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Resolvable in i-topological Proximity Space

A Dissertation

Submitted to The Council of The College of Education for Pure Sciences in the University of Babylon in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Education / Mathematics

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بسم الله الرحمن الرحيم

﴿اللَّهُ وَلِيُّ الَّذِينَ ءَامَنُواْ يُخْرِجُهُم مِّنَ الظُّلُمَاتِ إِلَى النُّورِ وَالَّذِينَ
كَفَرُواْ أَوْلِيَآؤُهُمُ الطَّاغُوتُ يُخْرِجُونَهُم مِنَ النُّورِ إِلَى الظُّلُمَاتِ
أُوْلَئِكَ أَصْحَابُ النَّارِ هُمْ فِيهَا خَالِدُونَ﴾
صدق الله العلي العظيم

البقرة (257)

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Abstract

This Study aim to gathering three important concepts, to present the resolvable in i-topological proximity space, so we introduced new concept and improve the old ones, there fore we divided this work into four stages as below,

In the first stage, we study the i-topological subspace notion and investigate the fundamental relation of i-subspace then some of the properties alongside to i-topological proximity space

While in the second stage, we introduce new properties for the notion of occlusion set via $i - open$ set (focal function) in addition the concept of redirect, strips set which is based on the focal function was studied at this stage, where we studied the important property that may play an important role in thus study.

The third stage is we studied the boundary point with respect to the focal function and redirect set notions so we introduce the notation of the focal function frontier operator and redirect frontier operator with some properties.

Finally, in the fourth stage, we define the resolvable space with respect to the concepts (focal function, redirect) set and investigate some properties and relations that will be important to fill the gaps in this field of pure mathematics.

We also studied all of these concepts in i-subspace of i-topological proximity space, and this contributed to expending the scope of this study to define the boundary point and frontier operator and density of focal function and redirect set

List of Symbols

$I_{\oint}$ (x)	The set of all local set of the point x
$i-int$ (Ʉ).	The interior of a set Ʉ via $i-open$
$\oint_T$(Ʉ)	The occlusion a set Ʉ via $i-open$
Fd(Ʉ)	The derived set of Ʉ via the focal set
Fcl(Ʉ)	The closure of a set Ʉ via the focal set
δ	Proximity relation
i − d(Ʉ)	The derived set of Ʉ via the i – open set.
i − cl(Ʉ)	The closure set via i – open set
X^{δ}	Proximity space
δ_D	Discrete proximity
i − dense	The dense set via i – open set.
FOdense	The dense set via focal set.
FDdense	The focal-derived dense set
$i-TS$	i-Topological space
X_{TI}^{δ}	The i −topological proximity space
$Y_{T_Y I_Y}^{\delta_Y}$	Subspace of $i-$ topological proximity space
$\oint_T$ − dense	The occlusion dense set via i – open set.
ψ_T(A)	ψ − Operator via i – open set.
$\oint_I(\boldsymbol{x})$	All the focal sets containing x.
$J(X)$	Jaw set
$i-J_T(X)$	The collection of all Jaw and $i-open$ sets
$\psi_T.O.(X)$	The family of all $i-\psi_T-open$ subsets

$Ų_r$	The set of all redirect point of Ų
$Ų_s$	The set of all strips point of Ų
$r - dense$	The redirect dense set
$R_n(X)$	The set of all nowhere $r - dense$ set in X
$∯_{Fr} - frontier$	The focal function frontier set via $i - open$
$O_T(Ų)$	The occlusion outer function via $i - open$ set for Ų
$r - frontier$	The redirect frontier set
$O_r(Ų)$	The redirect outer function of a subset Ų
$∯_{\mathrm{T}} - resolvable$	The occlusion resolvable $i -$ topological proximity space via $i - open$ set
$r - resolvable$	The redirect resolvable $i -$ topological proximity space

Contents

Introduction

This study is based on three fundamental concepts proximity, i-topological space, and resolvability. So, we introduce all of those concepts from a historical point of view. To see the growth of this work we have to follow three historical lines and see the improvement that leads to this study.

We started with the first line, have to know that " Nears" in topological space is known to be the point to set as the definition of the neighborhood, the nears between sets were unknown in the mathematics research, until proximity space appeared, which is a finer shape than topology and offers an easy conceptual strategy to many essential topological problems.

Proximity started in 1952 by V. A. Efremovich [28], then he collaborated with some of his colleagues (J.M. Smirnov, N.S. Ramm, A.S. Svarc) to develop a lot of the basic properties of this concept. Also, in (1966) the work of Lodato improve this concept [18] Many researchers worked in this space and study certain types of that structure and their applications, such as the notation of connectedness and local connected and uniform space by Dimitrrij [20], and one can find the most comprehensive investigation for proximity space in the work of Naimpally [22] which included essential results. Recently in 2019 Abdulsadaa and Al-Swidi introduce the new set called the center set by the employing proximity space [3] and they use the center set theory to introduce center topology [4]. And in 2022 Qahtan and Al-Swidi studied the continuity in center topology [9,10]. The application of this concept was various and one of them is introduced recently in the field of machine learning by Feng [23].

The second line about ideal topological space, the roots of ideal theory can be traced back to the ancient mathematicians, who worked on problems of Diophantine equations and arithmetic. In the 16th and 17th centuries, the work of mathematicians like François Viète and René Descartes led to the development of algebra as an independent branch of mathematics [14]. The theory of ideals worked of significant development in the late 19th and early 20th centuries. Notably, David Hilbert's work on the theory of algebraic number fields and the class number problem had a profound impact on ideal theory. Hilbert and his contemporaries such as Emmy Noether made seminal contributions to ring theory and the study of ideals in commutative rings [11]. Throughout the 20th and 21st centuries, the ideal theory continued to evolve, and its applications spread to various fields of mathematics and beyond, including algebraic number theory, coding theory, cryptography, and algebraic topology. [12]

Since ideal applications pose different problems, interest in applications of ideal concepts in different spaces has grown rapidly in recent years. The ideal is the basic idea of topological space, which plays an important role in solving topological problems and has been studied since the 20th century. Kuratowski, K. [17] in 1966 and Vidyanathaswamy, R. [21] in 1960 were the first batch of mathematicians who proposed the idea of ideal topological spaces, and then conducted extensive research in different fields and were widely used. Many researchers worked on ideal topological space and study different concepts with it like Jankovic, D. and T. R. Hamlet [13] in 1990, they defined the notion of an ideal as a nonempty collection closed by inherited properties and bounded unions. And Ekici and Noiri [8] in 2009 studied the connectedness in ideal topological space, and Ekici with Elmal in 2015 study the decomposition in ideal space[7] . In 2010 Vizvary and Lazarow defined $\Psi - \mathrm{I}$ density [5], while Modak studied new topology on ideal topological space in 2012 [25], Al-Omari and Noiri introduced the local closure function in an ideal topological space in 2013 [1,29]. In 2017, Maragathavalli and Vinodhini in [24] study the continuity in ideal topological space but the notation of $\psi - operator$ was introduced by Modak and Islam in (2018) [26,27].

The i-topological spaces represent another form of ideal topological spaces, which are the compensation of the family T and the ideal I defined on the space X. This was proposed by author Irina Zvina [40] in 2006 and is considered to be a special case of ideal topological spaces, and study more generalization of this space in 2011 [38.39]

The third line is about resolvability. Edwin Hewitt [6] started an extensive series of studies in 1943, and many scholars have followed this definition over the years: a topological space is resolvable if it contains complementary dense subsets. (Among other things, Hewitt proved that locally compact Hausdorff spaces and first countable spaces are always maximally resolvable. Furthermore, it was shown that in some cases the product of maximally resolvable spaces is maximally resolvable).

In 2006, Hosny and Tantawy introduced the concept of generalized proximity, which involves the use of ideals and proximity on a set. They also defined a local function operator on $P(X)$ with respect to an ideal I and proximity [19], and they studied various properties for this operator. Additionally, they introduced a new type of proximity and explored its relationship with newly generated topologies. The authors discussed how these concepts can be applied using a given proximity or topology and ideals.

AlTalkany's work which was the first one that makes the connection between the i-topological space and the proximity space, in 2020 the work of researchers AlTalkany and ALSwidi introduce the Focal Function in i-Topological proximity spaces, [30] and investigate some types of Proximity ψ-set in 2021[32] also, they introduce a new concept of dense sets in i-Topological proximity space, $P_n -$ crowded , $P_{na} - crowded$, congested set and not congested set. [36, 37]

This study aims to regroup these three lines together to present the resolvability in i-topological proximity space so we introduce new concepts and improve the old ones, therefore, we divide this dissertation into four chapters as follows:

In chapter one, we present some initial concepts underlying our thesis and it consists of three sections, first one is about the i-topological space with some details and priorities of basic definitions, such as open, closed, interior, and closure sets in this space. Also, we introduced the definition of all these primitives in the subspace of $i -$topological space and how the forms of sets and relationships will become with it. The second section was about the definition and characteristics of proximity space, with some of its basic objects we need. Then, in the third section, we covered more details about the focal set, focal derivative, and focal closure, we discussed, also, these primitive said by said in subspace $i -$topological space. Then we presented the concepts of occlusion set via $i - open$ set and $\psi_T - operator$.

In Chapter Two, we divide our work into three sections, the first one we improve the study of occlusion set via $i - open$ set and presented a new result for those concepts as the image and pre-image of occlusion set and $\psi_T - operator$ with studding them with respect to i-subspace of i-topological proximity space. Whiles section two spotlight on the construction of new type points by introducing the notion of redirect points and strips points, also, we studied their properties and study some relationships between them and we determined the relation between the occlusion set via $i - open$ set and members of ideal with these concepts. Also, we introduced the notation of the redirect set and strips set with respect to $i - subspace$ in $i -$ topological proximity space.

Finally, in section three we introduce the notion of redirect dense set and study some of its important properties and relations, as well as its definition w. r. t. $i - subspace$.

Chapter Three, we study the frontier point in i-topological proximity space in two sections, in section one we highlight the construction of the new type of boundary points for occlusion via $i - open$ set frontier set and we defined $\nsubseteq_* -$ operator, also we defined the outer function with highlighted the most important possible properties of them and their relationships. Further, section two highlighted the construction of redirect frontier points and the redirect outer function. and we highlighted the most important possible properties of them and their relationships.

Chapter Four is devoted to introducing the concept of resolvable space in the $i -$topological proximity space through what we are studied in the previous chapters, so it included two sections in the first section we introduce the concept of occlusion resolvable i-topological space via $i - open$ set and the second section we study the concept of redirect resolvable i-topological proximity space. We also focused on studying some of the characteristics and relationships related to these spaces.

Chapter One

Review of Necessary Backgrounds

Introduction

In this chapter, we focused on the elementary concepts underlying our research with the notion of i −topological space. The first section deals with the details and priorities of this concept definitions and the forms of open and closed sets within them, the same applied to interior and closure sets. Also, we defined during these primitives the subspace of i −topological space and how the forms of sets and relationships will become in it. The second section deals with the definition and characteristics of proximity space, with some of its basic objects needed by research. Then the third section deals with more details about the Focal set, which we will rely on in our definitions of many new concepts during this thesis, and we discussed these primitives in subspace i − topological space.

1.1 Study on the i – Topological Space

In this section, we described the details and priorities of the definition of i − topological space and the forms of open, closed, interior, closure, dense, nowhere dense set. and the definition of homeomorphism function with some properties. Also, in these primitives, we define the form of the i-subspaces of this space

Definition (1-1-1) [17]:

Anon empty family I of subsets of a set X is called an ideal on X if and only if it is satisfying the following conditions:

1. If $\mathrm{U} \in I$, and $\mathcal{K} \subseteq \mathrm{U}$, then $\mathcal{K} \in I$.
2. If $\mathrm{U}, \mathcal{K} \in I$, then $\mathrm{U} \cup \mathcal{K} \in I$.

   It is clear that $\emptyset \in I$, but $X \notin I$ in general.

Definition (1-1-2) [39]:

Let I be an ideal defined on a set X, and let $\mathrm{U}, \mathcal{K}$ are subsets of X, the relation α defined on X as follows: $\mathrm{U}\alpha\mathcal{K}$, if and only if $\mathrm{U} - \mathcal{K} \in I$. Also, a relation ≈ defined on X by: $\mathrm{U} \approx \mathcal{K}$, if and only if $(\mathrm{U} - \mathcal{K}) \cup (\mathcal{K} - \mathrm{U}) \in I$.

Example (1-1-3):

Let $X = \{\hbar, g, f\}$, $\mathrm{U} = \{\hbar, g\}, \mathcal{K} = \{g, f\}, C = \{g\}$, with the ideal $I = \{\emptyset, \{\hbar\}\}$, Then we have:

$\mathrm{U} - \mathcal{K} = \{\hbar, g\} - \{g, f\} = \{\hbar\} \in I$, so, we get that $\mathrm{U}\alpha\mathcal{K}$.

Also, $(\mathrm{U} - C) \cup (C - \mathrm{U}) = (\{\hbar, g\} - \{g\}) \cup (\{g\} - \{\hbar, g\}) = \{\hbar\} \in I$.

Thus $\mathrm{U} \approx \mathrm{C}$.

Proposition (1-1-4): [35]

Let I be any ideal defined on X and $\mathrm{U}, \mathcal{K}, C$ are subsets of X, then:

1. $\mathrm{U}\alpha\mathrm{X}$, for each subset U of X .
2. $\mathrm{U}\alpha\emptyset$, if and only if $\mathrm{U} \in I$.
3. If $\mathrm{U} \in I$, then $\mathrm{U}\ \alpha\ \mathcal{K}$ for each subset $\mathcal{K}$ of X.
4. If $C \subseteq \mathrm{U}$, such that $\mathrm{U}\ \alpha\ \mathcal{K}$ then $C\ \alpha\ \mathcal{K}$.
5. If $\mathcal{K} \subseteq D$, such that $\mathrm{U}\ \alpha\ \mathcal{K}$ then $\mathrm{U}\ \alpha\ D$.
6. If $\mathrm{U}\alpha\mathcal{K}_\lambda$, for each $\lambda \in \Lambda$,where Λ is any index, then
   i. $\mathrm{U}\alpha \cup_{\lambda\in\Lambda} \mathcal{K}_\lambda$ **ii.** $\mathrm{U}\alpha \cap \mathcal{K}_\lambda$,
7. If $\mathrm{U}_\lambda\alpha\mathcal{K}$, for each $\lambda \in \Lambda$, where Λ is any index then
   i. $\cap_{\lambda\in\Lambda} \mathrm{U}_\lambda\alpha\mathcal{K}$ **ii.** $\bigcup_{\lambda\in\Lambda} \mathrm{U}_\lambda\alpha\mathcal{K}$,
8. $\mathrm{U}\alpha\mathrm{U}$, for each subset U of X .
9. If $\mathrm{U}\alpha\mathcal{K}$, and $\mathcal{K}\alpha\mathrm{C}$ then $\mathrm{U}\alpha\mathrm{C}$.

Proposition (1-1-5): [35]

Let I be an ideal defined on X, and let $\text{Ų}, \mathcal{K}, C$ are subsets of X, then:

1. $\text{Ų} \approx \text{Ų}$ for each subset Ų of X .
2. $\text{Ų} \approx \emptyset$ for each $\text{Ų} \in I$.
3. $\text{Ų} \approx X$ for each $\text{Ų} \subseteq X$ such that $(X - \text{Ų}) \in I$.
4. If $\text{Ų} \approx \mathcal{K}$ and $\mathcal{K} \approx C$ then $\text{Ų} \approx C$.
5. $\text{Ų} \approx \mathcal{K}$, then $\mathcal{K} \approx \text{Ų}$.
6. If $\text{Ų} \approx C$,and $\mathcal{K} \approx C$,then $\text{Ų} \cup \mathcal{K} \approx C$.
7. If $\text{Ų}_j \approx \mathcal{K}_j$, $j \in \Lambda$,then

   i. $\cup\, v_j \approx \cap\, \mathcal{K}_j$ ii. $\cap\, \text{Ų}_j \approx \mathcal{K}_j$, for Λ is indexing set.

Now, we will define these relations in definition (1-1-2) on a subset Y of the set X, and we will investigate some properties on these relations.

Definition (1-1-6)

Let X be a set and Y be a subset of X, and let I_Y be an ideal with respect to Y defined by $I_Y = I \cap \{Y\}$, then we can define the relation α_Y on Y as following $\text{Ų}\alpha_Y\mathcal{K}$ if and only if $\text{Ų} \cap (Y - \mathcal{K}) \in I_Y$.
Also, we can define the relation $\approx_Y$ on Y as the following $\text{Ų} \approx_Y \mathcal{K}$ if and only if $(\text{Ų} \cap (Y - \mathcal{K})) \cup \big(\mathcal{K} \cap (Y - \text{Ų})\big) \in I_Y$

Example (1-1-7):

Let $X = \{\mathcal{k}, \mathcal{g}, \mathcal{f}\}$, with the ideal $I = \{\emptyset, \ \{\mathcal{k}\}\}$, If $Y = \{\mathcal{k}, \mathcal{g}\}$ and $\text{Ų} = \{\mathcal{k}\}, \ \mathcal{K} = \emptyset$, then $I_Y = I \cap \{Y\} = \{\emptyset, \ \{\mathcal{k}\}\}$ Then we have:

$\text{Ų} - \mathcal{K} = \{\mathcal{k}\} - \emptyset = \{\mathcal{k}\} \in I_Y$, so, we get that $\text{Ų}\alpha_Y\mathcal{K}$

Also, $(\text{Ų} - \mathcal{K}) \cup (\mathcal{K} - \text{Ų}) = (\{\mathcal{k}\} - \emptyset) \cup (\emptyset - \{\mathcal{k}\}) = \{\mathcal{k}\} \in I_Y$.

Thus $\text{Ų} \approx_Y \text{C}$.

Proposition (1-1-8):

Let I_Y be any ideal with respect to a subset Y of X and $\mathrm{U}, \mathcal{K}, C$ are subsets of Y, then:

1. $\mathrm{U}\alpha_{\mathrm{Y}}\mathrm{Y}$, for each subset U of Y .
2. $\mathrm{U}\alpha_{\mathrm{Y}}\emptyset$, if and only if $\mathrm{U} \in I_Y$.
3. If $\mathrm{U} \in I_Y$,then $\mathrm{U}\ \alpha_Y\ \mathcal{K}$ for each subset $\mathcal{K}$ of Y.
4. If $C \subseteq \mathrm{U}$,such that $\mathrm{U}\ \alpha_Y\ \mathcal{K}$ then $C\ \alpha_Y\ \mathcal{K}$.
5. If $\mathcal{K} \subseteq D$,such that $\mathrm{U}\ \alpha_Y\ \mathcal{K}$ then $\mathrm{U}\ \alpha_Y\ D$.
6. If $\mathrm{U}\,\alpha_{\mathrm{Y}}\mathcal{K}_\lambda$,for each $\lambda \in \Lambda$,where Λ is any index, then $\mathrm{U}\alpha_{\mathrm{Y}}(\cup_{\lambda\in\Lambda} \mathcal{K}_\lambda)$
7. If $\mathrm{U}_\lambda\alpha_{\mathrm{Y}}\mathcal{K}$,for each $\lambda \in \Lambda$,where Λ is any index, then $(\cap_{\lambda\in\Lambda} \mathrm{U}_\lambda)\alpha_{\mathrm{Y}}\mathcal{K}$
8. $\mathrm{U}\alpha_{\mathrm{Y}}\mathrm{U}$, for each subset U of Y .
9. If $\mathrm{U}\alpha_{\mathrm{Y}}\mathcal{K}$,and $\mathcal{K}\alpha_{\mathrm{Y}}\mathrm{C}$ then $\mathrm{U}\alpha_Y C$.
10. If $\mathrm{U}\alpha_{\mathrm{X}}\mathcal{K}$ then $\mathrm{U}\alpha_Y\mathcal{K}$

Proof:

1- $\mathrm{U} \cap (Y - Y) = \mathrm{U} \cap \emptyset = \emptyset \in I_Y$ so $\mathrm{U}\alpha_{\mathrm{Y}}\mathrm{Y}$

2- It is obvious that $\mathrm{U} \cap (Y - \emptyset) = \mathrm{U} \cap Y = \mathrm{U}$.

3- Since $\mathrm{U} \cap (Y - \mathcal{K}) \subseteq \mathrm{U}$ so $\mathrm{U} \cap (Y - \mathcal{K}) \in I_Y$.

4- Since $C \subseteq \mathrm{U}$ then $C \cap (Y - \mathcal{K}) \subseteq \mathrm{U} \cap (Y - \mathcal{K}) \in I_Y$, that means $C \cap (Y - \mathcal{K}) \in I_Y$ by definition (1-1-1), thus $\mathrm{C}\alpha_{\mathrm{Y}}\mathcal{K}$

5- Since $\mathcal{K} \subseteq D$,then we have $(Y - D) \subseteq (Y - \mathcal{K})$ and we will get $\mathrm{U} \cap (Y - D) \subseteq \mathrm{U} \cap (Y - \mathcal{K}) \in I_Y$, that means $\mathrm{U} \cap (Y - D) \in I_Y$, thus $\mathrm{U}\alpha_{\mathrm{Y}}\mathrm{D}$

6- Since $\mathcal{K}_\lambda \subseteq \cup_{\lambda\in\Lambda} \mathcal{K}_\lambda$, and $\mathrm{U}\alpha_{\mathrm{Y}}\mathcal{K}_\lambda$, then by (5) we will get $\mathrm{U}\alpha_{\mathrm{Y}}(\cup_{\lambda\in\Lambda} \mathcal{K}_\lambda)$.

7- Since $\cap_{\lambda\in\Lambda} \text{Ų}_\lambda \subseteq \text{Ų}_\lambda$ for any λ, and since $\text{Ų}_\lambda \alpha_Y \mathcal{K}$, then by (4) we will get $(\cap_{\lambda\in\Lambda} \text{Ų}_\lambda)\alpha_Y\mathcal{K}$

8- $\text{Ų} \cap (Y - \text{Ų}) = \emptyset \in I_Y$ so $\text{Ų}\alpha_Y\text{Ų}$,for each subset Ų of Y.

9- Since $[\text{Ų} \cap (Y - \mathcal{K})] \cup [\mathcal{K} \cap (Y - C)]$

$$= \left[\left(\text{Ų} \cap (Y - \mathcal{K})\right) \cup \mathcal{K}\right] \cap \left[\left(\text{Ų} \cap (Y - \mathcal{K})\right) \cup (Y - C)\right]$$

$$= [\text{Ų} \cup \mathcal{K}] \cap [(Y - \mathcal{K}) \cup \mathcal{K}] \cap [\text{Ų} \cup (Y - C) \cap (Y - \mathcal{K}) \cup (Y - C)]$$

$$= (\text{Ų} \cup \mathcal{K}) \cap \left(\text{Ų} \cup (Y - C)\right) \cap [(Y - \mathcal{K}) \cup (Y - C)]$$

But,

$$\text{Ų} \cap (Y - C) \subseteq \left[(\text{Ų} \cup \mathcal{K}) \cap \left(\text{Ų} \cup (Y - C)\right)\right] \cap [(Y - \mathcal{K}) \cup (Y - C)]$$

so, $\text{Ų} \cap (Y - C) \in I_Y$, thus $\text{Ų}\alpha_Y C$.

10- If $\text{Ų}\alpha_X\mathcal{K}$, then $\text{Ų} \cap (X - \mathcal{K}) \in I$, so $\text{Ų} \cap (X - \mathcal{K}) \cap Y \in I_Y$, but $(X - \mathcal{K}) \cap Y = Y - \mathcal{K}$, thus $\text{Ų} \cap (Y - \mathcal{K}) \in I_Y$. Hence $\text{Ų}\alpha_Y\mathcal{K}$.

From this proposition we can inclusion directly the following corollary.

Corollary (1-1-9):

Let I_Y be an ideal define on subset $Y$of a set X such that for $i = 1,2, \dots, n$ $\text{Ų}_i\alpha_Y\mathcal{K}_i$, then $(\bigcap_{i=1}^{n} \text{Ų}_i)\alpha_Y(\bigcup_{i=1}^{n} \mathcal{K}_i)$.

Proof: Directly from proposition (1-1-8) using part (6) and (7),

Proposition (1-1-10):

Let I_Y be an ideal defined on a subset Y of a set X, and let $\text{Ų}, \mathcal{K}, C$ are subset of Y then the following are true:

1. $\text{Ų} \approx_Y \text{Ų}$ for each subset Ų of Y .

2. $\text{Ų} \approx_Y \emptyset$ for each $\text{Ų} \in I_Y$.

3. $\text{Ꝋ} \approx_Y Y$ for each $\text{Ꝋ} \subseteq Y$ such that $Y - \text{Ꝋ} \in I_Y$.

4. If $\text{Ꝋ} \approx_Y \mathcal{K}$, then $\mathcal{K} \approx_Y \text{Ꝋ}$.

5. If $\text{Ꝋ} \approx_Y C$, and $\mathcal{K} \approx_Y C$,then $\text{Ꝋ} \cup \mathcal{K} \approx_Y C$.

Proof:

1- It is clear since $(\text{Ꝋ} - \text{Ꝋ}) \cup (\text{Ꝋ} - \text{Ꝋ}) = \emptyset \cup \emptyset = \emptyset \in I_Y$.

2- Since $\text{Ꝋ} \in I_Y$, and $\text{Ꝋ} = \big(\text{Ꝋ} \cap (Y - \emptyset)\big) \cup \big(\emptyset \cap (Y - \text{Ꝋ})\big) \in I_Y$, thus $\text{Ꝋ} \approx_Y \emptyset$.

3- Since $(Y - \text{Ꝋ}) \in I_Y$ then

$Y - \text{Ꝋ} = [\text{Ꝋ} \cap (Y - Y)] \cup [Y \cap (Y - \text{Ꝋ})] \in I_Y$. Thus $\text{Ꝋ} \approx_Y Y$

4- It is clear, since $\big(\text{Ꝋ} \cap (Y - \mathcal{K})\big) \cup \big(\mathcal{K} \cap (Y - \text{Ꝋ})\big)$

$$= \big(\mathcal{K} \cap (Y - \text{Ꝋ})\big) \cup \big(\text{Ꝋ} \cap (Y - \mathcal{K})\big)$$

5- Since $[\text{Ꝋ} \cap (Y - C)] \cup [C \cap (Y - \text{Ꝋ})] \in I_Y$, and

$[\mathcal{K} \cap (Y - C)] \cup [C \cap (Y - \mathcal{K})] \in I_Y$ then

$$[(\text{Ꝋ} \cap (Y - C)) \cup (C \cap (Y - \text{Ꝋ})] \cup [(\mathcal{K} \cap (Y - C)) \cup (C \cap (Y - \mathcal{K})] \in I_Y \ldots (*)$$

But,

$$\Big(\big(\text{Ꝋ} \cap (Y - C)\big) \cup \big(\mathcal{K} \cap (Y - C)\big)\Big) \cup \big((C \cap (Y - \text{Ꝋ}) \cup (C \cap (Y - \mathcal{K})\big)$$

$$= \big((\text{Ꝋ} \cup \mathcal{K}) \cap (Y - C)\big) \cup \Big(C \cap \big((Y - \text{Ꝋ}) \cup (Y - \mathcal{K})\big)\Big) \in I_Y \ldots (**)$$

And, $\big((\text{Ꝋ} \cup \mathcal{K}) \cap (Y - C)\big) \cup \Big(C \cap \big((Y - \text{Ꝋ}) \cap (Y - \mathcal{K})\big)\Big) \subseteq \big((\text{Ꝋ} \cup \mathcal{K}) \cap (Y - C)\big) \cup \Big(C \cap \big((Y - \text{Ꝋ}) \cup (Y - \mathcal{K})\big)\Big)$

So $\big((\text{Ꝋ} \cup \mathcal{K}) \cap (Y - C)\big) \cup \Big(C \cap \big((Y - \text{Ꝋ}) \cap (Y - \mathcal{K})\big)\Big) \in I_Y$,

Then $\big((\text{Ꝋ} \cup \mathcal{K}) \cap (Y - C)\big) \cup \big(C \cap (Y - (\text{Ꝋ} \cup \mathcal{K}))\big) \in I_Y$.

Thus $\text{Ꝋ} \cup \mathcal{K} \approx_Y C$.

Definition (1-1-11) [40]:

Let I be an ideal on X, an $i-$ topological space on X is a family T of subsets of X satisfies:

1. $X, \emptyset \in T$.
2. For any $U \subseteq T$, there exist $W \in T$ such that $\cup U \approx W$.
3. For any $U, W \in T$, there exist $H \in T$ such that $U \cap W \approx H$.
4. $I \cap T = \{\emptyset\}$.

Then (X, T, I) is called $i-$ topological space, and the elements of T is called $i-$ open set.

Example (1-1-12):

Let $X = \{\hslash, g, f\}, T = \{\emptyset, X, \{\hslash, g\}, \{\hslash, f\}\}$, with the ideal $I = \{\emptyset, \{f\}, \{g\}, \{f, g\}\}$, Then we have:

1- $\emptyset, X \in T$

2- Since $\emptyset \in T$, then $\exists \emptyset \in T$ such that $(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I$

Since $X \in T$, then $\exists X \in T$ such that $(X - X) \cup (X - X) = \emptyset \in I$

Since $\{\hslash, g\} \in T$, then $\exists \{\hslash, g\} \in T$ s.t

$(\{\hslash, g\} - \{\hslash, f\}) \cup (\{\hslash, f\} - \{\hslash, g\}) = \{g, f\} \in I$

Since $\{\hslash, f\} \in T$, then $\exists$ $\{\hslash, g\} \in T$ s.t

$(\{\hslash, g\} - \{\hslash, f\}) \cup (\{\hslash, f\} - \{\hslash, g\}) = \{g, f\} \in I$.

3- Since $\emptyset, X \in T$, and $\emptyset \cap X = \emptyset$, then $\exists \emptyset \in T$ such that

$(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I$

Since $\emptyset, \{\hslash, g\} \in T$, and $\emptyset \cap \{\hslash, g\} = \emptyset$, then $\exists \emptyset \in T$ such that

$(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I$

Since $\emptyset, \{\hslash, f\} \in T$, and $\emptyset \cap \{\hslash, f\} = \emptyset$, then $\exists \emptyset \in T$ such that

$(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I$

Since $X, \{\hslash, g\} \in T$, and $X \cap \{\hslash, g\} = \{\hslash, g\}$, then $\exists \{\hslash, g\} \in T$ such that $(\{\hslash, g\} - \{\hslash, g\}) \cup (\{\hslash, g\} - \{\hslash, g\}) = \emptyset \in I$.

Since $X, \{\mathscr{k}, \mathscr{f}\} \in T$, and $X \cap \{\mathscr{k}, \mathscr{f}\} = \{\mathscr{k}, \mathscr{f}\}$, then $\exists \{\mathscr{k}, \mathscr{f}\} \in T$ such that $(\{\mathscr{k}, \mathscr{f}\} - \{\mathscr{k}, \mathscr{f}\}) \cup (\{\mathscr{k}, \mathscr{f}\} - \{\mathscr{k}, \mathscr{f}\}) = \emptyset \in I$

Since $\{\mathscr{k}, \mathscr{g}\}, \{\mathscr{k}, \mathscr{f}\} \in T$, and $\{\mathscr{k}\} \cap \{\mathscr{k}, \mathscr{f}\} = \{\mathscr{k}\}$, then $\exists X \in T$ such that $(X - \{\mathscr{k}\}) \cup (\{\mathscr{k}\} - X) = \{\mathscr{g}, \mathscr{f}\} \in I$

4- $T \cap I = \{\emptyset\}$

Thus (X, T, I) will be i −topological space.

From example (1-1-12) we can conclude that the i −topological space it is not necessary to be topological space.

Definition (1-1-13) [40]:

Let $(\mathrm{X}, \mathrm{T}, \mathrm{I})$ be i −topological space and $Y \subseteq \mathrm{X}$ then (Y, T_Y, I_Y) is called i −subspace of $(\mathrm{X}, \mathrm{T}, \mathrm{I})$ such that $T_Y = \{Y \cap U \notin I, U \in T\} \cup \{\emptyset, Y\}$ for all $i - open$ set $U \subseteq \mathrm{X}$, and $I_Y = \{Y \cap \mathrm{U}, \mathrm{U} \in I\}$.

Since $T \cap I = \emptyset$ and $T_Y \cap I_Y \subseteq \mathrm{T} \cap \mathrm{I} = \emptyset$, then $T_Y \cap I_Y = \emptyset$

The condition $Y \cap U \notin I$ make sense because if $U_Y = Y \cap U \in I$, implies $U_Y \in T_Y$ and $U_Y \in I$, contradiction with definition (1-1-11).

Example (1-1-14):

From example (1-1-12) If $Y = \{\mathscr{k}, \mathscr{g}\}$, $T_Y = \{\emptyset, Y, \{\mathscr{k}\}\}$, with the ideal $I_Y = \{\emptyset, \{\mathscr{g}\}\}$, Then we have:

1- $\emptyset, Y \in T_Y$

2- Since $\emptyset \in T_Y$, then $\exists \emptyset \in T_Y$ such that $(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I_Y$

Since $Y \in T_Y$, then $\exists Y \in T_Y$ such that $(Y - Y) \cup (Y - Y) = \emptyset \in I_Y$

Since $\{\mathscr{k}\} \in T_Y$, then $\exists \{\mathscr{k}\} \in T_Y$ s.t $(\{\mathscr{k}\} - \{\mathscr{k}\}) \cup (\{\mathscr{k}\} - \{\mathscr{k}\}) = \emptyset \in I_Y$

3- Since $\emptyset, Y \in T_Y$, and $\emptyset \cap Y = \emptyset$, then $\exists \emptyset \in T_Y$ such that $(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I_Y$

Since $\emptyset, \{\mathscr{k}\} \in T_Y$, and $\emptyset \cap \{\mathscr{k}\} = \emptyset$, then $\exists \emptyset \in T_Y$ such that

$(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I_Y$

Since $Y, \{\mathcal{k}\} \in T_Y$, and $Y \cap \{\mathcal{k}\} = \{\mathcal{k}\}$, then $\exists \{\mathcal{k}\} \in T_Y$ such that $(Y - \{\mathcal{k}\}) \cup (\{\mathcal{k}\} - Y) = \{\mathcal{g}\} \in I_Y$.

4- $T_Y \cap I_Y = \{\emptyset\}$

Thus (Y, T_Y, I_Y) will be i −topological subspace.

Definition (1-1-15): [35]

Let (X, T, I) be an i– topological space. A point $x \in$ Ʉ is called i– interior point of Ʉ $\subseteq X$,if and only if there exist i – open set H such that $x \in H \subseteq$ Ʉ and the set of all i –interior point of Ʉ is denoted by i– int (Ʉ).

Example (1-1-16):

Let $X = \{\mathcal{k}, \mathcal{g}, \mathcal{f}\}$, $T = \{\emptyset, X, \{\mathcal{k}\}, \{\mathcal{g}\}\}$, with the ideal $I = \{\emptyset, \{\mathcal{f}\}\}$, Then for Ʉ $= \{\mathcal{k}, \mathcal{f}\}$ we have $i - int$(Ʉ) $= \{\mathcal{k}\}$

Proposition (1-1-17): [35]

Let (X, T, I) be an i– topological space and let Ʉ, B are subsets of X then:

1. i – int (Ʉ) =∪ {H ∈ T: H ⊆ Ʉ}.
2. If Ʉ ⊆ $\mathcal{K}$, then i – int(Ʉ) ⊆ i − int($\mathcal{K}$) .
3. i – int(Ʉ ∩ $\mathcal{K}$) ⊆ i − int(Ʉ) ∩ i − int($\mathcal{K}$) .
4. i – int(Ʉ) ∪ i − int($\mathcal{K}$) ⊆ i – int(Ʉ ∪ $\mathcal{K}$) .
5. if Ʉ ∈ T then Ʉ = i − int(Ʉ) .
6. i – int(Ʉ) ⊆ Ʉ .

Definition (1-1-18):

Let (X, T, I) be an $i - TS$ and Y be an i −subspasce of X for a subset Ʉ of Y , a point $\mathcal{k} \in$ Ʉ is called $i -$ interior point of Ʉ with respect to i −subspace Y if and only if there existe $i_Y - open$ set H_Y such that $\mathcal{k} \in H_Y \subseteq$ Ʉ. The set of all i_Y −interior points of Ʉ denoted by $i - int_Y$(Ʉ).

Example (1-1-19):

Let $X = \{\mathcal{k}, \mathcal{g}, \mathcal{f}\}$, $T = \{\emptyset, X, \{\mathcal{k}\}, \{\mathcal{g}\}\}$, with the ideal $I = \{\emptyset, \{\mathcal{f}\}\}$. Now let $Y = \{\mathcal{k}, \mathcal{f}\}$ then $T_Y = \{Y, \emptyset, \{\mathcal{k}\}\}$, $I_Y = \{\emptyset, \{\mathcal{f}\}\}$.

Then for $\mathcal{K} = \{\mathcal{k}\}$ we have $i - int_Y(\mathcal{K}) = \{\mathcal{k}\}$

Proposition (1-1-20):

Let Y be an i −subspace of $i - TS\ X$, then for any $Ư \subseteq Y$ we have $i - int(Ư) \cap Y \subseteq i - int_Y(Ư)$.

Proof: let $\mathcal{k} \in i - int(Ư) \cap Y$ so $\mathcal{k} \in i - int(Ư)$ and $\mathcal{k} \in Y$, so there exist $H \in T_X(\mathcal{k})$ such that $H \subseteq Ư$, but $H \cap Y = H_Y$ so $\mathcal{k} \in H_Y \subseteq Ư$. Hence $\mathcal{k} \in i - int_Y(Ư)$, when $T_X(\mathcal{k}) = \{H \in T_X, s.t. \mathcal{k} \in H\}$

Example (1-1-21):

Let $X = \{\mathcal{k}, \mathcal{g}, \mathcal{f}\}$, $T = \{\emptyset, X, \{\mathcal{k}, \mathcal{g}\}\}$, with the ideal $I = \{\emptyset, \{\mathcal{f}\}\}$, and let $Y = \{\mathcal{k}, \mathcal{f}\}$ then $T_Y = \{Y, \emptyset, \{\mathcal{k}\}\}$, $I_Y = \{\emptyset, \{\mathcal{f}\}\}$.

Then for $Ư = \{\mathcal{k}\} \subseteq Y$ we have $i - int_Y(Ư) = \{\mathcal{k}\}$.

But $i - int(Ư) = \emptyset$, so we included that the converse of proposition (1-1-20) not true in general which mean that $i - int_Y(Ư) \nsubseteq i - int(Ư) \cap Y$

Definition (1-1-22): [35]

Let (X, T, I) be an i–topological space and let Ư is a subset of X , then the i–closure of Ư is the intersection of all i – closed sets containing Ư ,and is denoted by $i - cl(Ư)$, i.e, $i - cl(Ư) = \cap \{H: H\ is\ i\text{–}\ closed\ set, Ư \subseteq H\}$.

Proposition (1-1-23): [35]

Let (X, T, I) be an i– topological space and let $Ư, \mathcal{K}$ are subset of X then:

1. $i - cl(Ư)$ is not necessary is i – closed set.
2. $i - int(Ư) \neq X - (i - cl(Ư))$.
3. If Ư is i – closed set then $Ư = i - cl(Ư)$ but not conversely.

4. $\text{Ʉ} \cup i - d(\text{Ʉ}) = i - cl(\text{Ʉ})$ and $\text{Ʉ} \subseteq i - cl(\text{Ʉ})$, s.t. $d(\text{Ʉ})$ is asset of limit points of Ʉ
5. $X - (i - cl(\text{Ʉ})) \subseteq i - cl(X - \text{Ʉ})$
6. $a \in i - cl(\text{Ʉ})$ if and only if $U \cap \text{Ʉ} \neq \emptyset$ for each i −open set U of a.
7. If $\text{Ʉ} \subseteq \mathcal{K}$, then $i - cl(\text{Ʉ}) \subseteq i - cl(\mathcal{K})$.
8. $i - cl(\text{Ʉ} \cup \mathcal{K}) = i - cl(\text{Ʉ}) \cup i - cl(\mathcal{K})$.
9. $i - cl(\text{Ʉ} \cap \mathcal{K}) \subseteq i - cl(\text{Ʉ}) \cap i - cl(\mathcal{K})$.

Definition (1-1-24):

Let (X, T, I) be an $i - TS$ and Y be i −subspasce of X. Then for a subset Ʉ of Y, the $i -$ closure of Ʉ with respect to i −subspace Y is the intersection of all $i_Y - closed$sets D_Y consisting Ʉ, and denoted by $i - cl_Y(\text{Ʉ})$. i.e., $i - cl_Y(\text{Ʉ}) = \cap \{D_Y : D\ is\ i_Y - closed\ set, \text{Ʉ} \subseteq D_Y\}$

Example (1-1-25):

Let $X = \{\mathscr{k}, \mathscr{g}, \mathscr{f}\}$, $T = \{\emptyset, X, \{\mathscr{k}\}, \{\mathscr{g}\}\}$, with the ideal $I = \{\emptyset, \{\mathscr{f}\}\}$. Now let $Y = \{\mathscr{k}, \mathscr{f}\}$ then $T_Y = \{Y, \emptyset, \{\mathscr{k}\}\}$, $I_Y = \{\emptyset, \{\mathscr{f}\}\}$.

Then for $\mathcal{K} = \{\mathscr{k}\}$ we have $i - cl_Y(\mathcal{K}) = Y$

Definition (1-1-26): [35]

Let (X, T, I) be an i–topological space, and let $\text{Ʉ} \subseteq X$ then Ʉ is called $i - dense$ set if $i - cl(\text{Ʉ}) = X$.

Example (1-1-27):

Let $X = \{\mathscr{k}, \mathscr{g}, \mathscr{f}\}$, $T = \{\emptyset, X, \{\mathscr{k}\}, \{\mathscr{g}\}\}$, with the ideal $I = \{\emptyset, \{\mathscr{f}\}\}$. Now let $\text{Ʉ} = \{\mathscr{k}, \mathscr{g}\}$ then $i - cl(\text{Ʉ}) = X$. So Ʉ is $i - dense$.

Definition (1-1-28) : [35]

Let (X, T, I) be an $i-$ topological space, then a subset Ų of X is called i – nowhere dense set if $i - int\big(i - cl(\text{Ų})\big) = \emptyset$.

And if $i - int\big(i - cl(\text{Ų})\big) \neq \emptyset$, we say that Ų is i – somewhere dense set.

The set of all i – nowhere dense set of (X, T, I) is denoted by $I_N(T, I)$.

Example (1-1-29):

Let $X = \{\mathscr{k}, \mathscr{g}, \mathscr{f}\}$, $T = \{\emptyset, X, \{\mathscr{k}\}, \{\mathscr{g}\}\}$, with the ideal $I = \{\emptyset,\ \{\mathscr{f}\}\}$. Now let $\mathcal{K} = \{\mathscr{f}\}$ then $i - cl(\mathcal{K}) = \{\mathscr{f}\}$, and $i - int(i - cl(\mathcal{K})) = \emptyset$. Thus $\mathcal{K}$ is i – nowhere dense set.

Definition (1-1-30): [35]

A function $f: (X, T_X, I_X) \to (Y, T_Y, I_Y)$ is called i – continuous function, if and only if the inverse image of each $i-$ open set in Y is i –open set in X.

Definition (1-1-31): [35]

A function $f: (X, T_X, I_X) \to (Y, T_Y, I_Y)$ is called i – open function, if and only if the image of each i – open set in X is i – open set in Y.

Definition (1-1-32): [35]

A function $f: (X, T_X, I_X) \to (Y, T_Y, I_Y)$ is called i – closed function, if and only if the image of each i – closed set in X is i – closed set in Y .

Definition (1-1-33): [35]

A function $f: (X, T_X, I_X) \to (Y, T_Y, I_Y)$ is called i – homeomorphism function, if and only if f is bijective, $i -$ continuous and f^{-1} is $i -$continuous function.

Theorem (1-1-34): [35]

If $f:(X,T_X,I_X)\to(Y,T_Y,I_Y)$ is i–closed function then

$f\big(i-cl(\text{U})\big)=i-cl\big(f(\text{U})\big)$ for each i –closed set U of X .

Theorem (1-1-35): [35]

Let $f:(X,T_X,I_X)\to(Y,T_Y,I_Y)$ be a bijective function, then each of the following are equivalent:

1. f is i – homeomorphism function.
2. f is i –open and i – continuous function.
3. f is i – closed and i – continuous function.

1.2 The Proximity Spaces

This Section is devoted to presenting the concept of proximity relationships, some of their features and advantages, and the topology that generated by this relation, called the proximity topology, which is an important element in our research. in addition to mentioning the relationship of non-proximity and its impact in defining the neighborhood in topological proximity space Furthermore, we introduced the concept of i-topology proximity space and i −subspace of this space.

Definition (1-2-1) [21]:

A binary relation δ defined on the power set of X is called proximity relation on X , if and only if it satisfies the following axioms:

1. $\text{Ų}\delta\mathcal{K}$ implies $\mathcal{K}\delta\text{Ų}$ for any $\mathcal{K}, \text{Ų} \subseteq \text{X}$
2. $(\text{Ų} \cup \mathcal{K})\delta C,$ if and only if $\text{Ų}\delta C$ or $\mathcal{K}\delta C$
3. $\text{Ų}\delta\mathcal{K}$ implies $\text{Ų} \neq \emptyset$ and $\mathcal{K} \neq \emptyset$
4. $\text{Ų} \cap \mathcal{K} \neq \emptyset$ implies $\text{Ų}\delta\mathcal{K}$
5. $\text{Ų}\bar{\delta}\mathcal{K}$ implies there exists a subset E of X such that $\text{Ų}\bar{\delta}E$ and $X - E\bar{\delta}\mathcal{K}$.

The pair (X, δ) is called a proximity space. and denoted by X^{δ}.

Example (1-2-2) [21]:

Let $X = \{\hbar, g, f\}$, and for any subsets of X, Ų and $\mathcal{K},$ we can define a discrete proximity relation δ_D such that $\text{Ų}\delta_D\mathcal{K}$ if and only if $\text{Ų} \cap \mathcal{K} \neq \emptyset$.

Some of axioms of proximity spaces listed in the following propositions below.

Proposition (1-2-3) [21] :

Let X^{δ} be a proximity space, then each of the following are exist:

1. If $U\bar{\delta}\mathcal{K}$, then $U\bar{\delta}C$ for each $C \subset \mathcal{K}$
2. If $U\bar{\delta}\mathcal{K}$, then $C\bar{\delta}\mathcal{K}$ for each $C \subset U$
3. If $U\bar{\delta}\mathcal{K}$, then $U \cap \mathcal{K} = \emptyset$
4. If $U\bar{\delta}\mathcal{K}$, then $\mathcal{K}\bar{\delta}U$
5. If $U\bar{\delta}\mathcal{K}$, then $\{x\}\bar{\delta}\mathcal{K}$ for each $x \in U$
6. If $U\bar{\delta}C$ and $\mathcal{K}\bar{\delta}C$, then $U \cup \mathcal{K}\bar{\delta}C$

Proposition (1-2-4)[21] :

Let X^{δ} be a proximity space, and let $U, \mathcal{K}, C, D$ are subsets of X then:

1. If $\mathcal{K}\delta C$, then $U\delta C$, where $\mathcal{K} \subseteq U$
2. If $\mathcal{K}\delta C$,then $\mathcal{K}\delta D$, where $C \subseteq D$
3. If $U\delta(X - D)$ or $D\delta\mathcal{K}$,then $U\delta\mathcal{K}$ for each $D \subseteq X$
4. If $U\delta\mathcal{K}$, then $\{x\}\delta\mathcal{K}$ for some $x \in U$ and also $U\delta\mathcal{K}$ then $U\delta\{y\}$ for some $y \in \mathcal{K}$.

Definition (1-2-5)[20] :

Let X^{δ_1} and X^{δ_2} be a proximity spaces such that $\delta_1 > \delta_2$, and let $U, \mathcal{K}$ are subsets of X then If $U\delta_1\mathcal{K}$ implies $U\delta_2\mathcal{K}$.

Definition (1-2-6) : [21]

A subset U of a proximity space X^{δ} is said to be δ-closed if and only if $\{x\}\delta U$ implies $x \in U$, The collection of complements of all δ-closed sets defined a topology $T(\delta)$ on X.

Definition (1-2-7) : [21]

Let X^{δ} be proximity space and Y subset of X . Then for subsets $Ų, \mathcal{K}$ of Y we define $Ų\delta_Y\mathcal{K}$ if and only if $Ų\delta\mathcal{K}$ then Y^{δ} is a subspace proximity on Y, and $T(\delta_Y)$ is the subspace topology of $T(\delta)$ generated on Y by δ_Y.

Definition (1-2-8)[20]:

A subset $\mathcal{K}$ of X^{δ} is called δ – neighborhood of Ų, if $Ų\bar{\delta}(X-\mathcal{K})$ and this is denoted by $Ų \ll \mathcal{K}$. .

Theorem (1-2-9) [21]:

In X^{δ} the relation $\ll$ has following properties:

1. $X \ll X$.
2. $\emptyset \ll Ų$ for any subset Ų of X .
3. $Ų \ll \mathcal{K}$ implies $Ų \subset \mathcal{K}$.
4. $Ų \subset \mathcal{K}$, $\mathcal{K} \ll C$ and $C \subseteq D$ imply $Ų \ll D$
5. $Ų \ll \mathcal{K}_i$, $i = 1, \dots, n$ if and only if $Ų \ll \cap \{\mathcal{K}_i : i = 1, \dots, n\}$
6. $Ų \ll \mathcal{K}$ implies $(X-\mathcal{K}) \ll (X-Ų)$
7. If $Ų \ll \mathcal{K}$ implies, there is $C \subset X$ such that $Ų \ll C \ll \mathcal{K}$
8. If δ is separated, then $x \ll (X-y)$ if and only if $x \neq y$
9. If $Ų_i \ll \mathcal{K}_i$, $i = 1, \dots, n$, then
   $\cup \{Ų_i : i = 1, \dots, n\} \ll \cap \{\mathcal{K}_i : i = 1, \dots, n\}$,
   and $\cup \{Ų_i : i = 1, \dots, n\} \ll \cup \{\mathcal{K}_i : i = 1, \dots, n\}$
10. $Ų \ll X$, for each subset Ų of X.
11. If $\{x\} \ll Ų$ then $x \in Ų$
12. $Ų \ll \mathcal{K}$, then $\{x\} \ll \mathcal{K}$ for all $x \in Ų$
13. Ų is $i-open$ iff $\forall x \in Ų, \{x\} \ll Ų$

Definition (1-2-10) [35]:

The X^{δ} is called σ – proximity if for any arbitrary family $\{\mu_{\lambda} \ ; \ \lambda \in \beta\}$ of subsets of X , β is indexing family, it has the following feature $\mathcal{K} \ \delta \cup_{\lambda \in \beta} \ \mu_{\lambda}$ iff $\mathcal{K} \ \delta \ \mu_{\lambda_0}$ for some $\lambda_0 \in \beta$.

Definition (1-2-11) [21] :

A mapping $f : X^{\delta} \to Y^{\delta}$ is said to be δ – continuous if $\mathrm{U}\delta_X\mathcal{K}$ then $f(\mathrm{U})\delta_Y f(\mathcal{K})$ for each $\mathrm{U}, \mathcal{K} \subseteq X$.

Proposition (1-2-12)[21] :

A mapping f from X^{δ} into Y^{δ} is δ – continuous if and only if for each $P, Q \subset Y \ P\bar{\delta}_Y Q$ implies $f^{-1}(P)\bar{\delta}_X f^{-1}(Q)$.

Corollary (1-2-13)[21] :

A mapping $f : X^{\delta} \to Y^{\delta}$ is δ –continuous, if and only if $P \ll_Y Q$ implies that $f^{-1}(P) \ll_X f^{-1}(Q)$ for each $, Q \subset Y$.

Corollary (1-2-14)[21] :

Let f be a mapping from X^{δ} into Y^{δ} then $\delta_X = f^{-1}(\delta_Y)$ is the coorsest proximity on X for which f is δ – continuous mapping.

Definition (1-2-15)[21] :

If $f: X^{\delta} \to Y^{\delta}$ is bijective δ–continous mapping, and $f^{-1} : Y^{\delta} \to X^{\delta}$ is δ – continuous mapping, then f is said to be is proximally isomorphism or δ – homeomorphism from X onto Y .

Definition (1-2-16) : [35]

The quadruple (X, T, I, δ) is called i– topological proximity space, where (X, T, I) is i– topological space and (X, δ) is a proximity space. And we will denote it by $i - TPS$

For this thesis we will use the nation X_{TI}^{δ} for any $i - TPS$, (X, T, I, δ)

Example (1-2-17):

Let $X = \{𝓀, ℊ, 𝒻\}$, $T = \{\emptyset, X, \{𝓀\}, \{ℊ\}\}$, $I = \{\emptyset, \{𝒻\}\}$ with δ_D . Then :

1- $\emptyset, X \in T$

2- Since $\emptyset \in T$, then $\exists \emptyset \in T$ such that $(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I$

Since $X \in T$, then $\exists X \in T$ such that $(X - X) \cup (X - X) = \emptyset \in I$

Since $\{𝓀\} \in T$, then $\exists \{𝓀\} \in T$ s.t

$(\{𝓀\} - \{𝓀\}) \cup (\{𝓀\} - \{𝓀\}) = \emptyset \in I$

Since $\{ℊ\} \in T$, then $\exists$ $\{ℊ\} \in T$ s.t

$(\{ℊ\} - \{ℊ\}) \cup (\{ℊ\} - \{ℊ\}) = \emptyset \in I.$

3- Since $\emptyset, X \in T$, and $\emptyset \cap X = \emptyset$, then $\exists \emptyset \in T$ such that

$(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I$

Since $\emptyset, \{𝓀\} \in T$, and $\emptyset \cap \{𝓀\} = \emptyset$, then $\exists \emptyset \in T$ such that

$(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I$

Since $\emptyset, \{ℊ\} \in T$, and $\emptyset \cap \{ℊ\} = \emptyset$, then $\exists \emptyset \in T$ such that

$(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I$

Since $X, \{𝓀\} \in T$, and $X \cap \{𝓀\} = \{𝓀\}$, then $\exists \{𝓀\} \in T$ such that

$(\{𝓀\} - \{𝓀\}) \cup (\{𝓀\} - \{𝓀\}) = \emptyset \in I$

Since $X, \{ℊ\} \in T$, and $X \cap \{ℊ\} = \{ℊ\}$, then $\exists \{ℊ\} \in T$ such that

$(\{ℊ\} - \{ℊ\}) \cup (\{ℊ\} - \{ℊ\}) = \emptyset \in I$

Since $\{𝓀\}, \{ℊ\} \in T$, and $\{𝓀\} \cap \{ℊ\} = \emptyset$, then $\exists \emptyset \in T$ such that

$(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I$

4- $T \cap I = \{\emptyset\}$

Thus, it is obvious that (X, T, I) is i-topological proximity space and with δ_D it will be $i - TPS$, X_{TI}^{δ}

Definition (1-2-18) :

The quadruple $Y^{\delta_Y}_{T_Y I_Y}$ is called $i\text{–}subspace$ of X^{δ}_{TI} topological proximity space, where (Y, T_Y, I_Y) is i– topological space and (Y, δ) is a proximity subspace.

Example (1-2-19):

Let $X = \{k, g, f\}$, $T = \{\emptyset, X, \{k\}, \{g\}\}$, $I = \{\emptyset, \{f\}\}$ with δ_D . And let $Y = \{g, f\}$ so $T_Y = \{\emptyset, X, \{g\}\}$, $I_Y = \{\emptyset, \{f\}\}$ with ${\delta_Y}_D$. Then

1- $\emptyset, Y \in T_Y$

2- Since $\emptyset \in T_Y$, then $\exists \emptyset \in T_Y$ such that $(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I_Y$
Since $Y \in T_Y$, then $\exists Y \in T_Y$ such that $(Y - Y) \cup (Y - Y) = \emptyset \in I_Y$
Since $\{g\} \in T_Y$, then $\exists$ $\{g\} \in T_Y$ s.t
$(\{g\} - \{g\}) \cup (\{g\} - \{g\}) = \emptyset \in I_Y$.

3- Since $\emptyset, Y \in T_Y$, and $\emptyset \cap Y = \emptyset$, then $\exists \emptyset \in T_Y$ such that
$(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I_Y$
Since $\emptyset, \{g\} \in T_Y$, and $\emptyset \cap \{g\} = \emptyset$, then $\exists \emptyset \in T_Y$ such that
$(\emptyset - \emptyset) \cup (\emptyset - \emptyset) = \emptyset \in I_Y$
Since $Y, \{g\} \in T_Y$, and $Y \cap \{g\} = \{g\}$, then $\exists \{g\} \in T_Y$ such that
$(\{g\} - \{g\}) \cup (\{g\} - \{g\}) = \emptyset \in I_Y$

4- $T_Y \cap I_Y = \{\emptyset\}$

Thus $Y^{\delta_Y}_{T_Y I_Y}$ is $i\text{–}subspace$ of X^{δ}_{TI}.

1.3 Study on the Focal Set

In this section, we will mention the concept of the focal set, as well as the concept of focal derived and focal closure, with the study of these concepts in subspace and mentioning some of their characteristics that have been studied. Then we will come to the concept of occlusion points for open set in i −topological proximity space, as well as the φ −operator. We also dealt with the study of these two concepts in $i - subspace$ with some characteristics and examples, and investigate the image and preimage under the i −homomorphesim function.

Definition (1-3-1) : [30]

Let X_{TI}^{δ} be an $i - TPS$, then a subset Ʉ is called a focal set of a point $x \in X$ if we have $U \in T(x)$ such that $U \alpha$ Ʉ.The system of all focal sets of a point x is denoted by $I_{\oint}(x) = \{Ʉ \subseteq X : \exists U \in T(x), U\alpha Ʉ\}$. Noted that X is a focal set for each $x \in X$. where $T(x) = \{U \in T, x \in U\}$.

Also, we can define the set of all focal set for some $x \in X$, by $\oint_{I}(x) = \{Ʉ \in I_{\oint}(x), x \in Ʉ\}$.

Example (1-3-2): **[30]**

Let $X = \{\&, g, f\}$, $T = \{\emptyset, X, \{\&, g\}\}$, $I = \{\emptyset, \{f\}\}$ with δ_D . Then $I_{\oint}(\&) = \{X, \{\&, g\}\} = I_{\oint}(g) = I_{\oint}(f)$

Theorem (1-3-3): [30]

Let X_{TI}^{δ} be an $i - TPS$, and let Ʉ, $\mathcal{K}$ are subsets of X, then each of the following properties are holds:

1. For each Ʉ $\in T(x)$, then Ʉ $\in I_{\oint}(x)$ and $\emptyset \notin I_{\oint}(x)$.
2. If $\mathcal{K} \in I_{\oint}(x)$ and $\mathcal{K} \subseteq$ Ʉ, then Ʉ is a focal set of x .
3. Ʉ, $\mathcal{K} \in I_{\oint}(x)$ if and only if Ʉ $\cap \mathcal{K} \in I_{\oint}(x)$.

4. For each $Ư \in I$,then $Ư \notin I_{\oint}(x)$.
5. If $Ư \in I_{\oint}(x)$, then $(X - Ư) \notin I_{\oint}(x)$.
6. If $Ư \in I$, then $(X - Ư) \in I_{\oint}(x)$, for each $x \in X$.
7. If $Ư, \mathcal{K} \in I_{\oint}(x)$, then $Ư \cup \mathcal{K} \in I_{\oint}(x)$.

Proposition (1-3-4): [30]

Let $X^{\delta}_{TI_i}$ $i = 1,2$ be an $i - TPS$, such that $I_1 \subseteq I_2$, then $I_{1\oint}(x) \subseteq I_{2\oint}(x)$.

Proposition (1-3-5): [30]

Let $X^{\delta}_{T_iI_i}$ $i = 1,2$ be an $i - TPS's$,such that T_2 is finer than T_1 and I_2 is finer than I_1 ,then :

1) $I_{\oint\, T_1}(x) \subseteq I_{\oint\, T_2}(x)$
2) $I_{1\oint T_1}(x) \subseteq I_{2\oint T_2}(x)$

Definition (1-3-6): [30]

Let $X^{\delta_1}_{T_1I_1}, Y^{\delta_2}_{T_2I_2}$ be $i - TPS's$. Then a function $f: X^{\delta_1}_{T_1I_1} \to Y^{\delta_2}_{T_2I_2}$, is called formatting mapping (simply $F-$ map) if satisfy that $f(U_x) \in I_{2\oint}\left(f(x)\right)$, for each $x \in X$ and $U \in I_{1\oint}(x)$.

Proposition (1-3-7): [35]

Let $X^{\delta_1}_{T_1I_1}, Y^{\delta_2}_{T_2I_2}$ be $i - TPS's$ and let $f: X^{\delta_1}_{T_1I_1} \to Y^{\delta_2}_{T_2I_2}$ be afunction if f is $i -$homeomorphism, then
$f^{-1}\left(U_{f(x)}\right) \in I_{1\oint}(x)$, for each $U_{f(x)} \in I_{2\oint}\left(f(x)\right)$,$x \in X$.

Our aim now is to introduce the definition for the notion of focal set with respect to $\mathrm{i} -$ subspace of $\mathrm{i} -$ TPS . with some of the properties and relations

Definition (1-3-8):

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ X^{\delta}_{TI}$, and let $Ư \subseteq Y$, $y \in Y$, then Ư is called focal set for y w. r. t. $i - subspace\ Y$ if there is $U_y \in T_Y(y)$ s.t. $U_y \alpha_Y Ư$ when $V \cap (Y - Ư) \in I$, i.e. $\exists\ V \in T_X(y), (V \cap Y)\alpha_Y Ư$ this implies, $V \cap Y \cap (Y - Ư) \in I_Y$ and we denoted by

$I_{Y\oint}\ (y) = \{Ư \subseteq Y : \exists U_Y \in T_Y(x), U_Y \alpha_Y Ư\}$ for some $y \in Y$.

Also, we can define the set of all focal set w. r. t. $i - subspace\ Y^{\delta_Y}_{T_Y I_Y}$ by $\oint_{\ I_Y} (y) = \{Ư \in I_{Y\oint}\ (y), y \in Ư\}$.

Proposition (1-3-9):

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ X^{\delta}_{TI}$, and for $y \in Y$. Then $I_{Y\oint}\ (y) = I_{X\oint}\ (y) \cap \{Y\}$

Proof: Let $Ư \in I_{Y\oint}\ (y)$ then $\exists\ U_Y \in T_Y(x)\ and\ U_Y \alpha_Y Ư$ which means that $U_Y \cap (Y - Ư) \in I_Y$. So $\exists\ V \in T_X(y)$ s.t.

$U_Y \cap (Y - Ư) = V \cap (X - Ư) \cap Y \in I_Y$ so $V \cap (X - Ư) \in I_X$. Thus $V \alpha_X Ư$ and hence $Ư \in I_{X\oint}\ (y)$, but $Ư \subseteq Y$, so $Ư \in I_{X\oint}\ (y) \cap \{Y\}$.

Now let $Ư \in I_{X\oint}\ (y) \cap \{Y\}$, then $Ư \in I_{X\oint}\ (y)$ and $Ư \subseteq Y$.

Since $Ư \in I_{X\oint}\ (y)$, then $\exists\ U \in T_X(y)$ s.t. $U \alpha_X Ư$, then by proposition (1-1-4) part (4) we have $(U \cap Y)\alpha_Y Ư$ hence $Ư \in I_{Y\oint}\ (y)$.

Example (1-3-10):

Let $X = \{\hbar, g, f\}$, $T = \{\emptyset, X, \{\hbar, g\}\}$, $I = \{\emptyset, \{f\}\}$ with δ_D .

And let $Y = \{\hbar, f\}$ so $T_Y = \{\emptyset, X, \{\hbar\}\}$, $I_Y = \{\emptyset, \{f\}\}$ with δ_{Y_D} will be i–$subspace Y^{\delta_Y}_{T_Y I_Y}$ of X^{δ}_{TI}. Then $I_{Y\oint}\ (\hbar) = \{Y, \{\hbar\}\} = I_{Y\oint}\ (g) = I_{Y\oint}\ (f)$

Theorem (1-3-11) :

Let $Y_{T_Y I_Y}^{\delta_Y}$ be $i-subspace$ of $i-TPS\ X_{TI}^{\delta}$, then for the subsets Ա, $\mathcal{K}$ of Y, the following statement are true for some $\mathscr{k} \in Y$

1) If Ա $\in T_Y(\mathscr{k})$, then Ա $\in I_{Y_\oint}(\mathscr{k})$ and $\emptyset \notin I_{Y_\oint}(\mathscr{k})$
2) If $\mathcal{K} \in I_{Y_\oint}(\mathscr{k})$ and $\mathcal{K} \subseteq$ Ա, then Ա $\in I_{Y_\oint}(\mathscr{k})$
3) If Ա, $\mathcal{K} \in I_{Y_\oint}(\mathscr{k})$, then Ա $\cap\, \mathcal{K} \in I_{Y_\oint}(\mathscr{k})$
4) For each $\mathcal{K} \in I_{Y_\oint}(\mathscr{k})$, then $\exists$Ա $\in T_Y(\mathscr{k})$ s.t. $\mathcal{K}\alpha_Y$Ա and Ա $\in I_{Y_\oint}(x)\ for\ each\ x \in$ Ա.
5) For each Ա $\in I_Y$, then Ա $\notin I_{Y_\oint}(\mathscr{k})$ for some $\mathscr{k} \in Y$.
6) If Ա $\in I_{Y_\oint}(\mathscr{k})$, then $Y -$ Ա $\notin I_{Y_\oint}(\mathscr{k})$.
7) If Ա $\in I_Y$, then $Y -$ Ա $\in I_{Y_\oint}(\mathscr{k})$
8) If Ա, $\mathcal{K} \in I_{Y_\oint}(\mathscr{k})$, then Ա $\cup\, \mathcal{K} \in I_{Y_\oint}(\mathscr{k})$

Proof:

1) Let Ա $\in T_Y(\mathscr{k})$, then $\exists U \in T_X(\mathscr{k})$ s.t. Ա $= U \cap Y$. Since $U \in T_X(\mathscr{k})$ then by proposition (1-3-3) part (1) we get $U \in I_{X_\oint}(\mathscr{k})$ which implies that Ա $= U \cap Y \in I_{X_\oint}(\mathscr{k}) \cap Y = I_{Y_\oint}(\mathscr{k})$.
2) Since $\mathcal{K} \in I_{Y_\oint}(\mathscr{k})$, then $\exists\ U_Y \in T_Y(\mathscr{k})$ s. t. $U_Y\alpha_Y\mathcal{K}$. Since $\mathcal{K} \subseteq$ Ա and by proposition (1-1-8) part (5) we get $U_Y\alpha_Y$Ա . Thus Ա $\in I_{Y_\oint}(\mathscr{k})$.
3) Since Ա, $\mathcal{K} \in I_{Y_\oint}(\mathscr{k})$, then we have Ա $= U_1 \cap Y$ s. t. $U_1 \in I_\oint(\mathscr{k})$ and $\mathcal{K} = U_2 \cap Y$ s. t. $U_2 \in I_\oint(\mathscr{k})$. Now Ա $\cap\, \mathcal{K} = U_1 \cap Y \cap U_2 \cap Y = (U_1 \cap U_2) \cap Y$, Since $U_1, U_2 \in I_\oint(\mathscr{k})$ so by proposition (1-3-3) part (3) we get $U_1 \cap U_2 \in I_\oint(\mathscr{k})$. Then Ա $\cap\, \mathcal{K} = U \cap Y$ s. t. $U = U_1 \cap U_2 \in I_\oint(\mathscr{k})$.

Thus $\text{Ⴎ} \cap \mathcal{K} \in I_{Y\oint}(\mathscr{k})$.

Conversely, let $\text{Ⴎ} \cap \mathcal{K} \in I_{Y\oint}(\mathscr{k})$, but $\text{Ⴎ} \cap \mathcal{K} \subseteq \text{Ⴎ}$ and $\text{Ⴎ} \cap \mathcal{K} \subseteq \mathcal{K}$, so by part (2) we have that $\text{Ⴎ}, \mathcal{K} \in I_{Y\oint}(\mathscr{k})$.

4) Let $\mathcal{K} \in I_{Y\oint}(\mathscr{k})$ and $\text{Ⴎ} \in T_Y(\mathscr{k})$ s. t. $\text{Ⴎ}\alpha_Y\mathcal{K}$, therefore by proposition (1-3-3) part (1) and for any $x \in \text{Ⴎ}$ we get $\text{Ⴎ} \in I_{Y\oint}(x)$.

5) Suppose that $\text{Ⴎ} \in I_Y$ and $\text{Ⴎ} \in I_{Y\oint}(\mathscr{k})$, so $\exists\, U_y \in T_Y(\mathscr{k})$ s. t. $U_y \cap (Y - \text{Ⴎ}) \in I_Y$, but $\text{Ⴎ} \in I_Y$, so $\left(U_y \cap (Y - \text{Ⴎ})\right) \cup \text{Ⴎ} \in I_Y$, which means that $U_y \cup \text{Ⴎ} \in I_Y$, then $U \in I_Y$ and that is a contradiction with the definition (1-1-13) .

6) Since $\text{Ⴎ} \in I_{Y\oint}(\mathscr{k})$ then by proposition (1-3-9) $\text{Ⴎ} \in I_{\oint}(\mathscr{k}) \cap \{Y\}$ so $\text{Ⴎ} \in I_{\oint}(\mathscr{k})$ then by proposition (1-3-3) part (5), $X - \text{Ⴎ} \notin I_{\oint}(\mathscr{k})$ hence $(X - \text{Ⴎ}) \cap \{Y\} \notin I_{\oint}(\mathscr{k}) \cap \{Y\}$. which means
$(Y - \text{Ⴎ}) \notin I_{\oint}(\mathscr{k}) \cap \{Y\}$, thus $Y - \text{Ⴎ} \notin I_{Y\oint}(\mathscr{k})$.

7) If possible, that $(Y - \text{Ⴎ}) \notin I_{Y\oint}(\mathscr{k})$, then $\forall\, U_Y \in T_Y(\mathscr{k})$, that mean $U_y \cap \left(Y - (Y - \text{Ⴎ})\right) \notin I_Y$, so $U_y \cap \text{Ⴎ} \notin I_Y$, but $U_y \cap \text{Ⴎ} \subseteq \text{Ⴎ} \in I_Y$, which a contradiction the definition (1-1-1) .

8) Let $\text{Ⴎ} \in I_{Y\oint}(\mathscr{k})$, but $\text{Ⴎ} \subseteq (\text{Ⴎ} \cup \mathcal{K})$, then by part (2) we conclude that $\text{Ⴎ} \cup \mathcal{K} \in I_{Y\oint}(\mathscr{k})$.

Proposition (1-3-12):

Let $X^{\delta}_{T_jI_j}$ $j = 1{,}2$ be an $i - TPS's$,such that T_2 is finer than T_1 and I_2 is finer than I_1 ,then for $i - subspace\ Y^{\delta_Y}_{T_{j_Y}I_{j_Y}}$ of $i - TPS\ X^{\delta}_{T_jI_j}$:

1) $I_{Y\oint\, T_{1_Y}}(x) \subseteq I_{Y\oint\, T_{2_Y}}(x)$

2) $I_{1_{Y\oint T_{1_Y}}}(x) \subseteq I_{2_{Y\oint T_{2_Y}}}(x)$

Proof:

1) Let $Ꮜ \in I_{Y \oint T_{1_Y}}(x)$, then $\exists\, U_Y \in T_Y(x)$ s. t. $U_Y \alpha_Y Ꮜ$, so we have $U_Y \cap (Y - Ꮜ) \in I_Y$, since $U_Y \in T_{2_Y}(x)$, then $U_Y \cap (Y - Ꮜ) \in I_Y$ w. r. t. T_2 . Thus $Ꮜ \in I_{Y \oint T_{2_Y}}(x)$.
2) Since $I_{1_Y} \subseteq I_{2_Y}$ so we get the result immediately by part (1).

Definition (1-3-13) : [30]

Let X^{δ}_{TI} be an $i - TPS$, and $Ꮜ \subseteq X$, $x \in X$,then x is called a focal limit point of Ꮜ, if and only if for each $U \in \oint_I(x), U_x \cap Ꮜ/_{\{x\}} \neq \emptyset$, and the set of all focal limit points is called the focal derived set and denoted by $Fd(Ꮜ)$, also the focal closure of the set Ꮜ denoted by $Fcl(Ꮜ)$ and defined by $Fcl(Ꮜ) = Ꮜ \cup Fd(Ꮜ)$.

Note that $Fcl(Ꮜ)$ is not necessary i– closed set.

Example (1-3-14) : [35]

Let $\mathrm{X} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$, $\mathrm{T} = \{\mathrm{X}, \emptyset, \{\mathrm{a}\}\}$, $\mathrm{I} = \{\emptyset, \{\mathrm{b}\}\}$,then $\mathrm{Fcl}\{\mathrm{a}\} = \{\mathrm{a}\}$ and $\{\mathrm{a}\}$ is not i – closed set. Also, if Ꮜ ={b}, then $Fcl(Ꮜ) = \{b\} \not\supseteq \cap \{\mathrm{H}: \mathrm{H} \text{ is i} - \text{closed set}, Ꮜ \subseteq \mathrm{H}\}$.

Proposition (1-3-15) : [30]

Let $(\mathrm{X}, \mathrm{T}, \mathrm{I})$ be an i – topological space, and $Ꮜ, \mathcal{K}$ are subsets of X. Then each of the following are holds:

1. If $Ꮜ \subseteq \mathcal{K}$,then $\mathrm{Fd}(Ꮜ) \subseteq \mathrm{Fd}(\mathcal{K})$.
2. $\mathrm{Fd}(Ꮜ) \cap \mathrm{Fd}(\mathcal{K}) \supseteq \mathrm{Fd}(Ꮜ \cap \mathcal{K})$.
3. $\mathrm{Fd}(Ꮜ \cup \mathcal{K}) = \mathrm{Fd}(Ꮜ) \cup \mathrm{Fd}(\mathcal{K})$.
4. $\mathrm{Fd}(\emptyset) = \emptyset$ and $\mathrm{Fd}(\mathrm{X}) = \mathrm{X}$.
5. $\mathrm{Fd}(Ꮜ) = \emptyset$, for each $Ꮜ \in \mathrm{I}$.
6. $\mathrm{U} \cap \mathrm{Fd}(Ꮜ) \subseteq \mathrm{Fd}(\mathrm{U} \cap Ꮜ)$, for each focal set U of X .

Corollary (1-3-16) : [30]

Let X_{TI}^{δ} be an $i-TP$, and let $\text{Ʉ},\mathcal{K}$ are subset of X then each of the following are existed:

1. *If* $\text{Ʉ} \subseteq \mathcal{K}$ *,then* $Fcl(\text{Ʉ}) \subseteq Fcl(\mathcal{K})$.
2. $Fcl(\text{Ʉ} \cap \mathcal{K}) \subseteq Fcl(\text{Ʉ}) \cap Fcl(\mathcal{K})$.
3. $Fcl(\text{Ʉ} \cup \mathcal{K}) = Fcl(\text{Ʉ}) \cup Fcl(\mathcal{K})$.
4. $Fcl(\text{Ʉ}) = \text{Ʉ}$, for each $\text{Ʉ} \in I$.
5. $Fcl(\text{Ʉ}) \subseteq \mathrm{i-cl}(\text{Ʉ})$

Proposition (1-3-17): [30]

Let X_{TI}^{δ} be an $i-TPS$, and let Ʉ be a subset of X, then

$\mathrm{Fcl}(\text{Ʉ}) = \text{Ʉ}$, for each i −closed set Ʉ of X.

Proposition (1-3-18): [30]

Let $X_{TI_j}^{\delta}, j = 1,2$ be an $i-TPS$,such that I_2 is finer than I_1 then

1. $Fd_{I_2}(\text{Ʉ}) \subseteq Fd_{I_1}(\text{Ʉ})$.
2. $Fcl_{I_2}(\text{Ʉ}) \subseteq Fcl_{I_1}(\text{Ʉ})$.

Corollary (1-3-19): [30]

Let $X_{T_jI}^{\delta}, j = 1,2$ be an $i-TPS$,such that T_2 is finer than T_1 then

1. $Fd_{T_2}(\text{Ʉ}) \subseteq Fd_{T_1}(\text{Ʉ})$.
2. $Fcl_{T_2}(\text{Ʉ}) \subseteq Fcl_{T_1}(\text{Ʉ})$.

Proposition (1-3-20) : [35]

Let X_{TI}^{δ} be an $i-TPS$, and Ʉ is a subset of X, then $Fcl(\text{Ʉ}) \subseteq i-cl(\text{Ʉ})$.

Definition (1-3-21) : [31]

Let X_{TI}^{δ} be an $i-TPS$, and let Ʉ is a subset of X ,then we say that Ʉ is focal derived dense set if and only if $Fd(\text{Ʉ}) = X$, and its denoted by FD dense set. Also, it is called focal dense if and only if $Fcl(\text{Ա}) = X$, and it is denoted by FOdense.

Proposition (1-3-22): [35]

Let $f: X_{TI}^{\delta} \to Y_{TI}^{\delta}$ be an $i-$ closed function, then $f\big(Fcl(\text{Ա})\big) = Fcl\big(f(\text{Ա})\big)$ for each $i-$ closed set Ա of X.

Our aim now is to introduce the definition for the notion of focal limit point and focal derivative set with respect to i − subspace of i − TPS, with some of the properties and relations

Definition (1-3-23) :

Let $Y_{T_Y I_Y}^{\delta_Y}$ be an $i-subspace$ of $i-TPS$ X_{TI}^{δ}, and let $\text{Ա} \subseteq Y$, then the focal limit point of Ա w. r. t. $i-subspace\, Y$ can be defined as the following every $y \in Y$ s. t. for each $U_Y \in \oint_{I_Y}(y), U_Y \cap \text{Ա}/_{\{y\}} \neq \emptyset$, and the set of all focal limit points w. r. t. $i-subspace\, Y$ is called the focal derived set in $i-subspace\, Y$ and define by $Fd_Y(\text{Ա}) = \bigcup\{\, y \in Y,$ $\forall\, U_Y \in \oint_{I_Y}(y),\ \exists\, z \neq y, s.t. z \in U_Y\ and\ \ z \in \text{Ա}\}$, also the focal closure of the set Ա w. r. t. $i-subspace\, Y$denoted by $Fcl_Y(\text{Ա})$

Example (1-3-24):

Let $X = \{\mathscr{k}, \mathscr{g}, \mathscr{f}\}$, $T = \{\emptyset, X, \{\mathscr{k}, \mathscr{g}\}\}$, $I = \{\emptyset, \{\mathscr{f}\}\}$ with δ_D . And let $Y = \{\mathscr{k}, \mathscr{f}\}$ so $T_Y = \{\emptyset, X, \{\mathscr{k}\}\}$, $I_Y = \{\emptyset, \{\mathscr{f}\}\}$ with δ_{Y_D} be $i-subspace$ Y of X_{TI}^{δ}. Then $I_{Y_{\oint}}(\mathscr{k}) = \big\{Y, \{\mathscr{k}\}\big\} = I_{Y_{\oint}}(\mathscr{g})$, but $I_{Y_{\oint}}(\mathscr{f}) = Y$. Now let $\text{Ա} = \{\mathscr{k}, \mathscr{f}\}$, then $Fd(\text{Ա}) = \{\mathscr{g}, \mathscr{f}\}$, and $Fd_Y(\text{Ա}) = \{\mathscr{f}\}$.

Proposition (1-3-25):

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i-TPS$ X^{δ}_{TI}, and let $Ա \subseteq Y$, Then $Fd_Y(Ա) = Fd_X(Ա) \cap \{Y\}$

Proof: Let $y \in Fd_Y(Ա)$, so $\forall\, U_Y \in \oint_{I_Y}(y)\ \exists\, z \neq y, s.t. z \in U_Y$ and $z \in Ա$ but $U_Y = U \cap$ Y s.t. $U \in \oint_I(y)$ so $z \in U \cap Ա$. Therefore $y \in Fd_X(Ա)$, and $y \in Y$ so, $y \in Fd_X(Ա) \cap \{Y\}$. Thus $Fd_Y(Ա) \subseteq Fd_X(Ա) \cap \{Y\}$.

Conversely, Let $y \in Fd_X(Ա) \cap \{Y\}$, then $y \in Fd_X(Ա)$, and $y \in Y$, and $\forall\, U \in \oint_I(y)$, $\exists\, z \neq y, s.t. z \in U \cap Ա$, but $Ա \subseteq$ Y, then $z \in Y$ which implies $z \in U \cap Y$, but $U_Y = U \cap Y$, therefore by definition (1-3-8) we have $y \in \oint_{I_Y}(y)$, so, $y \in Fd_Y(Ա)$. Thus $Fd_X(Ա) \cap \{Y\} \subseteq Fd_Y(Ա)$. And this completes the proof.

Proposition (1-3-26):

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i-TPS$ X^{δ}_{TI}, and let $Ա \subseteq Y$, Then $Fcl_Y(Ա) = Fd_Y(Ա) \cup Ա$

Proof: Since $Fcl_X(Ա) = Fd_X(Ա) \cup Ա$, by definition (1-3-13), so

$$Fcl_X(Ա) \cap \{Y\} = (Fd_X(Ա) \cup Ա) \cap \{Y\}$$

$$= (Fd_X(Ա) \cap \{Y\}) \cup Ա = Fd_Y(Ա) \cup Ա$$

Thus $Fcl_Y(Ա) = Fd_Y(Ա) \cup Ա$.

Proposition (1-3-27):

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i-TPS$ X^{δ}_{TI}, and let $Ա \subseteq Y$, Then $Fcl_Y(Ա) = \cup \left\{ y \in Y, \forall\, U_Y \in \oint_{I_Y}(y), \exists\, z \in U_Y \text{ and } z \in Ա \right\} \cup Ա$.

Proof: Let $y \in Fcl_Y(Ա)$, then $y \in Fd_Y(Ա)$ or $y \in Ա$.

If $y \in Fd_Y(\text{Ꝋ})$ then by definition (1-3-23) $\forall\ U_Y \in \oint_{I_Y}(y)\ \exists\ z \neq y, s.t. z \in U_Y\ and\ \ z \in \text{Ꝋ}$, then $y \in \bigcup \left\{y \in Y, \forall\ U_Y \in \oint_{I_Y}(y), \exists\ z \in U_Y\ \ and\ z \in \text{Ꝋ}\right\}$.

If $y \in \text{Ꝋ}$ and for any $U_Y \in \oint_{I_Y}(y)$, $y \in U_Y$, so we get $y \in U_Y \cap \text{Ꝋ}$ then $U_Y \cap \text{Ꝋ} \neq \emptyset$. Hence $y \in \bigcup \left\{y \in Y, \forall\ U_Y \in \oint_{I_Y}(y), \exists\ z \in U_Y\ \ and\ z \in \text{Ꝋ}\right\}$. Thus $Fcl_Y(\text{Ꝋ}) \subseteq \bigcup \left\{y \in Y, \forall\ U_Y \in \oint_{I_Y}(y), \exists\ z \in U_Y\ and\ z \in \text{Ꝋ}\right\}$

Conversely, let $y \in \bigcup \left\{y \in Y, \forall\ U_Y \in \oint_{I_Y}(y), \exists\ z \in U_Y\ \ and\ z \in \text{Ꝋ}\right\}$, then for any $U_Y \in \oint_{I_Y}(y)$, $y \in U_Y$, there exist $\exists\ z \neq y$, $z \in U_Y\ \ and\ z \in \text{Ꝋ}$. So $y \in Fd_Y(\text{Ꝋ})$, then $y \in Fcl_Y(\text{Ꝋ})$ which mean that

$\bigcup \left\{y \in Y, \forall\ U_Y \in \oint_{I_Y}(y), \exists\ z \in U_Y\ and\ z \in \text{Ꝋ}\right\} \subseteq Fcl_Y(\text{Ꝋ})$.

Proposition (1-3-28):

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ \ X^{\delta}_{TI}$, and let $\text{Ꝋ} \subseteq Y$, Then $Fcl_Y(\text{Ꝋ}) = Fcl_X(\text{Ꝋ}) \cap Y$

Proof: $Fcl_X(\text{Ꝋ}) \cap Y = [Fd_X(\text{Ꝋ}) \cup \text{Ꝋ}] \cap Y$

$$= (Fd_X(\text{Ꝋ}) \cap Y) \cup \text{Ꝋ}$$

$$= Fd_Y(\text{Ꝋ}) \cup \text{Ꝋ} = Fcl_Y(\text{Ꝋ}).$$

Definition (1-3-29):

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ X^{\delta}_{TI}$, and let Ꝋ be a subset of Y, then we say that Ꝋ is focal dense w. r. t. $i - subspace$ if and only if $Fcl_Y(\text{Ꝋ}) = Y$, and it is denoted by FO_Ydense.

Example (1-3-30):

Let $X = \{\mathscr{k}, \mathscr{g}, \mathscr{f}\}$, $T = \{\emptyset, X, \{\mathscr{k}, \mathscr{g}\}\}$, $I = \{\emptyset, \{\mathscr{f}\}\}$ with δ_D . And let $Y = \{\mathscr{k}, \mathscr{f}\}$ so $T_Y = \{\emptyset, X, \{\mathscr{k}\}\}$, $I_Y = \{\emptyset, \{\mathscr{f}\}\}$ with δ_{Y_D} be $i\text{–}subspace$ Y of X_{TI}^{δ}. Now let $\text{Ư} \subseteq Y$ s.t. $\text{Ư} = \{\mathscr{k}\}$, then $Fd_Y(\text{Ư}) = \{\mathscr{f}\}$, then we get $Fcl_Y(\text{Ư}) = \text{Ư} \cup Fd_Y(\text{Ư}) = \{\mathscr{k}\} \cup \{\mathscr{f}\} = Y$. Thus Ư is FO_Ydense.

Other concepts that introduced in the $i - TPS$ in [2], is the occlusion set and $\psi_T -$ operator so we will be showing the definitions, properties with some relations.

Definition (1-3-31): [30]

Let X_{TI}^{δ} be an $i - TPS$, and let Ư is a sub set of X, then the occlusion set of Ư with respect to $T(x)$ defined by $\oint_{\text{T}}(\text{Ư}) = \{x \in X: for\ each\ U \in T(x), U\ \delta\ \text{Ư}\}$.

Example (1-3-32) : [30]

In the i – topological proximity pace $(\text{X}, \text{T}_\text{i}, \{\emptyset\}, \delta)$,where δ is the discrete proximity then $\oint_{\text{T}}(\text{Ư}) = \text{X}$, for each subset Ư of X .

Proposition (1-3-33) : [30]

Let X_{TI}^{δ} be an $i - TPS$, then the following are true:

1. $\text{Ư} \subseteq \oint_{\text{T}}(\text{Ư})$.

2. $X - (\oint_{\text{T}}(\text{Ư})) \subseteq \oint_{\text{T}}(\text{X} - \text{Ư})$.

3. $\text{Ư} \subseteq \mathcal{K}$, then $\oint_{\text{T}}(\text{Ư}) \subseteq \oint_{\text{T}}(\mathcal{K})$.

4. $\oint_{\text{T}}(\text{Ư}) \cap \oint_{\text{T}}(\mathcal{K}) \supseteq \oint_{\text{T}}(\text{Ư} \cap \mathcal{K})$.

5. $\oint_{\text{T}}(\text{Ư}) \cup \oint_{\text{T}}(\mathcal{K}) = \oint_{\text{T}}(\text{Ư} \cup \mathcal{K})$.

6. $\oint_{\text{T}}(\text{Ư}) = \emptyset$, if and only if $\text{Ư} = \emptyset$, and $\oint_{\text{T}}(\text{X}) = \text{X}$.

7. $\oint_{\text{T}}(\oint_{\text{T}}(\text{Ư})) = \oint_{\text{T}}(\text{Ư})$, for each $\text{Ư} \in I_{\oint}(x)$,$x \in \text{X}$

8. $\oint_{\text{T}}(\text{Ư}) \in \text{I}_{\oint}$ (x) for each $\text{Ư} \in \text{I}_{\oint}$ (x) and for some $\text{x} \in \text{Ư}$.

9. $\text{i} - \text{cl}(\text{Ư}) \subseteq \oint_{\text{T}}(\text{Ư})$,for each $\text{Ư} \subseteq X$.

10. If $\oint_{T}(Ų) = \emptyset$,then $Fd(Ų) = \emptyset$.

11. $Fd(Ų) \subseteq Fcl(Ų) \subseteq \oint_{T}(Ų)$.

12. If $Fcl(Ų) = X$, then $\oint_{T}(Ų) = X$.

13. $\oint_{T}(Ų) = \oint_{T}(Ų - \mathcal{K}) = \oint_{T}(Ų \cup \mathcal{K})$, for each $\mathcal{K} \in I$ and $Ų \in I_{\oint}(x)$, for each $x \in X$.

14. $U \cap \oint_{T}(Ų) \subseteq \oint_{T}(U \cap Ų)$ for each $U \in \oint_{I}(x)$, $x \in X$.

15. Let (X, T_j, I, δ) be an i – topological proximity space such that $T_1 \subset T_2$,then $\oint_{T_2}(Ų) \subseteq \oint_{T_1}(Ų)$ for each subset Ų of X .

16. $\oint_{T}(Ų) \neq Ų$, for each $Ų \in I$.

Definition (1-3-34) : [31]

Let X_{TI}^{δ} be an $i - TPS$, then we say that a subset Ų of X is $\oint_{T}$ –dense set if and only if $\oint_{T}(Ų) = X$.

Definition (1-3-35): [34]

Let X_{TI}^{δ} be an $i - TPS$, an operator $\psi_T: P(X) \rightarrow P(X)$ defined as follow:
$\psi_T(Ų) = \{x \in X: there\ exist\ U \in T(x), U \ll Ų\}$

Example (1-3-36):

Let $X = \{k, g, f\}$, $T = \{\emptyset, X, \{k\}, \{g\}\}$, $I = \{\emptyset, \{f\}\}$ with δ_D. And let $Ų = \{k, g\}$, then $\psi_T(Ų) = \{k, g\}$.

Proposition (1-3-37) : [34]

Let X_{TI}^{δ} be an $i - TPS$, then each of the following are held for every subset $Ų, \mathcal{K}$ of X :

1. $\psi_T(Ų) \subseteq Ų$, for each $Ų \subseteq X$.
2. $\psi_T(Ų) = \emptyset$, for each $Ų \in I$.
3. If $Ų \subseteq \mathcal{K}$, then $\psi_T(Ų) \subseteq \psi_T(\mathcal{K})$.
4. $\psi_T(Ų) \cup \psi_T(\mathcal{K}) \subseteq \psi_T(Ų \cup \mathcal{K})$.

5. $\psi_T(\text{Ų} \cap \mathcal{K}) \subseteq \psi_T(\text{Ų}) \cap \psi_T(\mathcal{K})$ for each $\text{Ų}, \mathcal{K} \subseteq X$.

6. $\text{X} - \oint_{\text{T}}(\text{X} - \text{Ų}) = \psi_{\text{T}}(\text{Ų})$, for each subset Ų of X.

And $\oint_{\text{T}}(\text{Ų}) = \text{X} - \psi_{\text{T}}(\text{X} - \text{Ų})$,

7. $\psi_{\text{T}}\big(\psi_{\text{T}}(\text{Ų})\big) \subseteq \psi_{\text{T}}(\text{Ų})$, for each subset Ų of X.

8. $\psi_{\text{T}}(\text{Ų}) \subseteq \text{Fcl}(\text{Ų})$, for each subset Ų of X.

9. $\psi_{\text{T}}(X - \text{Ų}) \subseteq \text{X} - \big(\psi_{\text{T}}(\text{Ų})\big)$, for each subset Ų of X.

10. $\psi_{\text{T}}(\text{Ų}) \subseteq \oint_{\text{T}}(\text{Ų})$ for each subset Ų of X.

11. Ų is FOdense set, then $\psi_{\text{T}}(\text{X} - \text{Ų}) = \emptyset$.

12. Ų is $\oint_{\text{T}}-$dense set, if and only if $\psi_{\text{T}}(\text{X} - \text{Ų}) = \emptyset$.

Proposition (1-3-38) : [34]

Let $X^{\delta}_{T_j I}, j = 1{,}2$ be an $i - TPS$, such that T_2 is finer than T_1, then $\psi_{\text{T}_1}(\text{Ų}) \subseteq \psi_{\text{T}_2}(\text{Ų})$.

Chapter Two

Some Results in i-topological Proximity Space

2.1 Some Results in Occlusion Set via $i - open$ set

Our aim in this section is to study some results on the occlusion set with respect to $i - open$ set and investigate its image and preimage under the $\delta - symmetry$ function. Also, we study this notation $\psi_T - oerater$ notation with respect to $i - subspace$ of $i - TPS$, with some most important properties and relations.

Proposition (2-1-1):

Let $X_{TI_j}^{\delta_j}$ $be\ i - TPS\ \ j = 1,2$, such that $\mathrm{I}_1 \subseteq \mathrm{I}_2$ and for $\mathrm{Ц} \subseteq \mathrm{X}$ then:

1. If $I_1 \subseteq \mathrm{I}_2$ then $(\oint_{\mathrm{T}}(\mathrm{Ц}))_{I_1} \subseteq (\oint_{\mathrm{T}}(\mathrm{Ц}))_{I_2}$, where $(\oint_{\mathrm{T}}(\mathrm{Ц}))_{I}$ means $\oint_{\mathrm{T}}(\mathrm{Ц})$ w. r. t. ideal I.
2. If $\delta_2 < \delta_1$ then $(\oint_{\mathrm{T}}(\mathrm{Ц}))_{\delta_1} \subseteq (\oint_{\mathrm{T}}(\mathrm{Ц}))_{\delta_2}$ where $(\oint_{\mathrm{T}}(\mathrm{Ц}))_{\delta}$ means $\oint_{\mathrm{T}}(\mathrm{Ц})$ w. r. t. proximity relation δ.

Proof:

1. Let $k \in (\oint_{\mathrm{T}}(\mathrm{Ц}))_{I_1}$i.e., $\oint_{\mathrm{T}}(\mathrm{Ц})$ w. r. t. I_1, then $H\ \delta\ \mathrm{Ц}$, $\forall\ H \in T(k)$. So, $H \in \mathrm{I}_{1\oint}\ (k)$ by theorem (1-3-3) part (1), but by proposition (1-3-4) we have $\mathrm{I}_{1\oint}\ (k) \subseteq \mathrm{I}_{2\oint}\ (k)$, then $H\ \delta\ \mathrm{Ц}$, $\forall\ H \in T(k)$ w. r. t. I_2. Thus $k \in (\oint_{\mathrm{T}}(\mathrm{Ц}))_{I_2}$.
2. Let $k \in (\oint_{\mathrm{T}}(\mathrm{Ц}))_{\delta_1}$ i.e., $\oint_{\mathrm{T}}(\mathrm{Ц})$ w. r. t. δ_1, then $H\ \delta_1\ \mathrm{Ц}$, $\forall\ H \in T(k)$, but since $\delta_2 < \delta_1$ then by definition (1-2-5) we have $H\ \delta_2 \mathrm{Ц}$, $\forall\ H \in T(k)$. Thus $k \in (\oint_{\mathrm{T}}(\mathrm{Ц}))_{\delta_2}$.

Definition (2-1-2):

A function $f : X^{\delta} \to Y^{\delta}$ is called $\delta - symmetry$ if and only if f it is δ – homeomorphism $i - open$ map.

Proposition (2-1-3) :

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be a $\delta - symmetry$, then $f(\oint_{\mathrm{T}}(\mathrm{Ų})) = \oint_{\grave{\mathrm{T}}}\big(f(\mathrm{Ų})\big)$ for each subset Ų of X.

Proof: Let $k \in f(\oint_{\mathrm{T}}(\mathrm{Ų}))$, then $\exists\, x \in X$ s. t. $f(x) = k$ and $x \in \oint_{\mathrm{T}}(\mathrm{Ų})$, and $\forall\, H \in T(x), H\delta\mathrm{Ų}$, because f is $\delta - continuous$, then we get $f(H)\grave{\delta}f(\mathrm{Ų})\ \forall\, f(H) \in \grave{T}\big(f(x)\big) = \grave{T}(k)$,

which means $f(H)\grave{\delta}f(\mathrm{Ų}), \forall\, f(H) \in \grave{T}(k)$. Thus $k \in \oint_{\grave{\mathrm{T}}}(f(\mathrm{Ų}))$

Conversely, let $k \in \oint_{\grave{\mathrm{T}}}(f(\mathrm{Ų}))$, then $\forall\, H \in \grave{T}(k), H\grave{\delta}f(\mathrm{Ų})$ but f^{-1} is $\delta -$ continuous, f is $\delta -$ continuous and $i -$ open, then $f^{-1}(H)\delta\mathrm{Ų}$ $\forall\, f^{-1}(H) \in T\big(f^{-1}(k)\big)\ s.t.\ x = f^{-1}(k) \in \oint_{\mathrm{T}}(\mathrm{Ų})$.

Thus $k = f(x) \in f(\oint_{\mathrm{T}}(\mathrm{Ų}))$,and this completes the proof.

In the following proposition we will see that the pre-image of occlusion set via $i - open$ set under bijective $\delta -$continuous and $i -$open function is also occlusion set

Proposition (2-1-4):

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be $\delta - symmetry$ function, then

$$f^{-1}(\oint_{\grave{\mathrm{T}}}(\mathrm{Ų})) = \oint_{\mathrm{T}}\big(f^{-1}(\mathrm{Ų})\big).$$

Proof: Let $k \in f^{-1}(\oint_{\grave{\mathrm{T}}}(\mathrm{Ų}))$, for each subset Ų of Y then $\exists y \in Y$ s. t. $f(k) = y$ and $y \in \oint_{\grave{\mathrm{T}}}(\mathrm{Ų})$, then we will get $\forall\, H \in \grave{T}(y), H\grave{\delta}\mathrm{Ų}$. Since f^{-1} is $\delta -$continuous so $f^{-1}(H)\delta f^{-1}(\mathrm{Ų})\ \forall\, f^{-1}(H) \in T(k)$ which means that $k \in \oint_{\mathrm{T}}\big(f^{-1}(\mathrm{Ų})\big)$.

Conversely, let $k \in \oint_{\mathrm{T}}\big(f^{-1}(\mathrm{Ų})\big)$, then $\forall\, H \in T(k), H\delta f^{-1}(\mathrm{Ų})$ but f^{-1} is $\delta -$ continuous since f is $\delta -$ continuous and $i -$ open, so $f(H)\grave{\delta}\mathrm{Ų}$ $\forall\, f(H) \in \grave{T}\big(f(k)\big)\ so\ f(k) \in \oint_{\grave{\mathrm{T}}}(\mathrm{Ų})$, Thus $k \in f^{-1}(\oint_{\mathrm{T}}(\mathrm{Ų}))$.

Definition (2-1-5):

The proximity relation δ on a set X is called $\sigma - proximity$, if for any index Λ, and for any subsets A_λ, B of X $(\bigcup_{\lambda\in\Lambda} A_\lambda)\delta B$ if and only if $\exists\ \lambda_0 \in \Lambda$ such that $A_{\lambda_0}\delta B$.

Proposition (2-1-6):

Let X_{TI}^{δ} be an $i - TPS$ then for any subset Ա of X we have $(\oint_{\mathrm{T}}(\cup\ Ա_\lambda)) = \cup \oint_{\mathrm{T}}(Ա_\lambda)$.

Proof: Directly from proposition (1-3-33) part (5) .

In the following propositions we will see that the image (pre-image) of $\oint_{\mathrm{T}}$ −dense subset under $\delta - symmetry$ function is also $\oint_{\mathrm{T}}$ −dense subset

Proposition (2-1-7):

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be a $\delta - symmetry$ function, then if a subset Ա of X is $\oint_{\mathrm{T}}$ −dense in X, then $f(Ա)$ is $\oint_{\grave{\mathrm{T}}}$ −dense in Y.

Proof: Let Ա be $\oint_{\mathrm{T}}-$ dense in X, then $\oint_{\mathrm{T}}(Ա) = X$, so we get by proposition (2-1-3) $\oint_{\grave{\mathrm{T}}}(f(Ա)) = f(\oint_{\mathrm{T}}(Ա)) = f(X) = Y$ because f surjective. Thus $\oint_{\grave{\mathrm{T}}}(f(Ա))$ is $\oint_{\grave{\mathrm{T}}}$ −dense in Y.

Proposition (2-1-8) :

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be a $\delta - symmetry$ function, then if a subset $\mathcal{K}$ of Y is $\oint_{\grave{\mathrm{T}}}$ −dense in Y, then $f^{-1}(\mathcal{K})$ is $\oint_{\mathrm{T}}$ −dense in X.

Proof: Let $\mathcal{K}$ be $\oint_{\grave{\mathrm{T}}}-$ dense in Y, then $\oint_{\grave{\mathrm{T}}}(\mathcal{K}) = Y$, so we get by proposition (2-1-4) $\oint_{\mathrm{T}}(f^{-1}(\mathcal{K})) = f^{-1}(\oint_{\grave{\mathrm{T}}}(\mathcal{K})) = f^{-1}(Y) = X$.

Thus $f^{-1}(\mathcal{K})$ is $\oint_{\mathrm{T}}$ −dense in X.

Definition (2-1-9):

Let X^{δ} be a proximity space, a nonempty subset $\mathfrak{U}$ of X is called Jaw set if and only if $\mathfrak{U}\bar{\delta}(X-\mathfrak{U})$, the collection of all Jaw sets denoted by $J(X)$. In addition, the collection of all Jaw and $i-open$ sets denoted by $i-J_T(X)$.

Example (2-1-10):

Let $X=\{\mathscr{k},\mathscr{g},\mathscr{f}\}$, and for any subsets of X $A\ and\ B$, we can define an indiscrete proximity relation δ_i such that $A\delta_i B$ if and only if $A\neq\emptyset\ and\ B\neq\emptyset$ **[7]**. In this example it is clear that any nonempty subset of X is Jaw set.

The following proposition will study some properties of the Jaw set and $i-J_T(X)$ with some relations.

Proposition (2-1-11) :

Let X_{TI}^{δ} be an $i-TPS$ and for a subset $\mathfrak{U}$ of X if $\mathfrak{U}\in i-J_T(X)$, then:

1- $\mathfrak{U}=\psi_T(\mathfrak{U})$
2- $\mathfrak{U}\subseteq\psi_T(\oint_T(\mathfrak{U}))$
3- $\oint_T(\mathfrak{U})=X$
4- $\oint_T(\mathfrak{U})=\oint_T(\oint_T(\mathfrak{U}))$
5- $\oint_T(\mathfrak{U})=\oint_T(\psi_T(\mathfrak{U}))$
6- $\psi_T(\oint_T(\mathfrak{U}))\subseteq\oint_T(\psi_T(\mathfrak{U}))$

Proof:

1) Let $\mathscr{k}\in\mathfrak{U}$, since $\mathfrak{U}\in i-J_T(X)$, then $\mathfrak{U}\bar{\delta}(X-\mathfrak{U})$, so $\mathscr{k}\in\psi_T(\mathfrak{U})$. Hence $\mathfrak{U}\subseteq\psi_T(\mathfrak{U})$, but by proposition (1-3-37) part (1) we have $\psi_T(\mathfrak{U})\subseteq\mathfrak{U}$. Thus, $\mathfrak{U}=\psi_T(\mathfrak{U})$.

2) Since $\mathfrak{U} \subseteq \oint_{T}(\mathfrak{U})$ by proposition (1-3-33) part (1), then $\psi_T(\mathfrak{U}) \subseteq \psi_T(\oint_{T}(\mathfrak{U}))$ by proposition (1-3-37) part (3), but $\mathfrak{U} \in i - J_T(X)$, then by (1) we have $\mathfrak{U} = \psi_T(\mathfrak{U})$.
   Thus $\mathfrak{U} \subseteq \psi_T(\oint_{T}(\mathfrak{U}))$
3) It is clear that $\oint_{T}(\mathfrak{U}) \subseteq X$, we just need to prove that $X \subseteq \oint_{T}(\mathfrak{U})$, so let $\mathscr{k} \in X$ if $\mathscr{k} \notin \oint_{T}(\mathfrak{U})$, then $\exists\, H \in T(\mathscr{k})$ s.t. $H\bar{\delta}\mathfrak{U}$, so by proposition (1-2-3) part (3) $H \cap \mathfrak{U} = \emptyset$. But $H, \mathfrak{U} \in T$, so by definition (1-1-11) $\exists\, W \in T$ s. t.
   $\big((H \cap \mathfrak{U}) - W\big) \cup \big(W - (H \cap \mathfrak{U})\big) \in I$,
   i. e. $(\emptyset - W) \cup (W - \emptyset) = W \in I$ which a contradiction with the fact that $T \cap I = \emptyset$ in definition (1-1-11).
4) It is direct by proposition (1-3-33) part (7) and proposition (1-3-3) part (1)
5) It is easy by (1)
6) Since $\mathfrak{U} = \psi_T(\mathfrak{U})$ by part (1), then $\oint_{T}(\mathfrak{U}) = \oint_{T}(\psi_T(\mathfrak{U}))$ .Thus $\psi_T(\oint_{T}(\mathfrak{U})) = \psi_T(\oint_{T}(\psi_T(\mathfrak{U}))) \subseteq \oint_{T}(\psi_T(\mathfrak{U}))$ by proposition (1-3-37) part (1)

Corollary (2-1-12) :

Let X_{TI}^{δ} be an $i - TPS$ and for a subset $\mathfrak{U}$ of X if $\mathfrak{U}\ is\ i - closed$ and Jaw set, then $\oint_{T}(\mathfrak{U}) = \mathfrak{U}$ and Jaw set.

Proof: Since $\mathfrak{U}\ is\ i - closed$, then $X - \mathfrak{U}\ is\ i - open$ which is mean that $X - \mathfrak{U} \in i - J_T(X)$, and this implies by proposition (2-1-11) part (1) that $X - \mathfrak{U} = \psi_T(X - \mathfrak{U}) = X - \oint_{T}(\mathfrak{U})$

Thus $\oint_{T}(\mathfrak{U}) = \mathfrak{U}$ and $\oint_{T}(\mathfrak{U})$ jaw set and $i - closed.$

Definition (2-1-13) :

In $i - TPS\ X_{TI}^{\delta}$, the subset Ʉ $\subseteq X$ is called $i - \psi_T - open$ set if and only if Ʉ $\subseteq \psi_T(\oint_T(\mathfrak{U}))$, and the family of all $i - \psi_T - open$ subsets denoted by $\psi_T.O.(X)$.

Example (2-1-14) :

Let $X = \{\hbar, g, f\}$, $T = \{X, \emptyset, \{\hbar\}, \{g\}\}$, and $I = \{\emptyset, \{f\},\}$ with the discrete proximity δ_D, if Ʉ $= \{\hbar, g\}$ then $\oint_T$(Ʉ) $= X$ and $\psi_T(\oint_T$(Ʉ)$) = \psi_T(X) = X$. So, we get that Ʉ $\subseteq \psi_T(\oint_T$(Ʉ)$)$.

Proposition (2-1-15):

Let X_{TI}^{δ} be an $i - TPS$ and for a subset Ʉ of X. If Ʉ $\in i - J_T(X)$, then Ʉ $\in \psi_T.O.(X)$.

Proof: Obviously by proposition (2-1-11) part (2)

Proposition (2-1-16):

Let X_{TI}^{δ} be an $i - TPS$ and for a subset Ʉ of X, in $\psi_T.O.(X)$, $\psi_T(\oint_T$(Ʉ)$)$ is $i - \psi_T - open$

Proof: Let Ʉ $\in \psi_T.O.(X)$, then we have that Ʉ $\subseteq \psi_T(\oint_T$(Ʉ)$)$. So $\oint_T$(Ʉ) $\subseteq \oint_T(\psi_T(\oint_T$(Ʉ)$))$, then $\psi_T(\oint_T$(Ʉ)$) \subseteq \psi_T(\oint_T(\psi_T(\oint_T$(Ʉ)$)))$. Thus $\psi_T(\oint_T$(Ʉ)$)$ is $i - \psi_T - open$.

Theorem (2-1-17):

Let X_{TI}^{δ} be an $i - TPS$ and for $\hbar \in X$ and Ʉ $\subseteq X$. If Ʉ $\in i - J_T(\hbar)$ s. t. $i - J_T(\hbar) = \{A \in i - J_T(X), \hbar \in A\}$, then

$$\psi_T(\oint_T(\text{Ʉ})) = \psi_T(\oint_T(\psi_T(\oint_T(\text{Ʉ})))).$$

Proof: Let $W = \psi_T(\oint_T$(Ʉ)$) \in \psi_T.O.(X)$, by proposition (2-1-16) , then we have $W = \psi_T(\oint_T$(Ʉ)$) \subseteq \oint_T(\oint_T$(Ʉ)$) = \oint_T$(Ʉ) by theorem (1-3-3) part (1) and proposition (1-3-33) part (7).

Therefore, $\oint_{\mathrm{T}}(W) \subseteq \oint_{\mathrm{T}}(\oint_{\mathrm{T}}(\mathrm{U})) = \oint_{\mathrm{T}}(\mathrm{U})$, which implies that $\psi_T(\oint_{\mathrm{T}}(W)) \subseteq \psi_T(\oint_{\mathrm{T}}(\mathrm{U}))$, then $\psi_T(\oint_{\mathrm{T}}(W)) \subseteq W$.

Since $\psi_T(\oint_{\mathrm{T}}(\mathrm{U})) \in \psi_T.O.(X)$, so by proposition (2-1-16) we have $\psi_T(\oint_{\mathrm{T}}(\mathrm{U})) \subseteq \psi_T\big(\oint_{\mathrm{T}}\big(\psi_T(\oint_{\mathrm{T}}(\mathrm{U}))\big)\big)$, i.e. $W \subseteq \psi_T(\oint_{\mathrm{T}}(W))$

Thus $\psi_T(\oint_{\mathrm{T}}(\mathrm{U})) = \psi_T\big(\oint_{\mathrm{T}}\big(\psi_T(\oint_{\mathrm{T}}(\mathrm{U}))\big)\big)$.

Now we will study the nation of occlusion set via $i - open$ and the operator $\psi_T - operater$ with respect to $i - subspace$ of $i - TPS$, with some properties and relations

Definition (2-1-18) :

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS$ X^{δ}_{TI}, and let $\mathfrak{U} \subseteq$ Y, then the set of occlusion point via $i - open$ set w. r. t. $i - subspace\, Y$ defined by $\oint_{\mathrm{T_Y}}(\mathfrak{U}) = \{y \in Y: for\ each\ U_Y \in T_Y(y), U_Y\ \delta_Y\ \mathfrak{U}\}$.

Example (2-1-19):

Let $X = \{\mathscr{k}, \mathscr{g}, \mathscr{f}\}$, $T = \{\emptyset, X, \{\mathscr{k}\}, \{\mathscr{g}\}\}$, $I = \{\emptyset, \{\mathscr{f}\}\}$ with δ_D . And let $Y = \{\mathscr{k}, \mathscr{f}\}$ so $T_Y = \{\emptyset, Y, \{\mathscr{k}\}\}$, $I_Y = \{\emptyset, \{\mathscr{f}\}\}$ with δ_{Y_D} be i– $subspace$ Y of X^{δ}_{TI}. Now let $\mathrm{U} \subseteq Y$ s.t. $\mathrm{U} = \{\mathscr{f}\}$, then $\oint_{\mathrm{T_Y}}(\mathrm{U}) = \{\mathscr{f}\}$.

In the following proposition we will see that the occlusion set via $i - open$ with respect to $i - subspace$ of $i - TPS$, is a subset of the intersection of the occlusion set via $i - open$ w. r. t. $i - TPS$ with the subspace Y.

Proposition (2-1-20)

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i-TPS$ X^{δ}_{TI}, and let $\text{Џ} \subseteq Y$, Then $\oint_{T_Y}(\text{Џ}) \subseteq \oint_{T_X}(\text{Џ}) \cap Y$.

Proof: Let $y \in \oint_{T_Y}(\mathfrak{U})$, then $\forall\, U_y \in T_Y(y), U_Y\ \delta_Y\ \text{Џ}$, but $U_y = U \cap Y$ for any $U \in T_X(y)$, i.e. $U_y \subseteq U$. Thus $U\ \delta_Y\ \text{Џ}$, which means that $y \in \oint_T(\mathfrak{U})$, and since $y \in Y$ we get that $y \in \oint_T(\text{Џ}) \cap Y$.

The following example, will showing us that the equality of above proposition cannot satisfies in general.

Example (2-1-21):

Let $X = \{\hbar, g, f, \ell\}$, $T = \{\emptyset, X, \{\hbar\}, \{g, f\}\}$, $I = \{\emptyset, \{f\}, \{\ell\}, \{f, \ell\}\}$ with δ_D . And let $Y = \{\hbar, g, f\}$ so $T_Y = \{\emptyset, Y, \{\hbar\}, \{g, f\}\}$, $I_Y = \{\emptyset, \{f\}\}$ with δ_{Y_D} be $i\text{–}subspace$ Y of X^{δ}_{TI} . Now let $\text{Џ} \subseteq Y$ s.t. $\text{Џ} = \{\hbar\}$, then $\oint_{T_Y}(\text{Џ}) = \{\hbar\}$, but $\oint_{T_X}(\text{Џ}) = \{\hbar, f, \ell\}$.

Thus $\oint_{T_X}(\text{Џ}) \cap Y = \{\hbar, f\} \nsubseteq \oint_{T_Y}(\text{Џ}) = \{\hbar\}$.

The condition that makes the equality of proposition (2-1-20) satisfying is that the $i-TPS$ has to have intuition smoothing feature as we define in the following definition.

Definition (2-1-22)

The proximity relation it is said to have intuition smoothing feature if for any sets $A\delta B$ and for any $\emptyset \neq C \subseteq B$, then $A\delta C$.

Corollary (2-1-23)

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i-TPS$ X^{δ}_{TI}, and let $\text{Ư} \subseteq Y$, Then $\oint_{\mathrm{T_Y}}(\text{Ư}) = \oint_{\mathrm{T}}(\text{Ư}) \cap Y$ if and only if X^{δ}_{TI} have intuition smoothing feature.

Proof: We just need to proof that $\oint_{\mathrm{T}}(\text{Ư}) \cap Y \subseteq \oint_{\mathrm{T_Y}}(\text{Ư})$ so let $y \in \oint_{\mathrm{T}}(\text{Ư}) \cap Y$, then $\forall\ U \in T(y), U\delta\text{Ư}$, but $y \in Y$, so $y \in U \cap Y = U_y$. Since X is smooth $i-TPS$ and $U_y \subseteq U$ so we have $U_y\ \delta_Y\ \text{Ư}$, which means that $y \in \oint_{\mathrm{T_Y}}(\text{Ư})$.

Now we have to define the $\psi_T - operator$ with respect to $i-subspace$ as in the following definition.

Definition (2-1-24):

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i-TPS$ X^{δ}_{TI}, we can define an operator $\psi_{T_Y}: P(Y) \to P(Y)$ by $\psi_{T_Y}(\text{Ư}) = \{y \in Y : \exists\ U_y \in T_Y(y), U_y \bar{\delta}_Y\ (Y - \text{Ư})\}$.

Example (2-1-25):

Let $X = \{\hbar, g, f\}$, $T = \{\emptyset, X, \{\hbar\}, \{g\}\}$, $I = \{\emptyset, \{f\}\}$ with δ_D. And let $Y = \{\hbar, f\}$ so $T_Y = \{\emptyset, Y, \{\hbar\}\}$, $I_Y = \{\emptyset, \{f\}\}$ with δ_{Y_D} be $i\text{–}subspace$ Y of X^{δ}_{TI}. Now let $\text{Ư} \subseteq Y$ s.t. $\text{Ư} = \{\hbar\}$, then $\psi_{T_Y}(\text{Ư}) = \{\hbar\}$.

In the following proposition we will study the relation of $\psi_T - operator$ with respect to $i-subspace$ and the intersection of $\psi_T - operator$ with respect to $i-TPS$ with the $i-subspace$.

Proposition (2-1-26)

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i-TPS$ X^{δ}_{TI}, then for any subset Ư of Y. $\psi_T(\text{Ư}) \cap Y \subseteq \psi_{T_Y}(\text{Ư})$.

Proof: Let $y \in \psi_T(\text{Ʉ}) \cap Y$, then $y \in \psi_T(\text{Ʉ})$ and $y \in Y$, so $\exists\, U \in T(y), U\bar{\delta}(X - \text{Ʉ})$, but $y \in Y$, then $y \in U \cap Y = U_y$ and $U_y \subseteq U$ so we have $U_y\ \bar{\delta}_Y\ (Y - \text{Ʉ})$, which means that $y \in \psi_{T_Y}$ (Ʉ).

Proposition (2-1-27)

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i - TPS\ X^{\delta}_{TI}$, then for any subset Ʉ of Y. $\psi_{T_Y}\ (\text{Ʉ}) = [X - \oint_{\text{T}_\text{Y}}(Y - \text{Ʉ})] \cap Y$.

Proof: Let Ʉ be a subset of Y, then

$$\psi_{T_Y}\ (\text{Ʉ}) = Y - \oint_{\text{T}_\text{Y}}(Y - \text{Ʉ})$$

$$= \left[X - \left(\oint_{\text{T}_\text{X}}(Y - \text{Ʉ}) \cap Y\right)\right] \cap Y$$

$$= \left[\left(X - \oint_{\text{T}_\text{X}}(Y - \text{Ʉ})\right) \cup (X - Y)\right] \cap Y$$

$$= \left[\left(X - \oint_{\text{T}_\text{X}}(Y - \text{Ʉ})\right)\right] \cap Y$$

Definition (2-1-28) :

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ X^{\delta}_{TI}$, and for any subset Ʉ of Y, Ʉ is called occlusion denes set via $i - open$ set w.r.t. $i - subspace\ Y$, if for any $y \in Y$ and for all $H_y \in T_Y(y)$ then $H\delta$Ʉ. And we will be denoted by $\oint_{\text{T}_\text{Y}} - dense$.

Example (2-1-29):

Let $X = \{k, g, f\}$, $T = \{\emptyset, X, \{k, g\}, \{k, f\}\}$, $I = \{\emptyset, \{f\}\}$ with δ_D. And let $Y = \{k, g\}$ so $T_Y = \{\emptyset, Y\}$, $I_Y = \{\emptyset\}$ with δ_{Y_D} be $i\text{–}subspace\ Y$ of X^{δ}_{TI}. Now let $A \subseteq Y$ s.t. $A = \{k\}$, then $\oint_{\text{T}_\text{Y}}(A) = Y$ so A is $\oint_{\text{T}_\text{Y}} - denes$.

In the following proposition we will see that the $\oint_{\mathrm{T}} - denes$ set with respect to $i - TPS$, will be the $\oint_{\mathrm{T}} - denes$ with respect to $i - subspace$.

Proposition (2-1-30) :

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of smooth $i - TPS$ X^{δ}_{TI}, and for any subset Ʉ of Y, if Ʉ is $\oint_{\mathrm{T}} - dense$, then Ʉ is $\oint_{\mathrm{T_Y}} - dense$ in Y.

Proof: Since X^{δ}_{TI} is smooth, then by corollary (2-1-22) we will get $\oint_{\mathrm{T_Y}}(Ʉ) = \oint_{\mathrm{T}}(Ʉ) \cap Y$, for any subset Ʉ of Y. And since Ʉ is $\oint_{\mathrm{T}} - dense$ in X then $\oint_{\mathrm{T}}(Ʉ) = X$, so $\oint_{\mathrm{T_Y}}(Ʉ) = X \cap Y = Y$. Thus Ʉ is $\oint_{\mathrm{T_Y}} - dense$.

Example (2-1-31):

In example (2-1-29) we have $A = \{k\}$, then $\oint_{T}(A) = X$ so A is $\oint_{\mathrm{T}} - dense$, and $\oint_{\mathrm{T_Y}}(A) = Y$ so A is $\oint_{\mathrm{T_Y}} - dense$.

Remark (2-2-32):

The convers of proposition (2-1-30) not true in general and the following example show that.

Example (2-1-33):

In example (2-1-25) if $B = \{k\}$, then $\oint_{\mathrm{T_Y}}(B) = Y$ so B is $\oint_{\mathrm{T_Y}} - dense.$, but $\oint_{T_X}(B) = \{k, f\}$ so B is not $\oint_{\mathrm{T_X}} - dense$.

2.2 The Redirect and Strips set in i-Topological Proximity Spaces

By using the notation of occlusion set via $i-open$ set and the notation of $\psi_T - operater$ in i-topological proximity space, we spotlight in this section on the construction of new type points by introduce the notion of redirect points and strips points, also we showed their properties and study some relationships between them.

Definition (2-2-1) :

Let X_{TI}^{δ} be an $i-TPS$, the point $\hbar \in$ X is called redirect point of Ʉ $\subseteq X$ if and only if for all i −open set $\mathcal{H} \in$ T($\hbar$) we have $H\delta\oint_T(\mathfrak{U})$.

The set of all redirect point of Ʉ is called redirect set and denoted by $Ʉ_r$

Example (2-2-2) :

Let $X = \{\hbar, g, f\}$, $T = \{X, \emptyset, \{\hbar\}, \{g\}\}$, and $I = \{\emptyset, \{f\},\}$ with the discrete proximity δ_D, let Ʉ $= \{g\}$ then $\oint_T(Ʉ) = \oint_T(\{g\}) = \{g, f\}$.

So, we get that $Ʉ_r = \{g, f\}$.

Remarks (2-2-3) :

i) From example (2-2-2), we can conclude that $Ʉ_r$ is not $i-open$ set in general.

ii) If we replace the relation $\mathcal{H}\bar{\delta}\oint_T(Ʉ)$ instead of $\mathcal{H}\delta\oint_T(Ʉ)$ in the definition (2-2-1), then for any subset Ʉ ⊆ X we have the set $\{\hbar \in X, \forall H \in T(\hbar) s.t\, \mathcal{H}\bar{\delta}\oint_T(Ʉ)\} = \emptyset$.

In the following theorem, we studied some properties of redirect set and relationships between this concept and some concepts that mentioned previously

Theorem (2-2-4):

Let X_{TI}^{δ} be $i-TPS$, and let $\mathrm{U} \subseteq X$, then the following statement are hold:

1) $\mathrm{U} \subseteq \mathrm{U}_r$
2) $\emptyset_r = \emptyset,$ and $\mathrm{X}_r = \mathrm{X}$
3) $\oint_{\mathrm{T}}(\mathrm{U}) \subseteq \mathrm{U}_r$
4) If $\mathrm{U} \subseteq \mathcal{K}$ then $\mathrm{U}_r \subseteq \mathcal{K}_r$
5) $\mathrm{U}_r \subseteq (\mathrm{U}_r)_r$
6) $\mathrm{X} - \mathrm{U}_r \subseteq (\mathrm{X} - \mathrm{U})_r$
7) $(\mathrm{U} \cap \mathcal{K})_r \subseteq \mathrm{U}_r \cap \mathcal{K}_r$
8) $(\mathrm{U} \cup \mathcal{K})_r = \mathrm{U}_r \cup \mathcal{K}_r$
9) $(\bigcup_{\lambda\in\Lambda} \mathrm{U}_\lambda)_r \supseteq \bigcup_{\lambda\in\Lambda} \mathrm{U}_{\lambda_r}$

Proof:

1) Let $\hbar \in \mathrm{U}$ then for any $i-open$ set $\mathcal{H} \in \mathrm{T}(\hbar)$ we have $\mathcal{H}\delta\mathrm{U}$. But $\mathrm{U} \subseteq \oint_{\mathrm{T}}(\mathrm{U})$, so $\mathcal{H}\delta \oint_{\mathrm{T}}(\mathrm{U})$ by proposition (1-2-4) part (2), for any $\mathcal{H} \in \mathrm{T}(\hbar)$. Thus $\hbar \in \mathrm{U}_r$
2) For $\emptyset_r = \emptyset,$ if there exist any $\hbar \in \emptyset_r$ then for any set $\mathcal{H} \in \mathrm{T}(\hbar)$ we have $\mathcal{H}\delta \oint_{\mathrm{T}}(\emptyset) = \emptyset$, which a contradiction with proximity axiom in definition (1-2-1) part (3). So $\emptyset_r = \emptyset$.
   Now for $X_r = X$ it is clear that $X_r \subseteq X$, and by part (1) we get $X \subseteq \mathrm{X}_r$. Thus $X_r = X$.
3) Let $\hbar \in \oint_{\mathrm{T}}(\mathrm{U})$ then for any $\mathcal{H} \in \mathrm{T}(\hbar)$ we have $\mathcal{H}\delta\mathrm{U}$, but by proposition (1-3-33) part (1) we get $\mathrm{U} \subseteq \oint_{\mathrm{T}}(\mathrm{U})$ that is means $\mathcal{H}\delta \oint_{\mathrm{T}}(\mathrm{U})$ by proposition (1-2-4) part (2), which implies $\hbar \in \mathrm{U}_r$

4) Let $\hbar \in \mathrm{U}_r$ then for any set $\mathcal{H} \in \mathrm{T}(\hbar)$, $\mathcal{H}\delta \oint_{\mathrm{T}}(\mathrm{U})$, but $\mathrm{U} \subseteq \mathcal{K}$, so by proposition (1-3-33) part (3) we will get $\oint_{\mathrm{T}}(\mathrm{U}) \subseteq \oint_{\mathrm{T}}(\mathcal{K})$.

   Thus $\mathcal{H}\delta\oint(\mathcal{K})$ for any set $\mathcal{H} \in \mathrm{T}(\hbar)$ by proposition (1-2-4) part (2), that is mean $\hbar \in \mathcal{K}_r$

5) Since $\mathrm{U} \subseteq \mathrm{U}_r$, by (1), then $\mathrm{U}_r \subseteq (\mathrm{U}_r)_r$ by (4).

6) Let $\hbar \in \mathrm{X} - \mathrm{U}_r$ then $\hbar \notin \mathrm{U}_r$ which implies that $\hbar \notin \mathrm{U}$ (by 1), so $\hbar \in \mathrm{X} - \mathrm{U}$. Thus $\hbar \in (\mathrm{X} - \mathrm{U})_r$ by part (1).

7) Since $\mathrm{U} \cap \mathcal{K} \subseteq \mathrm{U}$ and $\mathrm{U} \cap \mathcal{K} \subseteq \mathcal{K}$, so $(\mathrm{U} \cap \mathcal{K})_r \subseteq \mathrm{U}_r$ and $(\mathrm{U} \cap \mathcal{K})_r \subseteq \mathcal{K}_r$ by (4). Thus $(\mathrm{U} \cap \mathcal{K})_r \subseteq \mathrm{U}_r \cap \mathcal{K}_r$.

8) $\hbar \in (\mathrm{U} \cup \mathcal{K})_r \Leftrightarrow \forall \mathcal{H} \in \mathrm{T}(\hbar), \mathcal{H}\delta \oint_{\mathrm{T}}(\mathrm{U} \cup \mathcal{K})$

   $\Leftrightarrow \mathcal{H}\delta(\oint_{\mathrm{T}}(\mathrm{U}) \cup \oint_{\mathrm{T}}(\mathcal{K}))$ by proposition (1-3-33) (5)

   $\Leftrightarrow \mathcal{H}\delta \oint_{\mathrm{T}}(\mathrm{U})$ or $\mathcal{H}\delta \oint_{\mathrm{T}}(\mathcal{K})$

   $\Leftrightarrow \hbar \in \mathrm{U}_r$ or $\hbar \in \mathcal{K}_r$

   $\Leftrightarrow \hbar \in \mathrm{U}_r \cup \mathcal{K}_r$

9) Directly by part (4).

Example (2-2-5) :

Let $X = \{\hbar, \mathscr{g}, \mathscr{f}\}$, $T = \{\mathrm{X}, \emptyset, \{\hbar\}, \{\mathscr{g}\}\}$, and $\mathrm{I} = \{\emptyset, \{\mathscr{f}\},\}$ with the discrete proximity δ_D, let $\mathrm{U} = \{\mathscr{g}\}$ and $\mathcal{K} = \{\hbar\}$ then $\mathrm{U}_r = \{\mathscr{g}, \mathscr{f}\}$, and $\mathcal{K}_{\mathrm{r}} = \{\hbar, \mathscr{f}\}$. So $\mathrm{U}_r \nsubseteq \mathrm{U}$, which means that the convers of part (1) in theorem (2-2-4) is not true in general.

Also, it is obvious that $\mathrm{U} \cap \mathcal{K} = \emptyset$, then by theorem (2-2-4) part (2) we will get $(\mathrm{U} \cap \mathcal{K})_r = \emptyset$. Thus $\mathrm{U}_{\mathrm{r}} \cap \mathcal{K}_{\mathrm{r}} = \{\mathscr{f}\} \nsubseteq (\mathrm{U} \cap \mathcal{K})_r = \emptyset$, and this shows that the convers of part (7) in theorem (2-2-4) is not always true. Now, if $B = \{\mathscr{f}\}$, so $X - B = \{\hbar, \mathscr{g}\}$, so $B_r = \{\mathscr{f}\}$ and $X - B_r = \{\hbar, \mathscr{g}\}$, but $(X - B)_r = (\{\hbar, \mathscr{g}\})_r = X$. Thus $(X - B)_r = X \nsubseteq X - B_r = \{\hbar, \mathscr{g}\}$ which explain way the convers of part (6) in theorem (2-2-4) cannot be true in general.

Example (2-2-6) :

Let $X = \{\hbar, g, f, \ell\}$, $T = \{X, \emptyset, \{\hbar\}, \{g, f\}\}$, and $I = \{\emptyset, \{f\}, \{\ell\}, \{f, \ell\}\}$ with the discrete proximity δ_D, let $U = \{\hbar\}$ then $\oint_T(U) = \{\hbar, f, \ell\}$, and $U_r = X$. So $U_r \nsubseteq \oint_T(U)$, which means that the convers of part (3) in theorem (2-2-4) is not true in general.

Remarks (2-2-7) :

i) The family of all redirect set in any $i - TPS$ be of the following form $R(x) = \{U_r, \forall U \subseteq X\}$.

ii) We can conclude from theorem (2-2-4) that the family $R(x)$ will be a subbase of the topology on X.

Our aim now is to introduce the concept of strips point that will play important role in our study.

Definition (2-2-8):

Let X_{TI}^{δ} be $i - TPS$. Then the point $\hbar \in X$ is called strip point of $U \subseteq X$ if and only if there exist an $i - open$ set $\mathcal{H} \in T(\hbar)$ s. t. $\mathcal{H} \bar{\delta} \oint_T(X - U)$.

The set of all strips point of U is called strips set and denoted by U_s

Example (2-2-9):

Let $X = \{\hbar, g, f\}$, $T = \{X, \emptyset, \{\hbar\}, \{g\}\}$, and $I = \{\emptyset, \{f\},\}$ with the discrete proximity δ_D, if $U = \{\hbar, f\}$ then $\oint_t(X - U) = \oint_T(\{g\}) = \{g, f\}$, so $U_s = \{\hbar\}$.

Also, if $\mathcal{K} = \{\hbar, g\}$ then $\oint_T(X - \mathcal{K}) = \oint_T(\{f\}) = \{f\}$, so $\mathcal{K}_s = \{\hbar, g\}$.

From example (2-2-8) we can conclude that U_s need not to be $i - open$ set in general.

In the following theorem we investigated some important properties and relationships for the concept of strips set .

Theorem (2-2-10):

Let X_{TI}^{δ} be $i-TPS$, and let $\text{Ʉ}, \mathcal{K} \subseteq X$ then the following statement are hold:

1) $\text{Ʉ}_s \subseteq \text{Ʉ}$
2) $\emptyset_s = \emptyset$, and $X_s = X$
3) $\text{Ʉ}_s \subseteq \oint_{\mathrm{T}}(\text{Ʉ})$
4) If $\text{Ʉ} \subseteq \mathcal{K}$ then $\text{Ʉ}_s \subseteq \mathcal{K}_s$
5) $(\text{Ʉ}_s)_s \subseteq \text{Ʉ}_s$
6) $(\mathrm{X} - \text{Ʉ})_s \subseteq \mathrm{X} - \text{Ʉ}_s$
7) $(\text{Ʉ} \cap \mathcal{K})_s = \text{Ʉ}_s \cap \mathcal{K}_s$
8) $\text{Ʉ}_s \cup \mathcal{K}_s \subseteq (\text{Ʉ} \cup \mathcal{K})_s$
9) $\text{Ʉ}_s \subseteq (\oint_{\mathrm{T}}(\text{Ʉ}))_s$

Proof:

1) Let $\hbar \in \text{Ʉ}_s$ then there exist a set $\mathcal{H} \in \mathrm{T}(\hbar)$ such that $\mathcal{H}\bar{\delta} \oint_{\mathrm{T}}(X - \text{Ʉ})$, So $\mathcal{H} \cap \oint_{\mathrm{T}}(X - \text{Ʉ}) = \emptyset$ That is mean $\hbar \notin \oint_{\mathrm{T}}(X - \text{Ʉ})$, but we have by proposition (1-3-33) part (1) $X - \text{Ʉ} \subseteq \oint_{\mathrm{T}}(X - \text{Ʉ})$, so $\hbar \notin X - \text{Ʉ}$. Thus $\hbar \in \text{Ʉ}$.
2) If there exist any point $\hbar \in \emptyset_s$, then there exist $i - open$ set $\mathcal{H} \in \mathrm{T}(\hbar)$, s. t. $\mathcal{H}\bar{\delta} \oint_{\mathrm{T}}(X - \emptyset)$ which means that $\mathcal{H}\bar{\delta} \oint_{\mathrm{T}}(X) = X$ and this a contradiction with the proximity axiom definition (1-2-1) part (3). Thus $\emptyset_s = \emptyset$.

Now, for X_s, it is clear that $X_s \subseteq X$ by (1), and to prove that $X \subseteq X_s$, let $\hbar \in X$, so $X \in T(\hbar)$ and since $X\bar{\delta}\emptyset$ then we get that $X\bar{\delta}\oint_T(\emptyset)$ so $X\bar{\delta}\oint_T(X - X)$. Thus $\hbar \in X_s$.

3) Let $\hbar \in \text{Ա}_s$ then there exist a set $\mathcal{H} \in T(\hbar)$ such that $\mathcal{H}\bar{\delta}\oint_T(X - \text{Ա})$, but by proposition (1-3-33) part (2) we have $X - \oint_T(\text{Ա}) \subseteq \oint_T(X - \text{Ա})$, so $\mathcal{H}\bar{\delta}(X - \oint_T(\text{Ա}))$, that is mean $\mathcal{H} \ll \oint_T(\text{Ա})$ but by part (3) of theorem (1-2-9) part (3) we will get that $\mathcal{H} \subseteq \oint_T(\text{Ա})$, hence $\hbar \in \oint_T(\text{Ա})$.
4) Let $\hbar \in \text{Ա}_s$ then there exist set $\mathcal{H} \in T(\hbar)$ such that $\mathcal{H}\bar{\delta}\oint_T(X - \text{Ա})$, but $\text{Ա} \subseteq \mathcal{K}$, so $X - \mathcal{K} \subseteq X - \text{Ա}$, and by proposition (1-3-33) part (3) we will get $\oint_T(X - \mathcal{K}) \subseteq \oint_T(X - \text{Ա})$. Thus by proposition (1-2-3) part (1) we get $\mathcal{H}\bar{\delta}\oint_T(X - \mathcal{K})$, that means $\hbar \in \mathcal{K}_s$.
5) Since $\text{Ա}_s \subseteq \text{Ա}$ by (1), then by (4) we will get $(\text{Ա}_s)_s \subseteq \text{Ա}_s$.
6) Since $\text{Ա}_s \subseteq \text{Ա}$ by (1), then $X - \text{Ա} \subseteq X - \text{Ա}_s$, Also, by (1) and (4) we have $(X - \text{Ա})_s \subseteq (X - \text{Ա}_s)_s \subseteq X - \text{Ա}_s$.
7) Since $\text{Ա} \cap \mathcal{K} \subseteq \text{Ա}$ and $\text{Ա} \cap \mathcal{K} \subseteq \mathcal{K}$, so by (4) we get $(\text{Ա} \cap \mathcal{K})_s \subseteq \text{Ա}_s$ and $(\text{Ա} \cap \mathcal{K})_s \subseteq \mathcal{K}_s$. So $(\text{Ա} \cap \mathcal{K})_s \subseteq \text{Ա}_s \cap \mathcal{K}_s$.
   Now, let $\hbar \in \text{Ա}_s \cap \mathcal{K}_s$, then there exist set $\mathcal{H} \in T(\hbar)$ such that $\mathcal{H}\bar{\delta}\oint_T(X - \text{Ա})$ and $\mathcal{H}\bar{\delta}\oint_T(X - \mathcal{K})$, so by proposition (1-2-3) part (6) $\mathcal{H}\bar{\delta}\oint_T(X - \text{Ա}) \cup \oint_T(X - \mathcal{K})$, so by proposition (1-3-3) part (5) $\oint_T(X - \text{Ա}) \cup \oint_T(X - \mathcal{K}) = \oint_T\big((X - \text{Ա}) \cup (X - \mathcal{K})\big)$, then we will get $\mathcal{H}\bar{\delta}\oint_T(X - (\text{Ա} \cap \mathcal{K}))$ which means that $\hbar \in (\text{Ա} \cap \mathcal{K})_s$, hence $\text{Ա}_s \cap \mathcal{K}_s \subseteq (\text{Ա} \cap \mathcal{K})_s$. Thus, $(\text{Ա} \cap \mathcal{K})_s = \text{Ա}_s \cap \mathcal{K}_s$
8) Since $\text{Ա} \subseteq \mathcal{K} \cup \text{Ա}$ and $\mathcal{K} \subseteq \text{Ա} \cup \mathcal{K}$, so by (4) we get $\text{Ա}_s \subseteq (\text{Ա} \cup \mathcal{K})_s$ and $\mathcal{K}_s \subseteq (\text{Ա} \cup \mathcal{K})_s$. Thus $\text{Ա}_s \cup \mathcal{K}_s \subseteq (\text{Ա} \cup \mathcal{K})_s$.
9) Since $\text{Ա} \subseteq \oint_T(\text{Ա})$, then by part (4) we have $\text{Ա}_s \subseteq (\oint_T(\text{Ա}))_s$.

The following example will be showing that the converse of part (9) of the theorem (2-2-10) is not true in general

Example (2-2-11):

In example (2-2-9), if $\text{Ư} = \{\hbar, f\}$ and $\text{Ư}_s = \{\hbar\}$, then $\text{Ư} \nsubseteq \text{Ư}_s$ in general Also, and if $\mathcal{K} = \{\hbar, g\}$ then $\oint_T(\mathcal{K}) = X$ and $\mathcal{K}_s = \{\hbar, g\}$, so it is clear that $(\oint_T(\mathcal{K}))_s \nsubseteq \mathcal{K}_s$.

In the next example, we will see that the converse of part (8) of the theorem (2-2-10) is not true in general.

Example (2-2-12):

In example (2-2-9), if $\text{Ư} = \{\hbar, f\}$ then $\text{Ư}_s = \{\hbar\}$ and if $\mathcal{K} = \{\hbar, g\}$ then $\mathcal{K}_s = \{\hbar, g\}$, so it is obvious that $(\text{Ư} \cup \mathcal{K})_s = X_s = X$, but in other hand we have $\text{Ư}_s \cup \mathcal{K}_s = \{\hbar\} \cup \{\hbar, g\} = \{\hbar, g\}$. Thus $(\text{Ư} \cup \mathcal{K})_s \nsubseteq \text{Ư}_s \cup \mathcal{K}_s$.

Also, if $\text{Ư} = \{\hbar, f\}$, then $X - \text{Ư} = \{g\}$, and $\text{Ư}_s = \{\hbar\}$, $(X - \text{Ư})_s = \{g\}$. Hence $X - \text{Ư}_s = \{g, f\} \nsubseteq (X - \text{Ư})_s = \{g\}$.

Remarks (2-2-13):

i) If we change the condition $\mathcal{H}\bar{\delta} \oint_T(X - \text{Ư})$ in definition (2-2-8), to the condition $\mathcal{H}\delta \oint_T(X - \text{Ư})$, then the family of all set in the form $\{\hbar \in X, \exists H \in T(\hbar) s.t\ \mathcal{H}\delta \oint_T(X - \text{Ư})\}$ will be construct of discrete topology on X.

ii) The family of all strips sets in any $i - TPS$ will be in the form $S(X) = \{\text{Ư}_s, \forall \text{Ư} \subseteq X\}$. And from theorem (2-2-10) especially part (2) and (7) we conclude that this family $S(X)$ will be a base of a topology on X.

In the next proposition, we will study the case that we have two $i-TPS$'s (ideals, proximity relations) such that one of them finer than the other.

Proposition (2-2-14) :

Let X_{TI}^{δ} be $i-TPS$, and let $\text{Ư} \subseteq X$ then the following statement are hold:

1) In $X_{T_jI}^{\delta}$, j=1,2 such that T_2 finer than T_1, then $(\text{Ư}_s)_{T_1} \subseteq (\text{Ư}_s)_{T_2}$, where $(\text{Ư}_s)_T$ means Ư_s w. r. t. $i-topological\ space\ T$.
2) In $X_{TI_j}^{\delta}$, j=1,2 such that I_2 finer than I_1, then $(\text{Ư}_s)_{I_2} \subseteq (\text{Ư}_s)_{I_1}$, where $(\text{Ư}_s)_I$ means Ư_s w. r. t. ideal I .
3) In $X_{TI}^{\delta_j}$, j=1,2 such that $\delta_2 < \delta_1$, then $(\text{Ư}_s)_{\delta_1} \subseteq (\text{Ư}_s)_{\delta_2}$, where $(\text{Ư}_s)_{\delta}$ means Ư_s w. r. t. proximity relation δ.

Proof:

1) Let $\hbar \in (\text{Ư}_s)_{T_1}$ then there exist set $\mathcal{H} \in T_1(\hbar)$ such that $\mathcal{H}\bar{\delta}\oint_{T_1}(X-\text{Ư})$, but since T_2 finer than T_1 we get $\mathcal{H} \in T_2(\hbar)$, which implies $\mathcal{H}\bar{\delta}\oint_{T_2}(X-\text{Ư})$, hence $\hbar \in (\text{Ư}_s)_{T_2}$.
2) Since $I_1 \subseteq I_2$, then by proposition (2-1-1-) part (1) we will get that $(\oint_T(X-\text{Ư}))_{I_1} \subseteq (\oint_T(X-\text{Ư}))_{I_2}$. Now, let $\hbar \in (\text{Ư}_s)_{I_2}$, then there exists a subset $\mathcal{H} \in T(\hbar)$ such that $\mathcal{H}\bar{\delta}(\oint_T(X-\text{Ư}))_{I_2}$, but we have $(\oint_T(X-\text{Ư}))_{I_1} \subseteq (\oint_T(X-\text{Ư}))_{I_2}$and by proposition (1-2-3) part (1) that implies $\mathcal{H}\bar{\delta}(\oint_T(X-\text{Ư}))_{I_1}$. Thus, $\hbar \in (\text{Ư}_s)_{I_1}$.
3) Let $\hbar \in (\text{Ư}_s)_{\delta_1}$, then there exists a subset $\mathcal{H} \in T(\hbar)$ such that $\mathcal{H}\overline{\delta_1}\oint_T(X-\text{Ư})$, but $\delta_2 < \delta_1$ so we will get by definition (1-2-5) that $\mathcal{H}\overline{\delta_2}\oint_T(X-\text{Ư})$. Thus, $\hbar \in (\text{Ư}_s)_{\delta_2}$.

Proposition (2-2-15) :

Let X_{TI}^{δ} be $i-TPS$, and let $\text{Ʉ} \subseteq X$ then $\text{Ʉ}_s \subseteq X - \oint_{\mathrm{T}}(X - \text{Ʉ})$.

Proof: Let $\hbar \in \text{Ʉ}_s$, then $\exists\, \mathcal{H} \in \mathrm{T}(\hbar)\ s.t. \mathcal{H}\bar{\delta} \oint_{\mathrm{T}}(X - \text{Ʉ})$, that means $\mathcal{H} \cap \oint_{\mathrm{T}}(X - \text{Ʉ}) = \emptyset$, so $\hbar \notin \oint_{\mathrm{T}}(X - \text{Ʉ})$. Therefore $\hbar \in \mathrm{X} - \oint(X - \text{Ʉ})$.

Corollary (2-2-16) :

Let X_{TI}^{δ} be $i-TPS$, and let $\text{Ʉ} \subseteq X$ then the following statements are true:

1) $\oint_{\mathrm{T}}(X - \text{Ʉ}) \subseteq X - \text{Ʉ}_s$
2) $\oint_{\mathrm{T}}(\text{Ʉ}) \subseteq X - (X - \text{Ʉ})_s$
3) $(X - \text{Ʉ})_s \subseteq X - \oint_{\mathrm{T}}(\text{Ʉ})$

Proof:

1) Directly from proposition (2-2-15)
2) It is obvious if we replace $X - \text{Ʉ}$ instead of Ʉ in part (1)
3) It is clear if we replace $X - \text{Ʉ}$ instead of Ʉ in proposition (2-2-15).

Theorem (2-2-17):

Let X_{TI}^{δ} be $i-TPS$, and let $\text{Ʉ} \subseteq X$ then the following statements are holds:

1) $\text{Ʉ}_s \subseteq \cup \{\mathrm{W} \in I_{\oint}(\hbar),\ \hbar \in \text{Ʉ}, \mathrm{W}\alpha\text{Ʉ}\}$
2) $\text{Ʉ}_s \subseteq \cup \{\mathrm{W} \in I_{\oint}(\hbar),\ \hbar \in \text{Ʉ}, \mathrm{W} \approx \text{Ʉ}\}$
3) $\text{Ʉ}_s \subseteq \cup \{\mathrm{H} \in \mathrm{T},\ H \ll \text{Ʉ}\}$.

Proof:

1) Let $\hbar \in \text{Ʉ}_s$ then by theorem (2-2-10) part (3) we get $\hbar \in \oint_{\mathrm{T}}(\text{Ʉ})$, but $\oint_{\mathrm{T}}(\text{Ʉ}) = \{\hbar \in X, \forall \mathrm{H} \in T(\hbar),\ \mathrm{H}\alpha\text{Ʉ}\}$, and by theorem (1-3-3) part (1) we have $\mathrm{H} \in I_{\oint}(\hbar)$ i.e $\hbar \in \{\hbar \in X, \text{for some } \mathrm{H} \in I_{\oint}(\hbar),\ \mathrm{H}\alpha\text{Ʉ}\}$. Thus $\hbar \in \cup \{\mathrm{W} \in I_{\oint}(\hbar),\ \hbar \in \text{Ʉ}, \mathrm{W}\alpha\text{Ʉ}\}$.

2) Let $\hbar \in \text{Ц}_s$, if $\hbar \notin \cup \{W \in I_{\oint}(\hbar),\ \hbar \in \text{Ц}, W \approx \text{Ц}\}$,
then $\hbar \notin \cup \{W \in I_{\oint}(\hbar), (W - \text{Ц}) \cup (\text{Ц} - W) \in I\}$, that will imply $\hbar \notin \cup \{W \in I_{\oint}(\hbar), W - \text{Ц} \in I\}$, which means $\hbar \notin \text{Ц}_s$ by (1) which a contradiction.
3) Let $\hbar \in \text{Ц}_s$, then there exist $H \in T(\hbar) s.t. \mathcal{H}\bar{\delta} \oint_T(X - \text{Ц})$, by theorem (1-3-33) part (1) we have $(X - \text{Ц}) \subseteq \oint_T(X - \text{Ц})$ which implies that $\mathcal{H}\bar{\delta}(X - \text{Ц})$ and that true if and only if $H \ll \text{Ц}$.
Therefore $\hbar \in \cup \{H \in T,\ H \ll \text{Ц}\}$.

The important question now, what is the relation between the occlusion set via $i - open$ set and members of ideal with the strips set, the answer will be determined by the following theorem.

Theorem (2-2-18):

Let X_{TI}^{δ} be $i - TPS$, and $\text{Ц} \subseteq X$ then the following statements are holds:

1) $(X - \text{Ц})_s = \emptyset$ for any $\text{Ц} \in I_{\oint}(\hbar)$, for some $\hbar \in X$.
2) If $\oint_T(\text{Ц}) = X\ \forall \text{Ц} \in I_{\oint}(\hbar)$, for some $\hbar \in X$, then $(X - \text{Ц})_s = \emptyset$
3) $\text{Ц}_s \subseteq Fcl(\text{Ц})$.

Proof:

1) If $\text{Ц} \in I_{\oint}(\hbar)$, then by proposition (1-3-3) part (2) and by part (1) of proposition (1-3-33) we get that $\oint_T(\text{Ц}) \in I_{\oint}(\hbar)$, but theorem (1-3-3) part (1) says that if $H \in T(\hbar)$ then $H \in I_{\oint}(\hbar)$, hence by part (3) of theorem (1-3-3) we will get $\mathcal{H} \cap \oint_T(\text{Ц}) \in I_{\oint}(\hbar)$.
Now, suppose that $(X - \text{Ц})_s \neq \emptyset$, so there exist $\hbar \in (X - \text{Ц})_s$ which means that there exist $H \in T(\hbar)\ s.t. \mathcal{H}\bar{\delta} \oint_T\big(X - (X - \text{Ц})\big)$, that implies $\mathcal{H} \cap \oint_T(\text{Ц}) = \emptyset \in I$. which a contradiction with proposition (1-3-3) part (4). Thus $(X - \text{Ц})_s = \emptyset$.

2) If $\oint_{T}(\text{U}) = X, \forall \text{U} \in I_{\oint}(\hbar)$, for some $\hbar \in$ X, then by part (3) of corollary (2-2-16) we have $(X - \text{U})_s \subseteq X - X = \emptyset$. Thus $(X - \text{U})_s = \emptyset$.
3) Let $\hbar \notin \text{Fcl}(\text{U})$, and by the definition (1-3-13) we have $\text{U} \subseteq \text{Fcl}(\text{U})$. Then $\hbar \notin \text{U}$, but $\text{U}_s \subseteq \text{U}$. Thus $\hbar \notin \text{U}_s$.

For studying the relationships between the redirect set and strips sets we can see the following proposition

Proposition (2-2-19):

Let X_{TI}^{δ} be $i - TPS$, and $\text{U} \subseteq X$ then the following statements are holds:

1) $\text{U}_s \subseteq \text{U}_r$
2) $\text{U}_s = X - (X - \text{U})_r$
3) $\text{U}_r = X - (X - \text{U})_s$

Proof:

1) It is direct result from theorem (2-2-4) part (1) with part (1) in theorem (2-2-10).
2) Let $\hbar \in X - (X - \text{U})_r \Leftrightarrow \hbar \notin (X - \text{U})_r$

$$\Leftrightarrow \exists\, \mathcal{H} \in \text{T}(\hbar)\ s.t. \mathcal{H} \bar{\delta} \oint_{T}(X - \text{U})$$

$$\Leftrightarrow \hbar \in \text{U}_s .$$

3) If we put $X - \text{U}$ instead of U in part (2) we will get $(X - \text{U})_s = X - \text{U}_r$, so that will imply $\text{U}_r = X - (X - \text{U})_s$.

For studying the relationships between the focal function and the redirect set we can see the following theorem.

Theorem (2-2-20):

Let X_{TI}^{δ} be $i-TPS$, and $\text{Ư} \subseteq X$ then the following statements are holds:

1) $\text{Ư}_r \subseteq (\oint_T(\text{Ư}))_r$
2) $X - \oint(X - \text{Ư}) \subseteq \text{Ư}_r$
3) $X - \text{Ư}_r \subseteq \oint_T(X - \text{Ư}) \subseteq (X - \text{Ư})_r$

Proof:

1) Since by proposition (1-3-33) part (1) we have $\text{Ư} \subseteq \oint_T(\text{Ư})$, then by proposition (2-2-4) part (4) we get $\text{Ư}_r \subseteq (\oint_T(\text{Ư}))_r$.
2) Let $\hbar \in X - \oint_T(X - \text{Ư})$, suppose if possible that $\hbar \notin \text{Ư}_r$, then by theorem (2-2-4) part (3) we get $\hbar \notin \oint_T(\text{Ư})$, so $\hbar \in X - \oint_T(\text{Ư})$, but by theorem (1-3-33) part (2) $X - \oint_T(\text{Ư}) \subseteq \oint_T(X - \text{Ư})$, therefore $\hbar \in \oint_T(X - \text{Ư})$ which a contradiction. Thus $\hbar \in \text{Ư}_r$.
3) By (2) we have $X - \oint_T(X - \text{Ư}) \subseteq \text{Ư}_r$, so $X - \text{Ư}_r \subseteq \oint_T(X - \text{Ư})$, but $\oint_T(\text{Ư}) \subseteq \text{Ư}_r$. Thus $\oint_T(X - \text{Ư}) \subseteq (X - \text{Ư})_r$.

To know the relationships between the focal set and the redirect set we can see the following theorem.

Proposition (2-2-21):

Let X_{TI}^{δ} be $i-TPS$, and $\text{Ư} \subseteq X$. If $\text{Ư} \in I_{\oint}(\hbar)$ for some $\hbar \in X$, then the following statements are holds:

1) $\text{Ư}_r = X$
2) $\text{Ư}_r \in I_{\oint}(\hbar)$
3) $(\text{Ư}_r)_r = \text{Ư}_r$
4) $X - \text{Ư} \subseteq \text{Ư}_r$.

Proof:

1) Let $\text{Ц} \in I_{\oint}(\hbar)$ for some $\hbar \in X$, then by theorem (2-2-18) part (1) we get $(X - \text{Ц})_s = \emptyset$, but by part (3) of theorem (2-2-19) we have that $\text{Ц}_r = X - (X - \text{Ц})_s$ which means $\text{Ц}_r = X$.
2) Let $\text{Ц} \in I_{\oint}(\hbar)$ then $\text{Ц}_r = X$ by (1). But $X \in I_{\oint}(\hbar)$ for any $\hbar \in X$, by theorem (1-3-3) part (1). Thus $\text{Ц}_r \in I_{\oint}(\hbar)$
3) Since $\text{Ц} \in I_{\oint}(\hbar)$. then by part (1) $\text{Ц}_r = X$ but by theorem (2-2-4) part (2) we have $X_r = X$. Thus $(\text{Ц}_r)_r = (X)_r = X = \text{Ц}_r$.
4) Directly by (1).

Proposition (2-2-22):

Let X_{TI}^{δ} be an $i - TPS$, and $\text{Ц}, \mathcal{K} \subseteq X$ such that $\mathcal{K} \in I$ and $\text{Ц} \in I_{\oint}(\hbar)$ then $\text{Ц}_r = (\text{Ц} - \mathcal{K})_r = (\text{Ц} \cup \mathcal{K})_r$.

Proof:

For $\text{Ц}_r = (\text{Ц} - \mathcal{K})_r$, it is obvious that $\text{Ц} - \mathcal{K} \subseteq \text{Ц}$ so by theorem (2-2-4) part (4) we get that $(\text{Ц} - \mathcal{K})_r \subseteq \text{Ц}_r$. Now, let $\hbar \in \text{Ц}_r$, if $\hbar \notin (\text{Ц} - \mathcal{K})_r$, then there exist $\mathcal{H} \in T(\hbar)$ such that $\mathcal{H} \bar{\delta} \oint_T (\text{Ц} \cap (X - \mathcal{K}))$, and by proposition (1-2-3) part (3) that implies $\mathcal{H} \cap \left(\text{Ц} \cap (X - \mathcal{K})\right) = \emptyset$, but since $\mathcal{K} \in I$ then by proposition (1-3-3) part (6) we get $(X - \mathcal{K}) \in I_{\oint}(\hbar)$ and since $\text{Ц} \in I_{\oint}(\hbar)$, then by theorem (1-3-3) part (3) we have $\left(\text{Ц} \cap (X - \mathcal{K})\right) \in I_{\oint}(\hbar)$, but by part (1) of the theorem (1-3-3) we will get that $\mathcal{H} \in I_{\oint}(\hbar)$ so again by theorem (1-3-3) part (3) we will have that $\mathcal{H} \cap \left(\text{Ц} \cap (X - \mathcal{K})\right) \in I_{\oint}(\hbar)$ and this a contradiction. Thus $\text{Ц}_r \subseteq (\text{Ц} - \mathcal{K})_r$.

Now to prove that $\text{Ц}_r = (\text{Ц} \cup \mathcal{K})_r$, it is obvious that $\text{Ц} \subseteq \text{Ц} \cup \mathcal{K}$ so by theorem (2-2-4) part (4) we get $\text{Ц}_r \subseteq (\text{Ц} \cup \mathcal{K})_r$. Now, let $\hbar \in (\text{Ц} \cup \mathcal{K})_r$, if $\hbar \notin \text{Ц}_r$ then there exist $\mathcal{H} \in T(\hbar)$ such that $\mathcal{H} \bar{\delta} \oint_T(\text{Ц})$, i.e. $\mathcal{H} \cap$

$\oint_T(Ų) = \emptyset$, which means that $\mathcal{H} \cap Ų = \emptyset$, but by theorem (1-3-3) parts (1,3) we have $\mathcal{H} \cap Ų \in I_{\oint}(\hbar)$ and this a contradiction.

Hence $(Ų \cup \mathcal{K})_r \subseteq Ų_r$, so $\hbar \in Ų_r$. Thus $Ų_r = (Ų \cup \mathcal{K})_r$.

Now we try to study in the following theorem the relationships between the redirect set, $i - close$ set and focal closure sets

Theorem (2-2-23):

Let X_{TI}^{δ} be $i - TPS$, and $Ų \subseteq X$, then the following statements are holds:

1) $i - cl(Ų) \subseteq Ų_r$
2) If $W \cap Ų \notin$ I, $\forall$ $W \in T(\hbar)$ for some $\hbar \in$ X, then $i - cl(Ų_r) = Ų_r$
3) $\mathrm{Fcl}(Ų) \subseteq Ų_r$

Proof:

1) If $\hbar \in i - cl(Ų)$, so by proposition (1-3-33) part (9) we get $\hbar \in \oint_T(Ų)$ which means for any $\mathcal{H} \in T(\hbar)$ $\mathcal{H} \cap \oint_T(Ų) \neq \emptyset$ and this implies $\mathcal{H}\delta \oint_T(Ų)$. Hence $\hbar \in Ų_r$.
2) Let $\hbar \in i - cl(Ų_r)$, so for any $\mathcal{H} \in T(\hbar)$ $\mathcal{H} \cap Ų_r \neq \emptyset$, suppose, if possible, that $\hbar \notin Ų_r$, then $\exists$ $W \in T(\hbar)$ s. t. $W\bar{\delta} \oint_T(Ų)$ which means that $W \cap \oint_T(Ų) = \emptyset \in$ I, which a contradiction with a assumption that $W \cap Ų \notin$ I . Therefore $\hbar \in Ų_r$, which means that $i - cl(Ų_r) \subseteq Ų_r$. Also, we have by proposition (1-1-23) part (4) $Ų_r \subseteq i - cl(Ų_r)$. Thus $i - cl(Ų_r) = Ų_r$
3) Directly by (1) and by Corollary (1-3-16)

In proposition (2-2-24) we will study the case that if we have two i-topological space (ideals, proximity) on the same set X which one of them finer than the other.

Proposition (2-2-24) :

Let X_{TI}^{δ} be $i-TPS$, and let Ų ⊆ X then the following statement are hold:

1) In $X_{T_jI}^{\delta}$, j=1,2 such that T_2 finer than T_1, then $(Ų_r)_{T_2} \subseteq (Ų_r)_{T_1}$, where $(Ų_r)_T$ means $Ų_r$ w. r. t. $i-topological\ space\ T$.
2) In $X_{TI_j}^{\delta}$, j=1,2 such that I_2 finer than I_1, then $(Ų_r)_{I_1} \subseteq (Ų_r)_{I_2}$, where $(Ų_r)_I$ means $Ų_r$ w. r. t. ideal I .
3) In $X_{TI}^{\delta_j}$, j=1,2 such that $\delta_2 < \delta_1$, then $(Ų_r)_{\delta_1} \subseteq (Ų_r)_{\delta_2}$, where $(Ų_r)_\delta$ means $Ų_r$ w. r. t. proximity relation δ.
4) In $X_{T_jI_j}^{\delta}$, j=1,2 such that T_2 finer than T_1 and I_2 finer than I_1, then
   i. $(Ų_r)_{T_1} \cap (Ų_r)_{T_2} \subseteq (Ų_r)_{T_1 \cap T_2}$,
   ii. $(Ų_r)_{I_1 \cap I_2} \subseteq (Ų_r)_{I_1} \cap (Ų_r)_{I_2}$.

Proof:

1) Let $\hbar \in (Ų_r)_{T_2}$ then $\forall \mathcal{H} \in T_2(\hbar)$ such that $\mathcal{H}\delta \oint_{T_2}(Ų)$, but since $T_1 \subseteq T_2$ so by theorem (1-3-33) part (15) we get $\oint_{T_2}(Ų) \subseteq \oint_{T_1}(Ų)$, which means that $\mathcal{H}\delta \oint_{T_1}(Ų)$, hence $\hbar \in (Ų_r)_{T_1}$.
2) Since $I_1 \subseteq I_2$, then by proposition (2-1-1-) part (1) we will get that $(\oint_T(Ų))_{I_1} \subseteq (\oint_T(Ų))_{I_2}$. Now, let $\hbar \in (Ų_r)_{I_1}$, then $\forall \mathcal{H} \in T(\hbar)$ $\mathcal{H}\delta(\oint_T(Ų))_{I_1}$, but we have $(\oint_T(Ų))_{I_1} \subseteq (\oint_T(Ų))_{I_2}$which implies by proposition (1-2-4) part (2) $\mathcal{H}\delta(\oint_T(Ų))_{I_2}$. Thus, $\hbar \in (Ų_r)_{I_2}$.
3) Let $\hbar \in (Ų_r)_{\delta_1}$, then $\forall \mathcal{H} \in T(\hbar)$ such that $\mathcal{H}\delta_1 \oint_T(Ų)$, and since $\delta_2 < \delta_1$, then by definition (1-2-5) we will get that $\mathcal{H}\delta_2 \oint_T(Ų)$. Thus, $\hbar \in (Ų_r)_{\delta_2}$.
4)
   i) Let $\hbar \in (Ų_r)_{T_1} \cap (Ų_r)_{T_2}$, then $\forall \mathcal{H} \in T_1(\hbar)$ and $\forall U \in T_2(\hbar)$ we have $\mathcal{H}\delta \oint_{T_1}(Ų)$ and $U\delta \oint_{T_2}(Ų)$, but since $T_1 \cap T_2 \subseteq T_1$,

then by theorem (1-3-33) part (15) we will get

$\oint_{T_1}(\text{Ų}) \subseteq \oint_{T_1 \cap T_2}(\text{Ų})$ and $\oint_{T_2}(\text{Ų}) \subseteq \oint_{T_1 \cap T_2}(\text{Ų})$, which means that $\mathcal{H}\delta \oint_{T_1 \cap T_2}(\text{Ų})$ and $\text{U}\delta \oint_{T_1 \cap T_2}(\text{Ų})$.

Hence $\hbar \in (\text{Ų}_r)_{T_1 \cap T_2}$.

ii) Since $I_1 \cap I_2 \subseteq I_1, I_2$, then by proposition (2-1-1-) part (1) we get $(\oint_T(\text{Ų}))_{I_1 \cap I_2} \subseteq (\oint_T(\text{Ų}))_{I_2}$ and $(\oint_T(\text{Ų}))_{I_1 \cap I_2} \subseteq (\oint_T(\text{Ų}))_{I_1}$. Now, let $\hbar \in (\text{Ų}_r)_{I_1 \cap I_2}$, then $\forall\, \mathcal{H} \in T(\hbar)$ $\mathcal{H}\delta(\oint_T(\text{Ų}))_{I_1 \cap I_2}$, then $\mathcal{H}\delta(\oint_T(\text{Ų}))_{I_1}$ and $\mathcal{H}\delta(\oint_T(\text{Ų}))_{I_2}$. Thus, $\hbar \in (\text{Ų}_r)_{I_2}$ and $\hbar \in (\text{Ų}_r)_{I_1}$ and this complete the proof.

The following proposition studied that effect of $\delta - symmetry$ function on the redirect set

Proposition (2-2-25) :

Let $f: X^{\delta}_{TI} \to Y^{\grave{\delta}}_{\grave{T}\grave{I}}$ be a $\delta - symmetry$ function, for any subset Ų of X, then $f(\text{Ų}_r) = \big(f(\text{Ų})\big)_r$

Proof: Let $\hbar \in f(\text{Ų}_r)$, then $\exists\, x \in X$ s. $f(x) = \hbar$ and $x \in \text{Ų}_r$. Since f is $\delta -$continuous then $\forall\, H \in T(x), H\delta \oint_T(\text{Ų})$, so we get $f(H)\grave{\delta} f(\oint_T(\text{Ų}))$ $\forall\, f(H) \in \grave{T}\big(f(x)\big) = \grave{T}(\hbar)$, which means $f(H)\grave{\delta} \oint_{\grave{T}}(f(\text{Ų}))\,, \forall\, f(H) \in \grave{T}(\hbar)$, by using proposition (2-1-3). Thus $\hbar \in \big(f(\text{Ų})\big)_r$

Conversely, let $\hbar \in \big(f(\text{Ų})\big)_r$, then $\forall\, H \in \grave{T}(\hbar), H\grave{\delta} \oint_T\big(f(\text{Ų})\big)$, therefor by using proposition (2-1-3) we have $,H\grave{\delta} f(\oint_T(\text{Ų}))$ but f^{-1} is $\delta -$continuous and f is $\delta -$continuous, then $f^{-1}(H)\delta \oint_T(\text{Ų})\ \forall\, f^{-1}(H) \in T\big(f^{-1}(\hbar)\big)\,.Thus\ \ f^{-1}(\hbar) \in \text{Ų}_r$.

Thus $\hbar \in f(\text{Ų}_r)$, and this completes the proof.

The following proposition show that effect of $\delta - symmetry$ invese function on the redirect set.

Proposition (2-2-26):

Let $f: X_{TI}^{\delta} \to Y_{\hat{T}\hat{I}}^{\hat{\delta}}$ be $\delta - symmetry$ function, and for any subset Ա of Y, then $f^{-1}(\text{Ա}_{\text{r}}) = \left(f^{-1}(\text{Ա})\right)_r$.

Proof: Let $k \in f^{-1}(\text{Ա}_{\text{r}})$, for each subset Ա of Y , then $f(k) \in \text{Ա}_{\text{r}}$, so $\forall\ H \in \hat{T}\left(f(k)\right), s.t.\ H\hat{\delta}\ ∯_{\hat{T}}(\text{Ա})$. Since f^{-1} is $\delta -$ continuous and by proposition (2-1-4) we will get $f^{-1}(H)\delta f^{-1}(∯_{\hat{T}}(\text{Ա})) = ∯_{\text{T}}\left(f^{-1}(\text{Ա})\right)$ for any $f^{-1}(H) \in T(k)$ which means that $k \in \left(f^{-1}(\text{Ա})\right)_r$

Conversely, let $k \in \left(f^{-1}(\text{Ա})\right)_r$, then $\forall\ H \in T(k), H\delta\ ∯_{\text{T}}\left(f^{-1}(\text{Ա})\right)$, and by proposition (2-1-4) we have $H\delta f^{-1}(∯_{\hat{T}}(\text{Ա}))$, but f^{-1} is $\delta -$continuous since f is $\delta -$ continuous and $i -$ open, so $f(H)\hat{\delta}\ ∯_{\hat{\text{T}}}(\text{Ա})\quad \forall\ f(H) \in \hat{T}\left(f(k)\right)\ so\ \ f(k) \in \text{Ա}_{\text{r}}$, Thus $k \in f^{-1}(\text{Ա}_{\text{r}})$. And this completes the proof.

Our aim now is to introduce the notation of redirect set and strips set w. r. t. $i - subspace$ in $i - TPS$ with some relations and properties.

Definition (2-2-27):

Let $Y_{T_Y I_Y}^{\delta_Y}$ be $i - subspace$ of $i - TPS\ X_{TI}^{\delta}$, then for any $k \in Y$, k is called redirect point with respect to $i - subspace\ Y_{T_Y I_Y}^{\delta_Y}$ for subset Ա of Y, if and only if for any $H_Y \in T_Y(k)$, then $H_Y \delta_y\ ∯_{\text{T}_{\text{y}}}(\text{Ա})$.

The set of all redirect points of Ա w. r. t. $i - subspace\ Y_{T_Y I_Y}^{\delta_Y}$ is called redirect set w. r. t. $i - subspace$ and denoted by $\text{Ա}_{\text{r}_{\text{Y}}}$.

Example (2-2-28):

Let $X = \{k, g, f\}$, $T = \{\emptyset, X, \{k\}, \{g\}\}$, $I = \{\emptyset, \{f\}\}$ with δ_D . And let $Y = \{k, g\}$ so $T_Y = \{\emptyset, Y, \{k\}, \{g\}\}$, $I_Y = \{\emptyset\}$ with δ_{Y_D} be $i\text{–}subspace$ Y of X_{TI}^{δ}. Now let $\mathcal{U} \subseteq Y$ s.t. $\mathcal{U} = \{g\}$, then $\mathcal{U}_{r_Y} = \{g\}$.

In the following proposition we will see that the redirect set with respect to $i - subspace$ of $i - TPS$, is a subset of the intersection of the redirect set with respect to $i - TPS$ with the $i - subspace\ Y$.

Proposition (2-2-29)

Let $Y_{T_Y I_Y}^{\delta_Y}$ be $i - subspace$ of $i - TPS\ X_{TI}^{\delta}$, and $\mathcal{U} \subseteq Y$. Then

$$\mathcal{U}_{r_Y} \subseteq \mathcal{U}_{r_X} \cap Y.$$

Proof: Let $y \in \mathcal{U}_{r_Y}$, then $\forall\ U_Y \in T_Y(y), U_Y\ \delta_Y\ \not\oint_{T_Y}(\mathcal{U})$, then we get for any $U_Y \in T_Y(y)$, there exist $V \in T_X(y)\ s.t. U_Y = V \cap Y$ for some $V \in T$, i.e. $(V \cap Y)\ \delta_Y\ \not\oint_{T_X}(\mathcal{U}) \cap Y$ by proposition (2-1-20). Thus $(V \cap Y)\delta_X\ \not\oint_{T_X}(\mathcal{U})$, which means that $y \in \mathcal{U}_r$, and since $y \in Y$ we get that $y \in \mathcal{U}_r \cap Y$.

In the following example we will see that the equality of above proposition cannot be satisfies in general.

Example (2-2-30):

Let $X = \{k, g, f, \ell\}$, $T = \{\emptyset, X, \{k\}, \{g, f\}\}$, $I = \{\emptyset, \{f\}, \{\ell\}, \{f, \ell\}\}$ with δ_D . And let $Y = \{k, g\}$ so $T_Y = \{\emptyset, Y, \{k\}, \{g\}\}$, $I_Y = \{\emptyset\}$ with δ_{Y_D} be $i\text{–}subspace$ Y of X_{TI}^{δ}. Now let $\mathcal{U} \subseteq Y$ s.t. $\mathcal{U} = \{k\}$, then $\mathcal{U}_{r_Y} = \{k\}$, but $\mathcal{U}_{r_X} = X$. Thus $\mathcal{U}_{r_X} \cap Y = Y \nsubseteq \mathcal{U}_{r_Y} = \{k\}$.

The condition that makes the equality of proposition (2-2-29) satisfies, that $i-TPS$ has to have intuition smoothing feature as in the next corollary.

Corollary (2-2-31):

Let $Y_{T_Y I_Y}^{\delta_Y}$ be $i-subspace$ of $i-TPS$ X_{TI}^{δ}, and let $Ա \subseteq Y$,

then $Ա_{r_Y} = Ա_{r_X} \cap Y$ if and only if X_{TI}^{δ} have intuition smoothing feature.

Proof: we need to proof that $Ա_{r_X} \cap Y \subseteq Ա_{r_Y}$, so let $y \in Ա_{r_X} \cap Y$, then $\forall\, U \in T_X(y), U\delta_X \oint_{T_X}(Ա)$. Since X is smooth so by definition (2-1-21) we get $(U \cap Y)\delta_X(\oint_{T_X}(Ա) \cap Y)$ but $y \in Y$, so $y \in U \cap Y = V_Y$, and $Ա \subseteq Y$ then $Ա \cap Y = Ա$ we will get $V_Y\delta_y(\oint_{T_X}(Ա) \cap Y)$. Also, since X is smooth and $Ա \subseteq (\oint_{T_X}(Ա) \cap Y)$. So, we have $V_Y\delta_y \oint_{T_Y}(Ա)$. Thus $y \in Ա_{r_Y}$.

Next, we will define the concept of strips point with respect to $i-subspace$ of $i-TPS$

Definition (2-2-32):

Let $Y_{T_Y I_Y}^{\delta_Y}$ be $i-subspace$ of $i-TPS$ X_{TI}^{δ}, and let $Ա \subseteq Y$, then the point $𝒽 \in \mathrm{Y}$ is called strip point of $Ա \subseteq \mathrm{Y}$ w. r. t. $i-subspace$ $Y_{T_Y I_Y}^{\delta_Y}$ if there exist an $i-open$ set $\mathcal{H}_\mathrm{Y} \in \mathrm{T_Y}(𝒽)$ such that $\mathcal{H}_\mathrm{Y}\overline{\delta_\mathrm{Y}} \oint_{\mathrm{T_Y}}(Y - Ա)$.

The set of all strips point of a subset $Ա$ is called strips set w. r. t.

$i-subspace$ $Y_{T_Y I_Y}^{\delta_Y}$, and denoted by $Ա_{S_Y}$

Example (2-2-33):

Let $\mathrm{X} = \{𝒽, ℊ, 𝒻\}$, $\mathrm{T} = \{\mathrm{X}, \emptyset, \{𝒽\}, \{ℊ\}\}$, and $\mathrm{I} = \{\emptyset, \{𝒻\}\}$ with the discrete proximity, if $\mathrm{Y} = \{𝒽, 𝒻\}$ then $\mathrm{T_Y} = \{\mathrm{Y}, \emptyset, \{𝒽\}\}$ and $\mathrm{I_Y} = \{\emptyset, \{𝒻\}\}$.

Now let $Ա = \{𝒽\}$, then it is clear that $\oint_{\mathrm{T_Y}}(Y - Ա) = \{𝒻\}$ so $Ա_{S_Y} = \{𝒽\}$.

In the following proposition, we will see that the intersection of the redirect set with respect to $i-TPS$ with the $i-subspace\ Y$, is a subset of. the strips set with respect to $i-subspace$ of $i-TPS$

Proposition (2-2-34)

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i-TPS\ X^{\delta}_{TI}$, then for any subset Ʉ of Y. $Ʉ_{s_X} \cap Y \subseteq Ʉ_{s_Y}$.

Proof: Let $y \in Ʉ_{s_X} \cap Y$, then $y \in Ʉ_{s_X}$ and $y \in Y$, so $\exists\, U \in T_X(y)$ s. t. $U\overline{\delta_X} \oint_{T_X}(X - Ʉ)$, but $y \in Y$ then $y \in U \cap Y = U_Y \in T_Y(y)$.

Also, we have $(Y - Ʉ) \subseteq (X - Ʉ)$ so by proposition (1-3-33) part (4) we have $\oint_{T_X}(Y - Ʉ) \subseteq \oint_{T_X}(X - Ʉ)$, and by using proposition (2-1-20) we have $\oint_{T_Y}(Y - Ʉ) \subseteq \oint_{T_X}(Y - Ʉ) \cap Y$. Thus, $U\overline{\delta_X} \oint_{T_X}(Y - Ʉ)$, and since $U_Y \subseteq U$ therefore by using proposition (1-2-3) part (2) we will get $U_Y\overline{\delta_X} \oint_{T_Y}(Y - Ʉ)$. Hens, $U_Y\overline{\delta_Y} \oint_{T_Y}(Y - Ʉ)$ which means that $y \in Ʉ_{s_Y}$.

Proposition (2-2-35)

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i-TPS\ X^{\delta}_{TI}$, then for any subset Ʉ of Y. $Ʉ_{s_Y} = \left[X - (Y - Ʉ)_{r_Y}\right] \cap Y$.

Proof: Let Ʉ be a subset of Y, then

$$Ʉ_{s_Y} = Y - (Y - Ʉ)_{r_Y}$$
$$= \left[X - \left((Y - Ʉ)_{r_X} \cap Y\right)\right] \cap Y$$
$$= \left[\left(X - (Y - Ʉ)_{r_X}\right) \cup (X - Y)\right] \cap Y$$
$$= \left[\left(X - (Y - Ʉ)_{r_X}\right)\right] \cap Y$$

2.3 Redirect Dense Set

Due to the fact that dense sets play an important role in mathematics in general, this section introduces the notion of redirect dense set and studies some of its important properties and relations, as well as its definition w. r. t. $i - subspace$.

Definition (2-3-1):

In $i - TPS\ X_{TI}^{\delta}$, a subset Ų ⊆ X is called redirect dense set if and only if for any $\hbar \in X, and\ for\ each\ H \in T(\hbar)$, then $H\delta \oint_{T}$(Ų), and denoted by $r - dense$ set.

Example (2-3-2):

Let $X = \{\hbar, g, f\}$, $T = \{X, \emptyset, \{\hbar\},\}$, and $I = \{\emptyset, \{g\},\}$ with the discrete proximity, if Ų $= \{\hbar, g\}$ then Ų$_r = X$ so Ų is $r - dense$ set.

For some relationships on this notion, we can see the following theorem which is easy to proof.

Proposition (2-3-3):

In $i - TPS\ X_{TI}^{\delta}$, and for Ų, $\mathcal{K} \subseteq X$ then the following statement are hold.

1) If Ų ⊆ $\mathcal{K}$ and Ų is $r - dense$ set, then $\mathcal{K}$ is $r - dense$ set
2) If Ų ∩ $\mathcal{K}$ is $r - dense$ set, then Ų and $\mathcal{K}$ are $r - dense$ set
3) Ų or $\mathcal{K}$ is $r - dense$ set if and only if Ų ∪ $\mathcal{K}$ is $r - dense$ set

Proof:

1) If Ų is $r - dense$ set, then for any $\hbar \in X, and\ \forall H \in T(\hbar)$, then $H\delta \oint_{T}$(Ų) , but Ų ⊆ $\mathcal{K}$ and by proposition (1-3-33) part (4) we have $\oint_{T}$(Ų) ⊆ $\oint_{T}(\mathcal{K})$, therefore by proposition (1-2-4) part (2) for any $\hbar \in X, and\ \forall H \in T(\hbar)$, then $H\delta \oint_{T}(\mathcal{K})$. Thus $\mathcal{K}$ is $r - dense$ set.

2) Since $\text{Ų} \cap \mathcal{K}$ is $r-dense$ set, and $\text{Ų} \cap \mathcal{K} \subseteq \text{Ų}, \text{Ų} \cap \mathcal{K} \subseteq \mathcal{K}$ then by (1) we get that Ų and $\mathcal{K}$ are $r-dense$ set.
3) Since $\text{Ų} \subseteq \text{Ų} \cup \mathcal{K}$ and $\mathcal{K} \subseteq \text{Ų} \cup \mathcal{K}$, so by (1) we will get if Ų or $\mathcal{K}$ is $r-dense$ set then $\text{Ų} \cup \mathcal{K}$ is $r-dense$ set.

   Conversely, let $\text{Ų} \cup \mathcal{K}$ be $r-dense$, then for any $\hbar \in X$ and for any $H \in T(\hbar)$, then $H\delta \oint_{\mathrm{T}}(\text{Ų} \cup \mathcal{K})$, but by proposition ((1-3-33) par (5) we will have $\oint_{\mathrm{T}}(\text{Ų} \cup \mathcal{K}) = \oint_{\mathrm{T}}(\text{Ų}) \cup \oint_{\mathrm{T}}(\mathcal{K})$, therefor we will get that $H\delta(\oint_{\mathrm{T}}(\text{Ų}) \cup \oint_{\mathrm{T}}(\mathcal{K}))$ which means $H\delta \oint_{\mathrm{T}}(\text{Ų})$ or $H\delta \oint_{\mathrm{T}}(\mathcal{K})$ for any $\hbar \in X$ and for any $H \in T(\hbar)$. Thus Ų or $\mathcal{K}$ is $r-dense$, and this complete the proof.

Next proposition will discuss the relationships between the $r-dense$ and other dense set that we mentioned in previous sections

Proposition (2-3-4):

In $i-TPS\ X_{TI}^{\delta}$, and for $\text{Ų} \subseteq X$ then the following statement are hold.

1) Every $i-dense$ set is $r-dense$ set.
2) Every $FD-dense$ set is $r-dense$ set.
3) Every $FO-dense$ set is $r-dense$ set.
4) Every $\oint_{\mathrm{T}}-dense$ set is $r-dense$ set.

Proof:

1) Let Ų is $i-dense$ set, then $i-cl(\text{Ų}) = X$, but $i-cl(\text{Ų}) \subseteq \text{Ų}_{\mathrm{r}}$ by theorem (2-2-23) part (1), so $\text{Ų}_{\mathrm{r}} = X$, hence Ų is $r-dense$.
2) Let Ų is $FD-dense$ set, then $Fd(\text{Ų}) = X$, but by proposition (1-3-33) part (11) and by theorem (2-2-23) part (3) we have $Fd(\text{Ų}) \subseteq \text{Ų}_{\mathrm{r}}$ so $\text{Ų}_{\mathrm{r}} = X$, hence Ų is $r-dense$.

3) Let Ꝋ is $FO-dense$ set, then $Fcl(Ꝋ)=X$, but $Fcl(Ꝋ)\subseteq Ꝋ_r$ by theorem (2-2-23) part (3), so $Ꝋ_r=X$, hence Ꝋ is $r-dense$.
4) Let Ꝋ is $\oint_T-dense$ set, then $\oint_T(Ꝋ)=X$, but by theorem (2-2-4) part (3) we have $\oint_T(Ꝋ)\subseteq Ꝋ_r$, so $Ꝋ_r=X$, hence Ꝋ is $r-dense$.

The important questions now is about the relationships between the r-dense set and the strips set, and what the effect of focal set on r-density.

Proposition (2-3-5):

In $i-TPS\ X_{TI}^{\delta}$, and for $Ꝋ\subseteq X$ then the following statement are hold.

1) If Ꝋ is $r-dense$, then $(X-Ꝋ)_s=\emptyset$
2) If $Ꝋ\in I_{\oint}(\hbar)$ for some $\hbar\in X$, then Ꝋ is $r-dense$

Proof:

1) Let Ꝋ be $r-dense$ set then $Ꝋ_r=X$, but by proposition (2-2-19) part (3) we have $Ꝋ_r=X-(X-Ꝋ)_s$,Then we will have $(X-Ꝋ)_s=X-Ꝋ_r=X-X=\emptyset$.
2) Since $Ꝋ\in I_{\oint}(\hbar)$ for some $\hbar\in X$, then by proposition (2-2-21) part (2) we have $Ꝋ_r=X$, Thus Ꝋ is $r-dense$ set.

In the next proposition we will study the $r-dense$ set in the case that we have two $i-TPS$'s (ideals, proximity relations) such that one of them finer than the other.

Proposition (2-3-6) :

Let X_{TI}^{δ} be an $i-TPS$, and let $Ꝋ\subseteq X$, then the following statement are hold:

1) In $X_{T_jI}^{\delta}$, j=1,2 such that T_2 finer than T_1, then every $r-dense$ set w. r. t. T_2 is $r-dense$ set w. r. t. T_1 .

2) In $X^{\delta}_{TI_j}$, j=1,2 such that I_2 finer than I_1, then every $r-dense$ set w. r. t. I_1 is $r-dense$ set w. r. t. I_2.
3) In $X^{\delta_j}_{TI}$, j=1,2 such that $\delta_2 < \delta_1$, then every $r-dense$ set w. r. t. δ_1 is $r-dense$ set w. r. t. δ_2 .

Proof:

1) Let Ų be $r-dense$ set w. r. t. T_2 then $(Ų_r)_{T_2} = X$, but by proposition (2-2-24) part (1) we have $(Ų_r)_{T_2} \subseteq (Ų_r)_{T_1}$. Thus $(Ų_r)_{T_1} = X$, which means that Ų is $r-dense$ set w. r. t. T_1.
2) Let Ų be $r-dense$ set w. r. t. I_1 then $(Ų_r)_{I_1} = X$, but by proposition (2-2-24) part (2) we have $(Ų_r)_{I_1} \subseteq (Ų_r)_{I_2}$. Thus $(Ų_r)_{I_2} = X$, which means that Ų is $r-dense$ set w. r. t. I_2.
3) Let Ų be $r-dense$ set w. r. t. δ_1 then $(Ų_r)_{\delta_1} = X$, but by proposition (2-2-24) part (3) we have $(Ų_r)_{\delta_1} \subseteq (Ų_r)_{\delta_2}$. Thus $(Ų_r)_{\delta_2} = X$, which means that Ų is $r-dense$ set w. r. t. δ_2.

Now we will introduce the notion of nowhere redirect dense set as in the following definition.

Definition (2-3-7):

In $i-TPS\ X^{\delta}_{TI}$, a subset Ų ⊆ X is called nowhere redirect dense set if $(Ų_r)_s = \emptyset$, and if $(Ų_r)_s \neq \emptyset$, then we say that Ų is somewhere redirect dense.

The set of all nowhere $r-dense$ set in X is denoted by $R_n(X)$.

Example (2-3-8):

Let $X = \{\hbar, g, f\}$, $T = \{X, \emptyset, \{\hbar\}, \{g\}\}$, and $I = \{\emptyset, \{f\},\}$ with the discrete proximity, if Ų $= \{f\}$ then $Ų_r = \{f\}$ and $Ų_s = \emptyset$, then we get that $(Ų_r)_s = (\{f\})_s = \emptyset$. Thus Ų is nowhere $r-dense$ set.

Proposition (2-3-9):

In $i-TPS\ X_{TI}^{\delta}$, and for $\text{Ц}, \mathcal{K} \subseteq X$ then the following statement are hold.

1) If $\text{Ц} \subseteq \mathcal{K}$ and $\mathcal{K}$ is nowhere $r-dense$ set, then Ц is nowhere $r-dense$ set
2) If Ц or $\mathcal{K}$ is nowhere $r-dense$ set, then $\text{Ц} \cap \mathcal{K}$ is nowhere $r-dense$ set.
3) If $\text{Ц} \cup \mathcal{K}$ is nowhere $r-dense$ set, then Ц and $\mathcal{K}$ are nowhere $r-dense$ set

Proof:

1) Let $\mathcal{K}$ be nowhere $r-dense$ set, then $(\mathcal{K}_{\mathrm{r}})_s = \emptyset$, but $\text{Ц} \subseteq \mathcal{K}$ and by proposition (2-2-4) part (4) we have $\text{Ц}_{\mathrm{r}} \subseteq \mathcal{K}_r$, and by Part (4) of proposition (2-2-10) we will get $(\text{Ц}_{\mathrm{r}})_s \subseteq (\mathcal{K}_{\mathrm{r}})_s$. Therefore Ц is nowhere $r-dense$ set.
2) Let Ц or $\mathcal{K}$ be nowhere $r-dense$ set, also we have $\text{Ц} \cap \mathcal{K} \subseteq \text{Ц}$ and $\text{Ц} \cap \mathcal{K} \subseteq \mathcal{K}$ then by (1) we get that $\text{Ц} \cap \mathcal{K}$ is nowhere $r-dense$ set.
3) Let $\text{Ц} \cup \mathcal{K}$ be nowhere $r-dense$ set, but we have $\text{Ц} \subseteq \text{Ц} \cup \mathcal{K}$ and $\mathcal{K} \subseteq \text{Ц} \cup \mathcal{K}$, so by (1) we will get Ц and $\mathcal{K}$ are nowhere $r-dense$.

In the next, proposition we will discuss the relationships between the nowhere $r-dense$ set and the strips set.

Proposition (2-3-10):

In $i-TPS\ X_{TI}^{\delta}$, and $\forall\ \text{Ц} \subseteq X$ if Ц is nowhere $r-dense$ set, then $\text{Ц}_s = \emptyset$

Proof: Let Ц be nowhere $r-dense$ set, then $(\text{Ц}_{\mathrm{r}})_s = \emptyset$, but by proposition (2-2-4) part (1) $\text{Ц} \subseteq \text{Ц}_r$ and by proposition (2-2-10) part (4) we will get $\text{Ц}_{\mathrm{s}} \subseteq (\text{Ц}_r)_s = \emptyset$.

Proposition (2-3-11):

Let X^{δ}_{TI} be $i-TPS$, then $T \cap R_n(X) = \emptyset$

Proof: Suppose, if possible, that $T \cap R_n(X) \neq \emptyset$, then $\exists$ Ա $\neq \emptyset$ s. t. Ա $\subseteq X$ and Ա $\in T \cap R_n(X)$ i.e., Ա is nowhere $r-dense$ set and Ա $\in$ T , so by proposition (1-3-3) part (1) and proposition (2-2-21) part (1) we will get $Ա_r = X$. Thus, by theorem (2-2-10) part (2) we will get $(Ա_r)_s = X$, which a contradiction.

In the following proposition we will see that the image of $r-dense$ set under $\delta-symmetry$ function will be $r-dense$.

Proposition (2-3-12) :

Let $f: X^{\delta}_{TI} \to Y^{\hat{\delta}}_{\hat{T}\hat{I}}$ be a $\delta-symmetry$ function, for any subset Ա of X, then if Ա is $r-dense$ set in X, then f(Ա) is $r-dense$ set in Y.

Proof: Let Ա be $r-dense$ set in X, then $Ա_r = X$, proposition (2-2-25) and $\delta-symmetry$ will give us that $f(Ա)_r = f(Ա_r) = f(X) = Y$. Thus f(Ա) is $r-dense$ set in Y.

In the following proposition we will see that the pre-image of $r-dense$ set under $\delta-symmetry$ function will be $r-dense$.

Proposition (2-3-13):

Let $f: X^{\delta}_{TI} \to Y^{\hat{\delta}}_{\hat{T}\hat{I}}$ be $\delta-symmetry$ function, and for any subset Ա of Y, if Ա is $r-dense$ set in Y, then f^{-1}(Ա) is $r-dense$ set in X.

Proof: Let Ա be $r-dense$ set in Y, then $Ա_r = Y$, so by $\delta-symmetry$ and proposition (2-2-26) we get $\left(f^{-1}(Ա)\right)_r = f^{-1}(Ա_r) = f^{-1}(\mathrm{Y}) = X$. Thus f^{-1}(Ա)is $r-dense$ set in X.

Definition (2-3-14) :

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace\ i-TPS\ X^{\delta}_{TI}$, and for any subset Ʉ of Y, then Ʉ is called $r-dense$ set w. r. t. $i-subspace\ Y$, if for any $y \in Y$ and for all $H_Y \in T_Y(y)$, then $H_Y \delta_Y \notin_{T_Y}$(Ʉ). And denoted by $r_Y - dense$

Example (2-3-15):

Let $X = \{ꝁ, ꞡ, ꞙ\}$, $T = \{\emptyset, X, \{ꝁ\}, \{ꞡ\}\}$, $I = \{\emptyset, \{ꞙ\}\}$ with δ_D. And let $Y = \{ꝁ, ꞙ\}$ so $T_Y = \{\emptyset, Y, \{ꝁ\}\}$, $I_Y = \{\emptyset, \{ꞙ\}\}$ with δ_{Y_D} be $i\text{–}subspace\ Y$ of X^{δ}_{TI}. Now let Ʉ $\subseteq Y$ s.t. Ʉ $= \{ꝁ\}$, then $Ʉ_{r_Y} = Y$. Thus Ʉ is $r_Y - dense$.

Proposition (2-3-16) :

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of smooth $i-TPS\ X^{\delta}_{TI}$, and for any subset Ʉ of Y, if Ʉ is $r_X - dense$, then Ʉ is $r_Y - dense$.

Proof: Let Ʉ is $r_X - dense$, then $Ʉ_{r_X} = X$, but X^{δ}_{TI} is smooth then by corollary (2-2-30) we have $Ʉ_{r_Y} = Ʉ_{r_X} \cap Y$. So $Ʉ_{r_Y} = X \cap Y = Y$.

Thus Ʉ is $r_Y - dense$ set.

Remark (2-3-17):

The convers of proposition (2-3-16) not true in general and the following example show that.

Example (2-3-18):

In example (2-3-15) if $B = \{ꝁ, ꞙ\}$, then $B_{r_Y} = Y$ so B is $r_Y - dense$., but $B_{r_X} = \{ꝁ, ꞙ\}$ so B is not $r_X - dense$.

Chapter Three

Frontier Operators in i-topological Proximity Space

Both boundary points and boundary sets are important topological concepts that have wide resonance and influence in applied mathematics, especially in metric spaces which have drawn researchers to them. Hence, we highlight in this chapter the construction of new types of boundary points and boundary sets in i-topological proximity space one of them by the occlusion set via $i-open$ set and the other one by redirect set. Also, we highlight the most important possible properties of them and their relationship of them.

3.1 Focal Frontier Operator

In this section, we highlight the construction of new type of boundary points and boundary sets in i-topological proximity space such as focal function frontier set and we defined $\oint_{*}-$operator, also we highlighted the most important possible properties of them and their relationships.

Definition (3-1-1)

Let X_{TI}^{δ} be an $i-TPS$, then we define the focal function frontier operator via $i-open$ set of a subset $\mathcal{H}$ by $\oint_{\mathrm{Fr}}(\mathcal{H})=\oint_{\mathrm{T}}(\mathcal{H})\cap\oint_{\mathrm{T}}(\mathrm{X}-\mathcal{H})$.

The point $\hbar\in X$ is said to be focal function frontier point of $\mathcal{H}$ if $\hbar\in\oint_{\mathrm{Fr}}(\mathcal{H})$, and we denoted by $\oint_{Fr}-\ frontier$ point.

Example (3-1-2):

Let $\mathrm{X}=\{\hbar,\mathscr{g},\mathscr{f}\}$, $\mathrm{T}=\{\mathrm{X},\emptyset,\{\hbar\},\{\mathscr{g}\}\}$, and $\mathrm{I}=\{\emptyset,\{\hbar\},\{\mathscr{g}\},\{\hbar,\mathscr{g}\}\}$ with the discrete proximity δ_D, if $\mathcal{H}=\{\hbar\}$, then $\oint_{\mathrm{T}}(\mathcal{H})=\{\hbar,\mathscr{f}\}$ and $\oint_{T}(X-\mathcal{H})=\oint_{T}(\{\mathscr{g},\mathscr{f}\})=\{\mathscr{g},\mathscr{f}\}$

$So\ \oint_{Fr}(\mathcal{H})=\oint_{T}(\mathcal{H})\cap\oint_{T}(X-\mathcal{H})=\{\hbar,\mathscr{f}\}\cap\{\mathscr{g},\mathscr{f}\}=\mathscr{f}.$

One of the characterizations of $\oint_{Fr}-\ frontier$ point is in the next theorem.

Theorem (3-1-3):

Let X_{TI}^{δ} be an $i-TPS$, and a subset Ų of X. Then

$$\oint_{\mathrm{Fr}}(\mathrm{Ų}) = \oint_{\mathrm{T}}(\mathrm{Ų}) - \psi_{\mathrm{T}}(\mathrm{Ų})$$

Proof: $\oint_{\mathrm{Fr}}(\mathrm{Ų}) = \oint_{\mathrm{T}}(\mathrm{Ų}) \cap \oint_{\mathrm{T}}(\mathrm{X} - \mathrm{Ų})$

$= \oint_{\mathrm{T}}(\mathrm{Ų}) \cap \left(\mathrm{X} - \psi_{\mathrm{T}}(\mathrm{Ų})\right)$ by proposition (1-3-37) part (6)

$= \oint_{\mathrm{T}}(\mathrm{Ų}) - \psi_{\mathrm{T}}(\mathrm{Ų})$.

Other characterization of $\oint_{\mathrm{Fr}} -$ frontier point is similar one:

Theorem (3-1-4) :

Let X_{TI}^{δ} be an $i-TPS$, and let $\mathrm{Ų} \subseteq X$, we have $\oint_{\mathrm{Fr}}(\mathrm{Ų}) = \emptyset$ if and only if $\oint_{\mathrm{T}}(\mathrm{Ų}) \subseteq \psi_{\mathrm{T}}(\mathrm{Ų})$.

Proof: Obviously, $\oint_{\mathrm{Fr}}(\mathrm{Ų}) = \emptyset$ if and only if $\oint_{\mathrm{T}}(\mathrm{Ų}) \cap \oint_{\mathrm{T}}(\mathrm{X} - \mathrm{Ų}) = \emptyset$, so by theorem (3-1-3) we have $\oint_{\mathrm{T}}(\mathrm{Ų}) - \psi_{\mathrm{T}}(\mathrm{Ų}) = \emptyset$.

Thus $\oint_{\mathrm{T}}(\mathrm{Ų}) \subseteq \psi_{\mathrm{T}}(\mathrm{Ų})$, and this complete the proof.

Theorem (3-1-5) :

Let X_{TI}^{δ} be an $i-TPS$, and let $\mathrm{Ų} \subseteq X$ then we have $\oint_{\mathrm{Fr}}(\mathrm{Ų}) = \oint_{\mathrm{T}}(\mathrm{X} - \mathrm{Ų})$ if and only if $X - \oint_{\mathrm{T}}(\mathrm{Ų}) \subseteq \psi_{\mathrm{T}}(\mathrm{Ų})$

Proof: Let $\oint_{\mathrm{Fr}}(\mathrm{Ų}) = \oint_{\mathrm{T}}(\mathrm{X} - \mathrm{Ų})$, then $\oint_{\mathrm{T}}(\mathrm{Ų}) \cap \oint_{\mathrm{T}}(\mathrm{X} - \mathrm{Ų}) = \oint_{\mathrm{T}}(\mathrm{X} - \mathrm{Ų})$. So, $\oint_{\mathrm{T}}(\mathrm{X} - \mathrm{Ų}) \subseteq \oint_{\mathrm{T}}(\mathrm{Ų})$, then $X - \oint_{\mathrm{T}}(\mathrm{Ų}) \subseteq X - \oint_{\mathrm{T}}(\mathrm{X} - \mathrm{Ų}) = \psi_{\mathrm{T}}(\mathrm{Ų})$. Thus $X - \oint_{\mathrm{T}}(\mathrm{Ų}) \subseteq \psi_{\mathrm{T}}(\mathrm{Ų})$.

Conversely, if $X - \oint_{\mathrm{T}}(\mathrm{Ų}) \subseteq \psi_{\mathrm{T}}(\mathrm{Ų})$ then, by theorem (3-1-3) we have $X - \oint_{\mathrm{T}}(\mathrm{Ų}) \subseteq X - \oint_{\mathrm{T}}(\mathrm{X} - \mathrm{Ų})$, so $\oint_{\mathrm{T}}(\mathrm{X} - \mathrm{Ų}) \subseteq \oint_{\mathrm{T}}(\mathrm{Ų})$. Therefore, $\oint_{\mathrm{T}}(\mathrm{Ų}) \cap \oint_{\mathrm{T}}(\mathrm{X} - \mathrm{Ų}) = \oint_{\mathrm{T}}(\mathrm{X} - \mathrm{Ų})$. Hence $\oint_{\mathrm{Fr}}(\mathrm{Ų}) = \oint_{\mathrm{T}}(\mathrm{X} - \mathrm{Ų})$ and this complete the proof.

Next theorem discusses the frontier point in case that the set is $\oint_{\mathrm{T}}$ −dense.

Theorem (3-1-6):

Let X_{TI}^{δ} be an $i - TPS$, and let $\text{Ų} \subseteq X$, if Ų is $\oint_{\mathrm{T}}$ −dense in X, then $\oint_{\mathrm{Fr}}(\text{Ų}) = \oint_{\mathrm{T}}(\mathrm{X} - \text{Ų})$.

Proof: By the definition (1-3-34), it is clear that $\oint_{\mathrm{T}}$ −dense set means $\oint_{\mathrm{T}}(\text{Ų}) = X$, so $\oint_{\mathrm{T}}(\mathrm{X} - \text{Ų}) \subseteq \oint_{\mathrm{T}}(\text{Ų})$. Thus, by theorem (3-1-5) we will get $\oint_{\mathrm{Fr}}(\text{Ų}) = \oint_{\mathrm{T}}(\mathrm{X} - \text{Ų})$.

Now, we will study some of $\oint_{\mathrm{Fr}}$ − frontier properties and also see that the $\oint_{\mathrm{Fr}}$ − frontier can give us a closure operator in the following theorem:

Theorem (3-1-7):

Let X_{TI}^{δ} be an $i - TPS$ and $\text{Ų}, \mathcal{K} \subseteq X$, then the following are hold.

1) $\oint_{\mathrm{Fr}}(\emptyset) = \emptyset$ and $\oint_{\mathrm{Fr}}(X) = \emptyset$
2) $\oint_{\mathrm{Fr}}(\text{Ų} \cup \mathcal{K}) \subseteq \oint_{\mathrm{Fr}}(\text{Ų}) \cup \oint_{\mathrm{Fr}}(\mathcal{K})$
3) $\oint_{\mathrm{Fr}}(\text{Ų}) \cup \oint_{\mathrm{Fr}}(\mathcal{K}) = [\text{Ų} \cap \oint_{\mathrm{Fr}}(\mathcal{K})] \cup [\oint_{\mathrm{Fr}}(\text{Ų} \cup \mathcal{K})] \cup [\oint_{\mathrm{Fr}}(\text{Ų}) \cap \mathcal{K}]$
4) $\oint_{\mathrm{Fr}}\left(\oint_{\mathrm{Fr}}(\text{Ų})\right) \subseteq \oint_{\mathrm{T}}\left(\oint_{\mathrm{Fr}}(\text{Ų})\right)$
5) $\oint_{\mathrm{Fr}}(\text{Ų}) = \oint_{\mathrm{T}}(\mathrm{X} - \text{Ų}) - \psi_{\mathrm{T}}(\mathrm{X} - \text{Ų})$
6) $\oint_{\mathrm{Fr}}(\mathrm{X} - \text{Ų}) = \oint_{\mathrm{Fr}}(\text{Ų})$
7) $X - \oint_{\mathrm{Fr}}(\text{Ų}) = \psi_{\mathrm{T}}(\mathrm{X} - \text{Ų}) \cup \psi_{\mathrm{T}}(\text{Ų})$
8) $X = \psi_{\mathrm{T}}(\mathrm{X} - \text{Ų}) \cup \psi_{\mathrm{T}}(\text{Ų}) \cup \oint_{\mathrm{Fr}}(\text{Ų})$
9) $X = \psi_{\mathrm{T}}(\mathrm{X} - \text{Ų}) \cup \psi_{\mathrm{T}}(\text{Ų}) \cup \oint_{\mathrm{Fr}}(\mathrm{X} - \text{Ų})$
10) $\oint_{\mathrm{Fr}}(\text{Ų}) = \oint_{\mathrm{T}}(\text{Ų}), \forall \text{Ų} \in I$
11) $\oint_{\mathrm{Fr}}(\text{Ų}) \cup \psi_{\mathrm{T}}(\text{Ų}) \cup \text{Ų} = \oint_{\mathrm{T}}(\text{Ų}) \cup \psi_{\mathrm{T}}(\text{Ų}) = \oint_{\mathrm{T}}(\text{Ų})$

Proof:

1) It is obvious by definition (3-1-1) and proposition (1-3-33) part (6).

2) $\oint_{Fr}(Ų \cup \mathcal{K}) = \oint_T(Ų \cup \mathcal{K}) - \psi_T(Ų \cup \mathcal{K})$ by theorem (3-1-3)

$$= \oint_T(Ų) \cup \oint_T(\mathcal{K}) - \psi_T(Ų \cup \mathcal{K}),$$

but, since $\psi_T(Ų) \cup \psi_T(\mathcal{K}) \subseteq \psi_T(Ų \cup \mathcal{K})$ (Proposition (1-3-37) part (4), so we will get,

$$\oint_T(Ų) \cup \oint_T(\mathcal{K}) - \psi_T(Ų \cup \mathcal{K}) \subseteq \oint_T(Ų) \cup \oint_T(\mathcal{K}) - (\psi_T(Ų) \cup \psi_T(\mathcal{K}))$$

Thus, $\oint_{Fr}(Ų \cup \mathcal{K}) \subseteq \oint_{Fr}(Ų) \cup \oint_{Fr}(\mathcal{K})$.

3) Note that, $Ų \cap \oint_{Fr}(\mathcal{K}) \cup [\oint_{Fr}(Ų \cap \mathcal{K})] \cup [\oint_{Fr}(Ų \cup \mathcal{K})]$

$$\subseteq \oint_{Fr}(\mathcal{K}) \cup \oint_{Fr}(Ų) \cup [\oint_{Fr}(Ų \cup \mathcal{K})] = \oint_{Fr}(\mathcal{K}) \cup \oint_{Fr}(Ų)$$

by part (2). Again, $\oint_{Fr}(\mathcal{K}) \cup \oint_{Fr}(Ų)$

$\subseteq \oint_{Fr}(\mathcal{K}) \cup \oint_{Fr}(Ų) \cup [\oint_{Fr}(Ų \cup \mathcal{K})] \cup Ų \cap \oint_{Fr}(\mathcal{K}) \cup [\oint_{Fr}(Ų \cap \mathcal{K})]$

$\subseteq \left[\oint_T(\mathcal{K}) \cap \oint_T(X - \mathcal{K})\right] \cup \left[\oint_T(Ų) \cap \oint_T(X - Ų)\right] \cup [\oint_{Fr}(Ų \cup \mathcal{K})] \cup$
$Ų \cap \oint_{Fr}(\mathcal{K}) \cup [\oint_{Fr}(Ų \cap \mathcal{K})]$.

$\subseteq \left[\oint_T(\mathcal{K}) \cup \oint_T(Ų)\right] \cap \left[\oint_T(X - \mathcal{K}) \cup \oint_T(X - Ų)\right] \cup [\oint_{Fr}(Ų \cup \mathcal{K})] \cup$
$Ų \cap \oint_{Fr}(\mathcal{K}) \cup [\oint_{Fr}(Ų \cap \mathcal{K})]$.

$= \left(\oint_T(\mathcal{K} \cup Ų)\right) \cap \left(\oint_T(X - (\mathcal{K} \cup Ų)\right) \cup \left(Ų \cap \oint_{Fr}(\mathcal{K})\right) \cup \left(\oint_{Fr}(Ų \cap \mathcal{K})\right)$

$= \oint_{Fr}(\mathcal{K} \cup Ų) \cup \left(Ų \cap \oint_{Fr}(\mathcal{K})\right) \cup \left(\oint_{Fr}(Ų \cap \mathcal{K})\right)$.

4) It is clear, since,

$$\oint_{Fr}\left(\oint_{Fr}(Ų)\right) = \oint_T\left(\oint_{Fr}(Ų)\right) \cap \left(\oint_T\left(X - \oint_{Fr}(Ų)\right)\right)$$
$$\subseteq \oint_T\left(\oint_{Fr}(Ų)\right)$$

5) $\oint_{Fr}(Ų) = \oint_T(Ų) \cap \oint_T(X - Ų)$

$= \oint_T(X - Ų) - [X - \oint_T(Ų)]$

$= \oint_T(X - Ų) - \psi_T(X - Ų)$, by proposition (1-3-37) part (6).

6) It is obvious, since $\oint_{\mathrm{T}}(\mathrm{U}) \cap \oint_{\mathrm{T}}(\mathrm{X}-\mathrm{U}) = \oint_{\mathrm{T}}(\mathrm{X}-\mathrm{U}) \cap \oint_{\mathrm{T}}(\mathrm{U})$.

7) $X - \oint_{\mathrm{Fr}}(\mathrm{U}) = X - [\oint_{\mathrm{T}}(\mathrm{U}) \cap \oint_{\mathrm{T}}(\mathrm{X}-\mathrm{U})]$

$$= \left(X - \oint_{\mathrm{T}}(\mathrm{U})\right) \cup \left(X - \oint_{\mathrm{T}}(\mathrm{X}-\mathrm{U})\right)$$

$$= \psi_{\mathrm{T}}(\mathrm{X}-\mathrm{U}) \cup \psi_{\mathrm{T}}(\mathrm{U}).$$

8) Directly by part (6) and part (7).

9) Directly by part (6) and part (7).

10) Let $\mathrm{U} \in \mathrm{I}$, then by proposition (1-3-37) part (2) we get $\psi_{\mathrm{T}}(\mathrm{U}) = \emptyset$.

So, by (5) we will have $\oint_{\mathrm{Fr}}(\mathrm{U}) = \oint_{\mathrm{T}}(\mathrm{U}) - \psi_{\mathrm{T}}(\mathrm{U}) = \oint_{\mathrm{T}}(\mathrm{U})$.

11) $\oint_{\mathrm{Fr}}(\mathrm{U}) \cup \psi_{\mathrm{T}}(\mathrm{U}) \cup \mathrm{U} = \left(\oint_{\mathrm{T}}(\mathrm{U}) - \psi_{\mathrm{T}}(\mathrm{U})\right) \cup \psi_{\mathrm{T}}(\mathrm{U}) \cup \mathrm{U}$

$$= \oint_{\mathrm{T}}(\mathrm{U}) \cup \psi_{\mathrm{T}}(\mathrm{U}) \cup \mathrm{U} = \oint_{\mathrm{T}}(\mathrm{U}).$$

by proposition (1-3-33) part (1) , and proposition (1-3-37) part (10).

Corollary (3-1-8):

Let X_{TI}^{δ} be an $i-TPS$, $\mathrm{U} \subseteq X$. Then $\oint_{\mathrm{Fr}}\left(\oint_{\mathrm{Fr}}(\mathrm{U})\right) \subseteq \oint_{\mathrm{Fr}}(\mathrm{U})$

Proof: By proposition (3-1-7) part (4) we have

$$\oint_{\mathrm{Fr}}\left(\oint_{\mathrm{Fr}}(\mathrm{U})\right) \subseteq \oint_{\mathrm{T}}\left(\oint_{\mathrm{Fr}}(\mathrm{U})\right)$$

$$= \oint_{\mathrm{T}}\left(\oint_{\mathrm{T}}(\mathrm{U}) \cap \oint_{\mathrm{T}}(\mathrm{X}-\mathrm{U})\right)$$

$$\subseteq \oint_{\mathrm{T}}\left(\oint_{\mathrm{T}}(\mathrm{U})\right) \cap \oint_{\mathrm{T}}\left(\oint_{\mathrm{T}}(\mathrm{X}-\mathrm{U})\right),$$ proposition (1-3-33) (4)

$$= \oint_{\mathrm{T}}(\mathrm{U}) \cap \oint_{\mathrm{T}}(\mathrm{X}-\mathrm{U}) = \oint_{\mathrm{Fr}}(\mathrm{U})$$

by using proposition (1-3-33) parts (7,8), and theorem (1-3-3) part (1) .

Example (3-1-9):

Let $X = \{\hbar, g, f\}$, $T = \{X, \emptyset, \{\hbar\}, \{g\}\}$, and $I = \{\emptyset, \{f\}\}$ with δ_D , now if $\mathcal{H} = \{g, f\}$ and $\mathfrak{G} = \{\hbar, f\}$ then $\oint_T(\mathcal{H}) = \{g, f\}$, $\oint_T(X - \mathcal{H}) = \oint_T(\{\hbar\}) = \{\hbar, f\}$, and $\oint_T(\mathfrak{G}) = \{\hbar, f\}$, $\oint_T(X - \mathfrak{G}) = \{g\}$.

So $\oint_{Fr}(\mathcal{H}) = f$ and $\oint_{Fr}(\mathfrak{G}) = f$.

Thus, $\oint_{Fr}(\mathcal{H}) \cup \oint_{Fr}(\mathfrak{G}) = f \not\subseteq \oint_{Fr}(\mathcal{H} \cup \mathfrak{G}) = \oint_{Fr}(X) = \emptyset$.

In example (3-1-9) we can see that the convers of part (2) in theorem (3-1-7) cannot be true in general.

Example (3-1-10):

In example (3-1-9), if $\mathcal{H} = \{g\}$and $\mathfrak{G} = \{f\}$, so $\mathcal{H} \cap \mathfrak{G} = \emptyset$.

So, we have $\oint_{Fr}(\mathcal{H}) = f$, $\oint_{Fr}(\mathfrak{G}) = f$ and $\oint_{Fr}(\mathcal{H} \cap \mathfrak{G}) = \emptyset$ but, we have $\oint_{Fr}(\mathcal{H}) \cap \oint_{Fr}(\mathfrak{G}) = f$. Thus, $\oint_{Fr}(\mathcal{H}) \cap \oint_{Fr}(\mathfrak{G}) \not\subseteq \oint_{Fr}(\mathcal{H} \cap \mathfrak{G})$.

Example (3-1-11) :

In example (1-3-32) we have $\oint_T(\mathcal{H}) = X$, for each non empty subset $\mathcal{H}$ of X, so $\oint_T(X - \mathcal{H}) = X$ also. Thus $\oint_{Fr}(\mathcal{H}) = X$ but by theorem (3-1-7) part (1) we have $\oint_{Fr}(X) = \emptyset$.

Thus, $\oint_{Fr}(\mathcal{H} \cap X) = \oint_{Fr}(\mathcal{H}) \not\subseteq \oint_{Fr}(\mathcal{H}) \cap \oint_{Fr}(X) = \emptyset$.

These examples above (3-1-10) and (3-1-11) we can conclude that there is no relation between the $\oint_{Fr} - frontier$ for the intersection of two sets and the intersection of their $\oint_{Fr} - frontier$.

Example (3-1-12) :

In example (3-1-11) we have, for each non empty subset $\mathcal{H}$ of X, $\oint_{Fr}(\mathcal{H}) = X$ but by theorem (3-1-7) part (1) we have $\oint_{Fr}(\emptyset) = \emptyset$.

So, it is clear that $\emptyset \subseteq \mathcal{H}$, but $\oint_{Fr}(\mathcal{H}) = X \nsubseteq \oint_{Fr}(\emptyset) = \emptyset$.

Also, $\oint_{Fr}(\mathcal{H}) = X$ but by theorem (3-1-7) part (1) we have $\oint_{Fr}(X) = \emptyset$,

So, it is clear $\mathcal{H} \subseteq X$, but $\oint_{Fr}(\mathcal{H}) = X \nsubseteq \oint_{Fr}(X) = \emptyset$.

From this example (3-1-12), we can conclude that if $\mathfrak{U} \subseteq \mathcal{H}$, then there is no relation between the $\oint_{Fr} - frontier$ sets for each of them.

Now, we will introduce an important theorem that study the relationships between the $\oint_{Fr} - frontier$ for the union of two sets and their intersections and their difference.

Theorem (3-1-13):

Let X_{TI}^{δ} be an $i - TPS$, and $\mathfrak{U}, \mathfrak{G} \subseteq X$ then

$$\oint_{\mathrm{Fr}}(\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}) = \oint_{\mathrm{Fr}}(\mathfrak{U} - \mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{U} \cap \mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G} - \mathfrak{U}) .$$

Proof: First, we Know that

$$\begin{aligned} \oint_{\mathrm{Fr}}(\mathfrak{U} \cap \mathfrak{G}) &= \oint_{\mathrm{Fr}}(\mathrm{X} - (\mathfrak{U} \cap \mathfrak{G})) \\ &= \oint_{\mathrm{Fr}}((\mathrm{X} - \mathfrak{U}) \cup (\mathrm{X} - \mathfrak{G})) \\ &\subseteq \oint_{\mathrm{Fr}}(\mathrm{X} - \mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathrm{X} - \mathfrak{G}) \text{ by theorem (3-1-7) part (2)} \\ &= \oint_{\mathrm{Fr}}(\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}) \text{ . by theorem (3-1-7) part (6) } \dots \dots \quad (1) \end{aligned}$$

Also, $\oint_{\mathrm{Fr}}(\mathfrak{U} - \mathfrak{G}) = \oint_{\mathrm{Fr}}(\mathfrak{U} \cap (\mathrm{X} - \mathfrak{G}))$

$$\begin{aligned} &\subseteq \oint_{\mathrm{Fr}}(\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathrm{X} - \mathfrak{G}) \\ &= \oint_{\mathrm{Fr}}(\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}) \text{ .} \quad \dots \dots (2) \end{aligned}$$

And, $\oint_{\mathrm{Fr}}(\mathfrak{G}-\mathfrak{U}) \subseteq \oint_{\mathrm{Fr}}(\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G})$. ……(3)

Thus from (1), (2) and (3) we will get

$$\oint_{\mathrm{Fr}}(\mathfrak{U}-\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{U}\cap\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}-\mathfrak{U}) \subseteq \oint_{\mathrm{Fr}}(\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G})$$

Further, by theorem (3-1-7) part (3) we have

$$\oint_{\mathrm{Fr}}(\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}) = \oint_{\mathrm{Fr}}[(\mathfrak{U}-\mathfrak{G}) \cup (\mathfrak{U}\cap\mathfrak{G})] \cup \oint_{\mathrm{Fr}}[(\mathfrak{G}-\mathfrak{U}) \cup (\mathfrak{U}\cap\mathfrak{G})]$$

$$\subseteq \oint_{\mathrm{Fr}}(\mathfrak{U}-\mathfrak{G}) \cup \oint_{\mathrm{Fr}}[(\mathfrak{U}\cap\mathfrak{G})] \cup \oint_{\mathrm{Fr}}[(\mathfrak{G}-\mathfrak{U})].$$

Therefore,

$$\oint_{\mathrm{Fr}}(\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{U}) = \oint_{\mathrm{Fr}}(\mathfrak{U}-\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{U}\cap\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}-\mathfrak{U}).$$

And, this completes the proof.

Now, we can easily apply theorem (3-1-13) to get the following result in theorem (3-1-14)

Theorem (3-1-14):

Let X_{TI}^{δ} be an $i-TPS$, and for $\mathfrak{U}, \mathfrak{G} \subseteq X$, then the following are hold

1) $\oint_{\mathrm{Fr}}(\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}) = \oint_{\mathrm{Fr}}(\mathfrak{U}\cap\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{U}-\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{U}\cup\mathfrak{G})$
2) $\oint_{\mathrm{Fr}}(\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}) = \oint_{\mathrm{Fr}}(\mathfrak{U}\cup\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}-\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{U}\cap\mathfrak{G})$.
3) $\oint_{\mathrm{Fr}}(\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}) = \oint_{\mathrm{Fr}}(\mathfrak{U}-\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}-\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{U}\cap\mathfrak{G})$.
4) $\oint_{\mathrm{Fr}}(\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{U}\Delta\mathfrak{G}) = \oint_{\mathrm{Fr}}(\mathfrak{U}-\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{U}\cap\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}-\mathfrak{U})$. (Where Δ denote the symmetric difference).
5) $\oint_{\mathrm{Fr}}(\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{U}\Delta\mathfrak{G}) = \oint_{\mathrm{Fr}}(\mathfrak{U}-\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{U}\cap\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}-\mathfrak{U})$.

Proof:

1) If the relation in theorem (3-1-13) takes $\mathrm{X}-\mathfrak{G}$ instead of $\mathfrak{G}$ we will get
$\oint_{\mathrm{Fr}}(\mathfrak{U}) \cup \oint_{\mathrm{Fr}}(\mathrm{X}-\mathfrak{G})$
$$= \oint_{\mathrm{Fr}}(\mathfrak{U}\cap\mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{U}\cap(\mathrm{X}-\mathfrak{G})) \cup \oint_{\mathrm{Fr}}((\mathrm{X}-\mathfrak{G})-\mathfrak{U}).$$

And by theorem (3-1-7) part (6) this will implies,

$\oint_{\mathrm{Fr}}(\mathrm{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}) = \oint_{\mathrm{Fr}}(\mathrm{U} \cap \mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathrm{U} - \mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathrm{U} \cup \mathfrak{G})$.

2) If the relation in theorem (3-1-13) takes $\mathrm{X} - \mathrm{U}$ instead of U we will get

$$\oint_{\mathrm{Fr}}(\mathrm{X} - \mathrm{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G})$$
$$= \oint_{\mathrm{Fr}}((\mathrm{X} - \mathrm{U}) - \mathfrak{G}) \cup \oint_{\mathrm{Fr}}((\mathrm{X} - \mathrm{U}) \cap \mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G} \cap \mathrm{U}).$$

And by theorem (3-1-7) part (6) this will implies,

$\oint_{\mathrm{Fr}}(\mathrm{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}) = \oint_{\mathrm{Fr}}(\mathrm{U} \cup \mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G} - \mathrm{U}) \cup \oint_{\mathrm{Fr}}(\mathrm{U} \cap \mathfrak{G})$.

3) If the relation in theorem (3-1-13) takes $\mathrm{X} - \mathrm{U}$ instead of U and $\mathrm{X} - \mathfrak{G}$ instead of $\mathfrak{G}$, then we will get

$$\oint_{\mathrm{Fr}}(\mathrm{X} - \mathrm{U}) \cup \oint_{\mathrm{Fr}}(\mathrm{X} - \mathfrak{G}) =$$
$$= \oint_{\mathrm{Fr}}((\mathrm{X} - \mathrm{U}) - (\mathrm{X} - \mathfrak{G})) \cup \oint_{\mathrm{Fr}}((\mathrm{X} - \mathrm{U}) \cap (\mathrm{X} - \mathfrak{G})$$
$$\cup \oint_{\mathrm{Fr}}(\mathrm{X} - \mathfrak{G}) - (\mathrm{X} - \mathrm{U}))$$

Which implies by theorem (3-1-7) part (6),

$\oint_{\mathrm{Fr}}(\mathrm{U}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G}) = \oint_{\mathrm{Fr}}(\mathfrak{G} - \mathrm{U}) \cup \oint_{\mathrm{Fr}}(\mathrm{U} \cup \mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathrm{U} - \mathfrak{G})$.

4) By theorem (3-1-13) we have takes $\mathrm{X} - \mathrm{U}$ instead of U and $\mathrm{X} - \mathfrak{G}$ instead of $\mathfrak{G}$, then we will get

$$\oint_{\mathrm{Fr}}(\mathrm{U}) \cup \oint_{\mathrm{Fr}}(\mathrm{U}\Delta\mathfrak{G}) =$$
$$= \oint_{\mathrm{Fr}}(\mathrm{U} - (\mathrm{U}\Delta\mathfrak{G})) \cup \oint_{\mathrm{Fr}}(\mathrm{U} \cap (\mathrm{U}\Delta\mathfrak{G}) \cup \oint_{\mathrm{Fr}}((\mathrm{U}\Delta\mathfrak{G}) - \mathrm{U})$$
$$= \oint_{\mathrm{Fr}}(\mathrm{U} \cap \mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathrm{U} - \mathfrak{G}) \cup \oint_{\mathrm{Fr}}(\mathfrak{G} - \mathrm{U})$$
$$= \oint_{\mathrm{Fr}}(\mathrm{U}) \cup \oint_{\mathrm{Fr}}(\mathrm{U}\Delta\mathfrak{G}).$$

5) Directly from (4).

Now we introduce the definition of the operator $\oint_{*}(\mathrm{A})$ on the i-topolgical space for a subset $\mathcal{H}$ of X , as in the following definition

Definition (3-1-15):

We can define the operator $\oint_{*}$ on an $i - TPS, X_{TI}^{\delta}$ for nay subset U of X, in the following way: $\oint_{*}(\mathrm{U}) = \oint_{\mathrm{T}}(\mathrm{U}) - \mathrm{U}$.

Example (3-1-16):

Let $X = \{\hbar, g, f\}$, $T = \{X, \emptyset, \{\hbar\}, \{g\}\}$, and $I = \{\emptyset, \{f\}\}$ with δ_D, now if $\mathcal{H} = \{\hbar\}$, then $\oint_T(\mathcal{H}) = \{\hbar, f\}$. Thus, $\oint_*(\mathcal{H}) = f$.

The important properties for this operator can be shown I the following theorem.

Theorem (3-1-17):

Let X_{TI}^{δ} be an $i - TPS$, and $\mathfrak{U}, \mathfrak{G} \subseteq X$, then the following statement are hold

1) $\oint_*(\emptyset) = \emptyset, \oint_*(X) = \emptyset$
2) $\mathfrak{U} \cap \oint_*(\mathfrak{U}) = \emptyset$;
3) $\oint_*(\mathfrak{U} \cup \mathfrak{G}) = (\oint_*(\mathfrak{U}) - \mathfrak{G}) \cup (\oint_*(\mathfrak{G}) - \mathfrak{U})$;
4) $\oint_*\left(\oint_*(\mathfrak{U})\right) \subseteq \mathfrak{U}$, for any $\mathfrak{U} \in T$.
5) If $\oint_*(\mathfrak{U}) = \emptyset$, then $\mathfrak{U} = i - cl(\mathfrak{U})$.

Proof: From the definition (3-1-15) we can directly proof (1) and (2). Now for other point we have :

3) $$\begin{aligned}\oint_*(\mathfrak{U} \cup \mathfrak{G}) &= \oint_T(\mathfrak{U} \cup \mathfrak{G}) - (\mathfrak{U} \cup \mathfrak{G}) \\ &= [\oint_T(\mathfrak{U}) \cup \oint_T(\mathfrak{G})] - (\mathfrak{U} \cup \mathfrak{G}) \\ &= [(\oint_T(\mathfrak{U}) - \mathfrak{U}) - \mathfrak{G}] \cup [(\oint_T(\mathfrak{G}) - \mathfrak{G}) - \mathfrak{U} \\ &= (\oint_*(\mathfrak{U}) - \mathfrak{G}) \cup (\oint_*(\mathfrak{G}) - (\mathfrak{U}).\end{aligned}$$

4) $$\begin{aligned}\oint_*\left(\oint_*(\mathfrak{U})\right) &= \oint_T\left(\oint_*(\mathfrak{U})\right) - \oint_*(\mathfrak{U}) \\ &= \oint_T\left(\oint_T(\mathfrak{U}) - \mathfrak{U}\right) - (\oint_T(\mathfrak{U}) - \mathfrak{U}) \\ &\subseteq \oint_T\left(\oint_T(\mathfrak{U})\right) - (\oint_T(\mathfrak{U}) - \mathfrak{U}) \\ &\subseteq \oint_T(\mathfrak{U}) - (\oint_T(\mathfrak{U}) - \mathfrak{U}) \subseteq \mathfrak{U}.\end{aligned}$$

5) $\oint_{*}(\text{Ų}) = \emptyset$, means that $\oint_{\text{T}}(\text{Ų}) - \text{Ų} = \emptyset$, i.e., $\oint_{\text{T}}(\text{Ų}) = \text{Ų}$, but we have $\text{Ų} \subseteq i - cl(\text{Ų})$, so $\oint_{\text{T}}(\text{Ų}) = \text{Ų} \subseteq i - cl(\text{Ų})$. and by proposition (1-3-33) part (9) we have $i - cl(\text{Ų}) \subseteq \oint_{\text{T}}(\text{Ų}) = \text{Ų}$.
Thus $\text{Ų} = i - cl(\text{Ų})$

Remark (3-1-18):

There is no subset $\text{Ų} \subseteq X$ satisfies that $\oint_{*}(\text{Ų}) = X$, if there exist such one then $\oint_{\text{T}}(\text{Ų}) - \text{Ų} = \text{X}$, which implies $\text{Ų} = \emptyset$, but $\oint_{\text{T}}(\text{Ų}) = \oint_{\text{T}}(\emptyset) = \emptyset$, which a contradiction

Now we can study the notion of focal function frontier $i - subspace\ Y^{\delta_Y}_{T_Y I_Y}$ of $i - TPS\ X^{\delta}_{TI}$ and study some of its properties and relations

Definition (3-1-19)

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ X^{\delta}_{TI}$ then we can define the focal function frontier set via $i - open$ set w. r. t. $i - subspace\ Y^{\delta_Y}_{T_Y I_Y}$ of a subset $\mathcal{H}$ of Y by $\oint_{\text{Fr}_\text{Y}}(\mathcal{H}) = \oint_{\text{T}_\text{Y}}(\mathcal{H}) \cap \oint_{\text{T}_\text{Y}}(\text{X} - \mathcal{H})$.

The point $\hbar \in Y$ is said to be focal function frontier point of $\mathcal{H}$ w. r. t. $Y^{\delta_Y}_{T_Y I_Y}$ if $\hbar \in \oint_{\text{Fr}_\text{Y}}(\mathcal{H})$, and we denoted by $\oint_{Fr_Y} -\ frontier$ point.

Example (3-1-20):

Let $\text{X} = \{\hbar, g, f\}$, $\text{T} = \{\text{X}, \emptyset, \{\hbar\}, \{g\}\}$, and $\text{I} = \{\emptyset, \{f\}\}$ with the discrete proximity δ_D, and let $Y = \{\hbar, f\}$ then $T_Y = \{\text{Y}, \emptyset, \{\hbar\}\}$ and $\text{I}_\text{Y} = \{\emptyset, \{f\}\}$. Now, if $\mathcal{H} = \{\hbar\}$, then $\oint_{\text{T}_\text{Y}}(\mathcal{H}) = \text{Y}$ and $\oint_{T_Y}(Y - \mathcal{H}) = \oint_{T_Y}(\{f\}) = \{f\}$

$So\ \oint_{Fr_Y}(\mathcal{H}) = \oint_{T_Y}(\mathcal{H}) \cap \oint_{T_Y}(Y - \mathcal{H}) = Y\ \cap \{f\} = f$.

One of the results we can discuss is the relation between $\oint_{Fr} - frontier$ set in X and $\oint_{Fr_Y} - frontier$ which shown in the next theorem.

Proposition (3-1-21)

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS \ X^{\delta}_{TI}$, and let $\mathfrak{U} \subseteq Y$, Then $\oint_{Fr_Y}(\mathfrak{U}) \subseteq \oint_{\mathrm{Fr_X}}(\mathfrak{U}) \cap Y$.

Proof:

It is clear by proposition (2-1-20) that $\oint_{\mathrm{T_Y}}(\mathfrak{U}) \subseteq \oint_{\mathrm{T_X}}(\mathfrak{U}) \cap Y$, so we have $\oint_{\mathrm{T_Y}}(Y - \mathfrak{U}) \subseteq \oint_{\mathrm{T_X}}(X - \mathfrak{U}) \cap Y$.

Now $\oint_{Fr_Y}(\mathfrak{U}) = \oint_{\mathrm{T_Y}}(\mathfrak{U}) \cap \oint_{\mathrm{T_Y}}(Y - \mathfrak{U})$

$$\subseteq \left(\oint_{\mathrm{T_X}}(\mathfrak{U}) \cap Y\right) \cap \left(\oint_{\mathrm{T_X}}(X - \mathfrak{U}) \cap Y\right)$$

$$= \left(\oint_{\mathrm{T_X}}(\mathfrak{U}) \cap \oint_{\mathrm{T_X}}(X - \mathfrak{U})\right) \cap Y$$

$$= \oint_{\mathrm{Fr_X}}(\mathfrak{U}) \cap Y$$

Corollary (3-1-22)

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS \ X^{\delta}_{TI}$, and let $\mathfrak{U} \subseteq Y$, Then $\oint_{Fr_Y}(\mathfrak{U}) = \oint_{\mathrm{Fr_X}}(\mathfrak{U}) \cap Y$. if and only if X^{δ}_{TI} have intuition smoothing feature.

Proof: Let X is smooth $i - TPS$ then by corollary (2-1-23) we have $\oint_{\mathrm{T_Y}}(\mathfrak{U}) = \oint_{\mathrm{T_X}}(\mathfrak{U}) \cap Y$ and this implies $\oint_{\mathrm{T_Y}}(Y - \mathfrak{U}) = \oint_{\mathrm{T_X}}(X - \mathfrak{U}) \cap Y$
So,$\oint_{Fr_Y}(\mathfrak{U}) = \oint_{\mathrm{T_Y}}(\mathfrak{U}) \cap \oint_{\mathrm{T_Y}}(Y - \mathfrak{U})$

$$= \left(\oint_{\mathrm{T_X}}(\mathfrak{U}) \cap Y\right) \cap \left(\oint_{\mathrm{T_X}}(X - \mathfrak{U}) \cap Y\right)$$

$$= \left(\oint_{\mathrm{T_X}}(\mathfrak{U}) \cap \oint_{\mathrm{T_X}}(X - \mathfrak{U})\right) \cap Y$$

$$= \oint_{\mathrm{Fr_X}}(\mathfrak{U}) \cap Y.$$

Our aim now is to highlight the concepts outer of the occlusion set via $i-open$ set in $i-TPS$

Definition (3-1-23)

In $i-TPS\ X_{TI}^{\delta}$, the occlusion outer function via $i-open$ set of a subset Ư of X defined as the following

$$O_T(\text{Ư}) = \{\mathscr{k} \in X;\ \exists \mathcal{H} \in T(\mathscr{k}): \mathcal{H}\bar{\delta} \oint_{\mathrm{T}}(\text{Ư})\}.$$

The relation between the outer function of occlusion set via $i-open$ set with the occlusion set via $i-open$ set and ψ_{T} −operator is given by the following theorem.

Theorem (3-1-24):

Let X_{TI}^{δ} be an $i-TPS$ and Ư subset of X, then

$$O_T(\text{Ư}) \subseteq \psi_{\mathrm{T}}(X-\text{Ư}) = X - \oint_{\mathrm{T}}(\text{Ư})$$

Proof: $\mathscr{h} \in O_T(\text{Ư})$ if and only if $\exists \mathcal{H} \in T(\mathscr{k}): \mathcal{H}\bar{\delta} \oint_{\mathrm{T}}(\text{Ư})$, and by part (1) of proposition (1-3-33) we have $\text{Ư} \subseteq \oint_{\mathrm{T}}(\text{Ư})$, then $\mathcal{H}\bar{\delta}\text{Ư}$ by proposition (1-2-3) part (1).

Thus, by definition (1-1-35) $\mathscr{h} \in \psi_{\mathrm{T}}(X-\text{Ư})$.

Remark (3-1-25):

Let X_{TI}^{δ} be an $i-TPS$ and Ư subset of X, then $O_T(\text{Ư}) = \psi_{\mathrm{T}}(X-\text{Ư})$ if and only if $\forall \mathscr{h} \in O_T(\text{Ư})\ \exists \mathcal{H} \in T(\mathscr{k}): \mathcal{H}\bar{\delta}\text{Ư}$.

Example (3-1-26):

Let $\mathrm{X} = \{\mathscr{h}, \mathscr{g}, \mathscr{f}\}$, $\mathrm{T} = \{\mathrm{X}, \emptyset, \{\mathscr{h}\}, \{\mathscr{g}\}\}$, and $\mathrm{I} = \{\emptyset, \{\mathscr{f}\}\}$ with the discrete proximity, if $\text{Ư} = \{\mathscr{g}, \mathscr{f}\}$, then $\oint_{\mathrm{T}}(\text{Ư}) = \{\mathscr{g}, \mathscr{f}\}$.

Hence $O_T(\text{Ư}) = \{\mathscr{h}\} = X - \oint_{\mathrm{T}}(\text{Ư})$.

In the following proposition we will study some properties of the occlusion outer function via $i-open$

Proposition (3-1-27):

In $i-TPS\ X_{TI}^{\delta}$, and for $\text{Ų}, \mathcal{K} \subseteq X$, the following statement are hold.

1) $O_T(\emptyset) = X$ and $O_T(X) = \emptyset$
2) $O_T(O_T(\text{Ų})) \subseteq \psi_T(\oint_T(\text{Ų}))$
3) If $\text{Ų} \subseteq \mathcal{K}$ then $O_T(\mathcal{K}) \subseteq O_T(\text{Ų})$
4) $O_T(\oint_T(\text{Ų})) \subseteq O_T(\text{Ų})$
5) $O_T(\text{Ų} \cup \mathcal{K}) \subseteq (X - \oint_T(\text{Ų})) \cap (X - \oint_T(\mathcal{K})$
   $= \psi_T(X - \text{Ų}) \cap \psi_T(X - \mathcal{K})$
6) $O_T(\text{Ų} \cup \mathcal{K}) \subseteq O_T(\text{Ų}) \cap O_T(\mathcal{K})$

Proof:

1) By theorem (3-1-24) we have $O_T(\text{X}) \subseteq X - \oint_T(\text{X}) = X - X = \emptyset$, so, $O_T(\text{X}) = \emptyset$. Now, for $O_T(\emptyset)$, we have by definition (3-1-23) $O_T(\emptyset) = \{\hbar \in X;\ \exists \mathcal{H} \in T(\hbar): \mathcal{H}\bar{\delta}\oint_T(\emptyset)\}$
   $= \{\hbar \in X;\ \exists \mathcal{H} \in T(\hbar): \mathcal{H}\bar{\delta}\emptyset\} = X.$
2) $O_T(O_T(\text{Ų})) \subseteq O_T\big(\psi_T(X - \text{Ų})\big)$
   $\subseteq \psi_T\Big(X - \big(\psi_T(X - \text{Ų})\big)\Big)$
   $= \psi_T\big(X - (X - \oint_T(\text{Ų}))\big)$
   $= \psi_T(\oint_T(\text{Ų})).$
3) Let $\hbar \in O_T(\mathcal{K})$, so $\exists \mathcal{H} \in T(\hbar): \mathcal{H}\bar{\delta}\oint_T(\mathcal{K})$, but by part (3) of proposition (1-3-33) we have $\oint_T(\text{Ų}) \subseteq \oint_T(\mathcal{K})$.
   Thus $\mathcal{H}\bar{\delta}\oint_T(\text{Ų})$ and hence $\hbar \in O_T(\text{Ų})$.
4) Since $\text{Ų} \subseteq \oint_T(\text{Ų})$, then by part (3) we will get $O_T(\oint_T(\text{Ų})) \subseteq O_T(\text{Ų})$.
5) $O_T(\text{Ų} \cup \mathcal{K}) \subseteq X - \oint_T(\text{Ų} \cup \mathcal{K})$
   $= X - (\oint_T(\text{Ų}) \cup \oint_T(\mathcal{K}))$
   $= (X - \oint_T(\text{Ų})) \cap (X - \oint_T(\mathcal{K}))$
   $= \psi_T(X - \text{Ų}) \cap \psi_T(X - \mathcal{K})$
6) Let $\hbar \in O_T(\text{Ų} \cup \mathcal{K})$, so $\exists \mathcal{H} \in T(\hbar): \mathcal{H}\bar{\delta}\oint_T(\text{Ų} \cup \mathcal{K})$ but by proposition (1-3-33) part (5) we have $\oint_T(\text{Ų} \cup \mathcal{K}) = \oint_T(\text{Ų}) \cup \oint_T(\mathcal{K})$ therefore $\mathcal{H}\bar{\delta}\oint_T(\text{Ų})$ and $\mathcal{H}\bar{\delta}\oint_T(\mathcal{K})$ which implies $\hbar \in O_T(\text{Ų})$ and $\hbar \in O_T(\mathcal{K})$, hence $\hbar \in O_T(\text{Ų}) \cap O_T(\mathcal{K})$.
   Thus $O_T(\text{Ų} \cup \mathcal{K}) \subseteq O_T(\text{Ų}) \cap O_T(\mathcal{K})$.

3.2 Redirect Frontier operator

In this section, we highlight the construction of redirect frontier operator and the redirect outer function. and we highlighted the most important possible properties of them and their relationships.

Definition (3-2-1):

Let X_{TI}^{δ} be an $i-TPS$ and Џ subset of X. Then the redirect frontier set of Џ is defined by

$$\mathrm{F_r}(Џ) = Џ_r \cap (\mathrm{X} - Џ)_r.$$

The point $\hbar \in X$ is said to be redirect frontier point of Џ if $\hbar \in \mathrm{F_r}(Џ)$, and denoted by $r-frontier$ point.

Example (3-2-2):

Let $\mathrm{X} = \{\hbar, g, f\}$, $\mathrm{T} = \{\mathrm{X}, \emptyset, \{\hbar\}, \{g\}\}$, and $\mathrm{I} = \{\emptyset, \{f\}\}$ with proximity δ_D, if Џ $= \{g\}$, then $Џ_r = \{g, f\}$ and $X -$ Џ $= \{\hbar, f\}$, so $(\mathrm{X} - Џ)_r = \{\hbar, f\}$. Thus $\mathrm{F_{rt}}(Џ) = \{f\}$

In the next theorem, we can find one of the characterizations for the $r-\ frontier$ point.

Theorem (3-2-3):

Let X_{TI}^{δ} be an $i-TPS$ and Џ subset of X. Then $\mathrm{F_r}(Џ) = Џ_r - Џ_s$

Proof:

$\mathrm{F_r}(Џ) = Џ_r \cap (\mathrm{X} - Џ)_r$ by definition (3-2-1)

$= Џ_r \cap (\mathrm{X} - Џ_s)$ by proposition (2-2-19)

$= Џ_r - Џ_s$

Other characterization of $r-\ frontier$ point is similar one can we see it in the next theorem:

Theorem (3-2-4):

In $i-TPS$, X_{TI}^{δ} and for $Ư \subseteq X$, $F_r(Ư) = \emptyset$ if and only if $Ư_r \subseteq Ư_s$.

Proof: Conspicuously, $F_r(Ư) = \emptyset$ if and only if $Ư_r - Ư_s = \emptyset$ (by Theorem (3-2-3)) which equivalent to the fact that $Ư_r \subseteq Ư_s$. And this completes the proof.

Theorem (3-2-5):

In i-TPS X_{TI}^{δ} and for $Ư \subseteq X$ we have $F_r(Ư) = (X - Ư)_r$ if and only if $X - Ư_r \subseteq Ư_s$

Proof: Suppose $F_r(Ư) = (X - Ư)_r$, then $Ư_r \cap (X - Ư)_r = (X - Ư)_r$, which implies $(X - Ư)_r \subseteq Ư_r$ so we get, $X - Ư_r \subseteq X - (X - Ư)_r = Ư_s$ Thus $X - Ư_r \subseteq Ư_s$.

Conversely, if $X - Ư_r \subseteq Ư_s = X - (X - Ư)_r$, then $(X - Ư)_r \subseteq Ư_r$ Thus $Ư_r \cap (X - Ư)_r = (X - Ư)_r$ Hence $F_r(Ư) = (X - Ư)_r$, and this complete the proof.

By the definition of $r - denes$ we can prove the next theorem conspicuously.

Theorem (3-2-6):

In $i - TPS$ X_{TI}^{δ} and for $r - denes$ subset $Ư \subseteq X$, then $F_r(Ư) = (X - Ư)_r$.

Proof: It is clear that $r - denes$ set means $Ư_r = X$, so we will get that $(X - Ư)_r \subseteq Ư_r = X$. then $F_r(Ư) = (X - Ư)_r$.

Now, we will see that the $r - frontier$ can give us some properties, some of them can proved easily by using the definition of $r - frontier$ and the others will be proved.

Theorem (3-2-7):

In $i - TPS$ X_{TI}^{δ} and for $Ư, \mathcal{K} \subseteq X$ then the following statement are hold.

1) $F_r(\emptyset) = \emptyset$ and $F_r(X) = \emptyset$
2) $F_r(Ư \cup \mathcal{K}) \subseteq F_r(Ư) \cup F_r(\mathcal{K})$
3) $F_r(Ư) \cup F_r(\mathcal{K}) = [Ư \cap F_r(\mathcal{K})] \cup [F_r(Ư \cup \mathcal{K})] \cup [F_r(Ư) \cap \mathcal{K}]$
4) $F_r(Ư) = (X - Ư)_r - (X - Ư)_s$
5) $F_r(X - Ư) = F_r(Ư)$
6) $X - F_r(Ư) = (X - Ư)_s \cup Ư_s$

7) $X = (X - U)_s \cup U_s \cup F_r(U)$
8) $X = (X - U)_s \cup U_s \cup F_r(X - U)$
9) $F_r(F_r(U)) \subseteq (F_r(U))_r$
10) $F_r(U) \cup U_s \cup U = U_r$

Proof:

1) It is clear by definition (3-2-1) and theorem (2-2-4) part (2).
2) $F_r(U \cup \mathcal{K}) = (U \cup \mathcal{K})_r - (U \cup \mathcal{K})_s$ by theorem (3-2-3)
$= U_r \cup \mathcal{K}_r - (U \cup \mathcal{K})_s$ by theorem (2-2-4) part (8)

But $U_s \cup \mathcal{K}_s \subseteq (U \cup \mathcal{K})_s$ by theorem (2-2-10) part (8), so we get
$F_r(U \cup \mathcal{K}) = U_r \cup \mathcal{K}_r \cap (X - (U \cup \mathcal{K})_s)$
$\subseteq U_r \cup \mathcal{K}_r \cap (((X - U_s) \cup (X - \mathcal{K}_s))$
$= \left(U_r \cap ((X - U_s) \cap (X - \mathcal{K}_s))\right) \cup \left(\mathcal{K}_r \cap ((X - \mathcal{K}_s) \cap (X - U_s))\right)$
$\subseteq (U_r \cap (X - U_s)) \cup (\mathcal{K}_r \cap (X - \mathcal{K}_s))$
$= F_r(U) \cup F_r(\mathcal{K})$

3) Note that
$[U \cap F_r(\mathcal{K})] \cup [F_r(U \cup \mathcal{K})] \cup [F_r(U) \cap \mathcal{K}]$
$\subseteq [F_r(\mathcal{K})] \cup [F_r(U \cup \mathcal{K})] \cup [F_r(U)]$
$\subseteq [F_r(\mathcal{K})] \cup [F_r(U)]$

Again, $[F_r(\mathcal{K})] \cup [F_r(U)]$
$\subseteq ([F_r(\mathcal{K})] \cup [F_r(U)]) \cup [U \cap F_r(\mathcal{K})] \cup [F_r(U \cup \mathcal{K})] \cup [F_r(U) \cap \mathcal{K}]$
$= ([(X - \mathcal{K})_r \cap \mathcal{K}_r] \cup [(X - U)_r \cap U_r])$
$\cup [U \cap F_r(\mathcal{K})] \cup [F_r(U \cup \mathcal{K})] \cup [F_r(U) \cap \mathcal{K}]$
$\subseteq ([U_r \cup \mathcal{K}_r] \cap [(X - U)_r \cup (X - \mathcal{K})_r])$
$\cup ([U \cap F_r(\mathcal{K})] \cup [F_r(U \cup \mathcal{K})] \cup [F_r(U) \cap \mathcal{K}]$
$= [(U \cup \mathcal{K})_r \cap (X - (U \cup \mathcal{K}))_r] \cup [U \cap F_r(\mathcal{K})] \cup [F_r(U) \cap \mathcal{K}]$
$= [F_r(U \cup \mathcal{K})] \cup [U \cap F_r(\mathcal{K})] \cup [F_r(U) \cap \mathcal{K}]$

4) $F_{rt}(U) = (X - U)_r \cap U_r = (X - U)_r \cap (X - (X - U)_s$
$= (X - U)_r - (X - U)_s.$

5) It is obvious since $(X - U)_r \cap U_r = U_r \cap (X - U)_r$
6) $X - F_r(U) = X - ((X - U)_r \cap U_r)$
$= (X - U_r) \cup (X - (X - U)_r)$

$= X - U_r \cup U_s$ by proposition (2-2-19) parts (2,3)
$= (X - U)_s \cup U_s$.

7) By part (6) we have $X - F_r(U) = (X - U)_s \cup U_s$.
So $X = (X - U)_s \cup U_s \cup F_r(U)$.
8) By part (7) we have $X = (X - U)_s \cup U_s \cup F_r(U)$, also by part (5) we have $F_r(U) = F_r(X - U)$. Thus, $X = (X - U)_s \cup U_s \cup F_r(X - U)$.
9) $F_r(F_r(U)) = F_r(U_r \cap (X - U)_r)$

$$= (U_r \cap (X - U)_r)_r \cap \big(X - (U_r \cap (X - U)_r)\big)_r$$
$$\subseteq (U_r \cap (X - U)_r)_r$$
$$= \big(F_r(U)\big)_r.$$

10) $F_r(U) \cup U_s \cup U = (U_r - U_s) \cup U_s \cup U$

$$= U_r \cup U_s \cup U$$
$$= U_r$$

by theorem (2-2-4) part (1) and theorem (2-2-19) part (1).

Corollary (3-2-8):

In $i - TPS$ X_{TI}^{δ} and for $U \subseteq X$, if U is $i - open$ set then

$$F_r(F_r(U)) \subseteq F_r(U)$$

Proof: By part (9) of theorem (3-2-7) we have Since

$$F_r(F_r(U)) \subseteq \big(F_r(U)\big)_r$$
$$= (U_r \cap (X - U)_r)_r$$
$$\subseteq (U_r)_r \cap ((X - U)_r)_r \text{ by theorem (2-2-4) part (7).}$$
$$= U_r \cap (X - U)_r$$
$$= F_r(U)$$

by proposition (2-2-21) part (3) and theorem (1-3-3) part (1).

In the following examples we will show that some parts of theorem (3-2-7) not satisfying the convers in general.

Example (3-2-9):

Let $X = \{\hbar, \mathscr{g}, \mathscr{f}\}$, $T = \{X, \emptyset, \{\hbar\}, \{\mathscr{g}\}\}$, and $I = \{\emptyset, \{\mathscr{f}\}\}$ with δ_D, now if $Ʉ = \{\mathscr{g}, \mathscr{f}\}$, so $X - Ʉ = \{\hbar\}$, then $Ʉ_r = \{\mathscr{g}, \mathscr{f}\}$, and $(X - Ʉ)_r = \{\hbar, \mathscr{f}\}$, Thus, $F_r(Ʉ) = \{\mathscr{f}\}$.

Now, if $\mathcal{H} = \{\hbar, \mathscr{f}\}$ so $X - \mathcal{H} = \{\mathscr{g}\}$, then $\mathcal{H}_r = \{\hbar, \mathscr{f}\}$ and $(X - \mathcal{H})_r = \{\mathscr{g}, \mathscr{f}\}$. Thus, $F_r(\mathcal{H}) = \{\mathscr{f}\}$.

Hence $F_r(\mathcal{H}) \cup F_r(Ʉ) = \{\mathscr{f}\} \nsubseteq F_r(\mathcal{H} \cup Ʉ) = F_r(X) = \emptyset$.

In this example we can see that the convers of part (2) in theorem (3-2-7) cannot be true in general.

Example (3-2-10):

Let $X = \{\hbar, \mathscr{g}, \mathscr{f}\}$, $T = \{X, \emptyset, \{\hbar, \mathscr{g}\}, \{\hbar, \mathscr{f}\}\}$, and $I = \{\emptyset, \{\mathscr{f}\}, \{\mathscr{g}\}, \{\mathscr{g}, \mathscr{f}\}\}$ with δ_D, now if $\mathcal{H} = \{\hbar\}$ and, so $\mathcal{H}_r = X$. also, we have $X - \mathcal{H} = \{\mathscr{g}, \mathscr{f}\}$, so $(X - \mathcal{H})_r = X$. Then, $F_r(\mathcal{H}) = X \cap X = X$.

Now, $F_r\big(F_r(\mathcal{H})\big) = F_r(X) = \emptyset$ by theorem (3-2-7) part (1).

Thus, $\big(F_r(\mathcal{H})\big)_r = X_r = X \nsubseteq F_r\big(F_r(\mathcal{H})\big) = \emptyset$.

Also, that mean the convers of part (9) in theorem (3-2-7) is not true in general.

Example (3-2-11) :

In example (3-2-9) if we take $\mathcal{H} = \{\mathscr{f}\}$ and $\mathfrak{G} = \{\mathscr{g}\}$, then $\mathcal{H}_r = \{\mathscr{f}\}$, $(X - \mathcal{H})_r = X$ so $F_r(\mathcal{H}) = \{\mathscr{f}\}$, also, we have $\mathfrak{G}_r = \{\mathscr{g}, \mathscr{f}\}$ and $(X - \mathfrak{G})_r = \{\hbar, \mathscr{f}\}$ so $F_r(\mathfrak{G}) = \{\mathscr{f}\}$. And $F_r(\mathfrak{G} \cap \mathcal{H}) = F_r(\emptyset) = \emptyset$, but $F_r(\mathfrak{G}) \cap F_r(\mathcal{H}) = \{\mathscr{f}\}$. Thus $F_r(\mathfrak{G}) \cap F_r(\mathcal{H}) \nsubseteq F_r(\mathfrak{G} \cap \mathcal{H}) = \emptyset$,

Example (3-2-12) :

In example (3-2-9) if we take , $\text{Ų} = \{\hbar, g\}$ and $\mathcal{K} = \{\hbar\}$, then we have $\text{Ų}_r = \{\hbar, g\}$, $(X - \text{Ų})_r = \{f\}$ so $\text{F}_\text{r}(\text{Ų}) = \emptyset$, also, we have $\mathcal{K}_\text{r} = \{\hbar, f\}$ and $(X - \mathcal{K})_\text{r} = \{g, f\}$ so $\text{F}_\text{r}(\mathcal{K}) = \{f\}$.

Thus $\text{F}_\text{r}(\text{Ų} \cap \mathcal{K}) = \{f\} \nsubseteq \text{F}_\text{r}(\mathcal{K}) \cap \text{F}_\text{r}(\text{Ų}) = \emptyset$.

From these examples above (3-2-11) and (3-2-12) we can conclude that there is no relation between the $r - frontier$ for the intersection of two sets and the intersection of their $r - frontier$.

Example (3-2-13) :

In example (1-3-32) we have $\oint_\text{T}(\mathcal{H}) = \text{X}$, for each non empty subset $\mathcal{H}$ of X , so $\mathcal{H}_r = \text{X}$ and $(X - \mathcal{H})_r = X$ also. Thus $\text{F}_\text{r}(\mathcal{H}) = X$ but by theorem (3-2-7) part (1) we have $\text{F}_\text{r}(\emptyset) = \emptyset$.

Thus, $\oint_{Fr}(\mathcal{H} \cap \text{X}) = \oint_{Fr}(\mathcal{H}) \nsubseteq \oint_{Fr}(\mathcal{H}) \cap \oint_{Fr}(\text{X}) = \emptyset$.

So, it is clear that $\emptyset \subseteq \mathcal{H}$, but $\text{F}_\text{r}(\mathcal{H}) = X \nsubseteq \text{F}_\text{r}(\emptyset) = \emptyset$.

Also, $\text{F}_\text{r}(\mathcal{H}) = X$ but by theorem (3-2-7) part (1) we have $\text{F}_\text{r}(X) = \emptyset$

So, it is clear $\mathcal{H} \subseteq X$, but $\text{F}_\text{r}(\mathcal{H}) = X \nsubseteq \text{F}_\text{r}(X) = \emptyset$.

From this example (3-2-13) we can conclude that if $\text{Ų} \subseteq \mathcal{H}$, then there is no relation between the $r - frontier$ sets for each of them.

Now we will introduce am important theorem that study the relationships between the $r - frontier$ for the union of two sets and their intersections and their difference.

Theorem (3-2-14):

In $i-TPS\ X_{TI}^{\delta}$ and for $\text{Ʉ},\mathcal{K} \subseteq X$ then

$$F_r(\text{Ʉ}) \cup F_r(\mathcal{K}) = F_r(\text{Ʉ}-\mathcal{K}) \cup F_r(\text{Ʉ}\cap\mathcal{K}) \cup F_r(\mathcal{K}-\text{Ʉ}) .$$

Proof: First, we know that

$$\begin{aligned} F_r(\text{Ʉ}\cap\mathcal{K}) &= F_r\big(X-(\text{Ʉ}\cap\mathcal{K})\big) \\ &= F_r((X-\text{Ʉ})\cup(X-\mathcal{K})) \\ &\subseteq F_r(X-\text{Ʉ}) \cup F_r(X-\mathcal{K}) \\ &= F_r(\text{Ʉ}) \cup F_r(\mathcal{K}) \qquad \dots\dots\dots \qquad (1) \end{aligned}$$

Also, $F_r(\text{Ʉ}-\mathcal{K}) = F_r\big(\text{Ʉ}\cap(X-\mathcal{K})\big)$

$$\begin{aligned} &\subseteq F_r(\text{Ʉ}) \cup F_r(X-\mathcal{K}) \\ &= F_r(\text{Ʉ}) \cup F_r(\mathcal{K}) \qquad \dots\dots\dots \qquad (2) \end{aligned}$$

And, $F_r(\mathcal{K}-\text{Ʉ}) \subseteq F_r(\text{Ʉ}) \cup F_r(\mathcal{K}) \qquad \dots\dots\dots \qquad (3).$

Then from (1), (2) and (3) we will get that

$$F_r(\text{Ʉ}-\mathcal{K}) \cup F_r(\text{Ʉ}\cap\mathcal{K}) \cup F_r(\mathcal{K}-\text{Ʉ}) \subseteq F_r(\text{Ʉ}) \cup F_r(\mathcal{K})$$

Further, we have

$$\begin{aligned} F_r(\text{Ʉ}) \cup F_r(\mathcal{K}) &= F_r[(\text{Ʉ}-\mathcal{K}) \cup (\text{Ʉ}\cap\mathcal{K})] \cup F_r[(\mathcal{K}-\text{Ʉ}) \cup (\text{Ʉ}\cap\mathcal{K})] \\ &\subseteq F_r(\text{Ʉ}-\mathcal{K}) \cup F_r[(\text{Ʉ}\cap\mathcal{K})] \cup F_r[(\mathcal{K}-\text{Ʉ})]. \end{aligned}$$

Therefore, we have

$$F_r(\text{Ʉ}) \cup F_r(\mathcal{K}) = F_r(\text{Ʉ}-\mathcal{K}) \cup F_r(\text{Ʉ}\cap\mathcal{K}) \cup F_r(\mathcal{K}-\text{Ʉ}).$$

If we applied theorem (3-2-14) in some way we can get the following theorem as a result

Theorem (3-2-15):

In $i-TPS\ X_{TI}^{\delta}$ and for $\text{Ʉ},\mathcal{K} \subseteq X$, the following are hold :

1) $F_r(\text{Ʉ}) \cup F_r(\mathcal{K}) = F_r(\text{Ʉ}\cap\mathcal{K}) \cup F_r(\text{Ʉ}-\mathcal{K}) \cup F_r(\text{Ʉ}\cup\mathcal{K})$
2) $F_r(\text{Ʉ}) \cup F_r(\mathcal{K}) = F_r(\text{Ʉ}\cup\mathcal{K}) \cup F_r(\mathcal{K}-\text{Ʉ}) \cup F_r(\text{Ʉ}\cap\mathcal{K})$.
3) $F_r(\text{Ʉ}) \cup F_r(\mathcal{K}) = F_r(\text{Ʉ}-\mathcal{K}) \cup F_r(\mathcal{K}-\text{Ʉ}) \cup F_r(\text{Ʉ}\cap\mathcal{K})$.
4) $F_r(\text{Ʉ}) \cup F_r(\text{Ʉ}\Delta\mathcal{K}) = F_r(\text{Ʉ}-\mathcal{K}) \cup F_r(\text{Ʉ}\cap\mathcal{K}) \cup F_r(\mathcal{K}-\text{Ʉ})$
   $= F_r(\text{Ʉ}) \cup F_r(\mathcal{K})$.
5) $F_r(\mathcal{K}) \cup F_r(\text{Ʉ}\Delta\mathcal{K}) = F_r(\text{Ʉ}-\mathcal{K}) \cup F_r(\text{Ʉ}\cap\mathcal{K}) \cup F_r(\mathcal{K}-\text{Ʉ})$.

Proof:

1) If the relation in theorem (3-2-14) takes $X - \mathcal{K}$ instead of $\mathcal{K}$ we will get

$$F_r(U) \cup F_r(X - \mathcal{K})$$
$$= F_r(U - (X - \mathcal{K})) \cup F_r\big(U \cap (X - \mathcal{K})\big) \cup F_r((X - \mathcal{K}) - U).$$

And by theorem (3-2-7) part (6) this will implies,

$$F_r(U) \cup F_r(\mathcal{K}) = F_r(U \cap \mathcal{K}) \cup F_r(U - \mathcal{K}) \cup F_r(U \cup \mathcal{K}).$$

2) If the relation in theorem (3-2-14) takes $X - U$ instead of U we will get

$$F_r(X - U) \cup F_r(\mathcal{K})$$
$$= F_r((X - U) - \mathcal{K}) \cup F_r((X - U) \cap \mathcal{K}) \cup F_r(\mathcal{K} - (X - U)).$$

And by theorem (3-2-7) part (6) this will implies,

$$F_r(U) \cup F_r(\mathcal{K}) = F_r(U \cup \mathcal{K}) \cup F_r(\mathcal{K} - U) \cup F_r(U \cap \mathcal{K}).$$

3) If the relation in theorem (3-2-14) takes $X - U$ instead of U and $X - \mathcal{K}$ instead of $\mathcal{K}$, then we will get

$$F_r(X - U) \cup F_r(X - \mathcal{K}) = F_r\big((X - U) - (X - \mathcal{K})\big)$$
$$\cup F_r((X - U) \cap (X - \mathcal{K})) \cup F_r\big((X - \mathcal{K}) - (X - U)\big)$$

Which implies by theorem (3-2-7) part (6),

$$F_r(U) \cup F_r(\mathcal{K}) = F_r(\mathcal{K} - U) \cup F_r(U \cup \mathcal{K}) \cup F_r(U - \mathcal{K}).$$

4) By theorem (3-1-14) if we have taken $X - U$ instead of U and $X - \mathcal{K}$ instead of $\mathcal{K}$, then we will get

$$F_r(U) \cup F_r(U \Delta \mathcal{K}) =$$
$$= F_r(U - (U \Delta \mathcal{K})) \cup F_r(U \cap (U \Delta \mathcal{K}) \cup F_r((U \Delta \mathcal{K}) - U)$$
$$= F_r(U \cap \mathcal{K}) \cup F_r(U - \mathcal{K}) \cup F_r(\mathcal{K} - U)$$
$$= F_r(U) \cup F_r(\mathcal{K})$$

5) Directly from (4).

Now we can define a new operator F_{op} based on conditions (1), (2) in theorem (3-2-7) and the corollary (3-2-8).

Definition (3-2-16):

In $i-TPS$ X_{TI}^{δ} and for $\mathfrak{U} \subseteq X$ we can define the operator $F_{op}(\mathfrak{U})$ that satisfies the conditions (1), (2) in theorem (3-2-7) and the corollary (3-2-8) such that $F_{op}(\mathfrak{U}) = \mathfrak{U} \cup F_r(\mathfrak{U})$

Example (3-2-17) :

In example (3-2-11) if we take $\mathfrak{G} = \{g\}$, $F_r(\mathfrak{G}) = \{f\}$, Then we have $F_{op}(\mathfrak{G}) = \mathfrak{G} \cup F_r(\mathfrak{G}) = \{g, f\}$.

In the following proposition we will study some properties and relations of the operator F_{op}.

Proposition (3-2-18) :

In $i-TPS$ X_{TI}^{δ} and for $\mathfrak{U} \subseteq X$ then the following statement are true

1) $F_{op}(\emptyset) = \emptyset$;
2) $\mathfrak{U} \subseteq F_{op}(\mathfrak{U})$;
3) $F_{op}\left(F_{op}(\mathfrak{U})\right) = F_{op}(\mathfrak{U})$
4) $F_{op}(\mathfrak{U} \cup \mathcal{K}) = F_{op}(\mathfrak{U}) \cup F_{op}(\mathcal{K})$

Proof:

We can get the proof of (1) and (2) directly by the definition (3-2-16) and theorem (3-2-7) part (1) . Now for the other parts proof we have:

3) $F_{op}\left(F_{op}(\mathfrak{U})\right) = F_{op}\big(\mathfrak{U} \cup F_r(\mathfrak{U})\big)$

$= \big(\mathfrak{U} \cup F_r(\mathfrak{U})\big) \cup F_r\big(\mathfrak{U} \cup F_r(\mathfrak{U})\big)$

$\subseteq \mathfrak{U} \cup F_r(\mathfrak{U}) \cup F_r(\mathfrak{U}) \cup F_r\big(F_r(\mathfrak{U})\big)$

by theorem (3-2-7) part (2)

$= \mathfrak{U} \cup F_r(\mathfrak{U}) \cup F_r(\mathfrak{U})$ by corollary (3-2-8)

$= F_{op}(\mathfrak{U})$

4) $F_{op}(\mathfrak{U} \cup \mathcal{K}) = (\mathfrak{U} \cup \mathcal{K}) \cup F_r(\mathfrak{U} \cup \mathcal{K})$

$\subseteq (\mathfrak{U} \cup \mathcal{K}) \cup \big(F_r(\mathfrak{U}) \cup F_r(\mathcal{K})\big)$

by theorem (3-2-7) part (2)

$= \big(\mathfrak{U} \cup F_r(\mathfrak{U})\big) \cup \big(\mathcal{K} \cup F_r(\mathcal{K})\big)$

$= F_{op}(\mathfrak{U}) \cup F_{op}(\mathcal{K})$

And, $\mathrm{F_{op}}(\mathfrak{U}) \cup \mathrm{F_{op}}(\mathcal{K}) = \big(\mathfrak{U} \cup \mathrm{F_r}(\mathfrak{U})\big) \cup \big(\mathcal{K} \cup \mathrm{F_r}(\mathcal{K})\big)$

$$= (\mathfrak{U} \cup \mathcal{K}) \cup \Big(\big(\mathfrak{U} \cap \mathrm{F_r}(\mathcal{K})\big) \cup \mathrm{F_r}(\mathfrak{U} \cup \mathcal{K}) \cup \big(\mathcal{K} \cap \mathrm{F_r}(\mathfrak{U})\big)\Big)$$

by theorem (3-2-7) part (3)

$$\subseteq (\mathfrak{U} \cup \mathcal{K}) \cup \mathrm{F_r}(\mathfrak{U} \cup \mathcal{K}) = \mathrm{F_{op}}(\mathfrak{U} \cup \mathcal{K}).$$

Now we will define the operator $\mathrm{F_{r_*}}$which passed on redirect set.

Definition (3-2-19):

In $i - TPS\ \mathrm{X}_{TI}^{\delta}$ and for $\mathfrak{U} \subseteq X$ we can define the operator $\mathrm{F_{r_*}}(\mathfrak{U})$ on X for any subset $\mathfrak{U}$ of X, in the following way: $\mathrm{F_{r_*}}(\mathfrak{U}) = \mathfrak{U}_r - \mathfrak{U}$.

Example (3-2-20):

In example (3-2-9) if we take $\mathfrak{G} = \{g\}$, $\mathfrak{G}_\mathrm{r} = \{g, f\}$, Then we have $\mathrm{F_{r_*}}(\mathfrak{G}) = \mathfrak{G}_\mathrm{r} - \mathfrak{G} = \{f\}$.

In the following proposition we will study some properties and relations of the operator $\mathrm{F_{r_*}}$.

Proposition (3-2-21) :

In $i - TPS\ \mathrm{X}_{TI}^{\delta}$ and for $\mathfrak{U}, \mathcal{K} \subseteq X$then the following statement are true

1) $\mathrm{F_{r_*}}(\emptyset) = \emptyset$, $\mathrm{F_{r_*}}(\mathrm{X}) = \emptyset$,
2) $\mathfrak{U} \cap \mathrm{F_{r_*}}(\mathfrak{U}) = \emptyset$;
3) $\mathrm{F_{r_*}}(\mathfrak{U} \cup \mathcal{K}) = \big(\mathrm{F_{r_*}}(\mathfrak{U}) - \mathcal{K}\big) \cup (\mathrm{F_{r_*}}(\mathcal{K}) - \mathfrak{U})$
4) $\mathrm{F_{r_*}}\big(\mathrm{F_{r_*}}(\mathfrak{U})\big) \subseteq \mathfrak{U}$ for any $\mathfrak{U} \in \mathrm{T}$
5) If $\mathrm{F_{r_*}}(\mathfrak{U}) = \emptyset$, then $\mathfrak{U} = \mathrm{i - cl}(\mathfrak{U})$.

Proof:

1) Directly from the definition (3-2-19) and proposition (2-2-4) part (2)
2) It is obvious by definition (3-2-19) .
3) $\mathrm{F_{r_*}}(\mathfrak{U} \cup \mathcal{K}) = (\mathfrak{U} \cup \mathcal{K})_r - (\mathfrak{U} \cup \mathcal{K})$

$$= [\mathfrak{U}_r \cup \mathcal{K}_r] - (\mathfrak{U} \cup \mathcal{K})$$
$$= [(\mathfrak{U}_r - \mathfrak{U}) - \mathcal{K}] \cup [(\mathcal{K}_r - \mathcal{K}) - \mathfrak{U}]$$
$$= \big(\mathrm{F_{r_*}}(\mathfrak{U}) - \mathcal{K}\big) \cup (\mathrm{F_{r_*}}(\mathcal{K}) - \mathfrak{U}).$$

4) $F_{r_*}\left(F_{r_*}(\text{Ų})\right) = \left(F_{r_*}(\text{Ų})\right)_r - F_{r_*}(\text{Ų}) = (\text{Ų}_r - \text{Ų})_r - (\text{Ų}_r - \text{Ų})$
$\subseteq (\text{Ų}_r)_r - (\text{Ų}_r - \text{Ų})$
$\subseteq \text{Ų}_r - (\text{Ų}_r - \text{Ų}) = \text{Ų}$

by proposition (2-2-21) part (3) and theorem (1-3-3) part (1).

5) If $F_{r_*}(\text{Ų}) = \emptyset$, then $\text{Ų}_r - \text{Ų} = \emptyset$, which implies $\text{Ų}_r = \text{Ų}$, but $\text{Ų} \subseteq i - cl(\text{Ų})$, so by theorem (2-2-23) part (1) we will get that $\text{Ų}_r = \text{Ų} \subseteq i - cl(\text{Ų}) \subseteq \text{Ų}_r$, and that complete the proof .

Remark (3-2-22):

There is no subset $\text{Ų} \subseteq X$ satisfies that $F_{r_*}(\text{Ų}) = X$, if there exist such subset then $\text{Ų}_r - \text{Ų} = X$, which means $\text{Ų} = \emptyset$, but $\text{Ų}_r = \emptyset_r = \emptyset$.

Theorem (3-2-23):

In $i - TPS$ X_{TI}^{δ} and for $\text{Ų}, \mathcal{K} \subseteq X$then the following statement are true

1) $F_{r_*}(\text{Ų}) \cup F_{r_*}(\mathcal{K}) = \left(\text{Ų} \cap F_{r_*}(\mathcal{K})\right) \cup F_{r_*}(\text{Ų} \cup \mathcal{K}) \cup (F_{r_*}(\text{Ų}) \cap \mathcal{K})$
2) $F_{r_*}(\text{Ų}_r) = \emptyset$ for any $\text{Ų} \in T$.

Proof:

1) By proposition (3-2-21) part (3) we have

$$\left(\text{Ų} \cap F_{r_*}(\mathcal{K})\right) \cup F_{r_*}(\text{Ų} \cup \mathcal{K}) \cup \left(F_{r_*}(\text{Ų}) \cap \mathcal{K}\right)$$

$$= \left(\text{Ų} \cap F_{r_*}(\mathcal{K})\right) \cup \left[\left(F_{r_*}(\text{Ų}) - \mathcal{K}\right) \cup \left(F_{r_*}(\mathcal{K}) - \text{Ų}\right)\right] \cup \left(F_{r_*}(\text{Ų}) \cap \mathcal{K}\right)$$

$$= \left(\text{Ų} \cap F_{r_*}(\mathcal{K})\right) \cup \left[\left(F_{r_*}(\text{Ų}) \cap (X - \mathcal{K})\right) \cup \left(F_{r_*}(\mathcal{K}) \cap (X - \text{Ų})\right)\right] \cup \left(F_{r_*}(\text{Ų}) \cap \mathcal{K}\right)$$

$$= \left(\text{Ų} \cap F_{r_*}(\mathcal{K})\right) \cup \left[F_{r_*}(\mathcal{K}) \cap (X - \text{Ų})\right] \cup \left(F_{r_*}(\text{Ų}) \cap (X - \mathcal{K})\right) \cup \left(F_{r_*}(\text{Ų}) \cap \mathcal{K}\right)$$

$$= F_{r_*}(\text{Ų}) \cup F_{r_*}(\mathcal{K}).$$

2) $F_{r_*}(\text{Ų}_r) = (\text{Ų}_r)_r - \text{Ų}_r$, but $\text{Ų} \in T$ so by proposition (2-2-21) part (3) and theorem (1-3-3) part (1) we have $(\text{Ų}_r)_r = \text{Ų}_r$.
Thus $F_{r_*}(\text{Ų}_r) = \text{Ų}_r - \text{Ų}_r = \emptyset$

Now we can define the operator $\mathrm{F}_{\mathrm{s}_*}$ on an $i-TPS, X$ for any subset Ա of X, based on the strips set of Ա in the following definition

Definition (3-2-24):

In $i-TPS\ \mathrm{X}_{TI}^{\delta}$ and for $\mathrm{U} \subseteq X$ we can define the operator $\mathrm{F}_{\mathrm{s}_*}(\mathrm{U})$ on X for any subset U of X, in the following way: $\mathrm{F}_{\mathrm{s}_*}(\mathrm{U}) = \mathrm{U} - \mathrm{U}_s$.

Example (3-2-25):

In example (2-2-8) we have $\mathrm{U} = \{\hbar, f\}$, $\mathrm{U}_{\mathrm{s}} = \{\hbar\}$, Then we have $\mathrm{F}_{\mathrm{s}_*}(\mathrm{U}) = \mathrm{U} - \mathrm{U}_{\mathrm{s}} = \{f\}$.

In the following theorem we will study some properties and relations of the operator $\mathrm{F}_{\mathrm{s}_*}$.

Theorem (3-2-26):

In $i-TPS\ \mathrm{X}_{TI}^{\delta}$ and for $\mathrm{U}, \mathcal{K} \subseteq X$, then the following are hold

1) $\mathrm{F}_{\mathrm{s}_*}(\emptyset) = \emptyset$, $\mathrm{F}_{\mathrm{s}_*}(\mathrm{X}) = \emptyset$.
2) $\mathrm{F}_{\mathrm{s}_*}(\mathrm{U}) \subseteq \mathrm{U}$;
3) $\mathrm{F}_{\mathrm{s}_*}(\mathrm{U} \cap \mathcal{K}) \subseteq \left(\mathrm{F}_{\mathrm{s}_*}(\mathrm{U}) \cap \mathcal{K}\right) \cup (\mathrm{F}_{\mathrm{s}_*}(\mathcal{K}) \cap \mathrm{U})$
4) $\mathrm{F}_{\mathrm{s}_*}\left(\mathrm{F}_{\mathrm{s}_*}(\mathrm{U})\right) \subseteq \mathrm{F}_{\mathrm{s}_*}(\mathrm{U})$.
5) $\mathrm{F}_{\mathrm{s}_*}(\mathrm{U}) \subseteq \mathrm{F}_{\mathrm{r}}((\mathrm{U})$
6) $\mathrm{F}_{\mathrm{s}_*}(\mathrm{U}) \cap \mathrm{F}_{\mathrm{r}_*}(\mathrm{U}) = \emptyset$.

Proof:

1) Directly from the definition (3-2-24) and theorem (2-2-10) part (2)
2) It is obvious by the definition (3-2-24).
3) $\mathrm{F}_{\mathrm{s}_*}(\mathrm{U} \cap \mathcal{K}) = (\mathrm{U} \cap \mathcal{K}) - (\mathrm{U} \cap \mathcal{K})_s$

$$= (\mathrm{U} \cap \mathcal{K}) \cap (\mathrm{X} - (\mathrm{U} \cap \mathcal{K})_s)$$
$$\subseteq (\mathrm{U} \cap \mathcal{K}) \cap \left(\mathrm{X} - (\mathrm{U}_s \cap \mathcal{K}_s)\right)$$
$$= (\mathrm{U} \cap \mathcal{K}) \cap (\mathrm{X} - \mathrm{U}_s) \cup (X - \mathcal{K}_s)$$
$$= (\mathrm{U} \cap (\mathrm{X} - \mathrm{U}_s) \cap \mathcal{K}) \cup \left(\mathrm{U} \cap \mathcal{K} \cap (X - \mathcal{K}_s)\right)$$
$$= \left((\mathrm{U} - \mathrm{U}_s) \cap \mathcal{K}\right) \cup \left(\mathrm{U} \cap (\mathcal{K} - \mathcal{K}_s)\right)$$
$$= \left(\mathrm{F}_{\mathrm{s}_*}(\mathrm{U}) \cap \mathcal{K}\right) \cup (\mathrm{F}_{\mathrm{s}_*}(\mathcal{K}) \cap \mathrm{U}).$$

4) $F_{s_*}\left(F_{s_*}(\text{U})\right) = F_{s_*}(\text{U}) - \left(F_{s_*}(\text{U})\right)_s$
$= (\text{U} - \text{U}_s) - (\text{U} - \text{U}_s)_s$
$= (\text{U} - \text{U}_s) - (\text{U} \cap (X - \text{U}_s)_s$
$\subseteq (\text{U} - \text{U}_s) - \left(\text{U}_s \cap (X - \text{U}_s)\right)_s$
$= (\text{U} - \text{U}_s) - \emptyset_s = (\text{U} - \text{U}_s) - \emptyset$
$= F_{s_*}(\text{U})$

5) Let $k \in F_{s_*}(\text{U})$, then $k \in \text{U} - \text{U}_s, i.e. k \in \text{U}$ and $k \notin \text{U}_s$. Since $k \in \text{U}$ then by theorem (2-2-4) part (1) we get $k \in \text{U}_r$ and since $k \notin \text{U}_s$ then $k \in X - \text{U}_s = (X - \text{U})_r$ by proposition (2-2-19) part (2). Thus, $k \in \text{U}_r \cap (X - \text{U})_r = F_r(\text{U})$.

6) $F_{s_*}(\text{U}) \cap F_{r_*}(\text{U}) = (\text{U} - \text{U}_s) \cap (\text{U}_r - \text{U})$
$= \left(\text{U} \cap (X - \text{U}_s)\right) \cap \left(\text{U}_r \cap (X - \text{U})\right)$
$= (\text{U} \cap (X - \text{U})_r) \cap \left(\text{U}_r \cap (X - \text{U})\right)$
$= \left(\text{U} \cap (X - \text{U})\right) \cap (\text{U}_r \cap (X - \text{U})_r) = \emptyset.$

Now we can study the notion of redirect frontier in $i - subspace$ $Y^{\delta_Y}_{T_Y I_Y}$ of $i - TPS\ X^{\delta}_{TI}$ and investigate some of its properties and relations

Definition (3-2-27)

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ X^{\delta}_{TI}$ then we can define the redirect frontier set w. r. t. $i - subspace\ Y^{\delta_Y}_{T_Y I_Y}$ of a subset $\mathcal{H}$ of Y as the following

$$F_{rY}(\mathcal{H}) = \left(\mathcal{H}_{r_Y}\right) \cap (X - \mathcal{H})_{r_Y}.$$

The point $h \in Y$ is said to be redirect frontier point of $\mathcal{H}$ w. r. t. $Y^{\delta_Y}_{T_Y I_Y}$ if $h \in F_{r_Y}(\mathcal{H})$, and we denoted by $F_{r_Y} - frontier$ point.

Example (3-2-28):

Let $X = \{h, g, f\}$, $T = \{X, \emptyset, \{h\}, \{g\}\}$, and $I = \{\emptyset, \{f\}\}$ with δ_D, and let $Y = \{h, f\}$ then $T_Y = \{Y, \emptyset, \{h\}\}$ and $I_Y = \{\emptyset, \{f\}\}$. Now, if $\mathcal{H} = \{h\}$, then $\mathcal{H}_{r_Y} = Y$ and $(Y - \mathcal{H})_{r_Y} = \{f\}$.

So $F_{r_Y}(\mathcal{H}) = \mathcal{H}_{r_Y} \cap (Y - \mathcal{H})_{r_Y} = Y \cap \{f\} = f.$

One of the results we can discus is the relation between $F_r - frontier$ set in X and $F_{r_Y} - frontier$ which shown in the next theorem.

Proposition (3-2-29)

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ X^{\delta}_{TI}$, and let $\mathfrak{U} \subseteq Y$. Then

$$F_{r_Y}(\mathfrak{U}) \subseteq F_{r_X}(\mathfrak{U}) \cap Y.$$

Proof: By proposition (2-2-29) we have that $\mathfrak{U}_{r_Y} \subseteq (\mathfrak{U}_{r_X}) \cap Y$, so we have $(Y - \mathfrak{U})_{r_Y} \subseteq (X - \mathfrak{U})_{r_X} \cap Y$.

Now, $F_{r_Y}(\mathfrak{U}) = \mathfrak{U}_{r_Y} \cap (Y - \mathfrak{U})_{r_Y}$

$$\subseteq (\mathfrak{U}_{r_X} \cap Y) \cap ((X - \mathfrak{U})_{r_X} \cap Y)$$

$$= (\mathfrak{U}_{r_X} \cap (X - \mathfrak{U})_{r_X}) \cap Y$$

$$= F_{r_X}(\mathfrak{U}) \cap Y$$

Corollary (3-2-30)

Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ X^{\delta}_{TI}$, and let $\mathfrak{U} \subseteq Y$, Then $F_{r_Y}(\mathfrak{U}) = F_{r_X}(\mathfrak{U}) \cap Y$ if and only if X^{δ}_{TI} have intuition smoothing feature.

Proof: Since X^{δ}_{TI} have intuition smoothing feature, then by corollary (2-2-30) we have $\mathfrak{U}_{r_Y} = (\mathfrak{U}_{r_X}) \cap Y$, so we get $(Y - \mathfrak{U})_{r_Y} = (X - \mathfrak{U})_{r_X} \cap Y$.

So, $F_{r_Y}(\mathfrak{U}) = \mathfrak{U}_{r_Y} \cap (Y - \mathfrak{U})_{r_Y}$

$$= (\mathfrak{U}_{r_X} \cap Y) \cap ((X - \mathfrak{U})_{r_X} \cap Y)$$

$$= (\mathfrak{U}_{r_X} \cap (X - \mathfrak{U})_{r_X}) \cap Y$$

$$= F_{r_X}(\mathfrak{U}) \cap Y .$$

Our aim now is to introduce the notation of the redirect outer function and study some of its properties.

Definition (3-2-31)

In $i-TPS$ X^{δ}_{TI} and for $Џ \subseteq X$, the redirect outer function of a subset Џ of X defined as the following

$$O_r(Џ) = \{\mathscr{k} \in X;\ \exists\, \mathcal{H} \in T(\mathscr{k}) : \mathcal{H}\bar{\delta}Џ_r\}.$$

Example (3-2-32):

Let $X = \{\mathscr{h}, \mathscr{g}, \mathscr{f}\}$, $T = \{X, \emptyset, \{\mathscr{h}\}, \{\mathscr{g}\}\}$, and $I = \{\emptyset, \{\mathscr{f}\}\}$ with the discrete proximity, if $Џ = \{\mathscr{h}, \mathscr{f}\}$ then $O_r(Џ) = \{\mathscr{g}\}$.

We can see the relation between redirect outer function, strips set and redirect set in the following proposition

Proposition (3-2-33):

In $i-TPS$ X^{δ}_{TI} and for $Џ \subseteq X$ then $O_r(Џ) \subseteq (X - Џ)_s = X - Џ_r$.

Proof: Let $\mathscr{k} \in O_r(Џ)$, then $\exists\, \mathcal{H} \in T(\mathscr{k}) : \mathcal{H}\bar{\delta}Џ_r$, so by theorem (2-2-4) part (3) we have $\mathscr{f}_T(Џ) \subseteq Џ_r$ which implies that $\mathcal{H}\bar{\delta}\,\mathscr{f}_T(Џ)$ by proposition (1-2-3) part (1). Thus $\mathscr{k} \in (X - Џ)_s$.

In the following proposition we will study some properties and relations of redirect outer function

Proposition (3-2-34):

In $i-TPS$ X^{δ}_{TI} and for $Џ, \mathcal{K} \subseteq X$ then the following statement are hold.

1) $O_r(\emptyset) = X$ and $O_r(X) = \emptyset$
2) $O_r(O_r(Џ)) \subseteq (Џ_r)_s$
3) If $Џ \subseteq \mathcal{K}$ then $O_r(\mathcal{K}) \subseteq O_r(Џ)$
4) $O_r(Џ_r) \subseteq O_r(Џ)$
5) $O_r(Џ \cup \mathcal{K}) \subseteq (X - Џ_r) \cap (X - \mathcal{K}_r) = (X - Џ)_s \cap (X - \mathcal{K})_s$
6) $O_r(Џ \cup \mathcal{K}) \subseteq O_r(Џ) \cap O_r(\mathcal{K}) \subseteq O_r(Џ \cap \mathcal{K})$

Proof:

1) It is clear by definition that $O_r(\emptyset) = X$,
   Now for $O_r(X) \subseteq (X - X_r) = X - X = \emptyset$
2) $O_r(O_r(Џ)) \subseteq O_r((X - Џ)_s) \subseteq (X - (X - Џ)_s)_s$
   $= (X - (X - Џ_r))_s = (Џ_r)_s$

3) Since $\mathrm{U} \subseteq \mathcal{K}$, then by theorem (2-2-4) part (4) we get $\mathrm{U_r} \subseteq \mathcal{K}_r$, so for $\mathscr{k} \in O_r(\mathcal{K})$, $\exists\, \mathcal{H} \in T(\mathscr{k}): \mathcal{H}\bar{\delta}\mathcal{K}_\mathrm{r}$, but $\mathrm{U_r} \subseteq \mathcal{K}_r$, then by proposition (1-2-3) part (1)we get $\mathcal{H}\bar{\delta}\mathrm{U_r}$, hence $\mathscr{k} \in O_r(\mathrm{U})$
4) Since $\mathrm{U} \subseteq \mathrm{U_r}$ by theorem (2-2-4) part (1) then by (3) we will get $O_r(\mathrm{U_r}) \subseteq O_r(\mathrm{U})$.
5) $O_r(\mathrm{U} \cup \mathcal{K}) \subseteq X - (\mathrm{U} \cup \mathcal{K})_r$
   $= X - (\mathrm{U_r} \cup \mathcal{K}_r)$
   $= (X - \mathrm{U_r}) \cap (\mathrm{X} - \mathcal{K}_r)$.
6) Since $\mathrm{U} \cap \mathcal{K} \subseteq \mathrm{U}, \mathcal{K}$, so by (3) we get $O_r(\mathrm{U}) \subseteq O_r(\mathrm{U} \cap \mathcal{K})$ and $O_r(\mathcal{K}) \subseteq O_r(\mathrm{U} \cap \mathcal{K})$, then $O_r(\mathrm{U}) \cap O_r(\mathcal{K}) \subseteq O_r(\mathrm{U} \cap \mathcal{K})$.
   Now, let $\mathscr{k} \in O_r(\mathrm{U} \cup \mathcal{K})$, $\exists\, \mathcal{H} \in T(\mathscr{k}): \mathcal{H}\bar{\delta}(\mathrm{U} \cup \mathcal{K})_\mathrm{r} = \mathrm{U_r} \cup \mathcal{K}_r$, by theorem (2-2-4) part (8), then $\mathcal{H}\bar{\delta}\mathrm{U_r}$ and $\mathcal{H}\bar{\delta}\mathcal{K}_\mathrm{r}$.
   Therefore, $\mathscr{k} \in O_r(\mathrm{U})$ and $\mathscr{k} \in O_r(\mathcal{K})$, then $\mathscr{k} \in O_r(\mathrm{U}) \cap O_r(\mathcal{K})$.
   Thus $O_r(\mathrm{U} \cap \mathcal{K}) \subseteq O_r(\mathrm{U}) \cap O_r(\mathcal{K})$, and this complete the proof.

Chapter Four

Resolvable i-Topological Proximity Spaces

We devote this chapter to introduce the concept of resolvable space in the $i-$topological proximity space through the concepts that were studied in the previous chapters, the most important of which is the concept of occlusion set via $i-open$ set and the concept of redirected set. We also focused on studying some of the characteristics and relationships related to these spaces.

4.1 Occlusion Resolvable Space via $i-open$ set

In this section, we introduce the notion of occlusion resolvable space via $i-open$ in $i-$toplogical proximity space and investigation some of the features and relationships of this space

Definition (4-1-1):

The $i-TPS$ X_{TI}^{δ} is called occlusion resolvable space via $i-open$ set if and only if there exist two $\oint_{\mathrm{T}}-dense$ subsets $\text{Ų},\mathcal{K}$ s. t. $\text{Ų}\bar{\delta}\mathcal{K}$ and $(X-\mathcal{K})\delta\text{Ų}$ and $X=\text{Ų}\cup\mathcal{K}$. We will denote it by $\oint_{\mathrm{T}}-resolvable$

Example (4-1-2) :

Let $X=\{\hbar,g,f\}$, $T=\{\mathrm{X},\emptyset,\{\hbar,g\},\{\hbar,f\}\}$, and $\mathrm{I}=\{\emptyset,\{f\},\}$ with the discrete proximity δ_D, if $\text{Ų}=\{\hbar\}$ then $\oint_{\mathrm{T}}(\text{Ų})=\oint_{\mathrm{T}}(\{\hbar\})=\mathrm{X}$ and let $\mathcal{K}=\{g,f\}$, then $\oint_{\mathrm{T}}(\mathcal{K})=\oint_{\mathrm{T}}(\{g,f\})=X$, so $\text{Ų},\mathcal{K}$, are $\oint_{\mathrm{T}}-dense$ sets and $\text{Ų}\bar{\delta}\mathcal{K}$, $(X-\mathcal{K})\delta\text{Ų}$. Thus X_{TI}^{δ} is $\oint_{\mathrm{T}}-resolvable.$

Some properties of $\oint_{\mathrm{T}}-resolvable$ space have been studied in the following proposition.

Proposition (4-1-3):

Let $X_{T_jI_j}^{\delta_j}$ be $i-TPS's$ for $j = 1,2$ s. t. $T_1 \subseteq T_2, I_1 \subseteq I_2$ and $\delta_1 < \delta_2$, then

1) If X is $\oint_{T_2} - resolvable$ then X is $\oint_{T_1} - resolvable$
2) If X is $\oint_{T_{I_1}} - resolvable$ then X is $\oint_{T_{I_2}} - resolvable$
3) If X is $\oint_{T_{\delta_1}} - resolvable$ then X is $\oint_{T_{\delta_2}} - resolvable$

Proof:

1) Let X be $\oint_{T_2} - resolvable$, then $\exists A_1, A_2$, $\oint_{T_2} - dense$ sets s. t. $A_1 \bar{\delta} A_2$, $(X - A_1)\delta A_2$ and $X = A_1 \cup A_2$. But, by theorem (1-3-33) part (15) we have $\oint_{T_2}(A) \subseteq \oint_{T_1}(A)$, so $\oint_{T_1}(A_1) = \oint_{T_1}(A_2) = X$, which means that A_1, A_2, $\oint_{T_1} - dense$ sets. Thus X is $\oint_{T_1} - resolvable$
2) Let X be $\oint_{T_{I_1}} - resolvable$, then $\exists A_1, A_2$, $\oint_{T_{I_1}} - dense$ sets s. t. $A_1 \bar{\delta} A_2$, $(X - A_1)\delta A_2$ and $X = A_1 \cup A_2$. But, by theorem (2-1-1) part (1) we have $\oint_{T_{I_1}}(A) \subseteq \oint_{T_{I_2}}(A)$, so A_1 and A_2 are $\oint_{T_{I_2}} - dense$ sets. Thus X is $\oint_{T_{I_2}} - resolvable$
3) Let X be $\oint_{T_{\delta_1}} - resolvable$, then $\exists A_1, A_2$, $\oint_{T_{\delta_1}} - dense$ sets s. t. $A_1 \overline{\delta_1} A_2$, $(X - A_1)\delta_1 A_2$ and $X = A_1 \cup A_2$. But, by theorem (2-1-1) part (2) we have $\oint_{T_{\delta_1}}(A) \subseteq \oint_{T_{\delta_2}}(A)$, so A_1 and $A_2$are $\oint_{T_{\delta_2}} - dense$ sets, s. t. $A_1 \overline{\delta_2} A_2$, $(X - A_1)\delta_2 A_2$. Thus X is $\oint_{T_{\delta_2}} - resolvable$

Proposition (4-1-4) :

In $\oint_T - resolvable$ $i - TPS$ X_{TI}^{δ} there exist a subset $A \subseteq X$ satisfy $\psi_T(A) = \psi_T(X - A) = \emptyset$

Proof: Let X_{TI}^{δ} be $∯_T - resolvable \ \ i - TPS$, then there exist two $∯_T - dense$ subset $A_1, A_2 \subseteq X \ s.t. X = A_1 \cup A_2, A_1 \bar{\delta} A_2, \ (X - A_1) \delta A_2$ which means $A_1 \cap A_2 = \emptyset$. Then $X - ∯_T(A_i) = \emptyset$, for $i = 1,2,$ but by proposition (1-3-37) part (6) we have $X - ∯_T(A_i) = \psi_T(X - A_i)$. Thus $\psi_T(X - A_i) = \emptyset$, and this complete the proof.

Now, we will introduce the definition of the $∯_T - resolvable$ subset concept in $i - TPS$.

Definition (4-1-5):

In $i - TPS \ X_{TI}^{\delta}$ and for $A \subseteq X$ we say that A is $∯_T - resolvable$ subsets if and only if there exist two subsets $Ų, \mathcal{K}$ s. t. $A \subseteq ∯_T(Ų)$ and $A \subseteq ∯_T(\mathcal{K})$, $A = Ų \cup \mathcal{K}$ and $Ų \cap \mathcal{K} = \emptyset$.

Example (4-1-6) :

Let $X = \{ \hbar, g, f \}$, $T = \{X, \emptyset, \{\hbar, g\}, \{\hbar, f\}\}$, and $I = \{\emptyset, \{f\},\}$ with the discrete proximity δ_D, let $A = \{\hbar, g\}$so if $Ų = \{\hbar\}$ then we have $∯_T(Ų) = ∯_T(\{\hbar\}) = X$ and if $\mathcal{K} = \{g\}$, then $∯_T(\mathcal{K}) = ∯_T(\{g\}) = X$, so $Ų \cap \mathcal{K} = \emptyset$ and $A = Ų \cup \mathcal{K}$ s. t. $A \subseteq ∯_T(Ų)$ and $A \subseteq ∯_T(\mathcal{K})$. Thus A is $∯_T - resolvable$ subset.

In the following proposition we will see that the union of $∯_T - resolvable$ subsets is also $∯_T - resolvable$ subset.

Proposition (4-1-7) :

The union of any collection of $∯_T - resolvable$ subset is $∯_T - resolvable$

Proof: Let $\{Ꞩ_\lambda, \lambda \in \Lambda\}$ be a family of $\oint_T - resolvable$ subsets in $i - TPS$ X^{δ}_{TI}. Then $\forall \lambda \in \Lambda, \exists\ A_{1_\lambda}, A_{2_\lambda}\ s.t. A_{1_\lambda} \cap A_{2_\lambda} = \emptyset,\ Ꞩ_\lambda = A_{1_\lambda} \cup A_{2_\lambda}$ s. t. $Ꞩ_\lambda \subseteq \oint_T(A_{1_\lambda})\ and\ Ꞩ_\lambda \subseteq \oint_T(A_{2_\lambda})$.

Therefore, by proposition (1-3-33) part (3) proposition (2-1-6) , and definition (2-1-5) we will get $\bigcup_{\lambda\in\Lambda} Ꞩ_\lambda \subseteq \bigcup_{\lambda\in\Lambda} \oint_T(A_{1_\lambda}) \subseteq \oint_T(\bigcup_{\lambda\in\Lambda} A_{1_\lambda})$

, $\bigcup_{\lambda\in\Lambda} Ꞩ_\lambda \subseteq \bigcup_{\lambda\in\Lambda} \oint_T(A_{2_\lambda}) \subseteq \oint_T(\bigcup_{\lambda\in\Lambda} A_{2_\lambda})$

and $\bigcup_{\lambda\in\Lambda} Ꞩ_\lambda = (\bigcup_{\lambda\in\Lambda} A_{1_\lambda}) \cup (\bigcup_{\lambda\in\Lambda} A_{2_\lambda})$.

This gives us that $\bigcup_{\lambda\in\Lambda} Ꞩ_\lambda$ is $\oint_T - resolvable$ subsets in X.

Definition (4-1-8) :

In $i - TPS\ X^{\delta}_{TI}$, a point $\mathscr{k} \in X$ is called has empty $\psi_T - tightness$ if $\forall\ \mathscr{k} \in \oint_T(Ꞩ), \exists D \subseteq Ꞩ$ s.t. $\mathscr{k} \in \oint_T(D)\ and\ \psi_T(D) = \emptyset$.

Example (4-1-9) :

In $i - TPS\ (X, T_i, \{\emptyset\}, \delta_D)$ then $\oint_T(A) = X$ and $\psi_T(X - A) = \emptyset$,for each subset A of X , so $\oint_T(X - A) = X$ and $\psi_T(A) = \emptyset$. Now let $A \subseteq Ꞩ$, then $\forall\ \mathscr{k} \in \oint_T(Ꞩ), \exists A \subseteq Ꞩ$ s.t. $\mathscr{k} \in \oint_T(A)\ and\ \psi_T(A) = \emptyset$.

The following property shows the relationship between the empty $\psi_T - tightness$ and $\oint_T - resolvable$ space.

Proposition (4-1-10) :

Any $\oint_T - resolvable\ i - TPS\ X^{\delta}_{TI}$ has empty $\psi_T - tightness$

Proof: Let X^{δ}_{TI} be $\oint_T - resolvable\ \ i - TPS$, then there exist two $\oint_T -\ dense$ subsets $Ꞩ, \mathcal{K} \subseteq X$ s.t. $X = Ꞩ \cup \mathcal{K}$ and $Ꞩ\bar{\delta}\mathcal{K}$, $(X - \mathcal{K})\delta Ꞩ$, i.e., $\text{Ʉ} \cap \mathcal{K} = \emptyset$, then $X - \mathcal{K} = \text{Ʉ}$ and $\oint_{\mathrm{T}}(\text{Ʉ}) = \oint_{\mathrm{T}}(\mathcal{K}) = X$, and by proposition (1-3-37) part (6) that implies $\psi_T(\text{Ʉ}) = \psi_T(\mathcal{K}) = \emptyset$.

Now let $A \subseteq \mathrm{X}$ and by using theorem (4-1-4) and for any $\mathscr{k} \in \oint_{\mathrm{T}}(\mathrm{A})$, $\psi_T(A \cap \mathcal{K}) = \emptyset$ and $\psi_T(A \cap \text{Ʉ}) = \emptyset$. Since

$A = A \cap X = A \cap (\text{Ʉ} \cup \mathcal{K}) = (A \cap \text{Ʉ}) \cup (\mathrm{A} \cap \mathcal{K})$ and by part (5) of proposition (1-3-33) we will get $\mathscr{k} \in \oint_{\mathrm{T}}(\mathrm{A}) = \oint_{\mathrm{T}}(A \cap \text{Ʉ}) \cup \oint_{\mathrm{T}}(\mathrm{A} \cap \mathcal{K})$,

so, either $\mathscr{k} \in \oint_{\mathrm{T}}(A \cap \text{Ʉ})$ or $\mathscr{k} \in \oint_{\mathrm{T}}(\mathrm{A} \cap \mathcal{K})$.

Thus $\mathscr{k}$ has empty $\psi_T - tightness$.

Definition (4-1-11) :

In $i - TPS\ X_{TI}^{\delta}$, then X is called $\oint_{\mathrm{T}} - submaximal$ if every $\oint_{\mathrm{T}} - dense$ set is $i - open$ set.

Definition (4-1-12) :

In $i - TPS\ X_{TI}^{\delta}$, then X is called $\oint_{\mathrm{T}} - hyper\ connected$ if and only if every nonempty $i - open$ set is $\oint_{\mathrm{T}} - dense$ set

Example (4-1-13) :

Let $X = \{\mathscr{h}, \mathscr{g}, \mathscr{f}\}$, $T = \{\mathrm{X}, \emptyset, \{\mathscr{h}, \mathscr{g}\}, \{\mathscr{h}, \mathscr{f}\}\}$, and $\mathrm{I} = \{\emptyset, \{\mathscr{f}\},\}$ with the discrete proximity δ_D, then we have $\oint_{\mathrm{T}}(\mathrm{X}) = X$, $\oint_{\mathrm{T}}(\{\mathscr{h}, \mathscr{g}\}) = \mathrm{X}$ and $\oint_{\mathrm{T}}(\{\mathscr{h}, \mathscr{f}\}) = X$, so every nonempty $i - open$ set is $\oint_{\mathrm{T}} - dense$ set. Thus X is $\oint_{\mathrm{T}} - hyper\ connected$.

Proposition (4-1-14) :

For any $\mathscr{k} \in X_{TI}^{\delta}$ and for any subset $A \subseteq \mathrm{X}$, if $\mathscr{k} \notin A$ and A is $\oint_{\mathrm{T}} - dense$, then $\{\mathscr{k}\}$ is not $i - open$.

Proof: Let A be $\oint_{\mathrm{T}} - dense$ subset in X_{TI}^{δ}, so $\oint_{\mathrm{T}}(A) = X$, i.e. $\forall\, x \in X\,, and\; \forall H \in T(x)$ then $H\delta A$ … (4.1) .

If $\{\mathscr{k}\}$ be $i - open$ then by (4.1) we have $\{\mathscr{k}\}\delta A$ … (4.2).

Since $\mathscr{k} \notin A$, then $A \subseteq \mathrm{X} - \{\mathscr{k}\}$ since $\{\mathscr{k}\}$ is $i - open$ then by (1-2-9) part (13) $\{\mathscr{k}\}\bar{\delta}(\mathrm{X} - \{\mathscr{k}\})$. Thus $\{\mathscr{k}\}\bar{\delta}A$ which a contradiction.

Recall that a topological space is said to be a door space if every subset is open or closed **[16]** so in $i - TPS$ we can define this concept by $i - door$ space if every subset is $i - open$ or $i - closed$.

Theorem (4-1-15):

If X_{TI}^{δ} is $\oint_{\mathrm{T}} - resolvabe$ and $i - door\; i - TPS$ space, then X_{TI}^{δ} is T_1.

Proof: Let X_{TI}^{δ} be $\oint_{\mathrm{T}} - resolvabe\; i - TPS$ space, then there exist two $\oint_{\mathrm{T}} - dense$ subsets $\mathrm{Џ}, \mathcal{K} \subseteq \mathrm{X}$ s.t. $X = \mathrm{Џ} \cup \mathcal{K}$ and $\mathrm{Џ}\bar{\delta}\mathcal{K}\,, (X - \mathcal{K})\delta\mathrm{Џ}$, so $\forall \mathscr{k} \in X$, either $\mathscr{k} \in \mathrm{Џ}$ then $\mathscr{k} \notin \mathcal{K}$, therefore by theorem (4-1-15) we get $\{\mathscr{k}\}$ is not $i - open$, but X_{TI}^{δ} is $i - door$ space, so $\{\mathscr{k}\}$ is $i - closed$.

Similarly, if $\mathscr{k} \in \mathcal{K}$ so $\mathscr{k} \notin \mathrm{Џ}$, then $\{\mathscr{k}\}$ is $i - closed$. Thus X_{TI}^{δ} is T_1 space.

In the next theorem, we will see the condition that makes the $i - TPS$ is $\oint_{\mathrm{T}} - irresolvabe$.

Theorem (4-1-16):

The $i - TPS\; X_{TI}^{\delta}$ is $\oint_{\mathrm{T}} - irresolvabe$ if there is no $\oint_{\mathrm{T}} - dense$ set D which $X - D$ is also $\oint_{\mathrm{T}} - dense$ set.

Proof: Suppose that X_{TI}^{δ} is $\oint_{\mathrm{T}} - resolvabe\ i - TPS$ space, then there exist two $\oint_{\mathrm{T}} - dense$ subsets $\mathrm{U}, \mathcal{K} \subseteq \mathrm{X}$ s.t. $X = \mathrm{U} \cup \mathcal{K}$ and $\mathrm{U}\bar{\delta}\mathcal{K}$, $(X - \mathcal{K})\delta\mathrm{U}$, i.e., $\mathrm{U} \cap \mathcal{K} = \emptyset$ which implies $\mathrm{U} = X - \mathcal{K}$, then $\oint_{\mathrm{T}}(\mathrm{U}) = \oint_{\mathrm{T}}(\mathrm{X} - \mathcal{K}) = X$, which a contradiction.

To study the $\oint_{Fr} - frontier$ set and the occlusion outer function via $i - open$ set in $\oint_{\mathrm{T}} - resolvabe$ we can see the following proposition

Proposition (4-1-17):

If X_{TI}^{δ} be $\oint_{\mathrm{T}} - resolvable\ i - TPS$ and for a subset $\mathrm{U} \subseteq X$ then

1) $\oint_{\mathrm{Fr}}(\mathrm{U}) = X$ for some $\mathrm{U} \subseteq X$
2) $\psi_T(\mathrm{U}) = \emptyset$ and $O_T(\mathrm{U}) = \emptyset$

Proof: Since X_{TI}^{δ} is $\oint_{\mathrm{T}} - resolvable\ i - TPS$, then there exist two $\oint_{\mathrm{T}} - dense$ subsets $\mathrm{U}, \mathcal{K} \subseteq \mathrm{X}$ s.t. $X = \mathrm{U} \cup \mathcal{K}$ and $\mathrm{U}\bar{\delta}\mathcal{K}$, $(X - \mathcal{K})\delta\mathrm{U}$, which implies $\mathcal{K} = X - \mathrm{U}$, then

$\oint_{\mathrm{Fr}}(\mathrm{U}) = \oint_{\mathrm{T}}(\mathrm{U}) \cap \oint_{\mathrm{T}}(\mathrm{X} - \mathrm{U}) = X \cap X = X$ which prove part (1).

Now to proof part (2) we have by proposition (4-1-4)

$\psi_T(\mathrm{U}) = \psi_T(X - \mathrm{U}) = \emptyset$. Hence $O_T(\mathrm{U}) = \emptyset$.

Definition (4-1-18):

An $i - TPS\ X_{TI}^{\delta}$, is called $weakly\ \oint_T - resolvable$ if and only if there exist two $\oint_{\mathrm{T}} - dense$ subsets $\mathrm{U}_1, \mathrm{U}_2 \subseteq \mathrm{X}$ s.t. $X = \mathrm{U}_1 \cup \mathrm{U}_2$, $\mathrm{U}_1 \cap \mathrm{U}_2$, i.e., $\mathrm{U}_1, \mathrm{U}_2$ are decomposition

Example (4-1-19):

In $i - TPS$ $(\mathrm{X}, \mathrm{T_i}, \{\emptyset\}, \delta_\mathrm{i})$ then $\oint_{\mathrm{T}}(\mathrm{A}) = \mathrm{X}$ for each subset A of X, so $\oint_{\mathrm{T}}(\mathrm{X} - \mathrm{A}) = \mathrm{X}$, s. t. $X = A \cup (\mathrm{X} - \mathrm{A})$, then $A \cap (\mathrm{X} - \mathrm{A}) = \emptyset$.

Thus X is $weakly\ \oint_T - resolvable.$

Remark (4-1-20):

It is obvious that every $\oint_T -resolvable$ is $weakly\ \oint_T -resolvable$, since for any $\text{Ɐ}, \mathcal{K}$ subsets of X, if $\text{Ɐ}\bar{\delta}\mathcal{K}$ then $\text{Ɐ} \cap \mathcal{K} = \emptyset$. But the convers need not to be true in general as in the example (4-1-19).

In the next proposition we will identify the weakly occlusion irresolvable $i - TPS$.

Proposition (4-1-21) :

The $i - TPS\ X_{TI}^{\delta}$ is $weakly\ \oint_{\mathrm{T}} - irresolvabe$ if and only if there is no $\oint_{\mathrm{T}} - dense$ set $D \subseteq \mathrm{X}$ which $X - D$ is also $\oint_{\mathrm{T}} - dense$ set.

Proof: It is obvious by theorem (4-1-16) and remark (4-1-20) that if there is no $\oint_{\mathrm{T}} - dense$ set $D \subseteq \mathrm{X}$ which $X - D$ is also $\oint_{\mathrm{T}} - dense$ set then X_{TI}^{δ} is $weakly\ \oint_{\mathrm{T}} - irresolvabe$ space.

Conversely, suppose that X_{TI}^{δ} is $weakly\ \oint_{\mathrm{T}} - irresolvabe$ space then if there is $\text{Ɐ} \subseteq \mathrm{X}$ s. t. $\oint_{\mathrm{T}}(\text{Ɐ}) = \oint_{\mathrm{T}}(\mathrm{X} - \text{Ɐ}) = X$, but $X = \text{Ɐ} \cup (\mathrm{X} - \text{Ɐ})$ and $\text{Ɐ} \cap (X - \text{Ɐ}) = \emptyset$, so by definition (4-1-18) this indeed leads to that X is $weakly\ \oint_{\mathrm{T}} - resolvabe$ which a contradiction.

Next step is to show is that the $\psi_T - operator$ will be empty for some subsets of $weakly\ \oint_{\mathrm{T}} - resolvabe$.

Proposition (4-1-22):

The $i - TPS\ X_{TI}^{\delta}$ is $weakly\ \oint_{\mathrm{T}} - resolvable$ if and only if there exist a subset $A \subseteq X$ satisfy $\psi_T(A) = \psi_T(X - A) = \emptyset$.

Proof: Let X_{TI}^{δ} be $weakly\ \oint_{\mathrm{T}} - resolvable\ i - TPS$, then there exist two $\oint_{\mathrm{T}} - dense$ subset $\mathrm{A}_1, A_2 \subseteq X\ s.t. X = \mathrm{A}_1 \cup A_2, \mathrm{A}_1 \cap A_2 = \emptyset$, so $A_2 = X - \mathrm{A}_1$ and $A_1 = X - \mathrm{A}_2$. Then $X - \oint_{\mathrm{T}}(\mathrm{A_i}) = \emptyset$, for $i = 1,2$, but by proposition (1-3-37) part (6) we have $X - \oint_{\mathrm{T}}(\mathrm{A_i}) = \psi_T(X - A_i)$. Thus $\psi_T(X - A_i) = \emptyset$. Thus $\psi_T(A_1) = \psi_T(A_2) = \emptyset$.

Conversely, let $\mathrm{A} \subseteq X$, s. t. $\psi_T(A) = \psi_T(X - A) = \emptyset$, so by proposition (1-3-37) part (6) we get that $X - \oint_{\mathrm{T}}(\mathrm{X} - \mathrm{A}) = \emptyset$, hence $\oint_{\mathrm{T}}(\mathrm{X} - \mathrm{A}) = X$ and $\oint_{\mathrm{T}}(\mathrm{A}) = X$, but $X = A \cup (X - A)$, and $A \cap (X - A) = \emptyset$.

Thus X_{TI}^{δ} is $weakly\ \oint_{\mathrm{T}} - resolvable.$

In the following propositions and corollaries, we are going to show that the image and pre-image for $\oint_{\mathrm{T}} - resolvabe$ space ($weakly\ \oint_{\mathrm{T}} - resolvabe$ space) also will be $\oint_{\mathrm{T}} - resolvabe$ space ($weakly\ \oint_{\mathrm{T}} - resolvabe$space).

Proposition (4-1-23):

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be a $\delta - symmetry$, if X_{TI}^{δ} is $\oint_{\mathrm{T}} - resolvable$ then $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ is $\oint_{\grave{T}} - resolvable$.

Proof: Let X_{TI}^{δ} is $\oint_{\mathrm{T}} - resolvable$ space, then $\exists A_1, A_2$, $\oint_T - dense$ sets s. t. $A_1 \bar{\delta} A_2$ and $(X - A_1)\delta A_2$. Now by proposition (2-1-7) we get that $f(A_1)$ and $f(A_2)$ are $\oint_{\mathrm{T}}$ −denes in Y, since $A_1 \bar{\delta} A_2$ and $(X - A_1)\delta A_2$ then we get that $f(A_1)\bar{\grave{\delta}} f(A_2)$ and $f(X - A_1)\grave{\delta} f(A_2)$. Finally, we have $f(A_1) \cup f(A_2) = f(A_1 \cup A_2) = f(X) = Y$. Thus Y is $\oint_{\grave{T}} - resolvable.$

Corollary (4-1-24) :

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be a $\delta - symmetr$, if X_{TI}^{δ} is $weakly\ \oint_{\mathrm{T}} - resolvable$ then $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ is $weakly\ \oint_{\grave{T}} - resolvable$.

Proof: directly by applied remark (4-1-18) and theorem (4-1-23) on the definition (4-1-16).

Proposition (4-1-25):

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be a $\delta - symmetry$, if $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ is $\oint_{\grave{T}} - resolvable$ then X_{TI}^{δ} is $\oint_{\mathrm{T}} - resolvable$.

Proof: Let $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be $\oint_{\grave{\mathrm{T}}} - resolvable$ space, then $\exists A_1, A_2$, $\oint_{\grave{T}} - dense$ sets s. t. $A_1 \bar{\grave{\delta}} A_2$ and $(X - A_1)\grave{\delta} A_2$. Now by proposition (2-1-8) we get that $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are $\oint_{\mathrm{T}} -$denes set in X, s. t. $f^{-1}(A_1)\bar{\delta} f^{-1}(A_2)$ and $f^{-1}(X - A_1)\delta f^{-1}(A_2)$.

Finally, $f^{-1}(A_1) \cup f^{-1}(A_2) = f^{-1}(A_1 \cup A_2) = f^{-1}(Y) = X$.

Thus X is $\oint_{\mathrm{T}} - resolvable$.

Corollary (4-1-26):

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be a $\delta - symmetr$, if $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ is $weakly\ \oint_{\grave{\mathrm{T}}} - resolvable$ then X_{TI}^{δ} is $weakly\ \oint_{\mathrm{T}} - resolvable$.

Proof: directly by applied remark (4-1-20) and theorem (4-1-25) on the definition (4-1-18) .

In the next propositions, we will see that the image and pre-image for $\oint_{\mathrm{T}} - resolvabe$ subsets are also will be $\oint_{\grave{\mathrm{T}}} - resolvabe$ subsets.

Proposition (4-1-27) :

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be $\delta - symmetry$, if A be $\oint_{T} - resolvable$ subset of X_{TI}^{δ} then $f(A)$ will be $\oint_{\grave{T}} - resolvable$ subset of $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$.

Proof: Let A be $\oint_{T} - resolvable$ subset of X_{TI}^{δ}, then $\exists A_1, A_2 \subseteq X$, s. t. $A \subseteq \oint_{T}(A_1)$, $A \subseteq \oint_{T}(A_2)$ and $A = A_1 \cup A_2$, so by proposition (2-1-3) we will get that :

$f(A) \subseteq f(\oint_{T}(A_1)) = \oint_{\grave{T}}(f(A_1))$ and $f(A) \subseteq f(\oint_{T}(A_2)) = \oint_{\grave{T}}(f(A_2))$.

Since f is one-one so $f(A_1) \cap f(A_2) = f(A_1 \cap A_2) = f(\emptyset) = \emptyset$ and $f(A_1) \cup f(A_2) = f(A_1 \cup A_2) = f(A)$.

Thus $f(A)$ is $\oint_{\grave{T}} - resolvable$ subset.

Proposition (4-1-28):

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be a $\delta - symmetry$, if B is $\oint_{\grave{T}} - resolvable$ subset in $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$, then $f^{-1}(B)$ will be $\oint_{T} - resolvable$ subsrt in X_{TI}^{δ}.

Proof: Let B be $\oint_{\grave{T}} - resolvable$ subset in $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$, then $\exists A_1, A_2 \subseteq Y$, s. t. $\oint_{\grave{T}} - dense$ sets s. t. $B \subseteq \oint_{\grave{T}}(A_1)$, $B \subseteq \oint_{\grave{T}}(A_2)$ and $B = A_1 \cup A_2$, so by proposition (2-1-4) we will get $f^{-1}(B) \subseteq f^{-1}(\oint_{\grave{T}}(A_1)) = \oint_{T}(f^{-1}(A_1))$ and $f^{-1}(B) \subseteq f^{-1}(\oint_{\grave{T}}(A_2)) = \oint_{T}(f^{-1}(A_2))$.

Since f is one-one so we will get that

$f^{-1}(A_1) \cap f^{-1}(A_2) = f^{-1}(A_1 \cap A_2) = f^{-1}(\emptyset) = \emptyset$, and

$f^{-1}(A_1) \cup f^{-1}(A_2) = f^{-1}(A_1 \cup A_2) = f^{-1}(A)$ and $f^{-1}(A_1)\bar{\delta}f^{-1}(A_2)$

Thus $f^{-1}(A)$ is $\oint_{T} - resolvable$ subset in X.

4.2 Redirect Resolvable Space

Another kind of resolvable spaces has been defined in i-topological proximity spaces in this section via the redirect dense sets and we investigated some of its properties and relations.

Definition (4-2-1):

The $i-TPS\ X_{TI}^{\delta}$ is called redirect resolvable space if and only if there exist two $r-\ dense$ subsets $\mathrm{U},\mathcal{K}$ s. t. $X=\mathrm{U}\cup\mathcal{K}$ and $\mathrm{U}\bar{\delta}\mathcal{K}$ and $(X-\mathcal{K})\delta\mathrm{U}$. We will denote it by $r-resolvable$ space.

Example (4-2-2) :

Let $X=\{\hbar,g,f\}$, $T=\{\mathrm{X},\emptyset,\{\hbar,g\},\{\hbar,f\}\}$, and $\mathrm{I}=\{\emptyset,\{f\},\}$ with the discrete proximity δ_D, if $\mathrm{U}=\{\hbar\}$ then $\mathrm{U_r}=\mathrm{X}$ and let $\mathcal{K}=\{g,f\}$, then $\mathcal{K}_r=X$, so $\mathrm{U},\mathcal{K}$, are $r-\ dense$ sets and $\mathrm{U}\bar{\delta}\mathcal{K}$, $(X-\mathcal{K})\delta\mathrm{U}$ also we have $X=\mathrm{U}\cup\mathcal{K}$. Thus X_{TI}^{δ} is $\mathrm{r}-resolvable.$

Proposition (4-2-3):

Let $X_{T_jI_j}^{\delta_j}$ be $i-TPS's$ for $j=1,2$ s. t. $T_1\subseteq T_2, I_1\subseteq I_2\ and\ \delta_1>\delta_2$, then

1) If X is $r_{T_2}-resolvable$ then X is $r_{T_1}-resolvable$
2) If X is $r_{I_1}-resolvable$ then X is $r_{I_2}-resolvable$
3) If X is $r_{\delta_1}-resolvable$ then X is $r_{\delta_2}-resolvable$

Proof:

1) Let X be $r_{T_2}-resolvable$, then $\exists A_1,A_2$, $\mathrm{r}_{\mathrm{T}_2}-dense$ sets s. t. $A_1\bar{\delta}A_2$, $(X-A_1)\delta A_2$ and $X=A_1\cup A_2$. But, by theorem (2-2-24) part (1) we have $(\mathrm{U_r})_{T_2}\subseteq(\mathrm{U}_r)_{T_1}$, so $(A_1)_{r_{T_1}}=(A_2)_{r_{T_1}}=X$, which means that A_1,A_2, $\mathrm{r}_{\mathrm{T}_1}-dense$ sets.
   Thus X is $r_{T_1}-resolvable.$

2) Let X be $r_{I_1} - resolvable$, then $\exists A_1, A_2, r_{I_1} - dense$ sets s. t. $A_1 \bar{\delta} A_2$, $(X - A_1)\delta A_2$ and $X = A_1 \cup A_2$. But, by theorem (2-2-24) part (2) we have $(\text{U}_r)_{I_1} \subseteq (\text{U}_r)_{I_2}$ for any subset U of X , so A_1 and A_2 will be $r_{I_2} - dense$ sets. Thus X is $r_{I_2} - resolvable$.
3) Let X be $r_{\delta_1} - resolvable$, then $\exists A_1, A_2$, $r_{\delta_1} - dense$ sets s. t. $A_1 \overline{\delta_1} A_2$, $(X - A_1)\delta_1 A_2$ and $X = A_1 \cup A_2$. But, by theorem (2-2-24) part (3) we have $(\text{U}_r)_{\delta_1} \subseteq (\text{U}_r)_{\delta_2}$, so A_1 and A_2 are $r_{\delta_2} - dense$ sets, s. t. $A_1 \overline{\delta_2} A_2$, $(X - A_1)\delta_2 A_2$. Thus X is $r_{\delta_2} - resolvable$.

Proposition (4-2-4) :

In $r - resolvable$ $i - TPS$ X_{TI}^{δ} there exist a subset $A \subseteq X$ satisfy $(A)_s = (X - A)_s = \emptyset$

Proof: Let X_{TI}^{δ} be $r - resolvable$ $i - TPS$, then there exist two $r - dense$ subset $A_1, A_2 \subseteq X$ $s.t. X = A_1 \cup A_2$, $A_1 \bar{\delta} A_2$, $(X - A_1)\delta A_2$ which means $A_1 \cap A_2 = \emptyset$, so $A_1 = X - A_2$ and $A_2 = X - A_1$.

Then $X - (A_i)_r = \emptyset$, for $i = 1,2$, but by proposition (2-2-19) part (3) we have $X - (A_i)_r = (X - A_i)_s$.

Thus $(X - A_i)_s = \emptyset$, Thus $(A_1)_s = (A_2)_s = \emptyset$, and this complete the proof.

Now we will introduce the definition of the $r - resolvable$ subset concept in $i - TPS$.

Definition (4-2-5):

In $i - TPS$ X_{TI}^{δ} and for $A \subseteq X$ we say that A is $r - resolvable$ subsets if and only if there exist two subsets U, $\mathcal{K} \subseteq X$ s. t. $A \subseteq \text{U}_r$, $A \subseteq \mathcal{K}_r$, and

$A = \text{U} \cup \mathcal{K}$ and $\text{U} \cap \mathcal{K} = \emptyset$.

Example (4-2-6) :

Let $X = \{\hbar, g, f\}$, $T = \{X, \emptyset, \{\hbar, g\}, \{\hbar, f\}\}$, and $I = \{\emptyset, \{f\},\}$ with the discrete proximity δ_D, let $A = \{\hbar, g\}$ so if $U = \{\hbar\}$ then we have $U_r = X$ and if $\mathcal{K} = \{g\}$, then $\mathcal{K}_r = X$, so $U \cap \mathcal{K} = \emptyset$ and $A = U \cup \mathcal{K}$ also we have $A \subseteq U_r$ and $A \subseteq \mathcal{K}_r$. Thus A is $r - resolvable$ subset.

In the following proposition we will see that the union of $r - resolvable$ subsets is also $r - resolvable$ subset.

Proposition (4-2-7) :

The union of any $r - resolvable$ subset is $r - resolvable$

Proof: Let $\{U_\lambda, \lambda \in \Lambda\}$ be a family of $r - resolvable$ subsets in $i - TPS$ X_{TI}^{δ}. Then $\forall \lambda \in \Lambda, \exists\ A_{1_\lambda}, A_{2_\lambda}\ s.t. A_{1_\lambda} \cap\ A_{2_\lambda} = \emptyset$, $U_\lambda = A_{1_\lambda} \cup\ A_{2_\lambda}$ s. t. $U_\lambda \subseteq \left(A_{1_\lambda}\right)_r$ *and* $U_\lambda \subseteq \left(A_{2_\lambda}\right)_r$.

Therefore, by proposition (2-2-4) part (8) and part (9) we will get

$$\bigcup_{\lambda\in\Lambda} U_\lambda \subseteq \bigcup_{\lambda\in\Lambda}\left(\left(A_{1_\lambda}\right)_r\right) \subseteq \left(\bigcup_{\lambda\in\Lambda} A_{1_\lambda}\right)_r, \text{ and}$$

$$\bigcup_{\lambda\in\Lambda} U_\lambda \subseteq \bigcup_{\lambda\in\Lambda}\left(\left(A_{2_\lambda}\right)_r\right) \subseteq \left(\bigcup_{\lambda\in\Lambda} A_{2_\lambda}\right)_r$$

and $\bigcup_{\lambda\in\Lambda} U_\lambda = (\bigcup_{\lambda\in\Lambda} A_{1_\lambda}) \cup (\bigcup_{\lambda\in\Lambda} A_{2_\lambda})$.

This gives us that $\bigcup_{\lambda\in\Lambda} U_\lambda$ is $r - resolvable$ subsets in X.

Definition (4-2-8) :

In $i - TPS\ \ X_{TI}^{\delta}$, a point $k \in X$ is called has empty $s - tightness$ if $\forall\ k \in U_r, \exists\ D\ \subseteq U$ s.t. $k \in D_r$ *and* $D_s = \emptyset$.

Example (4-2-9) :

In $i-TPS$ $(\mathrm{X}, \mathrm{T_i}, \{\emptyset\}, \delta_\mathrm{i})$ then $A_r = \mathrm{X}$ and $(X - A)_s = \emptyset$,for each subset A of X , so $(\mathrm{X} - \mathrm{A})_r = \mathrm{X}$ and $A_s = \emptyset$. Now let $A \subseteq \mathrm{Ц}$, then $\forall\, k \in \mathrm{Ц_r}$, $\exists A \subseteq \mathrm{Ц}$ s.t. $k \in \mathrm{A_r}$ $and\ A_s = \emptyset$.

The following property shows the relationship between the empty $s - tightness$ and $\mathrm{r} - resolvable$ space.

Proposition (4-2-10) :

Any $r - resolvable\ i - TPS\ X_{TI}^{\delta}$ has empty $s - tightness$

Proof: Let X_{TI}^{δ} be $r - resolvable\ \ i - TPS$, then there exist two $\mathrm{r} - dense$ subsets $\mathrm{Ц}, \mathcal{K} \subseteq \mathrm{X}$ s.t. $X = \mathrm{Ц} \cup \mathcal{K}$ and $\mathrm{Ц}\bar{\delta}\mathcal{K}$, $(X - \mathcal{K})\delta\mathrm{Ц}$, i.e., $\mathrm{Ц} \cap \mathcal{K} = \emptyset$, then $X - \mathcal{K} = \mathrm{Ц}$ and $\mathrm{Ц_r} = \mathcal{K}_r = X$, and by proposition (2-2-19) part (3) that implies $\mathrm{Ц_s} = \mathcal{K}_s = \emptyset$.

Now let $A \subseteq \mathrm{X}$ and by using theorem (4-2-4) and for any $k \in A_r$, $(A \cap \mathcal{K})_s = \emptyset$ and $(A \cap \mathrm{Ц})_s = \emptyset$.

Since $A = A \cap X = A \cap (\mathrm{Ц} \cup \mathcal{K}) = (A \cap \mathrm{Ц}) \cup (\mathrm{A} \cap \mathcal{K})$ and by part (8) of proposition (2-2-4) we will get $k \in \mathrm{A_r} = (A \cap \mathrm{Ц})_r \cup (\mathrm{A} \cap \mathcal{K})_r$, so, either $k \in (A \cap \mathrm{Ц})_r$ or $k \in (\mathrm{A} \cap \mathcal{K})_r$. Thus k has empty $s - tightness$.

Definition (4-2-11) :

Let X_{TI}^{δ} be an $i - TPS$. Then X is called $r - submaximal$ if every $\mathrm{r} -$ dense set is $i - open$ set.

Definition (4-2-12) :

Let X_{TI}^{δ} be an $i-TPS$. Then X is called $r-$ $hyper\ connected$ if and only if every nonempty $i-open$ set is $r-dense$ set

Example (4-2-13):

Let $X=\{\hbar, g, f\}$, $T=\{X, \emptyset, \{\hbar, g\}, \{\hbar, f\}\}$, and $I=\{\emptyset, \{f\},\}$ with the proximity δ_D, then we have $X_r = X$, $\{\hbar, g\}_r = X$ and $\{\hbar, f\}_r = X$, so every nonempty $i-open$ set is $r-dense$ set.

Thus X is $r-$ $hyper\ connected$.

In the next theorem, we will see the condition that makes the $i-TPS$ is $r-irresolvabe$.

Theorem (4-2-14) :

The $i-TPS\ X_{TI}^{\delta}$ is $r-irresolvabe$ if there is no $r-dense$ set D which $X-D$ is also $r-dense$ set.

Proof: Suppose that X_{TI}^{δ} is $r-resolvabe\ i-TPS$ space, then there exist two $r-$ $dense$ subsets $\mathrm{U}, \mathcal{K} \subseteq X$ s.t. $X=\mathrm{U}\cup\mathcal{K}$ and $\mathrm{U}\bar{\delta}\mathcal{K}$, then we have $\mathrm{U}\cap\mathcal{K}=\emptyset$, so $\mathrm{U}=X-\mathcal{K}$, that means $\mathrm{U}_r=(X-\mathcal{K})_r = X$, which a contradiction. Thus X_{TI}^{δ} is $r-irresolvabe$.

To study the $r-$ $frontier$ set and the redirect outer function in $r-resolvabe$ space we can see the following proposition

Proposition (4-2-15) :

If X_{TI}^{δ} be $r-resolvable\ i-TPS$ and for a subset $\mathrm{U}\subseteq X$ then

1) $\mathrm{Fr}(\mathrm{U})=X$ for some $\mathrm{U}\subseteq X$
2) $\mathrm{U}_s=\emptyset$ and $O_r(\mathrm{U})=\emptyset$

Proof: Since X_{TI}^{δ} is $r-resolvable\ i-TPS$, then there exist two $r-dense$ subsets $\text{Ų}, \mathcal{K} \subseteq X$ s.t. $X = \text{Ų} \cup \mathcal{K}$ and $\text{Ų}\bar{\delta}\mathcal{K}$, $(X-\mathcal{K})\delta\text{Ų}$, which implies $\mathcal{K} = X - \text{Ų}$, then

$\text{Fr}(\text{Ų}) = (\text{Ų}_r) \cap (X - \text{Ų})_r = X \cap X = X$ which prove part (1).

Now to proof part (2) we have by proposition (4-2-4)

$(\text{Ų}_s) = (X - \text{Ų})_s = \emptyset$. Hence $O_r(\text{Ų}) = \emptyset$.

Definition (4-2-16):

An $i-TPS\ X_{TI}^{\delta}$, is called $weakly\ r-resolvable$ if and only if there exist $r-dense$ subsets $\text{Ų}_1, \text{Ų}_2 \subseteq X$ s.t. $X = \text{Ų}_1 \cup \text{Ų}_2, \text{Ų}_1 \cap \text{Ų}_2 = \emptyset$, i.e., $\text{Ų}_1, \text{Ų}_2$ are decomposition

Example (4-2-17):

In $i-TPS$ $(X, T_i, \{\emptyset\}, \delta_i)$ then $A_r = X$ for each subset A of X , so $(X-A)_r = X$, s. t. $X = A \cup (X - A)$, then $A \cap (X - A) = \emptyset$.

Thus X is $weakly\ r-resolvable.$

Remark (4-2-18):

It is obvious that every $r-resolvable$ is $weakly\ r-resolvable$, since for any $\text{Ų}, \mathcal{K}$ subsets of X, if $\text{Ų}\bar{\delta}\mathcal{K}$ then $\text{Ų} \cap \mathcal{K} = \emptyset$. The convers need not to be true as in example (4-2-17).

In the next proposition we will identify the $weakly\ r-irresolvable\ i-TPS.$

Proposition (4-2-19):

The $i-TPS\ X_{TI}^{\delta}$ is $weakly\ \ r-irresolvabe$ if and only if there is no $\mathrm{r}-dense$ set $D\subseteq \mathrm{X}$ which $X-D$ is also $\mathrm{r}-dense$ set.

Proof: It is obvious by theorem (4-2-14) and remark (4-2-18) that if there is no $\mathrm{r}-dense$ set $D\subseteq \mathrm{X}$ which $X-D$ is also $\mathrm{r}-dense$ set then X_{TI}^{δ} is $weakly\ r-irresolvabe$ space.

Conversely, suppose that X_{TI}^{δ} is $weakly\ r-irresolvabe$ space then if there is $\mathrm{U}\subseteq \mathrm{X}$ s. t. $\mathrm{U_r}=(\mathrm{X}-\mathrm{U})_r=X$, but $X=\mathrm{U}\cup(\mathrm{X}-\mathrm{U})$ and we have $\mathrm{U}\cap(X-\mathrm{U})=\emptyset$, so by definition (4-2-16) this indeed leads to that X is $weakly$ $\mathrm{r}-resolvabe$ which a contradiction.

Next step is to show is that the $strips$ set will be empty for some subsets of $weakly$ $\mathrm{r}-resolvabe$.

Proposition (4-2-20):

The $i-TPS\ X_{TI}^{\delta}$ is $weakly\ r-resolvable$ if and only if there exist a subset $A\subseteq X$ satisfy $A_s=(X-A)_s=\emptyset$.

Proof: Let X_{TI}^{δ} be $weakly\ r-resolvable\ i-TPS$, then there exist two $\mathrm{r}-dense$ subset $\mathrm{A}_1, A_2\subseteq X\ s.t. X=\mathrm{A}_1\cup A_2, \mathrm{A}_1\cap A_2=\emptyset$, so we get that $A_2=X-\mathrm{A}_1$ and $A_1=X-\mathrm{A}_2$. Then $X-(\mathrm{A_i})_r=\emptyset$, for $i=1,2$, but by proposition (2-2-19) part (3) we have $X-(\mathrm{A_i})_r=(X-A_i)_s$. Thus $(X-A_i)_s=\emptyset$. Thus $(A_1)_s=(A_2)_s=\emptyset$.

Conversely, let $\mathrm{A}\subseteq X$, s. t. $A_s=(X-A)_s=\emptyset$, so by part (3) of proposition (2-2-19) we get that $X-(\mathrm{X}-\mathrm{A})_r=\emptyset$, hence $(\mathrm{X}-\mathrm{A})_r=X$ and $\mathrm{A_r}=X$, but $X=A\cup(X-A)$, and $A\cap(X-A)=\emptyset$.

Thus X_{TI}^{δ} is $weakly\ r-resolvable$.

In the following propositions and corollaries, we are going to show that the image and pre-image for any r $- resolvabe$ space ($weakly\ r - resolvabe$ space) also will be r $- resolvabe$ space ($weakly$ r $- resolvabe$space).

Proposition (4-2-21) :

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be a $\delta - symmetry$, if X_{TI}^{δ} is $r - resolvable$space then $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ is $r - resolvable$ space.

Proof: Let X_{TI}^{δ} is r $- resolvable$ space, then $\exists A_1, A_2, r - dense$ sets s. t. $A_1 \bar{\delta} A_2$ and $(X - A_1)\delta A_2$. Now by proposition (2-3-12) we get that $f(A_1)$ and $f(A_2)$ are r $-$denes in Y, since $A_1 \bar{\delta} A_2$ and $(X - A_1)\delta A_2$ then we get that $f(A_1)\bar{\grave{\delta}} f(A_2)$ and $f(X - A_1)\grave{\delta} f(A_2)$. Finally, we have $f(A_1) \cup f(A_2) = f(A_1 \cup A_2) = f(X) = Y$. Thus Y is r $- resolvable.$

Corollary (4-2-22) :

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be a $\delta - symmetry$, if X_{TI}^{δ} is $weakly\ r - resolvable$ then $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ is $weakly$ r $- resolvable$.

Proof: directly by applied remark (4-2-18) and theorem (4-2-21) on the definition (4-2-16) .

Proposition (4-2-23):

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be a $\delta - symmetry$, if $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ is $r - resolvable$ space then X_{TI}^{δ} is $r - resolvable$.

Proof: Let $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be r $- resolvable$ space, then $\exists A_1, A_2$, r $- dense$ sets s. t. $A_1 \bar{\grave{\delta}} A_2$ and $(X - A_1)\grave{\delta} A_2$. Now by proposition (2-3-13) we get that

$f^{-1}(A_1)$ and $f^{-1}(A_2)$ are r −denes set in X, s. t. $f^{-1}(A_1)\bar{\delta}f^{-1}(A_2)$ and $f^{-1}(X-A_1)\delta f^{-1}(A_2)$.

Finally, $f^{-1}(A_1)\cup f^{-1}(A_2)=f^{-1}(A_1\cup A_2)=f^{-1}(Y)=X$.

Thus X is r $-resolvable$ space.

Corollary (4-2-24) :

Let $f: X^{\delta}_{TI}\to Y^{\grave{\delta}}_{\grave{T}\grave{I}}$ be a $\delta-symmetr$, if $Y^{\grave{\delta}}_{\grave{T}\grave{I}}$ is $weakly$ r $-resolvable$ space then X^{δ}_{TI} is $weakly$ r $-resolvable$.

Proof: directly by applied remark (4-2-18) and theorem (4-2-23) on the definition (4-2-16) .

In the next propositions, we will see that the image and pre-image for $r-resolvabe$ subsets are also will be $r-resolvabe$ subsets.

Proposition (4-2-25) :

Let $f: X^{\delta}_{TI}\to Y^{\grave{\delta}}_{\grave{T}\grave{I}}$ be $\delta-symmetry$, if A be $r-resolvable$ subset of X^{δ}_{TI} then $f(A)$ will be r $-resolvable$ subset of $Y^{\grave{\delta}}_{\grave{T}\grave{I}}$.

Proof: Let A be $r-resolvable$ subset of X^{δ}_{TI}, then $\exists A_1, A_2\subseteq$ X, s. t. $A\subseteq (A_1)_r, A\subseteq (A_2)_r$ and $A=A_1\cup A_2$, so by proposition (2-2-25) we will get that :

$f(A)\subseteq f((A_1)_r)=\left(f(A_1)\right)_r$ and $f(A)\subseteq f((A_2)_r)=\left(f(A_2)\right)_r$.

Since f is one-one so $f(A_1)\cap f(A_2)=f(A_1\cap A_2)=f(\emptyset)=\emptyset$ and $f(A_1)\cup f(A_2)=f(A_1\cup A_2)=f(A)$.

Thus $f(A)$ is $r-resolvable$ subset in Y.

Proposition (4-2-26):

Let $f: X_{TI}^{\delta} \to Y_{\grave{T}\grave{I}}^{\grave{\delta}}$ be a $\delta - symmetry,$ if B is $r - resolvable$ subset in $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$, then $f^{-1}(B)$ will be $\mathrm{r} - resolvable$ subsrt in X_{TI}^{δ}.

Proof: Let B be $r - resolvable$ subset in $Y_{\grave{T}\grave{I}}^{\grave{\delta}}$, then $\exists A_1, A_2 \subseteq \mathrm{Y},$ s. t. $B \subseteq (A_1)_r, A \subseteq (A_2)_r$ and $B = A_1 \cup A_2$, so by proposition (2-2-26) we will get $f^{-1}(B) \subseteq f^{-1}((A_1)_r) = \left(f^{-1}(A_1)\right)_r$

and $f^{-1}(B) \subseteq f^{-1}((A_2)_r) = \left(f^{-1}(A_2)\right)_r$.

Since f is one-one so we will get that

$f^{-1}(A_1) \cap f^{-1}(A_2) = f^{-1}(A_1 \cap A_2) = f^{-1}(\emptyset) = \emptyset$, and

$f^{-1}(A_1) \cup f^{-1}(A_2) = f^{-1}(A_1 \cup A_2) = f^{-1}(A)$ and $f^{-1}(A_1)\bar{\delta}f^{-1}(A_2)$

Thus $f^{-1}(A)$ is $\mathrm{r} - resolvable$ subset in X.

Remark (4-2-27) :

It is not necessary that any $i - Subspace$ $Y_{T_Y I_Y}^{\delta_Y}$ of $r - resolvable$ $i - TPS$ X_{TI}^{δ} be $r - resolvable$ and the following example showing that.

Example (4-2-28) :

In example (4-2-2) we have $X = \{\hbar, g, f\}$, $T = \{\mathrm{X}, \emptyset, \{\hbar, g\}, \{\hbar, f\}\}$, and $\mathrm{I} = \{\emptyset, \{f\},\}$ with the discrete proximity δ_D is $\mathrm{r} - resolvable$. Now let $Y = \{\hbar, f\}$, $T = \{\mathrm{Y}, \emptyset, \{\hbar\}\}$, and $\mathrm{I} = \{\emptyset, \{f\},\}$ with the discrete proximity δ_{Y_D}. Then we have only two decomposition subsets of Y , $\mathrm{U} = \{\hbar\}$ and $\mathcal{K} = \{f\}$ and obviously we can find that $\mathrm{U_r} = Y$ and $\mathcal{K}_r = \{f\}$, so there is no decomposition $r_Y - dens$ subset. Thus $Y_{T_Y I_Y}^{\delta_Y}$ is not $r - resolvable\ i - subspace$

Proposition (4-2-29) :

Every $\oint_{\mathrm{T}} - resolvable\ i - TPS$ is $r - resolvable$

Proof: Let X_{TI}^{δ} be $\oint_{\mathrm{T}} - resolvable\ \ i - TPS$, then there exist two $\oint_{\mathrm{T}} -\ dense$ subsets $\text{Ա}, \mathcal{K} \subseteq \mathrm{X}$ s.t. $X = \text{Ա} \cup \mathcal{K}$ and $\text{Ա}\bar{\delta}\mathcal{K}$, $(X - \mathcal{K})\delta\text{Ա}$, , then by proposition (2-3-4) part (4) we have $\text{Ա}, \mathcal{K}$ are $r - dense$ subsets. Thus X_{TI}^{δ} is $r - resolvable.$

Corollary (4-2-30) :

Every $weakly\ \oint_{\mathrm{T}} - resolvable\ i - TPS$ is $weakly\ r - resolvable$

Proof: Directly by using remark (4-1-20) and remark (4-2-18) with proposition (4-2-29).

Proposition (4-2-31) :

Every $r - irresolvable\ i - TPS$ is $\oint_{\mathrm{T}} - irresolvable$

Proof: Let X_{TI}^{δ} be $r - irresolvable\ i - TPS$, then by theorem (4-2-14) there is no $r - dense$ set D such that $X - D$ is also $r - dense$, that is means there is no $\oint_{\mathrm{T}} -\ dense$ subsets D such that $X - D$ is also $r - dense$. Thus, by theorem (4-1-16) X_{TI}^{δ} is $\oint_{\mathrm{T}} - irresolvable.$

Corollary (4-2-32) :

Every $weakly\ r - irresolvable\ i - TPS$ is $weakly\ \oint_{\mathrm{T}} - irresolvable$

Proof: Directly by using proposition (4-1-21) and proposition (4-2-19) with proposition (4-2-231).

Chapter Five

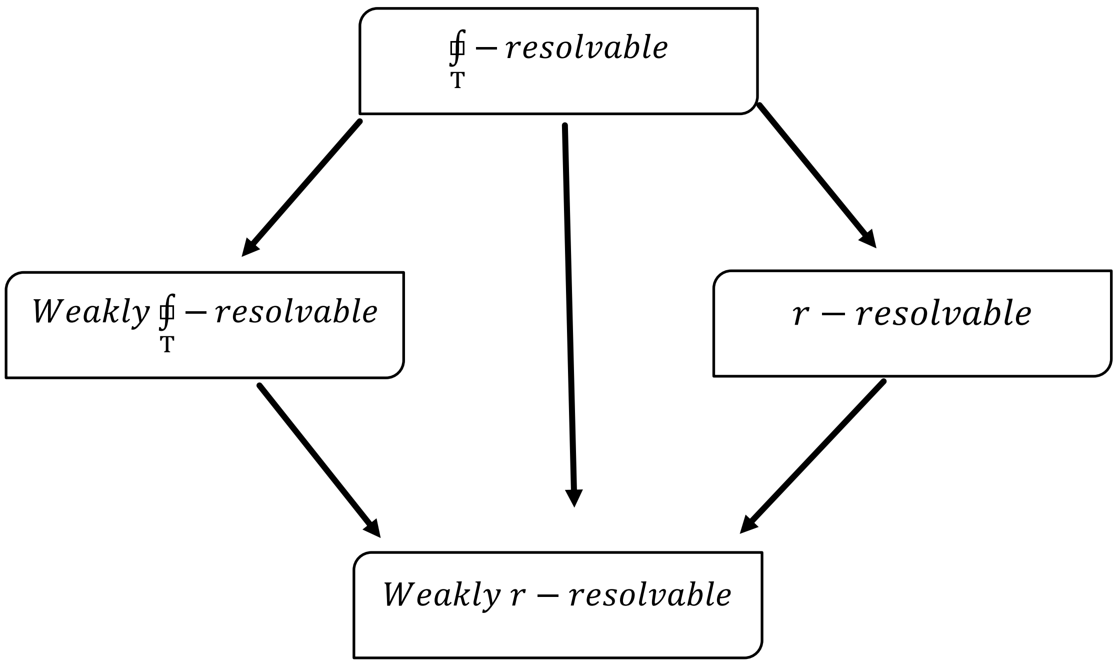
Conclusion
and
Future Work

Conclusion and future work

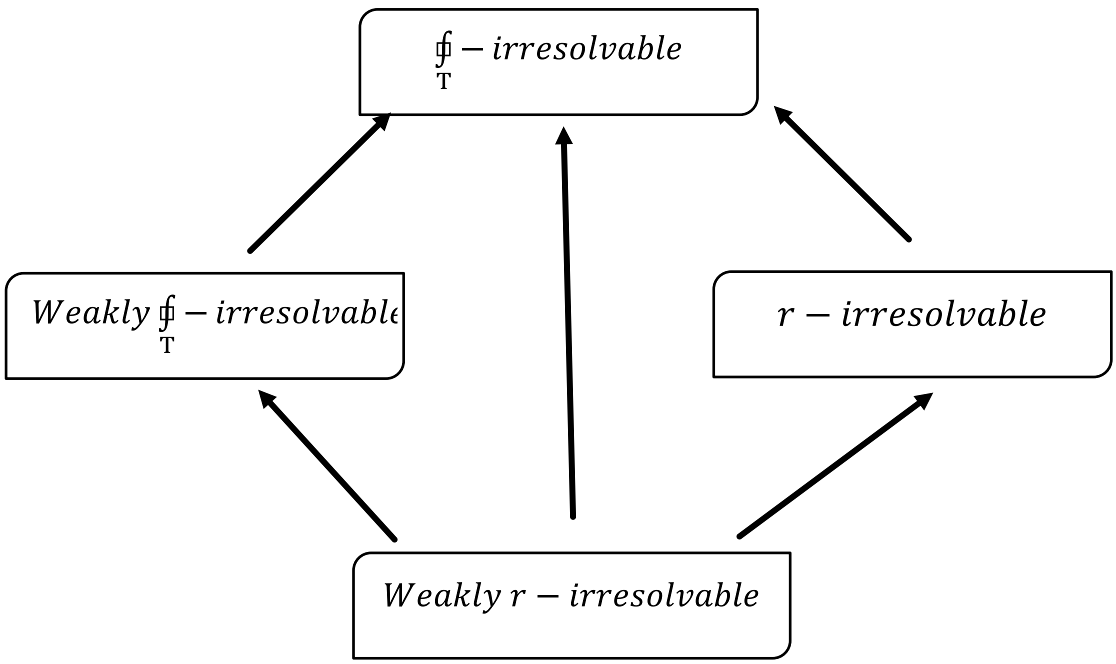
Through this work which is dependent on the focal function via $i-open$ set, in the i-topological space by the proximity space, we concluded that some properties of this concept were not clear in the previous study and not appropriate to its importance, but one of important result that makes the focal function is that the proximity relation plays an important role in congested sets. So we use this concept and duplicate the proximity relation in the definition of redirect set so we get a more denseness that will give us the ability to study the resolvableness in this space with all of these concepts.

While we study this new property, we felt that we need to see what the result if we studied them in the $i-subspace$ that we had to study more properties with respect to $i-subspace$ through all of the above concepts and the basic concepts Which contributed to expanding the scope of the study to define the boundary points and density for the main concepts (focal function, redirect sets) in $i-topologica\ proximity\ space$ and in $i-subspace$ to achieve more and great results in resolvable $i-topological\ proximity\ space$ and through this study we see the following facts :

The first one, there is a large and conspicuous effect of proximity relation on $r-dense$ set which studies the accumulation of the points in the space. Another fact is the redirect set saves its property under the i-homomorphism function also in $i-subspace$. The third fact is that the empty strip sets show us the nowhere dense in space which makes it irresolvable. The fourth one is the proximity relation gives a direct effect on the resolvableness more than the ideal dose. Also, we found that the redirect frontier operator has great effect in irresolvable space, We can find some important results of resolvable space in the following diagram :



Also, we can find some important results of irresolvable space in the following diagram :



Through this work, we face some open problems so we left them as future work, and some of them are listed below:

1- Can we define resolvableness dependent on the occlusion set via focal set, and what are the different results we will get?
2- Is there any relationship between the resolvable spaces which we studied and other kinds of resolvable spaces just as strong resolvable, maximal resolvable, and almost resolvable space?
3- Is the product space of two $r - resolvable\ i - toplogical\ proximity\ space$ will be resolvable or not?
4- What is the compactification of $r - resolvable\ i - toplogical\ proximity\ space$?
5- The definition of the fuzzy (Intuitionistic, Neutrosophic, and soft) $r - resolvable\ in\ i - toplogical\ proximity\ space.$

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المستخلص

يعتمد العمل في هذه الأطروحة على مفهومين مهمين، الأول هو فضاء القرب، والثاني هو الفضاء التبولوجي — i ، وأحد مفاهيمه هو مفهوم الدالة المحورية، لذلك قمنا ببناء مفهوم جديد يعتمد على المفاهيم التي ذكرناها أعلاه وقسمنا هذا العمل إلى أربع مراحل على النحو التالي:

في المرحلة الأولى درسنا مفهوم الفضاء الجزئي التبولوجي **i**- وبحثنا في العلاقة الأساسية في الفضاء الجزئي **i** ثم بعض الخصائص جنبًا إلى جنب مع فضاء القرب التبولوجي. بينما في المرحلة الثانية، قدمنا خصائص جديدة لمفهوم مجموعة الانسداد عبر المجموعة المفتوحة -**i** والتي تسمى الدالة المحورية وتمت دراسة مجموعة اعادة التوجيه بالإضافة الى مجموعة الشرائط التي تعتمد على الدالة المحورية في هذه المرحلة، حيث درسنا الخواص المهمة، والتي قد تلعب دوراً مهماً فيما حققناه لاحقاً.

كانت المرحلة الثالثة مخصصة لدراسة النقاطة الحدودية للمفاهيم السابقة لذلك قدمنا مفهوم النقطة الحدودية للدالة المحورية مع بعض الخصائص، وكذلك النقاط الحدودية للمجموعة اعادة التوجيه وخصائصها.

أخيراً، حدنا الفضاءات القابلة للحل فيما يتعلق بتلك المفاهيم التي تحدد بعض الخصائص والعلاقات التي ستكون مهمة لملئ الفجوات في هذا المجال من الرياضيات البحتة.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

قابل للحل
في فضاءات القرب التبولوجية-i

أطروحة مقدمة إلى مجلس كلية التربية للعلوم الصرفة في جامعة بابل

جزءاً من متطلبات نيل درجة الدكتوراه فلسفة في التربية/ الرياضيات

من قبل

علي خالد حسن محمد

بإشراف

أ. د. لؤي عبد الهاني السويدي

1445هـ 2023م