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On Fuzzy Markov Chains

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I certify that the preparation for this thesis entitled " On Fuzzy Markov Chains " was made under my supervision at the university of Babylon , College of Education as partial fulfillment of requirements of degree of Master of Science in Mathematics .

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To My Family

List of Symbols

Symbol	Definition
$\overline{A}$	Fuzzy set
$\overline{A}(x)$	Membership function
$S\left(\overline{A}\right)$	Support of fuzzy set
$\overline{A}[\alpha]$	α -cut of fuzzy set
$\overline{A}=(a_1, a_2, a_3)$	Triangular fuzzy number
(Ω, Σ, p)	Probability space
$k(x, y)$	Transition kernel
$\overline{k}(x, y)$	Fuzzy transition kernel
$P=(k(x, y))_{x, y\in\Omega}$	Transition matrix of Markov chain
$\overline{P}=\left(\overline{k}(x, y)\right)_{x, y\in\Omega}$	Fuzzy transition matrix of fuzzy Markov chain
π	Invariant probability measure
$\overline{\pi}$	Invariant fuzzy probability measure
$L^p(\pi)$	Space of real valued (simple) function
$L^p\left(\overline{\pi}\right)$	Fuzzy space of real valued (simple) function

$\left\lVert \cdot \right\rVert_p$ $\left\lVert \cdot \right\rVert_\infty$	Types of norms on $L^p(\pi)$
$\overline{\left\lVert \cdot \right\rVert}_p$	Fuzzy norm on $L^p\left(\overline{\pi}\right)$
K	Markov operator
$\overline{\mathbf{K}}$	Fuzzy Markov operator
$\left\lVert \cdot \right\rVert_{p\to q}$	Operator norm
$\overline{\left\lVert \cdot \right\rVert}_{p\to q}$	Fuzzy operator norm
$k^n(x,y)$	Iterated transition kernel
$\overline{k}^{\,n}(x,y)$	Iterated fuzzy transition kernel
E	Expected value
$\overline{\boldsymbol{E}}$	Fuzzy expected value
var	Variance
$\overline{\mathrm{var}}$	Fuzzy variance
Ent	Entropy
$\overline{Ent}$	Fuzzy entropy
$Ent(\mu)$	Relative entropy
$\overline{Ent}\left(\overline{\mu}\right)$	Fuzzy relative entropy
$L^{p'}(\pi)$	Conjugate space

$L^{p'}\left(\overline{\pi}\right)$	Conjugate fuzzy space
K^{*}	Adjoint Markov operator
$\overline{\mathrm{K}}^{*}$	Adjoint fuzzy Markov operator
$\kappa(x,y)$	Kernel of a Markov operator with respect to π
$\overline{\kappa}(x,y)$	Fuzzy kernel of a fuzzy Markov operator with respect to $\overline{\pi}$
H_t	Markov semi-group
$\overline{H}_t$	Fuzzy Markov semi-group
$H_t(x,y)$	Kernel of Markov semi-group
$\overline{H}_t(x,y)$	Fuzzy kernel of fuzzy Markov semi-group
$h_t(x,y)$	Kernel of Markov semi-group with respect to π
$\overline{h}_t(x,y)$	Fuzzy kernel of fuzzy Markov semi-group with respect to $\overline{\pi}$
H_t^{*}	Adjoint Markov semi-group
$\overline{H}_t^{*}$	Adjoint fuzzy Markov semi-group
β_i	Eigen-value of Markov operator
$\overline{\beta}_i$	Fuzzy eigen-value of fuzzy Markov operator
$\exp(-t\lambda_i)$	Eigen-value of Markov semi-group

$\exp\left(-t\,\overline{\lambda}_i\right)$	Fuzzy eigen-value of fuzzy Markov semi-group
λ	Spectral gap
$\overline{\lambda}$	Fuzzy spectral gap
ℓ	Dirichlet form
$\overline{\ell}$	Fuzzy Dirichlet form
$\lVert\cdot\rVert_{TV}$	Total variation distance
$\wp$	Log- Sobolev constant
$\overline{\wp}$	Fuzzy log-Sobolev constant
$q(t)\leq 1+e^{ct}\ ,0\leq c$	Hypercontractivity equation

جمهورية العراق

وزارة التعليم العالي و البحث العلمي

جامعة بابل – كلية التربية

قسم الرياضيات

حول سلاسل ماركوف الضبابية

رسالة مقدمة

إلى كلية التربية ، جامعة بابل

كجزء من متطلبات نيل درجة الماجستير

في علوم الرياضيات

من قبل

عبد الحميد قحطان عبود

بإشراف

د. كريمه عبد الله كاظم

٢٠٠٨

بسم الله الرحمن الرحيم

* الله نور السماوات والأرض مثل نوره كمشكاة فيها مصباح ۖ المصباح
في زجاجة ۖ الزجاجة كأنها كوكب دري يوقد من شجرة مباركة زيتونة
لا شرقية ولا غربية يكاد زيتها يضيء ولو لم تمسسه نار ۚ نور على نور يهدي
الله لنوره من يشاء ۚ ويضرب الله الأمثال للناس ۗ والله بكل شيء
عليم *

صدق الله العلي العظيم

النور الآية 35

المستخلص

لكون سلاسل ماركوف الضبابية من المواضيع المهمة التي حظيت اهتماما كبيرا من قبل الباحثين و خاصة قي الفترة الأخيرة ولتطبيقاته المهمة للرياضيات نفسها وللحياة ،حاولنا تسليط الضوء على هذا الموضوع و المشاركة فيه بعملنا البسيط هذا.

تكونت رسالتنا من خمسة فصول:-

لرسم صورة متكاملة في اذهاننا عن الموضوع ومن اجل أن نهيئ بعض المعلومات الأساسية التي نحتاجها حتى يصبح العمل أكثر وضوحا،استذكرنا في الفصل الأول بعض التعاريف والمعلومات الأساسية التي تدور حول المجموعات الضبابية .

كما تناولنا في الفصل الثاني مفهوم سلاسل ماركوف الضبابية , مؤثر ماركوف الضبابي , شبه- زمرة مؤثرات ماركوف الضبابية , ومفاهيم أساسية أخرى.

في الفصل الثالث درسنا معكوس سلاسل ماركوف الضبابية وكيفية إيجاد القيم الذاتية الضبابية لمؤثرات ماركوف الضبابية .

أما الفصل الرابع فقد برهنا بعض الحقائق حول صيغة درشليت الضبابية ، مسافة التباين الضبابية الكلية ، مسافة مربع- كاي الضبابية والخطأ النسبي، والتي استفدنا منها في برهان نظرياتنا الأساسية في الفصل الخامس .

في الفصل الخامس برهنا متباينات وثوابت سوبولوف اللوغارتمية الضبابية والانكماش الأعلى لشبه- زمرة مؤثرات ماركوف الضبابية ، وذلك لدراسة القيود الكمية لتقارب النواة الضبابية لشبه- زمرة مؤثرات ماركوف الضبابية إلى دالة القياس الاحتمالية الضبابية غير المتغيرة بدلالة دالة مسافة تباين الضبابية الكلية. ولاحظنا إن التقارب يعتمد على ثوابت سوبولوف اللوغارتمية الضبابية.

Contents

Abstract

Since fuzzy Markov chains is an important topic in the recent years , then we try to shat light on this topic by this our humble contribution .

Our thesis consists five chapters:-

In chapter one we have introduced basic concepts of fuzzy sets, arithmetic operation on fuzzy numbers , and lattice of fuzzy numbers .

Chapter two concerning fuzzy Markov chains , fuzzy Markov operator , fuzzy Markov semi-group , and other concepts .

In chapter three we have studied the reversibility of fuzzy Markov chains , and how to fined fuzzy eigen-values for a fuzzy Markov operator and fuzzy Markov semi-group .

In chapter four we have proved some results referring to the fuzzy Dirichlet forms , total variation distance , fuzzy chi-squared distance , and relative error that we need for our main theorems in chapter five .

Our achievement in chapter five is that we introduce the fuzzy logarithmic Sobolev inequalities , fuzzy logarithmic Sobolev constant , and the hypercontractivity of the fuzzy Markov semi-group , to give the quantative bounds for the convergence of the fuzzy kernel of the fuzzy Markov semi-group to stationary fuzzy probability measure using the total variation distance . These bounds depend on the fuzzy logarithmic Sobolev constant .

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Introduction

Because of importance and utility of fuzzy Markov chains , there are many connections that can be drawn to questions in analysis .

The main aim of our thesis is to give the quantative bounds for the convergence of the fuzzy kernel of the fuzzy Markov semi-group to stationary fuzzy probability measure using the total variation distance and these bounds depend on the fuzzy logarithmic Sobolev constant.

Our method in this thesis is theoretical .

In 1907 [12] , A.A. Markov began the study of an important new type of chance process . In this process , the outcome of a given experiment can affect the outcome of the next experiment , this type of process is called a Markov chains .

Fuzzy sets have been introduced by Lotfi Zadeh in 1965 . [23] as an extension of the classical notion of set . In classical set theory , the membership of elements in a set is assessed in binary terms according to a bivalent condition – an element either belongs or does not belong to the set . By contrast , fuzzy set theory permits the gradual assessment of the membership of elements in a set , this is described with the aid of a membership function valued in the real unite interval $[0,1]$. Fuzzy set generalizes classical sets , since the indictor functions of classical sets are special cases of membership functions of fuzzy sets , if the latter only take values 0 or 1.

In 2005 [2] J.J. Buckley and E . Eslami defined the fuzzy Markov chains by using a restricted fuzzy matrix multiplication through a transition probability $k(x,y)$ replaced by a triangular fuzzy number $\overline{k}(x,y)$ and restriction on $\overline{k}(x,y)$:- There are $k(x,y) \in \overline{k}(x,y)[1]$.

The theory of operator semi-group originates from the study of the equation

$$T(t+s) = T(t).T(s) \ , \ T(0) = I ....(1)$$

Where $T(t)$is an operator –valued function taking values in the set of bounded linear operators acting on a suitable functions space . This problem was independently studied by Hill and Yoside around 1948 [13] . The equation (1)bears a resemblance to the exponential Cauchy equation

$$g(t+s) = g(t) . g(s) \ , \ g(0) = 1.........(2)$$

Where $g(t)$is a non-negative function from R to R . The solution to the exponential Cauchy equation is well known :-It is the family exponential functions

$$g(t) = e^{r\,t} \ , \ r \in R$$

However this family represents all possible solutions only if additional assumption of continuity is made . The assumption that $g(t)$ is continuous from right in the origin is already sufficient to make the functions $g(t)=e^{rt}$, $r \in R$ the only solutions of (2).

Logarithmic Sobolev inequalities were introduced in 1975 [6] as a way of isolating smoothing properties of Markov semi-group in infinite-dimensional settings .They were defined by

$$\zeta(f) \leq c\ell(f, f)$$

Where f is a real valued function . The entropy ζ is

$$\zeta(f)=\sum_{x}|f(x)|^2 \log\left(\frac{|f(x)|^2}{\|f\|_2^2}\right)\pi(x)$$

And ℓ is Dirichlet form , such that

$$\ell(f, f)=\frac{1}{2}\sum_{x,y}|f(x)-f(y)|^2 k(x,y)\pi(x)$$

c is constant .

Logarithmic Sobolev constant of the Markov chains is

$$\wp=\inf\left\{\frac{\ell(f, f)}{\zeta(f)} : \zeta(f) \neq 0\right\}$$

And $\dfrac{1}{\wp}$ is the smallest constant c .

In 1996 [6] Diaconis and Saloff used logarithmic Sobolev constant for bounding rates of convergence of Markov chains on a finite state space to there stationary distributions , where the problem is discussed as follows :-

They worked with a finite state space Ω and an irreducible Markov kernel

$$k(x,y) \geq 0 \ \ , \ \sum_{y} k(x,y) = 1$$

The continuous time semi-group associated to a Markov operator is $H_t = \exp(-t(I - \mathrm{K}))$. Its kernel is denoted by $H_t^x(y) = H_t(x,y)$ which is the distribution at time $t > 0$ of the process started at x . It has a unique stationary probability measure π and $H_t^x(y) \to \pi(y)$ as t tend to infinite . They got quantative bounds on this convergence for instance in total variation distance $\left\| H_t^x - \pi \right\|_{TV}$, by using log- Sobolev constant .

We will discuss this convergence through fuzzy logic by using the triangular fuzzy number , where we will prove convergence of the fuzzy kernel of the fuzzy Markov semi-group $\overline{H}_t^x$ to invariant probability fuzzy measure $\overline{\pi}$ in total variation distance namely

$$\left\| \overline{H}_t^x - \overline{\pi} \right\|_{TV}$$

Our main theorems are

Theorem I

Let $\left(\overline{k},\overline{\pi}\right)$ be a fuzzy finite Markov chain with a fuzzy log-Sobolev constant $\overline{\wp}$.

$i)$ Assume that there exists $\beta > 0$ such that

$$\overline{\left\|\overline{H}_t\right\|_{2\to q}} < 1$$

for all $t > 0$ and $2 \le q(t) < \infty$ satisfying $q(t) - 1 \le e^{\beta t}$, then $\beta\overline{\zeta}(f) \le \overline{\ell}(f,f)$ and thus $\wp > \beta$.

$ii)$ Assume that $\left(\overline{k},\overline{\pi}\right)$ is reversible , then $\overline{\left\|\overline{H}_t\right\|_{2\to q}} < 1$ for all $t > 0$ and $2 \le q(t) < \infty$ satisfying $q(t) - 1 \le e^{4\wp t}$.

$iii)$ For non reversible chains , we still have $\overline{\left\|\overline{H}_t\right\|_{2\to q}} < 1$ for all $t > 0$ and $2 \le q(t) < \infty$ satisfying $q(t) - 1 \le e^{2\wp t}$.

Theorem II

Let $\overline{k}$ be a fuzzy finite Markov chain with invariant fuzzy probability measure $\overline{\pi}$ and fuzzy log-Sobolev constant $\overline{\wp}$. Then for any fuzzy probability measure $\overline{\mu} = f\,\overline{\pi}$ on a state space Ω, we have

$$\overline{Ent}\left(\overline{\mu}\,\overline{H}_t\right) \le e^{-2\overline{\wp}t}\ \overline{Ent}\left(\overline{\mu}\right), t > 0\cdot$$

Further , if we assume that $\left(\overline{k}, \overline{\pi}\right)$ is reversible , then

$$\overline{Ent}\left(\overline{\mu}\,\overline{H}_t\right) \le e^{-4\overline{\wp}t}\ \overline{Ent}\left(\overline{\mu}\right), t > 0\cdot$$

Corollary III

Let $\left(\overline{k}, \overline{\pi}\right)$ be a fuzzy finite Markov chain , and $\overline{\wp}$ be a fuzzy log-Sobolev constant we have

$$2\left\|\overline{H}_t^x - \overline{\pi}\right\|_{TV}^2 \le \left(\log \frac{1}{\overline{\pi}(x)}\right) e^{-2\overline{\wp}t}$$

If we assume that $\left(\overline{k}, \overline{\pi}\right)$ is reversible then

$$2\left\|\overline{H}_t^x - \overline{\pi}\right\|_{TV}^2 \le \left(\log \frac{1}{\overline{\pi}(x)}\right) e^{-4\overline{\wp}t}$$

Corollary IV

Let $\left(\overline{k}, \overline{\pi}\right)$ be a fuzzy finite Markov chain , then

$$4\left\|\overline{H}_t^x - \overline{\pi}\right\|_{TV}^2 \le \frac{1}{\overline{\pi}(x)} e^{-2\overline{\lambda}t}$$

Corollary V

Let $\left(\overline{k}, \overline{\pi}\right)$ be a fuzzy finite Markov chain , then

$$\left\| \overline{H}_t^x - \overline{\pi} \right\|_{TV} \leq \left(\log\left(\frac{1}{\overline{\pi}(x)} \right) \right)^{1/2} e^{-\overline{\wp} t}$$

Further , if $\left(\overline{k}, \overline{\pi}\right)$ is reversible , then

$$\left\| \overline{H}_t^x - \overline{\pi} \right\|_{TV} \leq \left(\log\left(\frac{1}{\overline{\pi}(x)} \right) \right)^{1/2} e^{-2\overline{\wp} t}$$

Theorem VI

Let $\left(\overline{k}, \overline{\pi}\right)$ be a fuzzy finite Markov chain . Assume that $\overline{\pi}(x) \leq \frac{1}{e}$, then

$$\overline{\left\| \overline{h}_t^x - 1 \right\|_2} \leq e^{1-\overline{\lambda} t} \quad , t = \frac{1}{2\wp} \log\log \frac{1}{\pi(x)} + c , c > 0$$

For reversible fuzzy chains , the inequality holds for

$$t = (4\wp)^{-1} \log\log \frac{1}{\pi(x)} + c \ , \ c > 0$$

Corollary VII

Let $\left(\overline{k}, \overline{\pi}\right)$ be a fuzzy finite Markov chain , then

$$\left\| \overline{H}_t^{x} - \overline{\pi} \right\|_{TV} \leq e^{1-2\overline{\wp} t} \ , \ t = \frac{1}{2\wp} \log\log \frac{1}{\pi(x)} + c, c > 0$$

For reversible fuzzy chains , the inequality holds for

$$t = (4\wp)^{-1} \log\log \frac{1}{\pi(x)} + c \ , \ c > 0$$

Corollary VIII

Let $\left(\overline{k}, \overline{\pi}\right)$ be a fuzzy finite Markov chain , then the maximal relative error is

$$\sup_{x,\, y} \left| \overline{h}_t(x,y) - 1 \right| \leq e^{2-\left(\overline{\lambda} + \overline{\lambda}^{*}\right) t}$$

For

$$t = \frac{1}{2\wp} \log\log \frac{1}{\pi_*(x)} + c \ , \pi_*(x) = \min \pi(x), c > 0$$

Further , if $\left(\overline{k}, \overline{\pi}\right)$ is reversible , then

$$\sup_{x, y} \left| \overline{h}_t(x,y) - 1 \right| \leq e^{2\left(1-\overline{\lambda}\, t\right)}$$

For

$$t = (4\wp)^{-1} \log\log \frac{1}{\pi_*(x)} + c, \pi_*(x) = \min \pi(x), c > 0$$

Corollary IX

Assume that $\left(\overline{k}, \overline{\pi}\right)$ is reversible chain and $\overline{\pi}(x) \leq \frac{1}{e}$. Set $\lambda_* = \min\{\lambda, 1+\beta_{\min}\}$

$$\overline{\left\|\overline{\kappa}^{\,n}_{\,x} - 1\right\|}_2 \leq \left(1+2e^2\right)^{1/2} e^{\frac{-n}{2}\overline{\lambda}}$$

For $$n \geq \frac{1}{4\wp} \log\log \frac{1}{\pi(x)} + \frac{c}{\lambda_*} + 1 \text{ , } c > 0 \cdot$$

Further , setting $\pi_*(x) = \min_x \pi(x)$ we setting

$$\sup_{x,y}\left|\overline{\kappa}^{\,2n}(x,y) - 1\right| \leq \left(1+2e^2\right) e^{-n\overline{\lambda}}$$

For $$n \geq \frac{1}{4\wp} \log\log \frac{1}{\pi_*(x)} + \frac{c}{\lambda_*} + 1 \text{ , } c > 0 \cdot$$

From our work we conclude that:- If $\overline{H_t^x}$ is the fuzzy kernel of the fuzzy Markov semi-group and $\overline{\pi}$ is invariant fuzzy probability measure , then we get quantative bounds on convergence $\overline{H_t^x}$ to $\overline{\pi}$ in total variation distance by using the fuzzy log-Sobolev constant $\overline{\wp}$.

CHAPTER ONE

Fuzzy Sets and Fuzzy Logic : An Overview

In order to draw a meaningful picture in our minds for our work and prepare the background for our work and motivate our results ; we understand the need to recall definitions and some results related to the basic concepts for our work .

1.1.The Fuzzy Sets

Definition 1.1.1.[24]

If X is a collection of objects denoted generically by x, then a fuzzy set $\overline{A}$ in X is a set of order pairs :-

$$\overline{A} = \left\{ \left(x, \overline{A}(x) \right) : x \in X \right\}$$

$\overline{A}(x)$ is called the membership function or grade of membership of x in $\overline{A}$ that maps X to the unite interval $[0, 1]$.

Example 1.1.2.[24]

$\overline{A} =$ "real numbers close to 10 "

Where

$$\overline{A} = \left\{ \left(x, \overline{A}(x) \right) : \overline{A}(x) = \left(1 + (x - 10)^{-2}\right)^{-1} \right\}$$

Definition 1.1.3.[23]

The standard intersection of fuzzy sets $\overline{A}$ and $\overline{B}$ is defined as

$$\left(\overline{A} \cap \overline{B} \right)(x) = \min \left\{ \overline{A}(x), \overline{B}(x) \right\}$$

$$= \overline{A}(x) \wedge \overline{B}(x)$$

For all $x \in X$.

Definition 1.1.4.[23]

The standard union of fuzzy sets $\overline{A}$ and $\overline{B}$ is defined as

$$\left(\overline{A} \bigcup \overline{B}\right)(x) = \max \left\{\overline{A}(x), \overline{B}(x)\right\}$$

$$= \overline{A}(x) \vee \overline{B}(x)$$

For all $x \in X$.

Definition 1.1.5.[23]

The standard complement of a fuzzy set $\overline{A}$ is defined as

$$\left(\neg \overline{A}\right)(x) = 1 - \overline{A}(x)$$

Remarks 1.1.6

A closely related pair of properties which hold in ordinary set theory are the law of excluded middle

$$A \vee \neg A = X$$

and the law of contradiction principle

$$A \wedge \neg A = \phi$$

Proposition 1.1.7.[23]

The law of excluded middle and contradiction are not satisfied in fuzzy logic .

Proof :-Let $\overline{A}(x) = \frac{1}{2}$ for all $x \in R$, then

$$\left(\neg \overline{A} \vee \overline{A}\right)(x) = \max\left\{\neg \overline{A}(x), \overline{A}(x)\right\}$$
$$= \max\left\{1 - \frac{1}{2}, \frac{1}{2}\right\}$$
$$= \frac{1}{2} \neq 1$$

And

$$\left(\neg \overline{A} \wedge \overline{A}\right)(x) = \min\left\{\neg \overline{A}(x), \overline{A}(x)\right\}$$
$$= \min\left\{1 - \frac{1}{2}, \frac{1}{2}\right\}$$
$$= \frac{1}{2} \neq 0$$

Definition 1.1.8.[23]

let $\overline{A}$ be a fuzzy set of X, the support of $\overline{A}$ denoted $S\left(\overline{A}\right)$ is the crisp set of X whose elements all have non zero membership grades in $\overline{A}$, that is

$$S\left(\overline{A}\right) = \left\{x \in X : \overline{A}(x) > 0\right\}$$

Definition 1.1.9.[23]

(α -cut) An α -level set of a fuzzy set $\overline{A}$ of X is a non fuzzy (crisp) set denoted by $\overline{A}[\alpha]$, such that

$$\overline{A}[\alpha]=\begin{cases}\left\{x\in X:\overline{A}(x)\geq\alpha\right\}, if \alpha>0\\ cl\left(S\left(\overline{A}\right)\right) \qquad , if \alpha=0\end{cases}$$

Where $cl(S(\overline{A}))$ denotes closure of the support of $\overline{A}$.

Example 1.1.10.[23]

Assume $X=\{-2,-1,0,1,2,3,4\}$ and the fuzzy set $\overline{A}$ is

$\overline{A}=\{(-2, 0.0), (-1, 0.3), (0, 0.6), (1, 1.0), (2, 0.6), (3, 0.3), (4, 0.0)\}$

It α -cuts

$$\overline{A}[\alpha]=\begin{cases}\{1\} & ,\alpha=1.0\\ \{0,1,2\} & ,\alpha=0.6\\ \{-1,0,1,2,3\} & ,\alpha=0.3\end{cases}$$

And

$$S\left(\overline{A}\right)=\{-1,0,1,2,3\}$$

Theorem 1.1.11.[5]

Let $\overline{A}$ be a fuzzy set in X with the membership function $\overline{A}(x)$.

Let $\overline{A}[\alpha]$ be the α -cuts of $\overline{A}$ and $\chi_{\overline{A}[\alpha]}(x)$ be the characteristic function of the crisp set $\overline{A}[\alpha]$ for all $\alpha \in [0,1]$. Then

$$\overline{A}(x) = \sup_{\alpha \in [0,1]} \left(\alpha \wedge \chi_{\overline{A}[\alpha]}(x) \right), x \in X$$

Proof :-Since $\chi_{\overline{A}[\alpha]}(x)$ is the characteristic function of the crisp set $\overline{A}[\alpha]$, it takes the value 1 if $x \in \overline{A}[\alpha]$ and takes the value 0 if $x \notin \overline{A}[\alpha]$, therefore

If $x \in \overline{A}[\alpha]$ then $\chi_{\overline{A}[\alpha]}(x) = 1$, $\left(\overline{A}(x) \geq \alpha \right)$

And

If $x \notin \overline{A}[\alpha]$ then $\chi_{\overline{A}[\alpha]}(x) \neq 0$, $\left(\overline{A}(x) < \alpha \right)$

Now

$$\sup_{\alpha \in [0,1]} \left(\alpha \wedge \chi_{\overline{A}[\alpha]}(x) \right) = \left(\sup_{\alpha \in \left[0, \overline{A}(x)\right]} \left(\alpha \wedge \chi_{\overline{A}[\alpha]}(x) \right) \right) \vee \left(\sup_{\alpha \in \left(\overline{A}(x), 1\right]} \left(\alpha \wedge \chi_{\overline{A}[\alpha]}(x) \right) \right)$$

$$= \left(\sup_{\alpha \in \left[0, \overline{A}(x)\right]} (\alpha \wedge 1) \right) \vee \left(\sup_{\alpha \in \left(\overline{A}(x)\ ,\ 1 \right]} (\alpha \wedge 0) \right)$$

$$= \sup_{\alpha \in \left[0, \overline{A}(x)\right]} \alpha$$

$$= \overline{A}(x)$$

■

Remarked 1.1.12

Given a fuzzy set $\overline{A}$ in X , one consider a special fuzzy set denoted $\alpha \overline{A}[\alpha]$ for $\alpha \in [0,1]$ whose membership function is defined as

$$\overline{A}_{\alpha \overline{A}[\alpha]}(x) = \left(\alpha \wedge \chi_{\overline{A}[\alpha]}(x) \right),\ x \in X$$

And the set

$$\Lambda_{\overline{A}} = \left\{ \alpha : \overline{A}(x) = \alpha,\ x \in X \right\}$$

is called the level set of $\overline{A}$. Then the above theorem states that the fuzzy set $\overline{A}$ can be expressed in the form

$$\overline{A} = \bigcup_{\alpha \in \Lambda_{\overline{A}}} \left(\alpha \overline{A}[\alpha] \right)$$

Where $\cup$ denotes the standard fuzzy union . This result is called the resolution principle of fuzzy sets . The essence of resolution principle is that a fuzzy set $\overline{A}$ can be decomposed in to fuzzy sets $\alpha\,\overline{A}[\alpha]$, $\alpha \in [0, 1]$.

Definition 1.1.13.[5]

A fuzzy set $\overline{A}$ of a classical set X is called normal , if there exists an $x \in X$, such that $\overline{A}(x) = 1$. Otherwise $\overline{A}$ is subnormal .

Example 1.1.14.

The fuzzy set in the Example 1.1.10 is normal , since $\overline{A}(1) = 1$.

Definition 1.1.15.[23]

A fuzzy set $\overline{A}$ of X is called convex , if $\overline{A}[\alpha]$ is a convex subset of X , for all $\alpha \in [0,1]$. That is , for any x , $y \in \overline{A}[\alpha]$, and for any $\lambda \in [0,1]$ then

$$\lambda\, x + (1 - \lambda)\, y \in \overline{A}[\alpha]$$

The convex means that any α -cut which parallel to the horizontal axis through interval .

Definition 1.1.16.[3]

A fuzzy set $\overline{A}$ whose $S(\overline{A})$ contains a single point $x \in X$, with $\overline{A}(x) = 1$, is referred to as a fuzzy set singleton .

Definition 1.1.17.[3]

The empty fuzzy set of X is defined as

$$\Phi = \{(x,0) : \forall x \in X\}$$

Definition 1.1.18.[3]

The largest fuzzy set in X is defined as

$$I_X = \{(x,1) : \forall x \in X\}$$

Definition 1.1.19.[3]

The concept of continuity is same as in other functions, that say, a function f is continuous at some number c if

$$\lim_{x \to c} f(x) = f(c)$$

for all x in range of f, that require existing $f(c)$ and $\lim_{x \to c} f(x)$.

In fuzzy set theory the condition will be

$$\lim_{x \to c} \overline{A}(x) = \overline{A}(c)$$

With x and $c \in \overline{A}$.

Definition 1.1.20.[23]

A fuzzy set $\overline{A}$ is said to be a bounded fuzzy set , if it α -cuts $\overline{A}[\alpha]$ are (crisp) bounded sets , for all $\alpha \in [0,1]$.

Definition 1.1.21.[23]

A fuzzy number $\overline{A}$ is a fuzzy set of the real line with a normal ,(fuzzy) convex , and continuous membership function of bounded support .

Example 1.1.22.[23]

The following fuzzy set is fuzzy number approximately

$$"5" = \{(3,0.2),(4,0.6),(5,1.0),(6,0.7),(7,0.1)\}$$

Proposition 1.1.23

Let $\overline{A}$ be a fuzzy number , then $\overline{A}[\alpha]$ is a closed , convex , and compact subset of R, for all $\alpha \in [0,1]$.

Proof :-We have

$$\overline{A}[\alpha] = \left\{ x \in X : \overline{A}(x) \geq \alpha \right\}$$

Which implies that $[\alpha,1]$ is a closed interval , that is , for any sequence of degrees $\langle\beta\rangle, \alpha \leq \beta \leq 1$, converges to a degree γ in

$[\alpha,1]$, then for any sequence in $\overline{A}[\alpha]$, associate $\langle\beta\rangle$ converges

to an element in $\overline{A}[\alpha]$ with degree γ . So $\overline{A}[\alpha]$ is a closed subset of R **.**

Since $\overline{A}$ is a fuzzy number , so $\overline{A}$ is a convex fuzzy set ,

which implies that $\overline{A}[\alpha]$ is convex .

Since $\overline{A}[\alpha] \subseteq S\left(\overline{A}\right)$ for any $\alpha \in [0,1]$, then $\overline{A}[\alpha]$ is bounded .

By Hein-Borel theorem [15][Every bounded closed set in Euclidean

space is compact] . Thus $\overline{A}[\alpha]$ is compact ■

Remark 1.1.24

We shall use the notation

$$\overline{A}[\alpha] = [a_1(\alpha), a_2(\alpha)]$$

Where $\overline{A}[\alpha]$ is an α -cut of the fuzzy number $\overline{A}$, and

$$a_1 : [0,1] \to R \quad , \quad a_1(\alpha) = \min \overline{A}[\alpha] \, ,$$

is left hand side function which monotone, increasing and continuous.

$$a_2 : [0,1] \rightarrow R \quad , \ a_2(\alpha) = \max \overline{A}[\alpha]$$

is right hand side function which monotone decreasing and continuous.

Proposition 1.1.25

If $\alpha \leq \beta$, then $\overline{A}[\alpha] \supset \overline{A}[\beta]$.

Proof :- We have

$$\overline{A}[\alpha] = [a_1(\alpha), a_2(\alpha)]$$

And

$$\overline{A}[\beta] = [a_1(\beta), a_2(\beta)]$$

Since $\alpha \leq \beta$,then

$$a_1(\alpha) = \min \ \overline{A}[\alpha] \leq \min \ \overline{A}[\beta] = a_1(\beta)$$

And

$$a_2(\alpha) = \max \overline{A}[\alpha] \geq \max \overline{A}[\beta] = a_2(\beta)$$

Hence

$$\overline{A}[\alpha] \supset \overline{A}[\beta] \ \blacksquare$$

Proposition 1.1.26

The support of a fuzzy number is an open interval $(a_1(0), a_2(0))$.

Proof :- Suppose that $S\left(\overline{A}\right)$ is a closed interval , that is , $S\left(\overline{A}\right)$ contains all limit points , so

$$S\left(\overline{A}\right) = cl\left(S\left(\overline{A}\right)\right) = \left\{x \in X \ : \overline{A}(x) \geq 0\right\}$$

but this contradict definition 1.1.8 , therefore $S\left(\overline{A}\right)$ is an open interval $(a_1(0), a_2(0))$ ■

Definition 1.1.27.[3]

A fuzzy number $\overline{A}$ is called a triangular fuzzy number , where $\overline{A}$ is defined by three numbers $a_1 < a_2 < a_3$ if :-

$i)$ $\overline{A}(x) = 1$ at $x = a_2$, ($\overline{A}$ is normal)

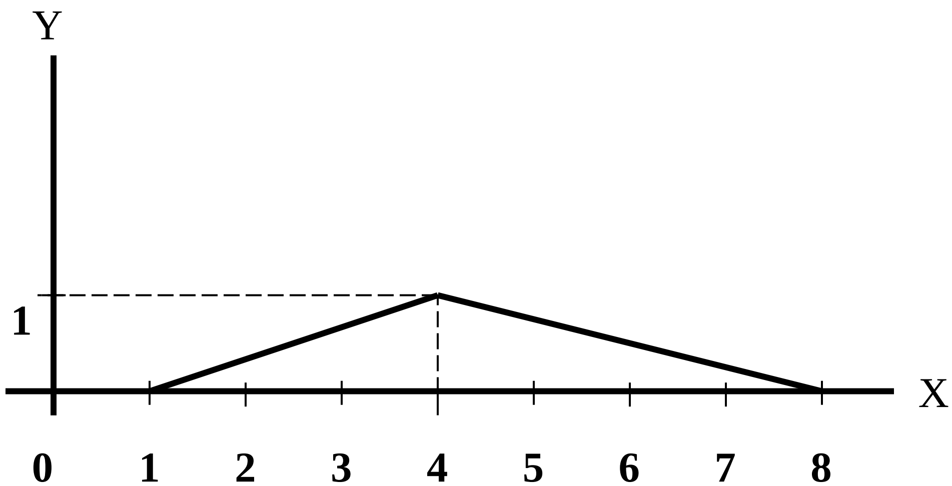
$ii)$ The graph of $y = \overline{A}(x)$ on $[a_1, a_2]$ is straight line from $(a_1,0)$ to $(a_2,1)$, also on $[a_1, a_2]$ the graph of $y = \overline{A}(x)$ is straight line from $(a_2,1)$ to $(a_3,0)$.

$iii)$ $\overline{A}(x) = 0$ for $x \leq a_1$ or $x \geq a_3$.

We write $\overline{A} = (a_1, a_2, a_3)$ for triangular fuzzy number and its α -cut $\overline{A}[\alpha] = [(a_2 - a_1)\alpha + a_1, (a_2 - a_3)\alpha + a_3]$ for all $\alpha \in [0,1]$.

Example 1.1.28.[23]

$\overline{A} = (1, 4, 8)$ is triangular fuzzy number , where



triangular fuzzy number

1.2. Arithmetic Operations on Fuzzy Numbers

We will define the arithmetic operations on fuzzy numbers based on resolution principle (α -cuts) .

Definition 1.2.1.[5]

Let $\overline{A}$ and $\overline{B}$ be two fuzzy numbers and $\overline{A}[\alpha]=[a_1(\alpha),a_2(\alpha)]$, $\overline{B}[\alpha]=[b_1(\alpha),b_2(\alpha)]$ be α -cuts , $\alpha\in[0,1]$ of $\overline{A}$ and $\overline{B}$ respectively. Then the operation ($*$ denoted any of the arithmetic operations $(+),(-),(\cdot),(\div),\wedge,\vee$) on fuzzy numbers $\overline{A}$ and $\overline{B}$ denoted by $\overline{A}*\overline{B}$ gives a fuzzy number in R , where

$$\overline{A}*\overline{B}=\bigcup_{\alpha}\alpha\left(\overline{A}*\overline{B}\right)[\alpha]$$

And

$$\left(\overline{A}*\overline{B}\right)[\alpha]=\overline{A}[\alpha]*\overline{B}[\alpha],\ \alpha\in[0,1]$$

Here it may be remarked that the reason for $\overline{A}*\overline{B}$ to be a fuzzy number , and not just a general fuzzy set , is that $\overline{A}$ and $\overline{B}$ being fuzzy numbers , the sets $\overline{A}[\alpha]$, $\overline{B}[\alpha]$, $\left(\overline{A}*\overline{B}\right)[\alpha]$, are all closed intervals for all $\alpha\in[0,1]$.

In particular

$$\overline{A}[\alpha](+)\overline{B}[\alpha]=[a_1(\alpha)+b_1(\alpha)\,,a_2(\alpha)+b_2(\alpha)]$$

$$\overline{A}[\alpha](-)\overline{B}[\alpha]=[a_1(\alpha)-b_2(\alpha)\,,a_2(\alpha)-b_1(\alpha)]$$

Further , for fuzzy numbers $\overline{A}$ and $\overline{B}$ in R_+

$$\overline{A}[\alpha](\cdot)\overline{B}[\alpha]=[a_1(\alpha)\cdot b_1(\alpha)\,,a_2(\alpha)\cdot b_2(\alpha)]$$

$$\overline{A}[\alpha](:)\overline{B}[\alpha]=\left[\frac{a_1(\alpha)}{b_2(\alpha)},\frac{a_2(\alpha)}{b_1(\alpha)}\right],\ 0\notin[b_1(\alpha),b_2(\alpha)]$$

The multiplication of a fuzzy number $\overline{A}$ in R by a real number $c>0$ is

$$\left(c\cdot\overline{A}\right)[\alpha]=c\cdot\overline{A}[\alpha]=[ca_1(\alpha),ca_2(\alpha)]$$

Example 1.1.30[9]

Consider two triangular fuzzy numbers $\overline{A}$ and $\overline{B}$ defined as

$$\overline{A}(x)=\begin{cases}0 & for\ x\le -1\ ,\ x\ge 3\\ \dfrac{(x+1)}{2} & for\ -1<x\le 1\\ \dfrac{(3-x)}{2} & for\ 1<x<3\end{cases}$$

$$\overline{B}(x) = \begin{cases} 0 & for\ x \le 1\ ,\ x \ge 5 \\ \dfrac{(x-1)}{2} & for\ 1 < x \le 3 \\ \dfrac{(5-x)}{2} & for\ 3 < x < 5 \end{cases}$$

Their α -cuts are

$$\overline{A}[\alpha] = [2\alpha - 1\ ,\ 3 - 2\alpha]$$

$$\overline{B}[\alpha] = [2\alpha + 1\ ,\ 5 - 2\alpha]$$

Now

$$\left(\overline{A} + \overline{B}\right)[\alpha] = [4\alpha\ ,\ 8 - 4\alpha],\ \alpha \in [0,1]$$

The resulting fuzzy number is then

$$\overline{A} + \overline{B}(x) = \begin{cases} 0 & for\, x \le 0\ ,\ x \ge 8 \\ \dfrac{x}{4} & for\ 0 < x \le 4 \\ \dfrac{(8-x)}{4} & for\, 4 < x < 8 \end{cases}$$

1.3.Lattice of Fuzzy Numbers [9]

Let $\mathfrak{R}$ denote the set of all fuzzy numbers . Then the operations MIN and MAX are functions of the form $\mathfrak{R} \times \mathfrak{R} \longrightarrow \mathfrak{R}$ such that :-

$$i)\ MIN\left(\overline{A}, \overline{B}\right) = MIN\left(\overline{B}, \overline{A}\right)$$

$$MAX\left(\overline{A}, \overline{B}\right) = MAX\left(\overline{B}, \overline{A}\right)$$ (commutativity)

$$ii)\ MIN\left[MIN\left(\overline{A}, \overline{B}\right), \overline{C}\right] = MIN\left[\overline{A}, MIN\left(\overline{B}, \overline{C}\right)\right]$$

$$MAX\left[MAX\left(\overline{A}, \overline{B}\right), \overline{C}\right] = MAX\left[\overline{A}, MAX\left(\overline{B}, \overline{C}\right)\right]$$

(associativity).

$$iii)\ MIN\left(\overline{A}, \overline{A}\right) = \overline{A}$$

$$MAX\left(\overline{A}, \overline{A}\right) = \overline{A}$$ (idempotence)

$$iv)\ MIN\left[\overline{A}, MAX\left(\overline{A}, \overline{B}\right)\right] = \overline{A}$$

$$MAX\left[\overline{A}, MIN\left(\overline{A}, \overline{B}\right)\right] = \overline{A}$$ (absorption)

$$v)\ MIN\left[\overline{A}, MAX\left(\overline{B}, \overline{C}\right)\right] = MAX\left[MIN\left(\overline{A}, \overline{B}\right), MIN\left(\overline{A}, \overline{C}\right)\right]$$

$$MAX\left[\overline{A}, MIN\left(\overline{B}, \overline{C}\right)\right] = MIN\left[MAX\left(\overline{A}, \overline{B}\right), MAX\left(\overline{A}, \overline{C}\right)\right]$$

(distributivity).

The triple $(\Re, MIN, MAX)$ is called lattice of fuzzy numbers .

The triple $(\Re, MIN, MAX)$ can be expressed as the pair $(\Re, \leq)$, where $\leq$ is a partial ordering defined as

$$\overline{A} \leq \overline{B} \quad iff \quad MIN\left(\overline{A}\,,\overline{B}\right) = \overline{A}$$

Or , alternatively

$$\overline{A} \leq \overline{B} \quad iff \quad MAX\left(\overline{A}\,,\overline{B}\right) = \overline{B}$$

for any $\overline{A}\,,\overline{B} \in \Re$.

Now , this partial ordering can be defined in terms of the relevant α -cuts :-

$$\overline{A} \leq \overline{B} \quad iff \quad MIN\left(\overline{A}[\alpha]\,,\overline{B}[\alpha]\right) = \overline{A}[\alpha]$$

$$\overline{A} \leq \overline{B} \quad iff \quad MAX\left(\overline{A}[\alpha],\overline{B}[\alpha]\right) = \overline{B}[\alpha]$$

for any $\overline{A}\,,\overline{B} \in \Re$ and $\alpha \in [0,1]$,Where $\overline{A}[\alpha]$ and $\overline{B}[\alpha]$ are closed intervals , then

$$MIN\left(\overline{A}[\alpha],\overline{B}[\alpha]\right)=[MIN(a_1,b_1),MIN(a_2,b_2)]$$

$$MAX\left(\overline{A}[\alpha],\overline{B}[\alpha]\right)=[MAX(a_1,b_1),MAX(a_2,b_2)]$$

If we define the partial ordering of closed intervals , that is

$$[a_1,a_2]\leq[b_1,b_2] \;\; iff \;\; a_1\leq b_1 \;,\; a_2\leq b_2$$

Then for any $\overline{A}\,,\overline{B}\in\Re$, we have

$$\overline{A}\leq\overline{B} \;\; iff \;\; \overline{A}[\alpha]\leq\overline{B}[\alpha]$$

for all $\alpha\in[0,1]$. For example , we have in example 1.2.2

that $\overline{A}\leq\overline{B}$ since $\overline{A}[\alpha]\leq\overline{B}[\alpha]$ for all $\alpha\in[0,1]$.

CHATER TWO

Fuzzy Markov Semi-Group

In this chapter we have introduced definitions of fuzzy Markov chains , fuzzy spaces , fuzzy Markov operator, fuzzy Markov semi-group and other concepts .

Definition 2.1.[15]

Let S be a subset of R ,we say that $f:S \to R$ is a μ - measurable function if for every open set G in R, then the set $f^{-1}(G) \subseteq S$ is measurable set .

Definition 2.2.[15]

A function f is said to be simple , if it is μ -measurable and takes no more than countably many distinct values .

Definition 2.3.[15]

Let f be a simple function , that is , a μ -measurable function taking no more than countably many distinct values

$$y_1, y_2, \ldots, y_n$$

Then the lebesgue integral of f over the set A , denoted by

$$\int_A f(x) d\mu$$

We mean the quantity

$$\sum_n y_n \mu(A_n)$$

Where

$$A_n = \{x \in A : f(x) = y_n\}.$$

Definition 2.4.[10]

A probability trio $(\Omega, \Sigma, \mathrm{P})$ is defined by the probability space , where Ω (state space) the set of all possible outcomes of an experiment . Σ (Borel field) the set of subsets of Ω which satisfy three properties:-

$i)\ \Omega \in \Sigma$

$ii)\ A \in \Sigma$ then $A^C = \Omega / A \in \Sigma$

$iii)\ A_n \in \Sigma$ then $\bigcup_{n \in N} A_n \in \Sigma$

P is a probability measure which a function from Σ to the real number that assigns to each event a probability between 0 and 1 such that:-

$i)\ \mathrm{P}(\Omega) = 1$

$ii)\ \mathrm{P}(A_n) \geq 0 \quad A_n \in \Sigma$

$iii)\ \mathrm{P}\left(\bigcup_n A_n\right) = \sum_{n=1}^{\infty} \mathrm{P}(A_n)$, $A_i \cap A_j = \phi$, $i \neq j$

Definition 2.5.[6]

A Markov chain on a finite state space Ω with cardinality $|\Omega|$ can be described through its transition kernel k which is a function on $\Omega \times \Omega$ satisfying

$$k(x, y) \geq 0 \quad , \quad \sum_y k(x, y) = 1$$

Note that :-If we substitute a transition kernel $k(x, y)$ by a triangular fuzzy number $\overline{k}(x, y)$ such that

$$\overline{k}(x, y) = (k_1(x, y), k_2(x, y), k_3(x, y))$$

Its α -cut

$$\overline{k}(x, y)[\alpha] = [k_1(x, y), k_2(x, y)]$$

And as $k_\alpha(x, y) \in \overline{k}(x, y)[\alpha]$ the α -cuts of triangular fuzzy numbers $\overline{k}(x, y)$ satisfy

$$k_\alpha(x, y) > 0 \quad , \quad \sum_y k_\alpha(x, y) = 1$$

Then we can define a fuzzy Markov chain through its fuzzy transition kernel $\overline{k}$.

Not all the transition kernel $k(x, y)$ need to be fuzzy , some can be crisp (a real number) . If $k(x, y) = 0$,or $k(x, y) = 1$ then we assume that there is no uncertainty in this value . If $0 < k(x, y) < 1$, and there is uncertainty in its value , then we assume that $0 < \overline{k}(x, y) < 1$ also. We put restriction on $\overline{k}(x, y)$:- there are $k_2(x, y) \in \overline{k}(x, y)[1]$, so that the Markov chain defined through its transition kernel with $\sum_y k(x, y) = 1$.

Definition 2.6.[1]

Let P be the transition matrix of the Markov chain where

$$\mathrm{P} = \left(k(x,y)\right)_{x,\,y\in\Omega}$$

Since each row of P has entries that sum to 1, then each row of P is a probability measure denoted by π.

Note that :- If we replace $k(x,y)$ by $\overline{k}(x,y)$, then the fuzzy transition matrix $\overline{\mathrm{P}}$ of the fuzzy Markov chain is

$$\overline{\mathrm{P}} = \left(\overline{k}(x,y)\right)_{x,\,y\in\Omega}$$

Since $\sum_{y} k_{\alpha}(x,y) = 1$ for all $k_{\alpha}(x,y) \in \overline{k}(x,y)[\alpha]$ of $\overline{k}(x,y)$, we have π_{α} a probability measure with $\pi_{\alpha} \in \overline{\pi}[\alpha]$ the α -cut of the triangular fuzzy number $\overline{\pi} = (\pi_1, \pi_2, \pi_3)$. We will call $\overline{\pi}$ the fuzzy probability measure of the fuzzy Markov chain .

Definition 2.7.[1]

A probability measure π for the transition kernel k is called invariant (stationary equilibrium) probability measure if satisfying

$$\sum_x \pi(x)k(x,y) = \pi(y)$$

such a measure always exists and unique .

Note that :-If we assume that $\pi_\alpha(x) \in \overline{\pi}(x)[\alpha]$ of $\overline{\pi}(x)$ for a transition kernel $k(x,y) \in \overline{k}(x,y)[\alpha]$ of $\overline{k}(x,y)$ satisfies

$$\sum_x \pi_\alpha(x)k_\alpha(x,y) = \pi_\alpha(y)$$

where $\pi_\alpha(y) \in \overline{\pi}(y)[\alpha]$ the α -cut of the fuzzy probability measure $\overline{\pi}(y) = (\pi_1(y), \pi_2(y), \pi_3(y))$. Then $\overline{\pi}$ will call invariant fuzzy probability measure .

Definition 2.8.[6]

The space of real valued (simple) functions L^p is defined by

$$L^p = L^p(\pi) = \left\{ f : \| f \|_p \le 1, 1 \le p \le \infty \right\}$$

Where

$$\| f \|_p = \left(\sum_x | f(x) |^p \pi(x) \right)^{1/p}, \quad \| f \|_\infty = \max_x | f(x) |$$

$\sum_x | f(x) |^p \pi(x)$ is Lebesgue Integral with respect to π .

Note that :- If $0 < \| f \|_p < 1 \, , 1 \leq p < \infty$ and $\pi_\alpha \in \overline{\pi}[\alpha]$ of the invariant fuzzy probability measure $\overline{\pi}$ .Then the norm at degree α is

$$\| f \|_{\alpha, p} = \left(\sum_x | f(x) |^p \pi_\alpha(x) \right)^{1/p}$$

and $\|\cdot\|_{\alpha, p} \in \overline{\|\cdot\|}_p[\alpha]$ the α - cut of the triangular fuzzy number $\overline{\|\cdot\|}_p = \left(\|\cdot\|_{1,p}, \|\cdot\|_{2,p}, \|\cdot\|_{3,p} \right)$. We will call $\overline{\|\cdot\|}_p$ the fuzzy norm and the space $L^p\left(\overline{\pi}\right)$ with the fuzzy norm $\overline{\|\cdot\|}_p$ the fuzzy space of simple functions , such that

$$L^p = L^p\left(\overline{\pi}\right) = \left\{ f : \overline{\| f \|}_p < 1 \, , 1 \leq p < \infty \right\}$$

$$\overline{\| f \|}_p = \left(\sum_x | f(x) |^p \, \overline{\pi}(x) \right)^{1/p}$$

and $\sum_x | f(x) |^p \, \overline{\pi}(x)$ is fuzzy Lebesgue integral with respect to $\overline{\pi}$.

Definition 2.9.[14]

Given a finite probability space (Ω, Σ, π).A linear operator $\mathrm{K} : L^p(\pi) \to L^q(\pi)$, $1 \le p \le q \le \infty$ is called a Markov operator if :-

$i)$ $f \ge 0$ then $0 \le \mathrm{K}\, f$

$ii)$ $f \ge 0$ then $\|\mathrm{K}\, f\|_q \le \|f\|_p$

Where

$$\mathrm{K}\, f(x) = \sum_y k(x,y) f(y)$$

Note that :- If $\mathrm{K} > 0$ and we have a transition kernel $k_\alpha \in \overline{k}[\alpha]$ of $\overline{k}$, then the Markov operator at degree α is

$$\mathrm{K}_\alpha : L^p(\pi_\alpha) \to L^q(\pi_\alpha),\ 1 \le p \le q < \infty$$

Satisfies

$i)$ $f \ge 0$ then $0 < \mathrm{K}_\alpha f$

$ii)$ $f \ge 0$ then $\|\mathrm{K}_\alpha f\|_{\alpha,q} \le \|f\|_{\alpha,p}$

Where

$$\mathrm{K}_\alpha f(x) = \sum_y k_\alpha(x,y) f(y)$$

This implies $\mathrm{K}_{\alpha} \in \overline{\mathrm{K}}[\alpha]$ the α - cut of the triangular fuzzy number $\overline{\mathrm{K}} = (\mathrm{K}_1, \mathrm{K}_2, \mathrm{K}_3)$ which will call the fuzzy Markov operator , such that

$$\overline{\mathrm{K}} : L^p\left(\overline{\pi}\right) \to L^q\left(\overline{\pi}\right), 1 \le p \le q < \infty \, , \, 0 < \overline{\mathrm{K}}$$

Example 2.10[14]

Let $\overline{\mathrm{T}}$ be a fuzzy measure preserving transformation on $\left(\Omega, \Sigma, \overline{\pi}\right)$.

The fuzzy operator

$$\overline{F} : L^p\left(\overline{\pi}\right) \to L^q\left(\overline{\pi}\right), 1 \le p \le q < \infty$$

Satisfying

$$\sum_{x \in A} \overline{F}\, f(x) \overline{\pi}(x) = \sum_{x \in \overline{\mathrm{T}}^{-1}(A)} f(x) \overline{\pi}(x)$$

Where $A \in \Sigma$, $\overline{\mathrm{T}}^{-1}$ is inverse of $\overline{\mathrm{T}}$. Its called the fuzzy Perron – Frobenius operator associated to $\overline{\mathrm{T}}$. This fuzzy operator is a fuzzy Markov operator .

Definition 2.11.[6]

The operator norm K is

$$\|\mathrm{K}\|_{p\to q} = \sup \frac{\|\mathrm{K}f\|_q}{\|f\|_p}$$
$$= \sup_{\|f\|_p \leq 1} \|\mathrm{K}f\|_q$$

Note that :-If $0 < \|\mathrm{K}\|_{p\to q} < 1$, and we have a Markov operator $\mathrm{K}_\alpha \in \overline{\mathrm{K}}[\alpha]$ of $\overline{\mathrm{K}}$, $\|\cdot\|_{\alpha,q} \in \overline{\|\cdot\|}_q[\alpha]$ of $\overline{\|\cdot\|}_q$, then the operator norm of K_α at degree α is

$$\|\mathrm{K}_\alpha\|_{\alpha,\, p\to q} = \sup_{\|f\|_{\alpha,p} < 1} \|\mathrm{K}_\alpha\ f\|_{\alpha,q}$$

And $\|\cdot\|_{\alpha,p\to q} \in \overline{\|\cdot\|}_{p\to q}[\alpha]$ the α -cut of the triangular fuzzy number $\overline{\|\cdot\|}_{p\to q} = \left(\|\cdot\|_{1,p\to q}, \|\cdot\|_{2,p\to q}, \|\cdot\|_{3,p\to q}\right)$, which will call the fuzzy operator norm of $\overline{\mathrm{K}}$.

Remark 2.12

If $\|f\|_p = 1$, then the operator norm of $\mathrm{K}_\alpha \in \overline{\mathrm{K}}[\alpha]$ of $\overline{\mathrm{K}}$ is

$$\|\mathrm{K}_\alpha\|_{\alpha,\, p\to q} = \sup_{\|f\|_p = 1} \|\mathrm{K}_\alpha\ f\|_{\alpha,q}$$

Definition 2.13.[6]

The iterated transition kernel k^n is

$$k^n(x,y)=\sum_{z\in\Omega}k^{n-1}(x,z)k(z,y)$$

Note that :-If $k_\alpha^{n-1}(x,z)\in \overline{k}^{n-1}(x,z)[\alpha]$ of the fuzzy transition kernel $\overline{k}^{n-1}(x,z)$, $k_\alpha(z,y)\in \overline{k}(z,y)[\alpha]$ of the fuzzy transition kernel $\overline{k}(z,y)$,then the iterated transition kernel at degree α is

$$k_\alpha^n(x,y)=\sum_{z\in\Omega}k_\alpha^{n-1}(x,z)k_\alpha(z,y)$$

and $k_\alpha^n(x,y)\in \overline{k}^n(x,y)[\alpha]$ the α -cut of the triangular fuzzy number

$$\overline{k}^n(x,y)=\left(k_1^n(x,y),k_2^n(x,y),k_3^n(x,y)\right)$$

We will call $\overline{k}^n(x,y)$ the iterated fuzzy transition kernel .

Definition 2.14.[6]

The transition kernel of a Markov operator K with respect to π is

$$\kappa(x,y)=\frac{k(x,y)}{\pi(y)}$$

Note that :-If we have $k_\alpha \in \overline{k}[\alpha]$ of $\overline{k}$ and $\pi_\alpha \in \overline{\pi}[\alpha]$ of $\overline{\pi}$. Then the transition kernel of a Markov operator $\mathrm{K}_\alpha \in \overline{\mathrm{K}}[\alpha]$ of $\overline{\mathrm{K}}$ with respect to π_α is

$$\kappa_\alpha(x, y) = \frac{k_\alpha(x, y)}{\pi_\alpha(y)}$$

and $\kappa_\alpha \in \overline{\kappa}[\alpha]$ the α -cut of the triangular fuzzy number $\overline{\kappa} = (\kappa_1, \kappa_2, \kappa_3)$ .We will call $\overline{\kappa}$ the fuzzy transition kernel of a fuzzy Markov operator $\overline{\mathrm{K}}$ with respect to $\overline{\pi}$.

Definition 2.15.[6]

The iterated transition kernel of a Markov operator K^n with respect to π is

$$\kappa^n(x, y) = \frac{k^n(x, y)}{\pi(y)}$$

Note that :-If we have $k_\alpha^n \in \overline{k}^n[\alpha]$ of $\overline{k}^n$ and $\pi_\alpha \in \overline{\pi}[\alpha]$ of $\overline{\pi}$, then the iterated transition kernel of a Markov operator $\mathrm{K}_\alpha^n \in \overline{\mathrm{K}}^n[\alpha]$ of $\overline{\mathrm{K}}^n$ with respect to π_α is

$$\kappa_\alpha^n(x,y)=\frac{k_\alpha^n(x,y)}{\pi_\alpha(y)}$$

and $\kappa_\alpha^n \in \overline{\kappa}^{\,n}[\alpha]$ of the triangular fuzzy number $\overline{\kappa}^{\,n} = \left(\kappa_1^n, \kappa_2^n, \kappa_3^n\right)$.

We will call $\overline{\kappa}^{\,n}$ the iterated fuzzy transition kernel of a fuzzy Markov operator $\overline{\mathrm{K}}^{\,n}$ with respect to $\overline{\pi}$.

proposition 2.16

The fuzzy Markov operator $\overline{\mathrm{K}}$ at $\overline{\kappa}$ can be given by

$$\overline{\mathrm{K}}\, f(x)=\sum_y \overline{\kappa}(x,y) f(y)\overline{\pi}(y)$$

for any function f .

Proof :- Let a Markov operator $\mathrm{K}_\alpha \in \overline{\mathrm{K}}[\alpha]$ of $\overline{\mathrm{K}}$ and f be any function , then

$$\begin{aligned}\mathrm{K}_\alpha f(x) &= \sum_y k_\alpha(x,y) f(y)\\ &= \sum_y \kappa_\alpha(x,y) f(y)\pi_\alpha(y)\end{aligned}$$

Since $\kappa_\alpha \in \overline{\kappa}[\alpha]$ of $\overline{\kappa}$ and $\pi_\alpha \in \overline{\pi}[\alpha]$ of $\overline{\pi}$, then

$$\overline{\mathrm{K}}\, f(x)=\sum_y \overline{\kappa}(x,y) f(y)\overline{\pi}(y) \blacksquare$$

Definition 2.17.[11]

Let $L^p(\pi), 1 \le p \le \infty$ be the space of simple functions defined on a state space Ω .The set of all bounded linear functional from $L^p(\pi)$ to R is called the conjugate space of $L^p(\pi), 1 \le p \le \infty$ denoted by $L^{p'}(\pi), 1 \le p' \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Note that :- If we have $L^p\left(\bar{\pi}\right), 1 \le p < \infty$ the fuzzy space of simple functions , then the set of all bounded linear functional on $L^p\left(\bar{\pi}\right)$ to R is called the conjugate fuzzy pace of $L^p\left(\bar{\pi}\right), 1 \le p < \infty$, denoted by $L^{p'}\left(\bar{\pi}\right), 1 \le p' < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1.$

Definition 2.18.[6]

Let $\mathrm{K} : L^p(\pi) \to L^q(\pi)$, $1 \le p \le q \le \infty$ be a Markov operator . The adjoint Markov operator is

$$\mathrm{K}^* : L^{q'}(\pi) \to L^{p'}(\pi), 1 \le q' \le p' \le \infty$$

such that

$$\mathrm{K}^* f(x) = \sum_y k^*(x, y) f(y)$$

And

$$k^*(x,y)=\frac{k(y,x)\pi(y)}{\pi(x)}$$

$L^{p'}(\pi),\ L^{q'}(\pi)$ are the conjugate spaces of $L^p(\pi)$ and $L^q(\pi)$ respectively .

Note that :-If we have a fuzzy Markov operator

$$\overline{\mathrm{K}}:L^p\left(\overline{\pi}\right)\to L^q\left(\overline{\pi}\right),1\le p\le q<\infty$$

Then the adjoint Markov operator at degree α is

$$\mathrm{K}_\alpha^*:L^{q'}(\pi_\alpha)\to L^{p'}(\pi_\alpha),1\le q'\le p'<\infty$$

such that

$$\mathrm{K}_\alpha^* f(x)=\sum_y k_\alpha^*(x,y)f(y)$$

and

$$k_\alpha^*(x,y)=\frac{k_\alpha(y,x)\pi_\alpha(y)}{\pi_\alpha(x)}$$

Since the transition kernel $k_\alpha\in\overline{k}[\alpha]$ of $\overline{k}$ and $\pi_\alpha\in\overline{\pi}[\alpha]$ of $\overline{\pi}$, then the adjoint transition kernel at degree α is $k_\alpha^*\in\overline{k}^*[\alpha]$ of the triangular fuzzy number $\overline{k}^*=(k_1^*,k_2^*,k_3^*)$. $\overline{k}^*$ will call the adjoint fuzzy transition kernel . In this case the adjoint Markov operator

$K_\alpha^* \in \overline{K}^*[\alpha]$ the α -cut of the triangular fuzzy number $\overline{K}^* = (K_1^*, K_2^*, K_3^*)$. .

$\overline{K}^*$ is called the adjoint fuzzy Markov operator such that

$$\overline{K}^* : L^{q'}\left(\overline{\pi}\right) \to L^{p'}\left(\overline{\pi}\right), 1 \le q' \le p' < \infty$$

$L^{p'}\left(\overline{\pi}\right), L^{q'}\left(\overline{\pi}\right)$ are the conjugate fuzzy spaces of $L^{p}\left(\overline{\pi}\right)$ and $L^{q}\left(\overline{\pi}\right)$ respectively .

Definition 2.19.[11]

Let $K : L^p(\pi) \to L^q(\pi), 1 \le p \le q \le \infty$ be a Markov operator. We say that K is a self-adjoint Markov operator if

$$K = K^*$$

Note that :- If we have a fuzzy Markov operator

$$\overline{K} : L^{p}\left(\overline{\pi}\right) \to L^{q}\left(\overline{\pi}\right), 1 \le p \le q < \infty$$

then K is a self-adjoint fuzzy Markov operator if

$$\overline{K} = \overline{K}^*$$

Definition 2.20.[6]

The transition kernel of an adjoint Markov operator K^* with respect to π is

$$\kappa^*(x,y)=\frac{k^*(x,y)}{\pi(y)}$$

Note that :-If we have $k_\alpha^* \in \overline{k}^*[\alpha]$ of $\overline{k}^*$ and $\pi_\alpha \in \overline{\pi}[\alpha]$ of $\overline{\pi}$, then the transition kernel of an adjoint Markov operator $\mathrm{K}_\alpha^* \in \overline{\mathrm{K}}^*[\alpha]$ of $\overline{\mathrm{K}}^*$ with respect to π_α is

$$\kappa_\alpha^*(x,y)=\frac{k_\alpha^*(x,y)}{\pi_\alpha(y)}$$

and $\kappa_\alpha^* \in \overline{\kappa}^*[\alpha]$ the α -cut of the triangular fuzzy number $\overline{\kappa}^* = (\kappa_1^*, \kappa_2^*, \kappa_3^*)$. We will call $\overline{\kappa}^*$ the fuzzy transition kernel of an adjoint fuzzy Markov operator $\overline{\mathrm{K}}^*$ with respect to $\overline{\pi}$.

Definition 2.21.[6]

The iterated transition kernel of an adjoint Markov operator K^{*^n} with respect to π is

$$\kappa^{*^n}(x,y)=\frac{k^{*^n}(x,y)}{\pi(y)}$$

Note that :-If we have $k_\alpha^{*^n} \in \overline{k}^{*^n}[\alpha]$ of $\overline{k}^{*^n}$ and $\pi_\alpha \in \overline{\pi}[\alpha]$ of $\overline{\pi}$, then the iterated transition kernel of an adjoint Markov operator $\mathrm{K}_\alpha^{*^n} \in \overline{\mathrm{K}}^{*^n}[\alpha]$ of $\overline{\mathrm{K}}^{*^n}$ with respect to π_α is

$$\kappa_\alpha^{*^n}(x,y) = \frac{k_\alpha^{*^n}(x,y)}{\pi_\alpha(y)}$$

And $\kappa_\alpha^{*^n} \in \overline{\kappa}^{*^n}[\alpha]$ the α -cut of the triangular fuzzy number $\overline{\kappa}^{*^n} = \left(\kappa_1^{*^n}, \kappa_2^{*^n}, \kappa_3^{*^n}\right)$. We will call $\overline{\kappa}^{*^n}$ the iterated fuzzy transition kernel of an adjoint fuzzy Markov operator $\overline{\mathrm{K}}^{*^n}$ with respect to $\overline{\pi}$.

Proposition 2.22

The adjoint fuzzy Markov operator $\overline{\mathrm{K}}^{*}$ at $\overline{\kappa}^{*}$ is given by

$$\overline{\mathrm{K}}^{*} f(x) = \sum_y \overline{\kappa}^{*}(x,y) f(y) \overline{\pi}(y)$$

For any function f .

Proof :-Let a Markov operator $K_\alpha^* \in \overline{K}^*[\alpha]$ of $\overline{K}^*$ and f be any function , then

$$K_\alpha^* f(x) = \sum_y k_\alpha^*(x,y) f(y)$$
$$= \sum_y \kappa_\alpha^*(x,y) f(y) \pi_\alpha(y)$$

Since $\kappa_\alpha^* \in \overline{\kappa}^*[\alpha]$ of $\overline{\kappa}^*$ and $\pi_\alpha \in \overline{\pi}[\alpha]$ of $\overline{\pi}$, then

$$\overline{K}^* f(x) = \sum_y \overline{\kappa}^*(x,y) f(y) \overline{\pi}(y) \quad \blacksquare$$

Proposition 2.23

If we have $\overline{\kappa}(x,y)$ is the fuzzy kernel of a fuzzy Markov operator $\overline{K}$ with respect to $\overline{\pi}$, then

$$\overline{\kappa}^*(x,y) = \overline{\kappa}(y,x)$$

Proof :- Let $\kappa_\alpha^* \in \overline{\kappa}^*[\alpha]$ of $\overline{\kappa}^*$ and $\kappa_\alpha \in \overline{\kappa}[\alpha]$ of $\overline{\kappa}$ then

$$\kappa_\alpha^*(x,y)=\frac{k_\alpha^*(x,y)}{\pi_\alpha(y)}$$
$$=\frac{1}{\pi_\alpha(y)}\;\frac{k_\alpha(y,x)\pi_\alpha(y)}{\pi_\alpha(x)}$$
$$=\frac{k_\alpha(y,x)}{\pi_\alpha(x)}$$
$$=\kappa_\alpha(y,x)$$

Hence $$\overline{\kappa}^*(x,y)=\overline{\kappa}(y,x)\;.\;\blacksquare$$

Definition 2.24.[6]

The expected value of a function f defined on a state space Ω is

$$Ef=\sum_x f(x)\pi(x)=\|f\|_1$$

Note that :-If $0<E<1$ and we have $\pi_\alpha\in\overline{\pi}[\alpha]$ of $\overline{\pi}$, then the expected value of f at degree α is

$$E_\alpha f=\sum_x f(x)\pi_\alpha(x)=\|f\|_{\alpha,1}$$

And $E_\alpha\in\overline{E}[\alpha]$ the α-cut of the triangular fuzzy number $\overline{E}=(E_1,E_2,E_3)$. We will call $\overline{\boldsymbol{E}}$ the fuzzy expected value of a function f, where $0<\overline{E}<1$.

Definition 2.25.[6]

The variance of a function f defined on a state space Ω is

$$\begin{aligned}\operatorname{var}(f) &= E\left(f(x) - Ef(x)\right)^2 \\ &= \sum_x \left| f(x) - Ef(x)\right|^2 \pi(x) \\ &= \| f - Ef \|_2^2\end{aligned}$$

Note that :-If $0 < \operatorname{var} < 1$ and we have $E_\alpha \in \overline{E}[\alpha]$ of $\overline{E}$, then the variance of f at degree α is

$$\begin{aligned}\operatorname{var}_\alpha(f) &= E_\alpha\left(f(x) - E_\alpha f(x)\right)^2 \\ &= \sum_x \left| f(x) - E_\alpha f(x)\right|^2 \pi_\alpha(x) \\ &= \| f - E_\alpha f \|_{\alpha,2}^2\end{aligned}$$

And $\operatorname{var}_\alpha \in \overline{\operatorname{var}}[\alpha]$ the α -cut of the triangular fuzzy number $\overline{\operatorname{var}} = (\operatorname{var}_1, \operatorname{var}_2, \operatorname{var}_3)$. We will call $\overline{\operatorname{var}}$ the fuzzy variance of a function f, such that $0 < \overline{\operatorname{var}} < 1$.

Definition 2.26.[4]

The entropy of a random variable x under a given probability measure π is

$$Ent(x) = E[x \log\ x] - E[x] \log E[x]$$

And the entropy of a positive function f such that $Ef = 1$ is

$$\begin{aligned} Ent(f) &= E[f(x)\log f(x)] \\ &= \sum_x [f(x)\log f(x)]\pi(x) \end{aligned}$$

Note that :- If we have $E_\alpha \in \overline{E}[\alpha]$ of $\overline{E}$, then , at degree α the entropy of a random variable x is

$$Ent_\alpha(x) = E_\alpha[x \log x] - E_\alpha[x] \log E_\alpha[x]$$

The entropy of a positive function f is

$$\begin{aligned} Ent_\alpha(f) &= E_\alpha[f(x)\log f(x)] \\ &= \sum_x [f(x)\log f(x)]\pi_\alpha(x) \end{aligned}$$

And $Ent_\alpha \in \overline{Ent}[\alpha]$ the α -cut of the triangular fuzzy number $\overline{Ent} = (Ent_1, Ent_2, Ent_3)$ which will call the fuzzy entropy .

Definition 2.27.[6]

The relative entropy of the probability measure $\mu = f\pi$ with respect to π is

$$Ent(\mu) = \sum_x \mu(x) \log \frac{\mu(x)}{\pi(x)}$$

Note that :-If we have $\pi_\alpha \in \overline{\pi}[\alpha]$ of $\overline{\pi}$, then at degree α we have $\mu_\alpha = f\pi_\alpha$ and the relative entropy is

$$Ent_\alpha(\mu_\alpha) = \sum_x \mu_\alpha(x) \log \frac{\mu_\alpha(x)}{\pi_\alpha(x)}$$

Where $\mu_\alpha \in \overline{\mu}[\alpha]$ the α -cut of the triangular fuzzy number $\overline{\mu} = (\mu_1, \mu_2, \mu_3)$, and $Ent_\alpha(\mu_\alpha) \in \overline{Ent}(\mu)[\alpha]$ the α -cut of the triangular fuzzy number $\overline{Ent}\left(\overline{\mu}\right) = \left(Ent_1(\mu_1), Ent_2(\mu_2), Ent_3(\mu_3)\right)$.

$\overline{Ent}\left(\overline{\mu}\right)$ will call the fuzzy relative entropy of the fuzzy probability measure $\overline{\mu} = f\,\overline{\pi}$.

Proposition 2.28

If we have $\overline{\mu} = f\,\overline{\pi}$ where $f \in L^p\left(\overline{\pi}\right)$ then

$$\overline{Ent}(f) = \overline{Ent}\left(\overline{\mu}\right)$$

Proof :-If we have $Ent_\alpha(f) \in \overline{Ent}(f)[\alpha]$ of $\overline{Ent}(f)$, then

$$\begin{aligned} Ent_\alpha(f) &= \sum_x \left[f(x) \log f(x) \right] \pi_\alpha(x) \\ &= \sum_x \left[f(x) \pi_\alpha(x) \log f(x) \right] \end{aligned}$$

$$= \sum_x \left[\mu_\alpha(x) \log \frac{\mu_\alpha(x)}{\pi_\alpha(x)} \right]$$
$$= Ent_\alpha(\mu_\alpha)$$

Since $Ent_\alpha(\mu_\alpha) \in \overline{Ent}\left(\overline{\mu}\right)[\alpha]$ of the fuzzy relative entropy $\overline{Ent}\left(\overline{\mu}\right)$. Thus

$$\overline{Ent}(f) = \overline{Ent}\left(\overline{\mu}\right) \blacksquare$$

Definition 2.29.[25]

Let $a, b \in R$ with $a < b$. Let $\phi : (a,b) \to R$ be an real valued function .We say that ϕ is a convex function if for all $x, y \in (a,b)$ and $t \in [0,1]$ we have:-

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

Definition 2.30.[25]

Jensen's Inequality states that any system of positive numbers $c_1, c_2, \ldots, c_n$ and for system of points $x_1, x_2, \ldots, x_n$ in (a,b)

$$\phi\left(\frac{\sum_n c_n x_n}{\sum_n c_n} \right) \leq \frac{\sum_n c_n \phi(x_n)}{\sum_n c_n}$$

Where ϕ is a convex function on (a,b).

Proposition 2.31

Jensen's Inequality implies that

$$\left|\overline{\mathrm{K}}\, f\right|^{p}(x) \leq \overline{\mathrm{K}}\left(|f|^{p}\right)(x)$$

Proof :-Let $\mathrm{K}_{\alpha} \in \overline{\mathrm{K}}[\alpha]$ of $\overline{\mathrm{K}}$. By Jensen's Inequality we have

$$\begin{aligned}
\left|\mathrm{K}_{\alpha} f\right|^{p}(x) &= \left|\sum_{y} k_{\alpha}(x, y) f(y)\right|^{p} \\
&\leq \sum_{y} k_{\alpha}(x, y)\left|f(y)\right|^{p} \\
&= \mathrm{K}_{\alpha}\left(|f|^{p}\right)(x)
\end{aligned}$$

Thus $$\left|\overline{\mathrm{K}}\, f\right|^{p}(x) \leq \overline{\mathrm{K}}\left(|f|^{p}\right)(x) \quad \blacksquare$$

Proposition 2.32

If $\overline{\pi}$ is invariant fuzzy probability measure , then

$$\overline{\left\|\overline{\mathrm{K}}\, f\right\|}_{q} < \overline{\|f\|}_{p}$$

Proof :-Let $\mathrm{K}_{\alpha} \in \overline{\mathrm{K}}[\alpha]$ of the fuzzy Markov operator $\overline{\mathrm{K}}$, then for any $f \in L^{p}\left(\overline{\pi}\right)$ we have

$$\left\|\mathrm{K}_{\alpha}\right\|_{\alpha,p\to q}=\sup\frac{\left\|\mathrm{K}_{\alpha}f\right\|_{\alpha,q}}{\left\|f\right\|_{\alpha,p}}$$

Then

$$\left\|\mathrm{K}_{\alpha}f\right\|_{\alpha,q}\leq\left\|\mathrm{K}_{\alpha}\right\|_{\alpha,p\to q}\left\|f\right\|_{\alpha,p}$$

Since $\left\|\mathrm{K}_{\alpha}\right\|_{\alpha.p\to q}<1$, then $\left\|\mathrm{K}_{\alpha}f\right\|_{\alpha,q}<\left\|f\right\|_{\alpha,p}$

Hence $$\overline{\left\|\overline{\mathrm{K}}\,f\right\|}_{q}<\overline{\left\|f\right\|}_{p}\quad\blacksquare$$

Definition 2.33.[6]

The action of the fuzzy Markov operator $\overline{\mathbf{K}}$ on the fuzzy probability measure $\overline{\mu}$ defined as

$$\left[\overline{\mu}\,\overline{\mathrm{K}}\right](u)=\overline{\mu}\left[\overline{\mathrm{K}}\,u\right]$$

For any fuzzy probability measure $\overline{\mu}$ and all function u .

Proposition 2.34

If $\overline{\mu}=f\,\overline{\pi}$, then $\overline{\mu}\,\overline{\mathrm{K}}$ has density $\overline{\mathrm{K}}^{*}f$.

Proof :- Let $\mathrm{K}_{\alpha}^{*}\in\overline{\mathrm{K}}^{*}[\alpha]$ of $\overline{\mathrm{K}}^{*}$, $f\in L^{p}\left(\overline{\pi}\right)$, $\mu_{\alpha}\in\overline{\mu}[\alpha]$ of $\overline{\mu}$, and $\pi_{\alpha}\in\overline{\pi}[\alpha]$ of $\overline{\pi}$ such that $\mu_{\alpha}=f\,\pi_{\alpha}$, then

$$
\begin{aligned}
\mathrm{K}_\alpha^* f(x) &= \sum_y k_\alpha^*(x,y) f(y) \\
&= \sum_y k_\alpha^*(x,y)\frac{\mu_\alpha(y)}{\pi_\alpha(y)} \\
&= \sum_y \kappa_\alpha^*(x,y)\mu_\alpha(y) \\
&= \sum_y \kappa_\alpha(y,x)\mu_\alpha(y) \\
&= \frac{1}{\pi_\alpha(x)}\sum_y k_\alpha(y,x)\mu_\alpha(y) \\
&= \frac{\mu_\alpha \mathrm{K}_\alpha(x)}{\pi_\alpha(x)}
\end{aligned}
$$

Therefore

$$
\overline{\mathrm{K}}^* f = \frac{\overline{\mu}\,\overline{\mathrm{K}}}{\overline{\pi}} \quad \blacksquare
$$

Definition 2.35.[6]

Let $L^p(\pi)$ be a space of simple functions . The continuous time semi-group H_t associated to a Markov operator K defined by

$$
H_t = e^{-t}\sum_0^{\infty}\frac{(t\mathrm{K})^n}{n!} = e^{-t(I-\mathrm{K})}
$$

Such that

$i)\ H_0 = 1$ $\qquad ii)\ H_{s+t} = H_s \cdot H_t$ for all $s,t \geq 0$

Note that :- If we have $L^p\left(\overline{\pi}\right), 1 \leq p < \infty$ the fuzzy space of simple functions , then the continuous time semi-group at degree α associated to a Markov operator $\mathrm{K}_\alpha \in \overline{\mathrm{K}}[\alpha]$ of $\overline{\mathrm{K}}$ defined as

$$H_{\alpha,t} = e^{-t} \sum_{0}^{\infty} \left(\frac{(t\mathrm{K}_\alpha)^n}{n!} \right) = e^{-t(I-\mathrm{K}_\alpha)}$$

and $H_{\alpha,t} \in \overline{H}_t[\alpha]$ the α -cut of the triangular fuzzy number $\overline{H} = \left(H_{1,t}, H_{2,t}, H_{3,t}\right)$.We will call $\overline{H}_t$ the fuzzy continuous time semi-group associated to a fuzzy Markov operator $\overline{\mathrm{K}}$.

Remark 2.36

$i)$ When elements of $\overline{H}_t$ satisfy (i) of definition 2.35 then $\overline{H}_t$ become fuzzy set singleton .

$ii)$ We will call $\overline{H}_t$ the fuzzy continuous time semi-group associated to a fuzzy Markov operator $\overline{\mathrm{K}}$, a fuzzy Markov semi-group to summarize .

Definition 2.37.[6]

The kernel of a Markov semi-group H_t can defined as

$$H_t(x,y) = H_t^x(y) = e^{-t} \sum_0^\infty \left(\frac{t^n}{n!} k^n(x,y) \right)$$

Note that :- If we have $H_{\alpha,t} \in \overline{H}_t[\alpha]$ of a fuzzy Markov semi-group $\overline{H}_t$, then the kernel of $H_{\alpha,t}$ at degree α is

$$H_{\alpha,t}(x,y) = H_{\alpha,t}^x(y) = e^{-t} \sum_0^\infty \left(\frac{t^n}{n!} k_\alpha^n(x,y) \right)$$

And $H_{\alpha,t}(x,y) \in \overline{H}_t(x,y)[\alpha]$ the α-cut of the triangular fuzzy number $\overline{H}_t(x,y) = (H_{1,t}(x,y), H_{2,t}(x,y), H_{3,t}(x,y))$.

We will call $\overline{H}_t(x,y)$ the fuzzy kernel of the fuzzy Markov semi-group $\overline{H}_t$.

Definition 2.38.[6]

The kernel of a Markov semi-group H_t with respect to π is

$$h_t(x,y) = h_t^x(y) = \frac{H_t(x,y)}{\pi(y)}$$

$$= e^{-t} \sum_0^\infty \frac{t^n}{n!} \kappa^n(x,y)$$

Note that :- If we have $H_{\alpha,t}(x,y) \in \overline{H}_t(x,y)[\alpha]$ of $\overline{H}_t(x,y)$, then the kernel of a Markov semi-group $H_{\alpha,t} \in \overline{H}_t[\alpha]$ of $\overline{H}_t$ with respect to $\pi_\alpha \in \overline{\pi}[\alpha]$ at degree α is

$$h_{\alpha,t}(x,y) = h^x_{\alpha,t}(y) = \frac{H_{\alpha,t}(x,y)}{\pi_\alpha(y)}$$

$$= e^{-t} \sum_0^\infty \frac{t^n}{n!} \kappa^n_\alpha(x,y)$$

And $h_{\alpha,t}(x,y) \in \overline{h}_t(x,y)[\alpha]$ the α -cut of the triangular fuzzy number $\overline{h_t}(x,y) = (h_{1,t}(x,y), h_{2,t}(x,y), h_{3,t}(x,y))$. $\overline{h}_t(x,y)$ will be called the fuzzy kernel of the fuzzy Markov semi-group $\overline{H}_t$ with respect to $\overline{\pi}$.

Remark 2.39

The fuzzy quantities $\overline{\kappa}^n(x,y)$ and $\overline{h}_t(x,y)$ are the densities of the fuzzy probability measures $\overline{k}^n(x,y)$ and $\overline{H}_t(x,y)$ with respect to $\overline{\pi}$ respectively.

Definition 2.40.[6]

The adjoint of the Markov semi-group H_t we can defined as

$$H_t^* = e^{-t}\sum_0^{\infty}\frac{(t\mathrm{K}^*)^n}{n!} = e^{-t(I-\mathrm{K}^*)}$$

Which associated to the adjoint Markov operator K^*.

Note that :- If we have $H_{\alpha,t} \in \overline{H}_t[\alpha]$ of $\overline{H}_t$, then the adjoint of a Markov operator $H_{\alpha,t}$ at degree α is

$$H_{\alpha,t}^* = e^{-t}\sum_0^{\infty}\frac{(t\mathrm{K}_\alpha^*)^n}{n!} = e^{-t(I-\mathrm{K}_\alpha^*)}$$

And $H_{\alpha,t}^* \in \overline{H}_t^*[\alpha]$ the α -cut of the triangular fuzzy number $\overline{H}_t^* = (H_{1,t}^*, H_{2,t}^*, H_{3,t}^*)$. We will call $\overline{H}_t^*$ the adjoint of the fuzzy Markov semi-group $\overline{H}_t$ which associated to the adjoint of the fuzzy Markov operator $\overline{\mathrm{K}}^*$.

Proposition 2.41

If $\overline{\mu} = f\,\overline{\pi}$, then the fuzzy measure $\overline{\mu}_t = \overline{\mu}\,\overline{H}_t$ has density $\overline{H}_t^* f$.

Proof :-Let $H^*_{\alpha,t} \in \overline{H}_t^*[\alpha]$ of $\overline{H}_t^*$, $f \in L^p\left(\overline{\pi}\right)$, $\mu_\alpha \in \overline{\mu}[\alpha]$ of $\overline{\mu}$ and $\pi_\alpha \in \overline{\pi}[\alpha]$ of $\overline{\pi}$ such that $\mu_\alpha = f\,\pi_\alpha$. By proposition 2.34 we have

$$H^*_{\alpha,t} f = e^{-t}\sum_0^\infty \frac{t^n}{n!}\mathrm{K}_\alpha^{*^n} f$$

$$= e^{-t}\sum_0^\infty \frac{t^n}{n!}\frac{\mu_\alpha \mathrm{K}_\alpha^n}{\pi_\alpha}$$

$$= \frac{\mu_\alpha}{\pi_\alpha} e^{-t}\sum_0^\infty \frac{t^n}{n!}\mathrm{K}_\alpha^n$$

$$= \frac{\mu_\alpha H_{\alpha,t}}{\pi_\alpha} = \frac{\mu_{\alpha,t}}{\pi_\alpha}$$

Since $H_{\alpha,t} \in \overline{H}_t[\alpha]$ of $\overline{H}_t$, then $\overline{H}_t^* f = \dfrac{\overline{\mu}_t}{\overline{\pi}}$ ■

Chapter three

The Reversibility of Fuzzy Markov chains

This chapter concerns with considering the reversibility of fuzzy Markov chains , and how to find fuzzy eigen-values of a fuzzy Markov operator and fuzzy Markov semi-group .

Definition 3.1.[7]

We say that (k,π) is reversible if

$$\frac{k(x,y)}{\pi(y)} = \frac{k(y,x)}{\pi(x)}$$

Note that :-If we have $k_\alpha \in \overline{k}[\alpha]$ of $\overline{k}$ and $\pi_\alpha \in \overline{\pi}[\alpha]$ of $\overline{\pi}$, then (k_α, π_α) is reversible if

$$\frac{k_\alpha(x,y)}{\pi_\alpha(y)} = \frac{k_\alpha(y,x)}{\pi_\alpha(x)}$$

for all $\alpha \in [0,1]$. This implies , that $\left(\overline{k}, \overline{\pi}\right)$ is reversible . In other words , we say that $\left(\overline{k}, \overline{\pi}\right)$ is reversible if $\overline{\kappa}(x,y)$ is symmetric .

Proposition 3.2

If $\left(\overline{k}, \overline{\pi}\right)$ is reversible , then $\frac{1}{2}\left(\overline{k} + \overline{k}^{*}\right)$ is also reversible fuzzy Markov kernel .

Proof :-We need to prove $\frac{1}{2}\left(\overline{\kappa} + \overline{\kappa}^{*}\right)$ is symmetric . Let $\kappa_\alpha \in \overline{\kappa}[\alpha]$

of $\overline{\kappa}$ and $\kappa_\alpha^* \in \overline{\kappa}^*[\alpha]$ of $\overline{\kappa}^*$, then

$$\begin{aligned}\frac{1}{2}\left(\kappa_\alpha(x,y)+\kappa_\alpha^*(x,y)\right) &= \frac{1}{2}\left(\kappa_\alpha(y,x)+\kappa_\alpha(y,x)\right)\\ &= \frac{1}{2}\left(\kappa_\alpha(y,x)+\kappa_\alpha(x,y)\right)\\ &= \frac{1}{2}\left(\kappa_\alpha(y,x)+\kappa_\alpha^*(y,x)\right)\end{aligned}$$

This leads $\frac{1}{2}\left(\kappa_\alpha + \kappa_\alpha^*\right)$ is symmetric .

Since $\frac{1}{2}\left(\kappa_\alpha + \kappa_\alpha^*\right) \in \frac{1}{2}\left(\overline{\kappa} + \overline{\kappa}^*\right)[\alpha]$ of $\frac{1}{2}\left(\overline{\kappa} + \overline{\kappa}^*\right)$. Thus

$\frac{1}{2}\left(\overline{\kappa} + \overline{\kappa}^*\right)$ is symmetric . Therefore $\frac{1}{2}\left(\overline{k} + \overline{k}^*\right)$ is reversible fuzzy Markov kernel . ■

Proposition 3.3

If $\left(\overline{k}, \overline{\pi}\right)$ is reversible , then the fuzzy Markov operator $\overline{\mathrm{K}}$ is a self-adjoint fuzzy operator on $L^2\left(\overline{\pi}\right)$.

Proof :-Let $f \in L^2\left(\overline{\pi}\right)$ and a Markov operator $\mathrm{K}_\alpha \in \overline{\mathrm{K}}[\alpha]$ of $\overline{\mathrm{K}}$, then

$$
\begin{aligned}
\mathrm{K}_\alpha f(x) &= \sum_y \kappa_\alpha(x,y)\, f(y)\pi_\alpha(y) \\
&= \sum_y \kappa_\alpha(y,x) f(y)\pi_\alpha(y) \\
&= \sum_y \kappa_\alpha^*(x,y) f(y)\pi_\alpha(y) \\
&= \mathrm{K}_\alpha^* f(x)
\end{aligned}
$$

that is $$\mathrm{K}_\alpha = \mathrm{K}_\alpha^*$$

Since $\mathrm{K}_\alpha^* \in \overline{\mathrm{K}}^*[\alpha]$ of $\overline{\mathrm{K}}^*$, then $\overline{\mathrm{K}} = \overline{\mathrm{K}}^*$ ■

Proposition 3.4

If $\overline{\mathrm{K}}$ is a self-adjoint fuzzy operator , then the fuzzy operator $\frac{1}{2}\left(\overline{\mathrm{K}}+\overline{\mathrm{K}}^*\right)$ is a self-adjoint fuzzy operator .

Proof :- Let $\frac{1}{2}\left(\mathrm{K}_\alpha + \mathrm{K}_\alpha^*\right) \in \frac{1}{2}\left(\overline{\mathrm{K}}+\overline{\mathrm{K}}^*\right)[\alpha]$ of $\frac{1}{2}\left(\overline{\mathrm{K}}+\overline{\mathrm{K}}^*\right)$. Then

$$
\begin{aligned}
\frac{1}{2}\left(\mathrm{K}_\alpha + \mathrm{K}_\alpha^*\right)^* &= \frac{1}{2}\left(\mathrm{K}_\alpha^* + \mathrm{K}_\alpha^{**}\right) \\
&= \frac{1}{2}\left(\mathrm{K}_\alpha^* + \mathrm{K}_\alpha\right) \\
&= \frac{1}{2}\left(\mathrm{K}_\alpha + \mathrm{K}_\alpha^*\right)
\end{aligned}
$$

Hence

$$\frac{1}{2}\left(\overline{\mathrm{K}}+\overline{\mathrm{K}}^{*}\right)^{*}=\frac{1}{2}\left(\overline{\mathrm{K}}+\overline{\mathrm{K}}^{*}\right)$$

This implies that $\frac{1}{2}\left(\overline{\mathrm{K}}+\overline{\mathrm{K}}^{*}\right)$ is a self-adjoint fuzzy operator ■

Representation of a Linear Operator by a Matrix [11]

Let X and Y be a linear finite dimensional vector spaces over the same field and $T : X \to Y$ is a linear operator . We choose a basis $E = \{e_1, e_2, \ldots, e_n\}$ for X and a basis $E' = \{e'_1, e'_2, \ldots, e'_n\}$ for Y , then every $x \in X$ has unique representation

$$x = \sum_{k=1}^{n} a_k e_k \ldots\ldots\ldots(1)$$

Since T is linear , then x has the image

$$y = Tx = T\left(\sum_{k=1}^{n} a_k e_k\right) = \sum_{k=1}^{n} a_k T e_k \ldots\ldots\ldots(2)$$

Since the representation (1) is unique , we have T is uniquely determined if the images $y_i = Te_i$ of n-basis vectors $e_1, e_2, \ldots, e_n$ are prescribed . Since y and $y_i = Te_i$ are in Y , they have unique representations of the form

$$\left.\begin{aligned} a) \qquad & y = \sum_{i=1}^{r} c_i\, e_i' \\ b) \qquad & Te_i = \sum_{i=1}^{r} b_{i\,j} e_i' \end{aligned}\right\} \ldots\ldots\ldots(3)$$

Substitution into (2) gives

$$\begin{aligned} y &= \sum_{i=1}^{r} c_i e_i' \\ &= \sum_{k=1}^{n} a_k T e_k \\ &= \sum_{k=1}^{n} a_k \sum_{i=1}^{r} b_{ik} e_i' \\ &= \sum_{i=1}^{r} \left(\sum_{k=1}^{n} b_{ik} a_k \right) e_i' \end{aligned}$$

Since the $b_i^{'}s$ form a linear independent set , then the coefficients of each e_i' on the left and on the right must be the same , that is

$$c_i = \sum_{k=1}^{n} b_{ik} a_k \ldots\ldots\ldots(4)$$

Thus the image $y = Tx = \sum_{i=1}^{r} c_i e_i'$ of $x = \sum_{k=1}^{n} a_k e_k$ can be obtained from (4). Hence the coefficients in (4) form a matrix $T_{E\,E'} = (b_{ik})$ which represents the operator T with respect to those bases .

Definition 3.6. [11]

An eigen-value of a square matrix $A = (a_{ij})$ is a number β, such that $Ax = \beta x$ has a solution $x \neq 0$ .This x is called an eigen-vector of A corresponding to that eigen-value β .

Definition 3.7.[11]

The set $\sigma(A)$ of all eigen-values of a square matrix $A(a_{ij})$ is called the spectrum of A .

Definition 3.8.[11]

Let $T : X \to Y$ be a linear operator on a normed space X of n-dimensional , and $T_{EE'} = (a_{ij})$ be the matrix representing T , where E , E' are basis's for X and Y respectively . Then the eigen-values of the matrix $T_{EE'}$ are called the eigen-values of the operator T , and the set $\sigma(T)$ of all eigen-values of $T_{EE'}$ is called the spectrum of a linear operator T .

Definition 3.9. [18]

The eigen-function of a linear operator T defined on the functions space is any non-zero function in that space satisfying $Tf = \beta f$ for some scalar β the corresponding eigen-value .

Proposition 3.10

If $\left(\overline{k},\overline{\pi}\right)$ is reversible , then the fuzzy eigen-values of the fuzzy Markov operator $\overline{\mathrm{K}}$ on $L^2\left(\overline{\pi}\right)$ are

$$-1<\overline{\beta}_{\min}=\overline{\beta}_{|\Omega|-1}\leq \ldots \leq \overline{\beta}_1<\beta_0=1$$

Proof :- Let ψ_i , $0\leq i\leq|\Omega|-1$ be a basis of real eigen-functions of $L^2\left(\overline{\pi}\right)$, and $\mathrm{K}_\alpha\in\overline{\mathrm{K}}[\alpha]$ of $\overline{\mathrm{K}}$, such that $\mathrm{K}_\alpha\psi_i=\beta_{\alpha,i}\psi_i$, for all $0\leq i\leq|\Omega|-1$, and $\psi_0=1$.

Now , when $i=0$, then $\mathrm{K}_\alpha\psi_0=\beta_{\alpha,0}\psi_0$

Then $\mathrm{K}\psi_0=\beta_0\psi_0$

Then $\beta_0 1=\mathrm{K}1=1$

Thus $\beta_0=1$

When ψ_i , $0<i\leq|\Omega|-1$, then $\mathrm{K}_\alpha\psi_i=\beta_{\alpha,i}\psi_i$ .We can represent K_α by a matrix $M=\left(\beta_{\alpha,i}\right)_{1\times 1}$ **.** By theorem(spectrum)[11]**[**the spectrum $\sigma(T)$ of a bounded linear operator $T:X\rightarrow Y$ on a complex Banach space X is compact and lies in the disk given by $|\beta_i|\leq\|T\|$**]**.

Then $\left|\beta_{\alpha,i}\right| \leq \left\|\mathrm{K}_{\alpha}\right\|_{\alpha,2\to 2}$

That is $\left|\beta_{\alpha,i}\right| < 1$

So $-1 < \beta_{\alpha,i} < 1$

Since $\beta_{\alpha,i} \in \overline{\beta}_i[\alpha]$ of $\overline{\beta}_i$, $0 < i \leq |\Omega| - 1$

Hence

$$-1 < \overline{\beta}_{\min} = \overline{\beta}_{|\Omega|-1} \leq \ldots \leq \overline{\beta}_1 < \beta_0 = 1 \quad \blacksquare$$

Remark 3.11

$i)$ If $\beta_j = 0$, we will consider there is no uncertainty in this value , that is , we don't use the fuzzy in this case .

$ii)$ We set the fuzzy parameter

$$\overline{\beta} = \max\left\{\left|\overline{\beta}_{\min}\right|, \overline{\beta}_1\right\}$$

Proposition 3.12

If ψ_i , $0 \leq i \leq |\Omega| - 1$ is a basis of real eigen-functions in $L^2\left(\overline{\pi}\right)$, then

$$\overline{\left(\sum_0^{|\Omega|-1} \psi_i^2(x)\right)} = \frac{1}{\overline{\pi}(x)}$$

Proof :- we have at degree α from [7]

$$\langle \delta_{\alpha,x}, \psi_i \rangle = \psi_i(x) \ldots\ldots\ldots(1)$$

Where

$$\delta_{\alpha,x}(y) = \begin{Bmatrix} \dfrac{1}{\pi_\alpha(x)} & , \; x = y \\ 0 & , \; x \neq y \end{Bmatrix}$$

Now , we have

$$\langle \delta_{\alpha,x}, \psi_i \rangle = \sum_y \delta_{\alpha,x}(y)\psi_i(y)\pi_\alpha(y)$$

$$= \delta_{\alpha,x}(x)\psi_i(x)\pi_\alpha(x) \ldots\ldots\ldots(2)$$

If we replace $\delta_{\alpha,x}(x)$ by $\left(\sum_0^{|\Omega|-1} \psi_i^2(x)\right)_\alpha$ in (2) , we get

$$\langle \delta_{\alpha,x}, \psi_i \rangle = \left(\sum_0^{|\Omega|-1} \psi_i^2(x)\right)_\alpha \psi_i(x)\pi_\alpha(x) \ldots\ldots\ldots(3)$$

If we put (3) in (1) then

$$\left(\sum_0^{|\Omega|-1} \psi_i^2(x)\right)_\alpha \psi_i(x)\pi_\alpha(x) = \psi_i(x) \ldots\ldots\ldots(4)$$

By multiplying (4) by $\psi_i^{-1}(x)$ we get

$$\left(\sum_{0}^{|\Omega|-1}\psi_i^2(x)\right)_\alpha \pi_\alpha(x)=1$$

So $$\left(\sum_{0}^{|\Omega|-1}\psi_i^2(x)\right)_\alpha = \frac{1}{\pi_\alpha(x)}$$

Since $\pi_\alpha \in \overline{\pi}[\alpha]$ of $\overline{\pi}$, then $\left(\sum_{0}^{|\Omega|-1}\psi_i^2(x)\right)_\alpha \in \overline{\left(\sum_{0}^{|\Omega|-1}\psi_i^2(x)\right)}[\alpha]$

the α -cut of the triangular fuzzy number

$$\overline{\left(\sum_{0}^{|\Omega|-1}\psi_i^2(x)\right)} = \left(\left(\sum_{0}^{|\Omega|-1}\psi_i^2(x)\right)_1, \left(\sum_{0}^{|\Omega|-1}\psi_i^2(x)\right)_2, \left(\sum_{0}^{|\Omega|-1}\psi_i^2(x)\right)_3\right)$$

Hence $$\overline{\left(\sum_{0}^{|\Omega|-1}\psi_i^2(x)\right)} = \frac{1}{\overline{\pi}(x)} \quad \blacksquare$$

Proposition 3.13

The fuzzy eigen-values of the fuzzy Markov semi-group $\overline{H}_t$ are $\exp\left(-t\overline{\lambda}_i\right),\ 0 \le i \le |\Omega|-1$ with the same corresponding eigen- functions to the fuzzy eigen-values $\overline{\beta}_i$ $,0 \le i \le |\Omega|-1$ where $\overline{\lambda_i} = 1-\overline{\beta_i}$.

Proof :-If we have $\mathrm{K}_{\alpha} \in \overline{\mathrm{K}}[\alpha]$ of $\overline{\mathrm{K}}$ and $\psi_i \quad 0 \le i \le |\Omega|-1$ is a basis of real eigen-functions , then

$$\mathrm{K}_{\alpha}\psi_i = \beta_{\alpha,i}\psi_i = (1-\lambda_{\alpha,i})\psi_i = \psi_i - \lambda_{\alpha,i}\psi_i$$

Then $$(I-\mathrm{K}_{\alpha})\ \psi_i = \lambda_{\alpha,i}\psi_i$$

Then we take ψ_i^{-1} to both side we get

$$I-\mathrm{K}_{\alpha} = \lambda_{\alpha,i}$$

Now we multiplying by $-t$ then

$$-t(I-\mathrm{K}_{\alpha}) = -t\lambda_{\alpha,i}$$

By taking the exponential we have

$$\begin{aligned} H_{\alpha,t} &= \exp(-t(I-\mathrm{K}_{\alpha})) \\ &= \exp(-t\lambda_{\alpha,i}) \end{aligned}$$

Which satisfying the equation

$$H_{\alpha,t}\psi_i = \exp(-t\lambda_{\alpha,i})\psi_i$$

So $\exp(-t\lambda_{\alpha,i}), 0 \le i \le |\Omega|-1$ is the eigen-value of the Markov semi-group $H_{\alpha,t}$. Since $\lambda_{\alpha,i} \in \overline{\lambda}_i[\alpha]$ of $\overline{\lambda}_i$, $0 \le i \le |\Omega|-1$, then $\exp(-t\lambda_{\alpha,i}) \in \exp\left(-t\,\overline{\lambda}_i\right)[\alpha]$ of the triangular fuzzy number

$$\exp\left(-t\,\overline{\lambda}_i\right) = \left(\exp(-t\lambda_{1,i}), \exp(-t\lambda_{2,i}), \exp(-t\lambda_{3,i})\right)$$

$0 \le i \le |\Omega| - 1$, which the fuzzy eigen-values of the fuzzy Markov semi-group $\overline{H}_t$ ■

Definition 3.14.[19]

Let $M = (a_{ij})_{n\times n}$ n row and n column matrix represent the linear operator $T : X \to Y$, then the spectral gap of the linear operator T is the same spectral gap of the matrix M , which defined by

$$d = d_0 - d_1$$

Where d_0, d_1 belong to the set of all eigen-values of M and $d_1 \le d_0$.

Proposition 3.15

The fuzzy spectral gap of the fuzzy Markov chains is

$$\overline{\lambda} = \overline{\lambda}_1 = 1 - \overline{\beta}_1$$

Proof :-We have by proposition 3.10 the fuzzy eigen-values of the fuzzy Markov operator $\overline{\mathrm{K}}$ are

$$-1 < \overline{\beta}_{\min} = \overline{\beta}_{|\Omega|-1} \le \ldots \le \overline{\beta}_1 < \beta_0 = 1$$

Thus the fuzzy spectral gap of the fuzzy Markov chains is

$$\overline{\lambda} = \overline{\lambda}_1 = 1 - \overline{\beta}_1 \ ■$$

Example 3.16.

Consider the fuzzy chain $\overline{k}$ on the symmetric group $X = S_d$ which the collection of all permutation of

$$S = \{id\} \cup \{(i, j)\ , 1 \le i < j \le d\}$$

This corresponding to randomly transposing pairs of cards with the identity thrown in with equal weight , where

$$\overline{k}(\sigma,\Theta) = \left(\left(\frac{1}{|S|+1} \right) I_S(\Theta^{-1}\sigma), \left(\frac{1}{|S|} \right) I_S(\Theta^{-1}\sigma), \left(\frac{1}{|S|-1} \right) I_S(\Theta^{-1}\sigma) \right)$$

And σ , Θ are permutations in S_d .

The lowest fuzzy eigen-value is

$$\overline{\beta}_{\min} = \left(-1 + \frac{1}{d^2+1}\ , -1 + \frac{1}{d^2}\ , -1 + \frac{1}{d^2-1} \right)$$

The second largest fuzzy eigen-value is

$$\overline{\beta}_1 = \left(1 - \frac{2}{d-1}\ , 1 - \frac{2}{d}\ , 1 - \frac{2}{d+1} \right)$$

The fuzzy spectral gap is

$$\overline{\lambda} = \overline{\lambda}_1 = 1 - \overline{\beta}_1 = \left(\frac{2}{d+1}, \frac{2}{d}, \frac{2}{d-1} \right).$$

Note :- This example came in [8] in classical form , where the kernel is

$$k(\sigma,\Theta) = \left(\frac{1}{|S|} \right) I_S(\Theta^{-1}\sigma)$$

The lowest eigen-value is

$$\beta_{\min} = -1 + \frac{1}{d^2}$$

The second largest eigen-value is

$$\beta_1 = 1 - \frac{2}{d}$$

The spectral gap is

$$\lambda = \lambda_1 = 1 - \beta_1 = \frac{2}{d} \ .$$

Lemma 3.17

If $\left(\overline{k}, \overline{\pi}\right)$ is reversible , and $\psi_i \ , 0 \le i \le |\Omega| - 1$ is an orthonormal basis of real eigen-functions in $L^2\left(\overline{\pi}\right)$ then

$$i) \ \overline{\kappa}^{\,n}(x,y) = \sum_{0}^{|\Omega|-1} \overline{\beta}_i^{\,n} \psi_i(x) \psi_i(y)$$

$$ii) \left\| \overline{\overline{\kappa}_x^{\,n} - 1} \right\|_2^2 = \sum_{1}^{|\Omega|-1} \overline{\beta}_i^{\,2n} \left|\psi_i(x)\right|^2 \le \frac{1-\pi(x)}{\pi(x)} \overline{\beta}^{\,2n}$$

$$iii) \ \overline{h}_t(x,y) = \sum_{0}^{|\Omega|-1} \exp\left(-t\,\overline{\lambda}_i\right) \psi_i(x) \psi_i(y)$$

$$iv) \left\| \overline{\overline{h}_t^x - 1} \right\|_2^2 = \sum_{1}^{|\Omega|-1} \exp\left(-2t\,\overline{\lambda}_i\right) \left|\psi_i(x)\right|^2 \le \frac{1-\pi(x)}{\pi(x)} \exp\left(-2t\,\overline{\lambda}\right)$$

Proof :-$i)$ We have at degree α

$$\mathrm{K}_{\alpha}^{n}\psi_{j}(x)=\beta_{\alpha,j}^{n}\psi_{j}(x)\ldots\ldots\ldots(1)$$

And

$$\mathrm{K}_{\alpha}^{n}\psi_{j}(x)=\sum_{y}\kappa_{\alpha}^{n}(x,y)\psi_{j}(y)\pi_{\alpha}(y)\ldots\ldots\ldots(2)$$

If we replace $\kappa_{\alpha}^{n}(x,y)$ by $\sum_{0}^{|\Omega|-1}\beta_{\alpha,i}^{n}\psi_{i}(x)\psi_{i}(y)$

in (2) we get

$$\begin{aligned}
\mathrm{K}_{\alpha}^{n}\psi_{j}(x) &= \sum_{y}\left(\sum_{0}^{|\Omega|-1}\beta_{\alpha,i}^{n}\psi_{i}(x)\psi_{i}(y)\right)\psi_{j}(y)\pi_{\alpha}(y) \\
&= \sum_{0}^{|\Omega|-1}\beta_{\alpha,i}^{n}\psi_{i}(x)\sum_{y}\psi_{i}(y)\psi_{j}(y)\pi_{\alpha}(y) \\
&= \sum_{0}^{|\Omega|-1}\beta_{\alpha,i}^{n}\psi_{i}(x)\langle\psi_{i},\psi_{j}\rangle \\
&= \underset{i=j}{\beta_{\alpha,j}^{n}\psi_{j}(x)}
\end{aligned}$$

Hence

$$\overline{\kappa}^{n}(x,y)=\sum_{0}^{|\Omega|-1}\overline{\beta}_{i}^{n}\,\psi_{i}(x)\psi_{i}(y)$$

$ii)$ We have at degree α

$$\left\|\kappa_{\alpha,x}^{n}-1\right\|_{\alpha,2}^{2}=\sum_{y}\left|\kappa_{\alpha}^{n}(x,y)-1\right|^{2}\pi_{\alpha}(y)$$

$$=\sum_{y}\left|\sum_{0}^{|\Omega|-1}\beta_{\alpha,\,i}^{n}\psi_{i}(x)\psi_{i}(y)-1\right|^{2}\pi_{\alpha}(y)$$

$$=\sum_{y}\left|\sum_{1}^{|\Omega|-1}\beta_{\alpha,\,i}^{n}\psi_{i}(x)\psi_{i}(y)\right|^{2}\pi_{\alpha}(y)$$

$$=\sum_{1}^{|\Omega|-1}\beta_{\alpha,i}^{2n}\left|\psi_{i}(x)\right|^{2}\sum_{y}\psi_{i}^{2}(y)\pi_{\alpha}(y)$$

$$=\sum_{1}^{|\Omega|-1}\beta_{\alpha,i}^{2n}\left|\psi_{i}(x)\right|^{2}\left\|\psi_{i}\right\|_{2}^{2}$$

$$=\sum_{1}^{|\Omega|-1}\beta_{\alpha,i}^{2n}\left|\psi_{i}(x)\right|^{2}$$

Hence

$$\overline{\left\|\overline{\kappa}_{x}^{n}-1\right\|_{2}^{2}}=\sum_{1}^{|\Omega|-1}\overline{\beta}_{i}^{2n}\left|\psi_{i}(x)\right|^{2}$$

Now

$$\sum_{1}^{|\Omega|-1}\beta_{\alpha,i}^{2n}\left|\psi_{i}(x)\right|^{2}\leq\beta_{\alpha}^{2n}\sum_{1}^{|\Omega|-1}\left|\psi_{i}(x)\right|^{2}$$

$$=\beta_{\alpha}^{2n}\left(\sum_{0}^{|\Omega|-1}\left|\psi_{i}(x)\right|^{2}-1\right)$$

$$= \beta_{\alpha}^{2n}\left(\frac{1}{\pi(x)}-1\right)$$

$$= \frac{1-\pi(x)}{\pi(x)}\beta_{\alpha}^{2n}$$

Thus

$$\sum_{1}^{|\Omega|-1}\overline{\beta}_i^{\,2n}\left|\psi_i(x)\right|^2 \le \frac{1-\pi(x)}{\pi(x)}\overline{\beta}^{\,2n}$$

This implies

$$\left\|\overline{\overline{\kappa}_x^{\,n}-1}\right\|_2^2 = \sum_{1}^{|\Omega|-1}\overline{\beta}_i^{\,2n}\left|\psi_i(x)\right|^2 \le \frac{1-\pi(x)}{\pi(x)}\overline{\beta}^{\,2n}$$

$iii)$ We have at degree α

$$\langle h_{\alpha,t}^{x},\psi_j\rangle = \exp\left(-t\lambda_{\alpha,j}\right)\psi_j(x)\ldots\ldots(1) \quad [7]$$

and

$$\langle h_{\alpha,t}^{x},\psi_j\rangle = \sum_{y} h_{\alpha,t}(x,y)\psi_j(y)\pi_{\alpha}(y)\ldots\ldots(2)$$

If we replace $h_{\alpha,t}(x,y)$ by $\sum_{0}^{|\Omega|-1}\exp\left(-t\lambda_{\alpha,i}\right)\psi_i(x)\psi_i(y)$

in (2) we get

$$\langle h_{\alpha,t}^{x},\psi_j\rangle = \sum_{y}\left(\sum_{0}^{|\Omega|-1}\exp\left(-t\lambda_{\alpha,i}\right)\psi_i(x)\psi_i(y)\right)\psi_j(y)\pi_{\alpha}(y)$$

$$= \sum_{0}^{|\Omega|-1}\exp\left(-t\lambda_{\alpha,i}\right)\psi_i(x)\langle\psi_i,\psi_j\rangle$$

$$= \exp\left(-t\lambda_{\underset{i=j}{\alpha,j}}\right)\psi_j(x)$$

Hence

$$\overline{h}_t(x,y)=\sum_{0}^{|\Omega|-1}\exp\left(-t\,\overline{\lambda}_i\right)\psi_i(x)\psi_i(y)\ .$$

$iv)$ We have at degree α

$$\begin{aligned}
\left\|h_{\alpha,t}^{x}-1\right\|_{\alpha,2}^{2} &=\sum_{y}\left|h_{\alpha,t}(x,y)-1\right|^{2}\pi_\alpha(y)\\
&=\sum_{y}\left|\sum_{0}^{|\Omega|-1}\exp(-t\lambda_{\alpha,i})\psi_i(x)\psi_i(y)-1\right|^{2}\pi_\alpha(y)\\
&=\sum_{y}\left|\sum_{1}^{|\Omega|-1}\exp(-t\lambda_{\alpha,i})\psi_i(x)\psi_i(y)\right|^{2}\pi_\alpha(y)\\
&=\sum_{1}^{|\Omega|-1}\left(\exp(-2t\lambda_{\alpha,i})|\psi_i(x)|^{2}\sum_{y}|\psi_i(y)|^{2}\pi_\alpha(y)\right)\\
&=\sum_{1}^{|\Omega|-1}\exp(-2t\lambda_{\alpha,i})|\psi_i(x)|^{2}\left\|\psi_i\right\|_{\alpha,2}^{2}\\
&=\sum_{1}^{|\Omega|-1}\exp(-2t\lambda_{\alpha,i})|\psi_i(x)|^{2}
\end{aligned}$$

Hence

$$\overline{\left\|\overline{h}_t^{x}-1\right\|_2^{2}}=\sum_{1}^{|\Omega|-1}\exp\left(-2t\,\overline{\lambda}_i\right)|\psi_i(x)|^{2}$$

Now

$$\sum_{1}^{|\Omega|-1}\exp\left(-2t\lambda_{\alpha,\,i}\right)\left|\psi_i\left(x\right)\right|^2 \le \exp\left(-2t\lambda_\alpha\right)\sum_{1}^{|\Omega|-1}\left|\psi_i\left(x\right)\right|^2$$

$$= \exp\left(-2t\lambda_\alpha\right)\left(\sum_{0}^{|\Omega|-1}\left|\psi_i\left(x\right)\right|^2 - 1\right)$$

$$= \exp\left(-2t\lambda_\alpha\right)\left(\frac{1}{\pi\left(x\right)} - 1\right)$$

$$= \frac{1-\pi\left(x\right)}{\pi\left(x\right)}\exp\left(-2t\lambda_\alpha\right)$$

Thus

$$\sum_{1}^{|\Omega|-1}\exp\left(-2t\,\overline{\lambda}_i\right) \le \frac{1-\pi\left(x\right)}{\pi\left(x\right)}\exp\left(-2t\,\overline{\lambda}\right)$$

This implies

$$\overline{\left\|\overline{h}_t^x - 1\right\|_2^2} = \sum_{1}^{|\Omega|-1}\exp\left(-2t\,\overline{\lambda}_i\right)\left|\psi_i\left(x\right)\right|^2 \le \frac{1-\pi\left(x\right)}{\pi\left(x\right)}\exp\left(-2t\,\overline{\lambda}\right) \quad \blacksquare$$

The following simple result gives a useful way of transferring results between discrete and continuous time .

Corollary.3.18

Assume that $\left(\overline{k}, \overline{\pi}\right)$ is reversible and set $\overline{\beta}_- = \max\left\{0, -\overline{\beta}_{\min}\right\}$ then

$$\left\| \overline{\kappa}_x^N - 1 \right\|_2^2 \leq \overline{\beta}_-^{2n} \left(1 + \left\| \overline{h}_{n'}^x - 1 \right\|_2^2 \right) + \left\| \overline{h}_N^x - 1 \right\|_2^2$$

For $N = n + n' + 1$.

Proof :-We have at degree α

$$\kappa_\alpha^{2n+1}(x, x) = \sum_0^{|\Omega|-1} \beta_{\alpha,i}^{2n+1} |\psi_i(x)|^2 > 0$$

Then

$$\sum_{i:\beta_{\alpha,i} < 0} \beta_{\alpha,i}^{2n+1} |\psi_i(x)|^2 + \sum_{i:\beta_{\alpha,i} > 0} \beta_{\alpha,i}^{2n+1} |\psi_i(x)|^2 > 0$$

Then

$$-\sum_{i:\beta_{\alpha,i} < 0} \beta_{\alpha,i}^{2n+1} |\psi_i(x)|^2 < \sum_{i:\beta_{\alpha,i} > 0} \beta_{\alpha,i}^{2n+1} |\psi_i(x)|^2$$

Hence

$$\sum_{i:\beta_{\alpha,i} < 0} \beta_{\alpha,i}^{2n+2} |\psi_i(x)|^2 < \sum_{i:\beta_{\alpha,i} > 0} \beta_{\alpha,i}^{2n} |\psi_i(x)|^2$$

Now , for those $\beta_{\alpha,i}$ that are positive we have

$$\beta_{\alpha,i}^{2n} = \exp\left(2n \log\left(1 - \lambda_{\alpha,i}\right)\right) \leq \exp\left(-2n\lambda_{\alpha,i}\right)$$

Then

$$\sum_{i:\beta_{\alpha.i}>0} \beta_{\alpha,i}^{2n} \left|\psi_i(x)\right|^2 \le \sum_{1}^{|\Omega|-1} \beta_{\alpha,i}^{2n} \left|\psi_i(x)\right|^2 + 1$$

$$\le \sum_{1}^{|\Omega|-1} \exp\left(-2n\lambda_{\alpha,i}\right)\left|\psi_i(x)\right|^2 + 1$$

$$= \left\|h_{\alpha,n}^{x} - 1\right\|_{\alpha,2}^{2} + 1$$

$$= \sum_{y} \left|h_{\alpha,n}(x,y) - 1\right|^2 \pi_\alpha(y) + 1$$

$$= \sum_{y} h_{\alpha,2n}(x,y)\pi_\alpha(y) - 2\sum_{y} h_{\alpha,n}(x,y)\pi_\alpha(y) + \sum_{y} \pi_\alpha(y) + 1$$

$$= \left\|h_{\alpha,n}^{x}\right\|_{\alpha,2}^{2} - 2\sum_{y} e^{-n} \sum_{0}^{\infty} \frac{n^m}{m!} \kappa_\alpha^m(x,y)\pi_\alpha(y) + 1 + 1$$

$$= \left\|h_{\alpha,n}^{x}\right\|_{\alpha,2}^{2} - 2e^{-n} \sum_{0}^{\infty} \frac{n^m}{m!} \sum_{y} k_\alpha^m(x,y) + 2$$

$$= \left\|h_{\alpha,n}^{x}\right\|_{\alpha,2}^{2} - 2e^{-n} \sum_{0}^{\infty} \frac{n^m}{m!} + 2$$

$$= \left\|h_{\alpha,n}^{x}\right\|_{\alpha,2}^{2} - 2(1) + 2$$

$$= \left\|h_{\alpha,n}^{x}\right\|_{\alpha,2}^{2}.$$

And

$$\sum_{\substack{i\neq 0\\ \beta_{\alpha,i}>0}} \beta_{\alpha,i}^{2n}\left|\psi_i(x)\right|^2 \leq \sum_{1}^{|\Omega|-1} \beta_{\alpha,i}^{2n}\left|\psi_i(x)\right|^2$$

$$\leq \sum_{1}^{|\Omega|-1} \exp\left(-2n\lambda_{\alpha,i}\right)\left|\psi_i(x)\right|^2$$

$$= \left\|h_{\alpha,n}^{x} - 1\right\|_{\alpha,2}^{2}$$

Putting these pieces together we get for $N = n + n' + 1$

$$\left\|\kappa_{\alpha,x}^{N} - 1\right\|_{\alpha,2}^{2} = \sum_{1}^{|\Omega|-1} \beta_{\alpha,i}^{2N}\left|\psi_i(x)\right|^2$$

$$= \sum_{i:\beta_{\alpha,i}<0} \beta_{\alpha,i}^{2N}\left|\psi_i(x)\right|^2 + \sum_{\substack{i\neq 0\\ \beta_{\alpha,i}>0}} \beta_{\alpha,i}^{2N}\left|\psi_i(x)\right|^2$$

$$= \sum_{i:\beta_{\alpha,i<0}} \beta_{\alpha,i}^{2n+2n'+2}\left|\psi_i(x)\right|^2 + \sum_{\substack{i\neq 0\\ \beta_{\alpha,i}>0}} \beta_{\alpha,i}^{2N}\left|\psi_i(x)\right|^2$$

$$\leq \beta_{-,\alpha}^{2n}\left(\sum_{i:\beta_{\alpha,i}<0} \beta_{\alpha,i}^{2n'+2}\left|\psi_i(x)\right|^2\right) + \sum_{\substack{i\neq 0\\ \beta_{\alpha,i}>0}} \beta_{\alpha,i}^{2N}\left|\psi_i(x)\right|^2$$

$$\leq \beta_{-,\alpha}^{2n}\left\|h_{\alpha,n'}^{x}\right\|_{\alpha,2}^{2} + \left\|h_{\alpha,N}^{x} - 1\right\|_{\alpha,2}^{2}$$

$$= \beta_{-,\alpha}^{2n}\left(1 + \left\|h_{\alpha,n'}^{x} - 1\right\|_{\alpha,2}^{2}\right) + \left\|h_{\alpha,N}^{x} - 1\right\|_{\alpha,2}^{2}$$

Therefore

$$\left\| \overline{\overline{\kappa}_x^{\,N} - 1} \right\|_2^2 \le \overline{\beta}_-^{\,2n} \left(1 + \left\| \overline{\overline{h}_{n'}^{x} - 1} \right\|_2^2 \right) + \left\| \overline{\overline{h}_N^{x} - 1} \right\|_2^2$$

For $N = n + n' + 1$ ■

As a direct application of above corollary we have the following corollary which allow us to separate out the effects of the smallest fuzzy eigen-value from those of the fuzzy spectral gap .

Corollary.3.19

Assume that $\left(\overline{k}, \overline{\pi}\right)$ is reversible and set $\lambda_* = \min\left\{\overline{\lambda}, 1 + \overline{\beta}_{\min}\right\}$.

Then

$$\left\| \overline{\overline{\kappa}_x^{\,n} - 1} \right\|_2 \le 3e^{-n\,\overline{\lambda}}$$

For $$n = \frac{1}{2\lambda} \log \frac{1}{\pi(x)} + \frac{1}{\lambda_*} + 1 .$$

Chapter Four

Fuzzy Dirichlet Forms and Distance to Equilibrium

We study in this chapter the fuzzy Dirichlet forms , total variation distance , fuzzy chi-square distance , and relative error . We will prove some propositions that we used in our main theorems in chapter five .

4.1. Fuzzy Dirichlet Forms

Definition 4.1.1.[6]

For a given chain k with invariant measure π the Dirichlet form is defined as

$$\ell(f,g) = \langle (I - \mathrm{K}) f, g \rangle$$
$$= \frac{1}{2} \sum_{x,y} (f(x) - f(y))(g(x) - g(y)) k(x,y) \pi(x)$$

Where f and g are two real-valued functions .

Also we have

$$\ell(f,f) = \langle (I - \mathrm{K}) f, f \rangle$$
$$= \frac{1}{2} \sum_{x,y} |f(x) - f(y)|^2 k(x,y) \pi(x)$$

Note that :- If we have a Markov operator $\mathrm{K}_\alpha \in \overline{\mathrm{K}}[\alpha]$ of $\overline{\mathrm{K}}$, then the Dirichlet form at degree α is

$$\ell_\alpha(f,g) = \langle (I - \mathrm{K}_\alpha) f, g \rangle$$
$$= \frac{1}{2} \sum_{x,y} (f(x) - f(y))(g(x) - g(y)) k_\alpha(x,y) \pi_\alpha(x)$$

And $\ell_\alpha \in \overline{\ell}[\alpha]$ the α -cut of the triangular fuzzy number $\overline{\ell} = (\ell_1, \ell_2, \ell_3)$, which will call the fuzzy Dirichlet form .

Proposition 4.1.2

If $\frac{1}{2}\left(\overline{k}+\overline{k}^{*}\right)$ is reversible fuzzy Markov kernel , then

$$\overline{\ell}(f,f)=\left\langle\left(I-\frac{\overline{\mathrm{K}}+\overline{\mathrm{K}}^{*}}{2}\right)f,f\right\rangle$$

Where f is a real-valued function .

Proof :-Let $\ell_{\alpha}\in\overline{\ell}[\alpha]$ of $\overline{\ell}$ then

$$\ell_{\alpha}(f,f)=\frac{1}{2}\sum_{x,y}|f(x)-f(y)|^{2}k_{\alpha}(x,y)\pi_{\alpha}(x)$$

$$=\frac{1}{2}\sum_{x,y}|f(x)-f(y)|^{2}\kappa_{\alpha}(x,y)\pi_{\alpha}(x)\pi_{\alpha}(y)$$

$$=\frac{1}{2}\sum_{x,y}|f(x)-f(y)|^{2}\left(\frac{1}{2}\,.\,2\kappa_{\alpha}(x,y)\right)\pi_{\alpha}(x)\pi_{\alpha}(y)$$

$$=\frac{1}{2}\sum_{x,y}|f(x)-f(y)|^{2}\,\frac{\kappa_{\alpha}(x,y)+\kappa_{\alpha}(x,y)}{2}\,\pi_{\alpha}(x)\pi_{\alpha}(y)$$

$$=\frac{1}{2}\sum_{x,y}|f(x)-f(y)|^{2}\,\frac{\kappa_{\alpha}(x,y)+\kappa_{\alpha}(y,x)}{2}\,\pi_{\alpha}(x)\pi_{\alpha}(y)$$

$$=\frac{1}{2}\sum_{x,y}|f(x)-f(y)|^{2}\,\frac{\kappa_{\alpha}(x,y)+\kappa_{\alpha}^{*}(x,y)}{2}\,\pi_{\alpha}(x)\pi_{\alpha}(y)$$

$$=\left\langle\left(I-\frac{\mathrm{K}_{\alpha}+\mathrm{K}_{\alpha}^{*}}{2}\right)f,f\right\rangle$$

Since $\frac{1}{2}\left(\mathrm{K}_{\alpha}+\mathrm{K}_{\alpha}^{*}\right) \in \frac{1}{2}\left(\overline{\mathrm{K}}+\overline{\mathrm{K}}^{*}\right)[\alpha]$ of $\frac{1}{2}\left(\overline{\mathrm{K}}+\overline{\mathrm{K}}^{*}\right)$, therefore

$$\overline{\ell}(f, f)=\left\langle\left(I-\left(\frac{\overline{\mathrm{K}}+\overline{\mathrm{K}}^{*}}{2}\right) f\right), f\right\rangle$$

Definition 4.1.3.[6]

The fuzzy spectral gap $\overline{\lambda}$ of $\overline{\mathrm{K}}$ can be defined by

$$\overline{\lambda}=\min \frac{\overline{\ell}(f, f)}{\overline{\operatorname{var}}(f)}$$

$$=\min \left\{\overline{\ell}(f, f):\|f\|_{2}=1, E(f)=0\right\}$$

Lemma 4.1.4

Let $\left(\overline{k}, \overline{\pi}\right)$ be a fuzzy Markov chain on a finite state space Ω then

$$\left\|\overline{\left(\overline{H}_{t}-E\right) f}\right\|_{2}^{2} \leq e^{-2 \overline{\lambda} t}\, \overline{\operatorname{var}}(f)$$

Where f is a real eigen-function .

Proof :-We have at degree α by elementary calculus

$$
\begin{aligned}
\partial_t \left\| H_{\alpha,t} f \right\|_{\alpha,2}^2 &= \partial_t \sum_x H_{\alpha,t}^2 f^2(x) \pi_\alpha(x) \\
&= -2 \sum_x \left(I - \mathrm{K}_\alpha\right)\left(H_{\alpha,t} f(x)\right)\left(H_{\alpha,t} f(x)\right) \pi_\alpha(x) \\
&= -2 \left\langle \left(I - \mathrm{K}_\alpha\right) H_{\alpha,t} f, H_{\alpha,t} f \right\rangle \\
&= -2 \ell_\alpha \left(H_{\alpha,t} f, H_{\alpha,t} f\right)
\end{aligned}
$$

We consider $Ef = 0$ then

$$
\begin{aligned}
\partial_t \left\| \left(H_{\alpha,t} - E\right) f \right\|_{\alpha,2}^2 &= \partial_t \left\| H_{\alpha,t} f - Ef \right\|_{\alpha,2}^2 \\
&= \partial_t \left\| H_{\alpha,t} f \right\|_{\alpha,2}^2 \\
&= -2 \ell_\alpha \left(H_{\alpha,t} f, H_{\alpha,t} f\right) \\
&= -2 \ell_\alpha \left(\left(H_{\alpha,t} - E\right) f, \left(H_{\alpha,t} - E\right) f\right) \\
&= -2 \left\langle \left(I - \mathrm{K}_\alpha\right)\left(H_{\alpha,t} - E\right) f, \left(H_{\alpha,t} - E\right) f \right\rangle \\
&= -2 \left(\left(1 - \beta_{\alpha,i}\right) \left\langle \left(H_{\alpha,t} - E\right) f, \left(H_{\alpha,t} - E\right) f \right\rangle\right) \\
&\le -2 \lambda_\alpha \left\langle \left(H_{\alpha,t} - E\right) f, \left(H_{\alpha,t} - E\right) f \right\rangle \\
&= -2 \lambda_\alpha \left\| \left(H_{\alpha,t} - E\right) f \right\|_{\alpha,2}^2 \\
&= -2 \lambda_\alpha \left\| H_{\alpha,t} f \right\|_{\alpha,2}^2 \\
&= -2 \lambda_\alpha \left\| e^{-\lambda_\alpha t} f \right\|_{\alpha,2}^2 \\
&= -2 \lambda_\alpha e^{-2\lambda_\alpha t} \left\| f \right\|_{\alpha,2}^2 \\
&= -2 \lambda_\alpha e^{-2\lambda_\alpha t} \left\| f - Ef \right\|_{\alpha,2}^2 \\
&= -2 \lambda_\alpha e^{-2\lambda_\alpha t} \operatorname{var}_\alpha(f)
\end{aligned}
$$

Integrating the both sides with respect to t implies

$$\left\|\left(H_{\alpha,t}-E\right)f\right\|_{\alpha,2}^{2} \leq e^{-2t\lambda_{\alpha}} \operatorname{var}_{\alpha}(f)$$

Hence

$$\left\|\overline{\left(\overline{H}_{t}-E\right)f}\right\|_{2}^{2} \leq e^{-2t\overline{\lambda}} \, \overline{\operatorname{var}}(f) \quad \blacksquare$$

Proposition 4.1.5

If we take the supremum over all functions f in lemma 4.1.4 such that $\|f\|_{2}=1$, then

$$\overline{\left\|\overline{H}_{t}-E\right\|}_{2\to 2} \leq \exp\left(-t\overline{\lambda}\right)$$

Proof :- By lemma 4.1.4 we have at degree α

$$\left\|H_{\alpha,t}-E\right\|_{\alpha,2\to 2}^{2} = \sup_{\|f\|_{2}=1}\left\|\left(H_{\alpha,t}-E\right)f\right\|_{\alpha,2}^{2}$$

$$\leq \sup_{\|f\|_{2}=1} e^{-2t\lambda_{\alpha}} \operatorname{var}(f)$$

$$= e^{-t\lambda_{\alpha}}$$

Hence

$$\overline{\left\|\overline{H}_{t}-E\right\|}_{2\to 2} \leq \exp\left(-t\overline{\lambda}\right) \quad \blacksquare$$

Proposition 4.1.6

In discrete time we can define $\overline{\beta} > 0$ by

$$\overline{\beta} = \overline{\left\| \overline{\mathrm{K}} - E \right\|}_{2\to 2}$$

When $Ef = 0$.

Proof :-Let f be a real eigen-function , then at degree α we have

$$\begin{aligned}
\left\| \mathrm{K}_{\alpha} - E \right\|_{\alpha,2\to 2} &= \sup_{\|f\|_{\alpha,2}<1} \left\| \left(\mathrm{K}_{\alpha} - E \right) f \right\|_{\alpha,2} \\
&= \sup_{\|f\|_{\alpha,2}<1} \left\| \mathrm{K}_{\alpha} f \right\|_{\alpha,2} \\
&= \sup_{\|f\|_{\alpha,2}<1} \left\| \beta_{\alpha} f \right\|_{\alpha,2} \\
&= \beta_{\alpha} \sup_{\|f\|_{\alpha,2}<1} \left\| f \right\|_{\alpha,2} \\
&= \beta_{\alpha}
\end{aligned}$$

This implies

$$\overline{\left\| \overline{\mathrm{K}} - E \right\|}_{2\to 2} = \overline{\beta} \quad \blacksquare$$

Proposition 4.1.7

If $\overline{\left\| \overline{\mathrm{K}} - E \right\|}_{2\to 2} = \overline{\beta}$, then $\overline{\left\| \overline{\mathrm{K}}^{n} - E \right\|}_{2\to 2} \le \overline{\beta}^{n}$

Proof :-We have at degree α

$$\left\|\mathrm{K}_{\alpha}^{n}-E\right\|_{\alpha,2\to 2}\leq\left\|\mathrm{K}_{\alpha}-E\right\|_{\alpha,2\to 2}^{n}$$
$$=\beta_{\alpha}^{n}$$

Thus

$$\overline{\left\|\overline{\mathrm{K}}^{n}-E\right\|}_{2\to 2}\leq\overline{\beta}^{n} \quad \blacksquare$$

Proposition 4.1.8

When $\left(\overline{k},\overline{\pi}\right)$ is reversible , the definitions of $\overline{\beta}$ in remark 3.11 and proposition 4.1.6 are equivalent .

Proof :-Let $\psi_i\ ,i=1,|\Omega|-1$ belong to a basis of real eigen-functions $\psi_i\ ,\ 0\leq i\leq|\Omega|-1$. Then at degree α

$$\begin{aligned}\left\|\mathrm{K}_{\alpha}-E\right\|_{\alpha,2\to 2}\psi_i &= \sup_{\|\psi_i\|_{\alpha,2}<1}\left\|\left(\mathrm{K}_{\alpha}-E\right)\psi_i\right\|_{\alpha,2}\psi_i\\ &= \sup_{\|\psi_i\|_{\alpha,2}<1}\left\|\mathrm{K}_{\alpha}\psi_i-E\psi_i\right\|_{\alpha,2}\psi_i\\ &= \sup_{\|\psi_i\|_{\alpha,2}<1}\left\|\mathrm{K}_{\alpha}\psi_i\right\|_{\alpha,2}\psi_i\\ &= \sup_{\|\psi_i\|_{\alpha,2}<1}\left\|\beta_{\alpha}\psi_i\right\|_{\alpha,2}\psi_i\\ &= \beta_{\alpha}(1)\psi_i\\ &= \beta_{\alpha}\psi_i\end{aligned}$$

Therefore

$$\overline{\left\| \overline{\mathrm{K}} - E \right\|}_{2\to 2} = \overline{\beta} = \max\left\{ \left| \overline{\beta}_{\min} \right|, \overline{\beta}_1 \right\} \blacksquare$$

Proposition 4.1.9

When $\left(\overline{k}, \overline{\pi}\right)$ is reversible , then the definitions of the fuzzy spectral gap $\overline{\lambda}$ in proposition 3.15 and definition 4.1.3 are equivalent .

Proof :- We have by the following from [6]

$$1 - \overline{\beta} = \min\left\{ \overline{\ell}\left(f, f\right), \| f \|_2 = 1, Ef = 0 \right\}$$

Where

$$\overline{\beta} = \overline{\left\| \overline{\mathrm{K}} - E \right\|}_{2\to 2}$$

So by proposition 4.1.8 we have

$$1 - \overline{\beta}_1 = 1 - \max\left\{ \left| \overline{\beta}_{\min} \right|, \overline{\beta}_1 \right\}$$

$$= 1 - \overline{\beta}$$

$$= \min\left\{ \overline{\ell}\left(f, f\right), \| f \|_2 = 1, Ef = 0 \right\}$$

$$= \overline{\lambda}$$

$\blacksquare$

Lemma 4.1.10

For any fuzzy chain $\left(\overline{k},\overline{\pi}\right)$ and $p\geq 1$, the fuzzy Dirichlet form $\overline{\ell}$ satisfies for all non-negative functions f

$i)$ $$\partial_t \left. \overline{\left\| \overline{H}_t\ f \right\|_p^p} \right|_{t=0} = -p\,\overline{\ell}\left(f, f^{p-1}\right)$$

$ii)$ $$\partial_t\, \overline{Ent}\left(\overline{H}_t\ f \right)\Big|_{t=0} = -\overline{\ell}\left(f, \log f\right)$$

Proof :- $i)$ We have at degree α

$$\begin{aligned}
\partial_t \left\| H_{\alpha,t}\, f \right\|_{\alpha,p}^p \Big|_{t=0} &= \partial_t \sum_x \left| H_{\alpha,t}\, f(x)\right|^p \pi_\alpha(x)\Big|_{t=0} \\
&= \partial_t \sum_x H_{\alpha,t}^p\, f^p(x)\pi_\alpha(x)\Big|_{t=0} \\
&= -p\sum_x \left(I-\mathrm{K}_\alpha\right)H_{\alpha,t}^p\, f(x) f^{p-1}(x)\pi_\alpha(x)\Big|_{t=0} \\
&= -p\sum_x \left(I-\mathrm{K}_\alpha\right) f(x) f^{p-1}(x)\pi_\alpha(x) \\
&= -p\left\langle \left(I-\mathrm{K}_\alpha\right) f, f^{p-1}\right\rangle \\
&= -p\ell_\alpha\left(f, f^{p-1}\right)
\end{aligned}$$

Hence

$$\partial_t \left. \overline{\left\| \overline{H}_t\ f \right\|_p^p} \right|_{t=0} = -p\,\overline{\ell}\left(f, f^{p-1}\right)$$

$ii)$ We have at degree α

$$\partial_t Ent_\alpha\left(H_{\alpha,t} f\right)\Big|_{t=0} = \partial_t \sum_x \left[H_{\alpha,t} f(x) \log H_{\alpha,t} f(x)\right] \pi_\alpha(x)\Big|_{t=0}$$

$$= \partial_t \sum_x \left[H_{\alpha,t} f(x)\left(t\left(\mathrm{K}_\alpha - I\right) + \log f(x)\right)\right] \pi_\alpha(x)\Big|_{t=0}$$

$$= \sum_x \left[H_{\alpha,t} f(x)\left(\mathrm{K}_\alpha - I\right) + \left(\mathrm{K}_\alpha - I\right) H_{\alpha,t} f(x) .\right.$$

$$\left.\left(t\left(\mathrm{K}_\alpha - I\right) + \log f(x)\right)\right] \pi_\alpha(x)\Big|_{t=0}$$

$$= \sum_x \left[-\left(I - \mathrm{K}_\alpha\right) f(x) - \left(I - \mathrm{K}_\alpha\right) f(x) \log f(x)\right] \pi_\alpha(x)$$

$$= -\sum_x \left(I - \mathrm{K}_\alpha\right) f(x)\left(1 + \log f(x)\right) \pi_\alpha(x)$$

$$= -\left\langle \left(I - \mathrm{K}_\alpha\right) f, 1 + \log f \right\rangle$$

$$= -\left(\left\langle \left(I - \mathrm{K}_\alpha\right) f, 1 \right\rangle + \left\langle \left(I - \mathrm{K}_\alpha\right) f, \log f \right\rangle\right)$$

$$= -\left(\ell_\alpha(f, 1) + \ell_\alpha(f, \log f)\right)$$

$$= 0 - \ell_\alpha(f, \log f)$$

$$= -\ell_\alpha(f, \log f)$$

Thus

$$\partial_t \overline{Ent}\left(\overline{H}_t f\right)\Big|_{t=0} = -\overline{\ell}(f, \log f) \quad \blacksquare$$

Lemma 4.1.11

Let $p \geq 2$. For any fuzzy chain $\overline{k}$ with invariant fuzzy measure $\overline{\pi}$ and any function $f \geq 0$.

$i)$ $$\overline{\ell}\left(f, f^{p-1}\right) \geq \frac{2}{p} \overline{\ell}\left(f^{p/2}, f^{p/2}\right)$$

$ii)$ If $\left(\overline{k}, \overline{\pi}\right)$ is reversible , then

$$\overline{\ell}\left(f, f^{p-1}\right) \geq \frac{4(p-1)}{p^2} \overline{\ell}\left(f^{p/2}, f^{p/2}\right)$$

for all $1 < p < \infty$.

Proof :- We follow[6]

$i)$ At degree α , when $p \geq 2$ the function $t \to t^{p/2}$ is convex on $[0, \infty)$. Now , for any smooth convex function ϕ

$$\phi(a) - \phi(b) \geq \phi'(b)(a-b)$$

So , we have

$$\left(a^{p/2} - b^{p/2}\right) \geq \left(p/2\right) b^{p/2 - 1}(a-b)$$

Multiplying by $-b^{p/2}$, we get

$$\left(b^{p/2} - a^{p/2}\right) b^{p/2} \leq \frac{p}{2} b^{p-1}(b-a)$$

for all $a, b \geq 0$. This gives

$$\left[(I - \mathrm{K}_\alpha) f^{p/2}\right] f^{p/2} \leq \frac{p}{2}\left[(I - \mathrm{K}_\alpha) f\right] f^{p-1}$$

That is

$$\ell_\alpha\left(f, f^{p-1}\right) \geq \frac{2}{p} \ell_\alpha\left(f^{p/2}, f^{p/2}\right)$$

Therefore

$$\overline{\ell}\left(f, f^{p-1}\right) \geq \frac{2}{p} \overline{\ell}\left(f^{p/2}, f^{p/2}\right)$$

$ii)$ At degree α , write for any $a,b \geq 0$

$$\left(\frac{a^{p/2}-b^{p/2}}{a-b}\right)^2=\left(\frac{p}{2(a-b)}\int_b^a t^{p/2-1}dt\right)^2$$

$$\leq \frac{p^2}{4(a-b)}\int_b^a t^{p-2}dt$$

$$=\frac{p^2}{4(p-1)}\frac{a^{p-1}-b^{p-1}}{a-b}$$

This shows that

$$\left(a^{p-1}-b^{p-1}\right)(a-b)\geq \frac{4(p-1)}{p^2}\left(a^{p/2}-b^{p/2}\right)^2$$

So

$$\left[(I-\mathrm{K}_\alpha)f^{\,p-1}\right]f \geq \frac{4(p-1)}{p^2}\left[(I-\mathrm{K}_\alpha)f^{\,p/2}\right]f^{\,p/2}$$

That is

$$\ell_\alpha\left(f,f^{\,p-1}\right)\geq \frac{4(p-1)}{p^2}\ell_\alpha\left(f^{\,p/2},f^{\,p/2}\right)$$

Therefore

$$\overline{\ell}\left(f,f^{\,p-1}\right)\geq \frac{4(p-1)}{p^2}\overline{\ell}\left(f^{\,p/2},f^{\,p/2}\right) \blacksquare$$

Lemma 4.1.12

For any fuzzy chain $\overline{k}$ with invariant fuzzy measure $\overline{\pi}$ and any functions $f \geq 0$

$i)$ $$\overline{\ell}(\log f, f) \geq 2\,\overline{\ell}\left(\sqrt{f}, \sqrt{f}\right)$$

$ii)$ Any reversible fuzzy chain $\left(\overline{k}, \overline{\pi}\right)$ satisfies

$$\overline{\ell}(\log f, f) \geq 4\,\overline{\ell}\left(\sqrt{f}, \sqrt{f}\right)$$

Proof :- We follow[6]

$i)$ At degree α , as $t \to -\log t^2$ is a convex function , then

$$-\left(\log a^2 - \log b^2\right) \geq \frac{-2}{b}(a-b)$$

for all $a, b > 0$. Multiplying by $-b^2$ yields

$$b^2\left(\log a^2 - \log b^2\right) \leq 2b(a-b)$$

for all $a, b > 0$. This shows that

$$f(x)\left[(\mathrm{K}_\alpha - I)\log f\right](x) \leq 2\sqrt{f(x)}\left[(\mathrm{K}_\alpha - I)\sqrt{f}\right](x)$$

Then

$$f(x)\left[(I - \mathrm{K}_\alpha)\log f\right](x) \geq 2\sqrt{f(x)}\left[(I - \mathrm{K}_\alpha)\sqrt{f}\right](x)$$

So $$\ell_\alpha(\log f, f) \geq 2\ell_\alpha\left(\sqrt{f}, \sqrt{f}\right)$$

Hence

$$\overline{\ell}(\log f, f) \geq 2\,\overline{\ell}\left(\sqrt{f}, \sqrt{f}\right) \blacksquare$$

$ii)$ At degree α , $a \geq b \geq 0$, we have

$$\left(\frac{a^{1/2}-b^{1/2}}{a-b}\right)^2 = \left(\frac{1}{2(a-b)}\int_b^a \frac{dt}{t^{1/2}}\right)^2$$

$$\leq \frac{1}{4(a-b)}\int_b^a \frac{dt}{t}$$

$$= \frac{1}{4}\frac{\log a - \log b}{a-b}$$

So

$$4\left(a^{1/2}-b^{1/2}\right)^2 \leq (\log a - \log b)(a-b)$$

And

$$4\left[(I-\mathrm{K}_\alpha)\sqrt{f}\,\right]\sqrt{f} \leq \left[(I-\mathrm{K}_\alpha)\log f\,\right]f$$

Then

$$4\ell_\alpha\left(\sqrt{f},\sqrt{f}\,\right) \leq \ell_\alpha(\log f, f)$$

Thus

$$4\,\overline{\ell}\left(\sqrt{f},\sqrt{f}\,\right) \leq \overline{\ell}(\log f, f)\ \blacksquare$$

Remark 4.1.13

The difficulty in proving of lemmas 4.1.11 and 4.1.12 comes from the fact that formula of the Dirichlet form , does not hold in general .

4.2.Distance to Equilibrium

Definition 4.2.1[22]

The total variation distance between two probability measures μ and ν is

$$\|\mu-\nu\|_{TV}=\sup_{A\subset\Omega}|\mu(A)-\nu(A)|$$
$$=\frac{1}{2}\sum_{x}|\mu(x)-\nu(x)|$$

Note that :-If we have $\mu_\alpha\in\overline{\mu}[\alpha]$ of the fuzzy probability measure $\overline{\mu}$, also $\nu_\alpha\in\overline{\nu}[\alpha]$ of the fuzzy probability measure $\overline{\nu}$, such that $\overline{\mu}$ and $\overline{\nu}$ are triangular fuzzy numbers , then the total variation distance between μ_α and ν_α is

$$\|\mu_\alpha-\nu_\alpha\|_{TV}=\sup_{A\subset\Omega}|\mu_\alpha(A)-\nu_\alpha(A)|$$
$$=\frac{1}{2}\sum_{x}|\mu_\alpha(x)-\nu_\alpha(x)|$$

Hence , the total variation distance between two fuzzy probability measures $\overline{\mu}$ and $\overline{\nu}$ is $\|\cdot\|_{TV}$.

Proposition 4.2.2

For the iterated fuzzy kernel $\overline{k}_x^{\,n}$ and invariant fuzzy probability measure $\overline{\pi}$, we have

$$2\left\|\overline{k}_x^{\,n}-\overline{\pi}\right\|_{TV}=\left\|\overline{\left\|\overline{\kappa}_x^{\,n}-1\right\|}\right\|_1$$

Proof :- We have at degree α

$$\begin{aligned}2\left\|k_{\alpha,x}^{n}-\pi_\alpha\right\|_{TV}&=\sum_y\left|k_{\alpha,x}^{n}(y)-\pi_\alpha(y)\right|\\&=\sum_y\left|\frac{k_{\alpha,x}^{n}(y)}{\pi_\alpha(y)}-1\right|\pi_\alpha(y)\\&=\sum_y\left|\kappa_{\alpha,x}^{n}(y)-1\right|\pi_\alpha(y)\\&=\left\|\kappa_{\alpha,x}^{n}-1\right\|_{\alpha,1}\end{aligned}$$

Thus

$$2\left\|\overline{k}_x^{\,n}-\overline{\pi}\right\|_{TV}=\left\|\overline{\left\|\overline{\kappa}_x^{\,n}-1\right\|}\right\|_1 \quad \blacksquare$$

Definition 4.2.3[22]

The chi-square distance between two probability measures μ and ν is the $L^2(\pi)$ distance between μ/ν and 1, that is

$$\left\| \frac{\mu}{\nu} - 1 \right\|_2 = \left(\sum_x \left| \frac{\mu(x)}{\nu(x)} - 1 \right|^2 \pi(x) \right)^{1/2}$$

Note that :- If we have $\mu_\alpha \in \overline{\mu}[\alpha]$ of $\overline{\mu}$, $\nu_\alpha \in \overline{\nu}[\alpha]$ of $\overline{\nu}$, such that $\overline{\mu}$ and $\overline{\nu}$ are triangular fuzzy numbers , then the chi-square distance at degree α between μ_α and ν_α is

$$\left\| \frac{\mu_\alpha}{\nu_\alpha} - 1 \right\|_{\alpha,2} = \left(\sum_x \left| \frac{\mu_\alpha(x)}{\nu_\alpha(x)} - 1 \right|^2 \pi_\alpha(x) \right)^{1/2}$$

and $\|\cdot\|_{\alpha,2} \in \overline{\|\cdot\|}_2[\alpha]$ of $\overline{\|\cdot\|}_2$. Hence the fuzzy chi-square distance between two fuzzy probability measures $\overline{\mu}$ and $\overline{\nu}$ is the $L^2\left(\overline{\pi}\right)$ distance between $\overline{\mu}/\overline{\nu}$ and 1.

Proposition 4.2.4

The total variation distance is dominated by the fuzzy chi-square distance , namely

$$2\left\| \overline{k}_x^n - \overline{\pi} \right\|_{TV} \le \left\| \overline{\kappa}_x^n - 1 \right\|_2$$

Proof :-We have by proposition 4.2.2

$$2\left\|\overline{k}_x^{\,n}-\overline{\pi}\right\|_{TV} = \overline{\left\|\overline{\kappa}_x^{\,n}-1\right\|_1}$$

By Jensen's inequality we have

$$\overline{\left\|\overline{\kappa}_x^{\,n}-1\right\|_1} \leq \overline{\left\|\overline{\kappa}_x^{\,n}-1\right\|_2}$$

Therefore

$$2\left\|\overline{k}_x^{\,n}-\overline{\pi}\right\|_{TV} \leq \overline{\left\|\overline{\kappa}_x^{\,n}-1\right\|_2} \quad \blacksquare$$

Definition 4.2.5.[22]

The relative error between two probability measures μ and ν is the $L^{\infty}(\pi)$ distance between μ/ν and 1, that is ,

$$\left\|\mu/\nu - 1\right\|_{\infty} = \sup_x \left|\frac{\mu(x)}{\nu(x)} - 1\right|$$

Note that :- If we have probability measures $\mu_\alpha \in \overline{\mu}[\alpha]$ of $\overline{\mu}$, $\nu_\alpha \in \overline{\nu}[\alpha]$ of $\overline{\nu}$, where $\overline{\mu}$ and $\overline{\nu}$ are triangular fuzzy numbers , then the relative error between μ_α and ν_α is

$$\left\| \mu_\alpha \big/ \nu_\alpha - 1 \right\|_\infty = \sup_x \left| \frac{\mu_\alpha(x)}{\nu_\alpha(x)} - 1 \right|$$

Hence the relative error between two fuzzy probability measures $\overline{\mu}$ and $\overline{\nu}$ is $L^\infty(\pi)$ distance between $\overline{\mu} \big/ \overline{\nu}$ and 1.

Proposition 4.2.6

The fuzzy chi-square distance $\overline{\left\| \overline{\kappa}_x^{\,n} - 1 \right\|}_2$ is dominated by the relative error $\sup_y \left| \overline{\kappa}_x^{\,n}(y) - 1 \right|$.

Proof :- We have by Jensen's inequality

$$\overline{\left\| \overline{\kappa}_x^{\,n} - 1 \right\|}_2 \leq \left\| \overline{\kappa}_x^{\,n}{}_{-1} \right\|_\infty = \sup_y \left| \overline{\kappa}_x^{\,n}(y) - 1 \right| \quad \blacksquare$$

.

Proposition 4.2.7

For reversibility fuzzy chains , the maximal relative error at time $2n$, that is , $\sup_{x,y} \left| \overline{\kappa}_x^{\,2n}(y) - 1 \right|$ is equal to the square maximal fuzzy chi-square distance , $\sup_x \overline{\left\| \overline{\kappa}_x^{\,n} - 1 \right\|}_2^2$ at time n .

Proof :-We have by lemma 3.17 at degree α

$$\sup_{x,y}\left|\kappa_{\alpha,x}^{2n}(y)-1\right| = \sup_{x,y}\left|\sum_{0}^{|\Omega|-1}\beta_{\alpha,i}^{2n}\psi_i(x)\psi_i(y)-1\right|$$

$$= \sup_{x,y}\left|\sum_{1}^{|\Omega|-1}\beta_{\alpha,i}^{2n}\psi_i(x)\psi_i(y)\right|$$

$$= \sup_{x}\left|\sum_{1}^{|\Omega|-1}\beta_{\alpha,i}^{2n}\psi_i^2(x)\right|$$

$$= \sup_{x}\sum_{1}^{|\Omega|-1}\beta_{\alpha,i}^{2n}\left|\psi_i(x)\right|^2$$

$$= \sup_{x}\left\|\kappa_{\alpha,x}^{n}-1\right\|_{\alpha,2}^{2}$$

Hence

$$\sup_{x,y}\left|\overline{\kappa}_x^{2n}(y)-1\right| = \sup_{x}\overline{\left\|\overline{\kappa}_x^{n}-1\right\|_2}^{2} \quad \blacksquare$$

Remark 4.2.8

Propositions 4.2.2 , 4.2.4 , 4.2.6 , and 4.2.7 can hold without changes in continuous time if we replace $\overline{\mathrm{K}}$ by $\overline{H}_t$, $\overline{\kappa}$ by $\overline{h}_t$, and n by t .

In the following lemma we state the simplest and most basic quantitative bounds on the fuzzy chi-square distance .

Lemma 4.2.9

Any finite fuzzy Markov chain $\overline{k}$ with invariant fuzzy probability measure $\overline{\pi}$ satisfies

$$i)\ \left\|\overline{\overline{\kappa}_x^{\,n}-1}\right\|_2 \leq \left(\overline{\pi}(x)\right)^{-1/2} \overline{\beta}^{\,n}$$

$$ii)\ \left\|\overline{\overline{h}_t^{x}-1}\right\|_2 \leq \left(\overline{\pi}(x)\right)^{-1/2} e^{-t\overline{\lambda}}$$

Proof :- $i)$ We define at degree α

$$\delta_{\alpha,x}(y)=\begin{Bmatrix} \dfrac{1}{\pi_\alpha(x)} & ,x=y \\ 0 & ,x\neq y \end{Bmatrix}$$

Observe that

$$\begin{aligned}\left(\mathrm{K}_\alpha^{*^n}-E\right)\delta_{\alpha,x}(y) &= \mathrm{K}_\alpha^{*^n}\delta_{\alpha,x}(y)-E\delta_x(y) \\ &= \sum_Z k_\alpha^{*^n}(y,z)\delta_{\alpha,x}(z)-\sum_y \delta_x(y)\pi(y) \\ &= \frac{k_\alpha^{*^n}(y,x)}{\pi_\alpha(x)}-1\end{aligned}$$

$$= \frac{1}{\pi_\alpha(x)} \frac{k_\alpha^n(x,y)\pi_\alpha(x)}{\pi_\alpha(y)} - 1$$

$$= \frac{k_\alpha^n(x,y)}{\pi_\alpha(y)} - 1$$

$$= \kappa_\alpha^n(x,y) - 1$$

So

$$\left\|\kappa_{\alpha,x}^n - 1\right\|_{\alpha,2} = \left\|\left(\mathrm{K}_\alpha^{*^n} - E\right)\delta_{\alpha,x}\right\|_{\alpha,2}$$

$$\leq \left\|\delta_{\alpha,x}\right\|_{\alpha,2} \left\|\mathrm{K}_\alpha^{*^n} - E\right\|_{\alpha,2\to 2}$$

$$= \left\|\delta_{\alpha,x}\right\|_{\alpha,2} \left\|\mathrm{K}_\alpha^n - E\right\|_{\alpha,2\to 2}$$

By proposition 4.1.7 we have

$$\left\|\kappa_{\alpha,x}^n - 1\right\|_{\alpha,2} \leq \left(\pi_\alpha(x)\right)^{-1/2} \beta_\alpha^n$$

Therefore

$$\left\|\overline{\overline{\kappa}_x^n - 1}\right\|_2 \leq \left(\overline{\pi}(x)\right)^{-1/2} \overline{\beta}^n .$$

$ii)$ As (i) we define $\delta_{\alpha,x}(y)$ at degree α , so

$$\left(H_{\alpha,t}^* - E\right)\delta_{\alpha,x}(y) = H_{\alpha,t}^*\delta_{\alpha,x}(y) - E\delta_x(y)$$

$$= e^{-t}\sum_0^\infty \frac{t^n}{n!} \mathrm{K}_\alpha^{*^n} \delta_{\alpha,x}(y) - 1$$

$$= e^{-t}\sum_0^\infty \frac{t^n}{n!} \sum_x k_\alpha^{*^n}(y,x)\delta_{\alpha,x}(x) - 1$$

$$= e^{-t}\sum_0^\infty \frac{t^n}{n!} k_\alpha^{*^n}(y,x)\frac{1}{\pi_\alpha(x)} - 1$$

$$= e^{-t}\sum_{0}^{\infty}\frac{t^n}{n!}\kappa_\alpha^{*^n}(y,x)-1$$

$$= e^{-t}\sum_{0}^{\infty}\frac{t^n}{n!}\kappa_\alpha^{n}(x,y)-1$$

$$= h_{\alpha,t}(x,y)-1$$

So

$$\left\|h_{\alpha,t}^{x}-1\right\|_{\alpha,2} = \left\|\left(H_{\alpha,t}^{*}-E\right)\delta_{\alpha,x}\right\|_{\alpha,2}$$

$$\leq \left\|\delta_{\alpha,x}\right\|_{\alpha,2}\left\|H_{\alpha,t}^{*}-E\right\|_{\alpha,2\to 2}$$

$$= \left\|\delta_{\alpha,x}\right\|_{\alpha,2}\left\|H_{\alpha,t}-E\right\|_{\alpha,2\to 2}$$

By proposition 4.1.5 we have

$$\left\|h_{\alpha,t}^{x}-1\right\|_{\alpha,2} \leq \left(\pi_\alpha(x)\right)^{-1/2} e^{-\lambda_\alpha t}$$

Thus

$$\overline{\left\|\overline{h_t^x}-1\right\|_2} \leq \left(\overline{\pi}(x)\right)^{-1/2} e^{-\overline{\lambda}t} \quad \blacksquare$$

.

Chapter Five

Fuzzy Logarithmic Sobolev Inequalities

In this chapter we will introduce concepts of fuzzy logarithmic Sobolev inequalities , fuzzy logarithmic Sobolev constant , hyper-contractivity of fuzzy Markov semi-group, and ergodicity of fuzzy Markov chains , and we prove essential results around quantative bounds on convergence fuzzy kernel of fuzzy Markov semi-group to stationary fuzzy distribution in total variation distance .

5.1.The Fuzzy Logarithmic Sobolev Constant

Definition 5.1.1.[12]

A Markov chain is called an irreducible chain if it possible to go from every state to every state **.**

Definition 5.1.2.[6]

Given an irreducible finite Markov chain k ,with invariant probability measure π . Consider the Dirichlet form

$$\ell(f,g)=\langle (I-\mathrm{K})f,g\rangle$$

And set the entropy

$$\zeta(f)=\sum_{x\in\Omega}|f(x)|^2\log\left(\frac{|f(x)|^2}{\|f\|_2^2}\right)\pi(x)$$

Then log-Sobolev inequality is

$$\zeta(f)\leq c\ell(f,f)$$

holding for any function f , c is constant .

Note that :-If we have $\ell_\alpha \in \overline{\ell}[\alpha]$of the fuzzy Dirichlet form $\overline{\ell}$,

$\pi_\alpha \in \overline{\pi}[\alpha]$of the invariant fuzzy probability measure $\overline{\pi}$, and

$$\zeta_\alpha(f)=\sum_{x\in\Omega}|f(x)|^2\log\left(\frac{|f(x)|^2}{\|f\|_{\alpha,2}^2}\right)\pi_\alpha(x)$$

Where $\zeta_\alpha \in \overline{\zeta}[\alpha]$ the α -cut of the triangular fuzzy number $\overline{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$, then the log-Sobolev inequality at degree α is

$$\zeta_\alpha(f) \leq c_\alpha \ell_\alpha(f, f)$$

holding for any function f , c_α is constant . Hence the fuzzy log-Sobolev inequality is

$$\overline{\zeta}(f) \leq \overline{c}\, \overline{\ell}(f, f)$$

.**Remark 5.1.3**

If we have $\|f\|_2 = 1$ then the fuzzy entropy $\overline{\zeta}$ become

$$\overline{\zeta}(f) = \sum_x |f(x)|^2 \log |f(x)|^2 \, \overline{\pi}(x)$$

Definition 5.1.4.[6]

Let $\frac{1}{\wp}$ be the smallest constant c , such that

$$\wp = \inf \left\{ \frac{\ell(f, f)}{\zeta(f)} : \zeta(f) \neq 0 \right\}$$

This called the log-Sobolev constant .

Note that :-If we deal with the fuzzy log-Sobolev inequality , then the fuzzy log-Sobolev constant is

$$\overline{\wp} = \inf \left\{ \frac{\overline{\ell}(f, f)}{\overline{\zeta}(f)} : \zeta_\alpha(f) \neq 0,\ for\, all\, \alpha \in [0,1] \right\}$$

Lemma 5.1.5

For any fuzzy chain $\overline{k}$, the fuzzy log-Sobolev constant $\overline{\wp}$ and the fuzzy spectral gap $\overline{\lambda}$ satisfy

$$2\overline{\wp} \leq \overline{\lambda}$$

Proof :-We follow [20] . Set $f = 1 + \varepsilon\, g$ and write ε small enough at degree α

$$\left|f\right|^2 \log\left|f\right|^2 = 2\left(1 + 2\varepsilon\, g + \varepsilon^2\left|g\right|^2\right)\left(\varepsilon\, g - \frac{\varepsilon^2\left|g\right|^2}{2} + O\left(\varepsilon^3\right)\right)$$

$$= 2\varepsilon\, g + 3\varepsilon^2\left|g\right|^2 + O\left(\varepsilon^3\right)$$

And

$$\left|f\right|^2 \log\left\|f\right\|_{\alpha,2}^2 = \left(1 + 2\varepsilon\, g + \varepsilon^2\left|g\right|^2\right)\Big(2\,\varepsilon\, E_\alpha g + \varepsilon^2\left\|g\right\|_{\alpha,2}^2$$

$$- 2\varepsilon^2\left(E_\alpha g\right)^2 + O\left(\varepsilon^3\right)\Big)$$

$$= 2\varepsilon\, E_\alpha g + 4\varepsilon^2 g\, E_\alpha g + \varepsilon^2\left\|g\right\|_{\alpha,2}^2 - 2\varepsilon^2\left(E_\alpha g\right)^2 + O\left(\varepsilon^2\right)$$

Thus

$$\left|f\right|^2 \log\frac{\left|f\right|^2}{\left\|f\right\|_{\alpha,2}^2} = 2\varepsilon\left(g - E_\alpha g\right) + \varepsilon^2\Big(3\left|g\right|^2 - \left\|g\right\|_{\alpha,2}^2 - 4gE_\alpha g$$

$$+ 2\left(E_\alpha g\right)^2 + O\left(\varepsilon^3\right)\Big)$$

And

$$\zeta_\alpha(f) = 2\varepsilon^2\left(\|g\|_{\alpha,2}^2 - (E_\alpha g)^2\right) + O(\varepsilon^3)$$
$$= 2\varepsilon^2 \operatorname{var}_\alpha(g) + O(\varepsilon^3)$$

Now

$$\begin{aligned}\ell_\alpha(f,f) &= \langle (I - \mathrm{K}_\alpha) f, f\rangle \\ &= \sum_{x,y} |f(x) - f(y)|^2 k_\alpha(x,y)\pi_\alpha(x) \\ &= \sum_{x,y} |1 + \varepsilon g(x) - (1 + \varepsilon g(y))|^2 k_\alpha(x,y)\pi_\alpha(x) \\ &= \varepsilon^2 \sum_{x,y} |g(x) - g(y)|^2 k_\alpha(x,y)\pi_\alpha(x) \\ &= \varepsilon^2 \langle (I - \mathrm{K}_\alpha) g, g\rangle \\ &= \varepsilon^2 \ell_\alpha(g,g)\end{aligned}$$

Hence

$$\begin{aligned}\wp_\alpha &= \inf\left\{\frac{\ell_\alpha(f,f)}{\zeta_\alpha(f)} : \zeta_\alpha(f) \neq 0\right\} \\ &= \inf\left\{\frac{\varepsilon^2 \ell_\alpha(g,g)}{2\varepsilon^2 \operatorname{var}_\alpha(g) + O(\varepsilon^3)} : 2\varepsilon^2 \operatorname{var}_\alpha(g) + O(\varepsilon^3) \neq 0\right\} \\ &\le \inf \frac{\ell_\alpha(g,g)}{2\operatorname{var}_\alpha(g)} \\ &= \frac{1}{2}\lambda_\alpha\end{aligned}$$

So $$2\wp_\alpha \le \lambda_\alpha$$

Therefore $$2\overline{\wp} \le \overline{\lambda} \ \blacksquare$$

Remark 5.1.6

In most examples where $2\overline{\wp}$ and $\overline{\lambda}$ are explicity known , they turn out to be equal .

Example 5.1.7

Assume that $\Omega = Z_2^2$ and $e_i\ , i = 1,2$ be the element of Z_2^2 with all coordinates 0 except the $i\,th$ which is 1. A fuzzy probability $\overline{Q}$ is defined $\overline{Q}(0) = \overline{Q}(e_i) = (0.2,0.3,0.4), i = 1,2$ and $Q(x) = 0$, otherwise . The associated random walk on Z_2^2 has fuzzy kernel

$$\overline{k}(x, y) = \overline{Q}(x - y)$$

The invariant fuzzy probability measure is

$$\overline{\pi} = (0.24, 0.25, 0.26)$$

The fuzzy spectral gap is

$$\overline{\lambda} = (0.5, 0.6, 0.7)$$

The fuzzy Log-Sobolev constant is

$$\overline{\wp} = \frac{\overline{\lambda}}{2} = (0.25, 0.30, 0.35) .$$

Note :- This example came in [8] in classical form , where the probability Q is

$$Q(0) = Q(e_i) = \frac{1}{3}, i = 1,2$$

The kernel is

$$k(x, y) = Q(x - y)$$

The invariant probability measure is

$$\pi = \frac{1}{4}$$

The spectral gap is

$$\lambda = \frac{2}{3}$$

The Log-Sobolev constant is

$$\wp = \frac{\lambda}{2} = \frac{1}{3} \quad .$$

5.2. The hypercontractivity of fuzzy Markov semi-group

Definition 5.2.1[17]

A Markov semi-group H_t on $L^2(\pi)$ to $L^q(\pi)$ is said to be hyper-contractive if there exist a finite $t > 0$ and a continuously differentiable function (hypercontractive equation) $q(t) \leq 1 + e^{ct}$ $, 2 \leq q(t) < \infty, 0 \leq c$ such that

$$\|H_t\|_{2 \to q} \leq 1$$

Note that :-If $\|H_t\|_{2 \to q} < 1$ then a Markov semi-group $H_{\alpha,t} \in \overline{H}_t[\alpha]$ of $\overline{H}_t$ is hypercontractive at degree α if there exist a finite $t > 0$, and a continuously differentiable function (hypercontractive equation) $q(t) \leq 1 + e^{ct}$ $, 2 \leq q(t) < \infty, 0 \leq c$ such that

$$\|H_{\alpha,t}\|_{\alpha, 2 \to q} < 1$$

Hence

$$\overline{\left\| \overline{H}_t \right\|}_{2 \to q} < 1$$

This implies $\overline{H}_t$ is hypercontractive on $L^2\left(\overline{\pi}\right)$.

Proposition 5.2.2

A fuzzy Markov semi-group $\overline{H}_t$ on $L^2\left(\overline{\pi}\right)$ is hypercontractive if and only if

$$\overline{\left\|\overline{H}_t\ f\right\|_q} < \overline{\|f\|}_2$$

For any $f \in L^2\left(\overline{\pi}\right)$

Proof :-First sense ,we have by definition 5.2.1 at degree α

$$\left\|H_{\alpha,t}\right\|_{\alpha,2\to q} < 1,\ q(t) \leq 1 + e^{ct},\ 0 \leq c$$

Since

$$\left\|H_{\alpha,t}\right\|_{\alpha,2\to q} = \sup \frac{\left\|H_{\alpha,t}\, f\right\|_{\alpha,q}}{\|f\|_{\alpha,2}}$$

Then

$$\left\|H_{\alpha,t}\, f\right\|_{\alpha,q} \leq \left\|H_{\alpha,t}\right\|_{\alpha,2\to q} \|f\|_{\alpha,2}$$

$$< \|f\|_{\alpha,2}$$

Hence

$$\overline{\left\|\overline{H}_t\ f\right\|_q} < \overline{\|f\|}_2$$

Second sense , we have at degree α

$$\left\|H_{\alpha,t}\, f\right\|_{\alpha,q} < \|f\|_{\alpha,2}$$

If we take the supremum over all f with $\|f\|_{\alpha,2} < 1$ we get

$$\left\|H_{\alpha,t}\right\|_{\alpha,2\to q} < 1$$

Therefore

$$\overline{\left\| \overline{H}_t \right\|_{2\to q}} < 1 \blacksquare$$

Example 5.2.3[17]

A fuzzy Markov semi-group $\overline{H}_t$ from $L^2\left(\overline{\pi}\right)$ to $L^4\left(\overline{\pi}\right)$ is hypercontractive since

$$\overline{\left\| \overline{H}_t \right\|_{2\to 4}} < 1 \ .$$

Remark 5.2.4

In case the log-Sobolev constant is the constant of the hyper-contractivity equation $q(t) \le 1 + e^{ct}$ $, 2 \le q(t) < \infty, 0 \le c$ then the fuzzy log-Sobolev inequality become

$$c\overline{\zeta}(f) \le \overline{\ell}(f, f)$$

Theorem 5.2.5

Let $\left(\overline{k}, \overline{\pi}\right)$ be a fuzzy finite Markov chain with log-Sobolev constant $\wp$.

$i)$ Assume that there exists $\beta > 0$ such that

$$\overline{\left\| \overline{H}_t \right\|_{2\to q}} < 1$$

for all $t > 0$ and $2 \le q(t) < \infty$ satisfying $q(t) - 1 \le e^{\beta t}$.

Then $\beta\overline{\zeta}(f) \le \overline{\ell}(f,f)$, and thus $\wp > \beta$.

$ii)$ Assume that $\left(\overline{k},\overline{\pi}\right)$ is reversible , then $\overline{\left\|\overline{H}_t\right\|}_{2\to q} < 1$ for all $t > 0$ and all $2 \le q(t) < \infty$ satisfying $q(t) - 1 \le e^{4\wp t}$.

$iii)$ For non reversible chains , we still have $\overline{\left\|\overline{H}_t\right\|}_{2\to q} < 1$ for all $t > 0$ and all $2 \le q(t) < \infty$ satisfying $q(t) - 1 \le e^{2\wp t}$.

Proof :-

$i)$ We set at degree α for $f > 0$, $F_\alpha = \left\|H_{\alpha,t} f\right\|_{\alpha,p(t)}$

Where $p(t) = 1 + e^{4\beta t}$ we compute the derivative of

$$F_\alpha(t) = \exp\left[\frac{\log G_\alpha(t)}{p(t)}\right], \; G_\alpha(t) = \left\|H_{\alpha,t} f\right\|_{\alpha,p(t)}^{p(t)}$$

That is

$$\begin{aligned} G_\alpha(t) &= \sum_x \left|H_{\alpha,t} f(x)\right|^{p(t)} \pi_\alpha(x) \\ &= \sum_x \left|e^{-t(I-\mathrm{K}_\alpha)} f(x)\right|^{1+e^{4\beta t}} \pi_\alpha(x) \\ &= \sum_x \exp\left[\left(1+e^{4\beta t}\right)\left(-t\left(I-\mathrm{K}_\alpha\right) + \log f(x)\right)\right]\pi_\alpha(x) \end{aligned}$$

$$G'_\alpha(t) = \sum_x \exp\left[\left(1+e^{4\beta t}\right)\left(-t\left(I-\mathrm{K}_\alpha\right) + \log f(x)\right)\right].$$
$$\left[-\left(I-\mathrm{K}_\alpha\right)\left(1+e^{4\beta t}\right) + 4\beta e^{4\beta t}\left(-t\left(I-\mathrm{K}_\alpha\right) + \log f(x)\right)\right]\pi_\alpha(x)$$

$$= \sum_x \exp\left[p(t)\log\, H_{\alpha,t} f(x)\right] \cdot$$

$$\left[-(I - \mathrm{K}_\alpha)p(t) + \frac{p'(t)}{p(t)} p(t) \log H_{\alpha,t} f(x)\right] \pi_\alpha(x)$$

$$= \sum_x \left(H_{\alpha,t} f(x)\right)^{p(t)} \left[-(I - \mathrm{K}_\alpha)p(t) + \frac{p'(t)}{p(t)} \log\left(H_{\alpha,t} f(x)\right)^{p(t)}\right] \pi_\alpha(x)$$

$$= -p(t) \sum_x (I - \mathrm{K}_\alpha)\left(H_{\alpha,t} f(x)\right)\left(H_{\alpha,t} f(x)\right)^{p(t)-1} \pi_\alpha(x)$$

$$+ \frac{p'(t)}{p(t)} \sum_x \left(H_{\alpha,t} f(x)\right)^{p(t)} \log\left(H_{\alpha,t} f(x)\right)^{p(t)} \pi_\alpha(x)$$

$$= -p(t) \ell_\alpha \left(H_{\alpha,t} f, \left(H_{\alpha,t} f\right)^{p(t)-1}\right)$$

$$+ \frac{p'(t)}{p(t)} \sum_x \left(H_{\alpha,t} f(x)\right)^{p(t)} \log\left(H_{\alpha,t} f(x)\right)^{p(t)} \pi_\alpha(x)$$

Now

$$F'_\alpha(t) = \exp\left[\frac{\log G_\alpha(t)}{p(t)}\right] \left[\frac{p(t)\dfrac{G'_\alpha(t)}{G_\alpha(t)} - p'(t)\log G_\alpha(t)}{p^2(t)}\right]$$

$$= F_\alpha(t)\left[\frac{-p'(t)\log G_\alpha(t)}{p^2(t)} + \frac{G'_\alpha(t)}{G_\alpha(t)p(t)}\right]$$

$$= F_\alpha(t)\left[\frac{-p'(t)\log\|H_{\alpha,t}f\|_{\alpha,p(t)}^{p(t)}}{p^2(t)} + \frac{G'_\alpha(t)}{(F_\alpha(t))^{p(t)}p(t)}\right]$$

$$= (F_\alpha(t))^{-p(t)+1}\left[\begin{array}{l} \dfrac{-p'(t)\|H_{\alpha,t}f\|_{\alpha,p(t)}^{p(t)}\log\|H_{\alpha,t}f\|_{\alpha,p(t)}^{p(t)}}{p^2(t)} \\ +\dfrac{p'(t)}{p^2(t)}\sum_x |H_{\alpha,t}f(x)|^{p(t)}\log|H_{\alpha,t}f(x)|^{p(t)}\pi_\alpha(x) \\ \qquad -\ell_\alpha\left(H_{\alpha,t}f,(H_{\alpha,t}f)^{p(t)-1}\right) \end{array}\right]$$

$$= (F_\alpha(t))^{-p(t)+1}\left[\begin{array}{l} \dfrac{p'(t)}{p^2(t)}\sum_x |H_{\alpha,t}f(x)|^{p(t)}\log\dfrac{|H_{\alpha,t}f(x)|^{p(t)}}{\|H_{\alpha,t}f\|_{\alpha,p(t)}^{p(t)}}\pi_\alpha(x) \\ -\ell_\alpha\left(H_{\alpha,t}f,(H_{\alpha,t}f)^{p(t)-1}\right) \end{array}\right]$$

$$= (F_\alpha(t))^{-p(t)+1}\left[\frac{p'(t)}{p^2(t)}\zeta_\alpha(H_{\alpha,t}f) - \ell_\alpha\left(H_{\alpha,t}f,(H_{\alpha,t}f)^{p(t)-1}\right)\right]$$

Since $\|H_{\alpha,t}f\|_{\alpha,p(t)} < \|f\|_{\alpha,2}$, $H_0 f = f$ and $p(0) = 2$, the derivative of $F_\alpha(t)$ at $t = 0$ must be negative , so

$$\beta\zeta_\alpha(f) \le \ell_\alpha(f,f)$$

That is

$$\beta\,\overline{\zeta}(f) \le \overline{\ell}(f,f) .$$

$ii)$ We assume that at degree α , (k_α,π_α) is reversible and satisfy the log-Sobolev inequality

$$\wp\,\zeta_\alpha(f)\le \ell_\alpha(f,f)$$

For $f\ge 0$, lemma 4.1.11 gives

$$\wp\,\zeta_{p,\alpha}(f)\le \ell_\alpha\left(f^{p/2},f^{p/2}\right)\le \frac{p^2}{4(p-1)}\ell_\alpha\left(f,f^{p-1}\right)$$

For any $1<p<\infty$.

If $p(t)=1+e^{4\wp t}$, then $p'(t)=4\wp(p(t)-1)$, and replacing f by $H_{\alpha,t}f$, we obtain

$$\frac{p'(t)}{p^2(t)}\zeta_{p(t),\alpha}(H_{\alpha,t}f)-\ell_\alpha\left(H_{\alpha,t}f,(H_{\alpha,t}f)^{p(t)-1}\right)\le 0$$

However , we use as in (i), the notation

$F_\alpha=\|H_{\alpha,\,t}f\|_{\alpha,\,p(t)}$ and then $F_\alpha{}'(t)\le 0$ for all $t\ge 0$.

Since $F_\alpha(0)=\|f\|_{\alpha,2}$, this implies that

$$\|H_{\alpha,t}f\|_{\alpha,\,p(t)}<\|f\|_{\alpha,2}$$

Thus

$$\overline{\left\|\overline{H}_t\ f\right\|}_{2\to p(t)}<\overline{\|f\|}_2$$

By proposition 5.2.2 we have

$$\overline{\left\|\overline{H}_t\right\|}_{2\to p(t)}<1\ .$$

$iii)$ We have at degree α , by lemma 4.1.11

$$\ell_\alpha\left(f^{p/2}, f^{p/2}\right) \le \frac{p}{2}\ell_\alpha\left(f, f^{p-1}\right) \le \frac{p^2}{2(p-1)}\ell_\alpha\left(f, f^{p-1}\right)$$

For all $p \ge 2$ .We set $p(t) = 1 + e^{2\wp t}$, $p'(t) = 2\wp(p(t) - 1)$

and replacing f by $H_{\alpha,t} f$, we obtain

$$\frac{p'(t)}{p^2(t)}\zeta_{p(t),\alpha}\left(H_{\alpha,t} f\right) - \ell_\alpha\left(H_{\alpha,t} f, \left(H_{\alpha,t} f\right)^{p(t)-1}\right) \le 0 .$$

However , we use as in (i), the notation

$F_\alpha = \left\|H_{\alpha,t} f\right\|_{\alpha,p(t)}$ and then $F_\alpha{}'(t) \le 0$ for all $t \ge 0$.

Since $F_\alpha(0) = \|f\|_{\alpha,2}$, so

$$\left\|H_{\alpha,t} f\right\|_{\alpha,p(t)} < \|f\|_{\alpha,2}$$

Thus

$$\overline{\left\|\overline{H}_t\ f\right\|}_{2\to p(t)} < \overline{\|f\|}_2$$

Hence

$$\overline{\left\|\overline{H}_t\right\|}_{2\to p(t)} < 1 \quad \blacksquare$$

5.3. Ergodicity of the Fuzzy Markov Chain .

We use the fuzzy log-Sobolev constant to discuses ergodicity through fuzzy entropy .

Proposition 5.3.1

If we have $\overline{\mu} = f^2 \, \overline{\pi}$ for a function $f \geq 0$, $\|f\|_2 = 1$, then

$$\overline{\zeta}(f) = \overline{Ent}\left(\overline{\mu}\right).$$

Proof :- We have at degree α

$$\begin{aligned}
\zeta_\alpha(f) &= \sum_x |f(x)|^2 \log \frac{|f(x)|^2}{\|f\|_2^2} \pi_\alpha(x) \\
&= \sum_x f^2(x) \log f^2(x) \pi_\alpha(x) \\
&= \sum_x \frac{\mu_\alpha(x)}{\pi_\alpha(x)} \log \frac{\mu_\alpha(x)}{\pi_\alpha(x)} \pi_\alpha(x) \\
&= \sum_x \mu_\alpha(x) \log \frac{\mu_\alpha(x)}{\pi_\alpha(x)} \\
&= Ent_\alpha(\mu_\alpha)
\end{aligned}$$

Since $\zeta_\alpha \in \overline{\zeta}[\alpha]$ of $\overline{\zeta}$, and $Ent_\alpha(\mu_\alpha) \in \overline{Ent}\left(\overline{\mu}\right)[\alpha]$ of $\overline{Ent}\left(\overline{\mu}\right)$,

then

$$\overline{\zeta}(f) = \overline{Ent}\left(\overline{\mu}\right) \blacksquare$$

Proposition 5.3.2

If $\overline{H}_t^{\,*} f$ is the density of $\overline{\mu}\,\overline{H}_t$, with respect to $\overline{\pi}$. Then

$$\overline{Ent}\left(\overline{\mu}\,\overline{H}_t\right)=\overline{Ent}\left(\overline{H}_t^{\,*} f\right)$$

Proof :- We have at degree α , by proposition 2.41

$$Ent_\alpha\left(H^*_{\alpha,t} f\right)=\sum_x H^*_{\alpha,t} f(x)\log\left(H^*_{\alpha,t} f(x)\right)\pi_\alpha(x)$$

$$=\sum_x \frac{\mu_\alpha H_{\alpha,t}(x)}{\pi_\alpha(x)}\log\left(\frac{\mu_\alpha H_{\alpha,t}(x)}{\pi_\alpha(x)}\right)\pi_\alpha(x)$$

$$=\sum_x \mu_\alpha H_{\alpha,t}(x)\log\left(\frac{\mu_\alpha H_{\alpha,t}(x)}{\pi_\alpha(x)}\right)$$

$$=Ent_\alpha\left(\mu_\alpha H_{\alpha,t}\right)$$

Thus

$$\overline{Ent}\left(\overline{\mu}\,\overline{H}_t\right)=\overline{Ent}\left(\overline{H}_t^{\,*} f\right) \quad \blacksquare$$

Theorem 5.3.3

Let $\overline{k}$ be a fuzzy finite Markov chain with invariant fuzzy probability measure $\overline{\pi}$ and fuzzy log-Sobolev constant $\overline{\wp}$. Then for any fuzzy probability measure $\overline{\mu}=f\,\overline{\pi}$ on a state space Ω , we have

$$\overline{Ent}\left(\overline{\mu}\,\overline{H}_t\right)\le e^{-2\overline{\wp}t}\,\overline{Ent}\left(\overline{\mu}\right),t>0$$

Further , if we assume that $\left(\overline{k},\overline{\pi}\right)$ is reversible , then

$$\overline{Ent}\left(\overline{\mu}\,\overline{H}_t\right)\le e^{-4\overline{\wp}t}\ \overline{Ent}\left(\overline{\mu}\right),t>0\cdot$$

Proof :- $i)$ We have at degree α , by proposition 5.3.2

$$Ent_\alpha\left(H^*_{\alpha,t}f\right)=Ent_\alpha\left(\mu_\alpha H_{\alpha,t}\right)$$

Lemma 4.1.10 gives

$$\partial_t Ent_\alpha\left(H^*_{\alpha,t}f\right)=-\ell\left(H^*_{\alpha,t}f,\log H^*_{\alpha,t}f\right)$$

Lemma 4.1.12 gives

$$\ell_\alpha\left(H^*_{\alpha,t}f,\log H^*_{\alpha,t}f\right)\ge 2\ell_\alpha\left(\left(H^*_{\alpha,t}f\right)^{1/2},\left(H^*_{\alpha,t}f\right)^{1/2}\right)$$

By log-Sobolev inequality we have

$$\wp_\alpha\zeta_\alpha\left(\left(H^*_{\alpha,t}f\right)^{1/2}\right)\le\ell_\alpha\left(\left(H^*_{\alpha,t}f\right)^{1/2},\left(H^*_{\alpha,t}f\right)^{1/2}\right)$$

And

$$\zeta_\alpha\left(\left(H^*_{\alpha,t}f\right)^{1/2}\right)=\sum_x\left|H^*_{\alpha,t}f(x)\right|\log\frac{\left|H^*_{\alpha,t}f(x)\right|}{\left\|\left(H^*_{\alpha,t}f\right)^{1/2}\right\|^2_{\alpha,2}}\pi_\alpha(x)$$

Since

$$\begin{aligned}\left\|\left(H^*_{\alpha,t}f\right)^{1/2}\right\|^2_{\alpha,2}&=\left\|H^*_{\alpha,t}f\right\|_{\alpha,1}\\&\le\left\|H^*_{\alpha,t}f\right\|_{\alpha,2}\\&\le\left\|H^*_{\alpha,t}\right\|_{\alpha,2\to2}\left\|f\right\|_{\alpha,2}\end{aligned}$$

By theorem 5.2.5-(iii) and $H^{*}_{\alpha,t}$ is adjoint of $H_{\alpha,t}$, we have

$$\left\| H^{*}_{\alpha,t} \right\|_{\alpha,2\to 2} < 1$$

So

$$\left\| \left(H^{*}_{\alpha,t} f \right)^{1/2} \right\|^{2}_{\alpha,t} < 1$$

Hence

$$\zeta_{\alpha}\left(\left(H^{*}_{\alpha,t} f \right)^{1/2} \right) \geq Ent_{\alpha}\left(H^{*}_{\alpha,t} f \right)$$

And

$$\begin{aligned} \partial_t Ent_{\alpha}\left(H^{*}_{\alpha,t} f \right) &\leq -2\ell_{\alpha}\left(\left(H^{*}_{\alpha,t} f \right)^{1/2}, \left(H^{*}_{\alpha,t} f \right)^{1/2} \right) \\ &\leq -2\wp_{\alpha}\zeta_{\alpha}\left(\left(H^{*}_{\alpha,t} f \right)^{1/2} \right) \\ &\leq -2\wp_{\alpha} Ent_{\alpha}\left(H^{*}_{\alpha,t} f \right) \end{aligned}$$

By the following from[16]

$$e^{-2\wp_{\alpha} t} Ent_{\alpha}(f) \leq Ent_{\alpha}\left(H^{*}_{\alpha,t} f \right)$$

We have

$$\partial_t Ent_{\alpha}\left(H^{*}_{\alpha,t} f \right) \leq -2\wp_{\alpha} e^{-2\wp_{\alpha} t} Ent_{\alpha}(f)$$

Integrating both two side with respect to t we get

$$Ent_{\alpha}\left(H^{*}_{\alpha,t} f \right) \leq e^{-2\wp_{\alpha} t} Ent_{\alpha}(f)$$

So

$$Ent_{\alpha}\left(\mu_{\alpha} H_{\alpha,t} \right) \leq e^{-2\wp_{\alpha} t} Ent_{\alpha}(\mu_{\alpha})$$

Therefore

$$\overline{Ent}\left(\overline{\mu}\, \overline{H}_t \right) \leq e^{-2\overline{\wp} t}\, \overline{Ent}\left(\overline{\mu} \right), t > 0 \, .$$

$ii)$ At degree α , we have by lemma 4.1.10 and 4.1.12 that

$$\partial_t Ent_\alpha\left(H^*_{\alpha,t} f\right) \le -4\wp_\alpha Ent_\alpha\left(H^*_{\alpha,t} f\right)$$

By the following from [21]

$$e^{-4\wp_\alpha t} Ent_\alpha(f) \le Ent_\alpha\left(H^*_{\alpha,t} f\right)$$

We have

$$\partial_t Ent_\alpha\left(H^*_{\alpha,t} f\right) \le -4\wp_\alpha e^{-4\wp_\alpha t} Ent_\alpha(f)$$

Integrating both two side with respect to t we get

$$Ent_\alpha\left(H^*_{\alpha,t} f\right) \le e^{-4\wp_\alpha t} Ent_\alpha(f)$$

So

$$Ent_\alpha\left(\mu_\alpha H_{\alpha,t}\right) \le e^{-4\wp_\alpha t} Ent_\alpha(\mu_\alpha)$$

Therefore

$$\overline{Ent}\left(\overline{\mu}\,\overline{H}_t\right) \le e^{-4\overline{\wp} t}\,\overline{Ent}\left(\overline{\mu}\right), t > 0 \quad \blacksquare$$

Corollary 5.3.4

Let $\left(\overline{k}, \overline{\pi}\right)$ be a fuzzy finite Markov chain , and $\overline{\wp}$ be a fuzzy log-Sobolev constant we have

$$2\left\|\overline{H}_t^x - \overline{\pi}\right\|_{TV}^2 \le \left(\log \frac{1}{\overline{\pi}(x)}\right) e^{-2\overline{\wp} t}$$

If we assume that $\left(\overline{k}, \overline{\pi}\right)$ is reversible then

$$2\left\|\overline{H}_t^x - \overline{\pi}\right\|_{TV}^2 \leq \left(\log \frac{1}{\overline{\pi}(x)}\right) e^{-4\overline{\wp} t}$$

Proof :-For the first inequality at degree α by the following from [6]

$$2\left\|H_{\alpha,t}^x - \pi_\alpha\right\|_{TV}^2 \leq Ent_\alpha\left(H_{\alpha,t}^x\right)$$

And by the following from [16]

$$Ent_\alpha\left(H_{\alpha,t}^x\right) \leq e^{-2\wp_\alpha t}\left(\log \frac{1}{\pi_\alpha(x)}\right)$$

We have

$$2\left\|H_{\alpha,t}^x - \pi_\alpha\right\|_{TV}^2 \leq \left(\log \frac{1}{\pi_\alpha(x)}\right) e^{-\wp_\alpha t}$$

Hence

$$2\left\|\overline{H}_t^x - \overline{\pi}\right\|_{TV}^2 \leq \left(\log \frac{1}{\overline{\pi}(x)}\right) e^{-2\overline{\wp} t}$$

For the second inequality at degree α by the following from [6]

$$2\left\|H_{\alpha,t}^x - \pi_\alpha\right\|_{TV}^2 \leq Ent_\alpha\left(H_{\alpha,t}^x\right)$$

And by the following from [21]

$$Ent_\alpha\left(H_{\alpha,t}^x\right) \leq e^{-4\wp_\alpha t}\left(\log \frac{1}{\pi_\alpha(x)}\right)$$

We have

$$2\left\|H_{\alpha,t}^{x}-\pi_{\alpha}\right\|_{TV}^{2}\le e^{-4\wp_{\alpha}t}\left(\log\frac{1}{\pi_{\alpha}(x)}\right)$$

Hence

$$2\left\|\overline{H}_{t}^{x}-\overline{\pi}\right\|_{TV}^{2}\le\left(\log\frac{1}{\overline{\pi}(x)}\right)e^{-4\overline{\wp}t}\ \blacksquare$$

The following corollary may be compared with corollary 5.3.4-case nonreversible

Corollary 5.3.5

Let $\left(\overline{k},\overline{\pi}\right)$ be a fuzzy finite Markov chain , then

$$4\left\|\overline{H}_{t}^{x}-\overline{\pi}\right\|_{TV}\le\frac{1}{\overline{\pi}(x)}e^{-2\overline{\lambda}t}$$

Proof :-We have at degree α

$$4\left\|H_{\alpha,t}^{x}-\pi_{\alpha}\right\|_{TV}^{2}\le\left\|h_{\alpha,t}^{x}-1\right\|_{\alpha,2}^{2}$$

By lemma 4.2.9 we have

$$\left\|h_{\alpha,t}^{x}-1\right\|_{\alpha,2}^{2}\le\frac{1}{\pi_{\alpha}(x)}e^{-2\lambda_{\alpha}t}$$

So

$$4\left\|H_{\alpha,t}^{x}-\pi_{\alpha}\right\|_{TV}^{2} \leq \frac{1}{\pi_{\alpha}(x)} e^{-2\lambda_{\alpha} t}$$

Thus

$$4\left\|\overline{H}_{t}^{x}-\overline{\pi}\right\|_{TV} \leq \frac{1}{\overline{\pi}(x)} e^{-2\overline{\lambda} t} \quad \blacksquare$$

Corollary 5.3.6

Let $\left(\overline{k},\overline{\pi}\right)$ be a fuzzy finite Markov chain , then

$$\left\|\overline{H}_{t}^{x}-\overline{\pi}\right\|_{TV} \leq \left(\log\left(\frac{1}{\overline{\pi}(x)}\right)\right)^{1/2} e^{-\overline{\wp} t}$$

Further , if $\left(\overline{k},\overline{\pi}\right)$ is reversible , then

$$\left\|\overline{H}_{t}^{x}-\overline{\pi}\right\|_{TV} \leq \left(\log\left(\frac{1}{\overline{\pi}(x)}\right)\right)^{1/2} e^{-2\overline{\wp} t}$$

Proof :- We have at degree α

$$\left\|H_{\alpha,t}^{x}-\pi_{\alpha}\right\|_{TV}^{2} \leq 2\left\|H_{\alpha,t}^{x}-\pi_{\alpha}\right\|_{TV}^{2}$$

By Corollary 5.3.4 we have

$$2\left\|H_{\alpha,t}^{x}-\pi_{\alpha}\right\|_{TV}^{2} \leq \left(\log\frac{1}{\pi_{\alpha}(x)}\right)e^{-2\wp_{\alpha}t}$$

So

$$\left\|H_{\alpha,t}^{x}-\pi_{\alpha}\right\|_{TV} \leq \left(\log\frac{1}{\pi_{\alpha}(x)}\right)^{1/2} e^{-\wp_{\alpha}t}$$

Hence

$$\left\|\overline{H}_{t}^{x}-\overline{\pi}\right\|_{TV} \leq \left(\log\left(\frac{1}{\overline{\pi}(x)}\right)\right)^{1/2} e^{-\overline{\wp}t}$$

For reversible chain , by corollary 5.3.4 we have

$$\left\|H_{\alpha,t}^{x}-\pi_{\alpha}\right\|_{TV} \leq \left(\log\frac{1}{\pi_{\alpha}(x)}\right)^{1/2} e^{-2\wp_{\alpha}t}$$

Hence

$$\left\|\overline{H}_{t}^{x}-\overline{\pi}\right\|_{TV} \leq \left(\log\left(\frac{1}{\overline{\pi}(x)}\right)\right)^{1/2} e^{-2\overline{\wp}t} \quad \blacksquare$$

Theorem 5.3.7

Let $\left(\overline{k},\overline{\pi}\right)$ be a fuzzy finite Markov chain . Assume that $\overline{\pi}(x)\leq\frac{1}{e}$, then

$$\left\|\overline{\overline{h}_t^x-1}\right\|_2\leq e^{1-\overline{\lambda}\,t}\quad , t=\frac{1}{2\wp}\log\log\frac{1}{\pi(x)}+c\,,c>0$$

For reversible fuzzy chains , the inequality holds for

$$t=(4\wp)^{-1}\log\log\frac{1}{\pi(x)}+c\ \ ,\ \ c>0$$

Proof :- At degree α , for $s>0$ set $q(s)=1+e^{2\wp s}$, theorem 5.2.5 - iii gives

$$\left\|H_{\alpha,s}\right\|_{\alpha,2\to q(s)}<1$$

Since $H^*_{\alpha,s}$ is adjoint of $H_{\alpha,s}$ then

$$\left\|H^*_{\alpha,s}\right\|_{\alpha,q'(s)\to 2}<1$$

Where $q'(s)$ is Holder conjugate of $q(s)$.

Consider the function

$$\delta_{\alpha,x}(y)=\begin{cases}\dfrac{1}{\pi_\alpha(x)} & , \ \ x=y\\ 0 & , \ \ x\neq y\end{cases}$$

We have from Lemma 4.2.9

$$\left(H_{\alpha,s}^{*}-E\right)\delta_{\alpha,x}(y)=h_{\alpha,s}(x,y)-1$$

Then

$$\begin{aligned}\left\|h_{\alpha,t+s}^{x}-1\right\|_{\alpha,2} &=\left\|\left(H_{\alpha,t+s}^{*}-E\right)\delta_{\alpha,x}\right\|_{\alpha,2}\\ &\leq\left\|H_{\alpha,s}^{*}\delta_{\alpha,x}\right\|_{\alpha,2}\left\|H_{\alpha,t}^{*}-E\right\|_{\alpha,2\to2}\\ &\leq\left\|\delta_{\alpha,x}\right\|_{\alpha,q'(s)}\left\|H_{\alpha,s}^{*}\right\|_{\alpha,q'(s)\to2}\left\|H_{\alpha,t}^{*}-E\right\|_{\alpha,2\to2}\end{aligned}$$

$$=\left(\left(\pi_{\alpha}(x)\right)^{-q'(s)}\pi_{\alpha}(x)\right)^{1/q'(s)}\left\|H_{\alpha,t}\right\|_{\alpha,2\to q(s)}\left\|H_{\alpha,t}-E\right\|_{\alpha,2\to2}$$

$$\leq\left(\pi_{\alpha}(x)\right)^{\frac{1-q'(s)}{q'(s)}}\left\|H_{\alpha,t}-E\right\|_{\alpha,2\to2}$$

By proposition 4.1.5 we have

$$\left\|h_{\alpha,t+s}^{x}-1\right\|_{\alpha,2}\leq\left(\pi_{\alpha}(x)\right)^{-1/q(s)}e^{-t\lambda_{\alpha}}$$

Choosing $s=\left(\frac{1}{2\wp}\right)\log\log\left(\frac{1}{\pi(x)}\right)$, we have $q(s)=1+\log\left(\frac{1}{\pi(x)}\right)$

So

$$\left\|h_{\alpha,t+s}^{x}-1\right\|_{\alpha,2}\leq e^{1-t\lambda_{\alpha}},\ t=\frac{1}{2\wp}\log\log\frac{1}{\pi(x)}+c,\ c>0$$

Hence

$$\overline{\left\|\overline{h_{t+s}^{x}}-1\right\|}_{2}\leq e^{1-\overline{\lambda}t},\ t=\frac{1}{2\wp}\log\log\frac{1}{\pi(x)}+c,\ c>0\,.$$

For the second part , at degree α , theorem 5.2.5-(ii) gives

$$\left\|H_{\alpha,s}\right\|_{\alpha,2\to q(s)}<1 \ , \ q(s)=1+e^{4\wp s}$$

By proposition 4.1.5 we have

$$\left\|h^{x}_{\alpha,t+s}-1\right\|_{\alpha,2}\leq\left(\pi_{\alpha}(x)\right)^{-1/q(s)}e^{-t\lambda_{\alpha}}$$

Choosing $s=\left(\frac{1}{4\wp}\right)\log\log\left(\frac{1}{\pi(x)}\right)$,we have $q(s)=1+\log\left(\frac{1}{\pi(x)}\right)$

So

$$\left\|h^{x}_{\alpha,t+s}-1\right\|_{\alpha,2}\leq e^{1-\lambda_{\alpha}t},t=\frac{1}{4\wp}\log\log\frac{1}{\pi(x)}+c,c>0$$

Therefore

$$\overline{\left\|\overline{h^{x}_{t+s}}-1\right\|_{2}}\leq e^{1-\overline{\lambda}t},t=\frac{1}{4\wp}\log\log\frac{1}{\pi(x)}+c,c>0 \ \blacksquare$$

Corollary 5.3.8

Let $\left(\overline{k},\overline{\pi}\right)$ be a fuzzy finite Markov chain , then

$$\left\|\overline{H}^{x}_{t}-\overline{\pi}\right\|_{TV}\leq e^{1-2\overline{\wp}t} \quad , \ t=\frac{1}{2\wp}\log\log\frac{1}{\pi(x)}+c,c>0$$

For reversible chains ,the inequality holds for

$$t=(4\wp)^{-1}\log\log\frac{1}{\pi(x)}+c \ , \ c>0$$

Proof :- We have at degree α

$$\left\| H_{\alpha,t}^{x} - \pi_{\alpha} \right\|_{TV} \le \left\| h_{\alpha,t}^{x} - 1 \right\|_{\alpha,2}$$

By theorem 5.3.7 we have

$$\left\| H_{\alpha,t}^{x} - \pi_{\alpha} \right\|_{TV} \le e^{1-\lambda_{\alpha} t} \quad , \quad t = \frac{1}{2\wp} \log\log \frac{1}{\pi(x)} + c \ , \ c > 0$$

By lemma 5.1.5 we have $2\wp_{\alpha} \le \lambda_{\alpha}$, so

$$\left\| H_{\alpha,t}^{x} - \pi_{\alpha} \right\|_{TV} \le e^{1-2\wp_{\alpha} t} \quad , \quad t = \frac{1}{2\wp} \log\log \frac{1}{\pi(x)} + c , c > 0$$

Then

$$\left\| \overline{H}_{t}^{x} - \overline{\pi} \right\|_{TV} \le e^{1-2\overline{\wp} t} \quad , \quad t = \frac{1}{2\wp} \log\log \frac{1}{\pi(x)} + c , c > 0$$

For reversible chain , we have at degree α

$$\left\| H_{\alpha,t}^{x} - \pi_{\alpha} \right\|_{TV} \le \left\| h_{\alpha,t}^{x} - 1 \right\|_{\alpha,2}$$

By theorem 5.3.7 and lemma 5.1.5 we have

$$\left\| H_{\alpha,t}^{x} - \pi_{\alpha} \right\|_{TV} \le e^{1-2\wp_{\alpha} t} \quad , \quad t = (4\wp)^{-1} \log\log \frac{1}{\pi(x)} + c \ , \ c > 0$$

Hence

$$\left\| \overline{H}_{t}^{x} - \overline{\pi} \right\|_{TV} \le e^{1-2\overline{\wp} t} \quad , \quad t = (4\wp)^{-1} \log\log \frac{1}{\pi(x)} + c \ , \ c > 0 \quad \blacksquare$$

Corollary 5.3.9

Let $\left(\overline{k},\overline{\pi}\right)$ be a fuzzy finite Markov chain , then the maximal relative error is

$$\sup_{x,\,y}\left|\overline{h}_t(x,y)-1\right|\le e^{2-\left(\overline{\lambda}+\overline{\lambda}^{*}\right)t}$$

For

$$t=\frac{1}{2\wp}\log\log\frac{1}{\pi_*(x)}+c\ ,\pi_*(x)=\min\pi(x),c>0$$

Further , if $\left(\overline{k},\overline{\pi}\right)$ is reversible , then

$$\sup_{x,y}\left|\overline{h}_t(x,y)-1\right|\le e^{2\left(1-\overline{\lambda}\,t\right)}$$

For

$$t=(4\wp)^{-1}\log\log\frac{1}{\pi_*(x)}+c,\pi_*(x)=\min\pi(x),c>0$$

Proof :- We have from [6]

$$\sup_{x,y}\left|\overline{h}_t(x,y)-1\right|\le\left(\sup_x\overline{\left\|\overline{h}^{x}_{\frac{t}{2}}-1\right\|}_2\right)\left(\sup_x\overline{\left\|\overline{h}^{x\,*}_{\frac{t}{2}}-1\right\|}_2\right)$$

By theorem 5.3.7 we have

$$\sup_{x,y}\left|\overline{h}_t(x,y)-1\right| \le \left(e^{1-\overline{\lambda}\,t}\right)\left(e^{1-\overline{\lambda}^{*}\,t}\right)$$

$$\le e^{2-\left(\overline{\lambda}+\overline{\lambda}^{*}\right)t}$$

For $t=\frac{1}{2\wp}\log\log\frac{1}{\pi_*(x)}+c\,,\pi_*(x)=\min \pi(x)\,,\ c>0$

For reversible chain , by proposition 4.2.7 and remark 4.2.8 we have

$$\sup_{x,y}\left|\overline{h}_t(x,y)-1\right| = \sup_{x}\overline{\left\|\overline{h}^{x}_{\frac{t}{2}}-1\right\|_2^2}$$

By theorem 5.3.7 we have

$$\sup_{x,y}\left|\overline{h}_t(x,y)-1\right| \le e^{2\left(1-\overline{\lambda}\,t\right)}$$

For

$$t=(4\wp)^{-1}\log\log\frac{1}{\pi_*(x)}+c\ ,\pi_*(x)=\min \pi(x)\,,\ c>0 \quad \blacksquare$$

Corollary 5.3.10

Assume that $\left(\overline{k},\overline{\pi}\right)$ is reversible chain and $\overline{\pi}(x)\le\frac{1}{e}$. Set $\lambda_*=\min\{\lambda,1+\beta_{\min}\}$ then

$$\left\| \overline{\kappa}^{n}_{x} - 1 \right\|_{2} \leq \left(1+2e^{2}\right)^{1/2} e^{\frac{-n}{2}\overline{\lambda}}$$

For $$n \geq \frac{1}{4\wp}\log\log\frac{1}{\pi(x)} + \frac{c}{\lambda_*} + 1\,,\; c > 0 \cdot$$

Further , setting $\pi_*(x) = \min_x \pi(x)$ we setting

$$\sup_{x,y}\left| \overline{\kappa}^{2n}(x,y) - 1 \right| \leq \left(1+2e^{2}\right)e^{-n\overline{\lambda}}$$

For $$n \geq \frac{1}{4\wp}\log\log\frac{1}{\pi_*(x)} + \frac{c}{\lambda_*} + 1, c > 0 \cdot$$

Proof :-At degree α , if we set $n = \frac{1}{2}n + \frac{1}{2}n + 1 - 1 = N - 1$

We have by corollary 3.18

$$\left\| \kappa^{n}_{\alpha,x} - 1 \right\|^{2}_{\alpha,2} \leq \beta^{2\left(\frac{n}{2}\right)}_{\alpha,-}\left(1 + \left\| h^{x}_{\alpha,\frac{n}{2}} - 1 \right\|^{2}_{\alpha,2}\right) + \left\| h^{x}_{\alpha,n} - 1 \right\|^{2}_{\alpha,2}$$

$$= \beta^{n}_{\alpha,-}\left(1 + \left\| h^{x}_{\alpha,\frac{n}{2}} - 1 \right\|^{2}_{\alpha,2}\right) + \left\| h^{x}_{\alpha,N-1} - 1 \right\|^{2}_{\alpha,2}$$

By theorem 5.3.7 we have

$$\left\| \kappa^{n}_{\alpha,x} - 1 \right\|^{2}_{\alpha,2} \leq e^{-n\lambda_\alpha}\left(1 + e^{2\left(1-\frac{n}{2}\lambda_\alpha\right)}\right) + e^{2\left(1-(N-1)\lambda_\alpha\right)}$$

$$= e^{-n\lambda_\alpha}\left(1 + e^{2-n\lambda_\alpha}\right) + e^{2\left(1-\frac{n}{2}\lambda_\alpha - \frac{n}{2}\lambda_\alpha\right)}$$

$$\leq e^{-n\lambda_\alpha}\left(1 + e^{2}\right) + e^{2-n\lambda_\alpha}$$

$$= e^{-n\lambda_\alpha}\left(1 + 2e^{2}\right)$$

So

$$\left\|\kappa_{\alpha,x}^{n}-1\right\|_{\alpha,2}\leq e^{\frac{-n}{2}\lambda_{\alpha}}\left(1+2e^{2}\right)^{1/2}$$

Therefore

$$\left\|\overline{\overline{\kappa}^{n}{}_{x}-1}\right\|_{2}\leq\left(1+2e^{2}\right)^{1/2}e^{\frac{-n}{2}\overline{\lambda}}$$

For $\quad n\geq\frac{1}{4\wp}\log\log\frac{1}{\pi\ (x)}+\frac{c}{\lambda_{*}}+1, c>0\cdot$

For the second part by proposition 4.2.7 we have

$$\sup_{x,y}\left|\overline{\kappa}^{2n}(x,y)-1\right|=\sup_{x}\left\|\overline{\overline{\kappa}^{n}{}_{x}-1}\right\|_{2}^{2}$$

$$\leq\left(1+2e^{2}\right)e^{-n\overline{\lambda}}$$

For $\quad n\geq\frac{1}{4\wp}\log\log\frac{1}{\pi_{*}(x)}+\frac{c}{\lambda_{*}}+1\ , c>0\ \blacksquare$

Conclusions:

From our work we conclude that

1) Let $\left(\overline{k},\overline{\pi}\right)$ be a fuzzy finite Markov chain , and $\overline{\wp}$ be a fuzzy log-Sobolev constant we have

$$2\left\|\overline{H}_t^x-\overline{\pi}\right\|_{TV}^2 \le \left(\log\frac{1}{\overline{\pi}(x)}\right)e^{-2\overline{\wp}t}$$

If we assume that $\left(\overline{k},\overline{\pi}\right)$ is reversible then

$$2\left\|\overline{H}_t^x-\overline{\pi}\right\|_{TV}^2 \le \left(\log\frac{1}{\overline{\pi}(x)}\right)e^{-4\overline{\wp}t}$$

2) Let $\left(\overline{k},\overline{\pi}\right)$ be a fuzzy finite Markov chain , then

$$4\left\|\overline{H}_t^x-\overline{\pi}\right\|_{TV}^2 \le \frac{1}{\overline{\pi}(x)}e^{-2\overline{\lambda}t}$$

3) Let $\left(\overline{k},\overline{\pi}\right)$ be a fuzzy finite Markov chain , then

$$\left\|\overline{H}_t^x-\overline{\pi}\right\|_{TV} \le \left(\log\left(\frac{1}{\overline{\pi}(x)}\right)\right)^{1/2} e^{-\overline{\wp}t}$$

Further , if $\left(\overline{k}, \overline{\pi}\right)$ is reversible , then

$$\left\| \overline{H}_t^x - \overline{\pi} \right\|_{TV} \leq \left(\log\left(\frac{1}{\overline{\pi}(x)} \right) \right)^{1/2} e^{-2\overline{\wp}t}$$

4) Let $\left(\overline{k}, \overline{\pi}\right)$ be a fuzzy finite Markov chain , then

$$\left\| \overline{H}_t^x - \overline{\pi} \right\|_{TV} \leq e^{1-2\overline{\wp}t} \text{ , } t = \frac{1}{2\wp} \log\log \frac{1}{\pi(x)} + c,\ c > 0$$

For reversible chains , the inequality holds for

$$t = (4\wp)^{-1} \log\log \frac{1}{\pi(x)} + c,\ c > 0$$

5) Let $\left(\overline{k}, \overline{\pi}\right)$ be a fuzzy finite Markov chain , then the maximal relative error is

$$\sup_{x,y} \left| \overline{h}_t(x,y) - 1 \right| \leq e^{2 - \left(\overline{\lambda} + \overline{\lambda}^*\right) t}$$

For

$$t = \frac{1}{2\wp} \log\log \frac{1}{\pi_*(x)} + c,\ \pi_*(x) = \min \pi(x), c > 0$$

Further , if $\left(\overline{k}, \overline{\pi}\right)$ is reversible , then

$$\sup_{x,y}\left|\overline{h}_t(x,y)-1\right| \le e^{2\left(1-\overline{\lambda}\,t\right)}$$

For

$$t=(4\wp)^{-1}\log\log\frac{1}{\pi_*(x)}+c\ ,\pi_*(x)=\min\pi(x),c>0$$

6) Assume that $\left(\overline{k}, \overline{\pi}\right)$ is reversible chain and $\overline{\pi}(x) \le \frac{1}{e}$. Set

$\lambda_* = \min\{\lambda, 1+\beta_{\min}\}$, then

$$\left\|\overline{\overline{\kappa}^{\,n}_{\ x}-1}\right\|_2 \le \left(1+2e^2\right)^{1/2} e^{\frac{-n}{2}\overline{\lambda}}$$

For $$n \ge \frac{1}{4\wp}\log\log\frac{1}{\pi\ (x)}+\frac{c}{\lambda_*}+1\ ,c>0$$

Further , setting $\pi_* = \min_x \pi(x)$ we setting

$$\sup_{x,y}\left|\overline{\kappa}^{2n}(x,y)-1\right| \le \left(1+2e^2\right)e^{-n\overline{\lambda}}$$

For $$n \ge \frac{1}{4\wp}\log\log\frac{1}{\pi_*(x)}+\frac{c}{\lambda_*}+1\ ,c>0\ \cdot$$

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