ضبابية كاما لبديهيات الفصل في القضاءات التبولوجية المضببة

احمد سعدي عبد عون

كلية التربية/رياضيات

ACKNOWLEDGEMENT

Thanks to Allah for helping and giving us the power to complete my thesis .I wish to express my deep appreciation and sincere thanks to my supervisor Assist . Prof . Dr. Luay A . A . AL−Swidi for his valuable suggestions , extensive discussion encouragement and continuous guidance .

Also, special thanks due to the head of the Department of Mathematics, Assist . Prof . Dr. Iftichar M .Talp . I would also like to thank all the members of faculty of the Mathematics department , specially , Assist . Prof . Dr . Eman Samir for their moral support .

Finally, I would like to thank all my friends, and my family , for their generosity and support at hard time .

Ahmed Saadi Abd Oon
April , 2009

المستخلص

الهدف الرئيسي من هذا العمل هو دراسة بديهيات فصل مضببة جديدة ، تدعى ضبابية گاما لبديهيات الفصل في الفضاءات التبولوجية المضببة . أدناه بعض النتائج الرئيسية التي تم الحصول عليها .

1) كل فضاء–FST_1 هو فضاء–FT_1 والعكس ليس صحيح .

2) أي X fts هو فضاء–$F\gamma T_2$ إذا وفقط إذا لكل نقطة ضبابية x_α في X ،

$$x_\alpha = \wedge\{\gamma\text{–}cl(V) , V \in \psi(x_\alpha)\}$$

ولأي $x , y \in X$ بحيث $x \neq y$ ، هناك $U \in \psi(x_1)$ بحيث $y \notin supp(\gamma\text{–}cl(U))$.

3) لأي X fts ، العبارات الآتية متكافئة :

(a) X هو فضاء–$F\gamma R$.

(b) لكل نقطة ضبابية x_λ في X وكل مجموعة مفتوحة ضبابية U بحيث أن $x_\lambda \in U$ ، هناك مجموعة مفتوحة–γ ضبابية V بحيث أن $x_\lambda \in V \le \gamma\text{–}cl(V) \le U$.

4) لأي X fts ، العبارات الآتية متكافئة :

(a) X هو فضاء–$F\gamma N$.

(b) لكل مجموعتين مغلقتينِ ضبابيتينِ A وB بحيث أن $A /q B$ ، هناك مجموعات مفتوحة–γ ضبابية U و V بحيث أن $A \le U$ و$B \le V$ و $U /q V$.

(c) لأي زوج من المجموعات المفتوحةِ الضبابية U و V بحيث $V^c /q U^c$ ، هناك مجموعات مغلقة–γ ضبابية A و B بحيث أن $A \le U$ و$B \le V$ و $B^c /q A^c$.

5) لتكن $f : X \to Y$ دالة متقابلة مفتوحة–γ ضبابية من فضاء– FT_i X إلى Y fts ، فان Y هو فضاء – $F\gamma T_i$ ، لأي $i = 0 , 1 , 2$.

6) إذا كانت $f : X \to Y$ دالة متقابلة مغلقة –γ بقوة ضبابية مستمرة من فضاء –$F\gamma R$ X إلى Y fts ، فان Y هو فضاء–$F\gamma R$.

7) إذا كانت $f : X \to Y$ دالة شاملة ضبابية مغلقة–γ ضبابية مستمرة من فضاء–FN X إلى Y fts ، فان Y فضاء–$F\gamma N$.

8) إذا كانت $f : X \to Y$ دالة متقابلة ضبابية مغلقة ضبابية مستمرة من فضاء–FT_4 X إلى Y fts ، فان Y فضاءِ–$F\gamma T_4$.

9) لأي فضاء–FST_1 X ، العبارات الآتية متكافئة :

(a) X هو فضاء–$F\gamma R$.

(b) لأي زوج النقطة الضبابية x_λ ومجموعة مغلقة ضبابية B في X بحيث أن $x_\lambda \in 1_X - B$ ، هناك مجموعات مفتوحة– γ Fg U و V في X بحيث أن $x_\lambda \in U$ و $B \le V$ و $U /q V$.

(c) لكل نقطة ضبابية x_λ وكل مجموعة مفتوحة ضبابية B تحوي x_λ ، هناك مجموعة مفتوحة –γ Fg U بحيث أن $cl(x_\lambda) \in U \le \gamma\text{–}cl(U) \le B$.

(d) لكل نقطة ضبابية x_λ وكل مجموعة مفتوحة – Fg B تحوي x_λ ، هناك مجموعة مفتوحة–γ ضبابية U بحيث أن $x_\lambda \in U \le \gamma\text{–}cl(U) \le int(B)$.

(e) لكل نقطة ضبابية x_λ وكل مجموعة مفتوحة – Fg B تحوي x_λ ، هناك مجموعة مفتوحة–γ Fg G بحيث أن $x_\lambda \in G \le \gamma\text{–}cl(G) \le int(B)$.

(f) لكل نقطة ضبابية x_λ وكل مجموعة مفتوحة ضبابية B تحوي x_λ ، هناك مجموعة مفتوحة –γ ضبابية U بحيث أن $cl(x_\lambda) \in U \le \gamma\text{–}cl(U) \le B$.

10) لأي فضاء تبولوجي ضبابي X ، العبارات الآتية متكافئة :

(a) X هو فضاء–$F\gamma N$.

(b) لأي زوج من المجموعات المغلقة الضبابية A و B في X بحيث أن $A /q B$ ، هناك مجموعات مفتوحة – $F\gamma g$ U , V في X بحيث أن $A \le U$, $B \le V$ و $U /q V$.

(c) لكل مجموعة مغلقة ضبابية A وكل مجموعة مفتوحة ضبابية B تحوي A ، هناك مجموعة مفتوحة – $Fg\gamma$ U بحيث أن $cl(A) \le U \le \gamma\text{–}cl(U) \le B$.

(d) لكل مجموعة مغلقة ضبابية A وكل مجموعة مفتوحة – Fg B تحوي A ، هناك مجموعة مفتوحة–γ ضبابية U بحيث أن $A \le U \le \gamma\text{–}cl(U) \le int(B)$.

(e) لكل مجموعة مغلقة ضبابية A وكل مجموعة مفتوحة – Fg B تحوي A ، هناك مجموعة مفتوحة –γ Fg G بحيث أن $A \le G \le \gamma\text{–}cl(G) \le int(B)$.

(f) لكل مجموعة مغلقة – Fg A وكل مجموعة مفتوحة ضبابية B تحوي A ، هناك مجموعة مفتوحة ضبابية –γ U بحيث أن $cl(A) \le U \le \gamma\text{–}cl(U) \le B$.

(g) لكل مجموعة مغلقة – Fg A و كل مجموعة مفتوحة ضبابية B تحوي A ، هناك مجموعة مفتوحة –γ Fg G بحيث أن $cl(A) \le G \le \gamma\text{–}cl(G) \le B$.

11) إذا كانت $f : X \to Y$ دالة متقابلة مفتوحة–γ F q من فضاء–$F\gamma T_i$ X إلى fts Y , فان Y هو فضاء–FT_i , لأي $i = 0 , 1 , 2$.

12) إذا كانت $f : X \to Y$ دالة متباينة γ–irresolute ضبابيةً من fts X إلى فضاء– $F\gamma T_i$ Y ، فان X هو فضاء– $F\gamma T_i$ ، لأي $i = 0 , 1 , 2$.

13) إذا كانت $f : X \to Y$ دالة متباينة مستمرة–gγ ضبابية من فضاء –FST_1 X إلى فضاء –FT_2 Y ، فان X هو فضاء–$F\gamma T_2$.

14) إذا كانت $f : X \to Y$ دالة متقابلة مستمرة–gγ ضبابية مغلقة ضبابية من فضاء –FST_1 X إلى فضاء–FR Y , فان X هو فضاء–FγR .

15) إذا كانت $f : X \to Y$ دالة متقابلة مستمرة–gγ–γ ضبابية مغلقة ضبابية من فضاء –FST_1 X إلى فضاء–FγR Y , فان X هو فضاء–FγR .

16) إذا كانت $f : X \to Y$ دالة متقابلة مستمرةً–gγ ضبابية مغلقة ضبابية من fts X إلى فضاء–FN Y , فان X هو فضاء–FγN .

17) إذا كانت $f : X \to Y$ دالة شاملة مغلقة–gγ F almost مستمرة ضبابية من فضاء –FN X إلى fts Y ، فان Y هو فضاء–FγN .

Introduction

The principle goal of this work is to define and study the concept of fuzzy γ–separation axioms in fuzzy topological spaces .

The concept of fuzzy set and fuzzy set operations were first introduced by L . A . Zadeh in 1965 [36] . After Zadeh's introduction of fuzzy sets , Chang [19] defined and studied the notion of fuzzy topological space in 1968 .Since then , much attention has been paid to generalize the basic concepts of classical topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed .In 1980 , P.P. Ming and L.Y. Ming [51] , introduced the concepts of quasi – coincidence and quasi – neighborhoods by which the extensions of functions in fuzzy setting can be carried out very interestingly and effectively. The following concepts were introduced in general topological spaces .

In 1937 , M . H . Stone [45] , introduced the concepts of regular open and regular closed sets . In 1963 , semiopen set was introduced and investigated by N . Levine [47] . In 1965 , α–open and α–closed sets were defined by O . Njastad [50] . In 1971 , semiclosed set was defined by Crossley and Hilde [69] . In 1982 , the concepts of preopen and preclosed sets were introduced and investigated by A .S . Mashhour , M . E . Abd EL – Monsef and S . N . EL – Deeb [8] . In 1996 , D.Andrijevi [21] defined the notion of b–open and b–closed sets which are found to be equivalent with the concepts of γ–open and γ–closed sets as defined by A . A . EL . Atik in 1997 [7] .

In 1970 , Levine [48] introduced the notion of generalized closed sets (g–closed set) in topological spaces as a generalization of closed sets .In 1987 , P . Bhattacharyya and B . K . Lahiki [53] invented the idea of sg–closed sets in topological spaces .In 1990 , the concept of gs–closed sets

, was defined by S . P . Arya and T . Nour [70] .In 1996 , H . Maki , J . Umehary and T . Noiri [31] , defined the notion of gp–closed sets .The concept of pg–closed sets was defined and investigated by K . Balachandran , P. Sundram , H . Maki and A . Rani [35] . In 2007 E . Ekici [24] defined the idea of gγ–closed and γg–closed sets in topological space . The following concepts were defined in fuzzy topological spaces .

In 1981 , fuzzy semiopen , fuzzy semiclosed , fuzzy regular open , fuzzy regular closed sets were defined and studied by K . K . Azad [33] .In 1991 , A . S . Bin Shahna [6] , introduced the idea of fuzzy α–open , fuzzy α–closed , fuzzy preopen , fuzzy preclosed sets .In 2002 , Noiri and O . R . Sayed [73] they defined the concepts of fuzzy γ–open , and fuzzy γ–continuity in fuzzifying topology .

In 1997 , fuzzy generalized closed set (Fg–closed set) was introduced by G . Balasubramania and P . Sundaram [28] .In 1998 , the notion of Fgs–closed set was defined and investigated by H . Maki el al [30] .In 2002 , O .Bedre Ozbakir [49] , defined the concept of fuzzy generalized strongly closed set . In this thesis , we call this set as fuzzy generalized α–closed set (Fgα–closed set) . In 2002 R . K .Saraf and S . Mishra [55] introduced and investigated the concept of fuzzy generalized α–closed set . This set is called fuzzy α–generalized closed set (Fαg–closed set) in this thesis .In 2006 , M . Caldas , G . N . Navagi and R. Saraf [42] introduced the notion of fuzzy pre–generalized preclosed set (Fpg–closed set) .In 2008 , fuzzy generalized preclosed set (Fgp–closed set) was defined by S . Muvugesan and P . Thangarel [68] .

Several fuzzy separation axioms have been defined in different ways and investigated by many authors . In 1975 , B . Hutton [12] defined the notion of fuzzy normality in fuzzy topological spaces . In 1980 B . Hutton and I . Reilly [11] they introduced the concept of separation axioms in fuzzy topological spaces . In 1985 , separation axioms and T_i– fuzzy continuity studied by S . Ganguly and S . Saha [60] .In 1986 , $\alpha-T_i$(i = 0 , 3 , 4) , $\alpha-T'_i$ (i = 0 , 1 , 2 , 3 , 4) , α–almost compact , α–nearly compact and α–continuous mappings were defined and studied in fuzzy topological spaces by A . S . Mashhour , M . H . Ghanim and M . A . Fath Alla [9] .In [22] , Dewan Muslim Ali , introduced and studied weak forms of fuzzy T_1 , fuzzy T_2 , fuzzy R_o ,fuzzy R_1 and fuzzy regular spaces in fuzzy topological spaces .In 1991, S . P . Sinha [65] , introduced some weak forms of fuzzy normality called fuzzy almost normality , fuzzy weak normality and fuzzy semi– normality , and the relations among them are investigated . E . Tsiporkora – B . De Baets (1997)[23] , studied T–compactness in separated fuzzy topological spaces . In 1998 , S . K .Cho and D . G . Chung [71] , studied the relations among the fuzzy T_2–axioms and presented some examples which shows that the axiom of fuzzy compactness , due to G – anguly and Saha , is not compatible with the fuzzy T_2–axioms . In 1999 , fuzzy separation axioms have been introduced and investigated with the help of fuzzy β – open sets by G . Balasubramanian [26] . In [18] , Bai Shi – Zhong introduced some new separation axioms , namely fuzzy semipreseparation axioms , and also established some of their characteristic properties .In 2005 , new class of sets called fuzzy semi–pre–generalized closed sets were introduced and its properties were studied .As application of this set , also the notion of $FspT_{1/2}$ –space , Fspg–continuity and Fspg–irresolute mappings are introduced by R . K . Saraf , G . Navalagi and M . Khanna [56] . In 2006 , M . E . EL–Shafei and A . Zakari [46] introduced the concepts of θ–generalized closed fuzzy sets and generalized Λ–fuzzy sets in fuzzy topological spaces .Furthermore generalized Λ–fuzzy sets are extended to θ–generalized Λ–fuzzy sets . Also , they introduced the concepts of fuzzy θ–generalized continuous and fuzzy θ–generalized mappings . M . Caldas , G . Navalagi , and R . Saraf(2006) [42] defined new weak and strong forms of fuzzy pre irresoluteness and fuzzy pre–closureness via the concept of fuzzy pg – closed sets which they called a P – Fp–irresolute functions and they used it to obtain a characterization of fuzzy pre – $T_{1/2}$ – spaces . In 2008 , S . Murugesan and P . Thangavelu [68] introduced the notion of fuzzy pre–semi–closed sets in fuzzy topological spaces and investigated their properties .Using fuzzy pre – semi – closed sets , equivalence of fuzzy regular open sets is established . As applications to fuzzy pre – semi – closed sets , they defined fuzzy spaces with new kinds of separation axioms , namely fuzzy pre–semi – $T_{1/2}$ – spaces , fuzzy pre – semi – $T_{3/4}$ – spaces , fuzzy pre – semi – $T_{1/3}$ – spaces and characterize them .

This thesis has three chapters :

In chapter one there are four sections : section one covers the principle definitions of fuzzy sets and fuzzy set operations .Section two covers the main definitions of fuzzy topological spaces with some of its concepts and results .Section three contains the main definitions of fuzzy continuity and its theorems .Finally , in section four ,we present and study the standard definitions of fuzzy separation axioms which are very needed throughout the work .

In chapter two ,we have three sections . In the first section ,we present some definitions and Theorems that are necessary to the work . In the second section , the concept of fuzzy γ–open sets and its properties are studied . In the last section , new class of fuzzy separation axioms called fuzzy γ–separation axioms is defined and studied .

In chapter three , we have four sections . In section one , we introduce and study new classes of functions , namely fuzzy γ–open and fuzzy strongly γ–open functions . Also , the relations between the standard definitions of FT_i–spaces and $F\gamma T_i$–spaces are investigated depending on these functions . In section two , we define and study the concepts of fuzzy generalized γ–closed and fuzzy γ generalized–closed sets in fuzzy topological space . In section three , new types of functions called fuzzy generalized γ–functions are defined and studied . Finally , section four have Preservation Theorems and other characterization of fuzzy γ–separation axioms are discussed .

Chapter one

Fuzzy Sets and Fuzzy Topological Spaces

This chapter contains the concept of fuzzy sets with some of its properties , fuzzy topological spaces and some concepts about fuzzy topological spaces , as a background for the material included in this thesis .

1.1 : Fuzzy Sets

This section will contain the concept of fuzzy sets with some of its properties that are necessary to the work .

Fuzzy sets theory , introduced by Lotfi . A . Zada in 1965 [36] , is the extension of classical set theory by allowing the membership of elements to range from 0 to 1 . Let X be the universe of a classical set of objects . Membership in a classical subset A of X is often viewed as a characteristic function μ_A from X into {0,1} , where

$$\mu_A(x) = \begin{cases} 1 & , \text{ for } x \in A \\ 0 & , \text{ for } x \notin A \end{cases} \qquad \text{(see [25])}$$

for any $x \in X$.

{0,1} is called a valuation set (see [74]) . If the valuation set is allowed to be the real interval [0,1] , A is called a fuzzy set in X . $\mu_A(x)$ (or simply A(x)) is the membership value (or degree of membership) of x in A . Clearly, A is a subset of X that has no sharp boundary . A fuzzy set A in X can be represented by the set of pairs : $A =\{(x , A(x)) , x \in X\}$.

Let $A : X \rightarrow [0,1]$ be a fuzzy set . If A(x)=1 , for each $x \in X$, we denote it by 1_X and if A(x)= 0 , for each $x \in X$, we denote it by 0_X . That is , by 0_X and 1_X , we mean the constant fuzzy sets taking the values 0 and 1 on X , respectively .[13]

Let I = [0,1] .The set of all fuzzy sets in X , denoted by I^X .[44]

Remark 1.1.1[58]

Since every subset A of an ordinary set X can be identified with its characteristic function $\mu_A : X \rightarrow \{0,1\}$ and such characteristic functions are fuzzy sets in X .Thus , an ordinary set is a special case of fuzzy set .

Definition 1.1.2:[58]

The characteristic functions of subsets of a set X are refered to as the crisp fuzzy sets in X .

Definition 1.1.3:[25 ,74]

Let A be a fuzzy set of a set X . The support of A is the elements x whose membership value is greater than 0 , i.e., supp(A) = $\{x \in X : A(x) > 0\}$.

Definition 1.1.4:[74]

Let A be a fuzzy set of a set X , then :

(1) The height of a fuzzy set A is the largest membership value of A and denoted as hgt(A) .

(2) The elements of x such that $A(x) = 1/2$ are called the crossover point of A .

(3) A is said to be normalized if there is $x \in X$, $A(x) = 1$.

Example 1.1.5:

Let $X =\{a, b ,c ,d\}$ be a set and the fuzzy set B of X is defined as follows :

$B(a) = 0.5$ $B(b) = 0$ $B(c) =1$ $B(d) = 0.8$.

Then the support of B is supp(B) = {a, c, d} , the height of B is hgt(B) = 1 , the element a is crossover point of B , and it is normalized because B(c) =1 .

Definition 1.1.6:[58 , 37]

For any A , B $\in I^X$, then :

(1) A is said to be contained in B if and only if $A(x) \le B(x)$, for each $x \in X$ and denoted as $A \le B$ ·

(2) A and B are equal if and only if $A(x) = B(x)$ for all $x \in X$, and denoted by $A = B$.

Definition 1.1.7:[26]

Let A and B be any two fuzzy sets in X .Then we define $A \vee B : X \to [0,1]$ as follows :

$$(A \vee B)(x) = \max \{A(x), B(x)\}$$

Also, we define $A \wedge B : X \to [0,1]$ as follows :

$$(A \wedge B)(x) = \min \{A(x), B(x)\} .$$

By $A \vee B$ $(A \wedge B)$,we mean the union (intersection) between two fuzzy sets A and B of X .

Definition 1.1.8:[25]

Let A be any fuzzy set in a set X . The complement of A , is denoted by $1_X - A$ or A^c and defined as follows :

$$A^c(x) = 1 - A(x) \quad , \text{for each } x \in X .$$

Remark 1.1.9:

From definition 1.1.7 and definition 1.1.8 , we have , if $A , B \in I^X$, then $A \vee B$, $A \wedge B$ and $1_X - A \in I^X$.

Proposition 1.1.10[74]

For any fuzzy sets A , B and C in a set X , the following hold :

(1) commutativity : $A \vee B = B \vee A$, $A \wedge B = B \wedge A$;

(2) Associativity : $(A \vee B) \vee C = A \vee (B \vee C)$, $(A \wedge B) \wedge C = A \wedge (B \wedge C)$;

(3) Idempotency : $A \vee A = A$, $A \wedge A = A$;

(4) Distributivity : $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$, $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$;

(5) Absorption : $A \vee 0_X = A$, $A \wedge 1_X = A$;

(6) De Morgan's law : $(A \vee B)^c = A^c \wedge B^c$, $(A \wedge B)^c = A^c \vee B^c$;

(7) Involution : $(A^c)^c = A$;

(8) Equivalence formula : $(A^c \vee B) \wedge (A \vee B^c) = (A^c \wedge B^c) \vee (A \wedge B)$;

(9) Symmetrical difference formula :

$$(A^c \wedge B) \vee (A \wedge B^c) = (A^c \vee B^c) \wedge (A \vee B) .$$

However , the excluded – middle law is no longer true :

$$A \vee A^c \neq 1_X \quad , \quad A \wedge A^c \neq 0_X .$$

Example 1.1.11:

Let $X = \{a, b, c\}$ be a set and A , B , C are fuzzy sets of X defined as follows :

$A(a) = 0.3$ $A(b) = 0.2$ $A(c) = 0.1$,

$B(a) = 0.4$ $B(b) = 0.7$ $B(c) = 0.5$,

$C(a) = 0.1$ $C(b) = 0.3$ $C(c) = 0.9$.

Then ,

$A \leq B$ because $A(x) \leq B(x)$ for all $x \in X$,

$(B \vee C)(a) = 0.4$ $(B \vee C)(b) = 0.7$ $(B \vee C)(c) = 0.9$,

$(A \wedge C)(a) = 0.1$ $(A \wedge C)(b) = 0.2$ $(A \wedge C)(c) = 0.1$,

$B^c(a) = 0.6$ $B^c(b) = 0.3$ $B^c(c) = 0.5$,

$(B \vee B^c)(a) = 0.6$ $(B \vee B^c)(b) = 0.7$ $(B \vee B^c)(c) = 0.5$, that is $B \vee B^c \neq 1_X$.

$(B \wedge B^c)(a) = 0.4$ $(B \wedge B^c)(b) = 0.3$ $(B \wedge B^c)(c) = 0.5$, that is $B \wedge B^c \neq 0_X$.

Definition 1.1.12 :[39]

The union (intersection) of the fuzzy sets A_i $(i \in J)$ is denoted by

$$\bigvee_{i \in J} A_i(x) = \sup\{A_i(x) : i \in J\} , \quad x \in X$$

(respectively , $\bigwedge_{i \in J} A_i(x) = \inf\{A_i(x) : i \in J\}$, $x \in X$) .

Theorem 1.1.13 [58]

For fuzzy sets V , A_i $(i \in J)$ of a set X :

(1) $V \wedge (\bigvee_{i \in J} A_i) = \bigvee_{i \in J} (V \wedge A_i)$,

(2) $V \vee (\bigwedge_{i \in J} A_i) = \bigwedge_{i \in J} (V \vee A_i)$,

(3) $1_X - \bigvee_{i \in J} A_i = \bigwedge_{i \in J} (1_X - A_i)$,

(4) $1_X - \bigwedge_{i \in J} A_i = \bigvee_{i \in J} (1_X - A_i)$.

Definition 1.1.14[51 , 32 , 38]

A fuzzy point x_λ in a set X is a fuzzy set defined as follows :

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x \\ 0 & \text{otherwise} \end{cases} ,$$

Where $0 < \lambda \le 1$. Now , $\text{supp}(x_\lambda) = \{y : x_\lambda(y) > 0\}$, but

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x \\ 0 & \text{otherwise} , \end{cases} \quad \text{and} \quad 0 < \lambda \le 1 \text{ . Then ,}$$

$\text{supp}(x_\lambda) = x$, so the value at x is λ , and call the point x its support of fuzzy point x_λ and λ is the height of x_λ .That is , x_λ has the membership degree 0 for all $y \in X$ except one , say $x \in X$.

Definition 1.1.15:[39]

Let x_λ and A be a fuzzy point and a fuzzy set , respectively , in a set X . Then x_λ is said to be contained in a fuzzy set A or x_λ belongs to A , denoted by $x_\lambda \in A$ if and only if $\lambda \leq A(x)$. $A(x)$ denoted the membership degree of x to A .

Definition 1.1.16:[59]

A fuzzy point in a set X with support x and membership value 1 is called crisp point , denoted by x_1 .

For any fuzzy set A in X , we have $x_1 \in A$ if and only if $A(x) = 1$.

Remark 1.1.17:

If x_λ is a fuzzy point contained in a fuzzy set A , then $x_\lambda \in A$ if and only if $x_\lambda \leq A$.

Proof :

Let x_λ be a fuzzy point contained in a fuzzy set A , so

$x_\lambda \in A$ if and only if $\lambda \leq A(x)$

if and only if $x_\lambda(x) \leq A(x)$, for all $x \in X$ by definition 1.1.14

if and only if $x_\lambda \leq A$ by definition 1.1.6.part 1 .

Definition 1.1.18 :[58 , 39]

Let X and Y be two sets ,and $f : X \rightarrow Y$ be a function . For a fuzzy set V in Y , the inverse image of V under f is the fuzzy set $f^{-1}(V)$ in X , denoted by the rule :

$f^{-1}(V)(x) = V(f(x))$ for $x \in X$

(I.e., $f^{-1}(V) = V\text{o}\, f$) .

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, swap, "f^{-1}(V)= Vof"] & Y \arrow[d, "V"] \\ & {[0,1]} \end{tikzcd}$$

For a fuzzy set U in X , the image of U under f is the fuzzy set f(U) in Y , defined , for $y \in Y$, by the rule :

$$f(U)(y) = \begin{cases} \sup\{U(z) : z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$$

for all y in Y , where $f^{-1}(y) = \{x \in X : f(x) = y\}$.

Remark 1.1.19:[2]

From the a above definition ,we see that :

(1) If f is an injective ,then $f(U)(y) = \begin{cases} U(z), z \in f^{-1}(y) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise ,} \end{cases}$

(2) If f is surjective , then $\forall\, x \in X$,

$f^{-1}(V)(x) = V(f(x)) = V(y)$, for each $y \in Y$, $x \in f^{-1}(y)$.

(3) If f is bijective , then $f(U)(y) = (U)(x) , \forall\, x = f^{-1}(y)$,

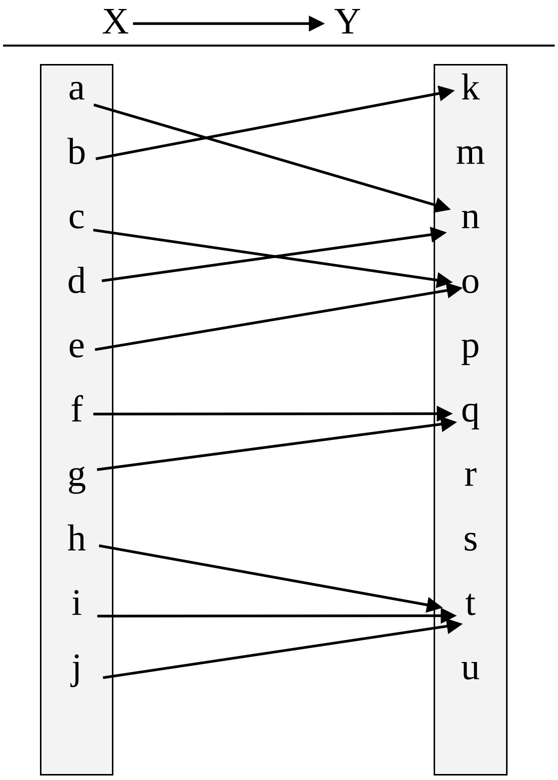
$f^{-1}(V)(x) = V(f(x)) = V(y) , \forall\, y \in Y , y = f(x)$.

Example 1.1.20:

Let A and B be two fuzzy sets in X and Y respectively as shown in the following table :

X	A(X)	Y	B(Y)
a	0.2	k	0.0
b	0.6	m	0.3
c	0.1	n	0.3
d	0.0	o	0.1
e	0.9	p	0.7
f	0.3	q	0.0
g	0.9	r	0.9
h	0.5	s	0.3
i	0.4	t	0.6
j	0.0	u	0.2

And let $f : X \rightarrow Y$ is a function defined by the following table :



Then $f^{-1}(B) = a_{0.3} \vee c_{0.1} \vee d_{0.3} \vee e_{0.1} \vee h_{0.6} \vee i_{0.6} \vee j_{0.6}$

and $f(A) = k_{0.6} \vee n_{\max\{0, 0.2\}} \vee o_{\max\{0.1, 0.9\}} \vee q_{\max\{0.3, 0.9\}} \vee t_{\max\{0, 0.4, 0.5\}}$

$= k_{0.6} \vee n_{0.2} \vee o_{0.9} \vee q_{0.9} \vee t_{0.5}$

Theorem 1.1.21:[76 , 77]

Let $f : X \to Y$ be a function , and A , B be two fuzzy sets in X and Y respectively , if x_λ , y_α be two fuzzy points in X and Y respectively **,** then the following hold :

(1) $f(x_\lambda) = (f(x))_\lambda$;

(2) If $x_\lambda \leq A$, then $f(x_\lambda) \leq f(A)$;

(3) If $y_{o\alpha} \leq f(A)$, then there is $x_o \in X$ such that $f(x_o) = y_o$ and $x_{o\alpha} \leq A$;

(4) If $y \in f(X)$, $y_\alpha \leq B$, then $\forall\, x \in f^{-1}(y)$, $x_\alpha \leq f^{-1}(B)$;

(5) If $x_\alpha \leq f^{-1}(B)$, then $f(x_\alpha) \leq B$.

Remark 1.1.22:

From the above theorem , we see that :

(1) If x_λ is a fuzzy point in X with supp x and value λ ,then $f(x_\lambda)$ is a fuzzy point in Y with supp f(x) and value λ . [2]

(2) If y_α is a fuzzy point in Y with supp y and value α , then $f^{-1}(y_\alpha)$ is in general a fuzzy set in X , not necessarily a fuzzy point because :

(a) If $f^{-1}(y) = \phi$, then $f^{-1}(y_\alpha) = 0_X$ .That is $f^{-1}(y_\alpha)$ is not fuzzy point .

(b) If there are a , b $\in f^{-1}(y)$, then $f^{-1}(y_\alpha)(a) = f^{-1}(y_\alpha)(b) = \alpha$. That is $f^{-1}(y_\alpha)$ is a fuzzy set in X , but not fuzzy point .

Thus if f is bijective function , then $f^{-1}(y_\alpha)$ is a fuzzy point in X .

(3) If f is an injective function and x_λ , y_α be two fuzzy points in X such that $\mathrm{supp}(x_\lambda) \neq \mathrm{supp}(y_\alpha)$, then $f(x_\lambda)$ and $f(y_\alpha)$ are fuzzy points in Y such that $\mathrm{supp}(f(x_\lambda)) \neq \mathrm{supp}(f(y_\alpha))$.

(4) If f is bijective function , and x_λ , y_α be two fuzzy points in Y such that $\mathrm{supp}(x_\lambda) \neq \mathrm{supp}(y_\alpha)$, then $f^{-1}(x_\lambda)$ and $f^{-1}(y_\alpha)$ be two fuzzy points in X such that $\mathrm{supp}(f^{-1}(x_\lambda)) \neq \mathrm{supp}(f^{-1}(y_\alpha))$.

Theorem 1.1.23:[19 , 37]

Let $f : X \to Y$ be a function , then :

(1) $f^{-1}(B^c) = (f^{-1}(B)^c$, for any fuzzy set B in Y ;

(2) If $B_1 \leq B_2$, then $f^{-1}(B_1) \leq f^{-1}(B_2)$, B_1 and B_2 are fuzzy sets in Y ;

(3) If $A_1 \leq A_2$, then $f(A_1) \leq f(A_2)$, A_1 and A_2 are fuzzy sets in X ;

(4) $A \leq f^{-1}(f(A))$, for any fuzzy set A in X ;

(5) $f(f^{-1}(B)) \leq B$, for any fuzzy set B in Y ;

(6) If f is an injective , then $f^{-1}(f(A)) = A$;

(7) If f is surjective , then $f(f^{-1}(B)) = B$;

(8) If f is bijective , then $f(A^c) = (f(A))^c$;

(9) If $f : X \to Y$, and $g : Y \to Z$, then

$(gof)^{-1}(c) = f^{-1}(g^{-1}(c))$, for any fuzzy set c in Z , where gof is the composition of g and f .

Definition 1.1.24:[51]

A fuzzy point x_λ is said to be quasi – coincident (q–coincident ,for short) with a fuzzy set A in X , denoted by $x_\lambda\, q\, A$ if and only if $\lambda > A^c(x)$ or $\lambda + A(x) > 1$.

Definition 1.1.25:[51]

A fuzzy set A is said to be q–coincident with a fuzzy set B in a set X , denoted by A q B if and only if there exists $x \in X$ such that $A(x) > B^c(x)$ or $A(x) + B(x) > 1$. If A is not q–coincident with B , then we write A /q B , i.e., $A(x) \leq B^c(x)$ or $A(x) + B(x) \leq 1$, for all $x \in X$.

Obviously if A and B are q–coincident at x , then both A(x) and B(x) are not zero and hence $x \in supp(A) \cap supp(B)$.

For two fuzzy sets A and B , $A \leq B$ if and only if A and B^c are not q–coincident , denoted by A /q B^c .

Example 1.1.26:

Let X = { a , b , c } be a set and $a_{0.5}$ be a fuzzy point in X and A , B and C are fuzzy sets in X defined as follows :

$$A(a) = 0.9 \qquad A(b) = 0.5 \qquad A(c) = 0.8\ ,$$
$$B(a) = 0.1 \qquad B(b) = 0.3 \qquad B(c) = 0.2\ ,$$
$$C(a) = 0.9 \qquad C(b) = 0.4 \qquad C(c) = 0.7\ .$$

Then $a_{0.5}$ is q–coincident with a fuzzy set A , since $0.5 + A(a) = 0.5 + 0.9 = 1.4 > 1$ by definition 1.1.24 . A and B are not q–coincident because $A(x) + B(x) \leq 1$ for each $x \in X$ by definition 1.1.25 . A and C are q–coincident , since $A(c) + C(c) = 0.8 + 0.7 = 1.5 > 1$ by definition 1.1.25 .

Proposition 1.1.27:[51 , 3 , 4]

Let A , B be two fuzzy sets in a set X , then :

(1) If $A \wedge B = 0_X$, then A /q B ,

(2) $A \leq B$ if and only if x_λ q B for each x_λ q A ,

(3) A /q B if and only if $A \leq B^c$,

(4) x_λ q ($\underset{i \in J}{\vee} A_i$) if and only if there is $i_o \in J$ such that x_λ q A_{i_o} ,

(5) $x_\lambda \in A$ if and only if x_λ is not q–coincident with A^c ,

(6) A /q A^c for each $A \in I^X$.

1.2 : Fuzzy Topological Spaces

Section two will be contain the standard definition of fuzzy topological space with some of its properties that are needed throughout the work .

Many areas of mathematics have been fuzzified , since Zadeh first introduced the notion of a fuzzy set.One of these notions has been topology .Several different definitions of fuzzy topology have been suggested .

In this thesis we will use one definition of fuzzy topological spaces .Chang's definition (1968) is the standard definition in this thesis .That definition is very similar to the usual definition for topological space .

The following definition of fuzzy topological space is due to Chang [19] .

Definition 1.2.1:

A fuzzy topology on a set X is a family δ of fuzzy sets in X which satisfies the following conditions :

(1) 0_X , $1_X \in \delta$,

(2) If A , B $\in \delta$, then $A \wedge B \in \delta$,

(3) If $\{A_i : i \in J\}$ is a family in δ , then $\underset{i\in J}{\vee} A_i \in \delta$.

δ is called a fuzzy topology for X and the pair (X,δ) (or simply X) is a fuzzy topological space or fts for short . Every element of δ is called δ–fuzzy open set (fuzzy open set , for short) . A fuzzy set is δ–fuzzy closed (or simply fuzzy closed) , if its complement is fuzzy open set .

As ordinary topologies , the indiscrete fuzzy topology on X contains only 0_X and 1_X (i.e., ϕ , X) , while the discrete fuzzy topology on X contains all fuzzy sets in X .

The following examples of fts .

Example:1.2.2:

Let X = {a , b , c} be a set and A , B and C are fuzzy sets on X defined as follows :

$A(a) = 0.3 \qquad A(b) = 0.2 \qquad A(c) = 0.7$,

$B(a) = 0.6 \qquad B(b) = 0.8 \qquad B(c) = 0.9$,

$C(a) = 0.1 \qquad C(b) = 0.5 \qquad C(c) = 0.8$.

Let $\delta_1 = \{0_X , A , B , 1_X\}$ and $\delta_2 = \{0_X , A \wedge C , 1_X\}$ .Then δ_1 (respectively, δ_2) is a fuzzy topology on X , since δ_1 (respectively, δ_2) is closed under arbitrary union and intersection . Then (X,δ_1) and (X,δ_2) are fuzzy topological spaces .

Example:1.2.3:

Let $X = [-1,1]$, and let B_1 , B_2 and B_3 are fuzzy sets in X defined as follows :

$$B_1(x) = \begin{cases} 1 & , \text{if} \quad -1 \leq x < 0 \\ 0 & , \text{if} \quad 0 \leq x \leq 1 \end{cases} ,$$

$$B_2(x) = \begin{cases} 0 & , \text{if} \quad -1 \leq x < 0 \\ 1 & , \text{if} \quad 0 \leq x \leq 1 \end{cases} ,$$

$$B_3(x) = \begin{cases} 0 & , \text{if} \quad -1 \leq x < 0 \\ 1/5 & , \text{if} \quad 0 \leq x \leq 1 \end{cases} .$$

Let $\delta = \{ 0_X , B_1 , B_2 , B_3 , B_1 \vee B_3 , 1_X \}$, then δ is a fuzzy topology on X , and (X,δ) is a fts .

Example:1.2.4:

Let $X = [0,1]$ and $\delta_1 = \{0_X, K , 1_X \}$ .Then δ_1 is a fuzzy topology on X , where $K : X \rightarrow [0,1]$ defined as : $K(x) = x^2$, for all $x \in X$, and (X, δ_1) is a fts .

Example:1.2.5:[5]

Let (X,T) be a topological space (in the ordinary sense) . If elements of T are identified with their characteristic functions , then (X,T) is a fts .

Example:1.2.6:[5]

The collection of all crisp fuzzy sets in X is a fuzzy topology on X .

Example:1.2.7:[58]

The collection of all constant fuzzy sets in X is a fuzzy topology on X .

Definition 1.2.8:[19 , 74]

Let A be any fuzzy set in a fts X . The interior of A is the union of all fuzzy open sets contained in A , denoted by int(A) .That is

$$\text{int}(A) = \vee\{B : B \text{ is fuzzy open set} , B \leq A\} .$$

Definition 1.2.9:[19,74]

Let A be any fuzzy set in a fts X . The closure of A is the intersection of all fuzzy closed sets containing A , denoted by cl(A) . That is , $\text{cl}(A) = \wedge\{B : B \text{ is fuzzy closed set} , B \geq A\}$.

The most important properties of the closure and interior of fuzzy sets are listed in the following propositions .

Proposition 1.2.10:[57]

If A be any fuzzy set in X then :

(1) A is a fuzzy open(closed) set if and only if A = int(A) (A = cl(A)) ,

(2) $\text{cl}(1_X - A) = 1_X - \text{int}(A)$,

(3) $\text{int}(1_X - A) = 1_X - \text{cl}(A)$.

Propositions 1.2.11:[74 , 75]

Let A , B be two fuzzy sets in a fts X .Then :

(1) $\text{int}(A) \leq A$, $\text{int}(\text{int}(A)) = \text{int}(A)$,

(2) $\text{int}(A) \leq \text{int}(B)$, whenever $A \leq B$,

(3) $\text{int}(A\wedge B) = \text{int}(A) \wedge \text{int}(B)$, $\text{int}(A\vee B) \geq \text{int}(A) \vee \text{int}(B)$.

(4) $A \leq cl(A)$, $cl(cl(A)) = cl(A)$,

(5) $cl(A) \leq cl(B)$, whenever $A \leq B$,

(6) $cl(A \wedge B) \leq cl(A) \wedge cl(B)$, $cl(A \vee B) = cl(A) \vee cl(B)$.

The following example shows that $A \not\leq int(A)$ and $cl(A) \not\leq A$ for any fuzzy set A in a fts X .

Example 1.2.12:

Let $X = \{a, b, c\}$ be a set , $\delta = \{0_X , a_{0.3} , a_{0.3} \vee b_{0.3} , 1_X\}$ .Then δ is a fuzzy topology on X . Suppose $A = a_{0.5}$, i.e., A is a fuzzy point in X(see definition 1.1.14) .Since 0_X and $a_{0.3}$ are only fuzzy open sets in X contained in A by definition 1.1.6 .Then $int(A) = int(a_{0.5}) = 0_X \vee a_{0.3} = a_{0.3}$ by definition 1.2.8 and definition 1.1.7 . That is $A \not\leq int(A)$. Now ,

$\xi = \{1_X , a_{0.7} \vee b_1 \vee c_1 , a_{0.7} \vee b_{0.7} \vee c_1 , 0_X\}$ be a collection of all δ–fuzzy closed sets in X . Since $(a_{0.7} \vee b_1 \vee c_1)$, $(a_{0.7} \vee b_{0.7} \vee c_1)$and 1_X are only fuzzy closed sets in X containing A , then $cl(A) = cl(a_{0.5}) = (a_{0.7} \vee b_1 \vee c_1) \wedge (a_{0.7} \vee b_{0.7} \vee c_1) \wedge 1_X = a_{0.7} \vee b_{0.7} \vee c_1$ by definition 1.2.9 and definition 1.1.7 . That is $cl(A) \not\leq A$.

Definition 1.2.13:[19 , 51]

A fuzzy set A in a fts (X,δ) is called fuzzy neighborhood (or simply F–nhd) of a fuzzy point x_λ if there is a $B \in \delta$ such that $x_\lambda \in B \leq A$.

The family of all F–nhds of a fuzzy point x_λ is called the F–nhd system of x_λ .

An F–nhd A is said to be F–open–nhd (respectively, F–closed–nhd) if and only if A is fuzzy open (respectively, closed) set .

Chang [19] extended the concept of F–nhd of fuzzy set as follows :

Definition 1.2.14:

A fuzzy set A in fts (X,δ) is a F–nhd of a fuzzy set B in X if there is $K \in \delta$ such that $B \leq K \leq A$ .The family consisting of all F–nhd of A is called the F–nhd system of A .

Definition 1.2.15:[51]

A fuzzy set A in a fts X is called q–neighborhood (in short q–nhd) of a fuzzy point x_λ if there exists fuzzy open set B in X such that x_λ q B and $B \leq A$ .That is $x_\lambda \, q \, B \leq A$.

Example 1.2.16:

Let $X = \{a, b, c\}$ be a set and $\delta = \{0_X , A , B , A\vee B , A\wedge B , 1_X\}$. where $c_{0.3}$ be a fuzzy point in X and A , B and E are fuzzy sets in X defined as follows :

$A(a) = 0.4$ $A(b) = 0.7$ $A(c) = 0.2$,

$B(a) = 0.3$ $B(b) = 0.1$ $B(c) = 0.8$,

$E(a) = 0.5$ $E(b) = 0.8$ $E(c) = 0.9$.

Then (X,δ) is fts and $c_{0.3}$ is q–coincident with a fuzzy set B , since $0.3 + B(c) = 0.3 + 0.8 = 1.1 > 1$ by definition 1.1.24 . Also , E is q–nhd of a fuzzy point $C_{0.3}$, because $B \in \delta$, $c_{0.3}$ q B and $B \leq E$ by definition 1.2.15 .

Theorem 1.2.17:[74]

A fuzzy point $x_\lambda \in cl(A)$ if and only if each q–nhd of x_λ is q–coincident with A .

1.3 : Fuzzy continuity

This section deals with the main definitions of fuzzy continuity and its theorem that are included throughout the work .

Definition 1.3.1:[59]

A function $f : X \to Y$ from a fts X to a fts Y is said to be fuzzy continuous at a fuzzy point x_λ of X if and only if for every F–nhd V of $f(x_\lambda)$, there exists a F–nhd U of x_λ such that $f(U) \leq V$. f is said to be fuzzy continuous on X , if it is so at each fuzzy point of X .

Definition 1.3.2:[19 , 37]

Let $f : (X,\delta_1) \to (Y,\delta_2)$ be a function from a fts (X,δ_1) into a fts (Y,δ_2) . The function is called fuzzy open (closed) (F open(closed) , for short) , if f(A) is fuzzy open (resp., fuzzy closed) set in Y , for each $A \in \delta_1$ (resp., $A^c \in \delta_1$) .

Example 1.3.3:

Let $X = \{a, b, c\}$, $Y = \{x, y\}$. Fuzzy sets A and B are defined as follows :

A (a) = 0.4 A(b) = 0.3 A (c) = 0.9 ,
B (x) = 0.9 B (y) = 0.4 .

Let $T_1 = \{0_X , A , 1_X\}$ and $T_2 = \{0_Y , B , 1_Y\}$. Then the function $f : (X, T_1) \to (Y, T_2)$ defined by f(a) = y and f(b) = f(c) = x is F open function .

Theorem 1.3.4:[52, 19 , 58]

Let (X,T) and (Y, δ) be two ftss , and Let $f : X \to Y$ be a function , then the following are equivalent :

(1) f is fuzzy continuous ,

(2) for every δ–fuzzy open set A , $f^{-1}(A)$ is T–fuzzy open set ,

(3) for every δ–fuzzy closed set A , $f^{-1}(A)$ is T–fuzzy closed set ,

(4) for any fuzzy set A in X , $f(cl(A)) \leq cl(f(A))$,

(5) for any fuzzy set B in Y , $f^{-1}(int(B)) \leq int(f^{-1}(B))$.

Example 1.3.5:

Let X = {a, b, c} , Y = {x, y, z} . Fuzzy sets A and B are defined as follows :

$A(a) = 0.7 \quad A(b) = 0.2 \quad A(c) = 0.1$,

$B(x) = 0.1 \quad B(y) = 0.7 \quad B(z) = 0.2$.

Let $T = \{0_X , A , 1_X\}$ and $\delta = \{0_Y , B , 1_Y\}$ .Then the function $f : (X,T) \rightarrow (Y,\delta)$ defined by f(a) = y , f(b) = z , f(c) = x is fuzzy continuous .

1.4 : Separation Axioms FT_i (i =0, 1,2,3,4) spaces

This section contains the concept of T_i –axioms for ftss .We choose a concept of fuzzy disjointness that agrees with ordinary set theoretic disjointness in the crisp case .Several fuzzy T_i –axioms , $i = 0 , 1 , 2 , 3 , 4$ have been defined in different ways and investigated by many authors , such as [60] , [61] , [62] , [27] , [11] , [63] , [64] , [71] . Now we consider the following standard definitions which are often needed throughout the work .

Definition 1.4.1:[1]

A fts X is called fuzzy T_o–space (FT_o–space, for short)if for each pair of fuzzy points x_λ and y_α , with $supp(x_\lambda) \neq supp(y_\alpha)$, there exists fuzzy open set U such that $x_\lambda \in U \leq (y_\alpha)^c$ or $y_\alpha \in U \leq (x_\lambda)^c$.

Proposition 1.4.2:

A fts X is FT_o–space if and only if for any pair of fuzzy points x_λ and y_α ,with $x \neq y$, there is a fuzzy open–nhd U of x_λ such that $U /q\, y_\alpha$ or U is a fuzzy open–nhd of y_α such that $U /q\, x_\lambda$.

Proof :

Let X be a FT_o–space , and x_λ , y_α be two fuzzy points in X ,with $x \neq y$. Then $supp(x_\lambda) \neq supp(y_\alpha)$ by definition 1.1.14 .As X is FT_o–space , by definition 1.4.1 , there exists fuzzy open set U such that $x_\lambda \in U \leq (y_\alpha)^c$ or $y_\alpha \in U \leq (x_\lambda)^c$ .That is , $x_\lambda \in U \leq U$ and $U \leq (y_\alpha)^c$ or $y_\alpha \in U \leq U$ and $U \leq (x_\lambda)^c$ .Therefore , U is a fuzzy open–nhd of x_λ such that $U /q\, y_\alpha$ or U is a fuzzy open–nhd of y_α such that $U /q\, x_\lambda$ by definition 1.2.13 and Proposition 1.1.27.part 3 .

Conversely , suppose that x_λ , y_α be two fuzzy points in X , with $supp(x_\lambda) \neq supp(y_\alpha)$ .Then $x \neq y$ by definition 1.1.14 .By hypothesis , there is a fuzzy open–nhd U of x_λ such that $U /q\, y_\alpha$ or U is a fuzzy open–nhd of

y_α such that $U \not q\ (x_\lambda)$. So $x_\lambda \in U \leq U$ and $U \leq (y_\alpha)^c$ or $y_\alpha \in U \leq U$ and $U \leq (x_\lambda)^c$ by definition 1.2.13 and Proposition 1.1.27.part 3 . So $x_\lambda \in U \leq (y_\alpha)^c$ or $y_\alpha \in U \leq (x_\lambda)^c$. Therefore , X is FT_o–space by definition 1.4.1 .

The following example of FT_o–space .

Example 1.4.3:

Let $X = \{x, y\}$ be a set and $\delta = \{0_X , x_1 , 1_X\}$ be a fts on X . Then x_1 is a crisp point in X (see definition 1.1.16) and (X, δ) is FT_o–space because , for any pair of fuzzy points x_λ and y_α , with $supp(x_\lambda) \neq supp(y_\alpha)$, x_1 is a fuzzy open set such that $x_\lambda \in x_1 \leq (y_\alpha)^c$.

Definition 1.4.4:[1]

A fts X is called fuzzy T_1–space (FT_1–space, for short) if every pair of fuzzy points x_λ and y_α ,with $supp(x_\lambda) \neq supp(y_\alpha)$, there exists fuzzy open sets U and V such that $x_\lambda \in U \leq (y_\alpha)^c$ and $y_\alpha \in V \leq (x_\lambda)^c$.

It follows from the above definition that every FT_1–space is FT_o–space but the converse is not true . for ,

Example 1.4.5:

Let $X = \{a, b\}$ be a set and $T = \{0_X , a_1 , 1_X\}$ be a fts on X . Then a_1 is a crisp point in X (see definition 1.1.16) and (X,T) is FT_o–space but not FT_1–space because , for any pair of fuzzy points a_λ and b_α such that $a \neq b$, then a_1 is a fuzzy open set such that $a_\lambda \in a_1 \leq (b_\alpha)^c$ and there is no fuzzy open set containing b_α not q–coincident with a_λ .

Proposition 1.4.6:

A fts X is FT_1–space if and only if for any pair of fuzzy points x_λ and y_α ,with $x \neq y$,there exists a fuzzy open–nhds U ,V of x_λ and y_α , respectively such that $U \,/q\, y_\alpha$ and $V \,/q\, x_\lambda$.

Proof : The proof is similar to Proposition 1.4.2 .

Definition 1.4.7:

A fts X is said to be fuzzy strongly T_1–space (FST_1–space, for short) if and only if every fuzzy point x_λ is a fuzzy closed set . In [60] , Ganguly and Saha called this space a fuzzy T_1 .

Theorem 1.4.8:

Every FST_1–space is a FT_1–space .

Proof :

Let X be a FST_1–space , and x_λ , y_α be two fuzzy points in X , such that $x \neq y$, since x_λ and y_α are fuzzy closed sets in X , $(x_\lambda)^c$ and $(y_\alpha)^c$ are fuzzy open sets .Then $y_\alpha \in (x_\lambda)^c$ and $x_\lambda \in (y_\alpha)^c$ such that $(x_\lambda)^c$ /q x_λ and $(y_\alpha)^c$ /q y_α by Proposition 1.1.27.part 6 . This shows that X is FT_1–space .

The following example shows that the converse of the above Theorem may not be true .

Example 1.4.9:

Let $X = \{x, y\}$ and $\delta = \{0_X , x_1 , y_1 , 1_X\}$ be a fts on X .Then x_1 , y_1 are crisp points in X (see definition 1.1.16) and (X , δ) is a FT_1–space , since for any pair of fuzzy points x_λ and y_α , with $x \neq y$, x_1 and y_1 are fuzzy open–nhds of x_λ and y_α resp., such that x_1 /q y_α and y_1 /q x_λ . But (X,δ) is not FST_1–space because , for example , $y_{0.3}$ is a fuzzy point in X but not fuzzy closed set .

Definition 1.4.10:[1]

A fts X is called fuzzy T_2–space (FT_2–space, for short), if for every pair fuzzy points x_λ and y_α , with $supp(x_\lambda) \neq supp(y_\alpha)$, there exists fuzzy open sets U and V such that $x_\lambda \in U \leq (y_\alpha)^c$, $y_\alpha \in V \leq (x_\lambda)^c$ and U /q V .

It follows from the above definition that every FT_2–space is FT_1–space .

Proposition 1.4.11:

A fts X is FT_2–space if and only if for any pair of fuzzy points x_λ and y_α , with $x \neq y$, there exists a fuzzy open–nhds U and V of x_λ and y_α , respectively , which are not q–coincident .

Proof :

Let X is FT_2–space , and x_λ , y_α be two fuzzy points in X ,with $x \neq y$ .Then $supp(x_\lambda) \neq supp(y_\alpha)$ by definition 1.1.14 .As X is FT_2–space by definition 1.4.10 , there exists fuzzy open sets U and V such that $x_\lambda \in U \leq (y_\alpha)^c$, $y_\alpha \in V \leq (x_\lambda)^c$ and U /q V .That is , $x_\lambda \in U \leq U$, $y_\alpha \in V \leq V$ and U /q V .Therefore , U and V are fuzzy open–nhds of x_λ and y_α , respectively , which are not q–coincident by definition 1.2.13 and definition 1.1.25 .

Conversely , suppose that x_λ , y_α be two fuzzy points in X , with $supp(x_\lambda) \neq supp(y_\alpha)$ .Then $x \neq y$ by definition 1.1.14 .By hypothesis , there exists a fuzzy open–nhds U and V of x_λ and y_α , respectively , which are not q–coincident .So $x_\lambda \in U$, $y_\alpha \in V$ and U /q V by definition 1.2.13 and definition 1.1.25 .That is , $U^c \leq (x_\lambda)^c$, $V^c \leq (y_\alpha)^c$ and $U \leq V^c$ by Proposition 1.1.27 . part 3 . So $x_\lambda \in U \leq V^c \leq (y_\alpha)^c$, $y_\alpha \in V \leq U^c \leq (x_\lambda)^c$ and U /q V .Therefore , X is FT_2–space by definition 1.4.10 .

The following example of FT_2–space .

Example 1.4.12:

Consider the fts (X,δ) of example 1.4.9 . By Proposition 1.4.11 , (X,δ) is FT_2–space , since for every fuzzy points x_λ and y_α , with $x \neq y$, x_1 and y_1 are fuzzy open–nhds of x_λ and y_α resp., such that x_1 /q y_1 .

Question 1.4.13:

Does there exists a fts which is FT_1–space and it is not FT_2–space ?

Definition 1.4.14:[2 , 23]

A fts X is called fuzzy regular space (FR–space , for short) , if for each fuzzy point x_λ and each fuzzy closed set F such that $x_\lambda \in 1_X - F$, there exists , two fuzzy open sets U_1 and U_2 such that $x_\lambda \in U_1$, $F \leq U_2$ and U_1 /q U_2 .

Proposition 1.4.15 :[2 , 23]

A fts X is FR–space if and only if for each fuzzy point x_λ in X and every fuzzy open set U such that $x_\lambda \in U$, there exists fuzzy open set V such that $x_\lambda \in V \leq cl(V) \leq U$.

The following example of FR–space .

Example 1.4.16:

Let X = {x, y} be a set and the fuzzy set V in X defined as follows :

$$V(x) = 0.5 \qquad V(y) = 0.5 .$$

Let T ={0_X , V , 1_X} be a fuzzy topology on X . Clearly ,V and 1_X are fuzzy open sets and fuzzy closed sets at the same time in X .That is , cl(V) = V and $cl(1_X) = 1_X$ by Proposition 1.2.10.part 1. So , for each fuzzy point x_λ in X such that $x_\lambda \in V$, there is a fuzzy open set V in X such that $x_\lambda \in V \leq cl(V) \leq V$ and for each fuzzy point x_λ in X such that $x_\lambda \in 1_X$, there is a fuzzy open set 1_X in X such that $x_\lambda \in 1_X \leq cl(1_X) \leq 1_X$ .Hence X is FR–space by Proposition 1.4.15 .

Theorem 1.4.17:[59]

Let (X,δ) be a FR–space .Then for any fuzzy closed set F and any fuzzy point $x_\lambda \in F^c$, there are U, V $\in \delta$ such that $x_\lambda \in U$, $F \leq V$ and $cl(U) \leq V^c$.

Definition 1.4.18:[64]

A fts X is called fuzzy T_3–space (FT_3–space , for short) , if and only if it is FST_1–space and FR–space .

Theorem 1.4.19:[2]

A FT_3–space is a FT_2–space .

Converse of the above Theorem is not true as seen in the following example .

Example 1.4.20:

Let $X = \{a, b\}$ be a set and $\delta = \{0_X , a_1 , b_1 , 1_X\}$ be a fuzzy topology on X . Then a_1 , b_1 are crisp points in X (see definition 1.1.16) and (X,δ) is FT_2–space but not FT_3–space , since it is not FST_1–space .

Definition 1.4.21:[12]

A fts X is called fuzzy normal space (FN–space ,for short) if and only if for any fuzzy closed set F and fuzzy open set U such that $F\leq U$, there exists , a fuzzy open set V such that $F\leq V \leq cl(V) \leq U$.

Theorem 1.4.22:[65]

For a fts X , the following are equivalent :

(1) X is FN–space .

(2) for two fuzzy closed sets F and G with $F \,/q\, G$, there exists a fuzzy open sets U and V such that $F \leq U$, $G \leq V$ and $U \,/q\, V$.

(3) for any two fuzzy closed sets F and G with $F \,/q\, G$, there exists a fuzzy open set U such that $F \leq U$ and $cl(U) \,/q\, G$.

The following example of FN–space .

Example 1.4.23:

Consider the fts (X,T) of example 1.4.16 .Then (X,T) is FN–space , because 0_X , V and 1_X are fuzzy open sets and fuzzy closed sets at the same time in X .

Definition 1.4.24:[2]

A fts X is called fuzzy FT_4–space (FT_4–space, for short), if and only if it is FST_1–space and FN–space .

The following example of FT_3–space and FT_4–space .

Example 1.4.25:

The discrete fuzzy topology in X = [–2 , 2] is FT_3–space and FT_4–space .

Theorem 1.4.26:[2]

A FT_4–space is a FT_3–space .

Question 1.4.27:

Does there exist fts which is FT_3–space and it is not FT_4–space ?

3.4: Preservation theorems and other characterization of fuzzy γ–separation axioms

In this section ,we present preservation theorems and other characterization of fuzzy γ–separation axioms depending on these new forms of fuzzy generalized γ–functions in section 3.3 .

Theorem 3.4.1:

For a FST$_1$–space X , the following are equivalent :

(a) X is FγR–space .

(b) for any pair of fuzzy point x_λ and fuzzy closed set B of X such that $x_\lambda \in 1_X - B$, there exists a Fgγ–open sets U and V of X such that $x_\lambda \in U$, $B \leq V$ and U /q V .

(c) for each fuzzy point x_λ and each fuzzy open set B containing x_λ , there exists a Fgγ–open set U such that $cl(x_\lambda) \in U \leq \gamma\text{–}cl(U) \leq B$.

(d) for each fuzzy point x_λ and each Fg–open set B containing x_λ , there exists a fuzzy γ–open set U such that $x_\lambda \in U \leq \gamma\text{–}cl(U) \leq int(B)$.

(e) for each fuzzy point x_λ and each Fg–open set B containing x_λ , there exists a Fgγ–open set G such that $x_\lambda \in G \leq \gamma\text{–}cl(G) \leq int(B)$.

(f) for each fuzzy point x_λ and each fuzzy open set B containing x_λ , there exists a fuzzy γ–open set U such that $cl(x_\lambda) \in U \leq \gamma\text{–}cl(U) \leq B$.

Proof :

(a) $\Rightarrow$(b) : Let x_λ and B be a fuzzy point and a fuzzy closed set ,resp., in X such that $x_\lambda \in 1_X - B$. By (a) ,there exists fuzzy γ–open sets U and V such that $x_\lambda \in U$, $B \leq V$ and U /q V .Since every fuzzy γ–open set is Fgγ–open by Proposition 3.2.16. part 1 . This proves that (b) is hold .

(b) $\Rightarrow$(a) : Let x_λ and B be a fuzzy point and a fuzzy closed set ,resp., in X such that $x_\lambda \in 1_X - B$ .By (b) ,there exists two Fgγ–open sets U and V such that $x_\lambda \in U$, $B \leq V$ and U /q V .Since X is FST$_1$–space ,then x_λ is fuzzy closed in X by definition 1.4.7 .Then x_λ and B are fuzzy closed sets contained in a Fgγ–open sets U and V respectively .So $x_\lambda \in \gamma$–int(U) and B $\leq \gamma$–int(V) by definition 3.2.14 and γ–int(U) /q γ–int(V) . Now, put $M_1 = \gamma$–int(U) and $M_2 = \gamma$–int(V) , then M_1, $M_2 \in F\gamma O(X)$ by Properties 2.2.13.part 2, such that $x_\lambda \in M_1$, $B \leq M_2$ and M_1/q M_2 .This shows that X is FγR–space by definition 2.3.19 .

(b)$\Rightarrow$(c) : Let x_λ be a fuzzy point and B be a fuzzy open set in X containing x_λ .So x_λ and 1_X–B be a fuzzy point and a fuzzy closed set , resp., in X such that $x_\lambda \in 1_X-(1_X-B)$. By (b) ,there exists a Fgγ–open sets U and V such that $x_\lambda \in U$, $1_X-B \leq V$ and U /q V .It follows that $x_\lambda \in U \leq 1_X-V \leq B$. As (1_X-V) is Fgγ–closed and B is fuzzy open set , we have γ–cl(U) $\leq \gamma$–cl(1_X–V) $\leq$ B and $x_\lambda \in U \leq \gamma$–cl(U) $\leq$ B . Since X is FST$_1$–space , x_λ is fuzzy closed set in X by definition 1.4.7 .This shows that cl(x_λ)$\in U \leq \gamma$–cl(U) $\leq$ B .

(c)$\Rightarrow$(b) : Let x_λ and B be a fuzzy point and a fuzzy closed set , resp., in X such that $x_\lambda \in 1_X-B$ .So 1_X–B is fuzzy open set containing x_λ . By (c), there exists a Fgγ–open set U such that cl(x_λ) $\in U \leq \gamma$–cl(U) $\leq 1_X$–B . Put V = 1_X–(γ–cl(U)) , so V is fuzzy γ–open set and B $\leq$ V .Since every fuzzy γ–open set is Fgγ–open ,V is Fgγ–open . Also since X is FST$_1$–space , then x_λ is fuzzy closed in X by definition 1.4.7 . It follows that there exists two Fgγ–open sets U and V such that x_λ = cl(x_λ) $\in$U , B $\leq$ V and U /q V .

(c)$\Rightarrow$(e) : Let x_λ and B be a fuzzy point and a Fg–open set ,resp., in X such that $x_\lambda \in B$ .Since X is FST$_1$–space , x_λ is fuzzy closed in X by definition 1.4.7 . Since B is Fg–open containing x_λ , so $x_\lambda \in$ int(B) by definition 2.1.18 . By (c) , there exists a Fgγ–open set U such that $x_\lambda \in U \leq \gamma$–cl(U) $\leq$ int(B) .

(e)$\Rightarrow$(c) : Let x_λ and B be a fuzzy point and a fuzzy open set , resp., in X such that $x_\lambda \in B$ .Since every fuzzy open set is Fg–open by Proposition 2.1.19 , then B is Fg–open set containing x_λ . By (e) , there exists a

Fgγ–open set U such that $x_\lambda \in U \leq \gamma$–cl(U) $\leq$ int(B) $\leq$ B . Since X is FST$_1$–space , x_λ is fuzzy closed in X by definition 1.4.7 , i.e., cl(x_λ) = x_λ by Proposition 1.2.10.part 1. It follows that cl(x_λ) $\in U \leq \gamma$–cl(U) $\leq$ B .

(d)$\Rightarrow$(e) : Let x_λ and B be a fuzzy point and a Fg–open set ,resp., in X such that $x_\lambda \in B$. By (d) ,there exists a fuzzy γ–open set U such that $x_\lambda \in U \leq \gamma$–cl(U) $\leq$ int(B) .Since every fuzzy γ–open set U is Fgγ–open by Proposition 3.2.16 .part 1.This proves that (e) is hold .

(e)$\Rightarrow$(d) : Let x_λ be a fuzzy point and B be a Fg–open set in X containing x_λ . By (e) , there exists a Fgγ–open set G such that $x_\lambda \in G \leq \gamma$–cl(G) $\leq$ int(B) . Since X is FST$_1$–space , x_λ is fuzzy closed in X by definition 1.4.7 .Since G is Fgγ–open set and x_λ is fuzzy closed , then $x_\lambda \in \gamma$–int(G) by definition 3.2.14 .Put U= γ–int(G) , then U$\in$ FγO(X) by Properties 2.2.13 . part 2 and $x_\lambda \in U \leq \gamma$–cl(U) $\leq$ int(B) .

(c)$\Rightarrow$(f) : Let x_λ be any fuzzy point and B be any fuzzy open set of X containing x_λ .By (c) there exists a Fgγ–open set G such that cl(x_λ) $\in$ G $\leq$ γ–cl(G) $\leq$ B . Since G is Fgγ–open set .Then cl(x_λ) $\in$ γ–int(G) by definition 3.2.14 .Put U = γ–int(G) , then U is fuzzy γ–open set by Properties 2.2.13 . part 2 and cl(x_λ) $\in U \leq \gamma$–cl(U) $\leq$ B .

(f)$\Rightarrow$(c) : Let x_λ be any fuzzy point and B be a fuzzy open set of X such that $x_\lambda \in B$. By (f) , there exists fuzzy γ–open set U such that cl(x_λ) $\in U \leq \gamma$–cl(U) $\leq$ B . Since every fuzzy γ–open set is Fgγ–open by Proposition 3.2.16 .part 1. So U is Fgγ–open .This shows that (c) is hold .

Theorem 3.4.2:

For a fuzzy topological space X , the following are equivalent :

(a) X is FγN–space .

(b) for any pair of fuzzy closed sets A and B of X such that A /q B , there exists a Fgγ–open sets U and V of X such that A $\leq$ U , B $\leq$ V and U /q V .

(c) for each fuzzy closed sets A and each fuzzy open set B containing A ,there exists a Fgγ–open set U such that cl(A) ≤ U ≤ γ–cl(U) ≤ B .

(d) for each fuzzy closed set A and each Fg–open set B containing A , there exists a fuzzy γ–open set U such that A ≤ U ≤ γ–cl(U) ≤ int(B) .

(e) for each fuzzy closed set A and each Fg–open set B containing A , there exists a Fgγ–open set G such that A ≤ G ≤ γ–cl(G) ≤ int(B) .

(f) for each Fg–closed set A and each fuzzy open set B containing A , there exists a fuzzy γ–open set U such that cl(A) ≤ U ≤ γ–cl(U) ≤ B .

(g) for each Fg–closed set A and each fuzzy open set B containing A , there exists a Fgγ–open set G such that cl(A) ≤ G ≤ γ–cl(G) ≤ B .

Proof :

(a)⇒(b) : Let A and B be two fuzzy closed sets in X such that A /q B . Since X is FγN–space , by Theorem 2.3.26 , there exists fuzzy γ–open sets U and V such that A ≤ U , B ≤ V and U /q V .Since every fuzzy γ–open set is F gγ–open . This shows that (b) is hold .

(b) ⇒(a) : Let A and B be two fuzzy closed sets in X such that A /q B . By (b) , there exists two fuzzy gγ–open sets U and V such that A ≤ U , B ≤ V and U /q V .Since A and B are fuzzy closed sets . So A ≤ γ–int(U) and B ≤ γ–int(V) by definition 3.2.14 .part 1 and γ–int(U) /q γ–int(V) . It follows that , X is FγN–space by Theorem 2.3.26 .

(b)⇒(c) : Let A be a fuzzy closed set and B be a fuzzy open set containing A .So A and $(1_X–B)$ are two fuzzy closed sets such that A /q $(1_X–B)$. By (b) , there exists a fuzzy gγ–open sets U and V such that A ≤ U , $1_X–B$ ≤ V and U /q V . It follows that A ≤ U ≤ $(1_X–V)$ ≤ B .

As $(1_X–V)$ is F gγ–closed and B is fuzzy open set , we have

γ–cl(U) ≤ γ–cl$(1_X–V)$ ≤ B and A ≤ U ≤ γ–cl(U) ≤ B . Since A is fuzzy closed set , cl(A) ≤ U ≤ γ–cl(U) ≤ B .

(c)$\Rightarrow$(b) : Let A and B be two fuzzy closed sets such that $A \leq 1_X - B$ .So $1_X - B$ be a fuzzy open set containing A .By (c) ,there exists a Fgγ–open set U such that $A \leq U \leq \gamma\text{–cl}(U) \leq 1_X - B$ .Put $V = \gamma\text{–int}(1_X - U)$, so V is fuzzy γ–open set by Properties 2.2.13 . part 2 and $B \leq V$. Since every fuzzy γ–open set is Fgγ–open by Proposition 3.2.16 .part 1 . It follows that there exists two Fgγ–open sets U and V such that $A \leq U$, $B \leq V$ and U /q V . This shows that (b) is hold .

(c)$\Rightarrow$(e) : Let A be a fuzzy closed set of X and B be a Fg–open set such that $A \leq B$. Since B is Fg–open set and A is fuzzy closed , $A \leq \text{int}(B)$. By (c) , there exists a Fgγ–open set U such that $A \leq U \leq \gamma\text{–cl}(U) \leq \text{int}(B)$.

(e)$\Rightarrow$(c) : Let A be a fuzzy closed set and B be a fuzzy open set containing A . Since every fuzzy open set is Fg–open by Proposition 2.1.19 , then B is a Fg–open set containing A . By (e), there exists aFgγ–open set U such that $A \leq U \leq \gamma\text{–cl}(U) \leq \text{int}(B) \leq B$.

(d)$\Rightarrow$(e) : Let A be a fuzzy closed set and B be a Fg–open set containing A . By (d) , there exists a fuzzy γ–open set U such that $A \leq U \leq \gamma\text{–cl}(U) \leq \text{int}(B)$ .Since every fuzzy γ–open set is Fgγ–open .This shows that (e) is hold .

(e)$\Rightarrow$(d) :Let A be any fuzzy closed set of X and B be a Fg–open set containing A . By(e) , there exists a Fgγ–open set G such that $A \leq G \leq \gamma\text{–cl}(G) \leq \text{int}(B)$. Since G is Fgγ–open , $A \leq \gamma\text{–int}(G)$. Put $U = \gamma\text{–int}(G)$, then U is fuzzy γ–open by Properties 2.2.13 . part 2 and $A \leq U \leq \gamma\text{–cl}(U) \leq \text{int}(B)$.

(c)$\Rightarrow$(g) : Let A be any Fg–closed set of X and B be a fuzzy open set such that $A \leq B$ .Then $\text{cl}(A) \leq B$ .Therefore ,there exists a Fgγ–open set U such that $\text{cl}(A) \leq U \leq \gamma\text{–cl}(U) \leq B$.

(g)$\Rightarrow$(c): Let A be a fuzzy closed set and B be a fuzzy open set containing A . Since every fuzzy closed set is Fg–closed by Proposition 2.1.14 .part 1 , so A is Fg–closed set containing B . By (g) , there exists a Fgγ–open set U such that $cl(A) \leq U \leq \gamma$–$cl(U) \leq B$.

(g)$\Rightarrow$(f) : Let A be any Fg–closed set of X and B a fuzzy open containing A .By (g) , there exists a Fgγ–open set G such that $cl(A) \leq G \leq \gamma$–$cl(G) \leq B$. Since G is Fgγ–open and $cl(A) \leq G$,we have $cl(A) \leq \gamma$–int(G) .Put U = γ–int(G) , then U is fuzzy γ–open set by Properties 2.2.13 . part 2 and cl(A) $\leq U \leq \gamma$–$cl(U) \leq B$.

(f)$\Rightarrow$(g) : Let A be a Fg–closed set and B be a fuzzy open set such that $A \leq B$. By(f) ,there exists a fuzzy γ–open set U such that $cl(A) \leq U \leq \gamma$–$cl(U) \leq B$. Since every fuzzy γ–open set is Fgγ–open set by Proposition 3.2.16.part 1 .This shows that (g) is hold .

Theorem 3.4.3:

Let $f : X \to Y$ be a Fgγ–open bijection function from a FT$_2$–space X into a FST$_1$–space Y . Then Y is FγT$_2$–space .

Proof :

Let x_λ and y_α are fuzzy points in Y such that $supp(x_\lambda) \neq supp(y_\alpha)$. Since f is bijective function .By Remark 1.1.22 .part 4 , $f^{-1}(x_\lambda)$ and $f^{-1}(y_\alpha)$ are fuzzy points in X such that $supp(f^{-1}(x_\lambda)) \neq supp(f^{-1}(y_\alpha))$. As X is FT$_2$– space , so by definition 1.4.10 , there exist fuzzy open sets U and V in X such that $f^{-1}(x_\lambda) \in U \leq (f^{-1}(y_\alpha))^c$, $f^{-1}(y_\alpha) \in V \leq (f^{-1}(x_\lambda))^c$ and U /q V . Since f is Fgγ–open function , f(U) and f(V) are Fgγ–open sets in Y by definition 3.3.1 . Again , since f is bijective ,we have $x_\lambda = f(f^{-1}(x_\lambda)) \in f(U) \leq f(f^{-1}(y_\alpha)^c) = (y_\alpha)^c$, $y_\alpha = f(f^{-1}(y_\alpha)) \in f(V) \leq f(f^{-1}(x_\lambda)^c) = (x_\lambda)^c$ and f(U) /q f(V) by Theorem 1.1.23 . Since Y is FST$_1$–space , by definition 1.4.7 , x_λ and y_α are fuzzy closed sets in Y contained in a Fgγ–open sets f(U) and f(V) resp., so $x_\lambda \in \gamma$–$int(f(U)) \leq (y_\alpha)^c$, $y_\alpha \in \gamma$–$int(f(V)) \leq (x_\lambda)^c$ by definition 3.2.14 and γ–int f(U) /q γ–int f(V) .Now, put $M_1 = \gamma$–int f(U) and $M_2 = \gamma$–int f(V) .Then M_1 , $M_2 \in F\gamma O(Y)$ by Properties 2.2.13 . part 2 such that $x_\lambda \in M_1 \leq (y_\alpha)^c$, $y_\alpha \in M_2 \leq (x_\lambda)^c$ and M_1 /q M_2 .Therefore , the space Y is $F\gamma T_2$–space by definition 2.3.14 .

Theorem 3.4.4:

Let f : X → Y be a Fgγ–closed bijection function from a FT_2–space X into a FST_1–space Y . Then Y is $F\gamma T_2$–space .

Proof : It follows from Lemma 3.3.2.part 2 ,and Theorem 3.4.3 .

Corollary 3.4.5:

Let f : X → Y be a Fγg–open bijection function from a FT_2–space X into a FST_1–space Y . Then Y is $F\gamma T_2$–space .

Proof :

The proof follows from Theorem 3.4.3 and Proposition 3.3.3 . part 1.

Corollary 3.4.6:

Let f : X → Y be a Fγg–closed bijection function from a FT_2–space X into a FST_1–space Y .Then Y is $F\gamma T_2$–space .

Proof :

It follows from Theorem 3.4.4 and Proposition 3.3.3 . part 1.

Theorem 3.4.7:

Let f : X → Y be a F q γ–open bijection function from a $F\gamma T_i$–space X into a fts Y . Then Y is FT_i–space , for i = 0 , 1, 2 .

Proof : we prove only i = 1 .

Let x_λ and y_α are fuzzy points in Y such that $supp(x_\lambda) \neq supp(y_\alpha)$. Since f is bijective function .By Remark 1.1.22 .part 4 , $f^{-1}(x_\lambda)$ and $f^{-1}(y_\alpha)$ are fuzzy points in X such that $supp(f^{-1}(x_\lambda)) \neq supp(f^{-1}(y_\alpha))$. As X is

$F\gamma T_1$–space , so by definition 2.3.6 , there exists fuzzy γ–open sets U and V in X such that $f^{-1}(x_\lambda)\in U\leq (f^{-1}(y_\alpha))^c$ and $f^{-1}(y_\alpha)\in V\leq(f^{-1}(x_\lambda))^c$ .Since f is q γ–open function ,we have f(U) and f(V) are fuzzy open sets in Y by definition 3.3.1 .Since f is surjective , $x_\lambda = f(f^{-1}(x_\lambda)) \in f(U) \leq f(f^{-1}(y_\alpha)^c) = (y_\alpha)^c$ and $y_\alpha = f(f^{-1}(y_\alpha)) \in f(V) \leq f(f^{-1}(x_\lambda)^c) = (x_\lambda)^c$ by Theorem 1.1.23. This proves that Y is FT_1–space by definition 1.4.4 .

Other cases can be proved in a similar manner .

Theorem 3.4.8:

Let $f : X \rightarrow Y$ be a F q γ–closed bijection function from a $F\gamma T_i$–space X into a fts Y . Then Y is FT_i–space , for i = 0 , 1, 2 .

Proof : It follows from Lemma 3.3.2.part 4 ,and Theorem 3.4.7 .

Theorem 3.4.9:

Let $f : X \rightarrow Y$ be Fq γ–closed bijection function from a FST_1–space X into a fts Y . Then Y is a FST_1–space .

Proof :

Suppose that X is FST_1–space and x_λ any fuzzy point in Y .Since f is bijective function . By Remark 1.1.22 .part 2 , $f^{-1}(x_\lambda)$ is fuzzy point in X . According to the assumption , $f^{-1}(x_\lambda)$ is fuzzy closed in X .But every fuzzy closed set is fuzzy γ–closed , so $f^{-1}(x_\lambda)$ is fuzzy γ–closed in X . Since f is surjective Fq γ–closed , then $x_\lambda = f(f^{-1}(x_\lambda))$ is fuzzy closed in Y by Theorem 1.1.23 .part 7 and definition 3.3.1 .Therefore ,Y is FST_1–space by definition 1.4.7 .

Theorem 3.4.10:

Let $f : X \rightarrow Y$ be a Fq γ–open bijection function from a FST_1–space X into a fts Y . Then Y is a FST_1–space .

Proof : It follows from Lemma 3.3.2.part 4 , and Theorem 3.4.9 .

Theorem 3.4.11:

Let $f : X \to Y$ be a $F\gamma$–$g\gamma$–open bijection function from a $F\gamma T_2$–space X into a FST_1–space Y . Then Y is $F\gamma T_2$–space .

Proof :

Let X is $F\gamma T_2$–space and let x_λ and y_α are fuzzy points in Y such that $supp(x_\lambda) \neq supp(y_\alpha)$. Since f is bijective function .By Remark 1.1.22 . part 4 , $f^{-1}(x_\lambda)$ and $f^{-1}(y_\alpha)$ are fuzzy points in X such that $supp(f^{-1}(x_\lambda)) \neq supp(f^{-1}(y_\alpha))$. As X is $F\gamma T_2$–space , so by definition 2.3.14 , there are U_1 , $U_2 \in F\gamma O(X)$ such that $f^{-1}(x_\lambda) \in U_1 \leq (f^{-1}(y_\alpha))^c$, $f^{-1}(y_\alpha) \in U_2 \leq (f^{-1}(x_\lambda))^c$ and U_1 /q U_2 . Again , since f is bijective $F\gamma$–$g\gamma$–open ,we have $f(U_1)$ and $f(U_2)$ are $Fg\gamma$–open sets in Y such that $x_\lambda = f(f^{-1}(x_\lambda)) \in f(U_1) \leq f(f^{-1}(y_\alpha)^c) = (y_\alpha)^c$,$y_\alpha = f(f^{-1}(y_\alpha)) \in f(U_2) \leq f(f^{-1}(x_\lambda)^c) = (x_\lambda)^c$ and $f(U_1)$ /q $f(U_2)$ by Theorem 1.1.23 and definition 3.3.1 . Since Y is FST_1–space , so by definition 1.4.7, x_λ and y_α are fuzzy closed sets in Y contained in a $Fg\gamma$–open sets $f(U_1)$ and $f(U_2)$ respectively .Then $x_\lambda \in \gamma$–$int(f(U_1)) \leq (y_\alpha)^c$, $y_\alpha \in \gamma$–$int(f(U_2)) \leq (x_\lambda)^c$ by definition 3.2.14 and γ–$int(f(U_1))$ /q γ–$int(f(U_2))$. Now, put $N_i = \gamma$–$int(f(U_i))$,then $N_i \in F\gamma O(Y)$ for i = 1 , 2 by Properties 2.2.13.part 2. So $x_\lambda \in N_1 \leq (y_\alpha)^c$, $y_\alpha \in N_2 \leq (x_\lambda)^c$ and N_1 /q N_2 .Therefore ,Y is $F\gamma T_2$–space by definition 2.3.14 .

Theorem 3.4.12:

Let $f : X \to Y$ be $F\gamma$–$g\gamma$–closed bijection function from a $F\gamma T_2$–space X into a FST_1–space Y . Then Y is $F\gamma T_2$–space .

Proof : It follows that from Lemma 3.3.2.part 5 ,and Theorem 3.4.11 .

Corollary 3.4.13:

Let $f : X \to Y$ be $F\gamma$–γg–open bijection function from a $F\gamma T_2$–space X into a FST_1–space Y . Then Y is $F\gamma T_2$–space .

Proof : It follows from Theorem 3.4.11 and Proposition 3.3.3 .part 4 .

Corollary 3.4.14:

Let f : X → Y be a Fγ–γg–closed bijection function from a $F\gamma T_2$–space X into a FST_1–space Y . Then Y is $F\gamma T_2$–space .

Proof : It follows from Theorem 3.4.12 and Proposition 3.3.3 .part 4 .

Theorem 3.4.15:

Let f : X → Y be a Falmost gγ–open bijection function from a FT_2–space X into a FST_1–space Y . Then Y is $F\gamma T_2$–space .

Proof :

Let X is FT_2–space and let x_λ and y_α are fuzzy points in Y such that $supp(x_\lambda) \neq supp(y_\alpha)$. Since f is bijective function ,so by Remark 1.1.22 . part 4 , $f^{-1}(x_\lambda)$ and $f^{-1}(y_\alpha)$ are fuzzy points in X such that $supp(f^{-1}(x_\lambda)) \neq supp(f^{-1}(y_\alpha))$ .As X is FT_2–space . By definition 1.4.10 , there are fuzzy open sets U_1 and U_2 such that $f^{-1}(x_\lambda) \in U_1$, $f^{-1}(y_\alpha) \in U_2$ and U_1 /q U_2 . Put Gi =int(cl(Ui)) , where i = 1 , 2 .Then Gi ∈ FRO(X) by definition 2.1.1.part 7 , $f^{-1}(x_\lambda) \in G_1$, $f^{-1}(y_\alpha) \in G_2$ and G_1 /q G_2 . Since f is Falmost gγ–open ,then $f(G_1)$ and $f(G_2)$ are Fgγ–open sets in Y . Again , since f is bijective , $x_\lambda \in f(G_1)$, $y_\alpha \in f(G_2)$ and $f(G_1)$ /q $f(G_2)$ by Theorem 1.1.23 . Since Y is FST_1–space .Then , by definition 1.4.7, x_λ and y_α are fuzzy closed sets contained in a Fgγ–open sets $f(G_1)$ and $f(G_2)$ respectively .Now , put Vi = γ–int$(f(G_i))$ for i = 1 , 2 .Then Vi ∈ FγO(Y) , $x_\lambda \in V_1$, $y_\alpha \in V_2$ and V_1 /q V_2 by Properties 2.2.13. part 2 and definition 3.2.14 .part 1 . Hence , we have Y is $F\gamma T_2$–space by definition 2.3.14 .

Theorem 3.4.16:

Let f : X → Y be a Falmost gγ–closed bijection function from a FT_2–space X into a FST_1–space Y . Then Y is $F\gamma T_2$–space .

Proof : It follows from Lemma 3.3.2.part 6 ,and Theorem 3.4.15 .

Theorem 3.4.17:

Let $f : X \to Y$ be a fuzzy γ–continuous an injection function from a fts X into a FT_i–space Y . Then X is $F\gamma T_i$–space , for $i = 0 , 1 , 2$.

Proof : we prove only $i = 0$.

Suppose that Y is FT_0–space , x_λ and y_α are fuzzy points in X such that $supp(x_\lambda) \neq supp(y_\alpha)$ .Since f is an injective function , so by Remark 1.1.22 . part 3 , $f(x_\lambda)$ and $f(y_\alpha)$ are fuzzy points in Y such that $supp(f(x_\lambda)) \neq supp(f(y_\alpha))$. According to the assumption there is a fuzzy open set U such that $f(x_\lambda) \in U \leq (f(y_\alpha))^c$ or $f(y_\alpha) \in U \leq (f(x_\lambda))^c$ .Suppose there is a fuzzy open set U such that $f(x_\lambda) \in U \leq (f(y_\alpha))^c$ .Since f is a fuzzy γ–continuous function , $f^{-1}(U)$ is fuzzy γ–open set in X by Theorem 3.3.8 .part 1 . Again , since f is an injective function , $x_\lambda = f^{-1}(f(x_\lambda)) \in f^{-1}(U) \leq f^{-1}(f(y_\alpha)^c) = (y_\alpha)^c$ by Theorem 1.1.23. part 6 .This shows that X is $F\gamma T_0$–space by definition 2.3.1 .

Other cases can be proved in a similar manner .

Theorem 3.4.18 :

Let $f : X \to Y$ be a fuzzy γ–irresolute an injection function from a fts X into a $F\gamma T_i$–space Y . Then X is $F\gamma T_i$–space , for $i = 0 , 1 , 2$.

Proof : we prove only $i = 1$.

Assume that Y is $F\gamma T_1$–space , x_λ and y_α are fuzzy points in X such that $supp(x_\lambda) \neq supp(y_\alpha)$ .Since f is an injective function ,so by Remark 1.1.22 . part 3 , $f(x_\lambda)$ and $f(y_\alpha)$ are fuzzy points in Y such that $supp(f(x_\lambda)) \neq supp(f(y_\alpha))$. As Y is $F\gamma T_1$–space . By definition 2.3.6 , there are U ,V $\in F\gamma O(Y)$ such that $f(x_\lambda) \in U \leq (f(y_\alpha))^c$ and $f(y_\alpha) \in V \leq (f(x_\lambda))^c$. Since f is fuzzy γ–irresolute , $f^{-1}(U)$ and $f^{-1}(V)$ are fuzzy γ–open sets in X by Theorem 3.3.8 .part 6 . Again , since f is an injective function ,we have $x_\lambda = f^{-1}(f(x_\lambda)) \in f^{-1}(U) \leq f^{-1}(f(y_\alpha)^c) = (y_\alpha)^c$ and $y_\alpha = f^{-1}(f(y_\alpha)) \in f^{-1}(V) \leq$

$f^{-1}(f(x_\lambda)^c) = (x_\lambda)^c$ by Theorem 1.1.23. part 6 .This proves that X is $F\gamma T_1$–space by definition 2.3.5 .

Other cases can be proved in the same manner .

Theorem 3.4.19:

Let $f : X \to Y$ be a fuzzy γ–irresolute an injection function from a fts X into a FT_i–space Y . Then X is $F\gamma T_i$–space , for $i = 0 , 1 , 2$.

Proof :

It follows from Theorem 3.4.18 and since every FT_i–space is $F\gamma T_i$–space ,where $i = 0 , 1, 2$.

Theorem 3.4.20:

Let $f : X \to Y$ be a fuzzy γ–gγ–continuous an injection function from a FST_1–space X into a $F\gamma T_2$–space Y . Then X is $F\gamma T_2$–space .

Proof :

Suppose that Y is $F\gamma T_2$–space ,and let x_λ and y_α are fuzzy points in X such that $supp(x_\lambda) \neq supp(y_\alpha)$. Since f is an injective function .By Remark 1.1.22 .part 3 , $f(x_\lambda)$ and $f(y_\alpha)$ are fuzzy points in Y such that $supp(f(x_\lambda)) \neq supp(f(y_\alpha))$. According to the assumption there are U , V $\in F\gamma O(Y)$ such that $f(x_\lambda) \in U \leq (f(y_\alpha))^c$, $f(y_\alpha) \in V \leq (f(x_\lambda))^c$ and U /q V . Since f is fuzzy γ–gγ–continuous function , $f^{-1}(U)$ and $f^{-1}(V)$ are fuzzy gγ–open sets in X by Theorem 3.3.8 .part 5 . Again , since f is an injective function , $x_\lambda = f^{-1}(f(x_\lambda)) \in f^{-1}(U) \leq f^{-1}(f(y_\alpha)^c) = (y_\alpha)^c$, $y_\alpha = f^{-1}(f(y_\alpha)) \in f^{-1}(V) \leq f^{-1}(f(x_\lambda)^c) = (x_\lambda)^c$ and $f^{-1}(U)$ /q $f^{-1}(V)$ by Theorem 1.1.23 . As X is FST_1–space , so by definition 1.4.7, x_λ and y_α are fuzzy closed sets in X contained in a Fgγ–open sets $f^{-1}(U)$ and $f^{-1}(V)$ respectively . So $x_\lambda \in \gamma\text{–}int(f^{-1}(U)) \leq (y_\alpha)^c$, $y_\alpha \in \gamma\text{–}int(f^{-1}(V)) \leq (x_\lambda)^c$ by definition 3.2.14 and $\gamma\text{–}int(f^{-1}(U))$ /q $\gamma\text{–}int(f^{-1}(V))$ .Now , put $N_1 = \gamma\text{–}int(f^{-1}(U))$ and $N_2 = \gamma\text{–}int(f^{-1}(V))$, then $N_1, N_2 \in F\gamma O(X)$ by Properties 2.2.13.part 2 . So x_λ

$\in N_1 \leq (y_\alpha)^c$, $y_\alpha \in N_2 \leq (x_\lambda)^c$ and N_1 /q N_2 .It follows that X is FγT$_2$–space by definition 2.3.14.

Corollary 3.4.21:

Let f : X $\rightarrow$ Y be a fuzzy γ–γg–continuous an injection function from a FST$_1$–space X into a FγT$_2$–space Y . Then X is FγT$_2$–space .

Proof : It follows from Theorem 3.4.20 and Proposition 3.3.12 . part 5.

Theorem 3.4.22:

Let f : X $\rightarrow$ Y be a fuzzy gγ–continuous an injection function from a FST$_1$–space X into a FT$_2$–space Y . Then X is FγT$_2$–space .

Proof :

Let Y be a FT$_2$–space , and let x_λ and y_α be two fuzzy points of X such that supp(x_λ) $\neq$ supp(y_α) . Since f is an injective function , so by Remark 1.1.22 .part 3 , f(x_λ) and f(y_α) are two fuzzy points in Y with supp(f(x_λ)) $\neq$ supp(f(y_α)) .As Y is FT$_2$–space .By definition 1.4.10, there are fuzzy open sets N_1 and N_2 of Y such that :

$f(x_\lambda) \in N_1 \leq (f(y_\alpha))^c$, $f(y_\alpha) \in N_2 \leq (f(x_\lambda))^c$ and N_1 /q N_2 .Since f is fuzzy gγ–continuous function , then $f^{-1}(N_1)$ and $f^{-1}(N_2)$ are fuzzy gγ–open sets of X by Theorem 3.3.8 .part 2 . Again , since f is an injective function , $x_\lambda = f^{-1}(f(x_\lambda)) \in f^{-1}(N_1) \leq f^{-1}(f(y_\alpha)^c) = (y_\alpha)^c$, $y_\alpha = f^{-1}(f(y_\alpha)) \in f^{-1}(N_2) \leq f^{-1}(f(x_\lambda)^c) = (x_\lambda)^c$ and $f^{-1}(N_1)$ /q $f^{-1}(N_2)$ by Theorem 1.1.23 . As X is FST$_1$–space ,then , by definition 1.4.7 , x_λ and y_α are fuzzy closed sets in X contained in a Fgγ–open sets $f^{-1}(N_1)$ and $f^{-1}(N_2)$ respectively .So $x_\lambda \in \gamma\text{–int}(f^{-1}(N_1)) \leq (y_\alpha)^c$, $y_\alpha \in \gamma\text{–int}(f^{-1}(N_2)) \leq (x_\lambda)^c$ by definition 3.2.14 and γ–int($f^{-1}(N_1)$) /q γ–int($f^{-1}(N_2)$) .Now, put $G_i = \gamma\text{–int}(f^{-1}(N_i))$ for i = 1, 2 .Then G_1 , G_2 $\in$FγO(X) by Properties 2.2.13.part 2 , $x_\lambda \in G_1 \leq (y_\alpha)^c$, $y_\alpha \in G_2 \leq (x_\lambda)^c$ and G_1 /q G_2 .Therefore , X is FγT$_2$–space by definition 2.3.14 .

Corollary 3.4.23:

Let f : X → Y be a fuzzy γg–continuous an injection function from a FST_1–space X into a FT_2–space Y . Then X is $F\gamma T_2$–space .

$F\gamma T_2$–space .

Proof : It is obvious from Theorem 3.4.22 and Proposition 3.3.12. part 2 .

Theorem 3.4.24:

Let f : X → Y be a fuzzy continuous F q γ–closed bijection function from a FγR–space X into a fts Y . Then Y is FR–space .

Proof :

Let X is FγR–space ,and let x_λ and G be a fuzzy point and a fuzzy closed set , resp., in Y such that $x_\lambda \in 1_Y - G$ .Since f is fuzzy continuous bijective , then $f^{-1}(x_\lambda)$ is fuzzy point in X by Remark 1.1.22.part 2 and $f^{-1}(G)$ is fuzzy closed set of X by Theorem 1.3. 4 . part 3 . such that $f^{-1}(x_\lambda) \in 1_X - f^{-1}(G)$ by Theorem 1.1.23 .part 2 and part 1. As X is FγR–space , so by definition 2.3.14 , there exist fuzzy γ–open sets U_1 and U_2 in X such that $f^{-1}(x_\lambda) \in U_1$, $f^{-1}(G) \leq U_2$ and $U_1 /q\ U_2$. Put $V_i = 1_Y - f(1_X - U_i)$ for i = 1 , 2 , then V_1 and V_2 are fuzzy open sets of Y by definition 3.3.1. part 2 . So $x_\lambda \in V_1$, $G \leq V_2$ and $f^{-1}(V_i) = U_i$ where i = 1 , 2 . Since $U_1 /q\ U_2$ and f is bijective , we have $V_1 /q\ V_2$ .Therefore , Y is FR–space by definition 1.4.14 .

Theorem 3.4.25:

Let f : X → Y be a fuzzy continuous F q γ–open bijection function from a FγR–space X into a fts Y . Then Y is FR–space .

Proof : It follows from Lemma 3.3.2.part 4 , and Theorem 3.4.24 .

From Theorem 3.4.24 (Theorem 3.4.25) and since every FR–space is FγR–space , the following corollary is true :

Corollary 3.4.26:

Let $f : X \rightarrow Y$ be a fuzzy continuous F qγ–closed(open) bijection from a FγR–space (resp., FR–space) X into a fts Y . Then Y is FγR–space .

Theorem 3.4.27:

Let $f : X \rightarrow Y$ be a fuzzy continuous F q γ–closed bijection function from a $F\gamma T_3$–space X into a fts Y . Then Y is FT_3–space .

Proof : It follows from Theorem 3.4.9 and Theorem 3.4.24 .

Theorem 3.4.28:

Let $f : X \rightarrow Y$ be a fuzzy continuous F q γ–open bijection function from a $F\gamma T_3$–space X into a fts Y . Then Y is FT_3–space .

Proof : It follows from Theorem 3.4.10 and Theorem 3.4.25 .

Corollary 3.4.29:

Let $f : X \rightarrow Y$ be a fuzzy continuous F q γ–closed(open) bijection from a $F\gamma T_3$–space (resp., FT_3–space)X into a fts Y .Then Y is $F\gamma T_3$–space .

Proof :

It follows from Theorem 3.4.27(Theorem 3.4.28) and since every FT_3–space is $F\gamma T_3$–space .

Lemma 3.4.30:

A function $f : X \rightarrow Y$ is Fγ–gγ–closed if and only if for each fuzzy set B of Y and each $U \in F\gamma O(X)$ containing $f^{-1}(B)$,there exists a Fgγ–open set V of Y such that $B \leq V$ and $f^{-1}(V) \leq U$.

Proof : The proof is similar to the proof of Lemma 3.1.17 .

Lemma 3.4.31:

If $f : X \rightarrow Y$ is Fγ–gγ–closed ,then for each fuzzy closed set M of Y and each $U \in F\gamma O(X)$ containing $f^{-1}(M)$, there exists a fuzzy γ–open set V of Y such that $M \leq V$ and $f^{-1}(V) \leq U$.

Proof :

Let M be any fuzzy closed set of Y and U a fuzzy γ–open in X containing $f^{-1}(M)$ .By Lemma 3.4.30 , there exists a Fgγ–open set N of Y such that $M \leq N$ and $f^{-1}(N) \leq U$ .Since M is fuzzy closed and N is Fgγ–open ,then $M \leq \gamma\text{–int}(N)$ and $f^{-1}(\gamma\text{–int}(N)) \leq f^{-1}(N) \leq U$ .Put $V = \gamma\text{–int}(N)$ .Then V is fuzzy γ–open set of Y such that $M \leq V$ and $f^{-1}(V) \leq U$.

Theorem 3.4.32:

Let $f : X \rightarrow Y$ be a fuzzy continuous Fγ–gγ–closed bijection function from a FγR –space X into a FST_1–space Y . Then Y is FγR–space .

Proof :

Let X be a FγR–space and let x_λ and N be a fuzzy point and a fuzzy closed set , resp., in Y such that $x_\lambda \in 1_Y – N$ .Since f is fuzzy continuous bijective ,then $f^{-1}(x_\lambda)$ is fuzzy point in X by Remark 1.1.22 .part 2 and $f^{-1}(N)$ is fuzzy closed set of X by Theorem 1.3.4 . part 3 such that $f^{-1}(x_\lambda) \in 1_X – f^{-1}(N)$ by Theorem 1.1.23. part 2 and part 1 . By fuzzy γ regularity of X , there exists a fuzzy γ–open sets U_1 and U_2 of X such that $f^{-1}(x_\lambda) \in U_1$, $f^{-1}(N) \leq U_2$ and $U_1 /q\ U_2$. Since Y is FST_1–space , then x_λ is fuzzy closed in Y by definition 1.4.7 . By Lemma 3.4.31 , there exist V_1 , $V_2 \in F\gamma O(Y)$ such that $x_\lambda \in V_1$, $N \leq V_2$, $f^{-1}(V_1) \leq U_1$ and $f^{-1}(V_2) \leq U_2$. Since f is surjective and $U_1 /q\ U_2$, we have $f^{-1}(V_1) \leq U_1 \leq (U_2)^c \leq f^{-1}(V_2)^c$ and $V_1 = f(f^{-1}(V_1)) \leq f(U_1) \leq f(U_2)^c \leq f(f^{-1}(V_2)^c) = (V_2)^c$. So $V_1 /q\ V_2$ .This proves that Y is FγR–space by definition 2.3.19 .

Corollary 3.4.33:

Let f : X →Y be a fuzzy continuous Fγ–γg–closed bijection function from a FγR–space X into a FST_1–space Y . Then Y is FγR–space .

Proof :

This is an immediate consequence from Theorem 3.4.32 and Proposition 3.3.3. part 4 .

Corollary 3.4.34:

Let f : X →Y be a fuzzy continuous Fγ–gγ–closed bijection function from a FR–space X into a FST_1–space Y . Then Y is FγR–space .

Proof :

It follows from Theorem 3.4.32 and since every FR–space is FγR–space .

Corollary 3.4.35:

Let f : X →Y be a fuzzy continuous Fγ–γg–closed bijection function from a FR–space X into a FST_1–space Y . Then Y is FγR–space .

Proof :

It follows from Corollary 3.4.33 and since every FR–space is FγR–space .

Theorem 3.4.36:

Let f : X →Y be a fuzzy continuous Fγ–gγ–open(resp., Fγ–γg–open) bijection function from a FγR–space X into a FST_1–space Y .Then Y is FγR–space .

Proof :

Directly from Theorem 3.4.32 (resp., Corollary 3.4.33) and Lemma 3.3.2 . part 5 (resp., Lemma 3.3.2 . part 3) .

From Theorem 3.4.36 and Since every FR–space is FγR–space , the following Corollary is true :

Corollary 3.4.37:

Let f : X →Y be a fuzzy continuous Fγ–gγ–open(resp., Fγ–γg–open) bijection function from a FR–space X into a FST_1–space Y .Then Y is FγR–space .

Theorem 3.4.38:

Let f : X →Y be a fuzzy closed fuzzy gγ–continuous bijection function from a FST_1–space X into a FR–space Y . Then X is FγR–space .

Proof :

Suppose that Y is FR–space , and x_λ be any fuzzy point in X and M be any fuzzy closed set of X such that $x_\lambda \in 1_X - M$. Since f is fuzzy closed bijective , then $f(x_\lambda)$ is a fuzzy point in Y by Remark 1.1.22.part 1 and f(M) is a fuzzy closed set in Y by definition 1.3.2 such that $f(x_\lambda) \in 1_Y - f(M)$ by Theorem 1.1.23 . part 3 and part 8 . By the fuzzy regularity of Y , there exists fuzzy open sets V_1 and V_2 in Y such that $f(x_\lambda) \in V_1$, $f(M) \leq V_2$ and V_1 /q V_2 . Since f is fuzzy gγ–continuous ,then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are Fgγ–open sets of X by Theorem 3.3.8 .part 2 .By Theorem 1.1.23 , we have $x_\lambda \in f^{-1}(V_1)$, $M \leq f^{-1}(V_2)$ and $f^{-1}(V_1)$ /q $f^{-1}(V_2)$. Since X is FST_1–space , so by Theorem 3.4.1 , X is FγR–space .

Corollary 3.4.39:

Let f : X →Y be a fuzzy closed fuzzy γ–gγ–continuous bijection function from a FST_1–space X into a FR–space Y . Then X is FγR–space .

Proof : Directly from Theorem 3.4.38 and Proposition 3.3.12 . part 7 .

Corollary 3.4.40:

Let f : X →Y be a fuzzy closed fuzzy γ –γg–continuous bijection function from a FST_1–space X into a FR–space Y . Then X is FγR–space .

Proof : It follows from Corollary 3.4.39 and Proposition 3.3.12 . part 5 .

Corollary 3.4.41

Let f : X →Y be a fuzzy closed fuzzy γg–continuous bijection function from a FST_1–space X into a FR–space Y . Then X is FγR–space .

Proof : It follows from Theorem 3.4.38 and Proposition 3.3.12 . part 2 .

Theorem 3.4.42:

Let f : X →Y be a fuzzy closed fuzzy γ–continuous bijection function from a fts X into a FR–space Y . Then X is FγR–space .

Proof :

Assume that Y is FR–space , and x_λ and W be a fuzzy point and a fuzzy closed set , resp., in X such that $x_\lambda \in 1_X - W$. Since f is fuzzy closed bijection , then $f(x_\lambda)$ and f(W) be a fuzzy point and a fuzzy closed set , resp., in Y such that $f(x_\lambda) \in 1_Y - f(W)$ by Remark 1.1.22.part 1 , definition 1.3.2 and Theorem 1.1.23 . part 3 and part 8. By the fuzzy regularity of Y , there are fuzzy open sets U_1 and U_2 such that $f(x_\lambda) \in U_1$, $f(W) \le U_2$ and U_1 /q U_2 . Since f is fuzzy γ–continuous , $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are fuzzy γ–open sets in X by Theorem 3.3.8 .part 1. Since f is an injective ,again by Theorem 1.1.23 .part 6 ,we have :

$x_\lambda = f^{-1}(f(x_\lambda)) \in f^{-1}(U_1)$, $W = f^{-1}(f(W)) \le f^{-1}(U_2)$ and $f^{-1}(U_1)$ /q $f^{-1}(U_2)$. Therefore , X is FγR–space by definition 2.3.19 .

Corollary 3.4.43:

Let f : X →Y be a fuzzy closed fuzzy continuous(resp., fuzzy α–continuous ,pre continuous ,semi continuous) bijection function from a fts X into a FR–space Y . Then X is FγR–space .

Proof : It follows from Theorem 3.4.42 , Remark 2.1.12 and Remark 3.3.9 .

Theorem 3.4.44:

Let f : X $\rightarrow$Y be a fuzzy closed fuzzy γ–gγ–continuous bijection function from a FST$_1$–space X into a FγR–space Y. Then X is FγR–space .

Proof :

Let Y is FγR–space ,and let x_λ be any fuzzy point and H be any fuzzy closed set in X such that $x_\lambda \in 1_X -$ H .Since f is fuzzy closed bijective , then f(x_λ) be a fuzzy point and f(H) be a fuzzy closed set in Y such that f(x_λ)$\in 1_Y$–f(H) by Remark 1.1.22 . part 1, definition 1.3.2 and Theorem 1.1.23 . part 3 and part 8 . By the fuzzy γ regularity of Y ,there exist fuzzy γ–open sets N_1 and N_2 in Y such that f(x_λ)$\in$ N_1 , f(H) $\leq$ N_2 and N_1 /q N_2 .Since f is fuzzy γ–gγ–continuous ,then $f^{-1}(N_1)$ and $f^{-1}(N_2)$ are Fgγ–open sets in X by Theorem 3.3.8 . part 5 such that $x_\lambda \in f^{-1}(N_1)$, H $\leq$ $f^{-1}(N_2)$ and $f^{-1}(N_1)$ /q $f^{-1}(N_2)$. Since X is FST$_1$–space . By Theorem 3.4.1 , X is FγR–space .

Corollary 3.4.45:

Let f : X $\rightarrow$Y be a fuzzy closed fuzzy γ–γg–continuous bijection function from a FST$_1$–space X into a FγR–space Y. Then X is FγR–space .

Proof : It is clear from Theorem 3.4.44 and Proposition 3.3.12 . part 5 .

Theorem 3.4.46:

Let f : X $\rightarrow$Y be a fuzzy closed fuzzy γ–irresolute bijection function from a fts X into a FγR–space Y . Then X is FγR–space .

Proof :

Let Y is FγR–space ,and let x_λ and G be a fuzzy point and a fuzzy closed set , resp., in X such that $x_\lambda \in 1_X$–G .Since f is fuzzy closed bijective ,then f(x_λ) is a fuzzy point and f(G) is a fuzzy closed set in Y , such that f(x_λ)$\in 1_Y$–f(G) by Remark 1.1.22 . part 1, definition 1.3.2 and Theorem 1.1.23 . part 3 and part 8. By the fuzzy γ regularity of Y ,there are U_1 , U_2 $\in$

FγO(Y) such that $f(x_\lambda) \in U_1$, $f(G) \leq U_2$ and U_1 /q U_2 .Since f is fuzzy γ–irresolute , then $f^{-1}(U_1)$, $f^{-1}(U_2)$ ∈FγO(X) by Theorem 3.3.8 .part 6 . Since f is an injective function , $x_\lambda = f^{-1}(f(x_\lambda)) \in f^{-1}(U_1)$, $G = f^{-1}(f(G)) \leq f^{-1}(U_2)$ and $f^{-1}(U_1)$ /q $f^{-1}(U_2)$ by Theorem 1.1.23 .part 6 and part 1 .This proves that X is FγR–space by definition 2.3.19 .

From Theorem 3.4.46 and since every FR–space is FγR–space , the following Corollary is true :

Corollary 3.4.47:

Let f : X →Y be a fuzzy closed fuzzy γ–irresolute bijection function from a fts X into a FR–space Y . Then X is FγR–space .

Lemma 3.4.48:

A function f : X →Y is fuzzy almost gγ–closed if and only if for each fuzzy set B of Y and each U ∈FRO(X) containing $f^{-1}(B)$, there exists a Fgγ–open set V of Y such that $B \leq V$ and $f^{-1}(V) \leq U$.

Proof : It can be proved by the same manner of Lemma 3.1.17 .

Lemma 3.4.49:

If f : X →Y is fuzzy almost gγ–closed ,then for each fuzzy closed set M of Y and for each U ∈FRO(X) containing $f^{-1}(M)$,there exists V ∈ FγO (Y) such that $M \leq V$ and $f^{-1}(V) \leq U$.

Proof : It can be proved in a similar manner of Lemma 3.4.31 .

Theorem 3.4.50:

If f : X →Y is a fuzzy continuous F almost gγ–closed bijection function from a FR–space X into a FST_1–space Y . Then Y is FγR–space .

Proof :

Let X is FR–space ,and let x_λ and H be a fuzzy point and a fuzzy closed set , resp., in Y such that $x_\lambda \in 1_Y - H$ .Since f is fuzzy continuous bijective , then $f^{-1}(x_\lambda)$ is fuzzy point in X and $f^{-1}(H)$ is fuzzy closed set of X such that $f^{-1}(x_\lambda) \in 1_X - f^{-1}(H)$ by Remark 1.1.22 .part 2 ,Theorem 1.3.4 .part 3 and Theorem 1.1.23.part 2 and part 1 .As X is FR–space , by definition 1.4.14 , there exists fuzzy open sets U_1 and U_2 such that $f^{-1}(x_\lambda) \in U_1$, $f^{-1}(H) \leq U_2$ and U_1 /q U_2 .Now, put $G_i = int(cl(U_i))$ for i = 1 , 2 . Then $G_i \in FRO(X)$ by definition 2.1.1.part 7 , $f^{-1}(x_\lambda) \in U_1 \leq G_1$, $f^{-1}(H) \leq U_2 \leq G_2$ and G_1 /q G_2 . Since Y is FST_1–space , then x_λ is fuzzy closed set in Y by definition 1.4.7 .By Lemma 3.4.49 ,there exist $V_i \in F\gamma O(Y)$ for i =1 , 2 such that $x_\lambda \in V_1$, $H \leq V_2$, $f^{-1}(V_1) \leq G_1$ and $f^{-1}(V_2) \leq G_2$ .Since G_1 /q G_2 and f is surjective ,we have V_1 /q V_2 .Thus , the space Y is FγR–space by definition 2.3.19 .

Corollary 3.4.51:

Let f : X →Y be a fuzzy continuous Fgγ–closed bijection function from a FR–space X into a FST_1–space Y . Then Y is FγR–space .

Proof : It follows from Theorem 3.4.50 and Proposition 3.3. 3 . part 2 .

Corollary 3.4.52:

Let f : X →Y be a fuzzy continuous Fγg–closed bijection function from a F R–space X into a FST_1–space Y . Then Y is FγR–space .

Proof : It follows from Corollary 3.4.51 and Proposition 3.3. 3 . part 1 .

Theorem 3.4.53:

Let f : X →Y be a fuzzy continuous F almost gγ–open (resp., Fgγ–open , Fγg–open) bijection function from a FR–space X into a FST_1–space Y . Then Y is FγR–space .

Proof :

It follows from Theorem 3.4.50 (resp., Corollary 3.4.51 , Corollary 3.4.52) and Lemma 3.3.2 . part 6 (resp., Lemma 3.3.2.part 2 , Lemma 3.3.2.part 1) .

Theorem 3.4.54:

If $f : X \rightarrow Y$ is a fuzzy continuous F q γ–closed surjection function from a FγN–space X into a fts Y .Then Y is FN–space .

Proof :

Let X is FγN–space ,and M_1 and M_2 be any fuzzy closed sets of Y such that M_1 /q M_2 .Since f is fuzzy continuous , then $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are fuzzy closed sets of X such that $f^{-1}(M_1)$ /q $f^{-1}(M_2)$ by Theorem 1.3.4.part 3 and Theorem 1.1.23.part 2 and part 1 .Since X is FγN–space , by Theorem 2.3.26 , there exist fuzzy γ–open sets U_1 and U_2 such that $f^{-1}(M_i) \leq U_i$ for i = 1 , 2 and U_1 /q U_2 . Put $V_i = 1_Y - f(1_X - U_i)$, then V_i is fuzzy open in Y , $M_i \leq V_i$ and $f^{-1}(V_i) \leq U_i$ for i = 1 , 2 . Since U_1 /q U_2 and f is surjective ,we have V_1 /q V_2 .Thus , by Theorem 1.4.22 , the space Y is FN–space .

Corollary 3.4.55:

If $f : X \rightarrow Y$ is a fuzzy continuous F q γ–closed surjection function from a FγN–space(resp., FN–space) X into a fts Y . Then Y is FγN–space .

Proof :

It follows from Theorem 3.4.54 and since every FN–space is FγN–space .

Theorem 3.4.56:

If $f : X \rightarrow Y$ is a fuzzy continuous Fq γ–open bijection function from a FγN–space X into a fts Y .Then Y is FN–space .

Proof : It follows from Theorem 3.4.54 and Lemma 3.3.2 . part 4 .

Corollary 3.4.57:

If f : X →Y is a fuzzy continuous Fq γ–open bijection function from a FγN–space(resp., FN–space) X into a fts Y . Then Y is FγN–space .

Proof :

It follows from Theorem 3.4.56 and since every FN–space is FγN–space .

Theorem 3.4.58:

If f : X →Y is a fuzzy continuous Fq γ–closed bijection function from a $F\gamma T_4$–space X into a fts Y . Then Y is FT_4–space .

Proof : It follows from Theorem 3.4.9 and Theorem 3.4.54 .

Corollary 3.4.59:

If f : X →Y is a fuzzy continuous Fq γ–closed bijection function from a $F\gamma T_4$ –space (resp., FT_4–space) X into a fts Y . Then Y is $F\gamma T_4$–space .

Proof :

The proof follows from Theorem 3.4.58 and since every FT_4–space is $F\gamma T_4$–space .

Theorem 3.4.60:

If f : X →Y is a fuzzy continuous Fq γ–open bijection function from a $F\gamma T_4$–space X into a fts Y . Then Y is FT_4–space .

Proof : It follows from Lemma 3.3.2.part 4 , and Theorem 3.4.58 .

From Theorem 3.4.60 and since every FT_4–space is $F\gamma T_4$–space , the following Corollary is true :

Corollary 3.4.61:

If f : X→Y is a fuzzy continuous Fq γ–open bijection function from a

FγT$_4$ –space (resp., FT$_4$–space) X into a fts Y . Then Y is FγT$_4$–space .

Theorem 3.4.62:

Let f : X$\rightarrow$Y be a fuzzy continuous Fγ–gγ–closed surjection function from a FγN–space X into a fts Y .Then Y is FγN–space .

Proof :

Let X is FγN–space ,and N_1 and N_2 be two fuzzy closed sets of Y such that N_1 /q N_2 .Since f is fuzzy continuous , then $f^{-1}(N_1)$ and $f^{-1}(N_2)$ are fuzzy closed sets in X such that $f^{-1}(N_1)$ /q $f^{-1}(N_2)$ by Theorem 1.3.4 .part 3 and Theorem 1.1.23 . part 2 and part 1 . As X is FγN–space , by Theorem 2.3.26 , there are U_1, U_2 $\in$FγO(X) such that $f^{-1}(N_i) \leq U_i$ for i = 1 , 2 . and U_1 /q U_2 . By Lemma 3.4.31 , there exist $V_i \in$ FγO(Y) such that $N_i \leq V_i$ and $f^{-1}(V_i) \leq U_i$, where i = 1 , 2 . Since U_1 /q U_2 and f is surjective ,
$V_1 = f(f^{-1}(V_1)) \leq f(U_1) \leq f((U_2)^c) \leq f(f^{-1}(V_2)^c) = (V_2)^c$. So V_1 /q V_2 .Thus , again by Theorem 2.3.26 , we have Y is FγN–space .

Corollary 3.4.63:

Let f : X$\rightarrow$Y be a fuzzy continuous Fγ–γg–closed surjection function from a FγN–space X into a fts Y .Then Y is FγN–space .

Proof : It is clear from Theorem 3.4.62 and Proposition 3.3.3 . part 4 .

Corollary 3.4.64:

Let f : X$\rightarrow$Y be a fuzzy continuous Fγ–gγ–closed (resp., Fγ–γg–closed) surjection function from a FN–space X into a fts Y .Then Y is FγN–space .

Proof :

This is an immediate consequence from Theorem 3.4.62(resp., Corollary 3.4.63) and since every FN–space is FγN–space .

Theorem 3.4.65:

Let f : X→Y be a fuzzy continuous Fγ–gγ–open (resp., Fγ–γg–open) bijection function from a FγN–space X into a fts Y .Then Y is FγN–space .

Proof :

It follows from Theorem 3.4.62(resp., Corollary 3.4.63) and Lemma 3.3.2.part .5 (resp., Lemma 3.3.2.part 3) .

From Theorem 3.4.65 and since every FN–space is FγN–space , the following Corollary is true :

Corollary 3.4.66:

Let f : X→Y be a fuzzy continuous Fγ–gγ–open (resp., Fγ–γg–open) bijection function from a FN–space X into a fts Y .Then Y is FγN–space .

Theorem 3.4.67:

Let f : X→Y be a fuzzy closed fuzzy gγ–continuous bijection function from a fts X into a FN–space Y . Then X is FγN–space .

Proof :

Let Y is FN–space ,and H_1 and H_2 be two fuzzy closed sets of X such that H_1 /q H_2 .Since f is a fuzzy closed bijective , then $f(H_1)$ and $f(H_2)$ are fuzzy closed sets in Y such that $f(H_1)$ /q $f(H_2)$ by definition 1.3.2 and Theorem 1.1.23 . part 3 and part 8 . As Y is FN–space , by Theorem 1.4.22 , there exist fuzzy open sets V_1 and V_2 such that $f(H_i) \leq V_i$ for i = 1 , 2 and V_1 /q V_2 . Since f is fuzzy gγ–continuous, then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are Fgγ–open of X such that $f^{-1}(V_1)$ /q $f^{-1}(V_2)$ and $H_i \leq f^{-1}(V_i)$ for i =1 , 2 by Theorem 3.3.8 .part 2 and Theorem 1.1.23 .part 2 , part 1 and part 5. Now , put $U_i = \gamma\text{–int}(f^{-1}(V_i))$ for i = 1 , 2 . Thus , $U_i \in F\gamma O(X)$ by Properties 2.2.13.part 2 , $H_i \leq U_i$ and U_1 /q U_2 .This proves that X is FγN–space by Theorem 2.3.26 .

Corollary 3.4.68:

Let f : X→Y be a fuzzy closed fuzzy γ–gγ–continuous bijection function from a fts X into a FN–space Y . Then X is FγN–space .

Proof : It follows from Theorem 3.4.67 and Proposition 3.3.12 . part 7 .

Corollary 3.4.69:

Let f : X→Y be a fuzzy closed fuzzy γ–γg–continuous bijection function from a fts X into a FN–space Y . Then X is FγN–space .

Proof : It follows from Corollary 3.4.68 and Proposition 3.3.12. part 5 .

Corollary 3.4.70:

Let f : X→Y be a fuzzy closed fuzzy γ–irresolute bijection function from a fts X into a FN–space Y . Then X is FγN–space .

Proof : It follows from Corollary 3.4.69 and Proposition 3.3.12. part 4 .

Corollary 3.4.71:

Let f : X→Y be a fuzzy closed fuzzy γg–continuous bijection function from a fts X into a FN–space Y . Then X is FγN–space .

Proof :

The proof follows from Theorem 3.4.67 and Proposition 3.3.12. part 2 .

Corollary 3.4.72:

Let f : X→Y be a fuzzy closed fuzzy γ–continuous bijection function from a fts X into a FN–space Y . Then X is FγN–space .

Proof : It follows from Corollary 3.4.71 and Proposition 3.3.12 . part 1 .

Corollary 3.4.73:

Let f : X→Y be a fuzzy closed fuzzy α–continuous (resp., fuzzy semi continuous , fuzzy pre continuous) bijection function from a fts X into a

FN–space Y . Then X is FγN–space .

Proof :

It is obvious from Corollary 3.4.72 , Remark 2.1.12 , and Remark 3.3.9 .

Theorem 3.4.74:

Let f : X→Y be a fuzzy closed fuzzy γ–gγ–continuous bijection function from a fts X into a FγN–space Y .Then X is FγN–space .

Proof :

Assume that Y is FγN–space ,and A_1 , A_2 be two fuzzy closed sets of X such that A_1 /q A_2 . Since f is fuzzy closed bijective , then $f(A_1)$ and $f(A_2)$ are fuzzy closed sets in Y such that $f(A_1)$ /q $f(A_2)$ by definition 1.3.2 and Theorem1.1.23 . part 3 and part 8 . As Y is FγN–space , by Theorem 2.3.26 , there exist fuzzy γ–open sets N_1 and N_2 in Y such that $f(A_i) \leq N_i$ where i = 1 , 2 and N_1/q N_2 . Since f is fuzzy γ–gγ–continuous , then $f^{-1}(N_1)$ and $f^{-1}(N_2)$ are Fgγ–open sets of X such that $f^{-1}(N_1)$ /q $f^{-1}(N_2)$ and $A_i \leq f^{-1}(N_i)$ for i = 1 , 2 by Theorem 3.3.8 .part 5 and Theorem 1.1.23. part 2 , part 1 and part 5 . By Theorem 3.4.2 , X is FγN–space .

Corollary 3.4.75:

Let f : X→Y be a fuzzy closed fuzzy γ–γg–continuous bijection function from a fts X into a FγN–space Y .Then X is FγN–space .

Proof : It is clear from Theorem 3.4.74 and Proposition 3.3.12.part 5.

Corollary 3.4.76:

Let f : X→Y be a fuzzy closed fuzzy γ–irresolute bijection function from a fts X into a FγN–space Y .Then X is FγN–space .

Proof : It is obvious from Corollary 3.4.75 and Proposition 3.3.12 . part 4 .

Theorem 3.4.77:

Let $f : X \rightarrow Y$ be a fuzzy continuous Falmost gγ–closed surjection function from a FN–space X into a fts Y . Then Y is FγN–space .

Proof :

Let X is FN–space ,and M_1 , M_2 be any two fuzzy closed sets of Y such that $M_1/q\ M_2$. Since f is fuzzy continuous , then $f^{-1}(M_1)$ and $f^{-1}(M_1)$ are fuzzy closed sets of X such that $f^{-1}(M_1)\ /q\ f^{-1}(M_1)$ by Theorem 1.3.4 . part 3 and Theorem 1.1.23.part 2 and part 1 . As X is FN–space , there exists fuzzy open sets U_1 and U_2 such that $f^{-1}(M_i) \leq U_i$ for i = 1 , 2 and $U_1/q\ U_2$. Now , put $G_i = int(cl(U_i))$ for i = 1 , 2 . Then $G_i \in FRO(X)$ by definition 2.1.1. part 7 , $f^{-1}(M_i) \leq U_i \leq G_i$ and $G_1/q\ G_2$. By Lemma 3.4.49 , there exist $V_i \in F\gamma O(Y)$ such that $M_i \leq V_i$ and $f^{-1}(V_i) \leq G_i$,where i = 1 , 2 . Since $G_1/q\ G_2$ and f is surjective , we obtain $V_1/q\ V_2$. Thus , by Theorem 2.3.26 , Y is FγN–space .

Corollary 3.4.78:

If $f : X \rightarrow Y$ is a fuzzy continuous Fgγ–closed surjection function from a FN–space X into a fts Y . Then Y is FγN–space .

Proof : It follows from Theorem 3.4.77 and Proposition 3.3.3 . part 2 .

Corollary 3.4.79:

If $f : X \rightarrow Y$ is a fuzzy continuous Fγg–closed surjection function from a FN–space X into a fts Y . Then Y is FγN–space .

Proof : It is clear from Corollary 3.4.78 and Proposition 3.3.3. part 1 .

Corollary 3.4.80:

If $f : X \rightarrow Y$ is a fuzzy continuous fuzzy closed surjection function from a FN–space X into a fts Y . Then Y is FγN–space .

Proof :

It is obvious from Corollary 3.4.79 and Proposition 3.3.3. part 11 and part 8 .

Corollary 3.4.81:

If f : X→Y is a fuzzy continuous fuzzy closed bijection function from a FT_4–space X into a fts Y . Then Y is Y is $F\gamma T_4$–space .

Proof : It follows from Corollary 3.4.80 and Theorem 3.1.12 .