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حول السلوك الفوضوي لأنظمة التفرع المحلية

من قبل

حسن كامل جاسم الحسيناوي

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On the Chaotic Behavior of Local Bifurcation Systems

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الخلاصة

نظرا لزيادة البحوث والدراسات التي تناقش موضوع الأنظمة الدينامية في حياتنا اليومية مما حدا بنا إلى تناول موضوع السلوك الفوضوي (Chaos) والتفرع (Bifurcation) للأنظمة الدينامية وخصوصا الأنظمة غير الخطية.

الهدف من عملنا هو البحث عن السلوك الفوضوي (Chaos) لأنواع من التفرع (Bifurcation) , حيث عممنا تعريف (Gulick) على الأنظمة الدينامية المستمرة ورمزنا له بالرمز (G.Gulick) ودرسنا التعريف الجديد على نموذج من الوباء ونظام أشنق من المعادلة

$$\dot{x} = a_i + \sum_{j=1}^3 \sum_{k=1}^3 c_{jk} x_j x_k, i = 1,2,3$$

من خلال حساب ثابت ليبنوف و وجدنا الاتي : إذا كان النظام يمتلك حساسية معتمدة على الشروط الابتدائية (sensitive dependence on initial conditions) فان ثابت ليبنوف له موجب (positive Lyapunov exponent) أما إذا النظام يمتلك ثابت ليبنوف موجب فليس من الضرورة أن يمتلك حساسية معتمدة على الشروط الابتدائية . ولهذا الغرض استخدمنا برنامج (Matlap) لحساب ثوابت ليبنوف.

Introduction



Introduction

The dynamical systems theory comprises a powerful mathematical language for systems described either by nonlinear ordinary differential equations or iterated functions. [4]

Also, the dynamical system theory is the branch of mathematics devoted to the motion of systems, evolved according to simple rules. It was developed originally in the 17th century by Newton to model the motions of their system, evolving under the rules of his new theory of universal gravitation. [2]

The basic goal of the dynamical systems is to understand the eventual or asymptotic behavior of an iterative process. If this process is differential equations whose independent variable is time, then the theory attempts to predict the ultimate behavior of solutions of the equation in either the distant future ($t \rightarrow \infty$) or the distant past ($t \rightarrow -\infty$). If this process is a discrete such as the iteration of maps, then the theory hopes to understand the eventual behavior of the points $x, f(x), f^2(x), \dots, f^n(x)$ as n becomes large [10]. If the differential equation or map depends on a parameter, the dynamics for solution of equation or map depends on the parameter as well. If the qualitative change in dynamics occurring upon a small change in a parameter, this change is called the bifurcation [28]. In case of the bifurcation there is a special parameter value μ_0 for which the following holds: the dynamics for near to but smaller than μ_0 is qualitatively different from the dynamics for near to but larger than μ_0 . This value μ_0 is called a bifurcation value. [4]

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The bifurcation occurs in both continuous systems (described by ordinary differential equations)[1,16,20,28,38], and discrete systems (described by maps)[10,18,34,35,38,39,40].

The bifurcation is divided into two principal classes: local bifurcations and global bifurcations. Local bifurcations, which can be analyzed entirely through changes in the local stability properties of equilibria, periodic orbit or other invariant sets as parameters cross through critical thresholds such as saddle node, transcritical, pitchfork, period-doubling (flip), Hopf and Neimark (secondary Hopf) bifurcation. Global bifurcations occur when larger invariant sets, such as periodic orbits, collide with equilibria. This causes changes in the topology of the orbits in phase space which cannot be confined to a small neighborhood, as is the case with Local bifurcations. Infact, the changes in topology extend out of an arbitrarily large distance (hence "global") such as homoclinic in which a limit cycle collides with a saddle point, and heteroclinic bifurcation in which a limit cycle collides with two or more saddle points, and infinite-periodic bifurcation in which a stable node and saddle point simultaneously occur on a limit cycle. [28]

These bifurcations happen when one varies a single parameter; such bifurcations are called codimension one. [38]

Also, there are five distinct codimensions two bifurcations for vector field. These correspond to the various ways a vector field can be made singular by varying such as Taken-Bogdanov bifurcation, Hopf-bifurcation, fold-Hopf or Gavrilov-Guckenheimer bifurcation, Bautin bifurcation and the cusp bifurcation. [28]

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In this work, we will recall sufficient conditions on one parameter of one dimensional vector field to undergo a saddle node bifurcation, transcritical bifurcation and pitchfork bifurcation. Also, we will give sufficient conditions on one parameter of two dimensional vector fields to undergo a Hopf bifurcation.

One of the important dynamical systems is the chaos. A dynamical system is chaotic on a given invariant set Λ for a flow φ when it satisfies certain properties. Of course, Λ could be a very small set in the phase space. And then the assertion of chaos on Λ would not necessarily be of much practical importance. [28]

Chaos theory was not known by this name until 1975 being paper (period three implies of chaos) of Y. Li and J. York which included the first use of the word chaos in the context of dynamical systems. [28]

In this thesis, we introduce many definitions of chaos which are Gulick [18], Devaney [10], Wiggins [38], Meiss [28] definition. Gulick's definition involves discrete dynamical systems. We will generalize this definition on continuous dynamical systems and calling this generalization (Gulick's generalization).

Many authors believe that the key element of deterministic chaos is the sensitive dependence on the initial conditions. As an example, the plot on the first 15 seconds of a demonstration of this effect for a horizontally driven pendulum with two different initial conditions is characteristics. Both initial conditions differ in one arcsec, only. Nevertheless, 10 seconds later they behave totally different. In the beginning, the distance of the trajectories increases on average exponentially (it is visible by the distance in phase space).

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The rate of divergence is measured by the largest Lyapunov exponent. A quantitative measure of the sensitive dependence on the initial conditions is the Lyapunov exponent. Aleksandr Mikhailovich Lyapunov was born in June, 1857 in Yaroslavl, Russia and died November, 1918 in Odessa, Russia. He was a Russian mathematician, mechanician and physician. Sometimes his name is also written as Ljapunov, Liapunov or Ljapunow, and often improperly pronounced “La-yapunov”. Lyapunov left us with more than just a simple note. He left a collection of papers on the equilibrium shape of rotating liquids, on probability, and on the stability of low dimensional dynamical systems. It was from his dissertation that the notion of Lyapunov exponent emerged. Lyapunov was interested in showing how to discover if a solution to a dynamical system is stable or not for all times [12]. It is the averaged rate of divergence (or convergence) of two neighboring orbits. Their number is equal to the dimension of the phase space. When speaking about the Lyapunov exponent, the largest one is meant [28].

The goal of our work is to calculate Lyapunov exponent to types of bifurcations, model of an epidemic and system taken from the formula:

$$\dot{x} = a_i + \sum_{j=1}^3 b_j x_j + \sum_{j=1}^3 \sum_{k=1}^3 c_{jk} x_j x_k, \quad i = 1, 2, 3,$$

and has three parameters, whereupon we found relation between sensitive dependence and Lyapunov exponent for these systems.

In this work, we use $\gamma = \sup_t \frac{1}{t} \log |\Phi(t, v)|$ in [28] to calculate Lyapunov exponent for the one dimensional vector field, and

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$\gamma_{\max}(T) = \frac{1}{T} \ln \frac{|v(T)|}{v_0}$ in [28] to calculate Lyapunov exponent for the two and three dimensional vector field.

This work is divided into four chapters: chapter one contains the basic definitions for this search. So, we study the bifurcation of one parameter vector fields, we enumerate four types of bifurcation. We give sufficient conditions on a one parameter vector fields to undergo these types of bifurcation and the end of this chapter we gave a brief discussion about the bifurcation points of Lorenz system and dual system to Lorenz system.

In chapter two, we explained the Lyapunov exponent definition and we introduce some its properties.

In chapter three, we introduce some definitions of chaos together with some simple Examples, at the end of this chapter we calculate Lyapunov exponent (chaos) to the four types of the bifurcation.

In chapter four, we study the bifurcation, also the chaos (in sense of G.Gulick) that is, positive Lyapunov exponent for nonlinear systems which posses three dimension autonomous differential equation as a model of epidemic, and also with quadratics of six or fewer dimension, whereupon we compare between Lyapunov exponent and sensitive dependence for these systems.

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

مَثَلُ الَّذِينَ يَبْذُلُونَ أَمْوَالَهُمْ فِي سَبِيلِ
اللَّهِ كَمَثَلِ حَبَّةٍ أَنْبَتَتْ سَبْعَ سَنَابِلٍ فِي
كُلِّ سَنَابِلَةٍ مِئَةٌ حَبَّةٌ وَاللَّهُ يُضَاعِفُ لِمَنْ
يَشَاءُ وَاللَّهُ وَاسِعٌ عَلِيمٌ.

صدق الله العظيم

سورة البقرة (الآية 261)

Abstract



Many recent research and studies are concerned with the dynamical systems because of its increasing importance for our life that lead to study the subject.

The goal of our work to study the chaotic behavior for types of bifurcations. We generalized the definition of Gulick to continuous dynamical system denoted by G.Gulick. We use G.Gulick to study the model of an epidemic and the system taken from the formula:

$$\dot{x} = a_i + \sum_{j=1}^3 \sum_{k=1}^3 c_{jk} x_j x_k, i = 1, 2, 3,$$

through calculate Lyapunov exponent. We found the following: If the system has sensitive dependence on initial conditions, then it has positive Lyapunov exponent. If the system has a positive Lyapunov exponent, then it is not necessary to has sensitive dependence. Thus, we use (Matlab) program to compute the Lyapunov exponents.

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Hassan Kamil Jassim

Appendixes

Appendixes

Appendix (1)

```
1. clc
2. clear
3. d=dsolve('Dx= $\mu - x^2$ ')
4. w=diff(d)
5. b=abs(w)
6. l=log(b)
7. t=input('t=')
8.  $\mu$  =input('μ =')
9. x=input('x=')
10.m=l/t
11.s=subs( m,t )
12.s1=subs(s,  $\mu$  )
13.s3=subs(s1, x )
```

Appendix (2)

```
1. clc
2. clear
3. d=dsolve('Dx= $\mu x - x^2$ ')
4. w=diff(d)
5. b=abs(w)
6. l=log(b)
7. t=input('t=')
8.  $\mu$  =input('μ =')
9. x=input('x=')
10.m=l/t
11.s=subs( m,t )
12.s1=subs(s,  $\mu$  )
13.s3=subs(s1, x )
```



Appendix (3)

```
1. clc
2. clear
3. d=dsolve('Dx=μx - x^3')
4. w=diff(d)
5. l=log(w)
6. t=input('t=')
7. μ =input('μ =')
8. x=input('x=')
9. m=l/t
10.s=subs(m,t)
11.s1=subs(s,μ)
12.s3=subs(s1,x)
```

A decorative banner with a central rectangular box containing the text "Appendix (4)". The banner has a ribbon-like shape with pointed ends on the left and right sides.

Appendix (4)

```
1. function mu= Lyapunov (tmax, r)
2. tstep= 1;
3. x0 = input (' x0 =')
4. v0 = [.1,.1];
5. v0 = horzcat (x0, v0);
6. scalefacto r = -log( norm (v0));
7. T = []; L = [];
8. for time = 1: tstep: tmax
9. [t, w]= ode45 (@LorenzOneJet, [time, time + tstep], w0, [], r);
10. Lyp = (scalefactor + 0.5*log (sum (w (:, 3: 4). ^2, 2))). /t;
11. T = [T; t]; % store integration output
12. L = [L; Lyp];
13. nm = norm (w (end, 1:4));
14. scalefactor = scalefactor + log (nm);
15. w0= horzcat (w (end, 1:2), w (end, 3:4)/nm);
16. end
17. plot (T,L);
18. mu = Lyp (end);
19. function wdot = LorenzOneJet (t, w, r)
20. wdot = zeros (4-1);
21. wdot(1) = r*w(1)-w(2)-w(1)*(w(1)^2+w(2)^2);
22. wdot(2) = w(1)+r*w(2)-w(2)*(w(1)^2+w(2)^2);
23. wdot(3) = (r-3*w(1)^2-w(2)^2)*w(3)+(-1-2*w(1)*w(2))*w(4);
24. wdot(4) = (1-2*w(1)*w(2))*w(3)-3*w(2)^2*w(4);
```

A decorative banner with a central rectangular box containing the text "Appendix (5)". The banner has pointed ends on both sides, resembling a ribbon or a stylized arrow.

Appendix (5)

```
1. function mu= Lyapunov(tmax, a, b, c)
2. tstep= 1;
3. x0=input('x0=')
4. v0= [.1,.1,.1];
5. w0= horzcat(x0,v0);
6. scalefactor= -log(norm(v0));
7. T=[]; L=[];
8. for time = 1:tstep:tmax
9. [t,w]= ode45(@LorenzOneJet,[time,time+tstep],w0,[],a,b,c);
10.Lyp = (scalefactor + 0.5*log(sum(w(:,4:6).^2,2)))./t;
11.T=[T; t]; % store integration output
12.L=[L; Lyp];
13.nm = norm(w(end,4:6));
14.scalefactor = scalefactor + log(nm);
15.w0 = horzcat(w(end,1:3), w(end,4:6)/nm);
16.end
17.plot(T,L);
18.mu = Lyp(end);
19.function wdot=LorenzOneJet(t,w,a,b,c)
20.wdot=zeros(6,1);
21.wdot(1)= -a*w(1)*w(2)+a*w(1)-w(3);
22.wdot(2)= a*w(1)*w(2)-b*w(2);
23.wdot(3)= b*w(2)+c*w(1);
24.wdot(4)= -a*w(2)*w(4)+a*w(4)-a*w(1)*w(5)-w(6);
25.wdot(5)= a*w(2)*w(4)+a*w(1)*w(5)-b*w(5);
26.wdot(6)= c*w(4)+b*w(5);
```



Appendix (6)

```
1. function mu= Lyapunov(tmax, a, b, c)
2. timestep= 1;
3. x0=input('x0=')
4. v0= [.1,.1,.1];
5. w0= horzcat(x0,v0);
6. scalefactor= -log(norm(v0));
7. T=[]; L=[];
8. for time = 1:timestep:tmax
9. [t,w]= ode45(@LorenzOneJet,[time,time+timestep],w0,[],a,b,c);
10. Lyp = (scalefactor + 0.5*log(sum(w(:,4:6).^2,2)))/t;
11. T=[T; t]; % store integration output
12. L=[L; Lyp];
13. nm = norm(w(end,4:6));
14. scalefactor = scalefactor + log(nm);
15. w0 = horzcat(w(end,1:3), w(end,4:6)/nm);
16. end
17. plot(T,L);
18. mu = Lyp(end);
19. function wdot=LorenzOneJet(t,w,a,b,c)
20. wdot=zeros(6,1);
21. wdot(1)= a*(w(2)-w(1));
22. wdot(2)= b*w(1)-w(3)*w(1);
23. wdot(3)= w(1)+w(1)*w(2)-2*c*w(3);
24. wdot(4)= a*(w(5)-w(4));
25. wdot(5)= b*w(4)-w(3)*w(4)-w(1)*w(3);
26. wdot(6)=w(4)+w(2)*w(4)+w(1)*w(5)-2*c*w(6);
```



Preliminaries

The purpose of this chapter is to introduce some elementary definitions and necessary theorems for our research. It consists of two sections. In section one; we recall some fundamental definitions and necessary theorems for this research. In section two, we recall the definition of bifurcation and some of its types.

1.1 Basic Definitions and Necessary Theorems.

Our goal in this section is to list the definitions and theorems that we use later in the research. We start with the general definition of a dynamical system.

Definition (1.1.1) [20]

A dynamical system is a map $\phi : K \times S \rightarrow S$; where S is an open set of Euclidean space and writing by $\phi(t, x) = \phi_t(x)$, the map $\phi_t : S \rightarrow S$ satisfies:

- (1) $\phi_0 : S \rightarrow S$ is the identity; that is $\phi_0(x) = x$, for all x in S .
- (2) The composition $\phi_t \circ \phi_s = \phi_{t+s}$, for each t, s in K .

In case K is $\mathbb{Z}$ the dynamical system is described to be discrete dynamical system. In case K is $\mathbb{R}$ the dynamical system is continuous dynamical system.

Now, we will give an example on the continuous dynamical system

Chapter one

Example (1.1.2) [22]

Suppose that $\phi_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} xe^{-t} \\ (xt + y)e^{-t} \end{bmatrix}$ for all

$x, y, t \in \mathbb{R}$.

(1) $\phi_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} xe^0 \\ (0 + y)e^0 \end{bmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. That is, the first condition is satisfied.

$$\begin{aligned} (2) \phi_t \circ \phi_s \begin{pmatrix} x \\ y \end{pmatrix} &= \phi_t \begin{bmatrix} xe^{-s} \\ (xs + y)e^{-s} \end{bmatrix} = \begin{bmatrix} xe^{-s}e^{-t} \\ (xe^{-s}t + (xs + y)e^{-s})e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} xe^{-(t+s)} \\ (x(t+s) + y)e^{-(t+s)} \end{bmatrix} \\ &= \phi_{t+s} \begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned}$$

for all x, y, t, s in $\mathbb{R}$. That is, the second condition is satisfied. Therefore, the map ϕ_t is dynamical system.

Definition (1.1.3) [10]

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a map. We denote the derivative of f at x by $f'(x)$, the second derivative by $f''(x)$, and r^{th} derivatives by $f^{(r)}(x)$. We say that f is of class C^r if $f^{(r)}$ exists and is continuous at all x in $I \subseteq \mathbb{R}$. The map $f(x)$ is C^∞ if all derivatives exist and are continuous.

Definition (1.1.4) [10]

The map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is C^1 , if all of its first partial derivatives exist and are continuous. f is C^∞ , if its mixed k^{th} partial derivatives exist and are continuous, for all $k \in \mathbb{Z}$.

Example (1.1.5) [14]

The map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x^2 + \frac{1}{4} \\ 4x - x^2 \end{bmatrix}$ is C^∞ .

Definition (1.1.6) [18]

Let $x_1, x_2, \dots, x_n$ be differentiable maps of time t on an interval J of real numbers, and let $f_1, f_2, \dots, f_n$ be maps of $x_1, x_2, \dots, x_n$ as well as t . Then n -differential equations

$$\left. \begin{aligned} \dot{x}_1 &= \frac{dx_1}{dt} = f_1(x_1, x_2, \dots, x_n, t) \\ \dot{x}_2 &= \frac{dx_2}{dt} = f_2(x_1, x_2, \dots, x_n, t) \\ &\cdot \\ &\cdot \\ &\cdot \\ \dot{x}_n &= \frac{dx_n}{dt} = f_n(x_1, x_2, \dots, x_n, t) \end{aligned} \right\} \dots\dots\dots(1-1)$$

form a system of differential equations. The over dot in (1-1) means " $\frac{d}{dt}$ ".

If $x : J \rightarrow \mathbb{R}^n$ is defined by $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ x_n(t) \end{pmatrix}$ such that x satisfies (1-1),

then x is a solution of the system.

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Definition (1.1.7) [18]

A differential equation of the form $\frac{dx}{dt} = f(t)x + g(t)$, where f and g are continuous maps of t on some given interval J , is called a linear differential equation.

Now we study some types of differential equations:

Definition (1.1.8) [7]

Recall the system

$$\left. \begin{aligned} \dot{x}_1 &= \frac{dx_1}{dt} = f_1(x_1, x_2, \dots, x_n, t) \\ \dot{x}_2 &= \frac{dx_2}{dt} = f_2(x_1, x_2, \dots, x_n, t) \\ &\vdots \\ \dot{x}_n &= \frac{dx_n}{dt} = f_n(x_1, x_2, \dots, x_n, t) \end{aligned} \right\} \dots\dots\dots(1-1)$$

which can be written in the vector form: $\dot{x} = f(x, t)$, where

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^n, x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, f(x, t) = \begin{pmatrix} f_1(x_1, x_2, \dots, x_n, t) \\ f_2(x_1, x_2, \dots, x_n, t) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n, t) \end{pmatrix}. \text{ If the}$$

system (1-1) does not explicitly contain the independent variable t , a system of this property is said to be autonomous, which can be written in the vector form

$$\dot{x} = f(x), x \in \mathbb{R}^n. \dots\dots\dots(1-2)$$

Definition (1.1.9) [19]

Consider a linear first order system of the form

$$\left. \begin{aligned} \dot{x}_1 &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + G_1(t) \\ \dot{x}_2 &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + G_2(t) \\ &\cdot \\ &\cdot \\ &\cdot \\ \dot{x}_n &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + G_n(t) \end{aligned} \right\} \text{.....(1-3)}$$

we say that the system (1-3) is homogeneous if the maps $G_1(t), G_2(t), \dots, G_n(t)$ are all identically zeros, otherwise it is nonhomogeneous.

Definition (1.1.10) [9]

Consider a system of differential equations in the form $\dot{x}_i = f_i(x_1, x_2, \dots, x_n, \mu)$ where $i = 1, 2, \dots, n$, and μ is a control Parameter. Geometric picture of the solution is useful to consider the vector $x(t) = (x_1, x_2, \dots, x_n)$ as a point of an n dimensional space called phase space. We view the variables μ as parameters.

Definition (1.1.11) [28]

A vector field is a map $f : M \rightarrow \mathbb{R}^n$ that defines a vector $v = f(x)$ at each point x in the phase space M .

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Definition (1.1.12) [8]

The Jacobian matrix is the matrix associated with a map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, having coordinate functions $f_1, f_2, \dots, f_m$, with the (i, j) entry $\frac{\partial f_i}{\partial x_j}(x_0)$, the first-order partial derivative of f_i with respect to x_j , $1 \leq i \leq m$, $1 \leq j \leq n$, at the point x_0 in the domain of f :

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \frac{\partial f_1}{\partial x_2}(x_0) & \dots\dots\dots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \frac{\partial f_2}{\partial x_1}(x_0) & \frac{\partial f_2}{\partial x_2}(x_0) & \dots\dots\dots & \frac{\partial f_2}{\partial x_n}(x_0) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial f_m}{\partial x_1}(x_0) & \frac{\partial f_m}{\partial x_2}(x_0) & \dots\dots\dots & \frac{\partial f_m}{\partial x_n}(x_0) \end{bmatrix}$$

and denoted by $Df(x_0)$.

Example (1.1.12)

Consider the system:

$$\dot{x} = -y - z$$

$$\dot{y} = x + ay$$

$$\dot{z} = b + z(x - c)$$

where a, b , and $c > 0$.

We have for $f_1 = -y - z, f_2 = x + ay$ and $f_3 = b + xz - cz$ that

$$\frac{\partial f_1}{\partial x} = 0, \frac{\partial f_1}{\partial y} = -1, \frac{\partial f_1}{\partial z} = -1, \frac{\partial f_2}{\partial x} = 1, \frac{\partial f_2}{\partial y} = a, \frac{\partial f_2}{\partial z} = 0, \frac{\partial f_3}{\partial x} = z, \frac{\partial f_3}{\partial y} = 0,$$

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$$\frac{\partial f_3}{\partial z} = x - c, \text{ so } Df \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z_0 & 0 & x_0 - c \end{bmatrix} \text{ for all } \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \text{ in } \mathbb{R}^3.$$

Definition (1.1.14) [1]

The geometrical representation of the qualitative behavior of the differential equation

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^n, \mu \in \mathbb{R} \quad \dots\dots\dots(1-4)$$

is called its phase portrait.

Example (1.1.15) [1]

Let $\dot{x} = \mu x$, $x, \mu \in \mathbb{R}$. Then $\dot{x} = \mu x = 0$, thus $x = 0$ is critical point.

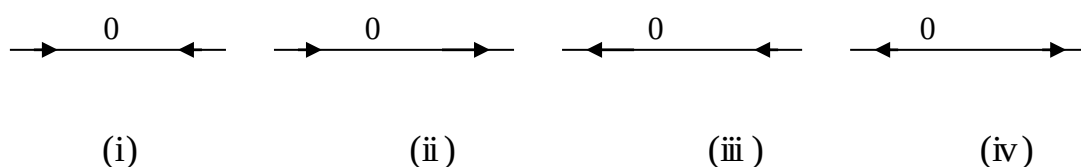


Figure (1)

(The four possible phase portraits of the differential equation $\dot{x} = \mu x$)

Definition (1.1.16) [38]

Consider a general autonomous vector field

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \quad \dots\dots\dots(1-5)$$

an equilibrium solution of (1-5) is a point $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) = 0$, i.e., a solution which does not change in time.

Remark (1.1.17) [20, 38]

Other terms often substituted for the term "equilibrium solution" are "fixed point," "stationary point," "rest point," "singularity," "critical point," or "steady state." In this research, we use term the critical point.

Now, we will introduce system of differential equations which is called Rossler system.

Example (1.1.18) [28]

Consider the Rossler system

$$\left. \begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned} \right\} \dots\dots\dots(1-6)$$

where the parameters a , b and c are positive. To find the critical points of system (1-6) by setting $\dot{x} = 0$, $\dot{y} = 0$ and $\dot{z} = 0$, we get

$$-y - z = 0 \dots\dots\dots(1-7)$$

$$x + ay = 0 \dots\dots\dots(1-8)$$

$$b + z(x - c) = 0 \dots\dots\dots(1-9)$$

from the equation (1-7), we have $y = -z$. Also from the equation (1-8), we have

$$x = az. \dots\dots\dots(1-10)$$

Now, we substitute (1-10) in the equation (1-9) to obtain

$$az^2 - cz + b = 0, \text{ thus } z = \frac{c \pm \sqrt{c^2 - 4ab}}{2a}. \text{ If } c^2 < 4ab, \text{ then there no}$$

critical points. If $c^2 > 4ab$, then there are two critical points which are

$$p_1 = \begin{pmatrix} \frac{c + \sqrt{c^2 - 4ab}}{2} \\ \frac{-c - \sqrt{c^2 - 4ab}}{2a} \\ \frac{c + \sqrt{c^2 - 4ab}}{2a} \end{pmatrix}, p_2 = \begin{pmatrix} \frac{c - \sqrt{c^2 - 4ab}}{2} \\ \frac{-c + \sqrt{c^2 - 4ab}}{2a} \\ \frac{c - \sqrt{c^2 - 4ab}}{2a} \end{pmatrix}.$$

Now, we introduce an important subject is the stability. The term stability is one that has a variety of different meanings within Mathematics. One often says that a problem is stable if it is “continuous with respect to the data”; i.e., a problem is stable if when we change the problem “slightly,” the solution changes only slightly. We make this precise below in the context of initial-value problems for ODEs. Another notion of Stability is that of “asymptotic stability.” [31]

Definition (1.1.19) [38]

$\bar{x}$ is said to be stable if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any other solution, $y(t)$, of vector field (1–5) satisfying $|\bar{x}(t_0) - y(t_0)| < \delta$, then $|\bar{x}(t) - y(t)| < \varepsilon$, for all $t > t_0$, $t_0 \in \mathbb{R}$.

Definition (1.1.20) [38]

The point $\bar{x}$ is said to be asymptotically stable if it is stable and if there exists a constant $\delta > 0$ such that, if $|\bar{x}(t_0) - y(t_0)| < \delta$, then

$$\lim_{t \rightarrow \infty} |\bar{x}(t) - y(t)| = 0, \text{ for all } t > t_0, t_0 \in \mathbb{R}.$$

Definition (1.1.21) [11,38]

The $\bar{x}$ is said to be unstable if it is not stable.

Example (1.1.22) [5]

Suppose that $\dot{x} = \frac{dx}{dt} = -x$. We have $x = 0$ is critical point and the solution of vector field is $\ln(x) = -t + c_1$; that is, $x(t) = ce^{-t}$, $c = e^{c_1}$, then for $\varepsilon > 0$ we have $|x(t) - 0| = |ce^{-t}| \leq |c| < \varepsilon$, for all $t \geq 0$. Let $\delta = \varepsilon$, by choosing $|x(0) - 0| = |c| < \delta$. then the critical point $x = 0$ is stable. Clearly $|ce^{-t}| \rightarrow 0$ as $t \rightarrow \infty$.i.e. $\lim_{t \rightarrow \infty} ce^{-t} = 0$, therefore the critical point 0 is asymptotically stable.

Example (1.1.23) [5]

Suppose that $\dot{x} = \frac{dx}{dt} = x$, we have $x(t) = ce^t$ and $x = 0$ is the critical point, for any $\varepsilon > 0$ clear as t increases infinitely this solution escapes out of any neighborhood of the origin. Therefore the critical point is unstable.

Remark (1.1.24) [1]

Every asymptotically stable critical point is stable, but the converse is not true.

Definition (1.1.25) [28]

The orbit (trajectory) is the sequence of states that follow from or lead to a given initial state.

Definition (1.1.26) [38]

Consider the vector field (1-5). A solution of (1-5) through the point x_0 is said to be periodic of period T if there exists $T > 0$ such that $x(t, x_0) = x(t + T, x_0)$, for all $t \in \mathbb{R}$.

Example (1.1.27) [38]

Consider the system of differential equations

$$\left. \begin{array}{l} \dot{x} = y \\ \dot{y} = -x \end{array} \right\} \dots\dots\dots(1-12)$$

where $(x, y) \in \mathbb{R} \times \mathbb{R}$. We have the solution of (1-12) passes through $(x, y) = (1, 0)$ at $t=0$ is given by $(x(t), y(t)) = (\cos(t), -\sin(t))$. The orbit passes through $(x, y) = (1, 0)$ at $t=0$ is given by the circle $x^2 + y^2 = 1$, as shown in figure (2). The solution of (1-12) passes through $(x, y) = (1, 0)$ is periodic of period 2π .

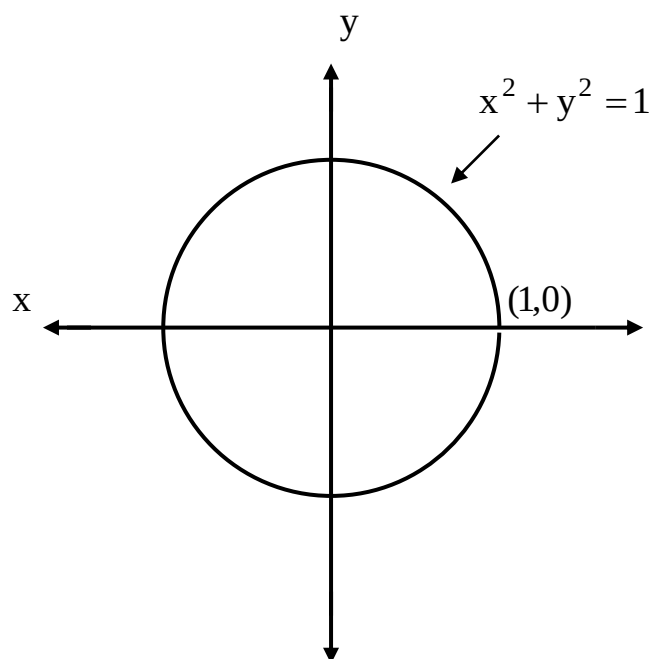


Figure (2)
(Orbit of $(1,0)$)

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Now, we will define important map is called Lyapunov map used in determinate of a stability critical point.

The Russian mathematician and engineer A. M. Lyapunov, in his 1892 doctoral thesis, found a very useful criterion for stability. It is a generalization of the idea that for a sink there is a norm on $\mathbb{R}^n$ such that $\|x(t) - \bar{x}\|$ decreases for solutions $x(t)$ near $\bar{x}$. Lyapunov showed that certain other map could be used instead of the norm to guarantee stability. [20]

Definition (1.1.28) [20]

Consider an autonomous system (1-2), where we assume $f(x)$ is continuous in an open set contain 0 and $f(0) = 0$, so that $x(t) = 0$ is a solution. A map $V(x)$, define in a neighborhood of $x = 0$ is called a Lyapunov map for above problem if:

- (1) V has continuous partial derivatives.
- (2) $V(x) \geq 0$ with $V(x) > 0$, if $x \neq 0$.
- (3) The orbit derivative $\dot{V}(x)$, along a solution is nonpositive. The orbit derivative along x of (3) is $\dot{V} = \frac{dV(x(t))}{dt}$.

Definition (1.1.29) [28]

A closed ball of radius r around a point x_0 in the phase space $\mathbb{R}^n$ is the closed set $B_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$.

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Definition (1.1.30) [23]

The sphere of radius r and center x in $\mathbb{R}^3$ is the set $\{x \in \mathbb{R}^3 : \|x - x_0\| = r\}$, and denoted by $S_r(x_0)$.

We can now state Lyapunov stability theorem.

Theorem (1.1.31) [38]

Let $\bar{x}$ be a critical point for (1-2), and let $V:U \rightarrow \mathbb{R}$ be a continuous map defined on a neighborhood U of $\bar{x}$, differentiable on $U - \{\bar{x}\}$, such that:

- (1) $V(\bar{x}) = 0$ and $V(x) > 0$ if $x \neq \bar{x}$.
- (2) $\dot{V}(x) \leq 0$ in $U - \{\bar{x}\}$. Then $\bar{x}$ is stable. Moreover, if
- (3) $\dot{V}(x) < 0$ in $U - \{\bar{x}\}$.

Then $\bar{x}$ is asymptotically stable.

Example (1.1.32) [20]

Consider the dynamical system on $\mathbb{R}^3$ described by the differential equations

$$\dot{x} = 2y(z-1)$$

$$\dot{y} = -x(z-1)$$

$$\dot{z} = xy$$

we have $(0,0,0)$ is critical point of the system. Let us investigate the origin for stability. The linear part of the system at $(0,0,0)$ is the matrix

$$\begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \text{ There are two imaginary eigenvalues and one zero}$$

eigenvalue. Let us look for a Lyapunov map for $(0,0,0)$ of the form

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$v(x, y, z) = ax^2 + by^2 + cz^2$, where the parameters a, b and c are positive.

For such a V , $\dot{V} = 2(ax\dot{x} + by\dot{y} + cz\dot{z})$ thus

$$\frac{1}{2}\dot{V} = 2axy(z-1) - bxy(z-1) + cxyz; \text{ we want to } \dot{V} \leq 0; \text{ this can be}$$

accomplished by setting $c=0$ and $2a=b$; this makes $\dot{V} = 0$. We conclude that $x^2 + 2y^2$ is a Lyapunov map for $(0,0,0)$; therefore the origin is a stable critical point. Moreover, the origin is not asymptotically stable, since Lyapunov map $V = x^2 + 2y^2$ is constant on orbits ($\dot{V}$ being 0), but not constant in any neighborhood of the origin.

Definition (1.1.33) [20, 38]

A map V satisfying hypotheses (1) and (2) of theorem (1.1.31) is called a weak Lyapunov map. If (2) is replaced by (3) then V is called strong Lyapunov map.

Example (1.1.34)

Consider the following vector field

$$\dot{x} = y$$

$$\dot{y} = -x + rx^2y$$

where r is a real number. To find the critical points: setting $\dot{x} = 0$ and

$$\dot{y} = 0, \text{ we get}$$

$$y = 0 \quad \dots\dots\dots(1-13)$$

$$-x + rx^2y = 0 \quad \dots\dots\dots(1-14)$$

we substitute (1-13) in (1-14), we get $x = 0$, so $(0,0)$ is the critical point.

Let $V(x, y) = \frac{x^2 + y^2}{2}$, then $V(0,0) = 0$, and $V(x, y) > 0$ in any

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neighborhood of $(0,0)$. Also, $\dot{V}(x, y) = xy + rx^2y^2 - xy = rx^2y^2$. Then, by theorem (1.1.31), $(0,0)$ is strong Lyapunov map for all $r < 0$.

Theorem (1.1.35) [38]

Suppose all of the eigenvalues of $Df(x)$ have negative real parts. Then the critical point $\bar{x}$ of the nonlinear vector field (1–5) is asymptotically stable.

Remark (1.1.36) [38]

A critical point $\bar{x}$ is said to be unstable if at least one eigenvalue of $Df(\bar{x})$ has positive real part.

Example (1.1.37) [38]

Consider the system

$$\dot{x} = y$$

$$\dot{y} = x - x^3 - \delta y, \quad \delta \geq 0.$$

To find the critical points of system setting $\dot{x} = 0$ and $\dot{y} = 0$, we obtain

$$y = 0 \quad \dots\dots\dots(1-15)$$

$$x - x^3 - \delta y = 0 \quad \dots\dots\dots(1-16)$$

we substitute (1–15) in (1–16), we get $x - x^3 = 0$, that is $x = 0, -1$ and 1 . Then the critical points are $(0, 0)$, $(1, 0)$ and $(-1, 0)$. The matrix

associated with the linearized vector field is given by $\begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -\delta \end{bmatrix}$.

To find the eigenvalues associated with the critical point $(0,0)$:

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$$\det \left[\begin{pmatrix} 0 & 1 \\ 1 & -\delta \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right] = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\delta - \lambda \end{bmatrix} = -\lambda(-\delta - \lambda) - 1$$

$$= \lambda^2 + \delta\lambda - 1 = 0, \text{ thus } \lambda = \frac{-\delta \pm \sqrt{\delta^2 + 4}}{2}. \text{ We have } \lambda_1 = \frac{-\delta}{2} + \frac{\sqrt{\delta^2 + 4}}{2}$$

$$\text{and } \lambda_2 = \frac{-\delta}{2} - \frac{\sqrt{\delta^2 + 4}}{2}. \text{ Hence, for } \delta > 0, (0,0) \text{ is unstable. To find the}$$

eigenvalues associated with the critical points $(1,0)$ and $(-1,0)$ are the

same for each point and are given by $\lambda_1 = \frac{-\delta}{2} + \frac{\sqrt{\delta^2 - 8}}{2}$ and

$$\lambda_2 = \frac{-\delta}{2} - \frac{\sqrt{\delta^2 - 8}}{2}.$$

Hence, for $\delta > 0$, $(1, 0)$ and $(-1, 0)$ are asymptotically stable.

Now, we indicate to the important type of critical points use in section two of chapter one.

Definition (1.1.38) [38]

Let $\bar{x}$ be a critical point of $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. Then $\bar{x}$ is called a hyperbolic critical point if none of the eigenvalues of $Df(\bar{x})$ have zero real part.

Example (1.1.38) [1]

Consider the following vector field

$$\dot{x} = y$$

$$\dot{y} = y(1 - x^2) - x$$

we have $(0,0)$ is critical point and $Df(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. To find the

eigenvalues associated with critical point $(0,0)$ is

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$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & 1 - \lambda \end{vmatrix} = -\lambda(1 - \lambda) + 1 = \lambda^2 - \lambda + 1 = 0, \text{ thus}$$

$\lambda = \frac{1 \pm \sqrt{1 - 4}}{2}$. Therefore, $\lambda_1 = \frac{1}{2} + 3i$ and $\lambda_2 = \frac{1}{2} - 3i$. Hence, the critical point (0,0) is hyperbolic.

Definition (1.1.40) [1]

A closed orbit C in a phase portrait is called a limit cycle if it is isolated from all other closed orbits; more precisely, if there is a tubular neighborhood of C which contains no other closed orbits.

Remark (1.1.41) [1]

There are three types of limit cycle:

- (1) The stable (or attracting) limit cycle, where orbits spiral into the closed orbit from both sides as $t \rightarrow \infty$.
- (2) The unstable (or repelling) limit cycle, where orbits spiral a way from the closed orbit on both sides as $t \rightarrow \infty$.
- (3) The semi-stable limits cycle, where orbits spiral towards the closed orbit from one side and spiral a way from it on the other.

Example (1.1.42) [1]

Consider the system

$$\dot{x} = -y + x(1 - \sqrt{x^2 + y^2})$$

$$\dot{y} = x + y(1 - \sqrt{x^2 + y^2})$$

In polar coordinates $x = r \cos(\theta)$, $y = r \sin(\theta)$, the system becomes

$$r^2 = x^2 + y^2 \text{ and } \tan(\theta) = \frac{y}{x}, x \neq 0, \text{ and differentiating with respect to } t.$$

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We find $2r\dot{r} = 2x\dot{x} + 2y\dot{y}$ and $\sec^2(\theta)\dot{\theta} = \frac{\dot{y}x - \dot{x}y}{x^2}$, we have

$$\begin{aligned} 2r\dot{r} &= 2x(-y + x(1-r)) + 2y(x + y(1-r)) \\ &= -2xy + 2x^2(1-r) + 2xy + 2y^2(1-r) \\ &= 2(1-r)(x^2 + y^2) = 2r^2(1-r). \text{ Therefore, } \dot{r} = r(1-r). \end{aligned}$$

Also,

$$\begin{aligned} \sec^2(\theta)\dot{\theta} &= \frac{\dot{y}x - \dot{x}y}{x^2} = \frac{(x + y(1-r))x - (-y + x(1-r))y}{x^2} \\ &= \frac{x^2 + xy(1-r) + y^2 - xy(1-r)}{x^2} \\ &= \frac{x^2 + y^2}{x^2} = \frac{r^2}{r^2 \cos^2(\theta)} = \sec^2(\theta) \end{aligned}$$

Therefore, $\dot{\theta} = 1$. Clearly, $r(t) = 1$ and $\theta(t) = t$ is a solution giving a closed orbit consisting of the circle $x^2 + y^2 = 1$. Hence, the limit cycle is $x^2 + y^2 = 1$. For $0 < r < 1$, $\dot{r}$ is positive and the trajectories in this region spiral outwards towards $r=1$. $r > 1$, $\dot{r}$ is negative and orbits spiral inwards as t increases. Hence, the limit cycle is stable when $r=1$.

Example (1.1.43) [1]

Consider the system

$$\dot{r} = r(1-r)^2$$

$$\dot{\theta} = 1$$

clearly, $r(t) = 1$ and $\theta(t) = t$ is a solution giving a closed orbit consisting of the circle $x^2 + y^2 = 1$. Hence, the limit cycle is $x^2 + y^2 = 1$.

However, $\dot{r}$ is positive for all $0 < r < 1$ and $r > 1$, so the limit cycle is semi stable.

Definition (1.1.44) [28]

The flow φ is the solution of the differential equation $\dot{x} = f(x)$.

Definition (1.1.45) [28]

For each point $x \in M$ (M is an n -dimensional phase space), let $T_x M$ denoted the collection of infinitesimal vectors attached to x , that is, the tangent vectors at x . The tangent space at x is an n -dimensional vector space, isomorphic to $\mathbb{R}^n$.

1.2 The Bifurcation and Some of Its Types.

A qualitative change in dynamics occurring upon a small change in a parameter is called bifurcation. [28]

As we have seen, systems of physical interest typically have parameters which appear in the defining systems of equations. As these parameters are varied, changes may occur in the qualitative structure of the solutions for certain parameter values. These changes are called bifurcations and the parameter values are called bifurcation value. For example when we press on the centre of a wooden piece and consider that the force by which we press is the parameter, we will see that the piece of wood will be as bow and change its shape until force arises at special degree called (bifurcation value) at which the shape of the piece changes then gets branch. The point at which this behavior appears called bifurcation point.

There are two types of bifurcation for vector fields and maps which are local bifurcation such as saddle node, transcritical, pitchfork, period-doubling, Hopf and other bifurcations, and global bifurcation such as homoclinic, heteroclinic and other bifurcations.

In this section, we study local bifurcation for vector fields, especially, saddle node, transcritical, pitchfork and Hopf bifurcation. We will introduce definitions and theorems about the bifurcation and its types with some examples.

Remark (1.2.1) [38]

By the term "local" we mean bifurcations occurring in a neighborhood of a critical point.

Definition (1.2.2) [38]

A critical point $(x, \mu) = (0, 0)$ of a one parameter family of one-dimensional vector fields

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}, \mu \in \mathbb{R} \dots\dots\dots(1-21)$$

is said to undergo a bifurcation at $\mu=0$ if the flow for μ near zero and x near zero is not qualitative the same as the flow near $x=0$ at $\mu=0$.

Remark (1.2.3) [38]

The phrase "qualitatively the same" is a bit vague. It can be made precise by substituting the term " C^0 -equivalent" and this is perfectly adequate for the study of the bifurcation of critical points of one-dimension vector fields.

Definition (1.2.4) [18]

The method of displaying the points at which a parameterized family of vector fields $f(x, \mu)$ bifurcates is called a bifurcation diagram, and is a graph for which the horizontal axis represents values of μ and the vertical axis represents higher iterates of the variable (normally x).

Now, we recall the implicit map theorem of calculus in two variables.

Theorem (1.2.5) [10]

Suppose $G: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a C^1 -map (i.e. both partial derivatives of G exist and are continuous.) Suppose further that

(1) $G(x_0, \mu_0) = 0$.

(2) $\frac{\partial G}{\partial \mu}(x_0, \mu_0) \neq 0$.

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Then there exist open intervals I about x_0 and J about μ_0 and a C^1 -map $P: I \rightarrow J$ satisfying:

- (1) $P(x_0) = \mu_0$
- (2) $G(x, p(x)) = 0$ for all $x \in I$.

Now, we have some types of bifurcation. We will start with the first type of bifurcation which is a saddle node bifurcation.

Definition (1.2.6) [38]

We say that a one parameter family of one dimensional vector fields $\dot{x} = f(x, \mu)$, $x \in \mathbb{R}, \mu \in \mathbb{R}$ has a saddle node bifurcation at $\mu = \mu_0$ if on one side of a parameter value there are no critical points and on the other side there are two critical points.

Remark (1.2.7) [38]

We say that $(0,0)$ is a nonhyperbolic critical point of vector field (1–21) if

(1) $f(0,0) = 0$ (1–22)

(2) $\frac{\partial f}{\partial x}(0,0) = 0$ (1–23)

where the equation (1–22) is simply critical point condition and equation (1–23) is the zero eigenvalue condition.

Remark (1.2.8) [28]

The bifurcation of differential equation is concerned with changes in the qualitative behavior of its phase portrait as a parameter varies.

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Example (1.2.9) [38]

Consider the vector field

$$\dot{x} = f(x, \mu) = \mu - x^2, \quad x \in \mathbb{R}, \mu \in \mathbb{R}. \quad \dots\dots\dots(1-24)$$

We have $f(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0) = 0$. Therefore, $(0,0)$ is a nonhyperbolic

critical point. To find the critical point setting $\dot{x} = \mu - x^2 = 0$ then $\mu = x^2$, thus $x = \pm\sqrt{\mu}$. This represents a parabola in the μ - x plane as shown in Figure (3). Thus, there two cases for μ :

Case(1): If $\mu < 0$, the vector field (1-24) has no critical points, and it is decreasing in x .

Case(2): If $\mu > 0$, the vector field has two critical points, one of the critical points is stable (represented by the solid branch of the parabola), and the other critical point is unstable (represented by the broken branch of the parabola). Therefore, the vector field (1-24) has a saddle node bifurcation at $\mu=0$.

This is an example for the bifurcation. We refer to $(x, \mu) = (0, 0)$ as a bifurcation point and the parameter value $\mu=0$ as a bifurcation value and the figure is referred to as a bifurcation diagram

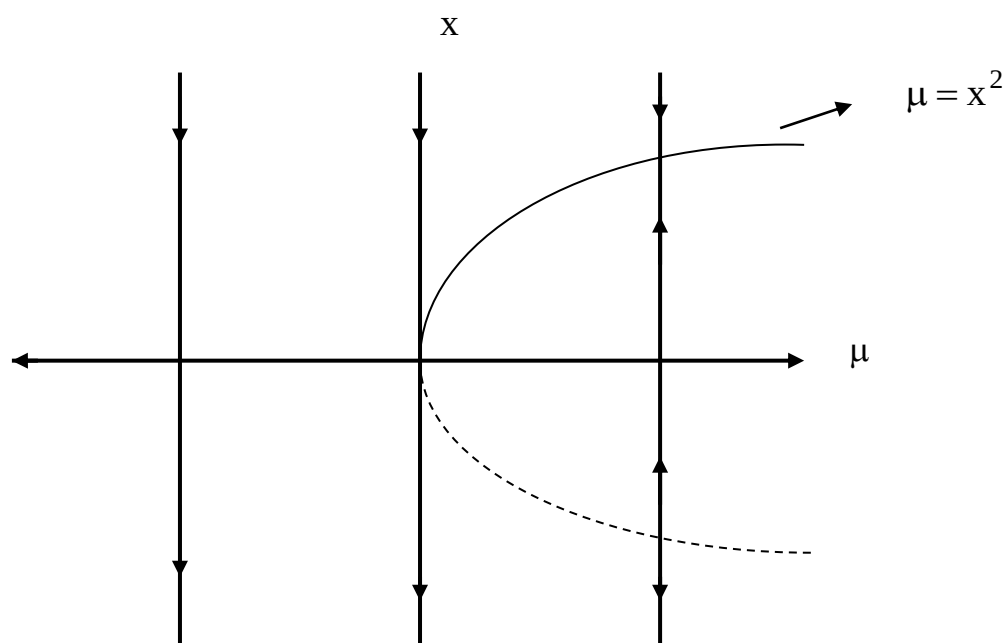


Figure (3)

Remark (1.2.10)

In example (1.2.9) a unique curve (or branch) of critical points passes through the origin. Moreover, the curve lay entirely on one side of $\mu=0$ in the μ - x plane.

Now, we want to drive conditions under which a general one-parameter family of one-dimensional vector fields will undergo a saddle node bifurcation exactly as in example (1.2.9). These conditions will involve derivatives of the vector field evaluated at the bifurcation point and are obtained by a consideration of the geometry of the curve of critical points in the μ - x plane in a neighborhood of the bifurcation point.

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Theorem (1.2.11) [38]

Consider a one parameter family of one-dimensional vector field
 (1-21) such that $f(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0) = 0$, then the sufficient conditions for the vector field to undergo a saddle node bifurcation at $(0,0)$ are:

$$(1) \frac{\partial f}{\partial \mu}(0,0) \neq 0.$$

$$(2) \frac{\partial^2 f}{\partial x^2}(0,0) \neq 0.$$

Proof:

The geometry of the curves of critical points associated with the bifurcation in example (1.2.9) had the following characterizes:

- (i) It was tangent to the line $\mu = 0$ at $x=0$.
- (ii) It lay entirely to one side of $\mu = 0$.

Now, if we have $\frac{\partial f}{\partial \mu}(0,0) \neq 0$, then by implicit map theorem, there exists

a unique function $\mu = \mu(x)$, $\mu(0) = 0$ such that $f(x, \mu(x)) = 0$. Now, we went to derive conditions in terms of derivatives of f evaluated at $(x, \mu) = (0, 0)$ so that we have

$$\frac{d\mu}{dx}(0) = 0 \dots\dots\dots(1-25)$$

$$\frac{d^2\mu}{dx^2}(0) \neq 0. \dots\dots\dots(1-26)$$

Equations (1-25) and (1-26), along with

$f(0,0) = 0, \frac{\partial f}{\partial x}(0,0) = 0, \frac{\partial f}{\partial \mu}(0,0) \neq 0$ imply that $(x, \mu) = (0,0)$ is a

bifurcation point at which a saddle node bifurcation occurs. We can

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derive expressions for (1-25) and (1-26) in terms of derivatives of f at the bifurcation point by implicitly differentiating f along the curve of critical

points. Using $\frac{\partial f}{\partial \mu}(0,0) \neq 0$, we have

$$f(x, \mu(x)) = 0. \quad \dots\dots\dots(1-27)$$

Differentiating (1-27) with respect to x gives

$$\begin{aligned} \frac{df}{dx}(x, \mu(x)) &= 0 \\ \frac{\partial f}{\partial x}(x, \mu(x)) + \frac{\partial f}{\partial \mu}(x, \mu(x)) \cdot \frac{d\mu}{dx}(x) &= 0. \quad \dots\dots\dots(1-28) \end{aligned}$$

Evaluating (1-28) at $(x, \mu) = (0,0)$, we obtain

$$\frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial \mu}(0,0) \cdot \frac{d\mu}{dx}(0) = 0, \text{ thus } \frac{d\mu}{dx}(0) = \frac{-\frac{\partial f}{\partial x}(0,0)}{\frac{\partial f}{\partial \mu}(0,0)}.$$

Thus, we see that $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial \mu}(0,0) \neq 0$ imply that

$$\frac{d\mu}{dx}(0) = 0. \quad \dots\dots\dots(1-29)$$

Now, let us differentiate (1-28) with respect to x , to obtain

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(x, \mu(x)) + 2 \frac{\partial^2 f}{\partial x \partial \mu}(x, \mu(x)) \cdot \frac{d\mu}{dx}(x) + \frac{\partial^2 f}{\partial \mu^2}(x, \mu(x)) \cdot \left(\frac{d\mu}{dx}(x)\right)^2 \\ + \frac{\partial f}{\partial \mu}(x, \mu(x)) \cdot \frac{d^2 \mu}{dx^2}(x) &= 0. \quad \dots\dots\dots(1-30) \end{aligned}$$

Evaluating (1-30) at $(x, \mu) = (0, 0)$ and (1-29) gives

$$\frac{\partial^2 f}{\partial x^2}(0,0) + \frac{\partial f}{\partial \mu}(0,0) \cdot \frac{d^2 \mu}{dx^2}(0) = 0, \text{ thus } \frac{d^2 \mu}{dx^2}(0) = \frac{-\frac{\partial^2 f}{\partial x^2}(0,0)}{\frac{\partial f}{\partial \mu}(0,0)}.$$

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Hence, $\frac{d^2\mu}{dx^2} \neq 0$ provided we have $\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$. ★

Now, we apply the theorem (1.2.11) in the following example.

Example (1.2.12)

Consider the vector field $\dot{x} = f(x, \mu) = \mu + x^2$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$, we have

$f(0,0) = 0$, $\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial \mu}(0,0) = 1 \neq 0$, and $\frac{\partial^2 f}{\partial x^2}(0,0) = 2 \neq 0$. Hence,

the vector field has a saddle node bifurcation at $(0,0)$ as shown in figure (4).

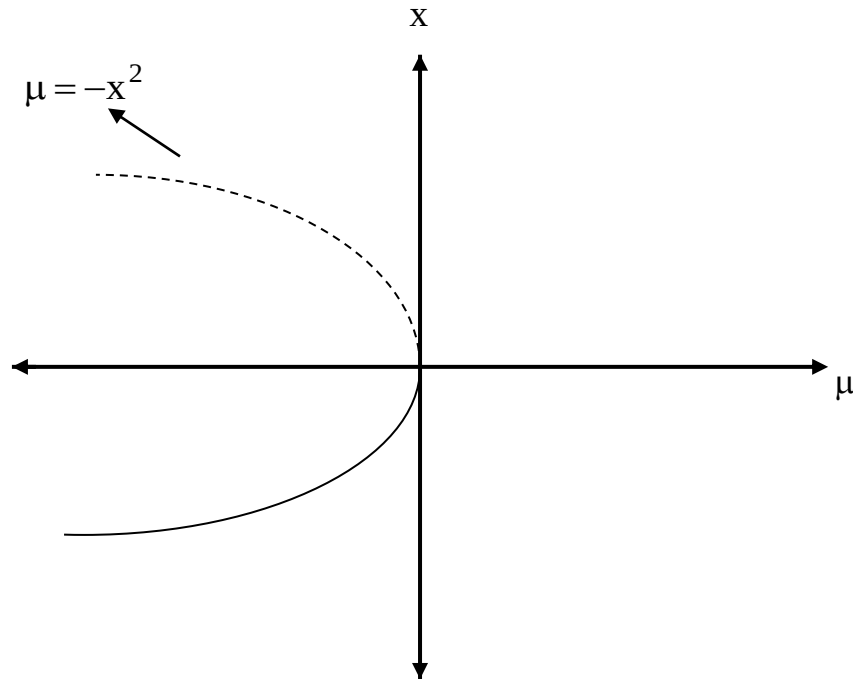


Figure (4)

Remark (1.2.13) [38]

The vector fields $\dot{x} = \mu \pm x^2$ can be viewed as the normal form for saddle node bifurcation.

Remark (1.2.14) [38]

In the mathematical area of bifurcation theory a saddle node bifurcation (or tangent bifurcating) is a local bifurcation in which two critical points (or equilibria) of a dynamical system collide and annihilate each other.

Remark (1.2.15) [39]

There are two types of a saddle node bifurcation, subcritical and supercritical saddle node bifurcation. The subcritical saddle node bifurcation occurs whenever $\frac{\partial^2 f}{\partial x^2}(0,0) \cdot \frac{\partial f}{\partial \mu}(0,0) > 0$, and the supercritical saddle node bifurcation occurs whenever $\frac{\partial^2 f}{\partial x^2}(0,0) \cdot \frac{\partial f}{\partial \mu}(0,0) < 0$.

Example (1.2.9) is a supercritical saddle node bifurcation and example (1.2.12) is a subcritical saddle node bifurcation.

Now, we will define the second type of bifurcation which is a transcritical bifurcation.

Definition (1.2.16) [38]

We say that one parameter family of one dimensional vector field (1 – 21) has a transcritical bifurcation at $\mu=\mu_0$, if it has two critical points and the stability of the two critical points is exchanged at the bifurcation value $\mu=\mu_0$.

Example (1.2.17) [38]

Consider the vector field $\dot{x} = f(x, \mu) = \mu x - x^2$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$. We have $f(0,0) = 0$, $\frac{\partial f}{\partial x}(0,0) = 0$. Therefore, $(0,0)$ is nonhyperbolic critical point. To find the critical points of the vector field setting $\dot{x} = \mu x - x^2 = 0$ thus $x(\mu - x) = 0$. Then, the critical points are $x = 0$ and $x = \mu$. There are three cases for μ :

Case(1) If $\mu < 0$, there are two critical points; $x = 0$ is stable and $x = \mu$ is unstable.

Case(2) If $\mu = 0$, these two critical points coalesce.

Case(3) If $\mu > 0$, there are two critical points; $x = 0$ is unstable and $x = \mu$ is stable.

Thus, an exchange of stability has occurred at $\mu = 0$. Hence, the vector field has a transcritical bifurcation at $(0,0)$. In figure (5) the stability represented by the solid branch of the line and the unstability represented by the broken of the line.

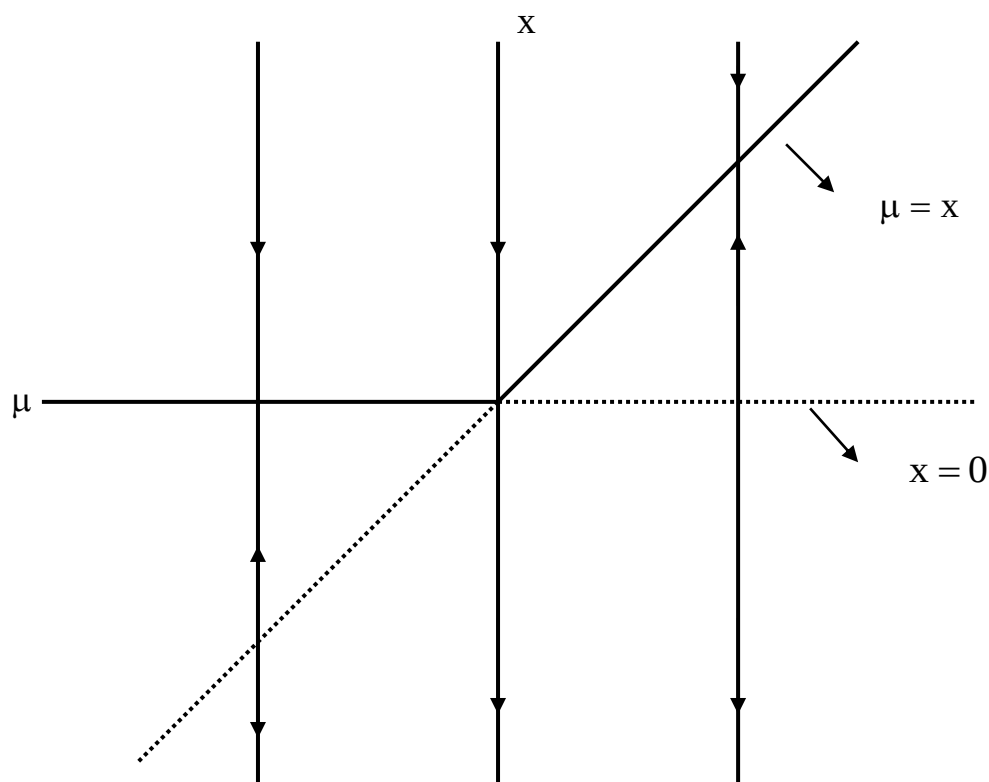


Figure (5)

Remark (1.2.18)

In example (1.2.17), curves of critical points intersected at the origin in the μ - x plane. Both curves existed on either side of $\mu=0$. However, the stability of the critical points along a given curve changed on passing through $\mu=0$.

Now, we want to drive conditions under which a general one-parameter family of one-dimensional vector fields will undergo a transcritical bifurcation exactly as in example (1.2.17). These conditions will involve derivatives of the vector field evaluated at the bifurcation point and are obtained by a consideration of the geometry of the curve of critical points in the μ - x plane in a neighborhood of the bifurcation point.

Theorem (1.2.19) [38]

Consider a one parameter family of one-dimensional vector field (1-21) such that $f(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0) = 0$, then the sufficient conditions for the vector field to undergo a transcritical bifurcation at $(0,0)$ are:

$$(1) \frac{\partial f}{\partial \mu}(0,0) = 0$$

$$(2) \frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$$

$$(3) \frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0$$

Proof:

The geometry of the curves of critical points associated with the bifurcation in example (1.2.17) had the following characterizes:

- (i) Two curves of critical points passed through $(x, \mu) = (0,0)$, one given by $x = 0$, the other by $x = \mu$.
- (ii) Both curves of critical points existed on both sides of $\mu = 0$.
- (iii) The stability along each curve of critical points changed on passing through $\mu = 0$.

We must have the condition $\frac{\partial f}{\partial \mu}(0,0) = 0$, in order to insure two curves of critical points passing through $(0, 0)$ (otherwise, the implicit function theorem would allow only one curve). We require $x=0$ to be a curve of critical point by assuming the vector field (1-21) has the form

$$\dot{x} = f(x, \mu) = xF(x, \mu), \quad x \in \mathbb{R}, \mu \in \mathbb{R}, \quad \dots\dots\dots(1-31)$$

where

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$$F(x, \mu) \equiv \begin{cases} \frac{f(x, \mu)}{x}, & x \neq 0 \\ \frac{\partial f}{\partial x}(0, \mu), & x = 0. \end{cases} \quad \dots\dots\dots(1-32)$$

Since $x = 0$ is a curve of critical points for (1-31), in order to obtain an additional curve of critical points passing through $(x, \mu) = (0, 0)$, we need conditions on F whereby F has a curve of zeros passing through $(x, \mu) = (0, 0)$ (that is not given by $x = 0$). Using (1-32), it is easy to verify the following:

$$F(0, 0) = 0 \quad \dots\dots\dots(1-33)$$

$$\frac{\partial F}{\partial x}(0, 0) = \frac{\partial^2 f}{\partial x^2}(0, 0) \quad \dots\dots\dots(1-34)$$

$$\frac{\partial^2 F}{\partial x^2}(0, 0) = \frac{\partial^3 f}{\partial x^3}(0, 0) \quad \dots\dots\dots(1-35)$$

$$\frac{\partial F}{\partial \mu}(0, 0) = \frac{\partial^2 f}{\partial x \partial \mu}(0, 0). \quad \dots\dots\dots(1-36)$$

Now, let us assume that (1-36) is not zero, then by the implicit map theorem there exists a map, $\mu(x)$, defined for x sufficiently small, such that

$$F(x, \mu(x)) = 0. \quad \dots\dots\dots(1-37)$$

Clearly, $\mu(x)$ is a curve of critical points of the vector field (1-31). In order for $\mu(x)$ to not coincide with $x = 0$ and to exist on both sides of

$\mu = 0$, we must require that $0 < \left| \frac{d\mu}{dx}(0) \right| < \infty$. Implicitly differentiating

(1-37) as in the case of the saddle node bifurcation, we obtain

$$\frac{dF}{dx}(x, \mu(x)) = 0 \quad \dots\dots\dots(1-38)$$

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$$\frac{\partial F}{\partial x}(x, \mu(x)) + \frac{\partial F}{\partial \mu}(x, \mu(x)) \cdot \frac{d\mu}{dx}(x) = 0. \dots\dots\dots(1-39)$$

Evaluating (1-39) at $(x, \mu(x)) = (0,0)$, to obtain

$$\frac{\partial F}{\partial x}(0,0) + \frac{\partial F}{\partial \mu}(0,0) \cdot \frac{d\mu}{dx}(0) = 0. \text{ Therefore,}$$

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial F}{\partial x}(0,0)}{\frac{\partial F}{\partial \mu}(0,0)}. \dots\dots\dots(1-40)$$

Using (1-33), (1-34), (1-35), (1-36) and (1-40) becomes

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial^2 f}{\partial x^2}(0,0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0,0)}. \star$$

Example (1.2.20)

Consider the vector field $\dot{x} = f(x, \mu) = \mu x + x^2$, $\mu \in \mathbb{R}$, $x \in \mathbb{R}$, we have

$f(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0) = 0$. Hence, $(0,0)$ is a nonhyperbolic critical point.

$\frac{\partial f}{\partial \mu}(0,0) = 0$, $\frac{\partial^2 f}{\partial x^2}(0,0) = 2 \neq 0$ and $\frac{\partial^2 f}{\partial x \partial \mu}(0,0) = 1 \neq 0$. Then, by theorem

(1.2.19) the vector field $\dot{x} = f(x, \mu) = \mu x + x^2$ has a transcritical bifurcation at $(0, 0)$ as shown in figure (6).

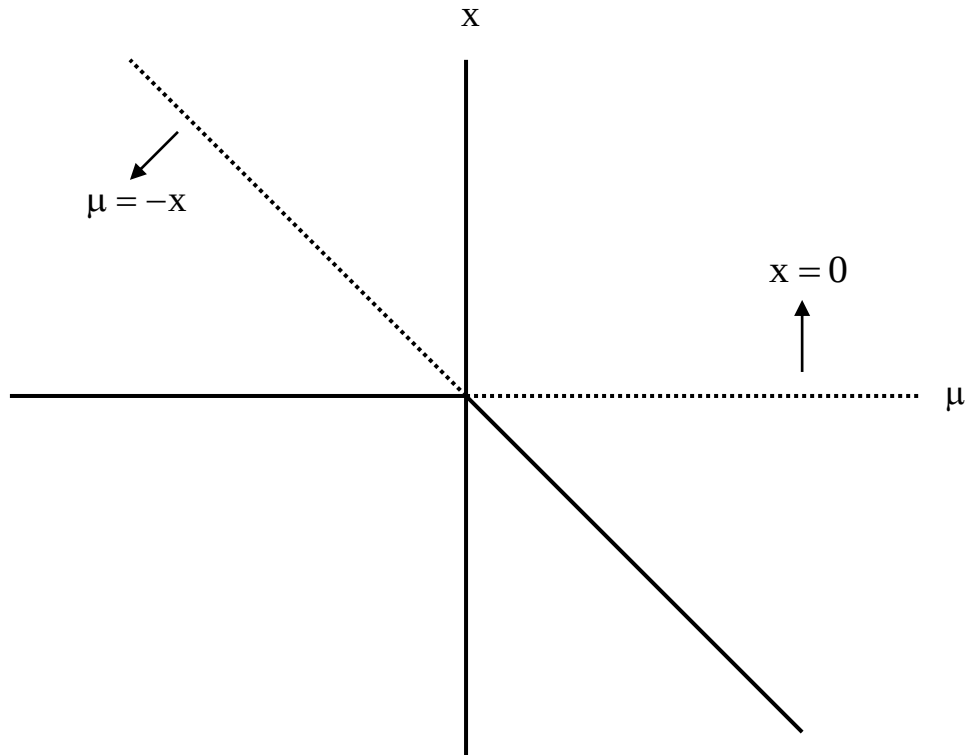


Figure (6)

Remark (1.2.21) [38]

The vector fields $\dot{x} = f(x, \mu) = \mu x \pm x^2$ can be viewed as a normal form for the transcritical bifurcation.

In bifurcation theory, a field within mathematics, a transcritical bifurcation is a particular kind of local bifurcation, meaning that it is characterized by an equilibrium having an eigenvalue whose real part passes through zero. Both before and after the bifurcation, there is one unstable and stable fixed point. However, their stability is exchanged when they collide. So, the unstable critical point becomes stable and vice versa. [34]

Now, we will define the third type of the bifurcation which is a pitchfork bifurcation.

Definition (1.2.22) [38]

We say that a one parameter family of one dimensional vector field $\dot{x} = f(x, \mu)$, $x \in \mathbb{R}, \mu \in \mathbb{R}$ has a pitchfork bifurcation at $\mu = \mu_0$ if it involves change from stable (or unstable) to an unstable (or stable) critical point, together with the birth of two stable (or unstable) critical points when μ passes through μ_0 .

Example (1.2.23) [38]

Consider the vector field $\dot{x} = f(x, \mu) = \mu x - x^3$, $\mu \in \mathbb{R}, x \in \mathbb{R}$, we have $f(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0) = 0$. Hence, $(0,0)$ is a nonhyperbolic critical point.

Moreover, the critical points of vector field are given by $\mu x - x^3 = 0$ thus $x(\mu - x^2) = 0$. Then $x=0$ and $x^2 = \mu$ are critical points and are plotted in Figure (7). Hence, there are two cases for μ :

Case (1) if $\mu < 0$, there is one critical point, $x=0$, which is stable.

Case (2) if $\mu > 0$, $x=0$ is still a critical point, but two new critical points have been created at $\mu=0$ and are given by $x^2 = \mu$. In process, $x=0$ has become unstable for $\mu > 0$, with the other two critical points stable.

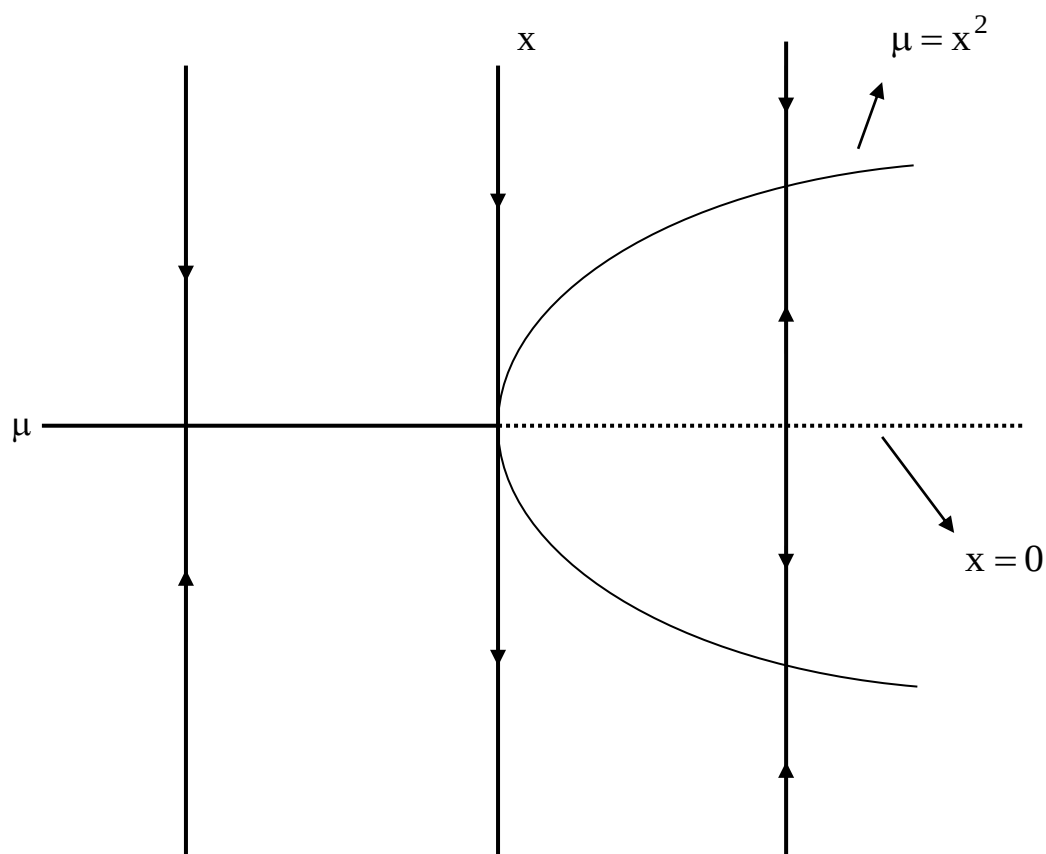


Figure (7)

Remark (1.2.24) [38]

In example (1.2.23) two curves of critical points intersected at the origin in the μ - x plane. Only one curve $x=0$ existed on both sides of $\mu=0$. However, its stability changed on passing through $\mu=0$. The other curve of critical points lay entirely to one side of $\mu=0$ and had a stability type that was the opposite of $x=0$ for $\mu > 0$.

Now, we want to drive conditions under which a general one-parameter family of one-dimensional vector fields will undergo a pitchfork bifurcation exactly as in example (1.2.23). These conditions will involve derivatives of the vector field evaluated at the bifurcation point and are obtained by a consideration of the geometry of the curve of critical points in the μ - x plane in a neighborhood of the bifurcation point.

Theorem (1.2.25) [38]

Consider a one parameter family of one dimensional vector field (1-21) such that $f(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0) = 0$, then the sufficient conditions for the vector field to undergo a pitchfork bifurcation at $(0,0)$ are:

$$(1) \frac{\partial f}{\partial \mu}(0,0) = 0$$

$$(2) \frac{\partial^2 f}{\partial x^2}(0,0) = 0$$

$$(3) \frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0$$

$$(4) \frac{\partial^3 f}{\partial x^3}(0,0) \neq 0$$

Proof:

The geometry of the curves of critical points associated with the bifurcation in example (1.2.23) had the following characterizes:

(i) Two curves of critical points passed through $(x, \mu) = (0,0)$, one given by $x = 0$, the other by $\mu = x^2$.

(ii) The curve $x = 0$ existed on both sides of $\mu = 0$, the curve $\mu = x^2$ existed on one side of $\mu = 0$.

(iii) The critical points on the curve $x = 0$ had different stability types on opposite sides of $\mu = 0$. The critical points on $\mu = x^2$ all had the same stability type.

We must have the condition $\frac{\partial f}{\partial \mu}(0,0) = 0$, in order to insure more than one curve of critical points passing through $(0, 0)$ (otherwise, the

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implicit map theorem would allow only one curve). We require $x=0$ to be a curve of critical points by assuming the vector field has the form

$$\dot{x} = xF(x, \mu), \quad x \in \mathbb{R}, \mu \in \mathbb{R} \quad \dots\dots\dots(1-41)$$

where

$$F(x, \mu) \equiv \begin{cases} \frac{f(x, \mu)}{x}, & x \neq 0 \\ \frac{\partial f}{\partial x}(0, 0), & x = 0 \end{cases} \quad \dots\dots\dots(1-42)$$

In order to have a second curve of critical points passing through $(x, \mu) = (0, 0)$, we must have $F(0, 0) = 0$, with

$$\frac{\partial F}{\partial \mu}(0, 0) \neq 0. \quad \dots\dots\dots(1-43)$$

Equation (1-43) insures that only one additional curve of fixed point's passes through at $(x, \mu) = (0, 0)$. Using (1-43) the implicit map theorem implies that for x sufficiently small, there exists a unique map $\mu(x)$ such that

$$F(x, \mu(x)) = 0. \quad \dots\dots\dots(1-44)$$

In order for the curve of critical points, $\mu(x)$, to satisfy the above mentioned characteristics, it is sufficient to have

$$\frac{d\mu}{dx}(0) = 0 \quad \dots\dots\dots(1-45)$$

$$\frac{d^2\mu}{dx^2}(0) \neq 0. \quad \dots\dots\dots(1-46)$$

The conditions for (1-45) and (1-46) to hold in terms of the derivatives of F evaluated at the bifurcation point. Now, we have by implicit differentiation of (1-44) with respect to x .

$$\frac{dF}{dx}(x, \mu(x)) = 0, \text{ thus}$$

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$$\frac{\partial F}{\partial x}(x, \mu(x)) + \frac{\partial F}{\partial \mu}(x, \mu(x)) \cdot \frac{d\mu}{dx}(x) = 0. \quad \dots\dots\dots(1-47)$$

Evaluate (1-47) at $(x, \mu) = (0, 0)$, we get $\frac{\partial F}{\partial x}(0, 0) + \frac{\partial F}{\partial \mu}(0, 0) \cdot \frac{d\mu}{dx}(0) = 0$.

$$\text{Therefore, } \frac{d\mu}{dx}(0) = \frac{-\frac{\partial F}{\partial x}(0, 0)}{\frac{\partial F}{\partial \mu}(0, 0)} = 0.$$

Let us differentiate (1-47) once more with respect to x , we get

$$\begin{aligned} & \frac{\partial^2 F}{\partial x^2}(x, \mu(x)) + 2 \frac{\partial^2 F}{\partial x \partial \mu}(x, \mu(x)) \cdot \frac{d\mu}{dx}(x) + \frac{\partial^2 F}{\partial \mu^2}(x, \mu(x)) \cdot \left(\frac{d\mu}{dx}(x)\right)^2 \\ & + \frac{\partial F}{\partial \mu}(x, \mu(x)) \cdot \frac{d^2 \mu}{dx^2}(x) = 0. \quad \dots\dots\dots(1-48) \end{aligned}$$

Evaluating (1-48) at $(x, \mu) = (0, 0)$, and using $\frac{d\mu}{dx}(0) = 0$ gives

$$\frac{\partial^2 F}{\partial x^2}(0, 0) + \frac{\partial F}{\partial \mu}(0, 0) \cdot \frac{d^2 \mu}{dx^2}(0) = 0, \text{ thus } \frac{d^2 \mu}{dx^2}(0) = \frac{-\frac{\partial^2 F}{\partial x^2}(0, 0)}{\frac{\partial F}{\partial \mu}(0, 0)} \neq 0$$

or in terms of the original vector field

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial^2 f}{\partial x^2}(0, 0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0, 0)} = 0 \text{ and } \frac{d^2 \mu}{dx^2}(0) = \frac{-\frac{\partial^3 f}{\partial x^3}(0, 0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0, 0)} \neq 0. \quad \star$$

Example (1.2.26)

Consider the vector field $\dot{x} = f(x, \mu) = \mu x + x^3$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$, we have

$$f(0, 0) = 0, \frac{\partial f}{\partial x}(0, 0) = 0, \frac{\partial f}{\partial \mu}(0, 0) = 0, \frac{\partial^2 f}{\partial x^2}(0, 0) = 0, \frac{\partial^2 f}{\partial x \partial \mu}(0, 0) = 1 \neq 0 \text{ and}$$

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$\frac{\partial^3 f}{\partial x^3}(0,0) = 6 \neq 0$. Hence, by theorem (1.2.25) the vector field has pitchfork bifurcation at $(x, \mu) = (0, 0)$ as shown in figure (8).

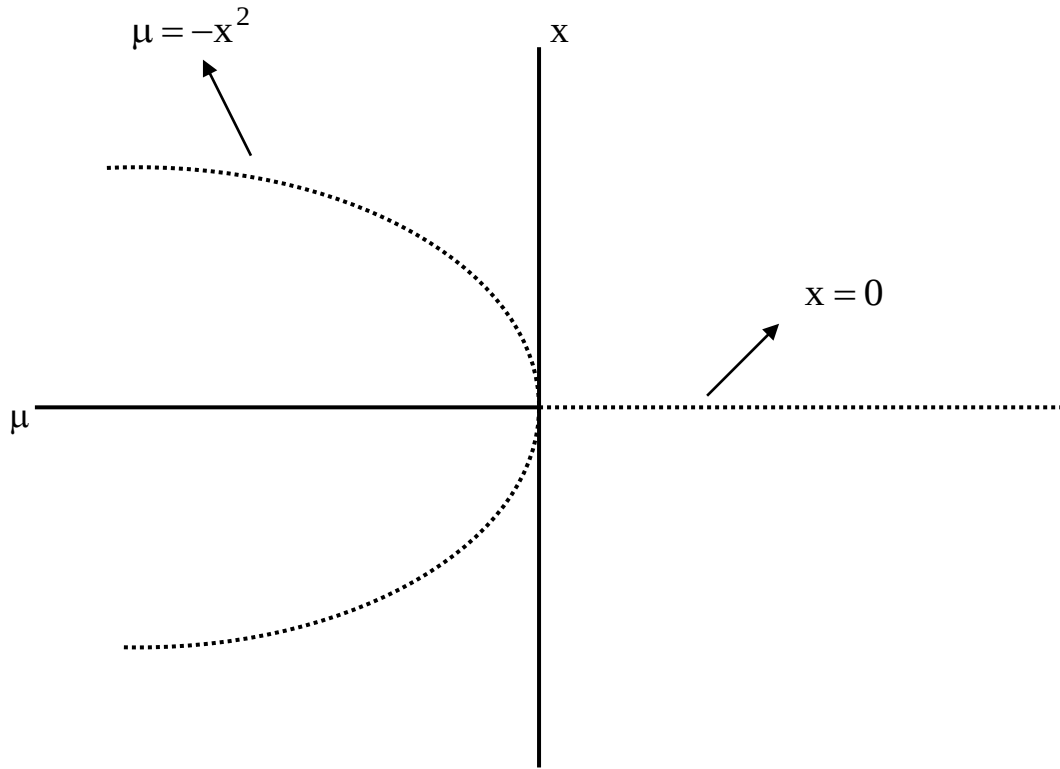


Figure (8)

Remark (1.2.27) [38]

The vector field $\dot{x} = f(x, \mu) = \mu x - x^3$ can be viewed as the normal form for pitchfork bifurcation.

Remark (1.2.28) [34]

In flows, (continuous dynamical system described by ordinary differential equation), pitchfork bifurcation occurs generically in systems with symmetry.

Remark (1.2.29) [38]

A critical point (x_0, μ_0) of one dimensional vector field is a bifurcation point if either more than one curve of critical points passes through (x_0, μ_0) in the x - μ plane or if only one curve of critical points passes through (x_0, μ_0) in the x - μ plane; then it (locally) lies entirely on one side of the line $\mu = \mu_0$ in the x - μ plane.

Remark (1.2.30) [21]

Changes in the number or stability-type of critical points can only happen when critical points are nonhyperbolic. The converse is not true as shown in the following example.

Example (1.2.31) [38]

Consider the vector field $\dot{x} = f(x, \mu) = \mu - x^3$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$. We have $f(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0) = 0$. Then $(0, 0)$ is nonhyperbolic critical point. To find the fixed points, setting $\dot{x} = \mu - x^3 = 0$ thus $\mu = x^3$, then dynamics of the vector field $\dot{x} = f(x, \mu) = \mu - x^3$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ are qualitatively the same for $\mu > 0$ and $\mu < 0$. Hence, there is not bifurcation.

Remark (1.2.32) [21, 38, 39]

There are two types of pitchfork bifurcation are subcritical and supercritical pitchfork bifurcation. The subcritical pitchfork bifurcation occurs at $\mu = \mu_0$ if it involves change from unstable to stable critical point, together with the birth of two unstable critical points when μ passes through μ_0 (or whenever $\frac{\partial^2 f}{\partial x \partial \mu}(x, \mu) \cdot \frac{\partial^3 f}{\partial x^3}(x, \mu) > 0$). The supercritical

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pitchfork bifurcation occurs at $\mu=\mu_0$ if it involves change from stable to unstable critical point, together with the birth of two stable critical points when μ passes through μ_0 (or whenever $\frac{\partial^2 f}{\partial x \partial \mu}(x, \mu) \cdot \frac{\partial^3 f}{\partial x^3}(x, \mu) < 0$). The vector field in Example (1.2.23) has supercritical pitchfork bifurcation at $\mu = 0$, and the vector field in example (1.2.26) has subcritical pitchfork bifurcation at $\mu = 0$.

Example (1.2.33) [21]

Consider the vector field $\dot{x} = f(x, \mu) = \mu x + ax^3$, $\mu \in \mathbb{R}$, $x \in \mathbb{R}$ and a is constant. Note that, if $a < 0$ the pitchfork bifurcation is subcritical and if $a > 0$ it is supercritical. Critical points are $x=0$ for all μ and $x = \pm \sqrt{-\frac{\mu}{a}}$ for all $\mu < 0$ (subcritical case) or for all $\mu > 0$ (supercritical case). The eigenvalue at $x=0$ is μ , so $x=0$ is stable (subcritical case) and unstable (supercritical case). The eigenvalues at $x = \pm \sqrt{-\frac{\mu}{a}}$ is -2μ , so these points are unstable in the subcritical case and stable in the supercritical cases.

Now, we will define the fourth type of bifurcation which is a Hopf bifurcation in two dimensional.

Definition (1.2.34) [1]

We say that the parameterized system

$$\left. \begin{aligned} \dot{x} &= f_1(x, y, \mu) \\ \dot{y} &= f_2(x, y, \mu) \end{aligned} \right\} \dots\dots\dots(1-49)$$

has a Hopf bifurcation at μ_0 if the eigenvalues of the linearization of

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(1-49) at μ_0 are complex conjugates λ_1, λ_2 and become purely imaginary at the bifurcation point $\mu = \mu_0$ and for $\mu > \mu_0$ a limit cycle exists.

Remark (1.2.35) [1]

The Hopf bifurcation is characterized by a stability change of the fixed point accompanied by the creation of a limit cycle.

Example (1.2.36) [1]

Consider the one parameter system

$$\left. \begin{aligned} \dot{x} &= \mu x - y - x(x^2 + y^2) \\ \dot{y} &= x + \mu y - y(x^2 + y^2) \end{aligned} \right\} \dots\dots\dots(1-50)$$

The origin is a critical point of the system for all μ . To find eigenvalues of the linearization of (1-50) at the origin.

$$\det \left(\begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \det \begin{bmatrix} \mu - \lambda & -1 \\ 1 & \mu - \lambda \end{bmatrix} = (\mu - \lambda)^2 + 1$$

$$= \lambda^2 - 2\mu\lambda + \mu^2 + 1 = 0.$$

$$\text{Thus } \lambda = \frac{2\mu \pm \sqrt{4\mu^2 - 4(\mu^2 + 1)}}{2} = \frac{2\mu \pm \sqrt{-4}}{2} = \mu \pm i. \text{ Hence,}$$

the eigenvalues of the linearization of (1-50) are $\lambda_1 = \mu + i$ and $\lambda_2 = \mu - i$. When $\mu=0$ the eigenvalues become purely imaginary $\lambda_1 = i$ and $\lambda_2 = -i$. There are two cases for μ :

Case (1) if $\mu < 0$, then $(0, 0)$ is stable

Case (2) if $\mu > 0$, then $(0, 0)$ is unstable.

To find limit cycle, we translate the system to coordinates r, θ . Let

$x = r \cos(\theta)$ and $y = r \sin(\theta)$, then $r^2 = x^2 + y^2$ and $\tan(\theta) = \frac{y}{x}$. We have

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$$\begin{aligned}
 2r\dot{r} &= 2x\dot{x} + 2y\dot{y} = 2x(\mu x - y - xr^2) + 2y(x + \mu y - yr^2) \\
 &= 2\mu x^2 - 2xy - 2r^2 x^2 + 2xy + 2\mu y^2 - 2r^2 y^2 \\
 &= 2\mu r^2 - 2r^4 \\
 &= 2r^2(\mu - r^2)
 \end{aligned}$$

Therefore, $\dot{r} = r(\mu - r^2)$ (1-51)

Now, $\tan(\theta) = \frac{y}{x}$ thus

$$\begin{aligned}
 \sec^2(\theta)\dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{x^2} = \frac{x(x + \mu y - yr^2) - y(\mu x - y - xr^2)}{x^2} \\
 &= \frac{x^2 + \mu xy - xyr^2 - \mu xy + y^2 + xyr^2}{x^2} \\
 &= \frac{x^2 + y^2}{x^2} = \frac{r^2}{r^2 \cos^2(\theta)} = \frac{1}{\cos^2(\theta)} = \sec^2(\theta).
 \end{aligned}$$

Therefore, $\dot{\theta} = 1$(1-52)

For $\mu > 0$, the maps $r(t) = \sqrt{\mu}$, $\theta(t) = t$ are solutions of (1-51) and (1-52) respectively and so the circle $r = \sqrt{\mu}$ forms a limit cycle. If $0 < r < \sqrt{\mu}$, then $\dot{r} > 0$, if $r > \sqrt{\mu}$, then $\dot{r} < 0$. Therefore, the limit cycle is a stable. Hence, the system (1-50) has a Hopf bifurcation at $\mu=0$.

Remark (1.2.37)

If the vector field has an eigenvalue is equal to zero, the bifurcation is a saddle node or transcritical or pitchfork bifurcation. If the eigenvalue is none zero but purely imaginary, this is a Hopf bifurcation.

Now, we want to consider conditions on general one parameter family of two dimensional vector fields to undergo a Hopf bifurcation.

Theorem (1.2.38) [1]

Suppose the parameterized system

$$\dot{x} = f_1(x, y, \mu)$$

$$\dot{y} = f_2(x, y, \mu)$$

has a critical point at the origin for all values of the real parameter μ . Furthermore, suppose the eigenvalues of the linearized system, $\lambda_1(\mu)$ and $\lambda_2(\mu)$, are purely imaginary when $\mu = \mu_0$. If real part of the eigenvalues $\text{Re}[\lambda_1(\mu)]$ $\{=\text{Re}[\lambda_2(\mu)]\}$, satisfies $\frac{d}{d\mu} \{ \text{Re}[\lambda_1(\mu)] \} > 0$ when $\mu = \mu_0$. And

the origin is asymptotically stable critical point when $\mu = \mu_0$, then:

- (1) $\mu = \mu_0$ is a bifurcation point of the system;
- (2) For $\mu \in (\mu_1, \mu_0)$, some $\mu_1 < \mu_0$, the origin is a stable;
- (3) For $\mu \in (\mu_0, \mu_2)$, some $\mu_2 > \mu_0$, the origin is an unstable, surrounded by a stable limit cycle, whose size increases with μ .

Example (1.2.39) [1]

Consider the parameterized system

$$\dot{x} = \mu x - 2y - 2x(x^2 + y^2)^2$$

$$\dot{y} = 2x - \mu y - y(x^2 + y^2)^2$$

we have, the origin is critical point of the system for all μ . The linearized equations are $\dot{x} = \mu x - 2y$ and $\dot{y} = 2x - \mu y$. To find the eigenvalues of the linearized equations at the origin:

$$\det \left(\begin{bmatrix} \mu & -2 \\ 2 & -\mu \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \lambda^2 - 2\mu\lambda + \mu^2 + 4 = 0 \text{ thus,}$$

$$\lambda = \frac{2\mu \pm \sqrt{-4}}{2} = \mu \pm 2i. \text{ Therefore, } \lambda_1(\mu) = \mu + 2i \text{ and } \lambda_2(\mu) = \mu - 2i. \text{ If}$$

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$\mu=0$, then $\lambda_1(\mu) = 2i$ and $\lambda_2(\mu) = -2i$ are purely imaginary. And

$\frac{d}{d\mu} \{\text{Re} [\lambda_1(\mu)]\} = 1 > 0$ when $\mu=0$. Let $V(x, y) = x^2 + y^2$, we have

$V(0, 0) = 0$ and $\dot{V}(x, y) = -2(2x^2 + y^2)(x^2 + y^2) < 0$, so that by theorem (1.1.31) the origin is asymptotically stable. By using theorem (1.2.38) the system bifurcate to give a stable limit cycle surrounding the origin for some interval of positive μ .

Remark (1.2.40) [38]

The normal form of the Hopf bifurcation was found to be

$$\left. \begin{aligned} \dot{x} &= \alpha(\mu)x - \omega(\mu)y + [a(\mu)x - b(\mu)y](x^2 + y^2) + O(|x|^5, |y|^5) \\ \dot{y} &= \omega(\mu)x + \alpha(\mu)y + [b(\mu)x + a(\mu)y](x^2 + y^2) + O(|x|^5, |y|^5) \end{aligned} \right\} \dots (1-53)$$

we will find it more convenient to work with (1-53) in polar coordinates.

In polar coordinates (1-53) is given by

$$\left. \begin{aligned} \dot{r} &= \alpha(\mu)r + a(\mu)r^3 + O(r^5) \\ \dot{\theta} &= \omega(\mu) + b(\mu)r^2 + O(r^4) \end{aligned} \right\} \dots (1-54)$$

because we are interested in the dynamics near $\mu=0$, it is natural to Taylor expand the coefficients in (1-54) about $\mu=0$. Equation (1-54) thus becomes

$$\left. \begin{aligned} \dot{r} &= \alpha'(0)\mu r + a(0)r^3 + O(\mu^2 r, \mu r^3, r^5) \\ \dot{\theta} &= \omega(0) + \omega'(0)\mu + b(0)r^2 + O(\mu^2, \mu r^2, r^4) \end{aligned} \right\} \dots (1-55)$$

where " ' " in this remark denotes differentiation with respect to μ . Our goal is to understand the dynamics of (1-55) for r small and μ small. By neglecting the higher order terms in (1-55) gives

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$$\left. \begin{aligned} \dot{r} &= d\mu r + ar^3 \\ \dot{\theta} &= \omega + c\mu + br^2 \end{aligned} \right\} \dots\dots\dots(1-56)$$

where $\alpha'(0) \equiv d, a(0) \equiv a, \omega(0) \equiv \omega, \omega'(0) \equiv c$ and $b(0) \equiv b$

Lemma (1.2.41) [38]

For $-\infty < \frac{\mu d}{a} < 0$ and μ sufficiently small

$(r(t), \theta(t)) = (\sqrt{\frac{-\mu d}{a}}, [\omega + (c - \frac{bd}{a})\mu]t + \theta_0)$ is a periodic orbit for (1-56).

Lemma (1.2.42) [38]

The periodic orbit $(r(t), \theta(t)) = (\sqrt{\frac{-\mu d}{a}}, [\omega + (c - \frac{bd}{a})\mu]t + \theta_0)$ is

- (1) Asymptotically stable for $a < 0$
- (2) Unstable for $a > 0$

Remark (1.2.43) [21, 38, 39, 40]

There are two types of a Hopf bifurcation, supercritical and subcritical Hopf bifurcation. In supercritical Hopf bifurcation, as the parameter μ varies a stable node with $\text{Re}(\lambda_{1,2}) < 0$ turns unstable simultaneously a stable limit cycle emerges. In subcritical Hopf bifurcation as the parameter μ varies an unstable node with $\text{Re}(\lambda_{1,2}) > 0$ turns stable simultaneously an unstable limit cycle disappears. (The symbol $\text{Re}(\lambda_{1,2})$ means the real part of eigenvalues λ_1 and λ_2).

Example (1.2.44) [39]

Consider the parameterized system

$$\dot{x} = -y + x(\mu - (x^2 + y^2))$$

$$\dot{y} = x + y(\mu - (x^2 + y^2)).$$

The system has a critical point at (0,0). The Jacobian matrix at (0,0)

is $\begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$, then the eigenvalues are $\mu \pm i$. For $\mu < 0$, (0,0) is stable and

$\text{Re}(\lambda_{1,2}) < 0$ and for $\mu > 0$, (0,0) is unstable and $\text{Re}(\lambda_{1,2}) > 0$. As μ increases past 0, the critical point (0,0) switches from stable to unstable and a stable limit cycle for $\mu > 0$ emerge. This example is supercritical Hopf bifurcation as shown in figure (9). Stable critical points are shown as bold lines, and unstable critical points as dotted lines.

Example (1.2.45) [39]

Consider the parameterized system

$$\dot{x} = -y + x(-\mu + (x^2 + y^2))$$

$$\dot{y} = x + y(-\mu + (x^2 + y^2)).$$

The system has a critical point at (0,0). The Jacobian matrix at (0,0) is

$\begin{bmatrix} -\mu & -1 \\ 1 & -\mu \end{bmatrix}$, then the eigenvalues at the origin are $-\mu \pm i$. For $\mu < 0$,

(0,0) is unstable and $\text{Re}(\lambda_{1,2}) > 0$ and for $\mu > 0$, (0,0) is stable and $\text{Re}(\lambda_{1,2}) < 0$. As μ increases past 0, the critical point (0,0) switches from unstable to stable and an unstable limit cycle for $\mu > 0$ disappears. This example is subcritical Hopf bifurcation as shown in figure (10). Stable critical points are shown as bold lines, and unstable critical points as dotted lines.

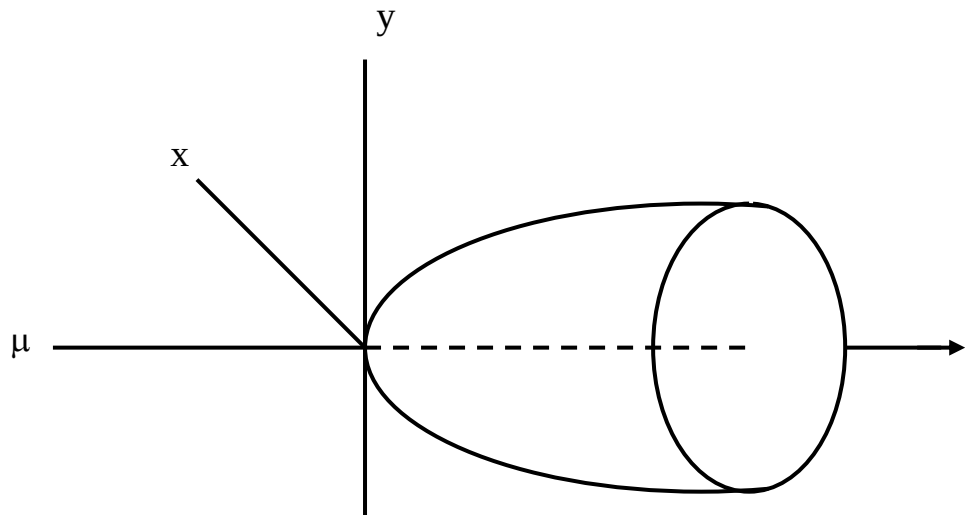


Figure (9)

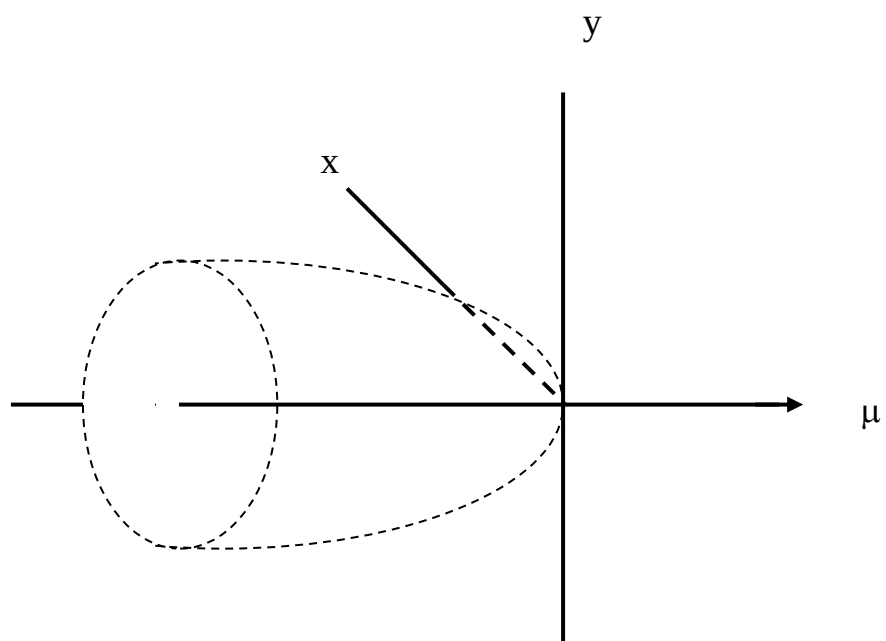


Figure (10)

Example (1.2.46) [36]

The Hopf bifurcation existed in a model of gene expression [29]. A section of the DNA molecule is copied (transcribed) onto messenger RNA (mRNA), which diffuses out of the nucleus of the cell into the cytoplasm, where it enters a subcellular structure called a ribosome. In the ribosome the genetic code on the mRNA produces a protein (a process called translation). The protein then diffuses back into the nucleus where it represses the transcription of its own gene.

Dynamically speaking, this process may result in a steady state equilibrium, in which case the concentrations of mRNA and protein are constant, or it may result in an oscillation. In this example we analyze a simple model previously proposed in the biological literature [29], and we show that the transition between equilibrium and oscillation is a Hopf bifurcation.

Remark (1.2.47) [41]

There are another types of local bifurcations for a one parameter vector fields.

1. Flip bifurcation

(i) Flip bifurcation (supercritical): a stable periodic point of order n transition to a stable periodic point of order $2n$ and an unstable periodic point of order n .

(ii) Flip bifurcation (subcritical): an unstable periodic point of order $2n$ and a stable periodic point of order n transition to an unstable periodic point of order n .

2. Niemark bifurcation

(i) Niemark bifurcation (supercritical): a stable periodic point transition to an unstable periodic point and a stable limit cycle.

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(ii)Niemark bifurcation (subcritical): a stable periodic point and unstable limit cycle transition to an unstable periodic point.

Now, we will give two general examples each of which contain some types of bifurcation which are the Lorenz system and a dual system to Lorenz system.

Example (1.2.48) [7,16]

In [1963] lorenz, a meteorologist working on the basis of earlier results due to Salzmann [1962], presented an analysis of a coupled set of three quadratic ordinary differential equations representing three mods (one in velocity and two in temperature) of the oberbeck-Boussinesq equations for fluid convection in a two- dimensional layer heated from below. The powers cited above provided more details. The equations are

$$\left. \begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \right\} \dots\dots\dots(1-57)$$

and contain the three parameters σ (the Parndtl number), r (the Rayleigh number), and b (an aspect ratio). The Lorenz equations also involve three parameters σ , r and b , all of which are real and positive. The parameters σ and b depend on the material and geometrical properties of the fluid layer. For the earth's atmosphere reasonable values of these

parameters are $\sigma = 10$ and $b = \frac{8}{3}$. The parameter r , on the other hand, is

proportional to the temperature difference ΔT , and our purpose is to investigate how the nature of the solutions of Eqs. (1-57) changes with r .

To find the critical points of the Lorenz equation setting $\dot{x} = 0, \dot{y} = 0$ and $\dot{z} = 0$. We get,

$$\sigma(y - x) = 0 \dots\dots\dots(1-58)$$

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$$rx - y - xz = 0 \dots\dots\dots(1-59)$$

$$xy - bz = 0 \dots\dots\dots(1-60)$$

from the equation (1-58) we have $x=y$. Then eliminating y from the equation (1-59) and (1-60), we obtain

$$x(r-1-z)=0 \dots\dots\dots(1-61)$$

$$-bz + x^2 = 0 \dots\dots\dots(1-62)$$

from equation(1-61), $x = 0$ or $z = r-1$. Then the equation (1-62)

requires that $x = \pm\sqrt{b(r-1)}$ and then $y = \pm\sqrt{b(r-1)}$.

If $0 < r < 1$ it is the only critical point $p_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, when $r > 1$, there are also

two other critical points $p_2 = \begin{pmatrix} \sqrt{b(r-1)} \\ \sqrt{b(r-1)} \\ r-1 \end{pmatrix}$ and $p_3 = \begin{pmatrix} -\sqrt{b(r-1)} \\ -\sqrt{b(r-1)} \\ r-1 \end{pmatrix}$.

Now, we will determine the local behavior of solutions in the neighborhood of each critical point. Although much of the following analysis can be carried out for arbitrary values of σ and b , we will simplify our work by using the values $\sigma = 10$ and $b = \frac{8}{3}$. Near the origin

(the critical point p_1) the approximating linear system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ r & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \dots\dots\dots(1-63)$$

The eigenvalues of (1-63) are determined from the equation

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$$\det \begin{pmatrix} -10-\lambda & 10 & 0 \\ r & -1-\lambda & 0 \\ 0 & 0 & -\frac{8}{3}-\lambda \end{pmatrix} = -\left(\frac{8}{3}+\lambda\right)[\lambda^2 + 11\lambda - 10(r-1)] = 0.$$

$$\text{Therefore, } \lambda_1 = -\frac{8}{3}, \lambda_2 = \frac{-11-\sqrt{81+40r}}{2} \text{ and } \lambda_3 = \frac{-11+\sqrt{81+40r}}{2}.$$

If $r < 1$, then all three eigenvalues are negative. For example, when $r = \frac{1}{2}$,

the eigenvalues are $\lambda_1 = -\frac{8}{3}, \lambda_2 = -10.52494, \lambda_3 = -0.47506$. Hence the

origin is asymptotically stable. However, λ_3 changes sign when $r = 1$ and is positive for $r > 1$. Hence, the origin is unstable for $r > 1$.

Now, let us consider the neighborhood of the critical point

$$p_2 = \begin{pmatrix} \sqrt{b(r-1)} \\ \sqrt{b(r-1)} \\ r-1 \end{pmatrix} \text{ for } r > 1. \text{ If } u, v \text{ and } w \text{ are perturbation from the critical}$$

point in the x, y and z direction, respectively, then the approximating linear system is

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{\frac{8(r-1)}{3}} \\ \sqrt{\frac{8(r-1)}{3}} & \frac{\sqrt{8(r-1)}}{3} & -\frac{8}{3} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \quad \dots\dots\dots(1-64)$$

The eigenvalues of the coefficient matrix of equation (1-64) are determined from the equation

$$3\lambda^3 + 41\lambda^2 + 8(r+10)\lambda + 160(r-1) = 0. \quad \dots\dots\dots(1-65)$$

The solutions of equation (1-65) depend on r in the following way:

(i) For $1 < r < r_1 \cong 1.3456$ there are three negative real eigenvalues.

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(ii) For $r_1 < r < r_2 \cong 24.737$ there are one negative real eigenvalue and two complex eigenvalues with negative real part.

(iii) For $r_2 < r$ there are one negative real eigenvalue and two complex eigenvalues with positive real part.

The same results are obtained for the critical point p_3 . Thus there are several different situations.

(i) For $0 < r < 1$ the only critical point is p_1 and it is asymptotically stable. All solutions approach this point (the origin) as $t \rightarrow \infty$.

(ii) For $1 < r < r_1$ the critical points p_2 and p_3 are asymptotically stable and p_1 is unstable. All nearby solutions approach one or the other of the points p_2 and p_3 exponentially.

(iii) For $r_1 < r < r_2$ the critical points p_2 and p_3 are asymptotically stable and p_1 is unstable. All nearby solutions approach one or the other of the points p_2 and p_3 most of them spiral inward to the critical point.

(iv) For $r_2 < r$ all three critical points are unstable. Most solutions near p_2 or p_3 spiral away from the critical point.

Therefore, at $r = 1$ a pitchfork bifurcation occurs and at $r = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} = \frac{470}{19} \approx 24.74$ a Hopf bifurcation occurs at the nontrivial critical points.

Example (1.2.48) [35]

At its origins, the Lorenz system was a model of the convection. Convection is close related to the dynamics of the atmosphere. Warm air with a lower density rises to higher altitudes where it becomes cool and its density increases. Falling again it creates the atmospheric currents. It was modeled in laboratory and it is known as Rayleigh-Benard

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experiment. Good notes on Lorenz system can be found in [33]. In the last decades the nonlinear system was intensively studied, grace to computers. The nonlinear phenomena are met in many areas, from engineering to human brain [13, 37] and heart disease. The dual system which will be investigated is described by the following three nonlinear differential equations

$$\left. \begin{aligned} \dot{x} &= a(y - x) \\ \dot{y} &= (c - a)x - axz \\ \dot{z} &= xy - bz \end{aligned} \right\} \dots\dots\dots(1-66)$$

where a, b, c with $a \neq 0$ are the real parameters of the system.

To find critical points setting $\dot{x} = 0, \dot{y} = 0$ and $\dot{z} = 0$, we get

$$ay - ax = 0$$

$$(c - a)x - axz = 0$$

$$xy - bz = 0$$

from the first equation we have $x = y$. then eliminating y from the third equation we obtain

$$x(c - a - az) = 0 \dots\dots\dots(1-67)$$

$$x^2 - bz = 0 \dots\dots\dots(1-68)$$

from equation (1-67), $x=0$. We can satisfy equation (1-67) by choosing

$$z = \frac{c - a}{a}. \text{ Then equation (1-68) requires that } x = y = \pm \sqrt{\frac{b}{a}(c - a)}.$$

If $\frac{b}{a}(c - a) < 0$ the system (1-66) has only one critical point $p_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, when

$\frac{b}{a}(c - a) > 0$ the system (1-66) has also two other critical points namely

$$p_2 = \begin{pmatrix} \sqrt{\frac{b}{a}(c-a)} \\ \sqrt{\frac{b}{a}(c-a)} \\ \frac{c-a}{a} \end{pmatrix} \quad \text{and} \quad p_2 = \begin{pmatrix} -\sqrt{\frac{b}{a}(c-a)} \\ -\sqrt{\frac{b}{a}(c-a)} \\ \frac{c-a}{a} \end{pmatrix}. \text{ Next we will determine the}$$

local behavior of solution in the neighborhood of each critical point. The Jacobian matrix of system (1-66) is

$$\begin{bmatrix} -a & a & 0 \\ c-a-az & 0 & -ax \\ y & x & -b \end{bmatrix}.$$

The Jacobian matrix of system (1-66) at the critical point p_1 is

$$\begin{bmatrix} -a & a & 0 \\ c-a & 0 & 0 \\ 0 & 0 & -b \end{bmatrix}. \quad \dots\dots\dots(1-69)$$

The Jacobian matrix of system (1-66) at the critical points p_2 and p_3 is

$$\begin{bmatrix} -a & a & 0 \\ 0 & 0 & -a\sqrt{\frac{b}{a}(c-a)} \\ \sqrt{\frac{b}{a}(c-a)} & \sqrt{\frac{b}{a}(c-a)} & -b \end{bmatrix}. \quad \dots\dots\dots(1-70)$$

The eigenvalues of Jacobian matrix at p_1 can be found from

$$(\lambda + b)(\lambda^2 + a\lambda + a^2 - ac) = 0, \text{ we get } \lambda_1 = -b, \lambda_2 = \frac{1}{2}(-a - \sqrt{4ac - 3a^2}) \text{ and}$$

$$\lambda_3 = \frac{1}{2}(-a + \sqrt{4ac - 3a^2}). \text{ The eigenvalues of Jacobian matrix at } p_2 \text{ and } p_3$$

can be found from $\lambda^3 + \lambda^2(a + b) + bc\lambda + 2ab(c - a) = 0$, we obtain

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$$\lambda_1 = \frac{2a^2 - 2ac}{c}, \lambda_2 = \sqrt{ac - 2a^2} \quad \text{and} \quad \lambda_3 = -\sqrt{ac - 2a^2}.$$

Now we study the stability of the critical points p_1 , p_2 and p_3 by examining the eigenvalues of (1-69) and (1-70).

1. The point $p_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is asymptotically stable if

(i) $a > 0, b > 0, 4c - 3a \leq 0,$

(ii) $a > 0, b > 0, 4c - 3a > 0, c \leq a.$

2. The point $p_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is unstable if

(i) $a < 0$, or

(ii) $b < 0$, or

(iii) $a > 0, b > 0, 4c - 3a > 0, c > a.$

3. The points $p_2 = \begin{pmatrix} \sqrt{\frac{b}{a}(c-a)} \\ \sqrt{\frac{b}{a}(c-a)} \\ \frac{c-a}{a} \end{pmatrix}$ and $p_2 = \begin{pmatrix} -\sqrt{\frac{b}{a}(c-a)} \\ -\sqrt{\frac{b}{a}(c-a)} \\ \frac{c-a}{a} \end{pmatrix}$ are

asymptotically stable if

(i) $a > 0, 4c - 3a > 0, c > a$

(ii) $a + b > 0, ab(c - a) > 0, b(2a^2 + bc - ac) > 0.$

Now we study the bifurcation of system (1-66).

(i) The system (1-66) has a pitchfork bifurcation at $c = a$ when $b > 0$.

(ii) The system (1-66) has a Hopf bifurcation at $b = \frac{ac - 2a^2}{c}.$



Chapter Two

Compute Lyapunov Exponents to Systems of Differential Equations.

Lyapunov exponents, or Lyapunov characteristic numbers, characterize growth rates of time dependent linear differential equations, and by linearizing about orbits measure rates of convergence or divergence of nearby orbits for nonlinear differential equations. For an n -dimensional problem there are n Lyapunov exponents: these are the natural generalization to time dependent linear differential equations of the eigenvalues for autonomous linear systems. For example, consider the linear scalar differential equation

$\dot{x} = (\sin(\ln(t)) + \cos(\ln(t)))x$, for $t > t_0 > 0$: the solution is
 $x(t) = \exp(t \sin(\ln(t)))k_0$, $k_0 = x(t_0)\exp(-t_0 \sin(\ln(t_0)))$, so that all growth rates in the interval $[-1, +1]$ are attained. [12]

The numerical techniques which have been developed to approximate Lyapunov exponents, with stability theory for Lyapunov exponents that was developed over 30 years ago. Characteristic exponents were developed by Lyapunov in his thesis [26] that was first published in 1892. Many of the ideas from Lyapunov's thesis and further developments on Lyapunov exponents are contained in the monograph of Adrianova that serves as an excellent accessible introduction to the use of Lyapunov exponents in stability theory [3].

In this chapter, we compute Lyapunov exponents to different systems of differential equations contain different types of the bifurcation by using Matlab.

This chapter consists of two sections. In section one, we recall some fundamental definitions and necessary theorems of Lyapunov exponent in discrete and continuous dynamical system. In section two, we recall some properties of Lyapunov exponent in continuous dynamical system.

2.1 Definitions and Theorems of Lyapunov Exponent

There are many definitions of Lyapunov exponent in discrete and continuous dynamical system. We start with discrete dynamical system.

Definition (2.1.1) [18]

Let J be a bounded interval, and $f: J \rightarrow J$ continuously differentiable on J . fix x in J , and let $\gamma(x)$ be defined by

$$\gamma(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln |(f^n)'(x)| \dots\dots\dots(2-1)$$

Provided that the limit exists. In this case $\gamma(x)$ is the Lyapunov exponent of f at x .

Remark (2.1.2) [18]

If $\gamma(x)$ is independent of x wherever $\gamma(x)$ is defined, then the common value of $\gamma(x)$ is denoted by γ , and is the Lyapunov exponent of f .

Example (2.1.3) [18]

Consider the tent map

$$T(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \leq 1. \end{cases}$$

If $\varepsilon > 0$ is small enough, then $|T(x + \varepsilon) - T(x)| = 2\varepsilon$ and

$|T^2(x + \varepsilon) - T^2(x)| = 2^2\varepsilon$. Now, let n be an arbitrary positive integer, we

have $|T^n(x + \varepsilon) - T^n(x)| = 2^n\varepsilon$. Next, we divide both sides by ε and let ε

approach 0, we obtain $\lim_{\varepsilon \rightarrow 0} \frac{|T^n(x + \varepsilon) - T^n(x)|}{\varepsilon} = 2^n$, then $|(T^n)'(x)| = 2^n$.

Therefore, $\gamma = \lim_{n \rightarrow \infty} \frac{1}{n} \ln 2^n = \ln 2$.

Definition (2.1.4) [10]

Let $\bar{x}$ be in the domain of f . The point $\bar{x}$ is a fixed point of f if $f(\bar{x}) = \bar{x}$. The point $\bar{x}$ is a periodic point of period n if $f^n(\bar{x}) = \bar{x}$. The point $\bar{x}$ is hyperbolic if $|(f^n)'(\bar{x})| \neq 1$.

Remark (2.1.5) [18]

Let $\bar{x}$ be in the domain of f . If $\bar{x}$ is a periodic point of period n , then the orbit of $\bar{x}$, which is $\{\bar{x}, f(\bar{x}), f^2(\bar{x}), \dots, f^{n-1}(\bar{x})\}$ is called a periodic orbit.

Definition (2.1.6) [10]

Let $\bar{x}$ be a hyperbolic periodic point of period n with $|(f^n)'(\bar{x})| < 1$.
the point $\bar{x}$ is called an attracting periodic point.

Remark (2.1.7) [18]

If $\bar{x}$ be attracting periodic point of period n , then the orbit of $\bar{x}$ $\{\bar{x}, f(\bar{x}), f^2(\bar{x}), \dots, f^{n-1}(\bar{x})\}$ is attracting.

Remark (2.1.8) [18]

To make general calculations of $\gamma(x)$ to be simpler, let $x_0 = x$, $x_1 = f(x)$ and $x_k = f^k(x)$, for $k = 1, 2, 3, \dots$ and by using law of logarithms,

$$\begin{aligned} \text{we have } \ln|(f^n)'(x)| &= \ln|f'(x_{n-1})f'(x_{n-2}) \dots f'(x_1)f'(x_0)| \\ &= \ln[|f'(x_{n-1})||f'(x_{n-2})| \dots |f'(x_1)||f'(x_0)|] \\ &= \sum_{k=0}^{n-1} \ln|f'(x_k)| \end{aligned}$$

Therefore (2 – 1) can be written as:

$$\gamma(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln|f'(x_k)| \dots \dots \dots (2 - 2)$$

Now we will use (2 – 2) in order to determine the Lyapunov exponent for quadratic map in the following example.

Example (2.1.9) [18]

Suppose that $F_\mu(x) = \mu x(1 - x)$, for $0 \leq x \leq 1$, where $1 < \mu < 3$ and $\mu \neq 2$.

Let x be arbitrary in $(0,1)$. Since F_μ has a fixed point $p_\mu = 1 - \frac{1}{\mu}$ that attracts all points in $(0,1)$. Therefore, $x_k = F_\mu^k(x) \rightarrow p_\mu$ as k increases without bound.

Since $F'_\mu(x) = \mu - 2\mu x$, it follows that

$$F'_\mu(x_k) \rightarrow F'_\mu(p_\mu) = \mu - 2\mu(1 - \frac{1}{\mu}) = 2 - \mu.$$

By hypothesis $\mu \neq 2$ so that $\ln|F'_\mu(x_k)| \rightarrow \ln|2 - \mu|$ as k increases without bound. Now, let $\varepsilon > 0$. Because the natural logarithm is continuous and x_k approaches p_μ as k increases, there is a positive integer N such that if $k > N$, then $\ln|2 - \mu| - \varepsilon < \ln|F'_\mu(x_k)| < \ln|2 - \mu| + \varepsilon$. For $n > N$, we have

$$\begin{aligned} \frac{n - N}{n}(\ln|2 - \mu| - \varepsilon) &= \frac{1}{n} \sum_{k=N+1}^n (\ln|2 - \mu| - \varepsilon) < \frac{1}{n} \sum_{k=N+1}^n \ln|F'_\mu(x_k)| < \\ &\frac{1}{n} \sum_{k=N+1}^n (\ln|2 - \mu| + \varepsilon) = \frac{n - N}{n}(\ln|2 - \mu| + \varepsilon) \end{aligned}$$

If n is sufficiently large and much larger than N , then $\frac{n - N}{n} \approx 1$, so that the preceding inequalities reduce to

$$\ln|2 - \mu| - \varepsilon < \frac{1}{n} \sum_{k=N+1}^n \ln|F'_\mu(x_k)| < \ln|2 - \mu| + \varepsilon$$

Moreover, for large enough n , we have $\left| \frac{1}{n} \sum_{k=0}^N \ln|F'_\mu(x_k)| \right| < \varepsilon$. Therefore,

$$\begin{aligned} \gamma(x) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^n \ln|F'_\mu(x_k)| = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{k=0}^N \ln|F'_\mu(x_k)| + \frac{1}{n} \sum_{k=N+1}^n \ln|F'_\mu(x_k)| \right) \\ &= \ln|2 - \mu|. \end{aligned}$$

Since the final expression is independent of the number x in $(0,1)$, we conclude that $\gamma = \ln|2 - \mu|$

Theorem (2.1.10) [24]

The Lyapunov exponent of an attracting periodic orbit is negative.

Proof:

We have $\gamma(x) = \frac{1}{n} \ln |(f^n)'(x)|$. We have seen that for an attractive periodic orbit $|(f^n)'(x)| < 1$, it follows that the Lyapunov exponent of an orbit which is attracted to a periodic orbit is negative. ★

Definition (2.1.11) [18]

Let I and J be intervals. The map $f : I \rightarrow J$ is a homeomorphism of I onto J provided that f is one-to-one and onto, and both f and f^{-1} are continuous. Thus we say the interval I is homeomorphic to J .

Example (2.1.12) [10]

The map $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$ such that $f(x) = \tan(x)$ is homeomorphism. Therefore, the open interval is homeomorphic to $\mathbb{R}$.

Definition (2.1.13) [18]

Let I and J be intervals, and suppose that $f : I \rightarrow I$ and $g : J \rightarrow J$. Then f and g are conjugate if there is a homeomorphism $h : I \rightarrow J$ such that $h \circ f = g \circ h$. In this case, we write $f \underset{h}{\approx} g$.

Example (2.1.14) [18]

Suppose that the quadratic map $F_4(x) = 4x(1-x)$ for $0 \leq x \leq 1$ and

$$\text{Tent map } T(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \leq 1. \end{cases}$$

Let $h:[0,1] \rightarrow [0,1]$ such that $h(x) = \sin^2(\frac{\pi}{2}x)$. It is clear h is homeomorphism.

$$\begin{aligned} (F_4 \circ h)(x) &= F_4(h(x)) = F_4(\sin^2(\frac{\pi}{2}x)) = 4(\sin^2(\frac{\pi}{2}x))(1 - \sin^2(\frac{\pi}{2}x)) \\ &= 4(\sin^2(\frac{\pi}{2}x))(\cos^2(\frac{\pi}{2}x)) \\ &= (2\sin(\frac{\pi}{2}x)\cos(\frac{\pi}{2}x))^2 \\ &= \sin^2(\pi x). \end{aligned}$$

Where the last equality results from a trigonometric double-angle formula. On the other hand, we find that:

$$\text{If } 0 \leq x \leq \frac{1}{2}, \text{ then } (h \circ T)(x) = h(T(x)) = h(2x) = \sin^2(\pi x).$$

$$\begin{aligned} \text{If } \frac{1}{2} < x \leq 1, \text{ then } (h \circ T)(x) &= h(2-2x) = \sin^2(\frac{\pi}{2}(2-2x)) \\ &= \sin^2(\pi - \pi x) \\ &= \sin^2(\pi x). \end{aligned}$$

Consequently, $(h \circ T)(x) = (F_4 \circ h)(x)$, for $0 \leq x \leq 1$, so that $h \circ T = F_4 \circ h$.

Therefore, the tents map T and the quadratic map F_4 are conjugates.

Theorem (2.1.15) [18]

Suppose that $f \underset{h}{\approx} g$. Then $f^n \underset{h}{\approx} g^n$.

Proof:

We will prove with method of induction. Since $f \underset{h}{\approx} g$, then there is a homeomorphism h such that $h \circ f = g \circ h$. Let us assume that

$$h \circ f^{n-1} = g^{n-1} \circ h.$$

$$\begin{aligned} \text{Then } h \circ f^n &= (h \circ f) \circ f^{n-1} \\ &= (g \circ h) \circ f^{n-1} \\ &= g \circ (h \circ f^{n-1}) \\ &= g \circ (g^{n-1} \circ h) \\ &= g^n \circ h. \end{aligned}$$

So that, $h \circ f^n = g^n \circ h$ for $n = 1, 2, 3, \dots$ ★

Theorem (2.1.16) [24]

If $\gamma(f, x)$ is the Lyapunov exponent of the orbit of f through x , and $\gamma(g, y)$ is the Lyapunov exponent of g through $y = h(x)$, and $f \underset{h}{\approx} g$, then $\gamma(f, x) = \gamma(g, y)$.

Proof:

Since $f \underset{h}{\approx} g$, then there is a homeomorphism h such that $h \circ f = g \circ h$. By using theorem (2.1.15), we get $h \circ f^n(x) = g^n \circ h(x)$, by derivative two sides with respect to x , we get $(g^n \circ h)'(x) = (h \circ f^n)'(x)$. By chain rule, we obtain

$$(g^n)'(h(x))h'(x) = h'(f^n(x))(f^n)'(x) \text{ thus } (g^n)'(y) = (f^n)'(x).$$

$$\text{Hence, } \gamma(g, y) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln |(g^n)'(y)| = \lim_{n \rightarrow \infty} \frac{1}{n} \ln |(f^n)'(x)| = \gamma(f, x). \quad \star$$

Now, we will give other definitions of the Lyapunov exponent, but in continuous dynamical systems.

Definition (2.1.17) [38]

Consider the vector field

$$\dot{x} = f(x), x \in \mathbb{R}^n \quad \dots\dots\dots(2-3)$$

and $\varphi_t(x)$ be an orbit of (2-3) satisfying $\varphi_0(x) = x_0$. Consider the linearization of (2-3) about $\varphi_t(x)$ given by

$$\dot{\xi} = Df(\varphi_t(x))\xi, \xi \in \mathbb{R}^n. \quad \dots\dots\dots(2-4)$$

Let $\Phi(t)$ be the fundamental solution matrix of (2-4) and let "**e**" be a vector in $\mathbb{R}^n$. Then the coefficient of expansion in the direction "**e**" along

the orbit through x_0 is defined by $\mu_t(x_0, e) = \frac{\|\Phi(t)e\|}{\|e\|}$, where

$\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ with $\langle \cdot, \cdot \rangle$ denoting the standard scalar product on $\mathbb{R}^n$. Then the

Lyapunov exponent in the direction "**e**" along the orbit through x_0 is defined by

$$\gamma(x_0, e) \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \log \mu_t(x_0, e).$$

Remark (2.1.18) [38]

An orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of R^n is called a normal basis if $\sum_{i=1}^n \gamma(x_0, e_i)$ attains its minimum on the basis.

Remark (2.1.19) [38]

The fundamental solution matrix $\Phi(t)$ is called regular as $t \rightarrow \infty$ if

1. $\lim_{t \rightarrow \infty} \frac{1}{t} \log |\det(\Phi(t))|$ exists and is finite,
2. For each normal basis $\{e_1, e_2, \dots, e_n\}$, $\sum_{i=1}^n \gamma(x_0, e_i) = \lim_{t \rightarrow \infty} \frac{1}{t} \log |\det(\Phi(t))|$.

Now we can state the main theorem due to Lyapunov 1966.

Theorem (2.1.20) [27]

If $\Phi(t)$ is regular as $t \rightarrow \infty$, then $\gamma(x_0, e) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \mu_t(x_0, e)$

Example (2.1.21) [38]

Consider the linear scalar vector field

$$\dot{x} = rx, \quad x \in R \quad \dots\dots\dots(2-5)$$

where r is a real number. The vector field (2-5) has three orbits $x = 0$, $x > 0$ and $x < 0$, the fundamental solution matrix associated with each orbit is given by $\Phi(t) = e^{at}$. The vector field (2-5) has only one Lyapunov exponent because it has one dimension. Therefore, Lyapunov exponent of each orbit is

$$\gamma = \lim_{t \rightarrow \infty} \frac{1}{t} \log \mu_t(x_0, e), \text{ where } \mu_t(x_0, e) = \frac{\|\Phi(t)e\|}{\|e\|} = \frac{|e^{at}|}{1} = e^{at}.$$

$$\text{Hence, } \gamma = \lim_{t \rightarrow \infty} \frac{1}{t} \log e^{at} = \lim_{t \rightarrow \infty} \frac{1}{t} \cdot at = a.$$

Remark (2.1.22) [38]

If the orbit is a critical point, then the Lyapunov exponents are the real parts of the eigenvalues associated with the matrix of the vector field linearized about the critical point.

Example (2.1.23) [38]

Consider the vector field

$$\left. \begin{array}{l} \dot{x} = x - x^3 \\ \dot{y} = -y \end{array} \right\} \dots\dots\dots(2-6)$$

The critical points $(0,0), (-1,0)$ and $(1,0)$ represent orbits of the vector field (2-6). Every orbit has two Lyapunov exponents because the vector field (2-6) has two dimensions. Let $e_1 = (1,0)$ denote a tangent vector in the x direction and $e_2 = (0,1)$ denote a tangent vector in the y direction. The tangent vectors e_1 and e_2 are a basis of $\mathbb{R}^2$. Therefore, Lyapunov exponents for the three fixed points are

- (i) $\gamma((0,0), e_1) = 1$ and $\gamma((0,0), e_2) = -1$
- (ii) $\gamma((-1,0), e_1) = -2$ and $\gamma((-1,0), e_2) = -1$
- (iii) $\gamma((1,0), e_1) = -2$ and $\gamma((1,0), e_2) = -1$

Remark (2.1.24) [38]

Lyapunov exponents are not continuous map of the orbits. This can be seen from example (2.1.23).

Remark (2.1.25) [38]

We can view the Lyapunov exponents of a given orbit as the long time average of the real parts of the eigenvalues of the fundamental solution matrix associated with the linearization of the vector field about the orbit. Therefore, they give us information concerning local expansion and construction of phase space only and nothing about twisting and folding.

Now, we introduce another definition of the Lyapunov exponent in the continuous dynamical systems.

Definition (2.1.26) [28]

Suppose that the vector field

$$\dot{x} = f(x), \quad x \in M^n \quad \dots\dots\dots(2-7)$$

Where M is an n -dimensional phase space. Let $\phi_t(x)$ be an orbit of (2-7), and let $\Phi(t, x)$ be the fundamental matrix solution of linearized vector field (2-7), and let v be a tangent vector in $T_x M$, the Lyapunov exponent defined as the supremum limit:

$$\gamma(x, v) = \lim_{t \rightarrow \infty} \sup \frac{1}{t} \log |\Phi(t, x)v|. \quad \dots\dots\dots(2-8)$$

Since this limit will occur often below, it is nice to give it a more compressed notation. For any function $f(t)$, define the characteristic exponent of f by

$$\chi(f) = \lim_{t \rightarrow \infty} \sup \frac{1}{t} \ln |f(t)| \dots\dots\dots(2-9)$$

That is, $\gamma(x, v) = \chi(\Phi(t, x)v)$.

We will use the two equations (2-8) and (2-9) to show properties of Lyapunov exponent in the next section.

2.2 The properties of Lyapunov exponents.

The goal of this section is to show some properties about Lyapunov exponents, and we compute the maximal Lyapunov exponent of some systems.

Lyapunov showed that the characteristic exponent (2-9) of any function obeys several simple properties (Adrianova 1995). [28]

Now, we will present a proposition, and the proof of it since it is not proved in any reference.

Proposition (2.2.1) [28]

For any function $f(t)$ and $g(t)$ and any constant $c \neq 0$, then

- (i) $\chi(cf) = \chi(f)$
- (ii) $\chi(fg) \leq \chi(f) + \chi(g)$
- (iii) $\chi(f + g) \leq \max\{\chi(f), \chi(g)\}$

Proof:

The equation (i) implies that, though the Lyapunov exponent may depend on the direction of the initial vector, it does not depend on its length.

$$(ii) \chi(fg) = \lim_{t \rightarrow \infty} \sup \frac{1}{t} \ln |fg(t)| = \lim_{t \rightarrow \infty} \sup \frac{1}{t} (\ln |f(t)| + \ln |g(t)|)$$

$$\begin{aligned}
 &= \lim_{t \rightarrow \infty} \sup \left(\frac{1}{t} \ln |f(t)| + \frac{1}{t} \ln |g(t)| \right) \\
 &\leq \lim_{t \rightarrow \infty} \sup \frac{1}{t} \ln |f(t)| + \lim_{t \rightarrow \infty} \sup \frac{1}{t} \ln |g(t)| \\
 &= \chi(f) + \chi(g)
 \end{aligned}$$

$$(iii) \quad \chi(f + g) = \lim_{t \rightarrow \infty} \sup \frac{1}{t} \ln |f(t) + g(t)|$$

$$\leq \begin{cases} \lim_{t \rightarrow \infty} \sup \frac{1}{t} \ln |f(t)|, & \text{If } \chi(f) > \chi(g) \\ \lim_{t \rightarrow \infty} \sup \frac{1}{t} \ln |g(t)|, & \text{If } \chi(g) > \chi(f) \end{cases}$$

Therefore, $\chi(f + g) \leq \max\{\chi(f), \chi(g)\}$. ★

Lemma (2.2.2) [28]

The Lyapunov exponent (2–8) is independent of the choice of norm on $\mathbb{R}^n$.

Proof:

Let $|v|_1$ and $|v|_2$ represent two different vector norms and γ_1, γ_2 be the corresponding Lyapunov exponents. As is well known, all norms on $\mathbb{R}^n$ are equivalent, meaning that there exist constants m and $M > 0$ such that for every vector v , $m|v|_1 \leq |v|_2 \leq M|v|_1$. Consequently,

$\chi(m|\Phi(t, x)v|_1) \leq \chi(|\Phi(t, x)v|_2)$, which implies, using (i) in proposition (2.2.1), that $\gamma_1 \leq \gamma_2$ and $\chi(|\Phi(t, x)v|_2) \leq \chi(M|\Phi(t, x)v|_1)$, which implies, using (i) in proposition (2.2.1), that $\gamma_2 \leq \gamma_1$, so that $\gamma_2 = \gamma_1$. ★

Lemma (2.2.3) [28]

If $\varphi_t(x)$ is a bounded orbit of a flow φ that is not forward asymptotically to a critical point, then it has a zero Lyapunov exponent.

Proof:

Consider the vector $v(t) = f(\varphi_t(x))$, where f is the vector field for φ . Differentiation gives

$\frac{d}{dt} v(t) = \frac{d}{dt} f(\varphi_t(x)) = Df(\varphi_t(x)) \frac{d}{dt} \varphi_t(x) = Df(\varphi_t(x)) v(t)$. Thus, $v(t)$ is a solution of $\dot{v} = Df(\varphi_t(x))v = A(t)v$ with initial condition $f(x)$. Since φ is bounded, then v is also bounded. Finally, since φ_t is not asymptotic to critical point, $\limsup_{t \rightarrow \infty} |v(t)| > 0$. Therefore, $\gamma(x, v) = 0$. ★

Theorem (Abel) (2.2.4) [28]

The determinate of the fundamental matrix is:

$\det(\Phi(t, t_0)) = \exp \int_{t_0}^t \text{tr}(A(s)) ds$. Where that $\text{tr}(A(s))$ is a scalar, so the exponential is the ordinary scalar exponential.

Remark (2.2.5) [28]

A constraining relation among the Lyapunov exponents can be obtained from Abel's theorem

$$\det(\Phi(t, x)) = \exp \int_0^t \text{tr}(Df(\varphi_s(x))) ds. \quad \dots\dots\dots (2-10)$$

Theorem (2.2.6) [28]

Suppose $\varphi_t(x)$ is a bounded orbit of a flow φ and $[v_1, v_2, \dots, v_n]$ is an independent set of vectors with Lyapunov exponents

$\gamma_i = \gamma(x, v_i)$, $i = 1, 2, \dots, n$. If the limit

$$\delta = \lim_{t \rightarrow \infty} \sup_t \frac{1}{t} \int_0^t \text{tr} D\varphi_s(x) ds \quad \dots\dots\dots (2-11)$$

exist, then $\delta \leq \sum_{i=1}^n \gamma_i$.

Proof:

Let $p(t) = [v_1, v_2, \dots, v_n] = \Phi(t, x)p(0)$; then according to (2-10) and (2-11), $\delta = \chi(\det(\Phi(t, x))) = \chi(\det(p(t)))$. The determinate of a $n \times n$ matrix is the sum of $n!$ terms, each of which is the product of n different elements of the matrix, one from each column. Using (iii) and (ii) in proposition (2.2.1) gives:

$$\chi(\det(p)) \leq \sum_{j=1}^n \max_{1 \leq i \leq n} \chi(p_{ij}) = \sum_{i=1}^n \chi(v_i). \star$$

Remark (2.2.7) [28]

Just as constant matrices may have degenerate eigenvalues, a time-dependent matrix may not have n distinct Lyapunov exponents. It is conventional to order the exponents so that

$$\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_n. \quad \dots\dots\dots (2-12)$$

Any set of independent vectors $\{v_1, v_2, \dots, v_n\}$ so that $\sum_{i=1}^n \gamma_i(x, v_i)$ is as small as possible is called a Lyapunov basis. Most bases are not Lyapunov bases, since each vector generally will contain some

component along the most rapidly growing direction; however, a Lyapunov bases can always be constructed.

Lemma (Lyapunov basis) (2.2.8) [28]

If $\Phi = [v_1, v_2, \dots, v_n]$ is any fundamental solution matrix of $\dot{v} = A(t)v$ obeying (2-12), then there is a special upper triangular matrix $U (u_{ji}) = 1$ such that ΦU is a Lyapunov basis.

Proof:

The columns w_i of ΦU are $w_i = v_i + \sum_{j=1}^{i-1} u_{ji} v_j$. consequently, $\chi(w_1) = \chi(v_1)$, and using (iii) in proposition (2.2.1) we get $\chi(w_2) = \chi(v_2 + u_{12}v_1) \leq \chi(v_1)$. we choose u_{12} to minimize $\chi(w_2)$; indeed we can always make $\chi(w_2) \leq \chi(v_2)$ (equality is achieved by setting $u_{12} = 0$). For each i , the u_{ji} are chosen to minimize $\chi(w_i)$. we claim that the sum $\sum_{i=1}^n \chi(w_i)$ is minimal. Indeed, the Lyapunov of any linear combination $\sum_{i=1}^k a_i w_i$ with $a_k \neq 0$ is the exponent of the combination $a_k v_k + \sum_{j=1}^{k-1} b_j v_j$, for some coefficients b_j depending upon a and u_{ji} . However, the minimal combination of these first k vectors was already selected. ★

To compute all of the Lyapunov exponents, it is necessary to find a Lyapunov basis. Consider the linear system $\dot{v} = A(t)v$, where $A(t)$ is continuous and uniformly bounded. Generalizing the basis change

$\Lambda = p^{-1} A p$ where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ to account for time dependence, $v = p(t)w$, the system $\dot{v} = A(t)v$ becomes

$$\dot{w} = (p^{-1} A p - p^{-1} \dot{p})w = B(t)w \dots \dots \dots (2-13)$$

If the transformation p is well behaved, the characteristic exponents of the new system are the same as those of the old.

Definition (2.2.9) [25]

We mean a list of numbers, say, $a_1, a_2, \dots, a_n$. Such a vector is denoted by $u = (a_1, a_2, \dots, a_n)$. The numbers a_i are called the components or entries of u . If all the $a_i = 0$, then u is called the zero vector.

Definition (2.2.10) [31]

Let $V_1, V_2, \dots, V_n$ and W be vector spaces. A map $f : V_1 \times V_2 \times \dots \times V_n \rightarrow W$ is called nonsingular if whenever $f(v_1, v_2, \dots, v_n) = 0$, at least one of the vectors $v_1, v_2, \dots, v_n$ is the zero vector.

Lemma (Lyapunov transformation) (2.2.11) [28]

If $p \in C^1$ and p, p^{-1} and $\dot{p}$ are bounded for all $t > 0$, then the Lyapunov exponents of the transformed system (2-13) are the same as those of the original system $\dot{v} = A(t)v$.

Proof:

If $A(t)$ is bounded and the hypotheses hold, then $B(t)$ is bounded so its Lyapunov exponents exist. Using $\|p(t)\| \leq M$ and the definition $v = pw$ give $\chi(v) \leq \chi(\|p(t)\|w(t)) = \chi(w)$. Applying the same analysis to $w = p^{-1}v$ and using $\|p^{-1}\| \leq M$, implies that $\chi(w) \leq \chi(v)$. thus these two characteristic exponents must be equal. Since p is nonsingular, all the exponents of B must be the same as those of A . ★

We have only hinted in this section at the various properties of Lyapunov exponents. For more information see [27].

Remark (2.2.12) [28]

To compute the maximal Lyapunov exponent of a system or ordinary differential equations we must integrate both the original system and its linearization $\dot{v} = A(t)v$. essentially any initial vector v_0 can be used because almost all vectors will have some component along the direction of the maximal Lyapunov direction. We can not compute the

limit in $\gamma(x, v) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log |\Phi(t, x)v|$ but instead simply integrate for

some "long" time T and estimate

$$\gamma_{\max}(T) = \frac{1}{T} \ln \frac{|v(T)|}{|v_0|} \dots\dots\dots (2-14)$$

This quantity will rapidly converge to the maximal exponent; to estimate the error in the computation, it is useful to plot $\gamma_{\max}$ as a function of T .

Definition (2.2.13) [12]

A system is regular (Lyapunov) if the time average of the trace has a finite limit and equality holds in $\sum_{i=1}^n \gamma_i \geq \lim_{t \rightarrow \infty} \sup \frac{1}{t} \int_0^t \text{tr} Df(\varphi_s(x)) ds$.

Example (2.2.14)

Consider the Lorenz system

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz$$

where the parameters r, σ and b are positive. The linearized equations for a vector $v \in T_x \mathbb{R}^3$ are

$$\dot{v} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & x \\ y & x & -b \end{bmatrix} v \dots \dots \dots (2-15)$$

To integrate these equations, we must simultaneously integrate the Lorenz system itself; a simple algorithm to do this and to compute $\gamma_{\max}$ is given in [28]. A plot of the time behavior of $\gamma_{\max}$ for two values of r .

(i) If $t = 1000, \sigma = 10, b = \frac{8}{3}$ and $r = 28$, then $\gamma_{\max} = 0.88$

(ii) If $t = 10000, \sigma = 10, b = \frac{8}{3}$ and $r = 28$, then $\gamma_{\max} = 0.90$

(iii) $t = 10000, \sigma = 10, b = \frac{8}{3}$ and $r = 23$, then $\gamma_{\max} = -0.05$

Even though only the largest Lyapunov exponent for the Lorenz system was computed, $\delta \leq \sum_{i=1}^n \gamma_i$ can be used to estimate the other two exponents,

the trace of the Jacobian matrix (2-15) is constant, so that $\delta = \text{tr}(\text{Df}) = -1 - \sigma - b$. Since one exponent vanishes, $\gamma_2 = 0$, for the standard parameters $\gamma_3 \geq \sigma - \gamma_1 = -13.66 - \gamma_1$. If the Lorenz were known to be regular, then the supremum limits could be replaced by ordinary limits and the inequality in $\delta \leq \sum_{i=1}^n \gamma_i$ would become an equality.

Therefore, $\gamma_3 = -13.66 - 0.90 = -14.6$

Remark (2.2.15)

There are many software packages for numerically computing Lyapunov exponents. See [15, 17, 32, 41]



Chaos in Dynamical Systems

In this chapter, we study two methods of describing the way in which iterates of neighboring points separate from one another: sensitive dependence on initial conditions and Lyapunov exponent. These points are fundamental to the concept of chaos.

In this chapter, we introduce some definitions of chaos together with some simple examples, also we calculate Lyapunov exponent to four types of the bifurcation saddle node, transcritical, pitchfork and Hopf bifurcation

3.1 The Chaos.

The word "chaos" is familiar in every day speech. It normally means a lack of order or predictability. Thus one says that the weather is chaotic, or that rising particles of smoke are chaotic, or that the stock market is chaotic. Both sensitive dependence on initial conditions and Lyapunov exponent qualify as measures of unpredictability. [18]

Li and Yorke first gave a mathematical definition of chaos in 1975; the definition that we use is due to Anulander and Yorke (1980).

In this section, we introduce some definitions of chaos in the sense of Gulick, Devaney, Wiggins and Meiss, with some simple examples.

We consider C^r ($r \geq 1$) autonomous vector fields and maps on $\mathbb{R}^n$ defined as follows:

vector field $\dot{x} = f(x)$ and map $x \rightarrow g(x)$.

Chapter Three

We denote the flow generated by vector field by $\varphi(t, x)$ and we assume that it exists for all $t \geq 0$. We assume that $\Lambda \subset \mathbb{R}^n$ is an invariant set under $\varphi(t, x)$ (resp. $g(x)$), i.e., $\varphi(t, \Lambda) \subset \Lambda$, for all $t \in \mathbb{R}$ (resp. $g^n(\Lambda) \subset \Lambda$ for all $n \in \mathbb{Z}$) [38]. Then we have the following definitions:

Definition (3.1.1) [38]

The flow $\varphi(t, x)$ (resp. $g(x)$) is said to have sensitive dependence on initial conditions on Λ , if there exists $\varepsilon > 0$ such that, for any $x \in \Lambda$ and any neighborhood U of x , there exists $y \in U$ and $t > 0$ (resp. $n > 0$) such that $|\varphi(t, x) - \varphi(t, y)| > \varepsilon$ (resp. $|g^n(x) - g^n(y)| > \varepsilon$).

We will give example on sensitive dependence on initial conditions in discrete and continuous dynamical system.

Example (3.1.2) [28]

A linear system $\dot{x} = Ax$ exhibits sensitive dependence on the invariant set $\mathbb{R}^n$ if any of the eigenvalues of A have a positive real part. Since the system is linear, the distance between any two points obeys the same equation. If $y = x + \varepsilon v$, where v is an eigenvector that corresponds to unstable eigenvalue, then $|\varphi_t(x) - \varphi_t(y)| = \varepsilon |v| e^{\operatorname{Re}(\lambda)t}$.

Example (3.1.3)[18]

The tent map in example (2.1.3) has sensitive dependence on initial conditions on $[0, 1]$.

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Definition (3.1.4) [38]

The flow $\varphi(t, x)$ (resp. $g(x)$) is said to be topologically transitive on Λ if for any two open sets $U, V \subset \Lambda$:

Flows: there is a real number t such that $\varphi_t(U) \cap V \neq \emptyset$,

Maps: there is an integer number n such that $g^n(U) \cap V \neq \emptyset$.

Example (3.1.5) [38]

Consider the vector field:

$$\dot{\theta}_1 = w_1$$

$$\dot{\theta}_2 = w_2$$

where $(\theta_1, \theta_2) \in S^1 \times S^1$, with $\frac{w_1}{w_2} = \text{irrational}$. Then, the flow generated by the vector field is topologically transitive on $S^1 \times S^1$.

Example (3.1.6) [18]

The tent map in example (2.1.3) is topologically transitive.

Definition (3.1.7) [18]

A subset B of the interval J is dense in J if B intersects every non-empty open subinterval of J .

We turn now to one of the main themes of this section, the notion of a chaotic dynamical system, some stronger and some weaker than ours.

Both sensitive dependence on initial conditions and the Lyapunov exponent qualify as measures of unpredictability. Thus, we have the following definition.

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Definition (3.1.8) [18]

Let J be a bounded interval. A map $f : J \rightarrow J$ is chaotic (in sense of Gulick) if it satisfies at least one of the following conditions:

1. f has a positive Lyapunov exponent at each point in its domain.
2. f has sensitive dependence on initial conditions on its domain.

Example (3.1.9) [18]

The tent map in example (2.1.3) has positive Lyapunov exponent $\ln(2)$. Therefore, it is chaotic in sense of Gulick.

Definition (3.1.10) [38]

Λ is said to be chaotic (in sense of Wiggins) if

1. $\varphi(t, x)$ (resp. $g(x)$) has sensitive dependence on initial conditions on Λ .
2. $\varphi(t, x)$ (resp. $g(x)$) is topologically transitive on Λ .

Some authors (for example Devaney 1986) add an additional requirement to definition (3.1.10).

Definition (3.1.11) [38]

Λ is said to be chaotic (in sense of Devaney) if

1. $\varphi(t, x)$ (resp. $g(x)$) has sensitive dependence on initial conditions on Λ .
2. $\varphi(t, x)$ (resp. $g(x)$) is topologically transitive on Λ .
3. The set of periodic orbits of $\varphi(t, x)$ (resp. $g(x)$) are dense in Λ .

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Example (3.1.12) [38]

Consider the following vector field $\dot{x} = ax$, $x \in \mathbb{R}$, with $a > 0$.

The flow generated by vector field is given by $\varphi(t, x) = e^{at}x$. From the flow we conclude the following.

1. $\varphi(t, x)$ has no periodic orbits.
2. $\varphi(t, x)$ is topologically transitive on the sets $(0, \infty)$ and $(-\infty, 0)$.
3. $\varphi(t, x)$ has sensitive dependence on initial conditions on $\mathbb{R}^1$.

Therefore, the flow $\varphi(t, x)$ is chaotic in sense of Wiggins, but it is not chaotic in sense of Devaney.

Example (3.1.13)

The tent map in example (2.1.3) is chaotic on $[0, 1]$ in sense of Wiggins and Devaney.

Definition (3.1.14) [28]

A flow φ is chaotic (in sense of Meiss) on an invariant set Λ if φ is transitive and exhibits sensitive dependence on Λ .

Example (3.1.14) [28]

Consider the Rossler system (Rossler 1967):

$$\dot{x} = -y - z$$

$$\dot{y} = x + ay$$

$$\dot{z} = b + z(x - c)$$

where a , b and c are positive numbers.

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The flow of this system exhibits sensitive dependence whenever $a = b = 0.2$ and $c = 5.7$, also it is topological transitive. Therefore, the Rossler system is chaotic in sense of Meiss.

Remark (3.1.15) [28]

We can option is to insist that sensitive dependence be replaced by "positive Lyapunov exponent" in definition of chaos.

Gulick gave definition to the chaos in discrete dynamical system. We generalize now its definition for continuous dynamical system (differential equations), and it's called Gulick generalization, that we will use to prove chaotic dynamical systems.

Definition (3.1.16) (Gulick generalization)

A flow φ is said to be chaotic in Λ if it satisfies at least one of the following conditions:

1. φ has a positive Lyapunov exponent at each point in Λ .
2. φ has a sensitive dependence on initial conditions on Λ .

3.2 Calculation of Lyapunov Exponent for Types of the Bifurcation.

In this section, we calculate Lyapunov exponents for four generic types of bifurcation which are saddle node, transcritical, pitchfork and Hopf bifurcation as shown in chapter one. We will use the equation (2-8) in the definition (2.1.28) to calculate Lyapunov exponent of saddle node, transcritical and pitchfork bifurcation, also we will use the equation (2-14) in remark (2.2.13) to calculate Lyapunov exponent of Hopf bifurcation.

1. Saddle node bifurcation.

As shown in chapter one, the normal form for saddle node bifurcation written as follows:

$$\dot{x} = f(x, \mu) = \mu \pm x^2, \quad x \in \mathbb{R}, \mu \in \mathbb{R} \dots\dots\dots(3-1)$$

If $\mu \leq -1$, then the vector field (3-1) has positive Lyapunov exponent for all $x \in \mathbb{R}$, otherwise it has negative Lyapunov exponent.

t	μ	x	γ
1000	-1	0	0.0012
1000	-1	0.1	0.0015
1000	-1	-0.1	0.0011
1000	-1	1	0.0019
1000	-1	-1	7.0042
1000	0.1	0	$-\infty$
1000	-0.1	0	-8.2382
1000	-0.1	0.1	-9.3600
1000	-1.1	0	3.3838

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See appendix (1). Therefore, the saddle node bifurcation is chaotic if $\mu \leq -1$.

2. Transcritical bifurcation.

As shown in chapter one, the normal form for transcritical bifurcation written as follows:

$$\dot{x} = f(x, \mu) = \mu x \pm x^2, \quad x \in \mathbb{R}, \mu \in \mathbb{R} \dots\dots\dots(3-2)$$

If $\mu \leq -1$, then the vector field (3-2) has positive Lyapunov exponent for all $x \in \mathbb{R}$, otherwise it has negative Lyapunov exponent. See appendix (2). We conclude that the Lyapunov exponent of saddle node bifurcation is equal to the Lyapunov exponent of transcritical bifurcation. Therefore, the transcritical bifurcation is chaotic if $\mu \leq -1$.

3. Pitchfork bifurcation.

As shown in chapter one, the normal form for pitchfork bifurcation written as follows:

$$\dot{x} = f(x, \mu) = \mu x \pm x^3, \quad x \in \mathbb{R}, \mu \in \mathbb{R} \dots\dots\dots(3-3)$$

The Lyapunov exponent of (3-3) is negative for all $\mu \in \mathbb{R}, x \in \mathbb{R}$. See appendix (3). Therefore, the pitchfork bifurcation is not chaotic.

4. The Hopf bifurcation.

As shown in chapter one, the normal form for Hopf bifurcation was found to be:

$$\dot{x} = \alpha(\mu)x - \omega(\mu)y + [a(\mu)x - b(\mu)y](x^2 + y^2) + O(|x|^5, |y|^5)$$

$$\dot{y} = \omega(\mu)x + \alpha(\mu)y + [b(\mu)x + a(\mu)y](x^2 + y^2) + O(|x|^5, |y|^5)$$

For example:

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$$\left. \begin{aligned} \dot{x} &= \mu x - y - x(x^2 + y^2) \\ \dot{y} &= x + \mu y - y(x^2 + y^2) \end{aligned} \right\} \dots\dots\dots(3-4)$$

The system (3-4) has only positive Lyapunov exponent at the point $(x, y) = (0,0)$ when $\mu > 0$, otherwise it has negative Lyapunov exponent.

t	μ	(x, y)	γ
1000	1	(0,0)	0.4992
1000	-1	(0,0)	-0.4990
1000	0	(0,0)	-2.6191
1000	1	(1,1)	-0.1193
1000	-1	(1,1)	-0.5007
1000	0	(1,1)	-0.0058
1000	1.1	(1,1)	-0.1298
1000	1.1	(0,0)	0.5494
1000	0.1	(0,0)	0.0499
1000	0.1	(0.1,0.1)	-0.0168
1000	-0.1	(0,0)	-0.0499

See appendix (4). Therefore, the Hopf bifurcation is chaotic at $(0,0)$ when $\mu > 0$.

Chapter Four

The Relation between the Sensitive Dependence and Lyapunov Exponent to Some Dynamical Systems

In this chapter, we apply the concept of Lyapunov exponent and bifurcation on a continuous dynamical system. We found relation between sensitive dependence and Lyapunov exponent, or in the other word if the system has sensitive dependence, is Lyapunov exponent positive?!

In this chapter, we will study the bifurcation and the chaos in sense of G.Gulick. In [5] the chaos studied by the sensitive dependence on initial conditions by varying the control parameters. Now, we study the chaos by using Lyapunov exponent which is defined in chapter two. In this work, we use Matlab to calculate Lyapunov exponent.

This chapter consists of three sections. In section one, we calculate Lyapunov exponent for model of epidemic. In section two, we calculate Lyapunov exponent for system which taken from the formula:

$\dot{x} = a_i + \sum_{j=1}^3 b_j x_j + \sum_{j=1}^3 \sum_{k=1}^3 c_{jk} x_j x_k$, $i = 1, 2, 3$. In section three, we introduce conclusions.

4.1 Lyapunov Exponent and The Bifurcation for Model of an Epidemic.

We will introduce now a new system a modify epidemic model. An Epidemic model is a simplified means of describing the transmission of communicable disease through individuals.

Suppose that, during an outburst of an infectious disease, the population can divided into three classes:

$x(t)$ = The number of healthy people;

$y(t)$ = the number of sick people;

$z(t)$ = the number of dead people.

The model will be based on the following three assumptions:

- (i) The total population remains constant in size, except for deaths due to the epidemic or deaths by other causes. That is, the epidemic so rapidly that we can ignore the slower changes in the populations due to births, emigration.
- (ii) Healthy people get sick at the rate proportional to the product of x and y . This would be true if healthy and sick peoples encounter each other at a rate proportional to their numbers, and if there were a constant probability that each such encounter would lead to transmission of the disease.
- (iii) Sick people die at a constant rate L .

Under these assumptions, the model is

$$\left. \begin{aligned} \dot{x} &= -kxy + kx - z \\ \dot{y} &= kxy - Ly \\ \dot{z} &= Ly + mx \end{aligned} \right\} \dots\dots\dots(4-1)$$

To find the critical points of system (4-1) setting $\dot{x} = 0, \dot{y} = 0$ and $\dot{z} = 0$, to get:

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$$-kxy + kx - z = 0$$

$$kxy - Ly = 0, \text{ thus } kx = L, \text{ we have } x = \frac{L}{k}$$

$$Ly + mx = 0, \text{ Thus } Ly + m \cdot \frac{L}{k} = 0. \text{ Therefore, } y = -\frac{m}{k}$$

$$-kxy + kx - z = 0, \text{ Thus } -k \cdot \frac{L}{k} \cdot \frac{-m}{k} + k \cdot \frac{L}{k} - z = 0, \text{ we have}$$

$$z = \frac{Lm + Lk}{k} = \frac{L(m + k)}{k}$$

Therefore, the critical points of system (4-1) are

$$p_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } p_2 = \begin{pmatrix} \frac{L}{k} \\ -\frac{m}{k} \\ \frac{L(m + k)}{k} \end{pmatrix}.$$

$$\text{The Jacobian matrix of the system (4-1) is } \begin{bmatrix} -ky + k & -kx & -1 \\ ky & kx - L & 0 \\ m & L & 0 \end{bmatrix}, \text{ and}$$

$$\text{the Jacobian matrix of the system (4-1) at } p_1 \text{ is } \begin{bmatrix} k & 0 & -1 \\ 0 & -L & 0 \\ m & L & 0 \end{bmatrix}, \text{ and the}$$

$$\text{Jacobian matrix of the system (4-1) at } p_2 \text{ is } \begin{bmatrix} m + k & -L & -1 \\ -m & 0 & 0 \\ m & L & 0 \end{bmatrix}.$$

To find the eigenvalues of the system at the points p_1 and p_2 , and also the stability of model see [5].

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Now, we study the chaotic behavior of system(4-1). In [5], they proved that the system(4-1) is chaotic because it has a sensitive dependence at

- (i) (0,2,2) and (0.02,2,2) with parameters $k = -3, L = -5, m = -3$.
- (ii) (0.2,0,0.01) and (0.2,0.01,0.01) with parameters $k = 3, L = 6, m = 5$.
- (iii) (0.2,0,0.01) and (0.2,0.03,0.01) with parameters $k = 3, L = 2, m = -5$.

We study the chaotic behavior of the system(4-1) by using Lyapunov exponent on the same conditions above.

- (i) The largest Lyapunov exponent of the system (4-1) at the points (0.02,2,2) and (0,2,2) with parameters $k = -3, L = -5, m = -3$ is positive number.

If $t = 10000$, then $\gamma = 0.0064$

- (ii) The largest Lyapunov exponent of the system (4-1) at the points (0.2,0.01,0.01) and (0.2,0,0.01) with parameters $k = 3, L = 6, m = 5$ positive too.

If $t = 10000$, then $\gamma = 0.0104$

- (iii) The largest Lyapunov exponent of the system (4-1) at the points (0.2,0.03,0.01) and (0.2,0,0.01) with parameters $k = 3, L = 2, m = -5$ is positive number.

If $t = 10000$, then $\gamma = 0.0157$

Therefore, the system (4-1) is chaotic in sense of G.Gulick. When we change the values of the parameters by taking $k = 2, L = 4, m = 1$, we get positive Lyapunov exponent at the point (0.2,0,0.01) as shown in appendix (5).

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Now, we will study bifurcation of model (4–1). In [5] they studied the stability of model and we found that, if $L > 0, k < 0, m = 0$ then p_1 stable and p_2 unstable, and if $L > 0, k > 0, m = 0$ then p_1 unstable and p_2 stable. Therefore, the system (4–1) has a transcritical bifurcation at k when $L > 0$.

4.2 Lyapunov Exponent and the Bifurcation for the System Taken From General Formula.

In this section, we will study one of the systems that taken from the formula:

$$\dot{x} = a_i + \sum_{j=1}^3 b_j x_j + \sum_{j=1}^3 \sum_{k=1}^3 c_{jk} x_j x_k, \quad i = 1, 2, 3,$$

and have three parameters a, b, c in the control space.

$$\left. \begin{aligned} \dot{x} &= ay - ax \\ \dot{y} &= bx - zx \\ \dot{z} &= x + xy - 2cz \end{aligned} \right\} \dots\dots\dots(4-2)$$

To find the critical points of system (4-2) by setting

$\dot{x} = 0, \dot{y} = 0$ and $\dot{z} = 0$, we get:

$ax - ay = 0$, Thus $x = y$. Also, $bx - xz = 0$, Therefore, $x = 0$ or $z = b$.

$x + xy - 2cz = 0$. We substitute $x = y$ and $z = b$, we have

$$x^2 + x - 2cb = 0, \text{ Thus } x = \frac{-1 \pm \sqrt{1+8cb}}{2}.$$

Hence, the critical points of system (4-2) are:

$$p_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{1+8cb}}{2} \\ -\frac{1}{2} + \frac{\sqrt{1+8cb}}{2} \\ b \end{pmatrix} \text{ and } p_2 = \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{1+8cb}}{2} \\ -\frac{1}{2} - \frac{\sqrt{1+8cb}}{2} \\ b \end{pmatrix}.$$

Let $k = \sqrt{1+8cb}$, then we have:

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$$p_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, p_2 = \begin{pmatrix} -\frac{1}{2} + \frac{k}{2} \\ -\frac{1}{2} + \frac{k}{2} \\ b \end{pmatrix} \text{ and } p_2 = \begin{pmatrix} -\frac{1}{2} - \frac{k}{2} \\ -\frac{1}{2} - \frac{k}{2} \\ b \end{pmatrix}.$$

The Jacobian matrix of system (4-2) is $\begin{pmatrix} -a & a & 0 \\ b-z & 0 & -x \\ 1+y & y & -2c \end{pmatrix}$, the Jacobian

matrix of system (4-2) at the point p_1 is $\begin{pmatrix} -a & a & 0 \\ b & 0 & 0 \\ 1 & 0 & -2c \end{pmatrix}$, and Jacobian

matrix of system (4-2) at the point p_2 is $\begin{pmatrix} -a & 0 & 0 \\ 0 & 0 & \frac{1}{2} - \frac{k}{2} \\ \frac{1}{2} + \frac{k}{2} & -\frac{1}{2} + \frac{k}{2} & -2c \end{pmatrix}$.

To find the eigenvalues and the stability of system (4-2). See [5]

Now, we will study bifurcation of system (4-2). They studied the bifurcation, and we find that system (4-2) has pitchfork bifurcation and transcritical bifurcation. See [5]

Now, we study the chaotic behavior of system (4-2). In [5] they proved that the system (4-2) is chaotic by using sensitive dependence on initial conditions through the following points by varying the control parameters (a, b, c):

- (i) (0, -0.01, 0.1) and (0.1, -0.01, 0.1) with parameters $a = 10, b = 8, c = 8$.
- (ii) (0, -0.01, 0.1) and (0.1, -0.01, 0.1) with parameters $a = 10, b = 8, c = 3$.
- (iii) (0.1, 0.1, 0.5) and (0, 0.1, 0.5) with parameters $a = -1, b = 0, c = 3$.

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(iv) $(0.1, 0, 0.01)$ and $(0, 0, 0.01)$ with parameters $a = -1, b = -1, c = -1$.

Also, they show that the system (4–2) has no sensitive dependence to the above initial conditions when we change the values of the parameters.

We will study the chaotic behavior of the system (4–2) by using Lyapunov exponent on the same conditions above.

1. The largest Lyapunov exponent of the system (4–2) at the points $(0, -0.01, 0.1)$ and $(0.1, -0.01, 0.1)$ with parameters $a = 10, b = 8, c = 8$ is a positive number.

If $t = 1000$, then $\gamma = 5.1613$

If $t = 10000$, then $\gamma = 5.1666$

2. The largest Lyapunov exponent of the system (4–2) at the points $(0, -0.01, 0.1)$ and $(0.1, -0.01, 0.1)$ with parameters $a = 10, b = 8, c = 3$ is a positive number.

If $t = 1000$, then $\gamma = 4.9087$

If $t = 10000$, then $\gamma = 4.9135$

3. The largest Lyapunov exponent of the system (4–2) at the points $(0.1, 0.1, 0.5)$ and $(0, 0.1, 0.5)$ with parameters $a = -1, b = 0, c = 3$ is a positive number.

If $t = 1000$, then $\gamma = 2.3560$

If $t = 10000$, then $\gamma = 2.3941$

4. The largest Lyapunov exponent of the system (4–2) at the points $(0, 0, 0.01)$ and $(0.1, 0, 0.01)$ with parameters $a = -1, b = -1, c = -1$ is a positive number.

If $t = 1000$, then $\gamma = 9.305$

If $t = 10000$, then $\gamma = 9.810$

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Therefore, the system (4 – 2) is chaotic in sense of G.Gulick, when we change the values of the parameters we get positive or negative Lyapunov exponent to the above initial conditions. For example, the point $(0, -0.01, 0.1)$ has positive Lyapunov exponent with parameters $a = 10, b = 10, c = 10$.

If $t = 1000$, then $\gamma = 5.6030$

If $t = 10000$, then $\gamma = 5.6080$

But, the point $(0, -0.01, 0.1)$ has negative Lyapunov exponent with parameters $a = 2, b = -4, c = 1$.

If $t = 1000$, then $\gamma = -0.9995$

If $t = 10000$, then $\gamma = -0.9999$

See appendix (6).

3.4 Conclusions.

1. The system (4 – 1) has a transcritical bifurcation.
2. The system (4 – 2) has a transcritical and pitchfork bifurcation.
3. The systems (4 – 1) and (4 – 2) have chaotic behavior.
4. The appearance of chaos depends on the values of the parameters.
5. If the system has sensitive dependence on initial conditions, then it has a positive Lyapunov exponent.
6. If the system has positive Lyapunov exponent, then it has not necessary that has sensitive dependence.

Examining Committee Certification

We certify that we have read this thesis entitled "**On the Chaotic Behavior of Local Bifurcation Systems**" and, as an Examining Committee, we examined the student in its content, and what is related to it, and that in our opinion it is adequate with standing as a thesis for the degree of Master of Science in Mathematics.

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(Member)

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I certify that this thesis was prepared under my supervision at the Department of Mathematics / College of Education / University of Babylon, as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Report of chairwoman of the Postgraduate Studies Committee

In view of the available recommendation, I forward this research for debate by the Examining Committee.

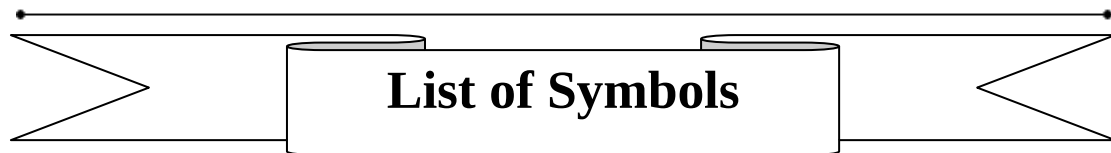
Signature:

Name: Dr. Iftichar Mudhar Talb

Title: Assist. Professor

Date:

List of Symbols



S	An open set of Euclidean space
$\mathbb{R}$	The set of real numbers
Λ	An invariant set
$\cap$	The intersection
∇	The gradient
f, g, h	Maps
$Df(x_0)$	Jacobian matrix of f at x_0
Φ	Fundamental solution matrix
φ	Flow
$(f^n)'$	The differentiating f^n with respect to x
λ	Eigenvalue
σ, r, μ, a, b, c	Parameters
$\dot{x}$	$\frac{dx}{dt}$
t	Time
γ	Lyapunov exponent
$\chi(f)$	The characteristic exponent of f
$\gamma(f, x)$	Lyapunov exponent of f through the point x
$\gamma(x_0, e)$	Lyapunov exponent in direction " e " through the orbit x_0
$ \quad $	The absolute value
$\ \quad \ $	Norm
P	Lyapunov transformation
V	Lyapunov map
M	Phase space

List of Symbols

$\det$	The determinant
$\operatorname{Re}(\lambda)$	The real part of λ
$\operatorname{Im}(\lambda)$	The imaginary part of λ
$v_1, v_2, \dots, v_n$	Vectors
ODEs	Ordinary differential equations
★	End of the proof

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To my family