-P_n''\|_{L_p(I_1)}.$$

Similarly except that $P_n(x_{n-1}^{-})=f(x_{n-1})=p_n(x_{n-1})$, for every $x\in I_n$,we have

$$|f(x)-P_n(x)|\leq cn^{-2}\omega_{3,2}^{\varphi}(f'',n^{-1})_p+cn^{-4}\|l_n-P_n''\|_{L_p(I_n)}.$$

Now , for $x\in I_1$

$$\begin{aligned}0\leq P_n''(x)-l_1(x)&\\ &=(x-x_1)\,(max\{0,[x_1,x_2,x_3;f'']\}-[x_1,x_2,x_3;f''])\\ &=(x-x_1)\,max\{0,-[x_1,x_2;f'']\}\\ &\quad+(x-x_1)(x-x_2)\ max\{0,-[x_1,x_2,x_3;f'']\}.\end{aligned}$$

Hence ,for $s\geq 2$,we conclude by Lemma 3.4.2 and Lemma 3.4.3,then

$$\begin{aligned}0\leq P_n''(x)-l_1(x)&\leq(x-x_1)cn^4\omega_{3,2}^{\varphi}(f'',n^{-1})_p\\ &\quad+(x-x_1)(x-x_2)cn^6\omega_{3,2}^{\varphi}(f'',n^{-1})_p\\ &\leq cn^2\omega_{3,2}^{\varphi}(f'',n^{-1})_p\ ,x\in I_1.\end{aligned}$$

For $s=1$, we apply Corollary 3.1.6 and similarly conclude that

$$0\leq P_n''(x)-l_1(x)\leq cn^2\omega_{3,2}^{\varphi}(f'',n^{-1})_p+c\omega_{2,2}^{\varphi}(f'',n^{-1})_p\ ,x\in I_1,$$

analogously ,for $x\in I_n$,

$$0\leq(-1)^s\big(P_n''(x)-l_n(x)\big)$$

$$= (x - x_{n-1})(-1)^{s+1} max\{0, (-1)^s [x_{n-1}, x_{n-2}; f'']\}$$

$$+(x - x_{n-1})(x - x_{n-2})\ (-1)^s\ max\{0, (-1)^{s+1}[x_{n-1}, x_{n-2}, x_{n-3}; f'']\}.$$

Hence ,for $s \geq 2$,by Lemma 3.4.2 and Lemma 3.4.3 instead Corollary 3.1.6 ,we get

$$0 \leq \left(P_n''(x) - l_n(x)\right) \leq cn^2 \omega_{3,2}^{\varphi}(f'', n^{-1})_p + c\omega_{2,2}^{\varphi}(f'', n^{-1})_p\ , x \in I_n.$$

Also, in the case $s = 1$,applying Lemma 3.4.1 ,instead Corollary 3.1.6 ,we have ,for $x \in I_1$

$$|P_n''(x) - l_1(x)|$$
$$\leq (x - x_1)|[x_1, x_2; f'']| + (x - x_1)(x - x_2)|[x_1, x_2, x_3; f'']|$$
$$\leq n^{-2}|[x_1, x_2; f'']| + n^{-4}|[x_1, x_2, x_3; f'']|$$
$$\leq cn^2 \omega_{3,2}^{\varphi}(f'', n^{-1})_p + cn^{-2}\|f''\|_{L_p\left[-\frac{1}{2},\frac{1}{2}\right]}\ ,$$

and the estimate for $\|l_n - P_n''\|_{L_p(I_n)}$ is derived analogously.

To summarize , in the case $s \geq 2$,we have

$$\|f - P_n\|_{L_p(I)} \leq cn^{-2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p\ ,$$

and in the case $s = 1$,we have

$$\|f - P_n\|_{L_p(I)} \leq cn^{-2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p + cn^{-6}\|f''\|_{L_p\left[-\frac{1}{2},\frac{1}{2}\right]}\ ,$$

and,

$$\|f - P_n\|_{L_p(I)} \leq cn^{-2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p + cn^{-4}\omega_{2,2}^{\varphi}(f'', n^{-1})_p\ .$$

By virtue of Lemma 3.2.5 and the estimate

$$\omega_5^{\varphi}(P_n, n^{-1})_p \leq c\|f - P_n\|_{L_p(I)} + c\ \omega_5^{\varphi}(f, n^{-1})_p$$
$$\leq c\|f - P_n\|_{L_p(I)} + cn^{-2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p\ ,$$

we conclude that there exists a polynomial $p_n \in \Delta^2(Y_s)$ of degree $\leq cn$ such that ,

$$\begin{aligned} \|f - p_n\|_{L_p(I)} &\leq \|f - P_n\|_{L_p(I)} + \|P_n - p_n\|_{L_p(I)} \\ &\leq \|f - P_n\|_{L_p(I)} + cn^{-2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p \,. \\ &\leq cn^{-2}\omega_{3,2}^{\varphi}(f'', n^{-1})_p + cn^{-6}\|f''\|_{L_p\left[-\frac{1}{2},\frac{1}{2}\right]} \quad \blacklozenge \end{aligned}$$

Future Works

Our strategy for future is to answer the following problems:

1- Coconvex Polynomial Approximation can be defined on $L_p(\mu)$, $1 \leq p \leq \infty$.

2- Study on Positive ,Copositive, Monotone, Comonotone ,Convex and Coconvex approximation on $L_p, 0 < p \leq \infty$ defined on the main-part modulus of smoothness Ω_φ^r ,that is

$$\Omega_\varphi^r(f,t)_p = sup_{0 \leq h \leq t} \left\| \Delta_{h\varphi}^r(f,.) \right\|_{L_p(J)},$$

Where, $J = [-1 + 2r^2h^2, 1 - 2r^2h^2]$.

3- Estimate the degree of coconvex approximation of f by splines defined on Principal Shift$-$Invariant space (PSI).

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