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Topological Algebra via Pseudo Differential Structures

A Thesis

Submitted to College of Education for Pure Sciences- University of Babylon in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Education/Mathematics

By

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بسم الله الرحمن الرحيم

الرَّحْمَنُ* عَلَّمَ الْقُرْآنَ* خَلَقَ الْإِنْسَانَ* عَلَّمَهُ الْبَيَانَ*

الشَّمْسُ وَالْقَمَرُ بِحُسْبَانٍ *وَالنَّجْمُ وَالشَّجَرُ يَسْجُدَانِ *

وَالسَّمَاءَ رَفَعَهَا وَوَضَعَ الْمِيزَانَ* أَلَّا تَطْغَوْا فِي الْمِيزَانِ*

وَأَقِيمُوا الْوَزْنَ بِالْقِسْطِ وَلَا تُخْسِرُوا الْمِيزَانَ*

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Dedication

I dedicate this humble work to my father's soul.

Faleh Abdul Mahdi Jaber

Acknowledgments

First and foremost I would like to thank Allah for giving me the strength and determination to carry on this dissertation.

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Faleh Abdul Mahdi Jaber

Abstract

The thesis introduced new structures over algebra by using the concepts of filters to define a new class of algebras that called pseudo-topological algebra, we introduce the conditions that make a pseudo topology compatible with algebra, we also mention the most important characteristics of it, morevor defined the locally convex algebra by pseudo- topological structures. In addition we generalize the concept of weak and strong topology with similar concepts in a pseudo topology when the underlying space of these structures is an algebra. These new concepts are called initial and final pseudo topological algebra, through it, we could be able to identify the pseudo topological structure on subalgebra, product of algebra, projective algebra and quotients algebra.

These structures will enable us to present generalize Frechet derivative on algebra without using norm, and that means expanding by using this derivative for structures that are broader than general topology. Finally we present some important applications of these structures.

List of Symbols and Terms

Symbol	Description
$\chi, \mathcal{F}$, Ƕ, ..	Filters.
$P(F(X))$	The power set of all filers on X
$\underline{E}$	The underlying space
$[B]$	The filter that generated by B .
$\wedge$	And
ʈ	neighborhood filter of zero of the field Ƙ.
N_x	neighborhood filter of x of the set A.
$\mathfrak{X} \downarrow_a E$	The filter $\mathfrak{X}$ convergence to a in E.
$\mathfrak{X} \downarrow E$	The filter $\mathfrak{X}$ convergence to 0 in E.
$C(X,Y)$	The space of all continuous mapping from X to Y.
$\mathcal{L}(X,Y)$	The space of all linear mapping from X to Y.
$H(X,Y)$	The space of all homomorphism from X to Y.
$\prod_{i\in I} A_i$	The product set of the A_i
$\tau(A)$	The locally convex modification of A.
$\varpi_{X\times Y}(\mathcal{K},\mathfrak{X})$	The evaluation filters

$C_\gamma(X,Y)$	A pseudo topological of class of all continuous function
r	The remainder map.
$R(A,B)$	The set of all remainder mapping between A and B.
$\approx$	Isomorphism.
${C^t}_\gamma$	The k -th derivative of a mapping exists and continuous.
${C^\infty}_\gamma$	The infinite derivative of a mapping exists and continuous.
nhd	neighborhood
V.S	Vector Space
T.V.S	Topological Vector Space
LMC-algebra	Locally multiplicative convex algebra

Publications

1. Faleh Abdul Mahdi Jaber, Al-Nafie Z. D, Locally Convex Algebra Via Pseudo Topological Structures, AIP Publishing, Conference on Advances in Physical Sciences and Materials (ICAPSM 2023).
2. Faleh Abdul Mahdi Jaber, Al-Nafie Z. D, Generalization of Gelfand-Mazur Theorem via Pseudo Topological Structures , AIP Publishing,Conference (ISSN: 0094-243x, 1551-7616).
3. Faleh Abdul Mahdi Jaber, Al-Nafie Z. D, Locally Convex Topological Algebra Associated with a Pseudo Topology Structures, IEEE Publishing, The fourth International-Seventh National Scientific Conference ISC-SNC-2023.

Introduction

It is well known that the functional analysis the calculus deals mainly with approximating functions to linear mapping, as confirmed by Dieudonni. The Frechet- derivative remains unchanged if it replace a standard norm with an equivalent one. It is known that the derivative of Frechet does not have a wide extension on the topology. This derivative on the Banach space depends mainly on the norm. Therefore, to obtain the greatest degree of generality for the application of that derivative, we must search through pseudo-topological spaces which is compatible with algebra structures.

The concept of pseudo topology so called convergence space[21, 23, 26, 28, 35]is a generalization of the general topology and finer than that topology space. It contains all topological spaces as well as many remarkable non-topological structures. There is much to be gained in such an analysis. Difficulties arise when one works exdusively with topological structures. Topological algebra is one of the most important structures in functional analysis. The term was coined by David van Dantzig[6,7, 29] it appears in the title of his doctoral dissertation (1931) as the sequences in the metric space are used to define the convergence, the filters, nets in analysis and topology they have an important role in defining convergence as well as the concept of limit to topological space. Filter concept, was first proposed in 1937 by H.Cartan [12, 14, 38, 39] and an excellent treatment of the subject can be found in Bourbaki work [8] in (1940). Many pseudo structures on topological vector spaces studied [24,25] in (1948). The concept of pseudo topology and a theory about its structures had been presented by Choquet [13], that used the concept of convergence of a filter to define the pre-topological structures on it. The basic convergence theory was developed by Fisher [19]. In functional analysis the foundations of pseudo structures are laid and also has been introduced the pseudo topological vector spaces by Frölicher A. and Walter B. [20] that used in [1, 2, 3] to construct a new topological vector space structures. Also in 2021 Yang Deng [16] was introduced and studied the convergent structures and some important topological properties and many mathematicians took up this topic and gave us important results in this field.

We introduce new structures over algebra by using the language of filters to define a new class of algebras that we will call pseudo- topological algebra and prove many most important conditions that make algebra compatible with a pseudo- topological. Also, defined the locally convex algebra by pseudo-topological structures.

Literature Review

- **Cartan [12] in 1937** study the convergence of sequences by using the concept of filters.

- **A. Frolicher and W. Bucher [20] in 1966** presented a detailed study of pseudo-topology compatible with vector space and calculus without the use of norms. They presented a concept of fundamental theorem of calculus

- **In 1972, Ernst Binz [17]** presented new results in the functional analysis of convergence spaces and their relations to modern analysis and algebra and in 1975, the same researcher **[18]** presented an extensive study on continuous convergence on $C(A)$. He studied the relationship between the space $C(A)$ and a completely regular topological space A, and provided a useful and important study about a special class of a pseudo topology spaces, this class of p-embedded spaces.

- **Narici, L. E. and Ch. Suffel in 1977 [40]** presented an extensive study on the subject of topological algebras and studied those topological structures from all aspects .

- **R. Beattie and H. Butzmann [9] in 2002** introduced study the uniform convergence structure of a convergence group and the dule of a pseudo topological vector space. Also, discussed applications of pseudotopology compatible with groups and vector spaces and they studied the duality and reflexivity for pseudo topological groups.

- **Harbi Intesar, and Z. D. Al-Nafie [25] In 2021** introduced study on metrizability of pseudo topological vector spaces.

All previous studies were limited to vector space without expansion when the space under the hand is algebra. The matter more complicated, because operations on algebra are more than vector space , in addition to vector space operations, we will have a third operation, which is vector multiplication, therefore, in order to define a synonym for topology, these processes must be continuous.

In this dissertation, we introduce a pseudo-topology compatible with algebra so that we can then extend the Frechet derivative on these structures and then find the higher order derivatives all the way to C^{∞} space.

Outline of the Dissertation

Through the pseudo-topological structures, we define the pseudo-topological algebra and then generalize many theorems and properties from the Banach space to those algebraic structures according to certain conditions, as follows;

Chapter 1 This chapter consists of three sections: In section one, we recall some concepts of filter, filter base, neighbourhood filter in topology, supermum filters, infimum filters, ultrafilter filters, proudict of filters, image of filter under the mapping f, and convergence of filters with some theorems and reselt. In section two, we mention the most important concepts that relate to algebra, subalgebra, homomorphism algebra, ideal of algebra, normed algebra, Banach algebra and topological algebra. In addition to mentioning the most important theorems and characteristics related to the above topics, which we will need in our study in this dissertation. While in te thid section, we recall the definition of pseudo topology space, the open and closed set in pseudo topology space, continuity of mapping in pseudo topology space and the product pseudo topology structure.

Chapter 2 This chapter consists of three sections as well:

In the first section we construct the concept of pseudo topological algebra, this is after setting the necessary conditions for the pseudo-topology to be compatible with the algebra so that the three operations of the algebra are continuous in the sense of continuity defined on the pseudo-topology, next, we take a special case, which is when we have a basis-filter of neighbourhoods of zero, also we present most important topological properties and some important generalizations regarding pseudo-algebra topologies. In section two, we construct a new pseudo topological algebra from given ones. The initial pseudo topology on a space from a family of a pseudo topology spaces and construct a final pseudo- topological algebra through another family of a pseudo topological algebra and give some examples of this, such as: The product of pseudo topological algebra on

$\prod_{i \in I} A_i$, a pseudo topological sub-algebra and inductive system of a pseudo topological algebras. In section three, we generate a locally convex topological algebra from a pseudo topology. We also study the relationship between convex algebras generated from pseudo topological structures and the convex algebra generated from the family of all continuous seminorms on that algebra.

Chapter 3 Consists of three sections:

In section one, we introduce a pseudo-topological γ on $C(X,Y)$ This structures consisting of the convergence of the filters are used to prove some properties on the space $C(X,Y)$ (algebra on class of all continuous function from two spaces). We study the relationship between the algebra $C_\gamma(A,B)$ and the algebra B. Also, we study the space $C_q(A,B)$ of all quasi bounded maps. In section two, we introduce the concepts of differentiability and derivatives on a pseudo topological algebra, we extend the concept of a Frechet derivative but without norm on algebra using pseudo-structures after introducing the appropriate conditions. In section three, we introduce the concepts of second derivative, higher derivative, and then we put the definition of C^∞- derivatives of the maps on a pseudo topological algebra.

Chapter 4 In chapter four, we presented some important applications that have an important role and practical applications, especially in physics. This chapter contains three sections, in section one, we expand the concepts of boundedly-convex and Lipschitz- continuous mapping from Banach spaces to pseudo-topology structures that are compatible with algebra A. While the second application in section two, is about the generalization of Gelfand Mazur theory from the Banach algebra to to the case in which we have the space under the hand is the algebra compatible to pseudo topological structures, because of that theory of great importance and very wide applications. While the third section, is devoted to the most important conclusions we obtained in this thesis.

Chapter One

Preliminaries

Introduction

This chapter consists of three sections:

The first section, we involves the basic information about the concept of filter and filter base. We give the most important characteristics related to these concepts . In section two, we introduced the basic information about the algebra and topological algebra. In section three, we deals with a pseudo topology (limit structure), the pseudo topological vector space. We also mention the most important characteristics related to this concept.

1.1 Filter and Filter Base

The pseudo-structures depend primarily on the language of convergence of filters, we will devote this section to recalling the concept of the filter and filter base and some properties of these.

Definition (1.1.1)[34]:

Let E be a set and K is field, we defined the operations $+$, $\bullet$ and $*$ as follws; for all $x, y \in E$ and $\lambda \in K$

i. $+: E \times E \to E$, such that $(x, y) \to x + y$
ii. $\bullet: K \times E \to E$, such that $(\lambda, y) \to x \bullet y$
iii. $*: E \times E \to E$, such that $(x, y) \to x * y$

And if we have $M, N \subseteq E$,we defined the operations $+$, $\bullet$ and $*$ on the subsets M and N as follws;

i. $M + N = \{m + n: m \in M \wedge n \in N\}$
ii. $\lambda \bullet N = \{\lambda \bullet n: \lambda \in K \wedge n \in N\}$
iii. $M * N = \{m * n: m \in M \wedge n \in N\}$

Definition (1.1.2)[8]:

A filter in E is a non-empty family $\mathcal{F}$ of subsets of E, such that the following three conditions are met;

i. The empty set is not belong to $\mathcal{F}$

ii. $X_1 \in \mathcal{F}$ and $X_1 \subset X_2$ then $X_2 \in \mathcal{F}$

iii. $X_1, X_2 \in \mathcal{F}$ then $X_1 \cap X_2 \in \mathcal{F}$.

Definition (1.1.3)[8]:

A filter-basis in a space E is a non-empty family β of subsets of E that satisfies the conditions:

i. $\emptyset \notin \beta$.

ii. For all $\mathcal{B}_1, \mathcal{B}_2 \in \beta$ there exists $\mathcal{B}_3 \in \beta$ such that $\mathcal{B}_1 \cap \mathcal{B}_2 \supseteq \mathcal{B}_3$.

Example (1.1.4)[37]:

Let $X \neq \emptyset$ and $N \subseteq X$.

Then the family $\mathcal{F} = \{F: F \supseteq N\}$ is a filter on X. This filter has a filter basis represented by $\{N\}$,

Especially if it is $x \in X$, then $\mathcal{F} = \{F: x \in F\}$ from filter on X, We will denote that filter with the symbol $[x]$.

Definition (1.1. 5)[12]:

Let (E, τ) be a topological space and let N_x be the collection of all τ-nhds of a point $x \in E$. Then N_x is a filter on E called the neighbourhood filter of x.

Example (1.1.6):

Let $E = \{a, b, c\}$ and

$\tau = \{\emptyset, \{a\}, \{a, b\}, E\}$.

Consider the filter $\mathcal{F} = \{\{b\},\{a,b\},\{b,c\},E\}$ then we have

$\mathcal{N}_a = \{\{a\},\{a,b\},\{a,c\},E\}$

$\mathcal{N}_b = \{\{a,b\},E\}$

$\mathcal{N}_c = \{E\}$

Definition (1.1.7)[12]:

If $\mathcal{F}_1,\mathcal{F}_2$ are filters in a set E, we say that $\mathcal{F}_2$ is finer than $\mathcal{F}_1$ if $\mathcal{F}_2 \supseteq \mathcal{F}_1$ and we can say also $\mathcal{F}_1$ is coarser than $\mathcal{F}_2$.

We can define the relation in E: $\mathcal{F}_1 \leq \mathcal{F}_2$ iff $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

Definition (1.1.8)[8]:

Let $\mathcal{F}_1$, $\mathcal{F}_2$be two filters. If $A \cap B \neq \emptyset$ for all $A \in \mathcal{F}_1 \wedge B \in \mathcal{F}_2$ then the supermum of the two filters is $\mathcal{F}_1 \vee \mathcal{F}_2 = \{A \cap B: A \in \mathcal{F}_1 \wedge B \in \mathcal{F}_2\}$.

Definition (1.1.9)[14]:

Let $\mathcal{F}_1,\mathcal{F}_2$ be a filters on E_1,E_2 respectively, then we define the direct proudect $\mathcal{F}_1 \times \mathcal{F}_2$ is the filter on $E_1 \times E_2$ generated by;

$\{F_1 \times F_2: F_1 \in \mathcal{F}_1 \wedge F_2 \in \mathcal{F}_2\}$.

If $f: E_1 \times E_2 \to E$ is a map, then

$f(\mathcal{F}_1,\mathcal{F}_2)$=$\{f(F_1,F_2): F_1 \in \mathcal{F}_1 \wedge F_2 \in \mathcal{F}_2\}$.

According to Definition(1.1.1), we get:

$\mathcal{F}_1 + \mathcal{F}_2 = \{F + H \ \ for\ all\ F \in \mathcal{F}_1 \wedge H \in \mathcal{F}_2\}$

$\lambda \bullet \mathcal{F}_1 = \{\lambda \bullet F \ \ for\ all\ F \in \mathcal{F}_1\}$

$\mathcal{F}_1 * \mathcal{F}_2 = \{F * H \ \ for\ all\ F \in \mathcal{F}_1 \wedge H \in \mathcal{F}_2\}$

proposition (1.1.10)[8]:

If $f: E \longrightarrow E_1$ be a mapping between two sets E and E_1, then;

1. If $\mathcal{F}$ is a filter on E then $\{ f(F): F \in \mathcal{F}\}$ is a filter basis on a space E_1,
2. If $f: E \longrightarrow E_1$, $g: E_1 \longrightarrow P$ then $(g \circ f)(\mathcal{F}) = g(f(\mathcal{F}))$,
3. $f([m]) = [f(m)]$, for all $m \in E$.
4. If $\mathcal{F}_1 \leq \mathcal{F}_2$ then, $f(\mathcal{F}_1) \leq f(\mathcal{F}_2)$.
5. Let $\mathcal{F} \in \mathrm{P}(F(E))$,then $\{f^{-1}(F)$: $F \in \mathcal{F}$ $\}$ is a filter base in E iff $f^{-1}(F) \neq \emptyset$, $\forall\, F \in \mathcal{F}$.

Definition (1.1.11)[37]:

Let $W \neq \emptyset$ be a class of filters on $X \neq \emptyset$. A filter $\mathcal{F}$ on X is said to be

(i) supremum of W if $\mathcal{F}$ is finer than each member of W

(ii) infimum of W if $\mathcal{F}$ is coarser than each member of W

(iii) ultrafilter if it is not properely contained in any other filter.

Theorem (1.1.12)[37]:

A filter $\mathcal{F}$ on a set X is an ultrafilter iff for all $M \subseteq X$, either $M \in \mathcal{F}$ or $X - M \in \mathcal{F}$.

Theorem (1.1.13)[37]:

If $g: X \to Y$ be a map and a filter $\mathcal{F}$ on a set X is an ultrafilter then $g(\mathcal{F})$ is an ultrafilter in Y.

Definition (1.1.14)[37]:

A filter χ on a set E is said to be eventually in a subset A of E iff $A \in \chi$.

Definition (1.1.15)[37]:

A filter χ on a set E is said to be frequently in a subset A of E iff A intersects every member of χ, that is $A \cap F \neq \varphi$ for every $F \in \chi$.

Definition (1.1.16)[8]:

Let (E, τ) be a topological space, and let $\mathcal{F}$ be a filter on a set E. Then $\mathcal{F}$ is said to be τ −converge or simply converge to a point $x \in E$ iff $\mathcal{F}$ is eventually in each nhd of x, that is iff every nhd of x is a member of $\mathcal{F}$ and we say that x is a limit point of $\mathcal{F}$ and write $\mathcal{F} \downarrow_x$.

Example (1.1.17):

From the previous Example (1.1.6), Since $N(b), N(c) \subseteq \chi$,
then b, c are limit points of χ.
From this example we note that, a filter may have more than one limit point.

Definition (1.1.18)[8]:

A filter base $\mathcal{B}$ on E is said to be converge to a point $x \in E$ if and only if the filter $\mathcal{F}$ which generated by $\mathcal{B}$ is converges to x and we say that x is a limit point of $\mathcal{B}$.

Theorem (1.1.19)[37]:

Let $\mathcal{F}$ be a filter on X and let $A \subset X$. Then there exists a filter $\mathcal{F}^*$ finer than $\mathcal{F}$ such that $A \in \mathcal{F}^*$ if and only if $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.

Theorem (1.1.20)[37]:

Let X be a topological space . Then the filter $\mathcal{F}$ converge to a point $x \in X$ if and only if for all ultrafilter containing $\mathcal{F}$ converge to x.

1.2 Algebra and Topological Algebra

In this section, we will mention the definition of algebra, homomorphism algebra, subalgebra, ideal of algebra, normed algebra, topological algebra, Banach algebra and Frechet algebra in addition to the most important properties and theorems that we will need in our study later. But before all this, it is okay to touch on vector space, since all algebra must be vector space.

For any operation $*$ on a set E, we say that $*$ is commutative if $a * b = b * a$, and if ther is an element $e \in E$ such that $a * e = e * a = a$, then this element is called identity element, also for any element $a \in E$, ther is an element $b \in E$, such that $a * b = b * a = e$, then b is said inverse of a.

Definition (1.2.1)[33]:

A vector space E over K is a set with two operations $+$ and $\bullet$ (addition and scalar multiplication) these operations are defined as follows:

1.$(E, +)$ is an abelian group

2. The operation $(\bullet)$ (called scalar multiplication) is defined between vectors, and satisfy the following conditions:

i. For all $x \in E$ and, for any $\lambda \in K$, then $\lambda \bullet x \in E$.

ii. For all $x, y \in E$ and $\in K$, then $\lambda \bullet (x + y) = \lambda \bullet x + \lambda \bullet y$.

iii. For all $\lambda,\ \alpha \in K$ and $x \in E$, $\lambda \bullet (\alpha \bullet x) = (\lambda \bullet \alpha) \bullet x$

iv. For any $\lambda,\ \alpha \in K$ and the vector $x \in E$, then

$$(\lambda + \alpha) \bullet x = \lambda \bullet x + \alpha \bullet x.$$

v. There exists $1 \in K$ such that for any $x \in E$ then

$$1 \bullet x = x \bullet 1 = x .$$

Example (1.2.2)[34]:

The set of complex numbers $\mathbb{C}$ over the field $\mathbb{R}$ (the set of real numbers), from a vector space, where the operations on a vector space $(\mathbb{C}, +, \bullet)$ defined as follws:

For all $(x_1 + y_1 i) \wedge (x_2 + y_2 i) \in \mathbb{C}$, and for all $\lambda \in \mathbb{R}$, then

$$(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2) i$$

$$\lambda \bullet (x_1 + y_1 i) = (\lambda \bullet x_1 + \lambda \bullet y_1 i)$$

Definition (1.2.3)[34]:

A topological vector space (T.V.S) is a vector space E and topology τ such that this topology compatible with vector space that makes vector space operations topologically continuous. That's mean:

i. For each $x, y \in E$, and for all open neighborhood W of $x + y$, there exist open neighborhood U_1 of x and U_2 of y, such that $U_1 + U_1 \subset$ W.

ii. For any $\alpha \in K$, $x \in E$ and for all open neighborhood W in E such that $\alpha \bullet x \in W$, $\exists$ nhd U of α and a nhd V of x, such that $U \bullet V \subset$ W.

Example (1.2.4)[34]:

Every normed vector space has a natural topological structure, the norm induces a metric induces a topology.

Definition (1.2.5)[33]:

Let M be a subset of a vector space E, then

1. M is said to be convex set if for all $a, b \in E$ and $\forall \alpha \in [0,1]$, then $\alpha \bullet a + (1 - \alpha) \bullet b \in E$.
2. M is said to be balanced set if $\alpha \bullet M \subseteq$ M for all $|\alpha| \leq 1$, and $\alpha \in K$.
3. M is absorbing set if $\forall x \in E$, there exists $s > 0$, s.t $\lambda \bullet x \in M$ for all $|\lambda| \leq s$.

Definition (1.2.6)[33] (semi-norm):

Let X be a vector space . Then the mapping $p: E \to R$ is said to be semi-norm if;

1. $p(a + b) \leq p(a) + p(b)$ and

2. $p(\alpha a) = |\alpha| p(a)$

for all $a, b \in X$ and all scalars α.

Definition (1.2.7)[33]:

Let X be a v.s over R. Then a map $\|.\|: E \to R,\ x \mapsto \|x\|$ is called a norm if:

1. For all $\alpha \epsilon R$ then $\|\alpha \bullet a\| = |\alpha| \|a\|$for all $a \in E$.
2. For all $x,\ y \in E$, $\|a + b\| \le \|a\| + \|b\|$.
3. $\|a\| \ge 0$ for all $a \in E$.
4. $\|a\| = 0$ if and only if $a = 0$.

A pair $(E, \|.\|)$ is called a normed space.

Definition (1.2.8)[33]:

Let X be a T.V.S over the filed K , then;

a. X is locally convex if there is a local base $\mathcal{B}$ whose members are convex.
b. X is metrizable if τ is compatible with some metric d.
c. A metric d on a vector space X will be called invariant if $d(x + z, y + z) = d(x, y)$ and $d(\lambda \bullet x, \lambda \bullet y) = |\lambda|\ d(x, y)$ for all $x, y, z\ in\ X$ and $\lambda \in K$.
d. X is an $F-$space if its topology τ is induced by a complete invariant metric d.

Definition(1.2.9)[34]:

A topological vector space X is a Fréchet space if X is a locally convex $F-$space.

Definition(1.2.10)[33] (via systems of semi-norms)

A Fréchet space X is a complete topological vector space X whose topology is induces by a countable family of semi-norms. To be more precise, there exists semi-norm functions

$P_n: X \to \mathbb{R}, n \in \mathbb{N}$

such that the collection of all balls

$$B^{(n)}{}_{\varepsilon}(x) = \{y \in X: P_n(x-y) < \varepsilon\}, \ \ x \in X, \varepsilon > 0, n \in \mathbb{N}$$

be from a base of the toplogy that is compatible with vector space.

Definition(1.2.11)[36]:

A topological vector space E is called normable if there exists a norm $\|.\|$ on E such that the canonical metric $(x, y) \longmapsto \|y - x\|$ induces the topology τ on E.

Theorem(1.2.12)[36]:

The topological vector space E is normable if and only if there is a bounded convex neighborhood of zero.

Definition (1.2.13)[7]:

A vector space X equipped with an additional binary operation, called vector multiplication and denoted by xy for $x, y \in X$, is called an algebra if X is a ring with respect to vector addition and vector multiplication, and $\alpha \bullet (x * y) = (\alpha \bullet x) * y = x * (\alpha \bullet y)$ for all scalars α and all $x, y \in X$.

Example (1.2.14)[34]:

If we add a third operation $*$ on the vector space $(\mathbb{C}, +, \bullet)$ in Example(1.2.2) it will become algebra $(\mathbb{C}, +, \bullet, *)$, which is the operation of multiplying complex numbers $\mathbb{C}$, that is;

for all $(x_1 + y_1 i) \wedge (x_2 + y_2 i) \in \mathbb{C}$, then

$$(x_1 + y_1 i) * (x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i$$

Definition (1.2.15)[33]:

Let A be an algebra, the dimension of A it is the same as the dimension of linear space A.

We note that if the $\dim(A)$ is finite , then the algebra dimension is also finite, and if the $\dim(A)$ is infinite, then the algebra dimension is also infinite.

Definition (1.2.16)[33] (Homomorphism algebra)

A linear map $f: X \to Y$ where X and Y are algebras is called an (algebra) homomorphism if $f(xy) = f(x)f(y)$ for all $x, y \in X$.

If f is also one to one it is referred to as (algebra) isomorphism or an embedding .

The null space of an algebra homomorphism, that is $f^{-1}(0)$, is renamed the kernel of f and is denoted by $\ker f$, where $\ker f = \{x \in X: f(x) = 0\}$.

Definition (1.2.17)[33]:

If A is without identity, we consider the direct product space A [e]=$A \oplus$ ₵ with multiplication defined by

$$(x, \alpha) \cdot (y, \beta) = (xy + \beta x + \alpha y, \alpha\beta)$$

It is a direct verification that this is indeed an algebra.

Theorem (1.2.18)[27]:

Let A be an algebra over the filed K, and $\mathfrak{H}(A) = \{x \in A: x\ has\ invers\ in\ A\}$,then;

1. $0 \notin \mathfrak{H}(A)$
2. $(\mathfrak{H}(A),*)$ is group
3. if $a \in \mathfrak{H}(A)$, then $-a \in \mathfrak{H}(A)$

4. $(-a)^{-1} = -a^{-1}$

Definition (1.2.19)[10]:

Let A be an algebra over the filed K, and $a \in A$, then

1. a has a left divisor of zero if there is an element $0 \neq b \in A$, such that $b * a = 0$.
2. a has a right divisor of zero if there is an element $0 \neq b \in A$, such that $a * b = 0$.
3. A is said to be have divisors of zero if there exists an elements $a, b \neq 0$ and $a * b = b * a = 0$.

Definition (1.2.20)[33]:

Let A be an algebra over the filed K, then $B \subseteq A$ is side to be subalgebra of A if B is also algebra over the filed K.

Definition (1.2.21)[11]:

Let $I \subset A$, where A be an algebra then;

1. I is said to be left ideal of A, if for all $x \in I$ and $a \in A$, then $ax \in I$
2. I is said to be right ideal of A, if for all $x \in I$ and $a \in A$, then $xa \in I$
3. I is said to be ideal of A, if for all $x \in I$ and $a \in A$, then $ax, xa \in I$

Remark (1.2.22)[10]:

Let A be an algebra over the filed K, then

1. Every ideal of A is a subalgebra of A but the opposite is not always true.

2. If A is commutative algebra, then every left ideal is right ideal and is therefore an ideal of A.

Example (1.2.23):

Let $A = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d, \in \mathbb{R}\right\} = M_2(\mathbb{R})\}$

And let $B = \{\begin{bmatrix} r & 0 \\ s & 0 \end{bmatrix} : r, s \in \mathbb{R}\}$

Then A is an algebra over the field $\mathbb{R}$ (set of all real numbers),

The identity element is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A is not commutative algebra

B is a subalgebra of A but it not ideal of A.

Lemma (1.2.24)[10] :

Let A an algebra with identity e, and I an ideal. Then

1. If $e \in I$, then $I = A$
2. If $x \in I$ and x is invertible, then $I = A$
3. when A is commutative: x is invertible if and only if $(x) = A$, where (x) means the ideal generated by element x.
4. when A is commutative: x is not invertible if and only if x is contained in a maximal ideal $M \subset A$.

Theorem (1.2.25)[7]:

Let B_1 and B_1 be two subalgebras of A , where A is an algebra over the field K, then;

1. $B_1 \cap B_2$ is subalgebras of A
2. $B_1 \cup B_2$ is subalgebras of A if and only if $B_1 \subseteq B_2$ or $B_1 \supseteq B_2$

Definition (1.2.26)[31] (Normed algebra)

A normed algebra $(A, \|.\|)$ is a normed vector space and an algebra, satisfying $\|xy\| \leq \|x\|\|y\| \quad \forall x, y \in A$

Some authors extend the definition by saying that if A has an identity e, then $\|e\|$=1.

For any normed algebra $(A, \|.\|)$ whose underlying vector space is a Banach space (complete relative to $\|.\|$) there exists an equivalent norm $\|.\|_e$ such that $\|.\|_e = 1$.

Definition (1.2.27)[31]

Let X be a non-empty set, then the function $f: N \to X$ such that for all $n \in N$ there exists only one element such that $f(n) = x_n$ is called a sequence in X. We denoted of f by $\{x_n\}$.

Definition (1.2.28)[31]

Let $\{x_n\}$ be a sequence in normed space X . Then $\{x_n\}$ is said to be converge in X if there exist $x \in X$ such that for any $\epsilon > 0$ there is $s \in Z^+$ such that $\|x_n - x\| < \epsilon$, for all $n > s$.

x is said to be a convergent point and denoted of this by $x_n \to x$.

Definition (1.2.29)[31]

Let $\{x_n\}$ be a sequence in normed space X . Then $\{x_n\}$ is said to be coushy sequence in X for any $\epsilon > 0$ there is $s \in Z^+$ such that $\|x_n - x_m\| < \epsilon$, for all $n, m > s$.

Lemma (1.2.30)[31]:

Let A be an algebra, then;

1. If $x_n \to x \ then \ yx_n \to yx \ and \ x_n y \to xy$ for all $y \in A$
2. If $x_n \to x \ and \ y_n \to y \ then \ x_n y_n \to xy$

3. If $I \subset A$ is an ideal, then the topological closure of I, denoted by $\bar{I}$, is a also an ideal
4. If $A_0 \subset A$ is a subalgebra, then $\overline{A_0}$ is also a subalgebra.

We note that if the normed space is a complete space, meaning that every Cauchy sequence is a convergent sequence to a point in the same space, then the normed algebra will be Banach algebra.

Definition (1.2.31)[31]:

If A is a normed algebra that is complet relative to the norm $\|.\|$(i.e A is Banach space) then A is called a Banach algebra.

If A is a Banach algebra without identity ,then A[e] is also a Banach algebra since ₵ is a Banach and A[e]=A⨁ ₵.

Definition (1.2.32)[7] (Topological Algebra)

A topological algebra A over the field K is an algebra with a topology τ which is compatible with algebra structure. Such that the three operations of an algebra are continuous.

The algebra A and the topology τ are compatible or τ is compatible with the algebraic structure of A when (A, τ) is a topological algebra.

In other words, in a ddition to what we mentioned in Definition( ), we also have;

The mapping $*: A \times A \to A$ is continuous at (x, y) if for all U_{x*y} (nhd of $x * y$) in A there are a nhd V_x of x and V_y of y in A such that $V_x * V_y \subseteq U_{x*y}$.

Definition (1.2.33)[7](Multiplicative Convexity)

A subset U of an algebra X is called;

1. Multiplicative (idempotent) if $U^2 = UU \subset U$

2. Multiplicatively- convex or m-convex if it is convex and Multiplicative
3. Absolutely m-convex if it is balanced and m-convex.

Properties (1.2.34)[7]:

Let X be an algebra . If $U \subset X$ is multiplicative, then so is;

1. its convex hull U_c
2. λU if U is balanced and $|\lambda| \leq 1$
3. its balanced hull U_b
4. its balanced convex hull U_{bc}
5. any direct or inverse homomorphic image
6. its closure $cl(U)$ if X is a topological algebra .

Definition (1.2.35)[7] (Multiplicative Seminorms)

A seminorm P on an algebra A is multiplicative if $P(xy) \leq p(x)p(y)$ for all $x, y \in A$

Lemma (1.2.36)[7] (Maxima of Multiplicative Seminorms)

If $p_1, p_2, \dots \dots p_n$ are multiplicative then max a_j, p_j is a multiplicative seminorm for any collection numbers $a_1, \dots \dots a_n$.

Theorem (1.2.37)[7]:

1. If p is a multiplicative seminorm, then $V_P = \{x: p(x) < 1\}$ is absolutely m-convex and absorbent.
2. If U is absolutely m-convex and absorbent then its gauge p_u, $p_u(x) = \inf\{a > 0: x \in aU\}$ is multiplicative seminorm.

Definition (1.2.38)[7]:

A topological algebra (X, τ) is a locally m-convex algebra (LMC algebra) if there is a base of m-convex sets for v(0).

We also say that τ is a locally m-convex or is an (LMC – topology). X is a locally convex algebra, if X is a topological algebra which carries a locally convex linear space structure.

In addition to being locally m-convex, if τ is Hausdorff, we say that X is an LMCH-algebra, and τ to be LMCH.

An LMC-algebra which is a complete metrizable topological space is a Frechet algebra.

Remark (1.2.39)[7]:

The topology of Frechet algebra can be generated by a sequence of multiplicative seminorms;

Let A be an algebra, and $(p_n)_n$ be a sequence of multiplicative seminorms on A such that;

1. $p_n(x) \leq p_{n+1}(x)$ for all $n \in N,\ x \in A$
2. For all $x \in A, x \neq 0$, $there\ is\ n \in N\ such\ that\ p_n(x) \neq 0$

then $(p_n)_n$ generates a topology on A in the following way:

for $x \in A,\ n \in N$ set $U_n(x) = \{y \in A: p_n(x-y) < \frac{1}{n}\}$

then $\Psi = (U_n(x))_n$ is a basis of neighbourhoods of x for a locally convex topology on A and A is metrizable.

Each $U_n(0)$ is an absolutely convex and multiplicative set.

Also A is an LMC-algebra,

That is the multiplication is continuous, so A becomes an Frechet algebra if A is complete.

Examples(1.2.40):

1. Each Banach algebra is an Frechet algebra.

2. let $C^{\infty}([0,1])$ denote to the algebra of all infinitely differentiable functions on the unit interval with pointwise operations. For n $\in N$ define $p_n(f) = 2^{n-1} \sup\{|f^{(k)}(x)| : x \in [0,1], k = 0,1,2, \dots n-1\}$, where $f^{(k)}$ denotes the k-th derivative of f.

Then $p_n(f)$ is a basis of neighbourhoods of x for a locally convex topology on $C^{\infty}([0,1])$ and $C^{\infty}([0,1])$ is metrizable.

Theorem (1.2.41)[7]:

Let X be an algebra, then the following expressions are equivalent:

1. X is an LMC-algebra;
2. X is a locally convex T.V.S and there is a base of multiplicative sets at 0;
3. X is a T.V.S and there exists a base of absolutely m-convex sets at 0;
4. X is a T.V.S and there exists a base of closed absolutely m-convex sets at 0.

Theorem (Tietze's Extension) (1.2.42)[32]:

Let X be a normed space, M be a closed subset of X , and f be a continuous function on M to the closed interval $[a, b] \subset R$. Then f has a continuous extension g to X such that $g(X) \subset [a, b]$.

Theorem (1.2.43)[32]:

Suppose K is a compact subset of the completely regular space X. If $f \in C(X)$,then there is $g \in C(X)$ extending f such that

$$\sup\{|g(x)|: x \in X\} = \|f\|_k$$

Definition (1.2.44)[30]:

Let f be a function on an open subset U of a Banach space X into the Banach space Y, we say f is Frechet differentiable at $x \in U$ if there is bounded and linear operator $T: X \to Y$ such that

$$\lim_{\|h\|\to 0} \frac{\|f(x+h)-f(x)-T(h)\|}{\|h\|} = 0$$

This derivative is usually denoted by $df\,(x) = T$.

Theorem (1.2.45)[7]:

The filterbase β in the algebra X determines a base at 0 for a compatible topology for X iff;

1. β is a neighborhood base at 0 for a topology which is compatible with $X's$ linear structure.
2. For each $v \in \beta$ there exists a $B \in \beta$ such that $BB \in v$.

Definition (1.2.46)[34]:

U is the filter of neighbourhoods of the origin for some vector space topology on E iff it has the following properties:

1. U has a balanced basis
2. the elements of U are absorbing
3. for every $V \in U$ some $V^{\sim} \in U$ can be found such that $V \supseteq V^{\sim} + V^{\sim}$.

Remarks(1.2.47)[34]:

i. Specially, if we have $\beta = \{U_\alpha\}$ is a basis of neighbourhoods of zero, then the set $x + U_\alpha, U_\alpha \in \beta$ from a basis for the neighborhood filter of x.

ii. Also if $x + U_\alpha, y + U_\alpha$ are the neighborhood filters of x and y respectively, then $xy + U_\alpha$ is the neighborhood filter of xy and we denote for this multiplication of neighborhood filters by:

$$(x+U_\alpha)*(y+U_\alpha)=xy+xU_\alpha+yU_\alpha+{U_\alpha}^2.$$

Definition (1.2.48)[7]:

U is the filter of neighbourhoods of the origin for some topological algebra on E iff it has the following properties:

1. U has a balanced basis
2. the elements of U are absorbing
3. for every V$\in U$ some $V^{\sim}\in U$ can be found such that $V\supseteq V^{\sim}+V^{\sim}$
4. for every $u\in U$ some $u^{\sim}\in U$ can be found such that $u^{\sim}*u^{\sim}\subseteq V$.

1.3 Pseudo Topology (Limit Structure)

In this section the concept of pseudo topology and the most important basic concepts about its structure, we will present with the mention of the most important theorem that will need later.

Definition (1.3.1)[20]:

Let X be a set, and $P(F(X))$ the set of all filters on X, and $\gamma\subseteq X\times P(F(X))$ a relation. We will write $\mathcal{F}\to_\gamma x$, (the filter $\mathcal{F}$ converges to x) if; $\mathcal{F}\to_\gamma x$ and $\mathcal{F}\subset$ Ƕ implies Ƕ $\to_\gamma x$, that is if $\mathcal{F}$ converges to x, then every finer filter also converges to x.

Definition (1.3.2)[20]:

Suppoe (X,γ) satisfies the Definition(1.3.1). Then $\mathrm{N}_\gamma(x)=\cap\{\mathcal{F}\in\mathcal{F}(X):\mathcal{F}\to_\gamma x\}$ is called the neighborhood filter of $x\in X$.

Definition (1.3.3)[20]:

Suppoe (X,γ) satisfies the Definition(1.3.1). Then (X,γ) is called a generalized convergence space if $[x] \rightarrow_{\gamma} x$ for all $x \in X$.

Definition (1.3.4)[20]:

Let (X,γ) be a generalized convergence space, then (X,γ) is called;

1. pretopological convergence space if $\mathcal{N}_{\gamma}(x) \rightarrow_{\gamma} x$.
2. Pseudotopological convergence space if $\mathcal{F} \rightarrow_{\gamma} x$ whenever all ultrafilters Ƕ that are finer than $\mathcal{F}$ coverge to x.
3. Limit space if for all $\mathcal{F} \rightarrow_{\gamma} x$ and Ƕ $\rightarrow_{\gamma} x$ implies $(\mathcal{F} \cap$ Ƕ$) \rightarrow_{\gamma} x$
4. Kent convergence space if $\mathcal{F} \rightarrow_{\gamma} x$ implies $(\mathcal{F} \cap [x]) \rightarrow_{\gamma} x$.

We will discuss in this thesis pseudotopological convergence space, and for ease we will call it pseudo topology, and if a filter $\mathcal{F}$ converges to x on X) we write $\mathcal{F} \downarrow_{x} X$.

The axiom can now be expressed as:

$\forall\, x \in X$ *we have*

1. $[x] \downarrow_{x} X$
2. $\mathcal{F}_i \downarrow_{x} X$ for $i = 1,2 \Longrightarrow \mathcal{F}_1 \cap \mathcal{F}_2 \downarrow_{x} X$
3. $\mathcal{F} \downarrow_{x} X$, $\mathcal{F} \leq$ Ƕ $\Longrightarrow$ Ƕ $\downarrow_{x} X$

Pseudo-topology becomes a topological space if the following condition is met;

For all $x \in E$ define φ_t such that $\mathcal{F} \in \varphi_t(x)$ if and only if $N_x \leq \mathcal{F}$ Where N_x is the filter of neighborhoods of x.

That is $\mathcal{F} \downarrow_{x} E$ if and only if $N_x \leq \mathcal{F}$

1. since $N_x \leq [x]$ then $[x] \downarrow_{x} E$
2. if $\mathcal{F} \in \varphi_t(x)$ and $\mathcal{F} \leq$ Ƕ, then $N_x \leq \mathcal{F} \leq$ Ƕ. Hence Ƕ $\in \varphi_t(x)$.
3. If the filters $\mathcal{F}_1, \mathcal{F}_2 \in \varphi_t(x)$, then $N_x \leq \mathcal{F}_1, \mathcal{F}_2$ and hence $N_x \leq \mathcal{F}_1 \cap \mathcal{F}_2$ therefor $\mathcal{F}_1 \cap \mathcal{F}_2 \in \varphi_t(x)$.

Example (1.3.5):

Let $X = \{a_1, a_2, a_3\}$, and

$\tau = \{\emptyset, \{a_1\}, \{a_1, a_2\}, X\}$ be a topology on X, then set of all filters on X;

$\chi_1 = \{\{a_1\}, \{a_1, a_2\}, \{a_1, a_3\}, X\}$

$\chi_2 = \{\{a_1, a_2\}, X\}$

$\chi_3 = \{\{a_1, a_3\}, X\}$

$\chi_4 = \{\{a_2\}, \{a_1, a_2\}, \{a_2, a_3\}, X\}$

$\chi_5 = \{\{a_2, a_3\}, X\}$

$\chi_6 = \{\{a_3\}, \{a_1, a_3\}, \{a_2, a_3\}, X\}$

$\chi_7 = \{X\}$

Now to define pseudo topology on X, we first find the neighbors of all points in X;

$N_{a_1} = \{\{a_1\}, \{a_1, a_2\}, \{a_1, a_3\}, X\}$

$N_{a_2} = \{\{a_1, a_2\}, X\}$

$N_{a_3} = \{X\}$

Then $\chi_1 \in \Psi(a_1)$

$\chi_1, \chi_2, \chi_4 \in \Psi(a_2)$

$\chi_1, \chi_2, \chi_3, \chi_4, \chi_5 \chi_6 \in \Psi(a_3)$

Definition (1.3.6)[13]:

Let (X, γ) be a pseudo- topological space, and M is subset of X, then;

1. $cl(M) = \{x \in X : \exists \mathcal{F} \in P(F(X))\ such\ that\ \mathcal{F} \downarrow_x\ X\}$
2. $int(M) = \{x \in M :\ such\ that\ \mathcal{F} \downarrow_x\ X\ implies\ M \in\ \mathcal{F}\}$

Definition (1.3.7)[13]:

Let (X, γ) be a pseudo- topological space, and M is subset of X, then;

1. M is open if $M = int(M)$
2. M is closed if $M = cl(M)$

Definition (1.3.8)[13]:

The structure of $E_1$is called finer than that of E_2, and we write $E_1 \leq E_2$, iff $\chi \downarrow_x E_1 \Longrightarrow \chi \downarrow_x E_2$

Definition (1.3.9)[20] (Continuity):

Let E_1, E_2 be pseudo- topological space and $f: E_1 \rightarrow E_2$ a map, then f is continuous at the point $a \in E_1$ iff $\forall\, \chi \downarrow_a E_1 \Longrightarrow f(\chi) \downarrow_{f(a)} E_2$.

Remark (1.3.10)[20]:

Let X, Y *and* Z be a pseudo topology space. If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are continuous map, then gof is continuous map.

Definition (1.3.11)[9]:

Let X be a set and $(X_i)_{i\in I}$ a collection of pseudo topology spaces and for all $i \in I$, $g_i: X \rightarrow X_i$ a mapping. A filter $\chi \downarrow_x X$ (In the initial pseudo topology) with respect to $(g_i)_{i\in I}$ if and only for each $i \in I$, $g_i(\chi) \downarrow_{g_i(x)} X_i$.

Definition (1.3.12)[9]:

Let X be a set and $(X_i)_{i\in I}$ a collection of pseudo topology spaces and for each $i \in I$, $g_i: X_i \rightarrow X$ a mapping. A filter $\chi \downarrow_x X$ (In the final pseudo topology) with respect to $(g_i)_{i\in I}$ if and only if $\chi = [x]$ or if there are finitely many indices i_1 , , ... i_n points $x_k \in X_{i_k}$ and filters $\chi_k \downarrow_{x_k} X_{i_k}$ such that $g_{i_k}(x_k) = x$ for all k and $g_{i_1}(\chi_1) \cap \ldots \ldots \cap g_{i_n}(\chi_n) \subseteq \chi$.

Definition (1.3.13)[9]:

Let X be a pseudo topology space and let Y be a subset of X. The subspace pseudo topology structure on Y is the initial pseudo topology with respect to the inclusion mapping $i: Y \to X$.

Definition (1.3.14)[9]:

Let $(X_i)_{i\in I}$ a collection of pseudo topology spaces. The product pseudo topology structure on $\prod_{i\in I} X_i$ is the initial pseudo topology on $\prod_{i\in I} X_i$ with respect to the projections $\pi_i: \prod_{i\in I} X_i \to X_i$.

Definition (1.3.15)[20]:

Let the underlying space $\underline{E}$ of a pseudo topology space E be a vector space . The compatibility with a pseudo topology and the vector space is achieved if the operations of vector space are continuous.

Remarks(1.3.16)[20]:

1. Continuity of addition on pseudo topological vector space implies that the translation are homeomorphisms.

   Therefor :$\mathfrak{X} \downarrow_a E \Leftrightarrow \mathfrak{X} - a \downarrow_0 E$ or $\mathfrak{X} + a \downarrow_a E \Leftrightarrow \mathfrak{X} \downarrow_0 E$

   $$g(\mathcal{H}, \mathfrak{X}) = \mathcal{H} - \mathfrak{X} \text{ if } g(h, x) = h - x$$

2. When the filter $\mathfrak{X}$ converges to $0 \in \underline{E}$, we will write $\mathfrak{X} \downarrow E$ instead of $\mathfrak{X} \downarrow_0 E$

3. Continuity of the operations implies the following compatibility conditions:

i. $\mathcal{F}_1 \downarrow E \ , \mathcal{F}_2 \downarrow E \implies \mathcal{F}_1 + \mathcal{F}_2 \downarrow E$

ii. $\mathcal{F} \downarrow E \ , \lambda \in K \implies \lambda \bullet \mathcal{F} \downarrow E$

iii. $x \in E \implies$ ƫ $\bullet [x] \downarrow E$

iv. $\mathcal{F} \downarrow E \implies$ ƫ $\bullet \mathcal{F} \downarrow E$

Definition (1.3.17)[20]:

i. On pseudo topological vector space, filters $\mathcal{B}$ with the property $\mathcal{V}.\mathcal{B} \downarrow E$ is called quasi-bounded filters on E.

ii. A filter χ on a vector space is called an equable filter if and only if it has the property $\mathcal{V}.\chi = \chi$.

Definition (1.3.18)[20]:

A pseudo topological vector space E is called equable iff for each χ with $\chi \downarrow E$ there exists an equable filter $\mathcal{Y} \leq \chi$ with $\mathcal{Y} \downarrow E$; that is . iff $\chi \downarrow E \Longrightarrow \chi \geq \mathcal{Y} = \mathcal{V}.\mathcal{Y} \downarrow E$.

We can introduce on any pseudo topological vector space $\underline{E}$ a new pseudo topology, thus obtaining a new pseudo topological vector space $E^{\#}$. It is defined as follows:

1. $\underline{E^{\#}} = \underline{E}$
2. $\chi \downarrow E^{*}$ if and only if there exists $\mathcal{Y}$ with $\chi \geq \mathcal{Y} = \mathcal{V}.\mathcal{Y} \downarrow E$.

lemma (1.3.19)[20]:

Let X be a normed vector space. A filter χ is quasi-bounded if and only if it contains a bounded set.

Definition (1.3.20)[20]:

A map $f: E_1 \to E_2$ is called a quasi-bounded map iff it sends quasi-bounded filters into quasi-bounded filters, that is,

$\mathcal{V}.\chi \downarrow E_1 \Longrightarrow \mathcal{V}.f(\chi) \downarrow E_2$

The space of all equably continuous and quasi- bounded maps from E_1 into E_1 is denoted by $\underline{C}\,(E_1; E_2)$.

We denote by $\underline{L}\,(E_1; E_2)$ the vector space fromed by the linear continuous map from E_1 into E_2.

It is clear that $\underline{L}\,(E_1; E_2) \subset \underline{C}\,(E_1; E_2)$.

Chapter Two

Topological Algebra via Pseudo Structures

Introduction

This chapter consists of three sections:

Section one introduce new structure of algebra by using the language of filters to define a new class of algebras that we will call pseudo-topological algebra, we also mentioned the most important conditions that make algebra compatible with a pseudo- topological. Also we will present the most important topological properties and some important generalizations regarding pseudo-algebra topologies.

In section two, we define the initial and final pseudo topological algebra. Through it, we be able to identify the pseudo topological structure on subalgebra, product of algebra, projective algebra, quotients algebra.

In section three, we generated a locally convex topological algebra from a pseudo topology. We also study the relationship between it and the locally convex topology on algebra A that generated from the family of all continuous semi norms on algebra A.

2.1 Pseudo Topological Algebra

In this section, we define the concepts of a pseudo- topological algebra and give some examples with some of their properties and results.

Suppose that Қ the field of real or complex numbers, ʈ denotes its zero neighborhood filter of the field Қ and Ɨ$= \{s \in$ Қ$: |s| \leq 1\}$ is the unit interval on real numbers and the unit disk on complex numbers.

Definition (2.1.3)

Let the underlying space $\underline{E}$ of a pseudo topology be an algebra space. The compatibility of algebra with pseudo topology is achieved if the three operations of algebra are continuous. A pseudo topological algebra is algebra compatible with a pseudo topology on it .

Remarks(2.1.4):

1. If the underlying space $\underline{E}$ of pseudo topological space E be algebra with identity element e , then the continuity of multiplication implies that:

   $\mathfrak{X} \downarrow_a E \Leftrightarrow \mathfrak{X} * a^{-1} \downarrow_e E$ or

   $\mathfrak{X} * a \downarrow_a E \Leftrightarrow \mathfrak{X} \downarrow_e E$

   By $\mathfrak{X} * a^{-1}$ we denote the image of $\mathfrak{X}$ under the translation map $x \to x \cdot a^{-1}$

   $h(\mathcal{H}, \mathfrak{X}) = \mathcal{H} * \mathfrak{X}$ if $g(h,x) = h.x$ for all $h \in \mathcal{H}$ and $x \in \mathfrak{X}$.

2. Continuity of the operations implies the following compatibility conditions[20]:

i. $\mathcal{F}_1 \downarrow E$, $\mathcal{F}_2 \downarrow E \Longrightarrow \mathcal{F}_1 + \mathcal{F}_2 \downarrow E$

ii. $\mathcal{F}_1 \downarrow E$, $\mathcal{F}_2 \downarrow E \Longrightarrow \mathcal{F}_1 * \mathcal{F}_2 \downarrow E$

iii. $\mathcal{F} \downarrow E$, $\lambda \in K \Longrightarrow \lambda \bullet \mathcal{F} \downarrow E$

iv. $x \in E \Longrightarrow$ ţ $\bullet [x] \downarrow E$

v. $\mathcal{F} \downarrow E \Longrightarrow$ ţ $\bullet \mathcal{F} \downarrow E$

It is clear that a pseudo topological algebra is an algebra which is a pseudo topological vector space such that the multiplication $*: E \times E \to E$ is continuous.

Example(2.1.5)

i. Every topological algebra is a pseudo topological algebra.

ii. If X is a pseudo topological space and A is a pseudo topological algebra, then $C_c(X, A)$ is a pseudo topological algebra. In particular, $C_c(X)$ is a pseudo topological algebra. Where $C_c(X, A)$ denoted to the set of all continuous mapping from X *to* A.

iii. Let X and Y be algebra, and let $C(X, Y)$ denoted the set of all continuous mapping from X to Y. Then we define the evaluation mapping $\varpi_{X\times Y}(f,x): C(X,Y) \times X \to Y$ by $\varpi_{X\times Y}(f,x) = f(x)$ for all $f \in C(X,Y)$ and $x \in X$. It is clear that the set of all evaluation mapping $\varpi_{X\times Y}$ is algebra if the underlying spaces X and Y be algebra. If $\mathcal{K}$ is the filter on $C(X,Y)$ and $\mathfrak{X}$ is a filter on X, then $\mathcal{K}(\mathfrak{X})$ is the filter on $C(X,Y) \times X$. Now if X and Y be a pseudo topological algebra, then we define a pseudo topological algebra γ on $C(X,Y)$ as follows : $\mathcal{K} \downarrow_f C_\gamma(X,Y)$ if and only if $\mathcal{K}(\mathfrak{X}) \downarrow_{f(x)} C(X,Y) \times X$ and $\mathfrak{X} \downarrow_x X$.

We will discuss this type of algebra in detail in Chapter three.

We first start by discussing a special case, when we have it base of neighborhood filter of 0.

Theorem(2.1.6)

The filterbase β in the algebra A determines a base at 0 for compatible with a pseudo topology for A iff ;

i. β is a neighborhood base at 0 for a pseudo topology which is compatible with A's linear structure
ii. for each $u \in \beta$ there exists a $v \in \beta$ such that $vv \subset u$.

Proof:

First, we mention a fact in functional analysis [33] where A is topological vector space, then every neighborhood of zero contains a balanced neighborhood of zero.

The first direction is clear therefore we need to prove the second direction Let $U(0)$ denote the filter of neighborhoods of 0 determined by β.

To prove that $(x,y) \to xy$ is continuous,

consider the neighborhood of xy is $xy + u$ where $u \in U(0)$,

and choose a balanced neighborhood B of 0 such that

$B + B + B \subset u$, then by (ii) there exists a $v \in \beta$ such that $vv \subset B$.

Choose a real number , $0 < a < 1$, such that $ax \in v$, $ay \in v$

Since$(x + av) * (y + av) = xy + axv + ayv + a^2vv$ (by Remarek 1.2.47)

$$\subset xy + vv + vv + a^2B$$

$$\subset xy + B + B + B$$

$$\subset xy + u.$$

This we have $(x + av)(y + av) \subset xy + u$.

Then $(x + v)(y + v) \subset xy + u$.

And so that $(x, y) \to xy$ is continuous.

■

Proposition(2.1.7):

Let A be an algebra over the field K and let M be a family of filters on A such that the following things are achieved:

i. If $\mathcal{F}, \mathcal{H} \in M$ then $\mathcal{F} \cap \mathcal{H} \in M$, $\mathcal{F} + \mathcal{H} \in M$ and $\mathcal{F} * \mathcal{H} \in M$
ii. If $\mathcal{F} \in M$ and $\mathcal{F} \subseteq \mathcal{H}$ then $\mathcal{H} \in M$
iii. If $\mathcal{F} \in M$ then ţ$\mathcal{F} \in M$
iv. If $\mathcal{F} \in M$ then $\alpha\mathcal{F} \in M$ for all $\alpha \in K$
v. ţ$a \in M$ for all $a \in A$

Then the mapping Ψ from the algebra A into the power set of all filters on A such that $\mathcal{F} \in \Psi(\mathrm{a})$ if and only if $\mathcal{F} - a \in M$ is a pseudo topology that compatible with algebra structure.

Proof:

It is clear that Ψ is a pseudo topology. In order to prove that the three operations of algebra are continuous;

1. Let $\mathcal{F} \downarrow_a A$ and $\mathcal{H} \downarrow_h A$ then $\mathcal{F} - a \in M$ and $\mathcal{H} - h \in M$

Since $\mathcal{F}+\mathcal{H}-(a+h)=(\mathcal{F}-a)+(\mathcal{H}-h)$

But $(\mathcal{F}-a)+(\mathcal{H}-h)\in M$,

this $\mathcal{F}+\mathcal{H}-(a+h)\in M$ and so that

$$\mathcal{F}+\mathcal{H}\downarrow_{a+h} A$$

2. To prove that the scalar multiplication is continuous, let $\mathcal{F}\downarrow_a A$ and s $\in K$, then ţ(s) $=$ s $+$ ţ denoted the neighborhood filter of s in K, since $\mathcal{F}\downarrow_a A$ then $\mathcal{F}-a\in M$

ţ(s)$\mathcal{F}-$s$a=$ (s $+$ ţ)(($\mathcal{F}-a)+a)-$sa

$=$ s$(\mathcal{F}-a)+$ ţ$(\mathcal{F}-a)+$s$a+$ţ$a-$sa

$=$ s$(\mathcal{F}-a)+$ ţ$(\mathcal{F}-a)+$ţa

The right side belongs to M and that's because;

s$(\mathcal{F}-a)\in M$, ţ$(\mathcal{F}-a)\in M$ and ţ$a\in M$

thus ţ(s)$\mathcal{F}-$s$a\in M$ and, therefore we get ţ(s)$\mathcal{F}\downarrow_{sa} A$.

3. Let $\mathcal{F}\downarrow_a A$ and $\mathcal{H}\downarrow_h A$ then $\mathcal{F}-a\in M$ and $\mathcal{H}-h\in M$

Since $\mathcal{F}\mathcal{H}-(ah)\supseteq(\mathcal{F}-a)(\mathcal{H}-h)$

But $(\mathcal{F}-a)(\mathcal{H}-h)\in M$, this $\mathcal{F}\mathcal{H}-(ah)\in M$ and so that

$$\mathcal{F}\mathcal{H}\downarrow_{ah} A$$

■

<u>Theorem (2.1.8)</u>

Let X and Y be an algebra. If $f: X\to Y$ a homomorphism algebra then X is a pseudo topological algebra if and only if Y is a pseudo topological algebra.

<u>Proof:</u>

Suppose that X is an algebra, Y is a pseudo topological algebra with neighborhood filter at 0 denoted by $v(0)$, and

$f: X\to Y$ a homomorphism.

We will prove that the filter $f^{-1}(v(0))$ determines a pseudo topology which is compatible with $X's$ algebra structure.

To see that it is compatible with the algebraic structure as well,

we first note that for any $v \in v(0)$,

there is a $B \in v(0)$ such that $BB \in v$ (by Theorem 2.1.6)

Hence $f^{-1}(B)f^{-1}(B) \subset f^{-1}(BB) \subset f^{-1}(v)$.

The pseudo topology determined by $f^{-1}(v(0))$ is called the initial pseudo topology induced by the homomorphism f.

Now suppose that X is a pseudo topological algebra with neighborhood filter at 0 denoted by $v(0)$, Y an algebra, , and $f: X \to Y$ a homomorphism. We will prove that the collection β of subsets U of Y such that the $f^{-1}(U) \in v(0)$ from a base at 0 for a pseudo topology compatible with Y's linear structure.

To see that for any $U \in \beta$, we may select $B \in v(0)$ such that $BB \in f^{-1}(U)$.

Hence $f(B)f(B) = f(BB) \subset f(f^{-1}(U)) \subset U$.

Since $f^{-1}(f(B)) \supset B \in v(0)$

It follows that $f^{-1}(f(B)) \in v(0)$.

■

Remarks(2.1.9)

Now we can construct subspaces, products, and projective limits by the methods involve initial and final pseudo topological algebra.

i. Let $(A_i)_{i\in I}$ be a collection of pseudo topological algebra and $\prod_{i\in I} A_i$ be the product set of the A_i. The product of pseudo topological algebra on $\prod_{i\in I} A_i$ is the initial of pseudo topological algebra with respect to the projection mapping $(P_i: \prod_{i\in I} A_i \to A_j)_{j\in I}$. Such that $\mathfrak{X} \downarrow_a \prod_{i\in I} A_i$ if and only if $P_i(\mathfrak{X}) \downarrow_{P_{i(a)}} A_i$.

ii. Let A be a pseudo topological algebra and let B be a subalgebra of A. Then we can construct a pseudo topological subalgebra on B is the initial pseudo topological algebra with respect to the inclusion mapping $i: B \to A$ such that $\chi \downarrow_x B$ if and only if $\chi \downarrow_x A$.

Proposition(2.1.10):

Let A and B be a pseudo topological algebra and let $f: A \to B$ be a continuous mapping. Also assume that $B^{\wedge} \subseteq B$ and $A^{\wedge} \subseteq f^{-1}(B)$ then the restriction map $f^{\wedge}: A^{\wedge} \to B^{\wedge}$ is continuous.

Proof:

Assume $\chi \downarrow_x A^{\wedge}$ then $\chi \downarrow_x A \Longrightarrow f^{\wedge}(\chi) = f(\chi) \downarrow_{f(x)} B^{\wedge} \Longrightarrow f^{\wedge}(\chi) \downarrow_{f(x)} B^{\wedge}$.

In particular $f / A^{\wedge}$ is continuous for all $A^{\wedge} \subseteq A$ and the codomain restriction $f: A \to f(A)$ is continuous.

■

Now we will present the most important topological properties and some important generalizations regarding pseudo-algebra topologies.

Definition (2.1.11)

Let M be subset of a pseudo topological algebra A, then M said to be multiplicative set if $M \cdot M \subseteq M$.

Definition (2.1.12)

A filter $\mathcal{F}$ on algebra A is said to be multiplicative filter if it is generated by the multiplicative sets of A and is said to be multiplicative convex filter when it generated by the multiplicative convex sets of A.

Proposition(2.1.13):

Let $\mathcal{F}$ be a multiplicatively convex filter on a pseudo topological algebra A. Then we associate a seminorm $p^{\mathcal{F}}$ with this filter $\mathcal{F}$ such that $p^{\mathcal{F}}(a) = \inf\{K \ni \lambda > 0: for\ all\ \mathcal{F} \downarrow_a A,\ \lambda\mathcal{F} \downarrow A\}$. This seminorm is multiplicative.

Proof:

It is clear that $p^{\mathcal{F}}$ is seminorm, so we have to prove that is a multiplicative seminorm,

Let $a, b \in A$ and there are $\lambda_1, \lambda_2 > 0$ such that

$p^{\mathcal{F}}(a) = \lambda_1$ and $p^{\mathcal{F}}(b) = \lambda_2$,

This we have $\lambda_1 \mathcal{F} \downarrow A$ and $\lambda_2 \mathcal{F} \downarrow A$

But by Remark(2.1.4), we get $\lambda_1 \mathcal{F}. \lambda_2 \mathcal{F} \downarrow A$

$\lambda_1 \lambda_2 \mathcal{F}. \mathcal{F} \downarrow A$ But, $\mathcal{F}. \mathcal{F} \subseteq \mathcal{F}$, this $\lambda_1 \lambda_2 \mathcal{F} \downarrow A$ and therefor

$$p^{\mathcal{F}}(ab) \leq \lambda_1 \lambda_2$$

■

Definition (2.1.14)

A filter $\mathcal{F}$ on a pseudo topological algebra A is said to be quasi-bounded if $ţ\mathcal{F} \downarrow A$.

Definition (2.1.15)

A set B on a pseudo topological algebra A is said to be bounded if $[B]$ is quasi-bounded, where $[B]$ means the filter that generated by B.

Definition (2.1.16)

A filter $\mathcal{F}$ on a pseudo topological algebra A is said to be equable if $ţ\mathcal{F} = \mathcal{F}$.

Definition (2.1.17)

A filter $\mathcal{F}$ in a pseudo topological algebra A is said to be Cauchy filter if $(\mathcal{F} - \mathcal{F}) \downarrow_0$ where $(\mathcal{F} - \mathcal{F})$ we mean by it the filter that is generated by $\{F - F' : \ F, F' \in \mathcal{F}\}$.

Definition (2.1.18)

A pseudo topological algebra A is said to be equable if for all filter $\mathcal{F} \downarrow A$ there exists a coarser equable filter $ƴ \leq \mathcal{F}$ such that $ţƴ \downarrow A$.

Proposition (2.1.19)

Let A be a pseudo topological algebra and let $A^{\#}$ define as fallows:

i. $\underline{A^{\#}} = \underline{A}$

ii. $\mathcal{F} \downarrow A^{\#}$ if and only if there exists a filter $\mathcal{G} \leq \mathcal{F}$ and $t\mathcal{G} = \mathcal{G}$ such that $t\mathcal{G} \downarrow A$.

Then $A^{\#}$ is equable.

Proof:

To prove that this filters on $A^{\#}$ satisfies the Definition (1.3.4), we prove the seconed and third conditions;

For the seconed condition, let $\mathcal{F}_1 \downarrow A^{\#}$, $\mathcal{F}_2 \downarrow A^{\#}$

Then there are $\mathcal{G}_1 \leq \mathcal{F}_1$, $\mathcal{G}_2 \leq \mathcal{F}_2$, such that $t\mathcal{G}_1 = \mathcal{G}_1$, $t\mathcal{G}_2 = \mathcal{G}_2$

And $t\mathcal{G}_1 \downarrow A$, $t\mathcal{G}_2 \downarrow A$

Since $\mathcal{G}_1 \cap \mathcal{G}_2 \subseteq \mathcal{F}_1 \cap \mathcal{F}_2$,

And $t\mathcal{G}_1 \cap t\mathcal{G}_2 = t(\mathcal{G}_1 \cap \mathcal{G}_2) = \mathcal{G}_1 \cap \mathcal{G}_2$

This we have $(\mathcal{F}_1 \cap \mathcal{F}_2) \downarrow A^{\#}$

To prove the third condition of Definition(1.3.4), let $\mathcal{F}_1 \downarrow A^{\#}$ and $\mathcal{F}_1 \leq \mathcal{F}_2$

Frome $\mathcal{F}_1 \downarrow A^{\#}$, we get there exists a $\mathcal{G}_1 \leq \mathcal{F}_1$, $t\mathcal{G}_1 = \mathcal{G}_1$ and $t\mathcal{G}_1 \downarrow A$

But $\mathcal{F}_1 \leq \mathcal{F}_2$, then $\mathcal{G}_1 \leq \mathcal{F}_1 \leq \mathcal{F}_2$ and so that $\mathcal{F}_2 \downarrow A^{\#}$.

■

Proposition (2.1.20)

A Pseudo- topological algebra A is a Hausdorff iff $\{0\}$ is closed set.

Proof :

Assume that A is a Hausdorff, then A is a $T_1 - space$, and hence $\{0\}$ is closed.

Conversely, let $\{0\}$ is closed, and suppose that $\mathcal{F} \downarrow_x A$ and $\mathcal{F} \downarrow_y A$

Then $\mathcal{F} - \mathcal{F} \downarrow_{x-y} A$ (by cointinuous of addition)

Since $\mathcal{F} - \mathcal{F} \subseteq [0] \Longrightarrow [0] \downarrow_{x-y} A$

This mean $x-y \in \overline{[0]}$, since $\{0\}$ is closed then $\overline{[0]} = \{0\}$ and hence $x-y=0 \Longrightarrow x=y \Longrightarrow A$ is Hausdorff.

■

Proposition(2.1.21):

Let A and B be two pseudo topological algebras. If $f: A \to B$ is continuous mapping and $\mathcal{O}$ is open set in B then $f^{-1}(\mathcal{O})$ is open set in A.

Proof:

Let $a \in f^{-1}(\mathcal{O})$, then $f(a) \in f(f^{-1}(\mathcal{O}))$

Since $f(f^{-1}(\mathcal{O})) \subseteq \mathcal{O}$

This we have $f(a) \in \mathcal{O}$

For all $\mathcal{F} \downarrow_a A$, then $f(\mathcal{F}) \downarrow_{f(a)} B$ (by continuous of f)

Since $\mathcal{O}$ is open in B

Then $\mathcal{O} \in f(\mathcal{F})$, and therefor, there exists a set such $F \in \mathcal{F}$ and $f(F) \subseteq \mathcal{O}$

Hence $F \subseteq f^{-1}(\mathcal{O})$

This we have $f^{-1}(\mathcal{O}) \in \mathcal{F}$.

This means $f^{-1}(\mathcal{O})$ belongs to every filter that converging to a

Hence $f^{-1}(\mathcal{O})$ is open in A.

■

Proposition(2.1.22):

Let $(A_i)_{i\in I}$ be a collection of pseudo topological algebra. A filter χ converge to $\varepsilon \in \prod_{i\in I} A_i$ or $\chi \downarrow_\varepsilon \prod_{i\in I} A_i$ if and only if for all $i \in I$ *there are filters* $\chi_i \downarrow_{\varepsilon(i)} A_i$ where $\varepsilon \in \prod_{i\in I} A_i$ such that $\prod_{i\in I} \chi_i \subseteq \chi$.

Proof:

Since $P_j(\prod_{i\in I} \chi_i) = \chi_j \ \forall\, j \in I$, then the product filter convergence if all component filters do

That is : If $\chi \downarrow_\varepsilon \prod_{i\in I} A_i$ then $\chi \supseteq \prod_{i\in I} P_j(\chi)$ and this gives the desired result.

■

Proposition(2.1.23):

Let A be a pseudo topological algebra. Then:

i. If B_1 is bounded subset of A and $B_2 \subseteq B_1$ then B_2 is also bounded.

ii. For every bounded subset B of A, αB is also bounded when $\alpha \in K$.

iii. If B_1, B_2 bounded subsets of A, then $B_1 \cup B_2$, $B_1 + B_2$ are bounded.

iv. If B_1 is multiplicative bounded subset of A, then $B_1 \cdot B_1$ is bounded subset of A

Proof:

i. Since $B_2 \subseteq B_1$, then $[B_1] \subseteq [B_2]$
then if $ţ[B_1] \downarrow A$ this achieves $ţ[B_2] \downarrow A$ and therefor B_2 is bounded.

ii. Since $ţB \downarrow A$, then $ţ\alpha B = \alpha ţB \downarrow A$, this αB is also bounded.

iii. If B_1, B_2, then $ţ[B_1] \downarrow A, ţ[B_2] \downarrow A$,
this $(ţ[B_1] \cup ţ[B_2]) \downarrow A$
and therefor $ţ([B_1] \cup [B_2]) \downarrow A$
since $([B_1] \cup [B_2]) \subseteq [B_1 \cup B_2]$,
then we have $ţ[B_1 \cup B_2] \downarrow A$
the proof of $B_1 + B_2$ is in the same way.

iv. Since $B_1 \cdot B_1 \subseteq B_1$, then $[B_1] \subseteq [B_1 \cdot B_1]$
If $ţ[B_1] \downarrow A$ then we have $ţ[B_1 \cdot B_1] \downarrow A$
and therefor $B_1 \cdot B_1$ is bounded

■

2.2 Initial and Final Pseudo Topological Algebra Structures

The formation of a pseudo-topological structure from other given structures that we be given in this section is called initial and final pseudo topological algebra. Through it, we be able to identify the pseudo topological structure on subalgebra, product of algebra, projective algebra, quotients algebra.

First we construct a new pseudo topological algebra from given ones. The initial pseudo topology on a space from a family of a pseudo topology spaces.

Definition (2.2.1):

Let X be a set and $(X_i)_{i\in I}$ a collection of pseudo topology spaces and for each $i \in I$, $g_i: X \to X_i$ a continuous mapping. A filter $\chi \downarrow_x X$ (In the initial pseudo topology) with respect to $(g_i)_{i\in I}$ if and only if for each $i \in I$, $g_i(\chi) \downarrow_{g_i(x)} X_i$.

Lemma (2.2.2):

Let A, B be a pseudo topology spaces and let $(B_i)_{i\in I}$ families of pseudo topology spaces and for each $i \in I$, $g_i: B \to B_i$ (B initial of pseudo topology spaces with respect to $(g_i)_{i\in I}$) be continuous mapping then the mapping $g: A \to B$ is continuous if and only if $g_i o g: A \to B_i$ is continuous for all $i \in I$.

Proof:

The first direction is clear because the composition of two continuous mapping is also continuous.

Conversely, suppose that $\mathcal{F} \downarrow_x A$ then $g_i o g(\mathcal{F}) \downarrow_x B_i$

Since B initial of pseudo topology space with respect to $(B_i)_{i\in I}$ and g_i then by the definition (2.2.1) we have $g(\mathcal{F}) \downarrow_x B$ and that g is continuous.

■

Proposition (2.2.3):

Let A be algebra, $(A_i)_{i\in I}$ be a family of a pseudo topological algebra and for $i \in I$, $g_i: A \to A_i$ be continuous algebra homomorphism. Then the initial pseudo topological structure on A with respect to $(g_i)_{i\in I}$ is pseudo topological algebra.

Proof:

By the Definition (2.2.1) A is pseudo topological structure, to prove that this pseudo topological structure compatible with algebra we must prove that the algebra operations; (+) the addition operation, (•) the scalar multiplication and (*) the vector multiplication are continuous.

i. In order to prove that $(+): A \times A \to A$ is continuous it is sufficient to show that $g_i \circ (+): A \times A \to A_i$ is continuous for all $i \in I$. (by using Lemma 2.2.2). This is shown through the commutatively of the following diagram

$$\begin{array}{ccc} (+): A \times A & \longrightarrow & A \\ \downarrow_{g_i \times g_i} & & \downarrow_{g_i} \\ (+): A_i \times A_i & \longrightarrow & A_i \end{array}$$

ii. The continuity of the scalar multiplication from the commutatively of the following diagram:

$$\begin{array}{ccc} (\bullet): K \times A & \longrightarrow & A \\ \downarrow_{id \times g_i} & & \downarrow_{g_i} \\ (\bullet): K \times A_i & \longrightarrow & A_i \end{array}$$

Since $g_i o (\bullet)$is continuouse

By using Lemma 2.2.2 we have $(\bullet)$ is continuouse .

iii. Now to prove the continuity of the vector multiplication

$$\begin{array}{ccc} (*): A \times A & \longrightarrow & A \\ \downarrow_{g_i \times g_i} & & \downarrow_{g_i} \\ (*): A_i \times A_i & \longrightarrow & A_i \end{array}$$

Since $g_i o (*)$is continuouse

By using Lemma 2.2.2 we have $(*)$ is continuouse .

■

Examples (2.2.4):

i. Let $(A_i)_{i\in I}$ be a collection of pseudo topological algebra and $\prod_{i\in I} A_i$ be the product set of the A_i. The product of pseudo topological algebra on $\prod_{i\in I} A_i$ is the initial of pseudo topological algebra with respect to the projection mapping $(P_i: \prod_{i\in I} A_i \to A_j)_{i\in I}$. Such that $\mathfrak{X} \downarrow_a \prod_{i\in I} A_i$ if and only if $P_i(\mathfrak{X}) \downarrow_{P_i(a)} A_i$.

ii. Let A be a pseudo topological algebra and B be a subalgebra of A. Then we can construct a pseudo topological sub-algebra on B is the initial pseudo topological algebra with respect to the inclusion mapping $i: B \to A$ such that $\chi \downarrow_x B$ if and only if $\chi \downarrow_x A$.

Proposition (2.2.5):

Suppose $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are families of pseudo topological algebra spaces. If for each $i \in I$, $g_i: A_i \to B_i$ be algebra homomorphism, then the mapping $g: \prod_{i\in I} A_i \longrightarrow \prod_{i\in I} B_i$ defined by $g(a) = (g_i(a_i))$ is continuous if for each g_i is continuous.

Proof:

Let $\alpha_j: \prod_{i\in I} A_i \to A_j$ and $\psi_j: \prod_{i\in I} B_i \to B_j$ the projection maps and assume the commutative diagram:

$$\begin{array}{ccc} g: \prod_{i\in I} A_i & \longrightarrow & \prod_{i\in I} B_i \\ \downarrow_{\alpha_j} & \searrow & \downarrow_{\psi_j} \\ g_j: A_j & \longrightarrow & B_j \end{array}$$

Then we have $\psi_j \circ g = g_j \circ \alpha_j$ and since each the mapping g_j and α_j are continuous then mapping $\psi_j \circ g$ is continuous, which means that g is continuous by Lemma (2.2.2).

We recall that a pseudo topological space become a topological space when the following condition is satisfied:

For all filter $\mathcal{F}$, $\mathcal{F} \downarrow_x A$ if and only if $N_x \subseteq \mathcal{F}$.

■

Proposition(2.2.6):

Let $(A_i)_{i\in I}$ be a collection of pseudo topological algebra and let $(g_i: A \to A_i)_{i\in I}$ be a family of continuous algebra homomorphism such that A have the initial of pseudo topological algebra with respect to $(g_i)_{i\in I}$. If each A_i is topological algebra, then A is topological algebra.

Proof:

We have to proof that $\mathcal{F} \downarrow_x A$ if and only if $N_x \subseteq \mathcal{F}$

let $\mathcal{F} \downarrow_x A$ and $B \in N_x \Longrightarrow there\ exists\ open\ set\ O \subseteq B$ and $x \in O$

Then for all $i \in I$ there exist open set $\omega_i \subseteq A_i$ such that $x \in {g_i}^{-1}(\omega_i) \subseteq O \subseteq B$

Since $\mathcal{F} \downarrow_x A \Longrightarrow g_i(\mathcal{F}) \downarrow_{g_i(x)} A_i \Longrightarrow \omega_i \in N_{g_i(x)} \subseteq g_i(\mathcal{F})$

Thus there exists $F \in \mathcal{F}$ such that $g_i(F) \subseteq \omega_i \Longrightarrow F \subseteq {g_i}^{-1}(\omega_i) \subseteq B \Longrightarrow N_x \subseteq \mathcal{F}$.

Conversely: Let $B \in N_{g_i(x)}$. Then there exists an open set O such that $g_i(x) \in O \subseteq B$

Since ${g_i}^{-1}(O)$ is open set and $x \in {g_i}^{-1}(O)$,

then ${g_i}^{-1}(B) \in N_x$

But $g({g_i}^{-1}(B)) \subseteq B \in g_i(N_x)$

Then we have $N_{g_i(x)} \subseteq g_i(N_x)$

$$\Longrightarrow g_i(\mathcal{F}) \downarrow_{g_i(x)} A_i \text{ for all } i \in I$$

Hence $\mathcal{F} \downarrow_x A$

■

We can now construct a final pseudo- topological algebra through another family of a pseudo topological algebra as follows:

Proposition (2.2.7):

Let A be algebra, $(A_i)_{i \in I}$ be a family of a pseudo topological algebra and let $g_i : A_i \to A$ be continuous algebra homomorphism, then $\mathcal{F} \downarrow_x A$ (as a final pseudo- topological algebra) if and only if for all $i \in I$ there exist a filter $\chi_i \downarrow_{g_i^{-1}(x)} A_i$ and $g_i^{-1}(\mathcal{F}) \subseteq \chi_i$.

Proof:

To prove that the filter set fulfills the conditions for a pseudo-topology definition, we will prove the first condition, and the rest of the two conditions are clear:

Now let $\mathcal{F}_1 \downarrow_x A$, $\mathcal{F}_1 \leq \mathcal{F}_2$, then $g_i^{-1}(\mathcal{F}_1) \leq g_i^{-1}(\mathcal{F}_2)$

since $\mathcal{F}_1 \downarrow_x A$ if and only if for all $i \in I$ there exist filters $\chi_i \downarrow_{g_i^{-1}(x)} A_i$ and $g_i^{-1}(\mathcal{F}_1) \subseteq \chi_i$,

and since $g_i^{-1}(\mathcal{F}_1) \leq g_i^{-1}(\mathcal{F}_2)$,

then $g_i^{-1}(\mathcal{F}_2) \subseteq \chi_i$, hence $\mathcal{F}_2 \downarrow_x A$.

To prove that pseudo- Topology is compatible with algebra, we must fulfill the five compatibility conditions, we will suffice to fulfill the first and second conditions, as for the rest of the conditions, they are clear

i. Let $\mathcal{F}_1 \downarrow A$ and $\mathcal{F}_2 \downarrow A$ then there are two sets of filters χ_i and Υ_i such that for all $i \in I$ $\chi_i \downarrow_{g_i^{-1}(0)} A_i$, $\Upsilon_i \downarrow_{g_i^{-1}(0)} A_i$
and $g_i^{-1}(\mathcal{F}_1) \subseteq \chi_i$, $g_i^{-1}(\mathcal{F}_2) \subseteq \Upsilon_i$ $\forall i \in I$.
$g_i^{-1}(\mathcal{F}_1)$+ $g_i^{-1}(\mathcal{F}_2) \subseteq \chi_i + \Upsilon_i$ $for\ all\ i \in I$
$g_i^{-1}(\mathcal{F}_1 + \mathcal{F}_2) \subseteq \chi_i + \Upsilon_i$ $for\ all\ i \in I$
Since each A_i is a pseudo topological algebra
then $\chi_i + \Upsilon_i \downarrow A_i$, hence $\mathcal{F}_1 + \mathcal{F}_2 \downarrow A$.

ii. We now apply the filters $\mathcal{F}_1, \mathcal{F}_2$as above (i) then we have:
$g_i^{-1}(\mathcal{F}_1) \subseteq \chi_i$, $g_i^{-1}(\mathcal{F}_2) \subseteq \Upsilon_i$ and this leads us to
$g_i^{-1}(\mathcal{F}_1) * g_i^{-1}(\mathcal{F}_2) \subseteq \chi_i * \Upsilon_i$ $for\ all\ i \in I$

$\Longrightarrow g_i^{-1}(\mathcal{F}_1 * \mathcal{F}_2) \subseteq \chi_i * \Upsilon_i \quad for\ all\ i \in I$

Since each filters χ_i, Υ_i from a pseudo topological which are compatible with algebra A_i, hence $\mathcal{F}_1 * \mathcal{F}_2 \downarrow A$.

■

Definition (2.2.8):

Let A be pseudo topological algebra, $(A_i)_{i\in I}$ be a family of a pseudo topological algebra and for $i \in I$, $g_i: A_i \to A$ be algebra homomorphism. If Ψ is the family of all the pseudo topological algebra on A making all g_i continuous, then the initial pseudo topological structure on A with respect to $id: A \to (A, \gamma)_{\gamma\in\Psi}$ is called the final pseudo topological algebra with respect to $(g_i: A_i \to A\ )_{i\in I}$.
Her γ denoted by the pseudo topological structurues

Example (2.2.9)

The linear subspace J of an algebra A is an ideal if for all $a \in A$ then $aJ \subset J$.
If A is a pseudo topology on algebra and $A\,/\,J$ carries the final pseudo topology induced by the canonical homomorphism from A onto $A\,/\,J$ $a \to a + J$, then $A\,/\,J$ is a pseudo topology on algebra by definition (2.2.8)
It is clear that if A is multiplicative set, then $A\,/\,J$ also.

Proposition(2.2.10):

Let A a pseudo topological algebra, $B \subseteq A$ a sub-algebra. Then A/B is a pseudo topological algebra.

Proof:

To prove that A/B is a pseudo topological algebra, we must prove the continuous of the algebra operations, (+) the addition operation, (•) the scalar multiplication and ($*$) the vector multiplication.

i. Let $p: A \to A/B$ is the projection map

Assume that $\mathcal{F}_1 \downarrow_{p(x_1)} A/B$ and $\mathcal{F}_2 \downarrow_{p(x_2)} A/B$

then there are $\chi_1 \downarrow_{x_1} A$ and $\chi_2 \downarrow_{x_2} A$

such that $p(\chi_1) \subseteq \mathcal{F}_1$ and $p(\chi_2) \subseteq \mathcal{F}_2$.

$p(\chi_1) \downarrow_{p(x_1)} A/B$ and $p(\chi_2) \downarrow_{p(x_2)} A/B$ $\Longrightarrow$ $p(\chi_1) + p(\chi_2) \downarrow_{p(x_1)+p(x_2)} A/B$

But $p(\chi_1) + p(\chi_2) \subseteq \mathcal{F}_1 + \mathcal{F}_2$

$\Longrightarrow$ $\mathcal{F}_1 + \mathcal{F}_2 \downarrow_{p(x_1)+p(x_2)} A/B$.

ii. Let $\mathcal{F} \downarrow_x A/B$

Then by Definition(2.2.7), there exists a filter χ such that $\chi \downarrow_{p^{-1}(x)} A$, and $p(\chi) \subseteq \mathcal{F}$, this we have $\lambda \cdot p(\chi) \subseteq \lambda \cdot \mathcal{F}$

Since A a pseudo topological algebra and $\chi \downarrow_{p^{-1}(x)} A$, then

$\lambda \cdot \chi \downarrow_{\lambda p^{-1}(x)} A$

iii. To prove that $(*)$ is continuous, it similar proof of (i)

■

Proposition(2.2.11):

Let A be pseudo topological algebra that holds the final pseudo topological algebra with respect to $(g_i: A_i \to A)_{i \in I}$. Then a homomorphism algebra $g: A \to B$, where B another pseudo topological algebra is continuous if and only if $gog_i: A_i \to B$ is continuous for all i.

Proof:

If g is continuous then gog_i is continuous for all i, since g_i is continuous and the composition of two continuous mapping is also continuous.

Now let gog_i is continuous for all i, and γ be the initial pseudo topological structures on A,

then γ is pseudo topological algebra by Proposition(2.2.3).

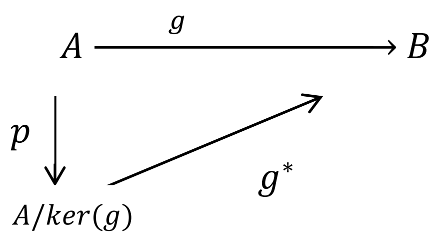
Also all $g_i: A_i \to (A, \gamma)$ are continuous.

Therefore $id: A \to (A, \gamma)$ is continuous hence $g: A \to B$ is continuous.

■

Proposition(2.2.12):

Let $g: A \to B$ be a continuous homomorphism algebra between two of pseudo topological algebras A and B. Then there is injective continuous homomorphism algebra $g^*: A/\ker(g) \to B$ such that the following diagram is commutes:



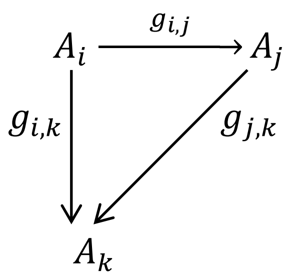
Proof:

It clear that $g^*: A/ker(g) \to B$ is injective homomorphism algebra.
By using Lemma (2.2.2),
The continuity to g^*: comes from the continuity of $g^* op = g$

■

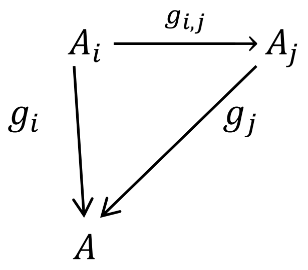
Definition (2.2.13)

Let $(A_i)_{i \in I}$ be a collection of a pseudo topological algebras and for all $i \le j$ let $g_{i,j}: A_i \to A_j$ be continuous mapping, then $((A_i)_{i \in I}, (g_{i,j}))$ is called inductive system of a pseudo topological algebras if the following diagram is commutes for all $\le j \le k$:



Definition (2.2.14)

A pseudo topological algebra A is said to be compatible with the inductive system $((A_i)_{i\in I},(g_{i,j}))$ if for all $i \in I$, $g_i: A_i \to A$ is a continuous homomorphism and the following diagram is commutes for all $i \leq j$



Proposition(2.2.15):

Let $((A_i)_{i\in I},(g_{i,j}))$ be inductive system of a pseudo topological algebras. Assume that A is algebra and for all $i \in I, g_i: A_i \to A$ is a continuous homomorphism such that $(A,(g_i))$ is compatible with the inductive system $((A_i)_{i\in I},(g_{i,j}))$. If A is algebra generated by $\bigcup_{i\in I} g_i(A_i)$. Then the final pseudo topological on A is a pseudo topological algebra.

Proof:

Let $a \in A$, then there are indices $\alpha_1, \alpha_2, \ldots \alpha_j$ and elements $b_k \in A_{\alpha k}$ such that $a = g_{\alpha 1}(b_1) + \cdots + g_{\alpha k}(b_k)$

$\Longrightarrow$ ţa = ţ$(g_{\alpha 1}(b_k) + \cdots + g_{\alpha k}(b_k))$

But $g_{\alpha 1}($ţ$b_1) + \cdots \ldots + g_{\alpha k}($ţ$b_k) \subseteq$ ţ$(g_{\alpha 1}(b_1) + \cdots + g_{\alpha k}(b_k))$

Since ţ$a \downarrow A_k$

Hence $(g_{\alpha 1}($ţ$b_1) + \cdots \ldots + g_{\alpha k}($ţ$b_k)) \downarrow A_k$

■

Definition (2.2. 16)

Let $(A_i)_{i\in I}$ be a family of a pseudo topological algebra and $\bigoplus_{i\in I} A_i$ be the algebraic coproduct of $(A_i)_{i\in I}$,

That is $\bigoplus_{i\in I} A_i = \{\omega \in \prod_{i\in I} A_i : \omega(i) \neq 0 \; for\ at\ most\ finitely\ many\ i\}$.

let $e_j: A_i \to \oplus_{i\in I} A_i$ the natural injection. Then the pseudo topology that endowed with $\oplus_{i\in I} A_i$ as the final pseudo topological algebra is called pseudo topology coproduct of $(A_i)_{i\in I}$.

Proposition(2.2.17):

Let $(A_i)_{i\in I}$ be a family of a pseudo topological algebra. For all finite subset $J \subseteq I$, let $e_j: \prod_{i\in J} A_i \to \oplus_{i\in I} A_i$ the natural injection. Then $\oplus_{i\in I} A_i$ carries the final pseudo topology structure with respect to e_j.

Proof:

Let $A = \oplus_{i\in I} A_i$,

We denote to the final pseudo topology structure on A by γ with respect to e_j

For all $J \subseteq K \subseteq I$ the natural injection $e_{J,K}: \prod_{i\in J} A_i \to \prod_{i\in K} A_i$

Then we have $(\prod_{i\in J} A_i, e_{J,K})$ is an inductive system

So that $(\oplus_{i\in I} A_i, e_J)$ compatible with an inductive system $(\prod_{i\in J} A_i, e_{J,K})$

Then γ is pseudo topological algebra

Since $e_i: A_i \to (A, \gamma)$ is continuous for all $i \in I$, and $i_d: A \to (A, \gamma)$ is continuous.

Let $\mathcal{F} \downarrow \prod_{i\in J} A_j$,

Then, there exists a filters $\mathcal{F}_j$ such that $\mathcal{F}_j \downarrow A_j$, and $\prod_{j\in J} \mathcal{F}_j \subseteq \mathcal{F}$

since $\sum_{j\in J} e_j(\mathcal{F}_j) = e_J(\prod_{j\in J} \mathcal{F}_j) \subseteq e_J(\mathcal{F})$

then $e_J(\mathcal{F}) \downarrow A$

this we have $e_J: \prod_{i\in J} A_j \to A$ is continuous,

and therefore $i_d: (A, \gamma) \to A$.

■

2.3. Locally Convex Algebra via Pseudo Topological Structure

In this section, we define the locally convex algebra without using a family of continuous semi norm, but by using concept of pseudo-topological.

We now create a locally convex topological algebra from a pseudo topology. We also study the relationship between it and the locally convex topology on Algebra A generated from the family of all continuous semi norms on algebra A.

Definition (2.3. 1)

Let $\mathcal{F}$ be a filter on a set M, we denote by $\mathcal{F}^{\wedge}$ the filter generated by the multiplicative convex sets of $\mathcal{F}$.

Since the intersection of two convex sets of $\mathcal{F}$ is also convex set of $\mathcal{F}$ then it clear that from a filter basis, then we have $A \in \mathcal{F}^{\wedge}$ if and only if there exists $F \in \mathcal{F}$ such that F convex and $F \subseteq A$.

Also we define $\mathcal{F}^{*} = (\mathcal{F} \cap [0])^{\wedge}$.

Then we have $\mathcal{F}^{*} \subseteq \mathcal{F}^{\wedge} \subseteq \mathcal{F}$.

Let $N_x = \bigcap\{\mathcal{F}: \mathcal{F} \downarrow_x M\}$ is the neighborhood filter of x and let $\eta = N_0{}^{*}$ then;

lemma(2.3. 2)

Let A be an algebra. The filter η has the following properties:

i. $\eta \subseteq [0]$

ii. $\eta \subseteq ţ \bullet [x]$ for all $x \in A$

iii. $\eta \subseteq \lambda \bullet \eta$ for all $\lambda \in K$

iv. $\eta \subseteq ţ \bullet \eta$

v. $\eta \subseteq \eta + \eta$

vi. $\eta * \eta \subseteq \eta$

proof:

i. since $[0] \downarrow A$,

$$then\ N_0 = \bigcap\{\mathcal{F}: \mathcal{F} \downarrow_0 A\} \subseteq [0].$$

But $\eta \subseteq N_0 \Longrightarrow \eta \subseteq [0]$

ii. since $ţ \bullet [x] \downarrow A$

then $N_0 \subseteq ţ \bullet [x]$ for all $x \in A$,

and we have $\eta \subseteq N_0$

$then \quad \eta \subseteq ţ \bullet [x]$

iii. If $\lambda = 0$, the proof is same as (i)

if $\lambda \neq 0$, we have from(Remark 2.1.4) the compatibility conditions of a pseudo topological algebra $\mathcal{F} \downarrow A$,

$\lambda \in K \Longrightarrow \lambda \bullet \mathcal{F} \downarrow A$

since F convex if and only if $\lambda \bullet F$ is convex

iv. Let $B \in \eta, \Longrightarrow B \in (\bigcap\{\mathcal{F}: \mathcal{F} \downarrow_0 A\} \cap [0])^*$

then there exists a convex set $W \subseteq B$

such that $W \in [0]$

If $Ɨ = \{s \in \text{K}: |s| \leq 1\} \in ţ$

$$\Longrightarrow Ɨ.\text{W} \in \text{B}$$

$$\Longrightarrow B \in ţ \bullet \eta$$

$$\Longrightarrow \eta \subseteq ţ \bullet \eta$$

v. Let $B \in \eta$ then there exists a convex set $W \subseteq B$

since $\frac{1}{2}W$ is also convex then $\frac{1}{2}W + \frac{1}{2}W \subseteq W \subseteq B,$

But $\frac{1}{2}W + \frac{1}{2}W \in \eta + \eta$

$$\Longrightarrow B \in \eta + \eta$$

$$\Longrightarrow \eta \subseteq \eta + \eta$$

vi. For all $\mathcal{F} \downarrow M, \mathcal{F} * \mathcal{F} \subseteq \mathcal{F},$

Then we have $(\mathcal{F} \cap [0])^{\wedge} * (\mathcal{F} \cap [0])^{\wedge} \subseteq (\mathcal{F} \cap [0])^{\wedge}$

This $\eta * \eta \subseteq \eta$

■

Remark (2.3. 3)

We note that Lemma (2.3.2) implies compatibility conditions of the Remarek (2.1.4)

Then we can define a new structure on algebra A. We denote A^o to this a new structure such that $\mathcal{F} \downarrow A^o$ if and only if $\eta \subseteq \mathcal{F}$.

If A a pseudo topological algebra then A^o is locally convex topological algebra.

As in general topology, to define locally convex algebra, we introduce a new concept that is synonymous with it, but on pseudo-topology, using some properties on filters, being the cornerstone of pseudo-topological structures, as follows;

Notation (2.3.4)

Let M be a subset of an algebra A then:

i. $co(M)$ is denote to the convex hull of M

ii. $\Gamma(M)$ is denote to the absolutely convex hull of M

Definition (2.3.5)

A pseudo topological algebra A is said to be locally convex if for every filter $\mathcal{F}$ with $\mathcal{F} \downarrow A$, then $co(\mathcal{F}) \downarrow A$, where $co(\mathcal{F})$ denoted for the filter that based on convex hull $F \in \mathcal{F}$, and A is said to be locally multiplicative convex if for all filter $\mathcal{F} \downarrow A$ has the properties $\mathcal{F}$ is multiplicative filter.

We know that a locally convex topological algebra can be generated from the family of all continuous semi norms on algebra A. let $\tau(A)$ denoted to the locally convex modification of A that is generated from the family of all continuous semi norms on algebra A. If we have a pseudo topology compatible with algebra A, what is the relationship between the two structures, this is what we will explain through the next proposition:

Proposition(2.3.6):

Let A be a pseudo topological algebra and $\tau(A)$ denoted to the locally convex modification of A, then:

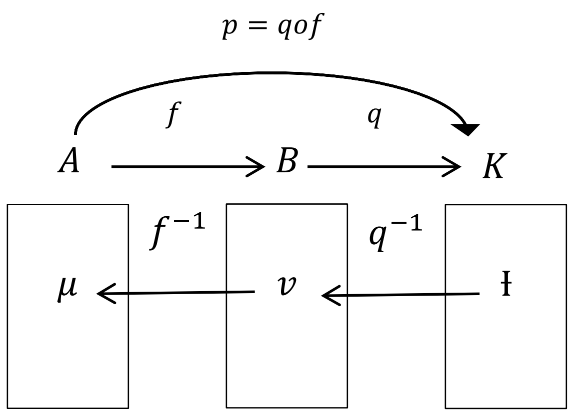
i. The identity mapping $i: A \to \tau(A)$ is continuous.

ii. If B is locally convex topological algebra, then the homomorphism $f: A \to B$ is continuous if and only if $f: \tau(A) \to B$ is continuous.

Proof:

i. Suppose that $\mathcal{F} \downarrow A$ and let μ the neighborhood of 0 in $\tau(A)$.
Since μ is open in $\tau(A)$,
then there is a seminorm p on A such that p is continuous and
$$p^{-1}(\text{ƚ}) \subseteq \mu \text{ where } \text{ƚ} = \{s \in K: |s| \leq 1\}$$
Since $p(\mathcal{F}) \downarrow K$ (as p is continuous)
Then there exists set $F \in \mathcal{F}$ such that $p(F) \subseteq$ ƚ
This we have $F \subseteq p^{-1}(\text{ƚ}) \subseteq \mu$, and there for $\mu \in \mathcal{F}$
Hence $\mathcal{F} \downarrow \tau(A)$.

ii. let $f: A \to B$ is continuous and let v the neighborhood of 0 in B
Since B is locally convex topological algebra,
then there is seminorm q on B such that q is continuous and $q^{-1}(\text{ƚ}) \subseteq v$.
Let $p = qof$, it is clear that p is seminorm, and since f is continuous, then p is continuous in A.
Hence $p^{-1}(\text{ƚ})$ is neighborhood of 0 in $\tau(A)$.
Put $\mu = p^{-1}(\text{ƚ})$, then we have $f(\mu) \subseteq v$, and therefor $f: \tau(A) \to B$ is continuous.



Conversely, suppose that $f: \tau(A) \rightarrow B$ is continuous.

From (i) we have the identity mapping $i: A \rightarrow \tau(A)$ is continuous,

And so that $f = foi: A \rightarrow B$ is continuous.

■

Corollary (2.3.7):

If $f: A \rightarrow B$ is continuous homomorphism from two pseudo topological algebra A and B then $f: \tau(A) \rightarrow \tau(B)$ is continuous.

Proof:

The proof is direct from Proposition(2.3.6).

■

To study the relationship between the convex topological algebra generated from the family of all continuous semi-norms and the convex algebra compatible with pseudo-topology, we will need the following result:

Lemma (2.3.8):

Let A *and* B are two of pseudo topological algebras then:

i. The identity mapping $id: A \rightarrow A^0$ is continuous.

ii. If $f: A \rightarrow B$ is continuous homomorphism algebra, so is $f: A^0 \rightarrow B^0$.

Proof:

Proof of (i) directly from the definition of A^0.

So we turn to proof (ii),

Suppose that $f: A \to B$ is continuous homomorphism algebra,

And let $\mathcal{F} \downarrow_a A$ this we have $(\mathcal{F}) \downarrow_{f(a)} B$.

The proof is complete if we prove that $\mu(f(x)) \subseteq f(\mu(x))$,

so that for all $W \in \mu(f(x))$, we have $W \in f(\mathcal{F})$,

this we get $f^{-1}(W) \in \mathcal{F}$,

and so that $f^{-1}(W) \in \mu(x)$,

$$\Rightarrow f(f^{-1}(W)) \subseteq f(\mu(x))$$

Since $f(f^{-1}(W)) \subseteq W$,

then $W \in f(\mu(x))$, and therefor $\mu(f(x)) \subseteq f\big(\mu(x)\big)$.

■

<u>Proposition(2.3.9):</u>

Let A be a pseudo topological algebra then the locally convex topological algebra $\tau(A)$ has the zero neighbourhood filter $\Gamma\big(\mu(0)\big)$.

Hint: $\Gamma\big(v(0)\big)$ the filter that has base $\{\Gamma(\mathrm{v}): \mathrm{v} \in \mu(0)\}$.

<u>Proof:</u>

Let $\aleph = \Gamma\big(\mu(0)\big)$, we have to prove that $\aleph$ is the zero neighbourhood filter of a locally convex topological algebra $\tau(A)$ that indused by family of all continuous seminorms on A.

for all $\lambda \in K$, the map $f: A \to A$

such that $f(a) = \lambda a \quad for\ all\ \ a \in A$ is homeomorphism

and so that $f: A^0 \to A^0$ by lemma (2.3.8).

This we have $\lambda\mu(0) = \mu(0)$, and so that $\lambda\aleph = \aleph$.

Now for all $a \in A$,

the map $g_a: K \to A \quad s.t\ g(\lambda) = \lambda a$ is continuous and also $g_a: K \to A^0$.

Then for all $v \in \mu(0)$ there exists $\lambda \neq 0$ such that $\lambda a \in v$.

Let $v \in \tau(A)$, where v be an absolutely convex neighbourhood of zero in $\tau(A)$.

Since $id: A \to \tau(A)$ is continuous,

then $id: A^0 \to \tau(A)$ is continuous by lemma(2.3.8)

Then there is $U \in \mu(0)$, such that $U \subseteq v$,

and therefor $\Gamma(U) \subseteq v$, this we have $\mu_{\tau(A)}(0) \subseteq \aleph$

■

Chapter Three

Differential Structures via Pseudo Topological Algebra

Introduction

This chapter, consists of three sections:

In section one, we introduce a pseudo-topological γ on $C(X,Y)$. These structures consisting of the convergence of the filters will be used to prove some properties on the space $C(X,Y)$ (algebra on class of all continuous function from two spaces).

In sections two, we introduce the concepts of differentiability and derivatives on a pseudo topological algebra, we will extend the concept of a Frechet derivative but without norm on algebra but using pseudo-structures after introducing the appropriate conditions.

In sections three, we will introduce concepts second derivative, higher derivative, and then we put the definition of C^{∞}- derivatives of the maps on a pseudo topological algebra.

3.1 Pseudo- Topological Algebra on Class of all Continuous Function $C(X,Y)$

The space $C(X,Y)$ is an algebra with point wise operations and if this space corresponds to a topological so that the operations are continuous, it becomes topological algebra. There are many studies and research on the relationship between the space X (when X is completely regular topological space) and the space $C(X,Y)$, but we know in general that the space $C(X,Y)$ is not complete.

However, there is an obstacle in the study of previous researchers, and this obstacle is that there is, in general, no topology compatible with algebra, which makes the map $\varpi(f,x): C(X,Y) \times X \to Y$ (that sends every ordered pair (f,x) into $f(x) \in Y$) is continuous mapping.

Accordingly, we will replace the concept of topology with a pseudo-topology based on the convergence of filters, which is finer than topology, which allows us to prove some important properties on a space $C(X,Y)$ as well as study the relationship between space Y and the space $C(X,Y)$. Finally, we will study the property of perfection on space $C(X,Y)$ and its connection with space Y when each of the two spaces represents algebra.

Definition (3.1.1)

Let X and Y be two algebras and $C(X,Y)$ the space of all continuous mapping between X,Y . We define the evaluation mapping:

$\varpi_{X\times Y}(f,x): C(X,Y)\times X\to Y$ by $\varpi_{X\times Y}(f,x)=f(x)$, for all $f\in C(X,Y)$ and $x\in X$.

If Ƕ is the filter on $C(X,Y)$ and $\mathfrak{X}$ is a filter on X, the evaluation filters $\varpi_{X\times Y}(Ƕ,\mathfrak{X}) = Ƕ(\mathfrak{X}) = \{G(F) : G \in Ƕ, F\in\mathfrak{X}\}$

Definition (3.1.2):

Let X and Y be a pseudo topological spaces. Then we define a pseudo topological γ on $C(X,Y)$ as follows :

Ƕ $\downarrow_f$ $C(X,Y)$ if and only if $\varpi_{X\times Y}(Ƕ,\mathfrak{X})$ $\downarrow_{f(x)}$ Y for all filter $\mathfrak{X}\downarrow_x$ X.

The space $(C(X,Y),\gamma)$ (for ease we use the symbol $C_\gamma(X,Y)$) is a pseudo topological of class of all continuous function from two spaces of pseudo-topology.

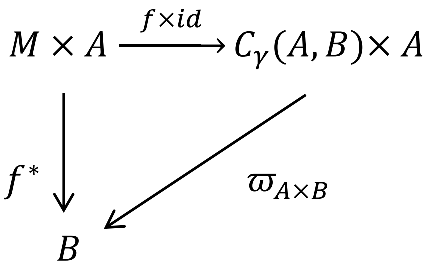
The next lemma will be the main basis for proving the next theorems:

Lemma (3.1.3):

Let be A,B and M be a pseudo topological spaces. Then the mapping $f: M\to C_\gamma(A,B)$ is continuous if and only if $f^*: M\times A\to B$ is continuous where $f^*(m,a)=f(m)(a)$.

Proof:

Let f be continuous map, by the following commutative diagram



since $f^{*} = \varpi_{X \times Y} o(f \times id)$,

and since id is continuous then $f \times id$ is also continuous

this we have $\varpi_{X \times Y} o(f \times id) = f^{*}$ is also continuous.

Now suppose that f^{*} is continuous and $\mathcal{F} \downarrow_{m} M$.

If we take any filter $\mathfrak{X}$ such that $\mathfrak{X} \downarrow_{a} X$,

then $f(\mathcal{F})(\mathfrak{X}) = f^{*}(\mathcal{F} \times \mathfrak{X})$ this implies $f^{*}(m, a) = f(m)(a)$.

■

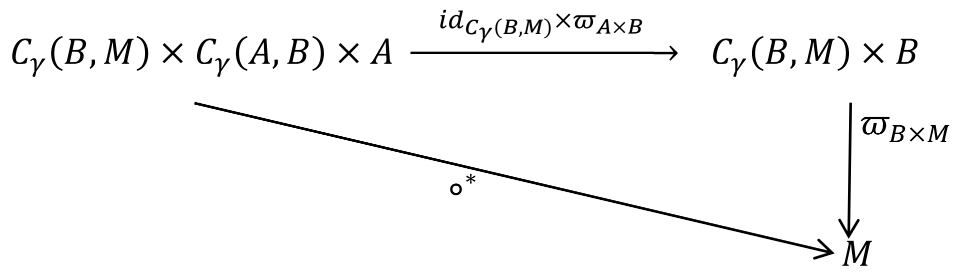
Proposition(3.1.4):

Let be A, B and M be a pseudo topological spaces, $C_{\gamma}(A, B), C_{\gamma}(A, M)$ and $C_{\gamma}(B, M)$ denoted to the pseudo topological spaces of a class of all continuous function on those spaces, each according to its location. Then the mapping (composition map)

$\circ: C_{\gamma}(B, M) \times C_{\gamma}(A, B) \rightarrow C_{\gamma}(A, M)$ is continuous.

Proof:

We prove it first the associated mapping $\circ^{*}: C_{\gamma}(B, M) \times C_{\gamma}(A, B) \times A \rightarrow M$ is continuous, but that we obtain through the following diagram:



And by using Lemma (3.1.3) when we replace f by($\circ$) and (f^*) by($\circ^*$) we will get the desired result.

To prove that pseudo-topology is compatible with $C_\gamma(A,B)$ space, there must be a pseudo-topology compatible with B where B is algebra. This is what we will prove in the next proposition based on Lemma (3.1.3) and Proposition(3.1.4)

■

<u>Proposition(3.1.5):</u>

Let A be a pseudo topology space and B be a pseudo topological algebra then $C_\gamma(A,B)$ is a pseudo topological algebra.

<u>Proof:</u>

To prove that $C_\gamma(A,B)$ is a pseudo topological algebra, we must apply a Definition (2.1.3)

That is we must prove the algebra operations.

$+: C_\gamma(A,B) \times C_\gamma(A,B) \to C_\gamma(A,B)$

$\bullet: K \times C_\gamma(A,B) \to C_\gamma(A,B)$

$*: C_\gamma(A,B) \times C_\gamma(A,B) \to C_\gamma(A,B)$ are continuous.

1. We define the map;

$$\varphi: C_\gamma(A,B) \times C_\gamma(A,B) \times A \to C_\gamma(A,B) \times A \times C_\gamma(A,B) \times A$$

Such that $\varphi(f,g)(a) = f(a) \cdot g(a)$

where $f,g \in C_\gamma(A,B)$ and $a \in A$

by using Proposition (3.1.4), it is clear that φ is continuous map

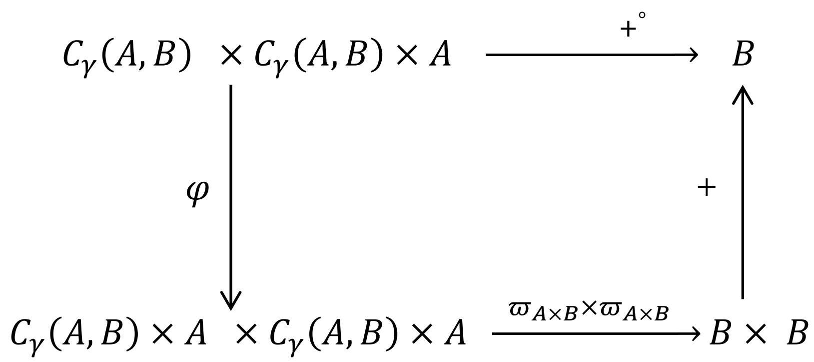
Also since B is a pseudo topological algebra

then $+: B \times B \to B$ is continuous operation

Also the map $\varpi_{A\times B}: C_\gamma(A,B) \times A \to B$ is continuous and therefore

$\varpi_{A\times B} \times \varpi_{A\times B}$ is continuous

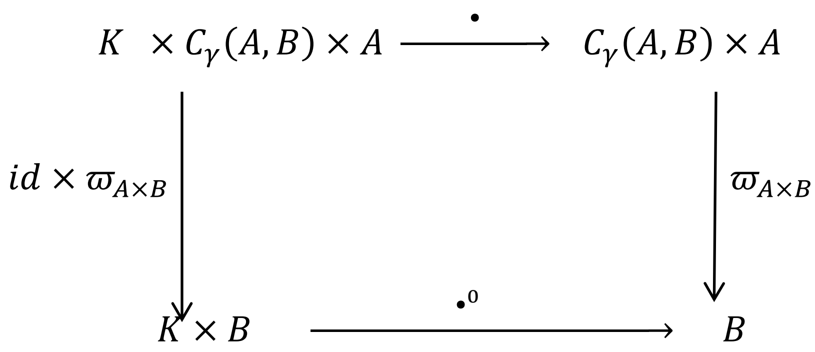
By the commutative diagram



We have $+^\circ: C_\gamma(A,B) \times C_\gamma(A,B) \times A \to B$ is continuous:

Now by Lemma(3.1.3) if we take (+) instead of f and $(+^\circ)$ instead of f^* then we get that (+) is continuous.

2. To prove the scalar multiplication is continuous on $C_\gamma(A,B)$ we can show that from the following diagram:



Then we have $(\bullet^0) o (id \times \varpi_{A\times B}) = (\varpi_{A\times B}) o (\bullet)$

Since $(\bullet^0) o (id \times \varpi_{A\times B})$ is continuous

Then $(\varpi_{A\times B}) o (\bullet)$ is continuous

By Lemma(2.2.2), we have (∙) is continuous.

3. Finally, we define the map

$$\varphi: C_\gamma(A,B) \times C_\gamma(A,B) \times A \to C_\gamma(A,B) \times A \times C_\gamma(A,B) \times A$$

Such that $\varphi(f,g)(a) = f(a) \cdot g(a)$ where $f,g \in C_\gamma(A,B)$ and $a \in A$

Then it is clear that φ is continuous map (Proposition(3.1.4))

Also, since B is a pseudo topological algebra then

$* : B \times B \to B$ is continuous operation

Also the map $\varpi_{A\times B}: C_\gamma(A,B) \times A \to B$ is continuous and therefor $\varpi_{A\times B} \times \varpi_{A\times B}$ is continuous

$$\begin{array}{ccc} C_\gamma(A,B) \times C_\gamma(A,B) \times A & \xrightarrow{*^0} & B \\ \Big\downarrow \varphi & & \Big\uparrow * \\ C_\gamma(A,B) \times A \times C_\gamma(A,B) \times A & \xrightarrow{\varpi_{A\times B} \times \varpi_{A\times B}} & B \times B \end{array}$$

And again we use Lemma (3.1.3) , if we take (*) instead of f and $(*^0)$ instead of f^*, then we get that we get the continuous of the vector multiplication algebra.

■

If we denote by the symbol $H(A,B)$ to indicate the space of all continuous homomorphism and by the symbol $\mathcal{L}(A,B)$ to denote the space of all linear mapping frome two of a pseudo topological space A and B, then referring to Proposition (2.2.3), we get the following Corollary, which we will mention without proof.

Corollary(3.1.6):

The subspaces $H(A,B)$ and $\mathcal{L}(A,B)$ of a pseudo topological algebra $C_\gamma(A,B)$ is also from a pseudo topological algebra and represent the initial pseudo topological structure with respect to the inclusion mapping.

Now we can study the relationship between the algebra $C_\gamma(A,B)$ and the algebra B.

Theorem(3.1.7):

If B be Hausdorff pseudo topological algebra then the space $C_\gamma(A,B)$ is also Hausdorff pseudo topological algebra.

Proof:

Let B is Hausdorff pseudo topological algebra and let $\mathcal{F} \downarrow_f \ C_\gamma(A,B)$ and $\mathcal{F} \downarrow_g \ C_\gamma(A,B)$

Then we have $\varpi(\mathcal{F} \times$ Ƕ$) \downarrow_{f(x)} \ B$ and $\varpi(\mathcal{F} \times$ Ƕ$) \downarrow_{g(x)} \ B$ for all Ƕ $\downarrow_x A$

Since B is Hausdorff , then $f(x) = g(x)$ for all x and therefore $C_\gamma(A,B)$ is Hausdorff pseudo topological algebra.

■

Theorem(3.1.8):

Let A and B be a pseudo topological algebras then B is homeomorphism to a sub-algebra of $C_\gamma(A,B)$.

Proof:

Let $\varphi: B \to C_\gamma(A,B)$ be a map,

define as $\varphi(b) = f_b$ where $f_b \in C_\gamma(A,B)$, such that for all $a \in A$

we have $f_b(a) = b$ (f_b is constant mapping in $C_\gamma(A,B)$).

To show that φ is injective map:

Let $\varphi(b_1) = \varphi(b_2)$ where $b_1, b_2 \in B$

Then $f_{b_1} = f_{b_1}$ and $f_{b_1}(a) = f_{b_2}(a)$, but $f_{b_1}(a) = b_1$, $f_{b_2}(a) = b_2$

this $b_1 = b_2$ and hence φ is injective.

It is clear that $\varphi^{-1}: C_\gamma(A, B) \to B$ is continuous

To prove that $\varphi: B \to C_\gamma(A, B)$ is continuous, by using lemma(3.2.2),

it suffices to prove that the associated mapping $\tilde{\varphi}: B \times A \to B$ is continuous.

But $\tilde{\varphi}$ is projective of $B \times A$ onto B where $\tilde{\varphi}(b, a) = \varphi(b)(a) = b$ and therefore $\tilde{\varphi}$ is continuous.

It remains to prove that φ is an embedding:

Suppose that $\mathcal{F} \downarrow_b B$ such that $\varphi(\mathcal{F}) \downarrow_{\varphi(b)} C_\gamma(A, B)$,

For all $a \in A$, $[a] \downarrow_b A$, where $[a]$ is the filter generated by a and therefore $\varphi(\mathcal{F}) \times [a] \downarrow_{(\varphi(b), a)} C_\gamma(A, B) \times A$,

hence $\mathcal{F} = \varphi(\mathcal{F})([a]) \Longrightarrow \varphi(b)(a) = b$.

■

The study of completion on space $C(A, B)$ requires us to have some concepts in this regard, and then we address the most important theorems in this section.

<u>Definition (3.1.9):</u>

A pseudo topological algebra A is said to be completeness, if for every Cauchy filter in A, then its filter converges to some element in A.

We recall the definition of Cauchy filter, which was mentioned in the second chapter(Definition 2.1.17), as the filter $\mathcal{F}$ in a pseudo topological algebra A is said to be Cauchy filter if $(\mathcal{F} - \mathcal{F}) \downarrow_0 A$.

A pseudo topological algebra $C_\gamma(A, B)$ is completeness provided that it is B completeness. This is what we will prove in the next theorem, which also shows the relationship between the $C(A,B)$ and the space B when carrying the structures of a pseudo topological algebra.

Theorem(3.1.10):

The space $C(A,B)$ of all continuous mapping from algebra A into algebra B is a complete space if it is done by pseudo topological structures γ and that make the evaluation mapping $\varpi_{A\times B}: C(A,B)\times A\to B$ is continuous and when B also complete pseudo topological algebra.

Proof:

Let $\mathcal{F}$ be a Cauchy filter on $C_\gamma(A,B)$, this means $\mathcal{F}-\mathcal{F}\downarrow_{\hat{0}} C_\gamma(A,B)$ where $\hat{0}$ we mean by it the zero mapping which transmits each element in A to the zero element in B,

Since $\mathcal{F}$ is a Cauchy filter then $\varpi_{A\times B}(\mathcal{F}\times \text{Ʊ})$ is also Cauchy filter in B

Since B is complete and A is a pseudo topological then we get $[a]\downarrow_a A$ where $[a]$ means the filter that generated of a

then $\varpi_{A\times B}(\mathcal{F}\times[a])\downarrow_{f(a)} B$ for all $a\in A$

we must prove that $f\in C_\gamma(A,B)$ (we must prove that f is continuous map)

Since $\mathcal{F}-\mathcal{F}\downarrow_{\hat{0}} C_\gamma(A,B)$, and $\forall a\in A$ such that Ʊ $\downarrow_a A$,

then there are $F\in\mathcal{F}$, $G\in$ Ʊ such that $\varpi_{A\times B}(\mathcal{F}\times[a])\downarrow_{f(a)} B$,

Then we have $\varpi((F-F)\times G)\subset u_0$ for u_0 it the zero neighborhood in B

For $s\in G$ we have $\varpi_{A\times B}(\mathcal{F}\times[s])\downarrow_{f(s)} B$

This means $f(s)\in\overline{\{g(s): g\in F\}}$

$\Rightarrow\{\,g(s)-f(s) : g\in F\,, a\in G\}\subset\overline{\varpi((F-F)\times G)}$

$\Rightarrow\{\,g(s)-f(s) : g\in F\,, a\in G\}\subset u_0$

Let $a\in G\Rightarrow\{(g(s)-f(s)-\big(g(a)-f(a)\big): g\in F, s\in G\}\subset u_0-u_0$

$\Rightarrow f(a)-f(s)\in u_0-u_0+g(a)-g(s)$

Since g is continuous, G` can be chosen to belong to the filter Ʊ such that

$g(a) - g(s) \subset u_0$ for $s \in G`$ and therefor $f(a) - f(s) \in u_0 - u_0 + u_0$

Then for any $\in G \cap G`$, we can choose u_0 such that $u_0 - u_0 + u_0 \subset Ɨ_\epsilon$ where $Ɨ_\epsilon = \{s \in K: |s| \leq \epsilon\}$

$\Rightarrow |f(a) - f(s)| < \epsilon,$

and so we proved that f is continuous map and so that

$f \in C(A, B)$

This means that for all Cauchy filter in $C_\gamma(A, B)$, then its filter converges to some element $f \in C(A, B)$.

■

Definition (3.1.11):

Let $f: X \to Y$ be a map from the pseudo topological space X to pseudo topological space Y then we say that f is quasi bounded iff ţ.$\mathcal{F} \downarrow_0 X \Longrightarrow$ ţ.$f(\mathcal{F}) \downarrow_0 Y$.

Definition (3.1.12):

Let X and Y be algebras, and let $C_q(X, Y)$ be denoted the set of all quasi bounded from X to Y. Then we define the evaluation mapping, $\varpi_{X\times Y}(f, x): C_q(X, Y) \times X \to Y$ by $\varpi_{X\times Y}(f, x) = f(x)$, for all $f \in C_q(X, Y)$ and $x \in X$.

It is clear that the set of all evaluation mapping $\varpi_{X\times Y_{C_q(X,Y)}}$ is algebra if the underlying spaces X and Y be algebras.

If $\mathcal{K}$ is the filter on $C_q(X, Y)$ and $\mathfrak{X}$ is a filter on X, then $\mathcal{K} \times \mathfrak{X}$ is the filter on $\varpi_{X\times Y_{C_q(X,Y)}}$.

Theorem(3.1.13):

Let A *and* B be the pseudo topology spaces. The space $C_q(A, B)$ of all quasi bounded maps is algebra.

Proof:

1. Let $f\ ,g \in C_q(A,B)$ and let ţ.$\mathcal{F} \downarrow_0 A$,

then we have ţ.$f(\mathcal{F}) \downarrow_0 B$ and ţ.$g(\mathcal{F}) \downarrow_0 B$

$$\Rightarrow ţ.f(\mathcal{F}) + ţ.g(\mathcal{F}) \downarrow_0 B$$

$$\Rightarrow ţ.(f+g)(\mathcal{F}) \downarrow_0 B$$

$$\Rightarrow f+g \in C_q(A,B)\ .$$

2. Let $\alpha \in$ K and $f \in C_q(A,B)$ and ţ.$\mathcal{F} \downarrow_0 A$

then ţ.$f(\mathcal{F}) \downarrow_0 B \Rightarrow \alpha($ţ.$f(\mathcal{F})) \downarrow_0 B$

$$\Rightarrow ţ.\alpha\, f(\mathcal{F}) \downarrow_0 B$$

$$\Rightarrow \alpha\, f \in C_q(A,B).$$

3. Let $f\ ,g \in C_q(A,B)$ and let ţ.$\mathcal{F} \downarrow_0 A$,

then we have ţ.$f(\mathcal{F}) \downarrow_0 B$ and ţ.$g(\mathcal{F}) \downarrow_0 B$

$$\Rightarrow ţ.f(\mathcal{F}).ţ.g(\mathcal{F}) \downarrow_0 B$$

$$\Rightarrow ţ^2.(f.g)(\mathcal{F}) \downarrow_0 B$$

Since ţ$^2 \subseteq$ ţ, then ţ$^2.(f.g)(\mathcal{F}) \subseteq$ ţ.$(f.g)(\mathcal{F})$

This, ţ.$(f.g)(\mathcal{F}) \downarrow_0 B$

This we get, $f.g \in C_q(A,B)$.

■

Proposition(3.1.14):

If $f \in C_q(A,B)$ and $g \in C_q(B,M)$ then $gof \in C_q(A,M)$.

Proof:

Let $f \in C_q(A,B)$ and let ţ.$\mathcal{F} \downarrow_0 A$,

then we have ţ.$f(\mathcal{F}) \downarrow_0 B$, but $g \in C_q(B,M)$

$$\Rightarrow ţ.\, g(f(\mathcal{F})) \downarrow_0 M$$

$$\Rightarrow ţ.\, (gof)(\mathcal{F}) \downarrow_0 M$$

$$\Rightarrow gof \in C_q(A, M).$$

■

<u>Remark(3.1.15):</u>

Using mathematical induction, we can generalize the facts that we reached in the previous theorem on

the space $C_q(A^n, B)$, where $A^n = A \times \dots \dots \times A$ (n factors) as following:

without loss of generality assume that n=2,

and let $a = (a_1, a_2) \in A^n = A \times A$

we must prove that $\mathcal{F}((ţ + [a_1]) \times (ţ + [a_2])) \downarrow_0 B$

since $(ţ + [a_1]) \times (ţ + [a_2]) = ţ \times ţ + ţ \times [a_2] + [a_1] \times ţ + [a_1] \times [a_2]$

this we have: $\mathcal{F}\big((ţ + [a_1]) \times (ţ + [a_2])\big)$

$$= \mathcal{F}(ţ \times ţ + ţ \times [a_2] + [a_1] \times ţ + [a_1] \times [a_2])$$

$$\geq \mathcal{F}(ţ \times ţ) + \mathcal{F}(ţ \times [a_2]) + \mathcal{F}([a_1] \times ţ) + \mathcal{F}([a_1] \times [a_2])$$

But $\mathcal{F}(ţ \times ţ) \downarrow_0 B$ and $\mathcal{F}([a_1] \times [a_2]) = \mathcal{F}([\mathrm{a}]) \downarrow_0 B$

Since $Ɨ_1 . ţ = ţ$, then we have:

$\mathcal{F}(ţ \times [a_2]) = \mathcal{F}(Ɨ_1 . ţ \times [a_2]) = \mathcal{F}(ţ \times Ɨ_1 . [a_2]) \downarrow_0 B$

in the same way we have $\mathcal{F}([a_1] \times ţ) \downarrow_0 B$,

and therefore

$(\mathcal{F}(ţ \times ţ) + \mathcal{F}(ţ \times [a_2]) + \mathcal{F}([a_1] \times ţ) + \mathcal{F}([a_1] \times [a_2])) \downarrow_0 B$

Hence $\mathcal{F}((ţ + [a_1]) \times (ţ + [a_2])) \downarrow_0 B$

Therefore, this can be generalized to all $n \in \mathbb{N}$.

■

<u>Definition (3.1.16):</u>

Let $f: A \to B$ be a mapping between two pseudo topological algebras $A\ and\ B$. Then we define a mapping $\Delta f: A \times A \to B$ associated with the mapping f as fallowing; $\Delta f(a, h) = f(a + h) - f(a)$.

Definition (3.1.17):

A map $f: A \to B$ between two pseudo topological algebras is called equably continuous if $\begin{cases} ţ \cdot \mathcal{F} \downarrow A \\ \mathcal{Y} \downarrow A \end{cases}$ then $\Delta f(\mathcal{F}, \mathcal{Y}) \downarrow B$

Proposition (3.1.18):

If map $f: A \to B$ is equably continuous, then it is continuous for every $a \in A$.

Proof:

Let $\mathcal{F} \downarrow_a A$

Since $ţ \cdot [a] \downarrow A$ by (Remark 2.1.4)

By translation of addition, we have $\mathcal{F} \downarrow_a A \Longrightarrow (\mathcal{F} - [a]) \downarrow A$

Since f is equably continuous, we get $\Delta f([a], \mathcal{F} - [a]) \downarrow B$

Since $\Delta f(s, m - s) = f(s + m - s) - f(s)$

$= f(m) - f(s)$

Then we get $f(\mathcal{F}) - f([a]) \geq \Delta f([a], \mathcal{F} - [a])$,

and hence $f(\mathcal{F}) - f(a) \downarrow B$

Once again by Remark (2.1.4), we get $f(\mathcal{F}) \downarrow_{f(a)} B$.

■

We will assume that A and B be Hausdorff locally convex algebra over the field of complex numbers $\mathbb{C}$. We also assume that Γ_A, Γ_B be denote the set of all continuous semi-norms as A and B respectively.

Definition (3.1.19):

Let A and B be Hausdorff locally convex topological algebra over the field $\mathbb{C}$ and $n \in N$, then

1. $\mathcal{L}^n(A, B)$ the algebra of all n-linear mapping from $A^n = A \times \ldots \times A$ into B.

2. $C^n(A,B)$ the algebra of all continuous n-linear mapping from $A^n = A \times \ldots \times A$ into B.
3. $H^n(A,B)$ the algebra of all continuous n-homomorphism from $A^n = A \times \ldots \times A$ into B.

Definition (3.1.20):

Let A and B be a locally convex topological algebra over the field K. A filter $\mathcal{F} \downarrow C^n(A,B)$, if for all Ƕ $\downarrow_{(a_1 \ldots a_n)} A^n$, then the evaluated filter $\mathcal{F}$(Ƕ) $\downarrow_{f(a_1 \ldots a_n)} B$, where $f \in F \in \mathcal{F}$.

Now we can generalize the previous Lemma (3.1.2), which can be written without proof as follows;

Lemma (3.1.21):

Let A, B be two locally convex topological algebras, and M be a pseudo topological spaces, $n \in N$. Then the mapping $f: M \to {C_\gamma}^n(A,B)$ is continuous if and only if $f^*: M \times A^n \to B$ is continuous.

Also, in the same context, we can generalize the previous Proposition(3.1.4);

Proposition(3.1.22):

Let A and B are two locally convex topological algebra, then ${C_\gamma}^n(A,B)$ is a pseudo topological algebra.

Definition (3.1.23): Topological space on $\boldsymbol{C^n(A,B)}$

Let A and B be a locally convex topological algebra over the field K, and let $\mathfrak{S}$ denotes to a collection of bounded subsets of A and that covers

A. We will denote by $\tau_{\mathfrak{S}}$ to the topology of $\mathfrak{S}$ −convergence on $C^n(A, B)$, this topological algebra will be symbolized by the symbol $(C^n(A, B), \tau_{\mathfrak{S}})$.

Proposition(3.1.24):

Let A be a topological algebra and B be a pseudo topological algebra. Then a filter $\mathcal{F} \downarrow_0 C_\gamma(A, B)$ iff:

1. $\mathcal{F} \in \tau(0)$
2. $\mathcal{F}(\mu) \downarrow_0 B$

Proof:

Assume that $\mathcal{F} \downarrow_0 C_\gamma(A, B)$ and let $a \in A$, then we have

$\mathcal{F}([\mathrm{a}]) \downarrow_0 B$ and $\mathcal{F}(\mu) \downarrow_0 B$.

Conversely, assume that 1 and 2 are true,

If $a \in A$, then $\mathcal{F}(\mu + [\mathrm{a}]) \geq \mathcal{F}(\mu) + \mathcal{F}([\mathrm{a}])$

But $\mathcal{F}(\mu) \downarrow_0 B$ and $\mathcal{F}([\mathrm{a}]) \downarrow_0 B$,

hence $(\mathcal{F}(\mu) + \mathcal{F}([\mathrm{a}])) \downarrow_0 B$ (by Remarek 2.1.4)

and therefor $\mathcal{F}(\mu + [\mathrm{a}]) \downarrow_0 B$.

As $a \in A$ is arbitrary element, this we have $\mathcal{F} \downarrow_0 C_\gamma(A, B)$.

■

Remark(3.1.25):

We can generalize the facts that we reached in the previous theorem on the space $C_\gamma(A^n, B)$, where $A^n = A \times \dots \dots \times A$ (n factors) as following:

without loss of generality assume that n=2,

and let $a = (a_1, a_2) \in A^2 = A \times A$

we must prove that $\mathcal{F}((\mu + [a_1]) \times (\mu + [a_2])) \downarrow_0 B$

since$(\mu + [a_1]) \times (\mu + [a_2]) = \mu \times \mu + \mu \times [a_2] + [a_1] \times \text{ţ} + [a_1] \times [a_2]$

this we have:

$$\mathcal{F}\big((\mu + [a_1]) \times (\mu + [a_2])\big)$$
$$= \mathcal{F}(\mu \times \mu + \mu \times [a_2] + [a_1] \times \mu + [a_1] \times [a_2])$$

$$\geq \mathcal{F}(\mu \times \mu) + \mathcal{F}(\mu \times [a_2]) + \mathcal{F}([a_1] \times \mu) + \mathcal{F}([a_1] \times [a_2])$$

But $\mathcal{F}(\mu \times \mu) \downarrow_0 B$ and $\mathcal{F}([a_1] \times [a_2]) = \mathcal{F}([\mathrm{a}]) \downarrow_0 B$

Since $Ɨ_1 . \mu = \mu$, then we have:

$$\mathcal{F}(\mu \times [a_2]) = \mathcal{F}(Ɨ_1 . \mu \times [a_2])$$

$$= \mathcal{F}(\mu \times Ɨ_1 . [a_2]) \downarrow_0 B$$

in the same way we have $\mathcal{F}([a_1] \times \mu) \downarrow_0 B$,

and therefore

$$(\mathcal{F}(\mu \times \mu) + \mathcal{F}(\mu \times [a_2]) + \mathcal{F}([a_1] \times \mu) + \mathcal{F}([a_1] \times [a_2])) \downarrow_0 B$$

Hence $\mathcal{F}((\mu + [a_1]) \times (\mu + [a_2])) \downarrow_0 B$

Therefore, this can be generalized to all $n \in \mathbb{N}$.

■

Proposition(3.1.26):

Let A and B be a locally convex topological algebra then $\mathcal{F} \downarrow_0 C_\gamma(A, B)$ if and only if:

1. $\mathcal{F} \in \tau(0)$
2. For all $q \in \Gamma_B$ there exists $p \in \Gamma_A$, $F \in \mathcal{F}$ such that $sup_{f \in F} |f|_{q,p} < \infty$.

Proof:

Assume that $\mathcal{F} \downarrow_0 C_q(A, B)$, then by Proposition (3.1.24) we have $\mathcal{F}(ţ) \downarrow_0 B$

Let $q \in S_B$, there exists $p \in \Gamma_A$ and $\delta > 0$ and $F \in \mathcal{F}$

such that, for $a \in A$

$|a|_p \leq \delta$ and $f \in F$, we have $|f(a)|_q \leq 1$

From above we have $sup_{f \in F} |f|_{q,p} < \delta^{-1} < \infty$.

Conversely, suppose that 1 and 2 holds,

Then for $q \in \Gamma_B$ and $\varepsilon > 0$, we can choose $p \in \Gamma_A$ and $F \in \mathcal{F}$ such that,

$g = sup_{f \in F} |f|_{q,p} < \infty$,

For all $f \in C(A, B)$ and $a \in A$, such that $f \in F$, $|a|_p \leq (\varepsilon g^{-1})^{-1}$

Then we have $|f(a)|_q \leq |f|_{q,p} . |a|_p \leq \varepsilon$

Since q and ε were arbitrary this $\mathcal{F}(ţ) \downarrow_0 B$

By Propoisition (3.1.24), we get the required proof.

■

3.2 Differentiability and Derivatives on a pseudo topological algebra

In this section, we will introduce the concepts of differentiability and derivatives on a pseudo topological algebra, using the concept of a Frechet derivative on Banachspaces, but this time on more general spaces using pseudo-topology structures.

Definition (3.2.1):

Let $r: A \to B$ be a mapping from a pseudo topological algebra A to a pseudo topological algebra B, and let $\varphi_r: K \times A \to B$ a new map that associated with r such that $\varphi_r(\gamma, a) = \begin{cases} 0 & if \quad \gamma = 0 \\ \frac{1}{\gamma} r(\gamma a) & if \ \gamma \neq 0 \end{cases}$

Then r is called remainder if it is holds;

If $ţ\mathcal{F} \downarrow A$ then $\varphi_r(ţ, \mathcal{F}) \downarrow B$.

We will denoted $R(A, B)$ to the set of all remainder mapping between A and B.

Lemma (3.2.2):

If we have three of a pseudo topological algebra A, B and M. If $r \in R(A, B)$ and $h \in H(B, M)$, then $hor \in R(A, M)$.

Proof:

Since $\varphi_{hor}(\gamma, a) = \frac{1}{\gamma}(hor)(\gamma a)$, this we have $\varphi_{hor}(\gamma, a) = h(\varphi_r(\gamma, a))$

If $ţ\mathcal{F} \downarrow A$, then $\varphi_r(ţ, \mathcal{F}) \downarrow B$

Since h is continuous homomorphism, then $h(\varphi_r(ţ, \mathcal{F})) \downarrow M$

This we have $hor \in R(A, M)$.

■

Proposition (3.2.3):

If $r \in R(A,B)$, then $r: A^{\#} \to B^{\#}$ is continuous

Proof:

Since $r(0) = 0$ and $\gamma\varphi_r(\gamma, a) = r(\gamma a)$

Let $\mathcal{F} \downarrow A^{\#}$, then $\mathcal{F} \geq \mathcal{H} = ţ\mathcal{H} \downarrow A$ (by Proposition 2.1.19)

This we have $r(\mathcal{F}) \geq r(\mathcal{H}) = r(ţ\mathcal{H}) \geq ţ\varphi_r(ţ, \mathcal{H})$

Let $\chi = ţ\varphi_r(ţ, \mathcal{H})$, then $\chi = ţ\chi \downarrow B$

Hence $r(\mathcal{F}) \downarrow B^{\#}$

■

Proposition (3.2.4):

Let A and B be two algebras. If $r_1, r_1 \in R(A,B)$, then;

1. $r_1 + r_1 \in R(A,B)$
2. $\alpha r_1 \epsilon R(A,B)$ for $\alpha \in K$.

Proof:

1. $$\varphi_{r_1+r_2}(\gamma, a) = \frac{1}{\gamma}(r_1 + r_2)(\gamma a)$$

$$= \frac{1}{\gamma} r_1(\gamma a) + \frac{1}{\gamma} r_2(\gamma a)$$

$$= \varphi_{r_1}(\gamma, a) + \varphi_{r_2}(\gamma, a)$$

This we have $\varphi_{r_1}(ţ, \mathcal{F}) + \varphi_{r_2}(ţ, \mathcal{F}) \leq \varphi_{r_1+r_2}(ţ, \mathcal{F})$

If $ţ\mathcal{F} \downarrow A$, then $\varphi_{r_1}(ţ, \mathcal{F}) \downarrow B$, $\varphi_{r_2}(ţ, \mathcal{F})$ and therefore

$(\varphi_{r_1}(ţ, \mathcal{F}) + \varphi_{r_2}(ţ, \mathcal{F})) \downarrow B$

Hence we have $\varphi_{r_1+r_2}(ţ, \mathcal{F}) \downarrow B$, and $r_1 + r_1 \in R(A,B)$.

2. The proof of this, by Lemma (3.2.2).

■

Proposition (3.2.5):

Let A, B *and* M be pseudo topological algebras. Assume that $r_{AB} \in R(A,B)$, $h \in H(A,B)$ and $r_{BM} \in R(B,M)$, then the mapping

$$r_{BM} o(h + r_{AB}) \in R(A,M)$$

Proof:

Since $(\varphi_{r_{BM}o(r_{AB}+h)})(\gamma,a)=\frac{1}{\gamma}(r_{BM}o(h+r_{AB}))(\gamma a)$

$$=\frac{1}{\gamma}r_{BM}\big(h(\gamma a)+r_{AB}(\gamma a)\big)$$

$$=\frac{1}{\gamma}r_{BM}\left(\gamma(h(a)+\varphi_{r_{AB}}(\gamma,a)\right)$$

$$=\varphi_{r_{BM}}\left(\gamma,(h(a)+\varphi_{r_{AB}}(\gamma,a)\right)$$

This we have $\varphi_{r_{BM}}\left(ţ,,(h(\mathcal{F})+\varphi_{r_{AB}}(ţ,\mathcal{F})\right)\leq\ \varphi_{r_{BM}o(r_{AB}+h)}(ţ,\mathcal{F})$

If $\mathfrak{X}=h(\mathcal{F})+\varphi_{r_{AB}}(ţ,\mathcal{F})$, then $ţ.\mathfrak{X}=ţ.(h(\mathcal{F})+\varphi_{r_{AB}}(ţ,\mathcal{F}))$

Now assume that $ţ\mathcal{F}\downarrow A$, we get $ţ\mathfrak{X}\downarrow B$, and hence $\varphi_{r_{AB}}(ţ,\mathfrak{X})\downarrow M$

This we get $(\varphi_{r_{BM}o(r_{AB}+h)})(ţ,\mathcal{F})\downarrow M$.

■

Proposition (3.2.6):

Let A *and* B be pseudo topological algebras. Then the only remainder map $r\in R(A,B)$ which is homomorphism is zero mapping whenever B is Hsusdorff.

Proof:

Let $a\in A$,

Since $ţ,[a]\downarrow A$ (by (3) of Remarek 2.1.4), then $\varphi_r(ţ,[x])\downarrow B$

Since $r\in R(A,B)$ is a homomorphism, then $\varphi_r(\gamma,x)=r(x)$,

Then we have $\varphi_r(ţ,[x])=r([x])=[r(x)]$

and therefore $[r(x)]\downarrow B$, hence $r(x)=0$

■

Proposition (3.2.7):

Let A and B be two pseudo topological algebras and $r\in R(A,B)$. Then; $r: A^0\to B^0$ is continuous

Proof:

Let $\mathcal{F}\downarrow A^0$, then from definition of A^0, there exists $\eta\leq\mathcal{F}$

From (Lemma 2.3.2), we get $\eta = ţ \cdot \eta$

Since $\gamma\varphi_r(\gamma, a) = \gamma \frac{1}{\gamma} r(\gamma a) = r(\gamma a)$

Let $\mathcal{F} \downarrow A^0$, then $\mathcal{F} \geq \eta = ţ\eta \downarrow A$

This we have $r(\mathcal{F}) \geq r(\eta) = r(ţ\eta) \geq ţ\varphi_r(ţ, \eta)$

Put $\eta' = ţ\varphi_r(ţ, \eta)$, then $\eta' = ţ\eta' \downarrow B$

Hence $r(\mathcal{F}) \downarrow B^0$

■

Definition (3.2.8)

Let $f: A \to B$ be a mapping between two of a pseudo topological algebras A and B. Then f is said to be differentiable at the point $a \in A$ if there exists linear mapping $l \in L(A, B)$ and $r \in R(A, B)$, such that

$$f(a + h) = f(a) + l(h) + r(h)$$

The map l is said to be derivative of f at the point a and we will denoted by $Df(a) = l$

Example (3.2.9):

Let A be pseudo topological algebra with identity 1. Let G be subset of A that contains all invertible elements. Let $g: G \to A$, $g(a) = a^{-1}$. Then g is differentiable at each point $a \in G$ and $Dg(a) = -a^{-1}ha^{-1}$.

Proof:

Since G is subalgebra,

By Proposition (2.3.2), then G it carries initial pseudo topological algebra

let $h \in A$ be small then

$$g(a + h) - g(a) = (a + h)^{-1} - a^{-1}$$

$$= \frac{1}{a + h} - \frac{1}{a}$$

$$= \frac{\frac{1}{a}}{1 + \frac{h}{a}} - \frac{1}{a}$$

$$= (1 - \frac{h}{a} + (\frac{h}{a})^2 - (\frac{h}{a})^3 + \cdots)\frac{1}{a} - \frac{1}{a}$$

$$= (1 - \frac{h}{a} + (\frac{h}{a})^2 - (\frac{h}{a})^3 + \cdots)\frac{1}{a} - \frac{1}{a}$$

$$= \frac{1}{a} - \frac{h}{a^2} + \frac{h^2}{a^3} - \frac{h^3}{a^4} + \cdots - \frac{1}{a}$$

$$= -\frac{h}{a^2} + \frac{h^2}{a^3} - \frac{h^3}{a^4} + \cdots$$

$$= -\frac{1}{a}(\frac{h}{a}) + \frac{h^2}{a^3} - \frac{h^3}{a^4} + \cdots$$

$$= -a^{-1}ha^{-1} + \frac{h^2}{a^3} - \frac{h^3}{a^4} + \cdots$$

Let $r(h) = \frac{h^2}{a^3} - \frac{h^3}{a^4} + \cdots$

Since $r(h) \to 0$, when $h \to 0$

Then we have $r(h) \in R(G;A)$

■

Proposition (3.2.10):

Let f: A → B be a differentiable at the point a, where A and B be two pseudo topological algebras then $f: A^{\#} \to B^{\#}$ is continuous at the point a.

Proof:

Since , $f(a+h) = f(a) + l(h) + r(h)$

$\ell \in \underline{L}(A;B)$ and $r \in R(A,B)$ then by Proposition (3.2.3), we get,

$f: A^{\#} \to B^{\#}$ is continuous at the point a.

■

Proposition (3.2.11):

Let A and B be a pseudo topological algebras. Then the continuous homomorphism $f: A \to B$ is differentiable at each point in A, also the remainder map is homomorphism and, $l(h) = f(h)$.

Proof:

$f\ (a+h) = f(a) + \ l(h) + r(h)$

Since f is homomorphism algebra then,

$f\ (a+h) = f\ (a) + f(h)$,

And by Proposition (3.2.6), we get $r(h) = 0$

$f\ (a) + f(h) = f(a) + \ l(h) + 0$

This we have $f(b) = \ l(b)$.

■

Proposition (3.2.12):

If g: A × B → M is bilinear continuous map, then g is differentiable at each point $a = (a, b) \in$ A × B, and $g'(a,b)(h_1, h_2)$= $g(h_1, b)$ + $\mathfrak{b}(a, h_2)$.

Proof:

Let $h = (h_1, h_2) \in A \times B$,

$g(a+h) = g(\text{ a} + h_1, b + h_2)$

$$= g(a,b) + g(h_1, b) + \ g(a, h_2) + g(h_1, h_2).$$

Therefore we have, with $l(h) = \ l(h_1, h_2) = \ g(h_1, b) + \ g(a, h_2)$,

and $r(h) = r(h_1, h_2) = g(h_1, h_2)$.

$g(m+h) = g(m) + \ l(h) + r(h)$.

l is homomorphism algebra, and since g is bilinear, and also continuous, since g is continuous. Therefore, $l \in \underline{H}$(A × B; M), and $r \in R$(A × B; M).

■

Proposition (3.2.13):

Let $A, B\ and\ M$ be pseudo topological algebras, and $f: A \to B$ is differentiable at $a \in A$, $g: B \to M$ is differentiable at $b = f(a) \in B$, then gof is differentiable at $a \in A$ such that $D(gof)(a) = Dg(b) \cdot Df(a))$.

Proof:

Since $f(a+h) = f(a) + l_1(h) + r_1(h)$

and $g(b+k) = g(a) + l_2(k) + r_2(k)$

such that $l_1 = Df(a) \in \mathcal{L}(A,B),\ l_2 = Dg(b) \in \mathcal{L}(B,M)$

$r_1 \in R(A,B),\ r_2 \in R(B,M)$.

If we combine the above two mapping, we get;

$$(gof)(a+h)$$
$$= (gof)(a) + l_2\big(l_1(h)\big) + l_2\big(r_1(h)\big) + r_2\big(l_1(h)\big) + r_2(r_1(h))$$
$$= (gof)(a) + l_2\big(l_1(h)\big) + l_2\big(r_1(h)\big) + r_2(l_1(h) + r_1(h))$$
$$l_2\big(l_1(h)\big) \in \mathcal{L}(A,M)$$

and by Lemma(3.2.2) we have $l_2\big(r_1(h)) \in R(A,M)\big)$

also by Proposition (3.2.5) we have $r_2(l_1(h) + r_1(h) \in R(A,M)$

using Proposition (3.2.4), we get

$l_2\big(r_1(h)\big) + r_2(l_1(h) + r_1(h)) \in R(A,M)$.

■

Proposition (3.2.14):

Let $A\ and\ B$ be two pseudo topological algebras, and let $f_1, f_2: A \to B$ two homomorphism algebra that are coincide in u_a (neighborhood filter of a). Then f_1 is differentiable at $a \in A$ if and only if f_2 is differentiable at a and $Df_1(a) = Df_2(a)$.

Proof:

Suppose f_1 is differentiable at $a \in A$, then for all $h \in A$ we have:

$f_1(a+h) = f_1(a) + l_1(h) + r_1(h)$, where $l_1 \in L(A,B), r_1 \in R(A,B)$
assume that $r_2: A \to B$ define by; $f_2(a+h) = f_2(a) + l_1(h) + r_2(h)$
we have to prove that $r_2 \in R(A,B)$
Since f_1, f_2 are coincide in u_a, then for all $x \in u_a$, we have:
$f_1(x) = f_2(x)$, and $r_1(h) = r_2(h)$ for all $h \in u_a - a$
since $(u_a - a)$ is neighborhood of 0 in A
Let $\omega = (u_a - a)$,
then ω is neighborhood of 0 in A, and so that:
$\omega \in \mathcal{F}$ for all $\mathcal{F} \downarrow A$.
Let ţǶ $\downarrow A$ (that Ƕ is quasi bounded filter on A)
This we have $\omega \in$ ţǶ
Therefor there exist ɨ $\in$ ţ and $\upsilon \in$ Ƕ such that ɨ $\cdot \upsilon \subset \omega$
So that $\varphi_{r_1}(\gamma, a) = \varphi_{r_2}(\gamma, a)$ for all $\gamma \in$ ɨ, $a \in \upsilon$
This we have φ_{r_1}(ţ, Ƕ) $= \varphi_{r_2}$(ţ, Ƕ)
Hence $r_2 \in R(A,B)$ if $r_1 \in R(A,B)$

Similarly, for the converse.

■

Proposition (3.2.15):

Suppose $f_1: A_1 \to B_1$, $f_2: A_2 \to B_2$ be two mappings of pseudo topological algebras. Then $f_1 \times f_1: A_1 \times A_2 \to B_1 \times B_2$ is differentiable at $a = (a_1, a_2) \in A_1 \times A_2$ if and only if f_1 is differentiable at a_1 and f_2 is differentiable at a_2. Also $D(f_1 \times f_1)(a_1, a_2) = Df_1(a_1) \times Df_2(a_2)$.

Proof:

Let $f_1(a_1+h)=f_1(a_1)+l_1(h)+r_1(h)$

$f_2(a_2+h)=f_2(a_2)+l_2(h)+r_2(h)$

Since $l_1\times l_2$ is linear if and only if both l_1 and l_2 are linear

Also $l_1\times l_2$ is continuous if and only if both l_1 and l_2 are continuous

Then we have $l_1\times l_2\in L(A_1\times A_2,B_1\times B_2)$ if and only if $l_1\in L(A_1,B_1)$ and $l_2\in L(A_2,B_2)$.

Now, we have to prove that $r_1\times r_2\in R(A_1\times A_2,B_1\times B_2)$ if and only if $r_1\in R(A_1,B_1)$ and $r_2\in R(A_2,B_2)$.

Let $r_1\in R(A_1,B_1)$ and $r_2\in R(A_2,B_2)$.

Let $\pi_j\colon B_1\times B_2\to B_j$, $p_j\colon A_1\times A_2\to A_j$ for $j=1,2$ the projections mapping .

Let $r_j\in R(A_j,B_j)$ for $j=1,2$

by the following commutative diagram

$$\begin{array}{ccc} A_1\times A_2 & \xrightarrow{r_1\times r_2} & B_1\times B_2 \\ {\scriptstyle p_j}\downarrow & & \downarrow{\scriptstyle \pi_j} \\ A_j & \xrightarrow{r_j} & B_j \end{array}$$

we have $r_j o p_j=\pi_j o(r_1\times r_2)$,

then from Lemma(3.2.2), we get;

$\pi_j o(r_1\times r_2)\in R(A_1\times A_2,B_j)$ for $j=1,2$.

Let $\text{ţǶ}\downarrow A_1\times A_2$, then $\varphi_{\pi_j o(r_1\times r_2)}(\text{ţ},\text{Ƕ})\downarrow B_j$ for $j=1,2$.

Since π_j is linear, then $\varphi_{\pi_j o(r_1\times r_2)}=\pi_j(\varphi_{(r_1\times r_2)})\downarrow B_j$ for $j=1,2$.

This we have $\varphi_{(r_1\times r_2)}(\text{ţ},\text{Ƕ})\downarrow B_1\times B_2$

Hence $r_1 \times r_2 \in R(A_1 \times A_2, B_1 \times B_2)$

Similarly for the converse , that is if $r_1 \times r_2 \in R(A_1 \times A_2, B_1 \times B_2)$, implies that $r_1 \in R(A_1, B_1)$ and $r_2 \in R(A_2, B_2)$.

■

The previous proposition can be generalized as follows;

Corollary (3.2.16):

Suppose $f_i: A_i \to B_i$, $i \in I$ be a family of mappings of a pseudo topological algebras. Then $\times f_i : \times A_i \to \times B_i$ is differentiable at $a = (a_i) \in \times A_i$ if and only if f_i is differentiable at each a_i Also $D(\times f_i)(a_i) = \times Df_i(a_i)$.

Proof:

The proof is direct from Proposition (3.2.15) by induction law.

■

Proposition (3.2.17):

Suppose $f_i: A \to A_i, i \in I$ be a family of mappings of a pseudo topological algebras. Then $\prod f_i : A \to \times A_i$ is differentiable at $a \in A$ if and only if $f_i: A \to A_i$ is differentiable at a for all $i \in I$. Also $D(\prod f_i)(a) = \prod Df_i(a)$.

Proof:

Let $\prod f_i : A \to \times A_i$ be differentiable at $a \in A$

Since $f_i = \pi_j o(\prod f_i)$ and from the continuity of the projective mapping π_j, and since it is linear mapping, we get the differentiability of f_i from the Proposition (3.2.15).

■

3.3 Higher Derivatives on Pseudo Topological Algebra

In this section, we will introduce concepts second derivative, higher derivative, and then we put the definition of C^∞- derivatives of the maps on a pseudo topological algebra.

We know that if $f: A_1 \times A_2 \to B$ be a bilinear and continuous, then b is differentiable at every point by Proposition(3.3.12), but is this possible for the next mapping $Df: A_1 \times A_2 \to L(A_1 \times A_2; B)$? this is what we will study in our next discussion.

In order to find the higher order derivatives on a pseudo topological algebra and to obtain more results we start with the following important definitions which we will need in this section.

Definition (3.3.1)

Let $\mathcal{F}$ be a filter on a pseudo topological algebra A, then we denote $cl(\mathcal{F})$ is the filter that is generated by the set $\{cl(F): F \in \mathcal{F}\}$.

It is clear that the set $\{cl(F): F \in \mathcal{F}\}$ from a falter base because $cl(F_1 \cap F_2) \subseteq cl(F_1) \cap cl(F_2)$.

Definition (3.3.2)

We call a pseudo topological algebra A, an admissible if and only if A^0 is separated, and for all $\mathcal{F} \downarrow A$, then we have $cl(\mathcal{F}) \downarrow A$ and $\mathcal{F}^0 \downarrow A$.

Lemma (3.3.3):

Let A, B and M be pseudo topological algebras. Then there is a linear isomorphism from $\mathcal{L}(A; \mathcal{L}(B; M))$ to $\mathcal{L}(A \times B; M)$.

Proof:

Let $g \in \mathcal{L}(A; \mathcal{L}(B; M))$

Consider the map $g \longmapsto \varphi(g)$, where $\varphi(g) = \mathbb{b}$,

Where $\mathbb{b}: A \times B \to M$ define as $\mathbb{b}(a, b) = \big(\varphi(g)\big)(a, b) = (g(a))(b)$

It is clear that $\mathbb{b}$ is bilinear map, and $\mathbb{b} \in \mathcal{L}(A \times B; M)$

We have to prove that φ is bijective and continuous;

1. Let $\mathbb{b} \in \mathcal{L}(A \times B; M)$,

For all $a \in A$, define the map $g_a: B \to M$ by $g_a(b) = \mathbb{b}(a, b)$

Since $\mathbb{b}: A \times B \to M$ is continuous, then $g_a: B \to M$ is continuous

This we have $g_a \in \mathcal{L}(B; M)$

Define $g = \varphi^*(\mathbb{b}): A \to \mathcal{L}(B; M)$, by $g(a) = g_a$

This we have $g = \varphi^*(\mathbb{b})$ is a linear map

Also $g: A \to \mathcal{L}(B; M)$ is continuous

This we have $g = \varphi^*(\mathbb{b}) \in \mathcal{L}(A; \mathcal{L}(B; M))$

And therefor $\varphi^*: \mathcal{L}(A \times B; M) \to \mathcal{L}(A; \mathcal{L}(B; M))$ is a mapping

But $\varphi\big(\varphi^*(\mathbb{b})\big) = \mathbb{b}$, for all $\mathbb{b} \in \mathcal{L}(A \times B; M)$

and $\varphi^*\big(\varphi(g)\big) = g$, for all $g \in \mathcal{L}(A; \mathcal{L}(B; M))$

This φ is bijective , and φ^* is inverse

2. Now we have to prove that φ is continuous

   Since φ is linear, It is sufficient to prove that φ is continuous at 0

   Let Ƕ $\downarrow \mathcal{L}(A; \mathcal{L}(B; M))$,

   we have to prove that $\varphi($Ƕ$) \downarrow \mathcal{L}(A \times B; M)$

   Since $\varphi($Ƕ$) \downarrow \mathcal{L}(A \times B; M)$ if for all ţ $\cdot \mathcal{F}_1 \downarrow A$ and ţ $\cdot \mathcal{F}_2 \downarrow B$

   Then $(\varphi($Ƕ$))(\mathcal{F}_1, \mathcal{F}_2) \downarrow M$

   But $\big(\varphi($Ƕ$)\big)(\mathcal{F}_1, \mathcal{F}_2) = ($Ƕ$(\mathcal{F}_1))(\mathcal{F}_2)$

   Then Ƕ$(\mathcal{F}_1) \downarrow \mathcal{L}(B; M)$

■

Remarek(3.3.4)

i. In the same way, we can prove that

   $\mathcal{L}^{\#}(A; \mathcal{L}^{\#}(B; M)) \approx \mathcal{L}^{\#}(A \times B; M)$.

ii. The above lemma can be generalized using mathematical induction to obtain $\mathcal{L}(A; \mathcal{L}_{p-1}(B; M)) \approx \mathcal{L}_p(A \times B; M)$

Lemma (3.3.5):

Let $n, m \in N$, then there is an isomorphism from $\mathcal{L}^m(A, \mathcal{L}^n(A, B))$ to $\mathcal{L}^{m+n}(A, B)$, defined by

$(\psi^{m,n} f) a_1 \dots \dots . a_{m+n} = (f a_1 \dots a_m) a_{m+1} \dots . a_{m+n}$.

Proof:

We can prove this lemma after referring to lemma(3.3.3) and using mathematical induction.

■

Definition (3.3.6)

Let A, B be a equable and admissible pseudo topological algebras, then the map $f: A \to B$ is said to be twice differentiable at $a \in A$, if and only if $Df: A \to \mathcal{L}^{\#}(A; B)$ exists for all neighborhood of a and is differentiable at a.

We will denote the second derivative by the symbol $D^2 f(a)$.

It is clear that $D^2 f(a) \in \mathcal{L}^{\#}(A; \mathcal{L}^{\#}(A; B))$.

But by Lemma (3.3.3), $\mathcal{L}^{\#}(A; \mathcal{L}^{\#}(A; B)) \approx \mathcal{L}^{\#}{}_2(A; B)$,

then we get $D^2 f(a) \in \mathcal{L}^{\#}{}_2(A; B)$.

Definition (3.3.7)

A map $f: A \to B$ of equable and admissible algebras is called t-times differentiable at $a \in A$ if and only $f^{t-1}: A \to \mathcal{L}^n{}_{t-1}(A, B)$ exists in a neighborhood of a and is differentiable at a.

This we have $D^{t-1} f(a) \in \mathcal{L}^{\#}(A; \mathcal{L}^{\#}{}_{t-1}(A; B))$

Since $\mathcal{L}^{\#}(A; \mathcal{L}^{\#}{}_{t-1}(A; B)) \approx \mathcal{L}^{\#}{}_t(A; B)$,

Then we will using $f^{(t)}(a)$ instead of $Df^{(t-1)}(a)$

Proposition(3.3.8):

Let $f: A \to B$ is t-times differentiable and $f^{(t)}: A \to \mathcal{L}^{\#}{}_{t}(A; B)$ is p-times differentiable, then $f: A \to B$ is (t+p)-times differentiable.

Proof:

Since $f^{(t)}: A \to \mathcal{L}^{\#}{}_{t}(A; B)$ is p-times differentiable,

then $(f^{(t)})^{(p-1)}: A \to \mathcal{L}^{\#}{}_{p-1}(A; \mathcal{L}^{\#}{}_{t}(A; B))$ is differentiable,

Using Lemma (3.3.3) we have,

$f^{(t+p-1)}: A \to \mathcal{L}^{\#}{}_{t+p-1}(A; B)$ is differentiable.

■

Proposition(3.3.9):

Let $f: A \to B_1$ and $g: A \to B_2$ be t-times differentiable at $a \in A$, then the mapping $(f, g): A \to B_1 \times B_2$ t-times differentiable at a and

$(f, g)^{(t)}(a) = (f^{(t)}(a), g^{(t)}(a))$

Proof:

We will prove that using mathematical induction,

For $t = 1$, this by Proposition(3.2.17)

Now suppose that our claim is true $t - 1$, and we want to prove that it is true for t-times.

For that, we assume that $f: A \to B_1$ and $g: A \to B_2$ are t-times differentiable at $a \in A$.

Since $\mathcal{L}_{t-1}(A; B_1 \times B_2) \approx \mathcal{L}_{t-1}(A; B_1) \times \mathcal{L}_{t-1}(A; B_2)$

The mapping $a \longmapsto (f, g)^{t-1}$ is differentiable at a and this derivative belongs to $\mathcal{L}_{t-1}(A; B_1 \times B_2)$

and therefor belongs to $\mathcal{L}_{t-1}(A; B_1) \times \mathcal{L}_{t-1}(A; B_2)$

$(f^{(t-1)}, g^{(t-1)})(a) = (f^{(t-1)}(a), g^{(t-1)}(a))$

$D(f^{(t-1)}, g^{(t-1)})(a) = (Df^{(t-1)}(a), Dg^{(t-1)}(a))$

$= (f^{(t)}(a), g^{(t)}(a))$

■

Definition (3.3.10):

The map $f: A \to B$ is C^k-map if f is K-times differentiable in A and $f^{(k)} \in C(A; \mathcal{L}^{\#}{}_K(A; B))$, where $k = 0,1,2 \dots.$

We write $f \in C_k(A; B)$, if f is K-times differentiable in A.

Proposition(3.3.11):

Let f be K-times differentiable in A. Then $f^{(t)} \in C^0(A; \mathcal{L}^{\#}{}_t(A; B))$, for $t = 0,1,2 \dots k.$

Proof:

If $k = 0$, the proof it is clear

If $k = 1$, let $f' \in C^0(A; \mathcal{L}^{\#}(A; B))$,

we have to prove that $f \in C^0(A; B)$,

Let ţ$\mathcal{F} \downarrow A$, and put $f_0(a) = f(a) - f(0)$,

Let $\beta = [0,1] \cdot \mathcal{F}$

Since $f_0{}' = f$ is quasi- bounded

From $\begin{cases} ţ \cdot \beta = ţ \cdot \mathcal{F} \downarrow A \\ ţ \cdot f_0{}'(\beta) \downarrow \mathcal{L}^{\#}(A; B) \end{cases}$,

we have $\left(ţ \cdot f_0{}'(\beta)\right)(\beta) = (ţ \cdot f_0)'(\beta) \downarrow B$

$$\begin{aligned} ţ \cdot \mathcal{F} &\supseteq ţ \cdot f_0(\mathcal{F}) + ţ \cdot f(0) \\ &\supseteq (ţ \cdot f_0)(\beta) + ţ \cdot f(0) \\ &\supseteq ((ţ \cdot f_0)'(\beta) \cdot (\beta))^0 + ţ \cdot f(0) \end{aligned}$$

■

Chapter Four

Applications and Conclusion

Introduction

The final chapter includes some applications and generalizations using pseudo-topology compatible with algebra.

This chapter consists of three sections:

The first section we present Lipschitz and boundedly convex function via pseudo topological algebra, while in the seconed section we will make a generalization for the concept of Gelfand-Mazur theory from the banach algebra to the pseudo-structures according to certain conditions. Finally, the third section includes the most important conclusions that we have reached through this thesis.

4.1 Lipschitzand Boundedly Convex Function via Pseudo Topological Algebra

It is known in functional analysis that the function $f: A \to R$ is called boundedly-convex function If the following condition is met;

$$\lambda_1 f(a_1) + \lambda_2 f(a_2) - f(\lambda_1 a_1 + \lambda_2 a_2) \leq \frac{1}{2}\lambda_1 \lambda_2 M \|a_1 - a_2\|^2$$

for all $a_1, a_2 \in A$ and $\lambda_1, \lambda_2 \in R$ such that $\lambda_1, \lambda_2 \geq 0,\ \lambda_1 + \lambda_2 = 1$.

J. Focke, had previously proved that any continuous boundedly-convex function f in Banach space A, the Frechet derivative $f': A \to L(A, R)$ is Lipschitz- continuous, which means that it, for all $a_1, a_2 \in A$, there exists $M > 0$, such that $\|f'(a_1) - f'(a_2)\| \leq M\|a_1 - a_2\|$.

In this section, we will expand the concepts of boundedly-convex and Lipschitz- continuous functions from Banach spaces to pseudo-topology structures that are compatible with algebra A.

Definition (4.1.1):

Let γ be a pseudo topology compatible with an algebra A over the filed $\mathbb{C}$.

The mapping $f: A \to \mathbb{C}$ is said to be boundedly convex if;

1. It is convex function (f is convex if and only if $f(\lambda_1 a_1 + \lambda_2 a_2) \leq \lambda_1 f(a_1) + \lambda_2 f(a_2)$)
2. There exists a continuous homogeneous function of degree 2 $\rho: A \to \mathbb{C}$ such that for all a_1, $a_2 \in A$, and for all λ_1, $\lambda_2 \in R$, λ_1, $\lambda_2 \geq 0 \; with \; \lambda_1 +, \lambda_2 = 1$ we have $0 \leq \lambda_1 f(a_1) + \lambda_2 f(a_2) - f(\lambda_1 a_1 + \lambda_2 a_2) \leq \lambda_1 \, \lambda_2 \rho(a_2 - a_1)$.

If ţ • $[\rho] \downarrow C_\gamma(A; R)$ we say for f is a γ- boundedly convex.

Note: We call the function $f: R^n \to R$ a homogeneous function of degree t if $f(\lambda a) = \lambda^t f(a)$

Remark (4.1.2):

It is clear that;

1. If f is a γ- boundedly convex, then $f + c$ is also γ- boundedly convex, for every constant c.
2. If f is a γ- boundedly convex and g is any anther linear mapping then $f + g$ is also γ- boundedly convex.
3. If f is a γ- boundedly convex, then the translation of f by any vector h is also γ- boundedly convex.

Definition (4.1.3):

We define the set $\mathfrak{F}_{R,A}$ as follows;

$\mathfrak{F}_{R,A} = \{f_{t,a}: t \in R, a \in A\}$

Where $f_{t,a}: A \to B$, such that $f_{t,a} = 0 \; if \; t = 0$, and

$f_{t,a}(h) = \frac{f(a+th)-f(a)}{t}$ if $t \neq 0$

Definition (4.1.4):

Let γ be a pseudo topology on $C(A;B)$, the mapping $f: A \to B$ is said to be γ- Lipschitz-continuous if the set $\mathfrak{F}_{R,A}$ is bounded in $C_\gamma(A,B)$, i.e $ţ\,\mathfrak{F}_{R,A} \downarrow C_\gamma(A,B)$.

Lemma (4.1.5):

Let A *and* B be a normed algebra. Then the mapping $g: A \to B$ is γ-Lipschitz-continuous if and only if there exists $\mathcal{M} > 0$ such that for all a_1, $a_2 \in A$, $\|g(a_2) - g(a_1)\| \leq \mathcal{M}\|a_1 - a_2\|$

Proof:

Let $\|g(a_2) - g(a_1)\| \leq \mathcal{M}\|a_1 - a_2\|$

Then for all $t > 0$, $a \in A$, we have

$$\left\|\frac{g(a+th) - g(a)}{t}\right\| = \frac{1}{t}\|g(a+th) - g(a)\|$$

$$\leq \frac{1}{t}\mathcal{M}\|a + th - a\|$$

$$= \frac{1}{t}\mathcal{M}\|th\| = \mathcal{M}\|h\|$$

And so that for any bounded set Q in A, the set

$\mathfrak{F}_{R,A}(Q) = \{\frac{g(a+th)-g(a)}{t} : t \in R, a \in A, h \in Q\}$ is bounded in B.

Conversely,

let the set $\mathfrak{F}_{R,A}(Q)$ is bounded set in B for any bounded set Q in A.

If we take unit ball as Q, and the norms of $\mathfrak{F}_{R,A}(Q)$ such that this norms not exceed $\mathcal{M}$, then for all $t \in R, a \in A, h \in Q$ we have;

We know that $h = \|h\|_e$, where $\|e\| = 1$.

If we choose $\|h\| = \varepsilon > 0$, then we have

$$\|g(a+h) - g(a)\| = \varepsilon\left\|\frac{g(a+\varepsilon h) - g(a)}{\varepsilon}\right\|$$

$$\leq \varepsilon\mathcal{M} = \mathcal{M}\|h\|.$$

■

We now present the main theorem in this section

Theorem (4.1.6):

Let γ be a pseudo topology on $C(A;B)$, and the continuous convex mapping $f: A \to R$ is γ- boundedly convex, such that the corresponding mapping ρ in $0 \le \lambda_1 f(a_1) + \lambda_2 f(a_2) - f(\lambda_1 a_1 + \lambda_2 a_2) \le \lambda_1 \lambda_2 \rho(a_2 - a_1)$ is also continuous, then f is γ- differentiable everywhere and its derivative $Df: A \to \mathcal{L}_\gamma(A, R)$ is γ- Lipschitz-continuous.

Proof:

Firstly, we have to prove that f is γ- differentiable everywhere,

from the convexity of the function f, the restriction of f on to the straight segment is aconvex continuous, then there exists one sided derivatives for all point of straight segment.

This f is differentiable for all point a in any direction h,

We denote to the maping $h \to D_h f(a) = \lim_{t\to 0} \frac{f(a+th)-f(a)}{t}$, $A \to B$ by $f'(a)$.

Now we will prove that $f'(a)$ is linear and continuous.

From the definition (4.1.1), we have

$0 \le \lambda_1 f(a_1) + \lambda_2 f(a_2) - f(\lambda_1 a_1 + \lambda_2 a_2) \le \lambda_1\, \lambda_2 \rho(a_2 - a_1)$

If we put $a_1 = th_1$, $a_2 = th_2$ such that $0 < t < 1$ and $h_1, h_2 \in A$, we get

$0 \le \lambda_1 f(th_1) + \lambda_2 f(th_2) - f(\lambda_1 th_1 + \lambda_2 th_2) \le \lambda_1\, \lambda_2 \rho(th_1 - th_2)$

Divide by t we get

$$0 \le \frac{\lambda_1 f(th_1) + \lambda_2 f(th_2) - f(\lambda_1 th_1 + \lambda_2 th_2)}{t} \le \frac{\lambda_1\, \lambda_2 \rho(th_1 - th_2)}{t}$$

Since ρ is homogeneous of degree 2,

$0 \le \lambda_1 \frac{f(th_1)}{t} + \lambda_2 \frac{f(th_2)}{t} - \frac{f(\lambda_1 th_1 + \lambda_2 th_2)}{t} \le \lambda_1\, \lambda_2 \frac{t^2 \rho(h_1 - h_2)}{t}$

for $t \to 0$, we have;

$0 \le \lambda_1 f'(0)h_1 + \lambda_2 f'(0)h_2 - f'(0)(\lambda_1 h_1 + \lambda_2 h_2) \le 0$

this $\lambda_1 f'(0)h_1 + \lambda_2 f'(0)h_2 - f'(0)(\lambda_1 h_1 + \lambda_2 h_2) = 0$

$$\lambda_1 f'(0)h_1 + \lambda_2 f'(0)h_2 = f'(0)(\lambda_1 h_1 + \lambda_2 h_2)$$

For all $\lambda_1\lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$

This shows that function $f'(0)$ is a linear function.

To prove that $f'(0)$ is continuous,

put $a_1 = 0$, $a_2 = h$, $\lambda_1 = 1 - t$, and $\lambda_2 = t$

$$0 \leq tf(h) - f(th) \leq t(1-t)\rho(h)$$

$$0 \leq -f(th) + tf(h) \leq t(1-t)\rho(h)$$

Dividing by t we get; $0 \leq -\frac{f(th)}{t} + f(h) \leq (1-t)\rho(h)$

And where $t \to 0$, we get $0 \leq -f'(0)h + f(h) \leq \rho(h)$

When $h \to 0$, then $f(h) \to 0$ and $\rho(h) \to 0$ (since f and ρ are continuous)

Then we get $f'(0)h \to 0$ if $h \to 0$, this means $f'(0)$ is continuous at 0.

Now, we have to prove that f is γ- differentiable at every point a.

Since the addition affine functions and translations do not disturb γ-differentiable, we can take $a = 0, f(0) = 0, f'(0) = 0$.

Now we have to prove that $r_t \downarrow C_\gamma(A;B)$ where $t \to 0$

$(r_t = h \to \frac{f(th)-f(0)}{t} = \frac{f(th)}{t})$

Since $0 \leq \lambda_1 f(a_1) + \lambda_2 f(a_2) - f(\lambda_1 a_1 + \lambda_2 a_2) \leq \lambda_1\lambda_2\rho(a_2 - a_1)$

Put $a_1 = -th$, $a_2 = th$, we get;

$0 \leq \frac{1}{2}f(-th) + \frac{1}{2}f(th) \leq \frac{1}{4}\rho(2th) = t^2\rho(h)$

Since $f(0) = 0, f'(0) = 0$, and f is convex, then $f(-th), f(th) \geq 0$

and therefor $0 \leq \frac{f(th)}{t} \leq 2t\rho(h)$,

this we have $0 \leq r_t \leq 2t\rho(h)$

when $t \to 0$, we get $t\rho(h) \to 0$, this means $t\rho \downarrow \gamma(A;B)$,

but ρ is $\gamma - bounded$, then $r_t \downarrow C_\gamma(A;B)$.

■

Theorem (4.1.7):

Let γ be a pseudo topology on $A \times B$, and the continuous convex mapping $f: A \to R$ is γ- boundedly convex, such that the corresponding mapping ρ in $0 \leq \lambda_1 f(a_1) + \lambda_2 f(a_2) - f(\lambda_1 a_1 + \lambda_2 a_2) \leq \lambda_1 \lambda_2 \rho(a_2 - a_1)$ is also continuous, then its derivative $Df: A \to \mathcal{L}_\gamma(A, R)$ is γ-Lipschitz-continuous.

Proof:

We have to prove that $ţ\, \mathfrak{F}'_{R,A} \downarrow \mathcal{L}_\gamma(A, \mathcal{L}_\gamma(A, R))$

Where $\mathfrak{F}'_{R,A} = \{f'_{t,a} : t \in R, a \in A\}$

Since $\mathcal{L}_\gamma(A, \mathcal{L}_\gamma(A, R)) \approx \mathcal{L}_\gamma(A \times A, R)$ by Lemma (3.3.3)

Therefore, it suffices to prove that $ţ\, \mathfrak{F}'^{*}_{R,A} \downarrow \mathcal{L}_\gamma(A \times A, R)$

Where $\mathfrak{F}'^{*}_{R,A}$ is the filter in $\mathcal{L}_\gamma(A \times A, R)$, corresponds to $\mathfrak{F}'_{R,A}$ in $\mathcal{L}_\gamma(A, \mathcal{L}_\gamma(A, R))$ by the canonical isomorphism.

For $a, h_1, h_2 \in A$, we have ;

$$f'_{t,a}(h_1, h_2) = f'_{t,a}(h_1) \cdot h_2$$

$$= \frac{f(a + th_1) - f(a)}{t} \cdot h_2$$

Take $a_1 = a$, $a_2 = a + th_1$, $a_3 = a + th_2$,

$$a_4 = a + th_1 - th_2, \quad a_0 = a + \frac{1}{2} th_1$$

It is clear that is the center of the parallelogram with vertices a_1, a_2, a_3, a_4

By Definition (4.1.1), we get;

$$\tfrac{1}{2} f(a_1) + \tfrac{1}{2} f(a_2) - f(a_0) \leq \tfrac{1}{4} t^2 \rho(h_1).$$

Since f is convex, we have;

$$\frac{1}{2} f(a_3) + \frac{1}{2} f(a_4) - f(a_0) \geq 0$$

From the previous theorem we get;

$$f(a_3) - f(a_1) - f'(a_1) \cdot (th_2) \leq \rho(th_2) = t^2 \rho(h_2)$$

$$f(a_4) - f(a_2) - f'(a_2) \cdot (-th_2) \leq \rho(th_2) = t^2 \rho(h_2)$$

From the previous four inequalities, after simplification, we conclude the following;

$$f'(a_2)\cdot th_2 - f'(a_1)\cdot th_2 \leq \frac{t^2}{2}\rho(h_1) + 2t^2\rho(h_2)$$

Dividing by t^2 we get;

$$\frac{f'(a+th_1)\cdot - f'(a)}{t}\cdot h_2 \leq \frac{1}{2}\rho(h_1) + 2\rho(h_2)$$

■

4.2 Generalization of Gelfand-Mazur theorem via pseudo Topological structures

The Gelfand-Mazur's theorem is one of the most important theories in Banach Algebra due to its many applications in other sciences, especially in physics. This theorem states that there is isomorphic from the commutative Banach Algebra with identity e and the field of complex numbers $\mathbb{C}$. The remainder of this section will be devoted to generalizing that important theory from Banach algebra to pseudo-topological structures, this means that generalization of the proof of this theory to the pseudo topology(convergence space) this structure without norm and that is compatible with Algebra structures.

We will mention some basic concepts that we will need later regarding spectrum theory on an algebra (it is not necessary to be a Banach algebra) Let A be an algebra over the field complex numbers $\mathbb{C}$ and $a \in A$ then:

i. a is a unit if a is invertible in A i.e $\exists b \in A$ such that $ab = e$
ii. $\sigma(a) = \{\lambda \in \mathbb{C}: (a - \lambda e)$ has no invers in A $\}$, $\sigma(a)$ is called spectrum of $a \in A$.

From now on, we assume that Algebra over the complex field $\mathbb{C}$.

Definition (4.2.1):

Let A be a pseudo topological algebra. If the map $a \to a^{-1}$ is continuous at e, then we say for A is a pseudo topological algebra with continuous invers.

Proposition(4.2.2):

Let A be a pseudo topological algebra with continuous invers. Then the map $a \to a^{-1}$ is continuous on every element contained in Q (where Q is the set of all elements that have an inverse).

Proof:

Let $\mathcal{F} \downarrow_a Q$,

we have to prove that $\mathcal{F}^{-1} \downarrow_{a^{-1}} Q$, where $\mathcal{F}^{-1} = \{F^{-1}: F \in \mathcal{F}\}$.

From the definition of algebra we know that if b has inverse then the map $a \to ab$ is homeomorphism of A onto A.

Therefore, which element a belongs to Q,

we have $b \to a^{-1}b$ is homeomorphism.

Since $\mathcal{F} \downarrow_a Q$, then $a^{-1}.\mathcal{F} \downarrow_e Q$.

From the continuity of the map $a \to a^{-1}$ at e,

we have $a.\mathcal{F}^{-1} \downarrow_e Q$,

and therefore $\mathcal{F}^{-1} \downarrow_{a^{-1}} Q$.

■

To complete what we want to reach and prove in this research, we need to generalize Liouville's theorem .In order to extend Liouville's theory from Banach algebra to pseudo topology that is compatible with an algebra structure.

Theorem (4.2.3): (Generalization of Liouville theorem)

Let A be a pseudo topological algebra and let $C(A)$ the space of all continuous homomorphism algebra from A to the filed $\mathbb{C}$ (complex number) such that $C(A)$ has the property; if $g(a) = 0$ for all $g \in C(A)$ leads to $a = 0$. Then if the mapping $f: \mathbb{C} \to A$ is bounded and analytic then f must be constant.

Proof:

For all $g \in C(A)$, if f is bounded, then gof is bounded also(since g is continuous homomorphism algebra and the image of bounded set is bounded).

Theorem Liouville can be applied to the mapping gof in order to satisfy the required conditions. To prove that f is a constant map, let $\lambda_1, \lambda_2 \in \mathbb{C}$ then we have:

$gof(\lambda_1) = gof(\lambda_2)$,

then $g(f(\lambda_1)) = g(f(\lambda_2))$,

therefor $g\big(f(\lambda_1)\big) - g\big(f(\lambda_2)\big) = 0$

and $g(f(\lambda_1) - f(\lambda_2)) = 0$,

this happens for all $g \in C(A)$,

using the property we put into the operative theorem,

then we get $f(\lambda_1) - f(\lambda_2) = 0$, so $f(\lambda_1) = f(\lambda_2)$

since λ_1, λ_2 are arbitrary element in $\mathbb{C}$, then f must be constant.

■

Theorem (4.2.4): (Generalization of Gelfand-Mazur Theorem)

Let A be a pseudo topological algebra that has continuous inverse. Then

i. $\sigma(a) \neq \varphi$ for any $a \in A$

ii. If for all $a \neq 0 \in A$ has inverse, then A is isomorphic to complex number.

Proof:

Suppose $\sigma(a) = \varphi$ for some $a \in A$, then for all $\lambda \in \mathbb{C}$, $(a - \lambda e)$ has inverse,

Define the map $\delta_a: \mathbb{C} \to A$ as $\delta_a(\lambda) = (a - \lambda e)^{-1}$. We have to prove that δ_a is analytic and bounded.

Let $\lambda_1, \lambda_2 \in \rho(a)$,

$$\delta_a(\lambda_1) - \delta_a(\lambda_1) = \frac{1}{(a-\lambda_1 e)} - \frac{1}{(a-\lambda_2 e)}$$

$$= \frac{\lambda_1 - \lambda_2}{(a - \lambda_1 e)(a - \lambda_2 e)}$$

$$= (\lambda_1 - \lambda_2)\delta_a(\lambda_1)\delta_a(\lambda_1),$$

This we have, $\frac{\delta_a(\lambda_1)-\delta_a(\lambda_1)}{\lambda_1-\lambda_2} = \delta_a(\lambda_1)\delta_a(\lambda_1)$

To prove $\delta_a(\lambda)$ is bounded, let the filter $\mathcal{F} = [F_n]$, (the filter that generated by the sets F_n), where $F_n = \{\lambda \in \mathbb{C}: |\lambda| \geq n\}$, $n = 1,2, ..$

Let $\chi \in \gamma(a)$, since $|\lambda| \geq n$,

then $|\lambda|^{-1} \leq n^{-1}$ for all $\lambda \in F_n \in \mathcal{F}$

This $|\lambda|^{-1}\chi \leq n^{-1}\chi$

Consider $\mathcal{F}^{-1} = [F_n{}^{-1}]$,

then we have $\mathcal{F}^{-1}[a] \downarrow A$,

hence $e - \mathcal{F}^{-1}[a] \downarrow_e A$

Since $e - \lambda^{-1}a$ must be have inverse and by continuous inverse property we get $(e - \mathcal{F}^{-1}[a])^{-1} \downarrow_{e^{-1}} A$,

but $e^{-1} = e$, hence $(e - \mathcal{F}^{-1}[a])^{-1} \downarrow_e A$,

and therefor δ_a is bounded.

So that and by generalization of liouvilles (theorem 4.2.3) we get that δ_a must be constant map.

Since $(e - \mathcal{F}^{-1}[a])^{-1} \downarrow_e A$, and A is Hausdorff,

then $(e - \lambda^{-1}a)^{-1} = e$ for all $\lambda \in \mathcal{F}$.

So we conclude that $\delta_a(\lambda) = e$ for all λ, but this is impossible.

Therefore hypothesis $\sigma(a) = \varphi$ has come to a contradiction.

ii. $\sigma(a) \neq \varphi$, according to the result obtained in (i),

this for all $a \in A$, there exists at least one $\lambda \in \mathbb{C}$ such that $a - \lambda e$ is singular.

This we have, for some $\lambda \in \mathbb{C}$, $a - \lambda e = 0$,

so that $a = \lambda e$ for some λ, and $A = \mathbb{C}e$.

It is clear that the mapping $a \to \lambda e$ is algebra homomorphism.

It remains for us to prove the continuity of this map in a pseudo topology γ that is compatible with algebra structure , for that we assume that the filter

$G \in \gamma(a), a \in A$, since there exists some $\lambda \in \mathbb{C}$ such that $a = \lambda e$, then $G \in \gamma(\lambda e)$.

■

4.3Conclusions

1. **1.** We introduced pseudo-topological structures through the language of convergence of filters when the space under hand is the algebra and the terms that make those structures compatible with the algebra by when the three operations of the algebra are continuous operations.
2. **2.** Expanding the concept of weak and strong topology on pseudo-topological structures via the so-called the concept initial and final pseudo topological algebra.
3. **3.** Locally convex algebra can be obtained via pseudo-topological structures rather than the way it is generated through of family of all continuous seminorms.
4. **4.** In proposition(2.3.8), the relationship between A^0(locally convex algebra that generated by pseudo-topological structures) and $\tau(A)$ (locally convex modification of A). Among the most important results were $f: A \to B$ is continuous if and only if $f: \tau(A) \to B$ is continuous. And If $f: A \to B$ is continuous homomorphism algebra, so is $f: A^0 \to B^0$.
5. **5.** The space $C(A, B)$ of all continuous mapping is a complete space if it is done by pseudo topological structures and that make the evaluation mapping $\varpi_{A\times B}: C(A, B) \times A \to B$ is continuous and when B also complete pseudo topological algebra. The concept of the Frechete derivative of Banach Algebra is extended to pseudo-topological structures without the use of norm. Also the concepts of second derivative, higher derivative, and the definition of C^{∞}- derivatives of the maps on a pseudo topological algebra.

6. Expand the concepts of boundedly-convex and Lipschitz- continuous functions from Banach spaces to pseudo-topology structures that are compatible with algebra A. That is, if γ be a pseudo topology on $C(A;B)$, and the continuous convex mapping $f: A \to R$ is $\gamma-$ boundedly convex, such that the corresponding mapping ρ in $0 \leq \lambda_1 f(a_1) + \lambda_2 f(a_2) - f(\lambda_1 a_1 + \lambda_2 a_2) \leq \lambda_1 \lambda_2 \rho(a_2 - a_1)$ is also continuous, then f is $\gamma-$ differentiable everywhere and its derivative $Df: A \to \mathcal{L}_\gamma(A, R)$ is γ −Lipschitz-continuous.

7. Generalizing the Gelfand-Mazur's theorem from Banach algebra to pseudo-topological structures, this means that generalization of the proof of this theory to the pseudo topology(convergence space) this structure without norm and that is compatible with Algebra structures.

Future Works

1. It is possible to find more generalizations and move from standard Banach spaces to structures that are broader than them, such as studying Frechet manifolds and others.
2. Extending the concept of the Frechet derivative on a power series of differentiable functions
3. Expand and study differential equations in a space of a pseudo topological algebra. This has a major role and wide applications in quantum mechanics.
4. Use these pseudo structures to approximate functions to polynomials without using a norm.

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المستخلص

في هذه الاطروحة ، قدمنا مفهوم الهياكل التبولوجية الزائفة المتوافقة مع الجبرا. وقد وضعنا الشروط الازمة للتبولوجيا الزائفة لتكون متوافقة مع الجبرا. قدمنا ايضا مفهوم الجبر التبولوجي الزائف الابتدائي والنهائي . ايضا تمكنا من توليد الجبر التبولوجي المحدب محليا من خلال هذه الهياكل وعلاقتها مع لجبر التبولوجي المحدب محليا المتولد من عائلة كل المقاييس الشبه معيارية المستمرة.

هذه الهياكل مكنتنا من التوسع في مفهوم مشتقة فريشية على فضاء بناخ الى فضاءات لجبر التبولوجي الزائف ومن ثم الانتقال الى المشتقات ذات الرتب العليا وصولا الى فضاء الدوال القابلة للاشتاق بعدد ن من المرات .

قدمنا ايضا بعض التطبيقات المهمة والتي كان اغلبها هو تعميم من فضاء معياري كامل الى تلك الفضاءات الجديدة ومن اهما توسيع مبرهنة جلفاند-مازور .

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

الجبر التبولوجي من خلال البنى التفاضلية الزائفة

أطروحة

مقدمة إلى مجلس كلية التربية للعلوم الصرفة في جامعة بابل

كجزء من متطلبات نيل درجة دكتوراه فلسفة في التربية \ الرياضيات

من قبل

فالح عبد المهدي جابر

بأشراف

أ. د. زاهر دبيس النافعي

1445هـ **2023م**