Convolutional Neural Networks Approximation on Quasi-Orlicz Spaces

A Thesis

Submitted to the Council of College of Education for Pure Sciences in University of Babylon as a Partial Fulfillment of the Requirements for Degree of Master in Education / Mathematics

By

Amna Manaf Ismael Fadhil Al-Janabi

Supervised by

Asst.Prof.Dr.Hawara Abbas Almurieb

2024 A.D.   1445 A.H

بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

﴿يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ ۚ وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ﴾

صدق الله العلي العظيم

سورة المجادلة الآية 11

Supervisor's Certification

I certify that the thesis entitled **"Convolutional Neural Networks Approximation"** by**" Amna Manaf Ismael Fadhil"** has been prepared under my supervision in University of Babylon / College of Education for Pure Sciences as a partial requirement for the degree of Master in Education / Mathematics.

Signature :

Name: **Dr. Hawara Abbas Almurieb**

Title: Assistant Professor

Date: /   / 2024

In view of the available recommendation, I forward this thesis for debate by the examining committee.

Signature:

Name: Dr. Azal Jaafar Musa

Head of Mathematics Department

Title: Assistant Professor

Date: /   / 2024

Certification of Scientific Expert

I certify that I have read the scientific content of this thesis **"Convolutional Neural Networks Approximation"** and I have approved this dissertation is qualified for debate

Signature:

Name:

Title:

Date: /   /2023

Certification of Linguistic Expert

I certify that I have read this thesis entitled **"Convolutional Neural Networks Approximation"** and corrected its grammatical mistakes; therefore, it has qualified for debate.

Signature:

Name:

Title:

Address:

Date: /   / 2023

Examining Committee Certification

We certify that we have read the thesis entitled **" Convolutional Neural Networks Approximation "** by**" Amna Manaf Ismael Fadhil "** and as a committee examined the student in its contents and, according to our opinion, it is accepted as a thesis for the degree of Master in Education /Mathematics.

Signature:

Name:

Title:

Date: /   / 2023

Chairman

Signature:

Name:

Title:

Date: /   / 2023

Member /

Signature:

Name:

Title:

Date: /   / 2023

Member

Signature:

Name:

Title:

Date: /   / 2023

Member / Advisor

Dedication

To My Family,

especially my compassionate father and my affectionate

mother, everyone who helped us

and for everyone who loves us.

Acknowledgments

Alhamdulillah, who gave me this opportunity, and let me in completing this work, and to have given me the right people to help me to do so. I would like to express my appreciation and gratitude to my supervisor **"Dr. Hawraa Abbas Almurieb"** to support me and make suggestions throughout my working period, it has the highest words of praise and appreciation. I would like to thank the staff of College of Education for Pure Sciences, especially my professors in Department of Mathematics. Also, I thank my family, especially my compassionate father... My role models, my ideals in life, Teach me how to live with dignity and pride. my affectionate mother, I do not find words that can give her the right, it is the epic of love, the pleasure of age, the example of dedication and giving. I thank my friends, and anyone who has supported, advised, or contributed, from near or far away.

Finally, repeatedly, Alhamdulillah

Contents

Abstract

It is essential to study approximation theoretical foundations of deep convolutional neural networks, because of its exciting developments in vital domains, despite its dependence upon approximation substantially. On the other hand, this relates to another matter of great importance, which is the choice of the space of functions. Studying that topic on a vast space of functions is excellent for covering as many functions as possible.

This thesis includes a new class of quasi-normed spaces, called Quasi-Orlicz Spaces as a space of functions for that is used convolutional neural network approximation. Moreover, to estimate the degree of best approximation with convolutional neural networks, we define a modulus of smoothness on the quasi–Orlicz space. In addition, we study the essential characteristics and the approximation abilities of deep convolutional neural networks produced by downsampling operators in quasi-Orlicz spaces, so we resulted in a degree of best approximation of quasi-Orlicz functions in terms of higher order modulus of smoothness.

List of Figures

Abbreviations

Abbreviation	Definition	Page
CCNs	Convolutional Neural Networks	25
ReLU	Rectified Activation Function	29
DCNN	Deep Convolutional Neural Networks	36
DDCNN	Deep Downsampling Convolutional Neural Networks	44

Table of Symbols

Symbol	Definition	Page
L_p	Lebesgue space	8
I^d	$I^d = I \times I \times \ldots \times I, d - \ times$	8
$\mathbb{R}^d$	$\mathbb{R}^d = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}, d - \ times$	8
$\|\cdot\|_p$	$p-$norm	8
L_Φ	Orlicz space with Φ Young function	8
$\|\cdot\|_\Phi$	Luxembourg norm	8
Φ	Young function	8
Ψ	Generator of modular unit ball	9
I_Φ	Modular	9
S_p	Outer Function	11
L_{Φ,S_p}	Quasi – Orlicz normed space	12
Δ_h^N	N -th symmetric difference	16
ω_N	modulus of smoothness of order N	16
$\mathbb{S}^{d-1}$	Spherical space of dimension $d-1$	22
$L_p(\mathbb{S}^{d-1})$	Lebegsue Spherical Space	23

$L_{\Phi,S_p}(\mathbb{S}^{d-1})$	Quasi-Orlicz Spherical Space	23
$\mathbb{R}$	The space of real numbers	25
$\mathbb{N}$	The space of natural numbers	25
f_N	Neural Network	25
$f \circ g$	Discrete convolution between two functions	27
$h^{(j)}$	Output	29
σ	Activation Function	30
J	Number of Hidden layers	31
$\mathcal{N}$	Collection of CNNs	31
$\mathfrak{I}$	Toeplitz Matrix	31
w	Weight	31
b, B	Bias vectors	31
$\mathfrak{D}_m$	Downsampling operator	36

Introduction

Convolutional neural networks (CNN) play a significant role in several applications, such as document analysis, understanding climate, and recognition of speech and images. The architecture of a convolutional neural network is a multi-layer neural network made by stacking many layers hidden over each other in order. This sequential design enables convolutional neural networks to learn hierarchical characteristics. Hidden layers are usually convolutional layers followed by activation layers, some followed by clustering layers.

Diversion of neural networks are characterized from other classical neural networks by their superior performance with acoustic signal or speech inputs, CNNs are commonly used to resolve spatial data issues such as images. In addition, regular neural networks are better suited to the analysis of temporal and sequential data, such as text or video. Approximate functions on subassemblies of $\mathbb{R}^d$ through shallow neural networks with their general form

$$f_N(x) = \sum_{k=1}^{N} c_k \ \sigma\left(\langle w_k, x \rangle - b_k\right), \ \ x \in \mathbb{R}^d$$

were well studied towards the end of the 1980s. Deep neural networks are

completely fully connected networks. It's very important to the study universality of CNNs, it gives a strong theoretical basis to understand and apply CNNs. However, proving universality is not as easy as that of classical neural networks, because of their complex structure and computation intensive.

The importance of neural networks comes from being function approximation. Researchers are interested in functional approximation through neural networks in various applied and theoretical fields such as,[1-6]. Zhou was the first who demonstrate the universality of functions approximation with convolution neural networks. He considered an applied family of deep convolutional neural network functions with the same order of computational complexity as the deep convolutional neural networks. Moreover, he showed that a deep convolutional neural network can be used to approximate any continuous function to an arbitrary accuracy when the depth of the neural network is large enough in his former paper [6]. He also showed the approximation theory of deep convolutional neural networks whose structures are induced by convolutions. To overcome the difficulty in theoretical analysis of the networks with linearly increased widths arising from convolutions, he introduced a downsampling operator to reduce the widths in this paper [4]. Fang, Feng, Huang, and Zhou[5] also studied uniform approximation of functions in the unit sphere of $\mathbb{S}^{d-1}$

of $\mathbb{R}^d$. In this thesis, we focus on the functions space that generates CNNs, it has to be more general than previous continuous functions space. The generated CNN is a best approximation for any function from a new-defined quasi-normed space, called *Quasi-Orlicz Space*. The order of approximation depends on a *Quasi-Orlicz Modulus of Smoothness* defined sequentially.

The thesis consists of four chapters, each of which has developments about the topic.

In Chapter One, we define a new class of quasi-normed spaces and name them quasi-Orlicz spaces. By proving that space is quasi-normed, we could identify the modulus of smoothness that we use later to study the degree of best approximation out of the space of CNNs generated by quasi -Orlicz functions.

The quasi-normed Orlicz space is defined by

$$L_{\Phi,S_p}(I^d) = \left\{ f \in L_p(I^d) \,\middle|\, \|f\|_{\Phi,S_p} < \infty \right\},$$

where

$$\| f \|_{\Phi,S_p} = \inf_{k>0} \frac{1}{k} \; (I_\Phi^2(kf) + 1)^{\frac{1}{p}},$$

and $I^d \subseteq \mathbb{R}^d$.

In Chapter Two, we introduce convolution neural networks with an approximation point of view. Few researchers studied them in terms of its approximation property. We mentioned above authors have studied that since 2018.

Moreover, we prove that a CNN is a universal approximation of quasi-Orlicz space defined in Chapter One. Our primary CNN is based on the construction of previous research and the degree of best approximation is calculated in the following theorem,

Theorem I

Let $f \in L_{\Phi, S_p}(I^d)$, $0 < p < 1$, then there exists w, b forming a CNN $f_j^{w,b}$ as follow

$$\mathcal{N}_j = \left\{ \sum_{k=1}^{d_J} c_k {h_k}^{(J)} : c_k \in R^{d_J} \right\}, \quad j = 1, 2, \dots, J$$

then

$$\left\| f - f_J^{w,b} \right\|_{\Phi, S_p} < \epsilon$$

As a development of previous researchers' work, to get a better degree of approximation, we define, in Chapter Three, a new CNN depending on a downsampling operator $\mathfrak{D}_m \colon \mathbb{R}^D \longrightarrow \mathbb{R}^{[D/m]}$ defined as follow

$$\mathfrak{D}_m(u) = (u_{im})_{i=1}^{[D/m]}, u \in \mathbb{R}^D$$

4

We prove that after using the downsampling operator and using special features, we can find the degree of approximation in terms of quasi-Orlicz modulus of smoothness.

For any $f \in L_{\Phi,S_p}(I^d)$, we define

$$\mathcal{L}_t\big(f(u)\big) = \sum_{i=1}^{d} f(x_i)\delta_i(u)$$

where

$$\delta_i(u) = \sum_{i=1}^{N} \binom{N}{i}(-1)^{N-i}\sigma(t_i - u)$$

In the following main results, we estimate the approximation error of our approximation. Both upper and lower bounds are resulted in terms of quasi-Orlicz modulus of smoothness.

Theorem II (Direct Theorem)

For any $g \in L_{\Phi,S_p}(I^d)$, there exists a CNN

$$L_t\big((g)(u)\big) = \sum_{i=2}^{2N+2} g(t_i)\,\delta_i(u), u \in [t_{i-1}, t_i]$$

where $|t_i - t_{i-1}| \sim \frac{1}{n}$, such that

$$\|L_t(g) - g\|_{\Phi,S_p}^p \le \frac{c}{n}\omega_N\left(g, \frac{1}{n}\right)_{\Phi,S_p},$$

where

$$\delta_i(u) = \sum_{i=1}^{N} \binom{N}{i} (-1)^{N-i} \sigma(t_i - u)$$

Theorem III (Inverse Theorem)

For any $g \in L_{\Phi,S_p}(I^d)$, then $L_t(g)$ satisfies

$$\omega_N(g,\delta)_{\Phi,S_p} \leq \frac{C}{n}\left[\|g\|_{\Phi,S_p} + E_m(g)_{\Phi,S_p}\right]$$

In Chapter Four, we build a better, more flexible CNN than those in Chapters Two and Three with the same degree of approximation. We benefit from the downsampling properties and add a finite number of layers as desired, or the user needs.

In addition, the order of modulus of smoothness that represents the degree of approximation matches the number of layers. That means that more additional hidden layers imply a better degree of approximation.

Theorem IV

Let $2 \leq S \leq d, d \geq 3, r > 0, m, n, N \in \mathbb{N}, \boldsymbol{f} \in L_{\Phi,S_p}(\mathbb{S}^{d-1})$. Let $J \geq \lceil \frac{md-1}{S-1} \rceil$. Then there is a CNN of J layers with filters of length S and bias vectors satisfying $b^{(j)}_{s^{(j)}+1} = b^{(j)}_{s^{(j)}+2} = \ldots = b^{(j)}_{d_j-1}$ followed by downsampling and k-th layers satisfying

$$\|h^{(J+k)} - \boldsymbol{f}\|_{\Phi,S_p} \leq C\,\omega_N\left(\boldsymbol{f}, \frac{1}{n}\right)_{\Phi,S_p}$$

Chapter One:
Quasi - Orlicz Spaces

1.1. A Short History about Orlicz Spaces

Orlicz spac is a generalization of Lebesgue space

$$L_p(I^d) := \{f : I^d \to \mathbb{R} \,|\, \|f\|_p < \infty\}, \qquad (1.1)$$

where $I^d \subseteq \mathbb{R}^d$, $d \in \mathbb{N}^N$

We define Luxembourg norm on measurable function f by

$$\|f\|_\Phi = \inf \left\{ \lambda : \int \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}$$

Where Φ is Young function, defined below,

Definition 1.1.1: [7] A function $\Phi : \mathbb{R} \to [0, \infty]$ is called a Young function if it satisfies the conditions

I. Φ is convex lower semicontinuous $[0, \infty]$-valued function on $\mathbb{R}$.

II. Φ is even and $\Phi(0) = 0$.

III. Φ is non-trivial: it is different from the constant function. $\Phi(x) = 0, x \in \mathbb{R}$ and it is a convex conjugate $\Phi^*(x) = \begin{cases} 0 & if \ x = 0 \\ +\infty & otherwise \end{cases}$

We define the Orlicz space L_Φ to be those measurable functions f with finite $\|f\|_\Phi$. We see that Orlicz space L_Φ is a generalization of L_p, when we take $\Phi(x) = \frac{x^p}{p}$.

The first who studied this type of spaces was the mathematician Orlicz in his paper [8]. As a generalization of Lebesgue integrable spaces L_p, Orlicz space forms an extended Banach space

$$L_\Phi(\mu) = \{f : f \text{ is } \mu - \text{measurable and } \|f\|_\Phi < \infty\}, \quad (1.2)$$

where the norm $\|\cdot\|_\Phi$ had been defined in many ways to cover that space.

Until today, the spaces of Orlicz have been examined by many authors in different ways. However, most of these studies used different definitions for the convex function that defines Orlicz spaces, beginning with Orlicz himself with his norm $\|\cdot\|_\Phi^0$,

$$\|f\|_\Phi^0 = \sup\left\{\int_T |f(t)y(t)|d\mu : y \in L_\Psi, I_\Psi(y) \le 1\right\} \quad (1.3)$$

Orlicz norm was defined depending on Young function Φ. Also, Ψ is the generator of the modular unit ball, which is defined by

$$\Psi(u) = \sup_{u \in R}\{| u | v - \Phi(v) : v \ge 0\} \quad (1.4)$$

In simultaneous times in the fifteenth of the last century, Nakano [9], Morse-Transue [10], and Luxembourg [11] investigated the Luxembourg norm, which has been defined, using the concept of functional Minikowski over a convex modular unit ball, $\{x : I(x) \le 1\}$, with

$$\|f\|_\Phi = \inf_{\lambda > 0} \left\{ I_\Phi \left(\frac{f}{\lambda} \right) \leq 1 \right\} \qquad (1.5)$$

Then not long later, Amemiya [9] defined the norm

$$\|f\|_\Phi^A = \inf_{\lambda > 0} \frac{1}{\lambda} \left(1 + I_\Phi(\lambda f) \right) \qquad (1.6)$$

In separated papers, Krasnoselskii and RutickiI [12], Nakano [9], and Luxembourg and Zaneen[13] proved under additional conditions on Φ, that Amemiy norm is exactly the Orlicz norm $\|\cdot\|^A = \|\cdot\|^0$. The only difference between the two Amemyia forms is the function under the infimum operation: for all $u \geqslant 0, s(u) = 1 + u$ (for the Orlicz norm) and $s(u) = \max\{1, u\}$ (for the Luxemburg norm). In 2000, Hudzik and Maligranda [14] suggested investigating the Amemiya formula generated by outer functions of the type $S_p(u) = (1 + u^p)^{\frac{1}{p}}$, where $1 \leqslant p \leqslant \infty$. They obtained a family of topologically equivalent norms (called the p-Amemiya norms and denoted by $\|\cdot\|_{\Phi,p}$), indexed by $1 \leqslant p \leqslant \infty$, and they satisfy the inequalities

$$\|f\|_\Phi = \|f\|_{\Phi,\infty} \leqslant \|f\|_{\Phi,p} \leqslant \|f\|_{\Phi,q} \leqslant \|f\|_{\Phi,1} = \|f\|_\Phi^0 \leqslant 2^{\frac{1}{p}} \|f\|_{\Phi,p},$$

for all $1 \leqslant q \leqslant p \leqslant \infty$.

Moreover, authors in [15] gave the basic results about the so called p −Amemiya norms and equipped spaces were presented. Also, in [16] Wisla showed an overview of the developments of the spaces defined

later. Moreover, he introduced more generalized $p-$Amemiya type norms by restricting his conditions about the outer function in [17].

Later, he introduced the concept of outer function s and get s-norms on Orlicz spaces

$$\|f\|_{\Phi,s} = \inf_{\lambda>0} \frac{1}{\lambda} s(I_\Phi(\lambda f)) \qquad (1.7)$$

1.2. Quasi – Orlicz spaces

All the definition were given for Orlicz spaces that form normed spaces. To generalize them, we define a quasi -normed spaces by extending the value of p in p-Amemiya norm that is given by

Definition 1.2.1: The family $\{S_p : 0 < p < 1\}$ of functions are defined by

$$S_p(u) = (1 + u^2)^{\frac{1}{p}}, \qquad (1.8)$$

where $u \in [0, \infty)$

In the Figure (1,1), the graphs of outer function S_p , $p \in [0, \infty)$. Light color represents the graphs of S_p, when the values of p are larger than one. Bold graphs represent graph of S_p with values of $p \leq 1$. Violet $p = \frac{1}{5}$ Light Violet $p = 5$, Green $p = \frac{1}{4}$ ‹Greenlight $p = 4$, Red $p = \frac{1}{3}$, Redlight $p = 3$, Turquoise $p = 1$ and Black $p = \infty$
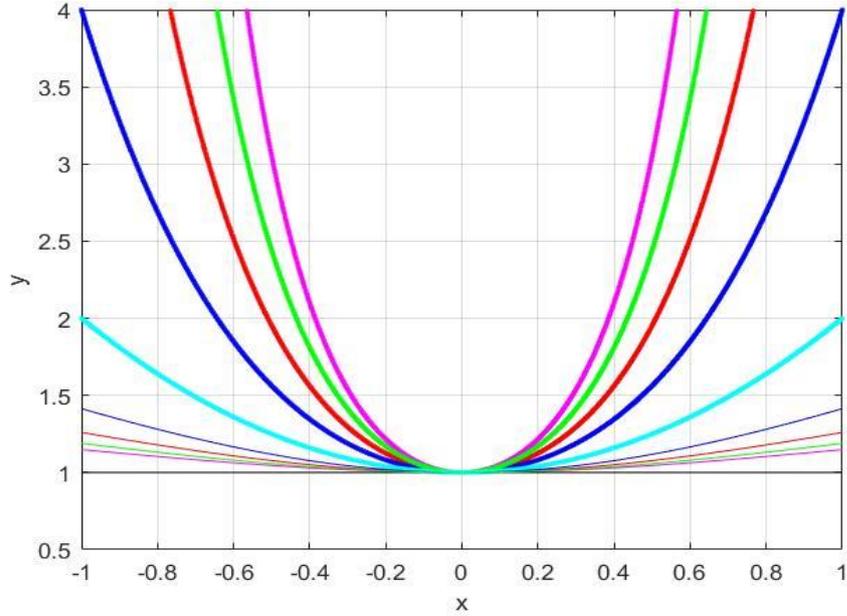
Figure 1. 1 S_p Functions

We investigate some important properties of S_p, they are

1. convex

2. all S_p, match at one point, that is $S_p(0) = 1$, for all p

3. strictly increasing on $[0, \infty)$

In our work, we focus on positive-values functions, so we define and prove Quasi-Orlicz norm depending on **Definition1.2.1** as follow

Definition 1.2.2: A Quasi - Orlicz norm for $0 < p < 1$ is defined depending on **Definition1.2.1** are follow,

$$\| f \|_{\Phi, S_p} = \inf_{k>0} \frac{1}{k} \ (I_\Phi^2(kf) + 1)^{\frac{1}{p}} \qquad (1.9)$$

Where Φ is young function, S_p is outer function from Definition 1.1.1, I_Φ is modular,

$$I_\Phi(x) = \int_T \Phi(x(t))du$$

So that the quasi Orlicz-normed spaces is linear space measurable functions given by

$$L_{\Phi,S_p}(I^d) = \left\{f : I^d \to \mathbb{R} \,\middle|\, \|f\|_{\Phi,S_p} < \infty\right\} \qquad (1.\,10)$$

The following figure shows the relationship between some related spaces. We notice that every function from L_p-space, $1 \leq p \leq \infty$ belongs to both Quasi -L_p space, $0 < p < 1$ and Orlicz space equipped with p-Amemiya, $1 \leq p \leq \infty$. But   Quasi -Orlicz space combine all above functions.
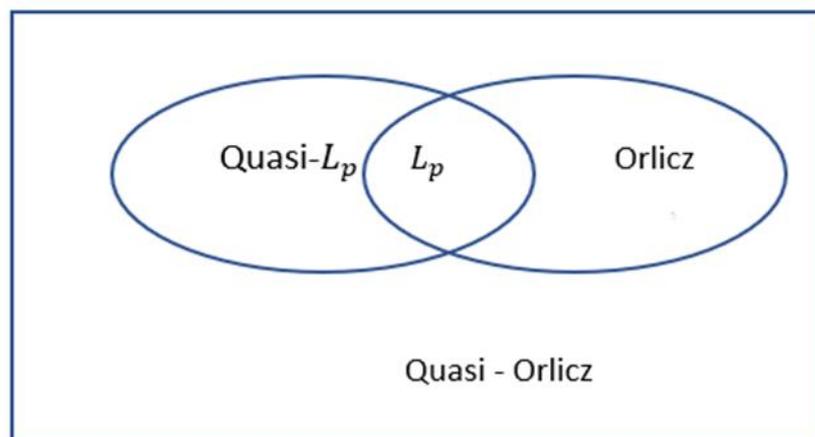

Figure 1. 2 Relations between spaces

In the following theorem, we prove our claim that (1.9) satisfies quasi-norm, so that (1.10) is quasi-normed space.

Theorem1.2.3: $\left(L_{\Phi,S_p}(I^d), \|\cdot\|_{\Phi,S_p} \right)$ is quasi- normed space, where Φ is young function, S_p is outer function.

Proof

Let $f \in L_{\Phi,S_p}$, then it is clear that $\| f \|_{\Phi,S_p} \geq 0$.

Also, $\| 0 \|_{\Phi,S_p} = \inf_{k>0} \frac{1}{k} = 0$

Now, for any positive λ, we have

$$\|\lambda f\|_{\Phi,S_p} = \inf_{k>0} \frac{1}{k} \left(1 + I_\Phi^2(\lambda k f) \right)^{\frac{1}{p}}$$

$$= |\lambda| \inf_{k>0} \frac{1}{k|\lambda|} \left(1 + I_\Phi^2(\lambda k f) \right)^{\frac{1}{p}}$$

$$= |\lambda| \; \|f\|_{\Phi,S_p}$$

Let $\epsilon > 0$, we choose $k, r > 0$ so that for any $f, g \in L_{\Phi,S_p}$,

$$\inf_{k>0} \frac{1}{k} \left(1 + I_\Phi^2(\lambda k f) \right)^{\frac{1}{p}} \leq \|f\|_{\Phi,S_p} + \frac{\epsilon}{2}$$

and

$$\inf_{r>0} \frac{1}{r} \left(1 + I_\Phi^2(\lambda r g) \right)^{\frac{1}{p}} \leq \|g\|_{\Phi,S_p} + \frac{\epsilon}{2}$$

Thus,

$$\|f+g\|_{\Phi,S_p} \le \inf_{k,r>0} \frac{k+r}{kr}\left(1+I_\Phi^2\left(\frac{kr}{k+r}(f+g)\right)\right)^{\frac{1}{p}}$$

Since S_p and Φ are convex , then

$$\|f+g\|_{\Phi,S_p} \le c\left(\inf_{k,r>0} \frac{k+r}{kr}\left(1+I_\Phi^2\left(\frac{kr}{k+r}f\right)\right)^{\frac{1}{p}}\right.$$

$$\left.+\inf_{k,r>0} \frac{k+r}{kr}\left(1+I_\Phi^2\left(\frac{kr}{k+r}g\right)\right)^{\frac{1}{p}}\right)$$

$$= C\left(\inf_{k>0}\frac{1}{k}\ (1+I_\Phi^2(\lambda kf))^{\frac{1}{p}}+\inf_{r>0}\frac{1}{r}\ (1+I_\Phi^2(\lambda rg))^{\frac{1}{p}}+\epsilon\right)$$

$$\le C\big(\|f\|_{\Phi,Sp}+\|g\|_{\Phi,Sp}+\epsilon\big) \qquad\qquad \square$$

One of the most important properties of any norm, or quasi – norm, strict convexity, since it implies uniqueness of the best approximation if exist. So, we prove that (1.9) is strictly convex quasi–norm in the following theorem,

Theorem1.2.4 $\ \|\cdot\|_{\Phi,S_p}$ is strictly convex norm, where Φ is young function, S_p is outer function

Proof

Let $0<\lambda<1, f,g\in L_{\Phi,S_p}$, s.t $\|f\|_{\Phi,S_p}=r=\|g\|_{\Phi,S_p}$, then

$$\|\lambda f + (1-\lambda)g\|_{\Phi,S_p} = \inf_{k>0} \frac{1}{k} \, S_p(\lambda f + (1-\lambda)g)$$

Since S_p are convex, then

$$\|\lambda f + (1-\lambda)g\|_{\Phi,S_p} \leq \inf_{k>0} \frac{1}{k} \left(\lambda S_p(f) + (1-\lambda)S_p(g) \right)$$

$$< \lambda \inf_{k>0} \frac{1}{k} \, S_p(f) + (1-\lambda) \inf_{k>0} \frac{1}{k} S_p(g)$$

$$< \lambda \|f\|_{\Phi,S_p} + (1-\lambda)\|g\|_{\Phi,S_p}$$

$$< \lambda r + (1-\lambda)r$$

$$= r \qquad\qquad \square$$

1.3. Orlicz Moduli of Smoothness

Moduli of smoothness had been studied widely in Orlicz spaces. The importance of this comes from the need to improve the degree of function approximation via direct approaches. Direct theorem approximation implies a faster convergence to zero of its degree of approximation than previous estimates. On the other hand, converse theorems give characterization of smoothness of functions depending on its degree of approximation in the direct theorem. First results were given by Jackson [18], and Bernstein [19], for direct and inverse theorems, respectively, for the space of continuous functions in terms of modulus of continuity , see[20] for details. Later, second and third orders moduli of smoothness were involved in many papers concerning

many wide generalizations of function's spaces [21-28]. However, the birth of modulus of smoothness is much deeper in history (see[29-32]). We concern here with moduli of smoothness as far as relates to Orlicz spaces. It's very hard to investigate the early works of this topic, since they were written in Russian. However, the oldest paper we obtained is attributed to Tsyganok [33] in 1966. He proved direct theorem as a continuity modulus for Orlicz spaces. Later, in the eighteens of the last century, Ramazanov generalized direct theorem in Orlicz spaces for modulus of smoothness with higher orders [34]. In 1985 authors in[30], generalized τ-modulus of smoothness, resulted by [31], for Orlicz spaces. In 1991, Garidi generalized those results of Ramazanov [34], in his paper [35]. Many other results were studied by [36-38], for weighted Orlicz spaces. More extensions were resulted to Orlicz spaces by Shidlich in [39, 40], which were combined by Chaichenko Shidlich and Abdullayev [38].

1.4. Quasi – Orlicz Modulus of Smoothness

All moduli of smoothness that studied previously delt with Orlicz norms. Let's now present the main definitions of modulus of smoothness. that deals with quasi Orlicz norms, presented before.

At the beginning, define the following N-th symmetric difference from [29],

$$\Delta_h^N(f(x)) = \sum_{i=1}^{N} (-1)^i \binom{N}{i} f(x - ih) \qquad (1.11)$$

Definition 1.4.1 The quasi - Orlicz Modulus of smoothness in terms of (1.11) and (1.9) is given by

$$\omega_N(f, \delta)_{\Phi, S_p} = \sup_{|h| \leq \delta} \left\| \Delta_h^N(f) \right\|_{\Phi, S_p} \qquad (1.12)$$

We study the main properties of the symmetric difference (1.11) and the modulus of smoothness (1.12) in the following section,

In the following theorem, we prove that (1.11) satisfies the following properties under our Orlicz norm (1.9)

Theorem 1.4.2 Let $f \in L_{\Phi, S_p}(I^d)$, then

1. $\left\| \Delta_h^N(f) \right\|_{\Phi, S_p} \leq k(N) \|f\|_{\Phi, S_p}$, where

$$k(N) = \sum_{i=0}^{\infty} \binom{N}{i} \leq 2^N, N = \inf\{k \in \mathbb{N} : k > N\}$$

2. $\left(\Delta_h^N \left(\Delta_h^M(f(x)) \right) \right) = \Delta_h^{N+M}(f(x))(a.e.)$

3. $\left\| \Delta_h^{N+M}(f(x)) \right\|_{\Phi, S_p} \leq 2^M \left\| \Delta_h^N(f(x)) \right\|_{\Phi, S_p}$

4. $\lim_{h \to 0} \left\| \Delta_h^N(f(x)) \right\|_{\Phi, S_p} = 0$

Proof

1. $\left\|\Delta_h^N(f(x))\right\|_{\Phi,S_p} = \inf_{\lambda>0} \frac{1}{\lambda} S_p\left(\Delta_h^N(f(x))\right)$

$$= \inf_{\lambda>0} \frac{1}{\lambda} S_p\left(\sum_{i=1}^{N}(-1)^i \binom{N}{i} f(x-ih)\right)$$

$$\leq C \inf_{\lambda>0} \frac{1}{\lambda} \sum_{i=0}^{\infty} \binom{N}{i} S_p(-1)^i f(x-ih)$$

$$\leq C \sum_{i=0}^{\infty} \binom{N}{i} \inf_{\lambda>0} \frac{1}{\lambda} \; S_p(f(x-ih))$$

$$\leq Ck(N)\|f\|_{\Phi,S_p}$$

2. $\left(\Delta_h^N\left(\Delta_h^M(f(x))\right)\right) = \sum_{i=1}^{N}(-1)^i \binom{N}{i}\left(\Delta_h^M(f(x-ih))\right)$

$$= \sum_{i=1}^{N}(-1)^i \binom{N}{i}\left(\sum_{i=1}^{M}(-1)^i \binom{M}{i}(f(x-ih))\right)$$

$$= \sum_{i=1}^{N}\sum_{i-1}^{M}(-1)^i \binom{N}{i}\binom{M}{i} f(x-hi)$$

$$= \sum_{i=1}^{N+M}(-1)^i \binom{N+M}{i} f(x-hi)$$

$$= \Delta_h^{N+M}(f(x))$$

3. By (2), we get

$$\left\|\Delta_h^{N+M}(f)\right\|_{\Phi,S_p} = \left\|\Delta_h^M\left(\Delta_h^N(f)\right)\right\|_{\Phi,S_p}$$

$$\leq k(M)\left\|\Delta_h^N(f)\right\|_{\Phi,S_p}$$

$$\leq 2^{(M)}\left\|\Delta_h^N(f)\right\|_{\Phi,S_p}$$

4. Let $\epsilon > 0$, choose $\delta = \delta(\epsilon, N)$, by theorem I

$$\left\|\Delta_h^N(f_J^{w,b})\right\|_{\Phi,S_p} < \frac{\epsilon}{2}$$

$$\left\|\Delta_h^N(f)\right\|_{\Phi,S_p} \leq C\left[\left\|\Delta_h^N(f - f_J^{w,b})\right\|_{\Phi,S_p} + \left\|\Delta_h^N(f_J^{w,b})\right\|_{\Phi,S_p}\right]$$

$$< C\left[2\ \left\|f - f_J^{w,b}\right\|_{\Phi,S_p} + \left\|\Delta_h^N(f_J^{w,b})\right\|_{\Phi,S_p}\right]$$

$$< C\left[2^N \cdot \frac{\epsilon}{2^{N+1}} + \frac{\epsilon}{2}\right] = \epsilon \qquad \qquad \square$$

In the following theorem, we give more properties for quasi-Orlicz modulus of smoothness,

Theorem 1.4.3 Let f be any function from (1.10), then

1. $\omega_N(f, \delta)_{\Phi,S_p}$ is a positive nondecreasing continuous function of δ on

$\quad (0, \infty)$

2. $\lim\limits_{\delta \to 0} \omega_N(f, \delta)_{\Phi,S_p} = 0$

3. $\omega_N(f + g, \delta)_{\Phi,S_p} \leq c\left(\omega_N(f, \delta)_{\Phi,S_p} + \omega_N(g, \delta)_{\Phi,S_p}\right)$

4. $\omega_N(f, \delta)_{\Phi,S_p} \leq 2^{N-M}\omega_M(f, \delta)_{\Phi,S_p}, \forall\ M \leq N$

$5. \omega_N(f, \delta)_{\Phi, S_p} \leq 2^N \|f\|_{\Phi, S_p}$

$6. \omega_N(f, \delta)_{\Phi, S_p} \leq \omega_N(f, \acute{\delta})_{\Phi, S_p}$, for $\delta \leq \acute{\delta}$

$7. \omega_N(f, \gamma\delta)_{\Phi, S_p} \leq (1 + \gamma)^k \omega_N(f, \delta)_{\Phi, S_p}$

Proof

1. and 2. are clear from Theorem 1.4.2

$3. \omega_N(f + g, \delta)_{\Phi, S_p} = \sup_{|h| \leq \delta} \left\| \Delta_h^N(f + g) \right\|_{\Phi, S_p}$

$= \sup_{|h| \leq \delta} \left\| \Delta_h^N(f(x)) + \Delta_h^N(g(x)) \right\|_{\Phi, S_p}$

$\leq c \left(\sup_{|h| \leq \delta} \left\| \Delta_h^N(f(x)) \right\|_{\Phi, S_p} + \sup_{|h| \leq \delta} \left\| \Delta_h^N(g(x)) \right\|_{\Phi, S_p} \right)$

$\leq c \left(\omega_N(f, \delta)_{\Phi, S_p} + \omega_N(g, \delta)_{\Phi, S_p} \right)$

By throrem1.4.2, we have

$4. \omega_N(f, \delta)_{\Phi, S_p} = \sup_{|h| \leq \delta} \left\| \Delta_h^N(f(x)) \right\|_{\Phi, S_p}$

$= \sup_{|h| \leq \delta} \left\| \Delta_h^{N-M} \left(\Delta_h^M(f(x)) \right) \right\|_{\Phi, S_p}$

$\leq k(N - M) \sup_{|h| \leq \delta} \left\| \Delta_h^M(f(x)) \right\|_{\Phi, S_p}$

$\leq 2^{N-M} \omega_M(f, \delta)_{\Phi, S_p}$

$5. \omega_N(f, \delta) = \sup_{|h| \leq \delta} \left\| \Delta_h^N(f) \right\|_{\Phi, S_p}$

$$\leq 2^N \sup_{|h|\leq\delta} \|f\|_{\Phi,S_p}$$

$$= 2^N \; \|f\|_{\Phi,S_p}$$

6. Let $\delta < \acute{\delta}$, then

$$\omega_N(f,\delta)_{\Phi,S_p} = \sup_{|h|\leq\delta} \left\| \Delta_h^N (f(x)) \right\|_{\Phi,S_p}$$

$$\leq \sup_{|h|\leq\acute{\delta}} \left\| \Delta_h^N (f(x)) \right\|_{\Phi,S_p}$$

$$= \omega_N(f,\acute{\delta})_{\Phi,S_p}$$

7. Let $\delta > 0$, and since $0 < \lambda < 1,$ then $\gamma\delta \leq \delta$

$$\omega_N(f,\gamma\delta)_{\Phi,S_p} \leq \omega_N(f,\delta)_{\Phi,S_p} \leq (1+\gamma)^k \omega_N(f,\delta)_{\Phi,S_p} \qquad \square$$

1.5. Quasi-Orlicz on Spherical Spaces

This section deals with approximation on the sphere $\mathbb{S}^d \subseteq \mathbb{R}^d$.[41] provides a cohesive account of recent developments in approximation theory and harmonic analysis on these domains. Analysis on the unit sphere appears as part of Fourier analysis [42], in the study of homogeneous spaces, and in several fields in applied mathematics, from numerical analysis to geoscience, and it has seen increased activity in recent years.

They studied the characterization of the best approximation by polynomials on the sphere. Best approximation of L_p , $p \geq 1$, functions

out of spherical polynomials was studied by Dai and Xu in their paper and book [41, 42]. Their big challenge was to define modulus of smoothness on $\mathbb{S}^{d-1}$, when $d \geq 3$. The difficulty was that multiplication on three or more-dimensional sphere is not commutative. They defined modulus of smoothness on $L_p(\mathbb{S}^{d-1}), 1 \leq p \leq \infty$, depending on that of $L_p(\mathbb{S}^1)$ and used them for trigonometric approximation in their paper [5].

In this section, and later in chapter 4, we define a modulus of smoothness on the quasi – Orlicz space $L_{\Phi,S_p}(\mathbb{S}^{d-1}) = \{f : \mathbb{S}^{d-1} \to \mathbb{R}^{d-1} : \|f\|_{\Phi,S_p} < \infty\}$, beginning with the $k - th$ symmetric difference from [29], For $r = 1,2,\ldots$, we use $Q_{i,j,\theta}$ to determine the symmetric difference operator

$$\Delta_{i,j,\theta}^r = \left(I - T(Q_{i,j,\theta})\right)^r, \quad 1 \leq i = j \leq d,$$

where $Q_{i,j,\theta}$ is a rotation by the angle θ in the plane (x_i, x_j), orientated in such a way that the rotation from the vector e_i at vector e_j is supposed to be positive.

Now we define norm and modulus of smoothness on space $L_{\Phi,S_p}(\mathbb{S}^d)$

Definition 1.5.1 For $r \in N, t > 0,$ and $f \in L_{\Phi,S_p}(\mathbb{S}^d), 0 < p < 1,$

$$\omega_{r,\mathbb{S}}(f,\delta)_{\Phi,S_p} := \max_{1 \leq i < j \leq d} \sup_{|\theta| \leq t} \left\|\Delta_{i,j,\theta}^r(f)\right\|_{\Phi,S_p}$$

In the following theorem, we present the main properties of the above modulus of smoothness. We omit the proof because it is similar to proof of Theorem 1.4.3.

Theorem 1.5.2. Let $f \in L_{\Phi,S_p}(\mathbb{S}^d)$, where Φ is young function, S_p is outer function , then

1. $\omega_{r,\mathbb{S}}(f,\delta)_{\Phi,S_p}$ is a positive nondecreasing continuous function of δ on

    $(0,\infty)$

2. $\lim\limits_{\delta\to 0}\omega_{r,\mathbb{S}}(f,\delta)_{\Phi,S_p} = 0$

3. $\omega_{r,\mathbb{S}}(f+g,\delta)_{\Phi,S_p} \le c\left(\omega_{r,\mathbb{S}}(f,\delta)_{\Phi,S_p} + \omega_{r,\mathbb{S}}(g,\delta)_{\Phi,S_p}\right)$

4. $\omega_{r,\mathbb{S}}(f,\delta)_{\Phi,S_p} \le 2^{r-v}\omega_{v,\mathbb{S}}(f,\delta)_{\Phi,S_p}$

5. $\omega_{r,\mathbb{S}}(f,\delta)_{\Phi,S_p} \le 2^r \|f\|_{\Phi,S_p}$

6. $\omega_{r,\mathbb{S}}(f;\delta)_{\Phi,S_p} \le \omega_{r,\mathbb{S}}(f,\acute{\delta})_{\Phi,S_p}$, for $\delta \le \acute{\delta}$

7. $\omega_{r,\mathbb{S}}(f,\gamma) \le (1+\gamma)^k\omega_{r,\mathbb{S}}(f,\delta)$

[41] studied the main properties of difference operator that are useful to our approximation in the following auxiliary lemma,

Lemma 1.5.3. [41] Let $r \in N$, then $\Delta^r f(x)$ satisfies

$$\Delta^r\big(f(x)g(x)\big) = \sum_{k=0}^{r}\binom{r}{k}\Delta^k f(x)\,\Delta^{r-k}g(x+k)$$

Chapter Two:
Approximation in Convolution Neural Networks

2.1 Convolutional neural networks (CNNs)

Deep learning is essential because of its ability to cope with large data in many quantities [38]. One of the most common in deep learning is neural networks. A neural network is a series of algorithms attempting to recognize the underlying relationships within a dataset through a process miming how the human brain functions. Neural networks can adapt to changes in input; thus, the network generates the best possible result without redesigning the exit criteria. The general mathematical form of a neural network is given by

$$f_N(x) = \sum_{k=1}^{N} c_k \, \sigma \, (w_k . x - b_k), \quad x \in \mathbb{R}^d \qquad (2.1)$$

where N corresponds to the number of neurons known as width, $w_k \in \mathbb{R}^d$, $c_k \in \mathbb{R}$ and $b_k \in \mathbb{R}$ where $k \in \{1, \dots N\}$. $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is the activation function that gives complexity to the neural network to deal with complex data. Moreover, a neural network without an activation function is just a simple linear function. However, nonlinearity and complexity should be manageable so that neural networks become applicable. For those reasons, choosing a suitable activation function for both mathematical and applicable purposes is essential. There are many activation functions but the most common types are sigmoidal and ReLU

functions. More details come forward in the following section when discussing convolution neural networks.

CNNs are used extensively for computer vision applications such as image recognition, segmentation and facial detection [43].

Since the 1950s, researchers have begun to work on a discovery through which visual data can be understood and developed called computer vision.

A CNN is a developed kind of artificial neural network that analyzes visual imagery. CNNs were developed for the first time in the 1980s. The maximum a CNN could do at that point was to recognize handwritten figures. In 2012, there was a qualitative leap in the development of synaptic neural networks where it was used Alex Krizhevsky neural networks where it provides a wide range of data, see [4, 5, 44-49]. Synthetic neural networks have become deep neural networks.

CNNs have three main layers: convolutional layers, pooling layers, and fully connected layers.

The first layer uses a mathematical operation known as convolution instead of general matrix multiplication in some of its layers. The most prominent characteristic of CNN is the convolutions that imposes the

network. The discrete convolution between two functions f and g is defined in this way

$$(f \circ g)(x) = \sum_t f(t)\, g(x + t). \qquad (2.2)$$

The following description shows the convolution procedure among matrices that construct CNNs. Simply, the convolution of the two matrices w and v is given by the sequence

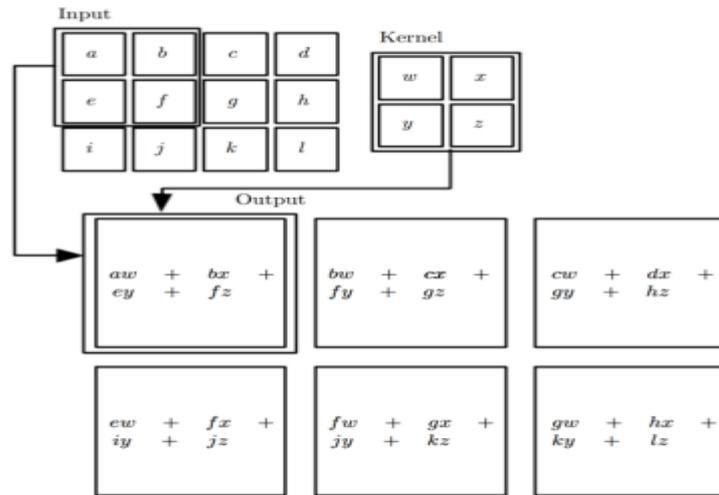$$(w * v)_i = \sum_{k=0}^{D-1} w_{i-k} v_k \qquad (2.3)$$

Figure 2. 1 Discrete Convolution between Two Matrices

The second layer is a pooling layer used in CNNs to decrease the spatial dimensions of the output quantity. There are two kinds of pooling layers, max pooling and average pooling.

The most common type of pooling is max pooling, which is a means of extracting the most significant characteristics of the output quantity; it works by dividing the input into a set of nonoverlapping rectangles and then taking the maximum value from each rectangle

$$s_j = \lambda \max_{i \in R_j} a_i + (1 - \lambda) \frac{1}{|R_j|} \sum_{i \in R_j} a_i \qquad (2.4)$$
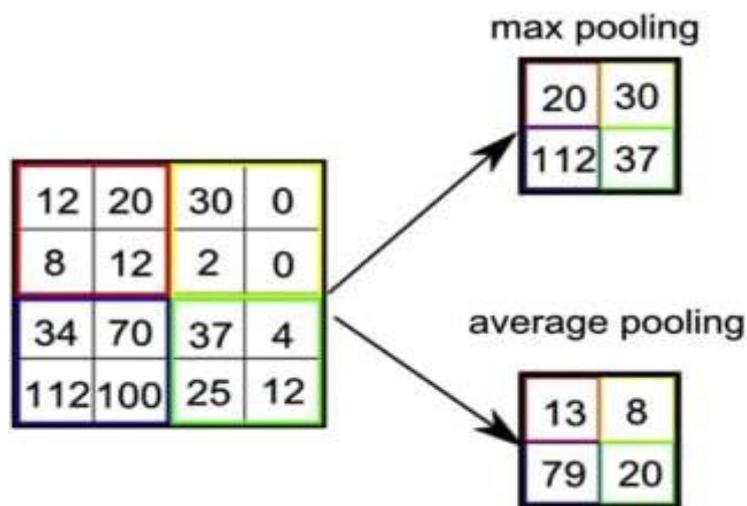

Figure 2.2 Pooling layer

Finally, a fully connected layer is a layer in a CNN where each input is connected to each output via a weight matrix. It is usually used in the final phase of a CNN, where outputs from the previous layer are introduced into the fully connected layer.
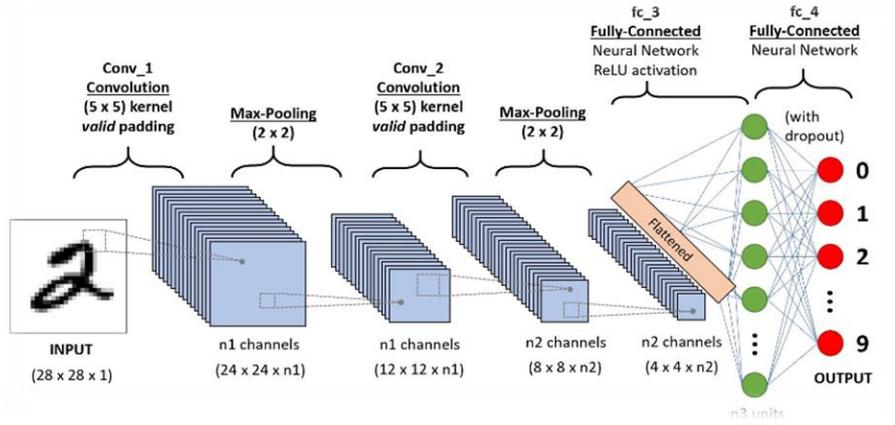
Figure 2. 3 Convolutional Neural Network (CNN)

A CNN has a different architecture than an ordinary neuronal network. The form of convolutional neural networks for estimating of function or treatment data on $\mathbb{R}^d$ is given by

$$h^{(j)}(x) = \sigma\big(\mathfrak{I}^{(j)} h^{(j-1)}(x) - b^j\big), \qquad (2.5)$$

where σ is the ReLU activation function, $\mathfrak{I}$ is the convolutional matrix, b is bias and $h^{(0)}(x) = x$. All of these features are examined later throughout this chapter

2.2. Activation function in CNNs

The activation function is a component of the convolutional layer that adds nonlinearity to the convolutional neural network. Each type of activation function has advantages and disadvantages, so the choice of the suitable activation function depends on its mathematical proper and functioning when implemented. In this work, we use Rectified activation Function **(ReLU)** for many reasons. The sigmoid and hyperbolic tangent

activation functions cannot be used in multi-layer arrays because of the disappearing gradient issue.

ReLU is the most used activation function in neural networks, especially in CNNs and multilayered perceptrons. ReLU is a linear and non-linear activation function that shows the entry directly if it is positive or it will show zero. The mathematics formula for ReLU is given by

$$\sigma(x) = max(0, x) \qquad\qquad (2.6)$$

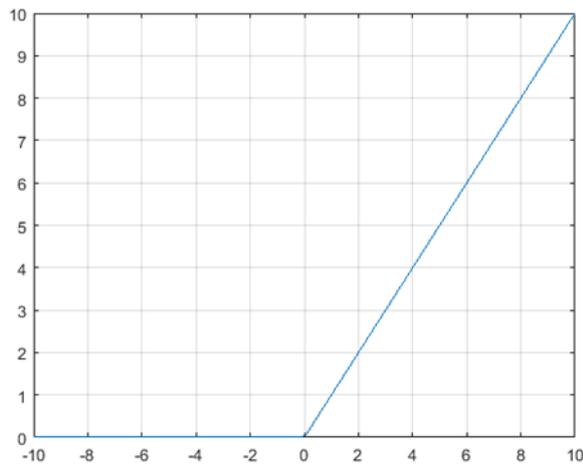The curve of σ is shown in Figure 2.4 below,

Figure 2. 4 ReLU activation function

2.3. Approximation of CNNs

The first who offer a theoretical approach about approximation through neural networks is Cybenko through his global approximation theorem [50]. Cybenko's theorem provides that any function from $C\ [a, b]$ can be approximated by a neural network generated by f, written $N_n f$ with

$$\| f - N_n f \| < \epsilon \qquad\qquad (2.7)$$

Afterward, Cybenko's theorem followed by many most profound theories having the same target with different points of view. Although the fully connected layer is known to have the universal approximation property, it is not known if CNNs inherit this property, especially when the kernel size in the convolution layer is small. The first who demonstrated their universality of approximation is Fang [5], they consider an applied family of deep convolutional neural networks functions of the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$.

2.4. Construction of CNNs

In [5], they use weights like rows to form $d_j \times d_{j-1}$ of matrix $\mathfrak{I}^{(j)}$ and vector $b^{(j)} = b_i^{(j)}, i = 1, \dots, d_j$ and used activation function σ, then DCNN is expressed with depth J as follow $h^{(j)} \colon \mathbb{R}^d \to \mathbb{R}^{d_j}$

$$h^{(j)}(x) = \sigma\big(\mathfrak{I}^{(j)} H^{(j-1)}(x) - b^{(j)}\big), \quad j = 1, \dots, J \qquad (2.8)$$

where iterations begin with $h^{(0)}(x) = x$

In DCNNs, it is important to induce sparsity by filter masks. Define $w = \{w^{(j)}\}_j$ to be a sequence of filter coefficients, called filter masks. Fix s to be the filter length with $w_k^{(j)} \neq 0$, so $0 \leq k \leq s$.

After $(D + s) \times D$ convolutions, we get the matrix $\mathfrak{I}$ that is called Toeplitz matrix:

$$\mathfrak{I} = \begin{bmatrix} w_0 & 0 & 0 & 0 & \cdots & 0 \\ w_1 & w_0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ w_s & w_{s-1} & \cdots & w_0 & 0 & 0 \\ 0 & w_s & \cdots & w_1 & w_0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & w_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & w_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & & 0 & w_s & \\ & & & & 0 & w_{s-1} \\ 0 & \cdots & \cdots & \cdots & & w_s \end{bmatrix}$$

The matrix $\mathfrak{I}$ is the main difference between CNNs and full connected

neural networks. Moreover, matrices $\mathfrak{I}$ has rows more than columns

which leads deep CNNs to use better functions than others.

For the deep CNN of depth J, define the following space as a collection of

CNNs which is given by

$$\mathcal{N}_j = \left\{ \sum_{k=1}^{d_J} c_k h_k^{(J)} : c_k = \in R^{d_J} \right\}, \quad j = 1,2,\dots,J \qquad (2.9)$$

In their papers [5, 51, 52], Zhou and his team proposed the

convolutional matrix as follow $\mathfrak{I}^{(J)} = \mathfrak{I}^{w^{(j)}}$, with $D = d_{j-1}$ and $s =$

$s^{(j)}$.

DCNN is given by convolutions of the activation functions $\sigma_{\mathfrak{I},b} \colon \mathbb{R}^{d_{j-1}} \to$

$\mathbb{R}^{d_{j-1}+s^{(j)}}$ of matrix $\mathfrak{I}$ and bias $b \in \mathbb{R}^{d_{j-1}+s^{(j)}}$.

To get DCNN $h^j \colon \mathbb{R}^d \to \mathbb{R}^{d_j}$, $j = 1,\dots,J$. $d_j = d + js, s^{(j)} \equiv s$, the

form

$$h^{(j)} = \sigma_{\mathfrak{J}^{(j)}, b^{(j)}} \circ \cdots \circ \sigma_{\mathfrak{J}^{(1)}, b^{(1)}}(x), \qquad\qquad (2.10)$$

where $\sigma_{\mathfrak{J},b}(u) = \sigma(\mathfrak{J}u - b), u \in \mathbb{R}^{d_{j-1}}, b \in \mathbb{R}^{d_{j-1}+s^{(j)}}$

After defining the suitable environment to approximate functions by CNNs, it is time to construct them using the major features, which are, the filter masks w and the bias b. To this purpose, w exists by the following lemma

Lemma2.4.1 Let $s \geq 2$, $M \geq 0$, any sequence $w = (w_k)_{-\infty}^{\infty}$ in $\{0, \dots, M\}, \exists$ a finite sequence of the filter mask $\{w^{(j)}\}_{j=1}^{J}$ with nonzero elements in $\{0, \dots, s\}$, where $J < \dfrac{M}{s-1} + 1$, that satisfies

$$\mathcal{W} = w^{(j)} * \dots * w^2 * w^1$$

2.5. CNN Direct Approximation Theorem

This is the time to approximate any function from $L_{\Phi, S_p}(I^d)$ with a CNN from $\mathcal{N}$ in the following theorem

Theorem 2.5.1. Let $f \in L_{\Phi, Sp}(I^d), 0 < p < 1$, then there exists $f_j^{w,b} \epsilon \mathcal{N}$ s.t

$$\left\| f - f_J^{w,b} \right\|_{\Phi, S_p} < \epsilon$$

Proof

We begin with some conclusions from[51]

34

For $2 \le s \le d, J \ge \frac{2d}{s-1},\ r \ge 2 + \frac{d}{2}$,

Let $a = \left\lceil \frac{(s-1)J}{d} - 1 \right\rceil \ge 1$ then $J \ge \frac{(a+1)}{s-1}$

To build the filter mask sequence w. set $\mathcal{W}$ in $\{0, \dots, (a+1)d - 1\}$ as follow

$$\left[\mathcal{W}_{(m=1)d-1} \ \dots \ \mathcal{W}_1 \mathcal{W}_0 \right] = \left[a_m^{\Im} \ \dots \ a_1^{\Im} a_0^{\Im} \right]$$

Then by **Lemma 2.4.1,** there exists a sequence of a filter masks $\mathcal{W} =$

$\left\{ w^{(j)} \right\}_{j=1}^{\hat{\jmath}}$ in $\{0, 1, \dots, s\}$ with $\hat{\jmath} < \frac{(a+1)d}{s-1} + 1$ and

$$\mathcal{W} = w^{(\hat{\jmath})} * w^{(\hat{\jmath}-1)} * \dots * w^{(2)} * w^{(1)},$$

where $\hat{\jmath} \le J$ because $\frac{(a+1)d}{s-1} \le J$

Take $w^{(\hat{\jmath}-1)}$ to get $\mathcal{W} = w^{(\hat{\jmath})} * w^{(\hat{\jmath}-1)} * \dots * w^{(2)} * w^{(1)}$ so from lemma

$$\Im^{(J)} \dots \Im^{(1)} = \left[\mathcal{W}_{\ell-k} \right]_{\ell=1,\dots,d_J, k=1,\dots,d},$$

where $\Im^{\mathcal{W}}$ is The $d_J \times d$ matrix.

To build b,we find $B^{(j)}$ as follow

Let $B^{(j)} = \left\| w^{(j)} \right\|_{\Phi, S_p} \dots \left\| w^{(1)} \right\|_{\Phi, S_p} \|x\|_{\Phi, S_p}$ for $j \ge 1$ then

$$\left\| \left(\Im^{(j)} \dots \Im^{(1)} x \right)_k \right\|_{\Phi, S_p} \le B^{(j)}, \qquad \forall k = 1, \dots, d$$

Take $b^{(1)} = -B^{(1)} \mathbf{1}_{d_1}$

$$b^{(j)} = B^{(j-1)}\mathfrak{I}^{(j)}\mathbf{1}_{d_{j-1}} - B^{(j)}\mathbf{1}_{d_j}, \qquad \forall j = 1, \dots, J-1$$

Hanse we have $h^{(j)}{}_{(x)} = \mathfrak{I}^{(j)} \dots \mathfrak{I}^{(1)}x + B^{(j)}\mathbf{1}_{d_j}$,

Now, let $f \in \left(\Phi, S_p[0,1]^d, \|.\|_{\Phi,S_p}\right)$, then

$$\|f - f_J{}^{w,b}\|_{\Phi,S_p} = \left\|f(x) - \sum_{k=1}^{d_J} c_k f_k{}^{(J)}(x)\right\|_{\Phi,S_p}$$

$$= \left\|\sum_{k=1}^{d_J} c_k(f - f_k)^{(J)}(x)\right\|_{\Phi,S_p}$$

$$\leq c \sum_{k=1}^{d_J} c_k \|f - f_k{}^{(J)}\|_{\Phi,S_p}$$

$$\leq c \sum_{k=1}^{d_J} c_k (1 + I_\Phi(f - f_k)^2)^{\frac{1}{p}}$$

$$\leq \frac{B^{(j)}}{J} \|f - f_+\|_{\Phi,S_p}$$

$$\leq \frac{C}{J} \|f\|_{\Phi,S_p},$$

where

$$f_k{}^{(J)} = \sigma\left(\mathfrak{I}^{(j)}\sigma\left(\mathfrak{I}^{(j-1)}\sigma\left(\mathfrak{I}^{(j)}\dots\dots\sigma\left(\mathfrak{I}^{(j-2)}f(x) - \sum_{m=0}^{j} b^{(m)}\right)\right)\right)\right)$$

Neural network performance can be improved using different functions or constants that we review in Chapter three

Chapter Three:
Downsampling Convolutional Neural Networks

3.1. Downsampling in CNNs

One of CNN's problems is big widths we came to downsampling. Zhou[4] provided the definition of downsampling operator for the Convolutional Neural Networks (DNN), he introduced a downsampled operation into DCNNs to avoid big widths in (2.12) that happens with pooling layers. The ℓ downsampled are introduced at layers $\mathcal{J} = \{J_k\}_{k=1}^{\ell}$ with $1 < J_1 \leq \cdots \leq J_\ell = J$. His concept of downsampling operators is induced from wavelets [53, 54]. The downsampling operation is defined below,

Definition3.1.1. Let $m \leq D$ be a scaling parameter, the function $\mathfrak{D}_m : \mathbb{R}^D \longrightarrow \mathbb{R}^{[D/m]}$ is called Downsampling operator, and it is given by

$$\mathfrak{D}_m(u) = (u_{im})_{i=1}^{\left[\frac{D}{m}\right]} , u \in \mathbb{R}^D, \tag{3.1}$$

where $[u]$ is the integer part of $u \in \mathbb{R}^+$.

Definition 3.1.2. Downsampling DCNN is defined iteratively for $k = 1, \dots, \ell$, beginning with $d_0 = d$ and filter lengths $[s^{(j)}]_{j=1}^{J}$ has widths $[d_j]_{j=0}^{J}$, where ℓ downsamplings at layers $\mathcal{J}$,

$$d_j = \begin{cases} d_{j-1} + s^{(j)} , & \text{if } J_{k-1} < j < J_k \\ [(d_{j-1} + s^{(j)})/d_{J_{k-1}}], & \text{if } j = J_k \end{cases} \tag{3.2}$$

As before, define the function vector sequence iteratively by $\{h^{(j)}(x) : \mathbb{R}^d \longrightarrow \mathbb{R}^{dj}\}_{j=1}^{J}$ by $h^{(0)}(x) = x$ and for $k = 1, \dots, \ell$

$$h^{(j)} = \begin{cases} \sigma_{\mathfrak{J}^{(j)}, b^{(j)}}\left(h^{(j-1)}(x)\right), & \text{if } J_{k-1} < j < J_k \\ \mathfrak{D}_{d_{J_{k-1}}} \circ \sigma_{\mathfrak{J}^{(j)}, b^{(j)}}\left(h^{(j-1)}(x)\right), & \text{if } j = J_k \end{cases} \tag{3.3}$$

we restrict the $\mathfrak{J}\ b^{(j)} \in \mathbb{R}^{d_{j-1}+s^{(j)}}$ to satisfy

$$b_{s^{(j)}+1}^{(j)} = b_{s^{(j)}+2}^{(j)} = \ldots = b_{d_j-1}^{(j)} \tag{3.4}$$

Definition 3.1.3: Uniform Downsampled DCNN is defined with uniform filter lengths for $k \in \{1, \ldots, \ell\}$, as follow

$$S = \{s^{[k]} \in N\}_{k=1}^{\ell} \text{ if } s^{(J_{k-1}+1)} = \ldots = s^{(J_k)} = s^{[k]}. \tag{3.5}$$

3.2. Construction of CNNs

For any input $x = (x_i)_{i=1}^{d}$, define the oprater $\mathcal{L}_x : L_{\Phi, S_p}(I^d) \to R^+$ as follow

$$\mathcal{L}_t\big(g(u)\big) = \sum_{i=1}^{d} g(x_i)\delta_i(u), \tag{2.11}$$

where $g \in L_{\Phi, S_p}, u \in I$ and δ_i is given by

$$\delta_{i,g}(u) = \sum_{i=2}^{d} (-1)^{N-i} \binom{N}{i} g(t_i)\sigma(t_i - u), \tag{2.12}$$

where $\sigma : I \to I$ is the ReLU activation function defined by (2.6).

The following lemma from [51] arises the relationship between the convolutions of filter masks $w^{(J)} * \ldots * w^{(1)}$, that implies that sequence $W^{(k)}$ and the Toeplitz matrices product $\mathfrak{J}^{(J_k)} \cdots \mathfrak{J}^{(J_{k-1}+1)}$, where $\mathfrak{J}^{(J_k)}$ is a $\left(d_{j-1} + S^{(j)}\right) \times d_{j-1}$ matrix.

Lemma3.2.1. For $k = 1, \ldots, \ell$, we have

$$\mathfrak{J}^{(J_k, J_{k-1}+1)} = \mathfrak{J}^{(J_k)} \ldots \mathfrak{J}^{(J_{k-1}+2)} \mathfrak{J}^{(J_{k-1}+1)}.$$

Then we immediately see how convolutional matrices are defined $\mathfrak{J}^{(j)} = \mathfrak{J}^{w^{(j)}}$ that for any $h : \Omega \to R^{d_{j-1}}$, we denote

$$\left\| \mathfrak{J}^{(j)} h \right\|_{\Phi, S_p} \leq \left\| w \right\|_{\Phi, S_p} \left\| h \right\|_{\Phi, S_p} \tag{3.6}$$

From our previous study, we use some thoughts from DCNNs without downsampling and select biases small enough for the vectors $\mathfrak{J}^{(j)} h^{(j-1)}(x) - b^{(j)}$ have non-negative entries.

Lemme 3.2.2. For any $k \in \{1, \ldots, \ell\}$, there exists $B \in R^+$, $\hat{B} \in [-B, B]$, then

$$\left\| h^{(J_{k-1})} - \hat{B} \mathbf{1}_{d_{j_{k-1}}} \right\|_{\Phi, S_p} \leq B \tag{3.7}$$

Proof

Set $b^{(J_{k-1}+1)} = \hat{B} \mathfrak{J}^{(J_{k-1}+1)} \mathbf{1}_{d_{J_{k-1}}} - B \left\| w^{((J_{k-1}+1)} \right\|_{\Phi, S_p} \mathbf{1}_{d_{J_{k-1}}+1}$

To prove $b^{(j)}_{s^{(j)}+1} = b^{(j)}_{s^{(j)}+2} = \ldots = b^{(j)}_{d_{j-1}}$ from

40

$$b^{(j)} = B\left(\Pi_{p=J_{k-1}+1}^{j-1}\|w^{(p)}\|_{\Phi,S_p}\right)\mathfrak{I}^{(j)}\mathbf{1}_{d_{j-1}} -$$

<div align="right">(3.8)</div>

$$B\left(\Pi_{p=J_{k-1}+1}^{j-1}\|w^{(p)}\|_{\Phi,S_p}\right)\mathbf{1}_{d_{j-1+s^{(j)}}}$$

For $j = J_{k-1} + 2, \ldots, J_{k-1}$, then for $J_{k-1} < j < J_k$ and $i = s^{(j)} + 1, \ldots, d_{j-1}$,

we have

$$\left(\mathfrak{I}^{(j)}\mathbf{1}_{d_{j-1}}\right)_i = \Sigma_{p=1}^{d_{j-1}}\left(\mathfrak{I}^{(j)}\right)_{i,p} = \Sigma_{p=1}^{d_{j-1}} w^{(j)}{}_{i-p}.$$

Notice that $w^{(j)}$ in $\{0, \ldots, s^{(j)}\}$. $p \in (-\infty, 0] \cup [d_{j-1} + 1, \infty)$ we have $i -$

$p \in [s^{(j)} + 1, \infty)$ which implies that $w^{(j)}{}_{i-p} = 0$. So,

$$\left(\mathfrak{I}^{(j)}\mathbf{1}_{d_{j-1}}\right)_i = \Sigma_{p=-\infty}^{\infty} w^{(j)}{}_{i-p} = \Sigma_{p=-\infty}^{\infty} w_p{}^{(j)}$$

for all $i = s^{(j)} + 1, \ldots, d_{j-1}$

Now we prove

$$h^{(j)}(x) = \mathfrak{I}^{(j)} \ldots \mathfrak{I}^{(J_{k-1}+1)}\left(h^{J_{k-1}}(x) - \hat{B}\mathbf{1}_{d_{J_{k-1}}}\right) + \quad (3.9)$$

$$B\left(\Pi_{p=J_{k-1}+1}^{j}\|w^{(p)}\|_{\Phi,S_p}\right)\mathbf{1}_{d_j}$$

By induction. for $j = J_{k-1} + 1$, we have

$$\mathfrak{I}^{(J_{k-1}+1)}h^{(J_{k-1})}(x) - b^{(J_{k-1}+1)}$$

$$= \mathfrak{I}^{(J_{k-1}+1)}\left(h^{J_{k-1}}(x) - \hat{B}\mathbf{1}_{d_{J_{k-1}}}\right) + B\|w^{(J_{k-1}+1)}\|_{\Phi,S_p}\mathbf{1}_{d_{J_{k-1}}+1}$$

the use σ is identical to the identity function on $[0, \infty)$, hence

$$h^{(J_{k-1}+1)}(x)$$

$$= \mathfrak{I}^{(J_{k-1}+1)}\left(h^{J_{k-1}}(x) - \hat{B}\mathbf{1}_{d_{J_{k-1}}}\right) + B\left\|w^{(J_{k-1}+1)}\right\|_{\Phi,S_p}\mathbf{1}_{d_{J_{k-1}}+1}$$

for $j = 1$ is verifies.

Suppose that it is holds for $J_{k-1}+1 \leq j \leq J_{k-1}$, then $d_{j-1} + s^{(j)} = d_j$

Through the induction hypothesis and selection of

$$b^{(j)} = B\left(\Pi_{p=J_{k-1}+1}^{j-1}\left\|w^{(p)}\right\|_{\Phi,S_p}\right)\mathfrak{I}^{(j)}\mathbf{1}_{d_{j-1}} -$$

$$B\left(\Pi_{p=J_{k-1}+1}^{j-1}\left\|w^{(p)}\right\|_{\Phi,S_p}\right)\mathbf{1}_{d_{j-1+s^{(j)}}}$$

of the bias vector, we have

$$h^{(j)}(x) = \sigma\left(\mathfrak{I}^{(j)}\mathfrak{I}^{(j-1)}\ldots\mathfrak{I}^{(J_{k+1})}\left(h^{(J_{k-1})}(x) - \hat{B}\mathbf{1}_{d_{J_{k-1}}}\right)\right.$$

$$\left. + B\left(\prod_{p=J_{k-1}+1}^{j}\left\|w^{(p)}\right\|_{\Phi,S_p}\right)\mathbf{1}_{d_j}\right)$$

Now to prove

$$h^{(J_k)}(x) = M^{(k)}\mathfrak{I}^{(J_k,J_{k-1}+1\,)}\left(h^{(J_{k-1})}(x) - \hat{B}\mathbf{1}_{d_{J_{k-1}}}\right)$$

$$+ B\left(\prod_{p=J_{k-1}+1}^{J_k}\left\|w^{(p)}\right\|_{\Phi,S_p}\right)\mathbf{1}_{d_{J_k}}$$

when $b^{(j)} = B\left(\Pi_{p=J_{k-1}+1}^{j-1}\left\|w^{(p)}\right\|_{\Phi,S_p}\right)\mathfrak{I}^{(j)}\mathbf{1}_{d_{j-1}} - B$

then

$$\mathfrak{J}^{(J_k)} h^{(J_{k-1})}(x) - b^{(J_k)}$$

$$= \mathfrak{J}^{(J_k)} \dots \mathfrak{J}^{(J_{k-1})} \left(h^{(J_{k-1})}(x) - \hat{B} \mathbf{1}_{d_{J_{k-1}}} \right)$$

$$+ B \left(\prod_{p=J_{k-1}+1}^{J_k} \left\| w^{(p)} \right\|_{\Phi, S_p} \right) \mathbf{1}_{d_{J_k}}$$

3.3. Approximation Abilities of CNNs in Quasi - Orlicz Spaces

The main purpose of this chapter is to study the degree of approximation of functions from Orlicz space by convolutional neural networks in terms of Quasi-Orlicz modulus of smoothness. The following direct theorem estimates the upper bound of the degree of the best approximation theorem

Theorem 3.3.1 (Direct theorem)

For any $g \in L_{\Phi, S_p}(I^d)$, there exists a CNN

$$L_t\big((g)(u)\big) = \sum_{i=2}^{2N+2} g(t_i)\,\delta_i(u), u \in [t_{i-1}, t_i]$$

where $|t_i - t_{i-1}| \sim \frac{1}{n}$, such that

$$\|L_t(g) - g\|_{\Phi, S_p}^p \leq \frac{c}{n} \omega_N \left(g, \frac{1}{n} \right)_{\Phi, S_p},$$

where

$$\delta_i(u) = \sum_{i=1}^{N} \binom{N}{i} (-1)^{N-i} \sigma(t_i - u)$$

Proof

$$\|L_t(g) - g\|_{\Phi,S_p}^p$$

$$\leq \left\| \sum_{i=2}^{2N+2} \left[(-1)^{N-i} \binom{N}{i} g(t_i)\sigma(t_i - u) - (-1)^{N-i} \binom{N}{i} g(u - ih) \right. \right.$$

$$\left. \left. + (-1)^{N-i} \binom{N}{i} g\left(u - \frac{i}{n}\right) - g(u) \right] \right\|_{\Phi,S_p}^p$$

$$\leq c \left(\left\| \sum_{i=2}^{2N+2} (-1)^{N-i} \binom{N}{i} g(t_i)\sigma(t_i - u) - (-1)^{N-i} \binom{N}{i} g\left(u - \frac{i}{n}\right) \right\|_{\Phi,S_p}^p \right.$$

$$\left. + \left\| (-1)^{N-i} \binom{N}{i} g(u - ih) - g(u) \right\|_{\Phi,S_p}^p \right)$$

$$\leq c \sum_{i=2}^{2N+2} \left\| (-1)^{N-i} \binom{N}{i} \left(hg(t_i) - g\left(u - \frac{i}{n}\right) \right) \right\|_{\Phi,S_p}^p + \omega_N(g,n)$$

$$\leq c \left(h + \omega_N \left(g, \frac{1}{n} \right)_{\Phi,S_p} \right)$$

$$\leq \frac{c}{n} \omega_N \left(g, \frac{1}{n} \right)_{\Phi,S_p} \qquad \square$$

To study the lower bound of a degree of best approximation, we prove

the following inverse theorem

Theorem3.3.2 (Inverse Theorem)

For any $g \in L_{\Phi,S_p}(I^d)$, then $L_t(g)$ satisfies

$$\omega_N(g,\delta)_{\Phi,S_p} \leq \frac{c}{n}\left[\|g\|_{\Phi,S_p} + E_m(g)_{\Phi,S_p}\right]$$

Proof

Let $b = \max_{1 \leq i \leq 2N+3}\{i; 2^i < n\}$,

$$g(t_{2N+3}) - g(t_1) = \left(g(t_{2N+3}) - g(t_{2^b})\right) + \left(g(t_{2^b}) - g(t_{2^{b-1}})\right) + \cdots +$$

$$\left(g(t_2) - g(t_1)\right),$$

for any $m < 2N+3$, suppose that

$$\|g(t_{2N+3}) - g(t_m)\|_{\Phi,S_p} \leq cE_m(g)_{\Phi,S_p},$$

where $E_m(g)_{\Phi,S_p} = \inf\|g - L_t(g)\|_{\Phi,S_p}$, then by Theorem (3.3.1),we

get

$$\omega_N(g,\delta)_{\Phi,S_p} \leq c(p)\left[\omega_N(g - L_t(g),\delta)_{\Phi,S_p} + \omega_N(L_t(g),\delta)_{\Phi,S_p}\right]$$

$$\leq c(p)\left[c(N)\|g - L_t(g)\|_{\Phi,S_p} + \omega_N\left(L_t(g,\delta)_{\Phi,S_p}\right)\right]$$

$$\leq c(p)\left[c(N)\omega_N(g,\delta)_{\Phi,S_p} + c(N)E_m(g)_{\Phi,S_p}\right]$$

$$\leq c(p,N)\left[\|g\|_{\Phi,S_p} + E_m(g)_{\Phi,S_p}\right] \qquad \square$$

As a conclusion, we introduce the following Comparison Theorem to prove that downsampling affects the CNN to get smaller widths and a better degree of approximation.

Theorem 3.3.3: (Comparison Theorem)

For any $k = 1, .., \ell$ define ℓ-layer fully connected neural network $H^{(k)} \colon \mathbb{R}^d \to \mathbb{R}^{n_k}$, $H^{(k)}(x) = \sigma\big(F^{(k)} H^{(k-1)}(x) - \hat{b}^{(k)}\big)$, Then there exists a uniform Deep Downsampling Convolutional Neural Networks (DDCNN) with ℓ downsampling layers J_k as follow $h^{(j)}(x) = \mathbb{R}^d \to \mathbb{R}^{d_j}$ where $b^{(j)} \in R^{d_{j-1}+s^{(j)}}$, $J_k = \sum_{j=1}^{k} \Delta_j$ where $\Delta_j \leq \left\lceil \frac{n_j n_{j-1} - 1}{s^{[j]} - 1} \right\rceil$, $s^{[k]} \in [2, n_k n_{k-1}]$ for each k s.t

$$h^{(j)}(x) = H^{(k)}(x)$$

Proof

Begin with $H^{(0)}(x) = x$ input layer with $k = 1, n_0 = d$

Suppose that for some $k \in \{1, \ldots, \ell\}$, $\left[w^{(j)}\right]_{j=1}^{J_k - 1}$ is filter masks and $\left(b^{(j)}\right)_{j=1}^{J_k - 1}$ bias vector in (3.9) validated have been built so that J_{k-1}th layer $h^{(J_{k-1})}(x) = (k-1)th$ layer $H^{(k-1)}(x)$ where $d_{J_{k-1}} = n_{k-1}$

Now we construct the DCNN layer $\left\{h^{(j)}(x) \colon \mathbb{R}^d \to \mathbb{R}^{d_j}\right\}_{j=J_{k-1}+1}^{J_k}$

With $\quad B = \left\|h^{(J_{k-1})}\right\|_{\Phi,s_p} = \left\|H^{(k-1)}\right\|_{\Phi,s_p}$ and $\hat{B} = 0$, when $J_k > J_{k-1} +$

1and satisfied the (3.8), bylemma 3.2.2, for $j = J_{K-1} + 1, \dots, J_{K-1}$, (3.4)satisfied and

$$h^{(J_{k-1})}(x) = \mathfrak{J}^{(J_{k-1})} \dots \mathfrak{J}^{(J_{k-1}+1)} H^{(k-1)}(x)$$

$$+ \left\|H^{(k-1)}\right\|_{\Phi,s_p} \left(\prod_{p=J_{k-1}+1}^{J_{k-1}} \left\|w^{(p)}\right\|_{\Phi,s_p}\right) 1_{d_{J_{k-1}}}$$

And we choose $b^{(J_k)}$ as

$$b^{(J_k)}$$

$$= \begin{cases} \left\|H^{(k-1)}\right\|_{\Phi,s_p} \left(\prod_{p=J_{k-1}+1}^{J_{k-1}} \left\|w^p\right\|_{\Phi,s_p}\right) \mathfrak{J}^{(J_k)} 1_{d_{J_{k-1}}} + \theta^{(k)} & ,\text{if } J_k > J_{k-1} + 1, \\ \theta^{(k)}, & \text{if } J_k = J_{k-1} + 1, \end{cases}$$

where $\theta^{(k)} \in R^{d_{J_{k-1}}+s^k}$ is an arbitrary vector satisfying $\mathfrak{D}_{d_{J_{k-1}}} \theta^{(k)} = \hat{b}^{(k)}$, then we have

$$h^{(J_k)}(x) = \mathfrak{D}_{d_{J_{k-1}}} \sigma\left(\mathfrak{J}^{(J_k)} \dots \mathfrak{J}^{(J_{k-1}+1)} H^{(k-1)}(x) - \theta^{(k)}\right)$$

By lemma 3.2.1

$$\mathfrak{J}^{(J_k)} \dots \mathfrak{J}^{(J_{k-1}+1)} = \mathfrak{J}^{(J_k,J_{k-1}+1)} = \mathfrak{J}^w$$

$$= \left(W_{q-i}\right)_{q=1,\dots d_{J_{k-1}}+n_k n_{k-1}-1, i=1,\dots,d_{J_{k-1}}}$$

Recall that $d_{J_{k-1}} = n_{k-1}$ and thereby $\mathfrak{D}_{d_{J_{k-1}}} = \mathfrak{D}_{n_{k-1}}$. So for $r \in$

$\{1, \dots, n_k\}$, the $rn_{k-1} - th$ row of the matrix $\mathfrak{I}^{(J_k)} \dots \mathfrak{I}^{(J_{k-1}+1)} =$

$\left[W_{rn_{k-1}-1} W_{rn_{k-1}-2} \dots W_{rn_{k-1}-n_{k-1}}\right]\square$

Chapter Four:
Approximation on Spherical Orlicz-Spaces

This chapter deals with approximation for quasi -Orlicz functions on the sphere $\mathbb{S}^d$, we touch on the rounding of functions by CNN on spherical space of Quasi-Orlicz space $L_{\Phi,S_p}(\mathbb{S}^{d-1})$ in Chapter One in section 1.5.

In [5], authors added two completely connected layers $h^{(J+1)}$, $h^{(J+2)}$ with widths $D_1, D_2 > 0$, respectively, after the final CNN layer. The connection matrices $\mathfrak{L}^{(J+1)}, \mathfrak{L}^{(J+2)}$ and bias vectors $b^{(J+1)}, b^{(J+2)}$, were determined.

Our aim here is to investigate approximation abilities after adding k layers to get more accurate approximation. By using downsampling properties, we add a finite number of layers to the CNN as desired, or the user needs. Moreover, the degree of the best approximation in terms of modulus of smoothness with order N, that is the number of layers.

4.1. Approximation Theorems

Theorem 4.1.1: Let $2 \leq S \leq d$, $d \geq 3$, $r > 0$, $m, n, N \in \mathbb{N}$, $\boldsymbol{f} \in L_{\Phi,S_p}(\mathbb{S}^{d-1})$. Let $J \geq \lceil \frac{md-1}{S-1} \rceil$. Then there is a CNN of J layers with filters of length S and bias vectors satisfying $b_{s^{(j)}+1}^{(j)} = b_{s^{(j)}+2}^{(j)} = \ldots = b_{d_{j}-1}^{(j)}$ followed by downsampling and k-th layers satisfying

$$\left\| h^{(J+k)} - f \right\|_{\Phi,S_p} \leq C\, \omega_N\left(f, \frac{1}{n} \right)_{\Phi,S_p}$$

Proof

Let $m \in \mathbb{N}$ *and* $\mathbf{f} \in L_{\Phi,S_p}(\mathbb{S}^{d-1})$ and W in $\{0, \dots, M\}$ given by $W_{(j-1)d+(d-i)} = (f_j)_i$, where $j \in \{1, \cdots, m\}$ and $i \in \{1, \cdots, d\}$. By theorem 2.5.1

We have taken $w^{(j)}$. by Lemma 3.2.1, we have

$$\mathfrak{I}^{(J_k, J_{k-1}+1)} = \mathfrak{I}^{(J_k)} \cdots \mathfrak{I}^{(J_{k-1}+2)} \mathfrak{I}^{(J_{k-1}+1)}$$

Now we construct bias vectors in the neural networks. We denote

$$\|w\|_{\Phi,S_p} = \inf_{\lambda > 0} \frac{1}{\lambda} (\lambda w_k^2 + 1)^{1/p}$$

Take $b^{(1)} = -\| w^{(1)} \|_{\Phi,S_p} \mathbf{1}_{d_0}$ and

$$b^{(j)} = \left(\prod_{p=1}^{j-1} \|w^{(p)}\|_{\Phi,S_p} \right) \mathfrak{I}^{(j)} \mathbf{1}_{d_{j-1}} - \left(\prod_{p=1}^{j} \|w^{(p)}\|_{\Phi,S_p} \right) \mathbf{1}_{d_{j-1}+s},$$

for $j = 2, \cdots, J$. The bias vectors satisfy $b_{s+1}^{(j)} = \dots = b_{dj-s}^{(j)}$. since that $\|g\|_{\Phi,S_p} \le 1$,define

$$\mathrm{h}^{(J)}(g(u)) = \mathcal{L}_t(g(u)) + B ,$$

Where $B^{(j)} = \|w^{(j)}\|_{\Phi,S_p} \dots \|w^{(1)}\|_{\Phi,S_p} \|x\|_{\Phi,S_p}$ for $j \ge 1$.

Applying the downsampling operator (3.1) leads to

$$\mathfrak{D}_d\big(h^{(J)}(\mathrm{g}(\mathrm{u}))\big) = \begin{bmatrix} \mathcal{L}_t\Big(g_i(t_{1_i})\Big)_{1d} \\ \vdots \\ \mathcal{L}_t\Big(g_i(t_{m_i})\Big)_{md} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + B^{(J)}\mathbf{1}_{\left\lfloor\frac{(d+JS)}{d}\right\rfloor}$$

Denote $\hat{d} = \lfloor\frac{(d + JS)}{d}\rfloor$. Since $J \geq \lceil\frac{md-1}{S-1}\rceil$, we have

$$\frac{d + JS}{d} \geq 1 + \frac{md - 1}{d}\,\frac{S}{S - 1} > 1 + \frac{md - 1}{d} \geq m.$$

Hence $\hat{d} \geq m$.

$$h^{(J+k)}(\boldsymbol{f}(\boldsymbol{u})) = h^{(J)}(\boldsymbol{f}) \circ \Delta^k(\boldsymbol{fg}),$$

where

$$\Delta^k(\boldsymbol{fg}) = \sum_{r=0}^{k} \binom{k}{r}\Delta^r \boldsymbol{f}(x)\Delta^{k-r}\boldsymbol{g}(x + k),$$

Thus,

$$h^{(J+k)}(\boldsymbol{f}(u)) = \mathcal{L}_t\big(\boldsymbol{g}(u)\big) \circ \Delta^k(\boldsymbol{fg}),$$

where

$$\mathcal{L}_t\big(\boldsymbol{g}(u)\big) = \sum_{i=1}^{d} \boldsymbol{g}(x_i)\delta_i(u),$$

and

$$\delta_k(u) = \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} g_k(t_{ki}) \sigma(u)$$

Now, with similar steps as Theorem 3. 3.1, we get

$$\left\| h^{(J+k)}(f)(u) - f \right\|_{\Phi,S_p} = \left\| \sum_{i=1}^{d} \mathcal{L}_t(g(u)) \circ \Delta^k(fg) - f \right\|_{\Phi,S_p}$$

$$\leq \left\| \sum_{i=2}^{2N+2} [g(t_i) \circ \Delta^k(\sigma) \circ \Delta^k(fg) - f] \right\|_{\Phi,S_p}^{p}$$

$$\leq \frac{c}{n} \omega_N \left(f, \frac{1}{n} \right)_{\Phi,S_p} \square$$

According to same techniques of Theorem 3.3.2, we get the following inverse theorem,

Corollary 4.1.2 (Inverse Theorem)

For any $f \in L_{\Phi,S_p}(I^d)$, then $L_t(f)$ of Theorem I satisfies

$$\omega_N(f,\delta)_{\Phi,S_p} \leq c(p,N) \left[\|f\|_{\Phi,S_p} + E_m(f)_{\Phi,S_p} \right]$$

Proof

Let $b = \max_{1 \leq i \leq 2N+3} \{i; 2^i < n\}$, define

$$f(t_{2N+3}) - f(t_1) = \left(f(t_{2N+3}) - f(t_{2^b}) \right) + \left(f(t_{2^b}) - f(t_{2^{b-1}}) \right) + \cdots +$$

$$\left(f(t_2) - f(t_1) \right),$$

For any $m < 2N + 3$, suppose that

$$\|\boldsymbol{f}(t_{2N+3}) - \boldsymbol{f}(t_m)\|_{\Phi,S_p} \leq cE_m(\boldsymbol{f})_{\Phi,S_p}$$

where $E_m(\boldsymbol{f})_{\Phi,S_p} = \inf \left\|\boldsymbol{f} - h^{(J+k)}(\boldsymbol{f})\right\|_{\Phi,S_p}$, we get by Lemma 2.1 and

Theorem 4.1.1, we easily get

$$\omega_N(\boldsymbol{f},\delta)_{\Phi,S_p} \leq c(p)\left[\omega_N\big(\boldsymbol{f} - h^{(J+k)}(\boldsymbol{f}),\delta\big)_{\Phi,S_p} + \omega_N\big(h^{(J+k)}(\boldsymbol{f}),\delta\big)_{\Phi,S_p}\right]$$

$$\leq c(p)\left[c(N)\left\|\boldsymbol{f} - h^{(J+k)}(\boldsymbol{f})\right\|_{\Phi,S_p} + \omega_N\Big(L_t(\boldsymbol{f},\delta)_{\Phi,S_p}\Big)\right]$$

$$\leq c(p)\left[c(N)\omega_N(\boldsymbol{f},\delta)_{\Phi,S_p} + c(N)E_m(\boldsymbol{f})_{\Phi,S_p}\right]$$

$$\leq c(p,N)\left[\|\boldsymbol{f}\|_{\Phi,S_p} + E_m(\boldsymbol{f})_{\Phi,S_p}\right] \qquad \square$$

Conclusion and Future Work

Conclusions

We studied the approximation theoretical foundations of deep convolutional neural networks, we used Orlicz space as functions space to approximate convolutional networks by defining quasi-Orlicz spaces. We found a better degree of approximation in terms of new-defined modulus of smoothness by convolutional neural network. Furthermore, we studied the basic characteristics and approximation abilities of deep-convolutional neural networks produced by downsampling operators in quasi-Orlicz spaces, and found a degree of best approximation of quasi-Orlicz functions in terms of higher order modulus of smoothness.

Future Works

We will study the approximation of the functions on CNN with other spaces and study more kinds of moduli of smoothness on them. We can approximate as we start by defining K- functional so that we use it to approximation and use the equivalence between K- functional and the modulus of smoothness until we get approximation.

References

[1]   I. Arel, D. C. Rose, and T. P. Karnowski, "Deep machine learning-a new frontier in artificial intelligence research [research frontier]," *IEEE computational intelligence magazine,* vol. 5, no. 4, pp. 13-18, 2010.

[2]   M. J. Perez-Ilzarbe, "Convergence analysis of a discrete-time recurrent neural network to perform quadratic real optimization with bound constraints," *IEEE transactions on neural networks,* vol. 9, no. 6, pp. 1344-1351, 1998.

[3]   P. Guo, K. Wang, and X. Zhou, "PILAE: A non-gradient descent learning scheme for deep feedforward neural networks," *arXiv preprint arXiv:1811.01545,* 2018.

[4]   D.-X. Zhou, "Theory of deep convolutional neural networks: Downsampling," *Neural Networks,* vol. 124, pp. 319-327, 2020.

[5]   Z. Fang, H. Feng, S. Huang, and D.-X. Zhou, "Theory of deep convolutional neural networks II: Spherical analysis," *Neural Networks,* vol. 131, pp. 154-162, 2020.

[6]   D.-X. Zhou, "Universality of deep convolutional neural networks," *Applied and computational harmonic analysis,* vol. 48, no. 2, pp. 787-794, 2020.

[7]   C. Léonard, "Orlicz spaces," *Unpublished notes, available on the Internet,* 2007.

[8]   W. Orlicz, "Über eine gewisse Klasse von Räumen vom Typus B," *Bull. Int. Acad. Pol. Ser. A,* vol. 8, no. 9, pp. 207-220, 1932.

[9]   H. Nakano, *Topology and linear topological spaces*. Maruzen Company, 1951.

[10]   M. Morse and W. Transue, "Functionals f bilinear over the product a× b of two pseudo-normed vector spaces: II. admissible spaces a," *Annals of Mathematics,* pp. 576-614, 1950.

[11]   W. A. J. Luxemburg, "Banach function spaces," 1955.

[12]   M. Kranoselkii and Y. B. Rutickii, "Convex functions and Orlicz spaces (translation)," ed: Groningen, 1961.

[13]   W. Luxemburg and A. Zaanen, "Conjugate spaces of Orlicz spaces," in *Indagationes Mathematicae (Proceedings)*, 1956, vol. 59: Elsevier, pp. 217-228.

[14]   H. Hudzik and L. Maligranda, "Amemiya norm equals Orlicz norm in general," *Indagationes Mathematicae,* vol. 11, no. 4, pp. 573-585, 2000.

[15]   Y. Cui, L. Duan, H. Hudzik, and M. Wisła, "Basic theory of p-Amemiya norm in Orlicz spaces ($1 \leq p \leq \infty$): extreme points and rotundity in Orlicz spaces endowed with these norms," *Nonlinear*

Analysis: Theory, Methods & Applications, vol. 69, no. 5-6, pp. 1796-1816, 2008.

[16] M. Wisła, "Geometric properties of Orlicz spaces equipped with p-Amemiya norms— results and open questions," *Commentationes Mathematicae,* vol. 55, no. 2, 2015.

[17] M. Wisła, "Orlicz spaces equipped with s-norms," *Journal of Mathematical Analysis and Applications,* vol. 483, no. 2, p. 123659, 2020.

[18] D. Jackson, *Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung.* Dieterich'schen Universität--Buchdruckerei, 1911.

[19] S. N. Bernstein, "On the best approximation of continuous functions by polynomials of a given degree," *Comm. Soc. Math. Kharkow, Ser,* vol. 2, no. 13, pp. 49-194, 1912.

[20] C. J. De La Vallée Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle.* Paris, 1919.

[21] A. Zygmund, "On the continuity module of the sum of the series conjugate to a Fourier series," *Prace Mat.-Fiz,* vol. 33, pp. 25-132, 1924.

[22] P. L. Butzer and R. J. Nessel, "Fourier analysis and approximation, Vol. 1," *Reviews in Group Representation Theory, Part A (Pure and Applied Mathematics Series, Vol. 7),* 1971.

[23] R. A. DeVore and G. G. Lorentz, *Constructive approximation*. Springer Science & Business Media, 1993.

[24] V. K. Dzyadyk and I. A. Shevchuk, *Theory of uniform approximation of functions by polynomials*. Walter de Gruyter, 2008.

[25] M. Timan, "Approximation and properties of periodic functions," *Nauk. dumka, Kiev,* 2009.

[26] M. D. Sterlin, "Exact constants in inverse theorems in the theory of approximations," in *Doklady Akademii Nauk*, 1972, vol. 202, no. 3: Russian Academy of Sciences, pp. 545-547.

[27] H. A. Almurieb and E. S. Bhaya, "Nearly Monotone Neural Approximation with Quadratic Activation Function," in *Journal of Physics: Conference Series*, 2021, vol. 1804, no. 1: IOP Publishing, p. 012098.

[28] H. A. Almurieb and E. S. Bhaya, "Monotone approximation by quadratic neural network of functions in Lp spaces for p< 1," *Iraqi Journal of Science,* pp. 870-874, 2020.

[29] Z. Ditzian and V. Totik, "Modulus of Smoothness," vol. 9, ed: Springer-Verlag New York, 1987.

[30] H. Musielak, "On the τ-modulus of smoothness in generalized Orlicz spaces," *Commentationes Mathematicae,* vol. 25, no. 2, 1985.

[31] V. Popov, "On the one-sided approximation on functions," *Constructive Functi,* pp. 465-468, 1980.

[32] V. A. Popov and V. K. Khristov, "Averaged moduli of smoothness for functions of several variables and function spaces generated by them," *Trudy Matematicheskogo Instituta imeni VA Steklova,* vol. 164, pp. 136-141, 1983.

[33] I. Tsyganok, "A generalization of a theorem of Jackson," *Matematicheskii Sbornik,* vol. 113, no. 2, pp. 257-260, 1966.

[34] A.-R. Ramazanov, "On approximation by polynomials and rational functions in Orlicz spaces," *Analysis Mathematica,* vol. 10, no. 2, pp. 117-132, 1984.

[35] W. Garidi, "On approximation by polynomials in Orlicz spaces," *Approximation Theory and its Applications,* vol. 7, no. 3, pp. 97-110, 1991.

[36] D. M. Israfilov and R. Akgün, "Approximation in weighted Smirnov-Orlicz classes," *Journal of mathematics of Kyoto University,* vol. 46, no. 4, pp. 755-770, 2006.

[37] R. Akgün, "Approximating polynomials for functions of weighted Smirnov-Orlicz spaces," *Journal of Function Spaces and Applications,* vol. 2012, 2012.

[38] S. Chaichenko, A. Shidlich, and F. Abdullayev, "Direct and inverse approximation theorems of functions in the Orlicz type spaces ?M," *Mathematica Slovaca,* vol. 69, no. 6, pp. 1367-1380, 2019.

[39] A. Shidlich and S. Chaichenko, "Some extremal problems in the Orlicz spaces," *Matematychni Studii,* vol. 42, no. 1, pp. 21-32, 2014.

[40] A. L. Shidlich and S. O. Chaichenko, "Approximative properties of diagonal operators in Orlicz spaces," *Numerical Functional Analysis and Optimization,* vol. 36, no. 10, pp. 1339-1352, 2015.

[41] F. Dai and Y. Xu, *Approximation theory and harmonic analysis on spheres and balls.* Springer, 2013.

[42] F. Dai and Y. Xu, "Moduli of smoothness and approximation on the unit sphere and the unit ball," *Advances in Mathematics,* vol. 224, no. 4, pp. 1233-1310, 2010.

[43] Z. Li, F. Liu, W. Yang, S. Peng, and J. Zhou, "A survey of convolutional neural networks: analysis, applications, and prospects," *IEEE transactions on neural networks and learning systems,* 2021.

[44] Y. LeCun, Y. Bengio, and G. Hinton, "Deep learning," *nature,* vol. 521, no. 7553, pp. 436-444, 2015.

[45] H. Panwar, P. Gupta, M. K. Siddiqui, R. Morales-Menendez, P. Bhardwaj, and V. Singh, "A deep learning and grad-CAM based color visualization approach for fast detection of COVID-19 cases using chest X-ray and CT-Scan images," *Chaos, Solitons & Fractals,* vol. 140, p. 110190, 2020.

[46] Y. LeCun, L. Bottou, Y. Bengio, and P. Haffner, "Gradient-based learning applied to document recognition," *Proceedings of the IEEE,* vol. 86, no. 11, pp. 2278-2324, 1998.

[47] S. B. Jadhav, "Convolutional neural networks for leaf image-based plant disease classification," *IAES International Journal of Artificial Intelligence,* vol. 8, no. 4, p. 328, 2019.

[48] S. Dittmer, E. J. King, and P. Maass, "Singular values for ReLU layers," *IEEE transactions on neural networks and learning systems,* vol. 31, no. 9, pp. 3594-3605, 2019.

[49] T. M. Kadarina, Z. Iklima, R. Priambodo, R. Riandini, and R. N. Wardhani, "Dental caries classification using depthwise separable convolutional neural network for teledentistry system," *Bulletin of Electrical Engineering and Informatics,* vol. 12, no. 2, pp. 940-949, 2023.

[50] U. o. I. a. U.-C. C. f. S. Research, Development, and G. Cybenko, *Continuous valued neural networks with two hidden layers are sufficient.* 1988.

[51] D.-X. Zhou, "Deep distributed convolutional neural networks: Universality," *Analysis and Applications,* vol. 16, no. 06, pp. 895-919, 2018.

[52] T. Mao, Z. Shi, and D.-X. Zhou, "Theory of deep convolutional neural networks III: Approximating radial functions," *Neural Networks,* vol. 144, pp. 778-790, 2021.

[53] I. Daubechies, *Ten lectures on wavelets*. SIAM, 1992.

[54] S. Mallat, "Understanding deep convolutional networks," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences,* vol. 374, no. 2065, p. 20150203, 2016.

الخلاصة

من الضروري دراسة الأسس النظرية التقريبية للشبكات العصبية التلافيفية العميقة، بسبب تطوراتها المثيرة للاهتمام في المجالات الحيوية، ويعود ذلك لاعتمادها على التقريب الدالي بشكل كبير. من ناحية أخرى، يتعلق هذا بمسألة أخرى ذات أهمية كبيرة، وهي اختيار فضاء الدوال. عملنا هنا على دراسة هذا الموضوع على فضاء واسع من الدوال لتغطية أكبر عدد ممكن من الدوال. عرفنا في هذا العمل فئة جديدة من فضاءات أورلسز تسمى فضاء شبه أورلسز كفضاء دوال لتقريب الشبكة العصبية التلافيفية. يتم تقدير درجة أفضل تقريب بواسطة الشبكة العصبية التلافيفية. علاوة على ذلك، ندرس الخصائص الأساسية وقدرات التقريب للشبكات العصبية التلافيفية العميقة الناتجة من تقليل الحجم في فضاء شبه أورلسز ، لذلك نتج درجة تقريب أفضل لتلك الدوال بدلالة مقياس نعومة أعلى.

تقريب الشبكات العصبية التلافيفية في فضاءات أورلسز الكاذبة

رسالة

مقدمة الى مجلس كلية التربية للعلوم الصرفة / جامعة بابل كجزء من متطلبات نيل

درجة الماجستير في التربية / الرياضيات

من قبل

آمنة مناف إسماعيل فاضل الجنابي

بإشراف

أ. م. د. حوراء عباس فاضل المرعب

1445 هـ   2024 م